Frequency-domain representation of signals: Fourier Series and Fourier Transforms

$\Updownarrow$

Representing signals as linear combination of sinusoids

Frequency-domain representation of signals: Fourier Series and Fourier Transforms

Goal: Write signals as a linear combination of sinusoids

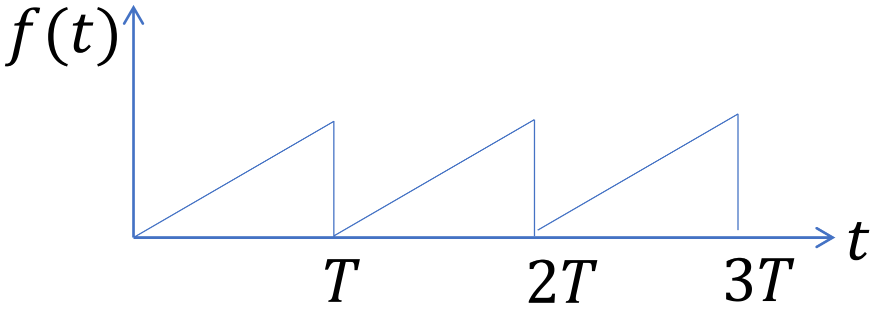
For Periodic Signals

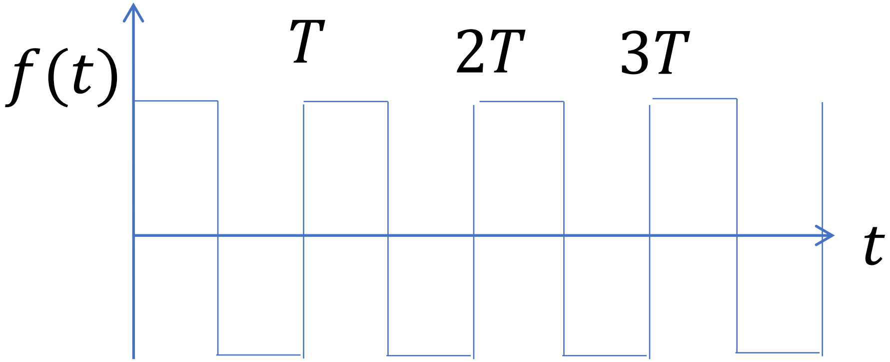
$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

For Aperiodic Signals

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Periodic Signals and Fourier Series





The general periodic signal with period $T = 2\pi/\omega$ can be represented as a linear superposition of harmonic functions [also known as Fourier Series]:

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

In other words, the claim is that periodic functions can be expressed as a linear combination of complex exponentials whose frequencies are integral multiples of the fundamental frequency ω

Periodic Signals and Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

To determine the "Fourier coefficients" a_m (includes m = 0):

Multiply both sides by $\cos(\tilde{m}\omega t)$ and integrate over one period. We get:

$$\int_0^T f(t)\cos(\tilde{m}\omega t)\,dt$$

$$= \int_0^T \frac{a_0}{2}\cos(\tilde{m}\omega t)\,dt + \sum_{m=1}^{\infty}\int_0^T a_m \cos(m\omega t)\cos(\tilde{m}\omega t)\,dt$$

$$+ \sum_{n=1}^{\infty}\int_0^T b_n \sin(n\omega t)\cos(\tilde{m}\omega t)\,dt$$

Periodic Signals and Fourier Series

$$\int_0^T \sin(n\omega t)\cos(\tilde{m}\omega t)\,dt = 0$$

cos a cos b = (1/2)[cos(a + b) + cos(a - b)]

$\cos^2\theta = (1 + \cos 2\theta) / 2$

$$\int_0^T a_m \cos(m\omega t)\cos(\tilde{m}\omega t)\,dt = 0 \quad \text{if } \tilde{m} \neq m$$

$$\int_0^T a_m \cos(m\omega t)\cos(\tilde{m}\omega t)\,dt = \frac{a_{\tilde{m}}}{2}T \quad \text{if } \tilde{m} = m$$

$$\int_0^T f(t)\cos(\tilde{m}\omega t)\,dt$$

sin(a)cos(b) = (1/2)[sin(a+b) + sin(a-b)]

$$= \int_0^T \frac{a_0}{2}\cos(\tilde{m}\omega t)\,dt + \sum_{m=1}^{\infty}\int_0^T a_m \cos(m\omega t)\cos(\tilde{m}\omega t)\,dt + \sum_{n=1}^{\infty}\int_0^T b_n \sin(n\omega t)\cos(\tilde{m}\omega t)\,dt$$

$$\int_0^\tau f(t)\cos(m\omega t)\,dt = \frac{a_m}{2}T$$

Or,

$$a_m = \frac{2}{T}\int_0^T f(t)\cos(m\omega t)\,dt$$

Periodic Signals and Fourier Series

$$a_m = \frac{2}{T}\int_0^T f(t)\cos(m\omega t)\,dt \qquad \text{for } m = 0, 1, 2 \ \ldots \ldots$$

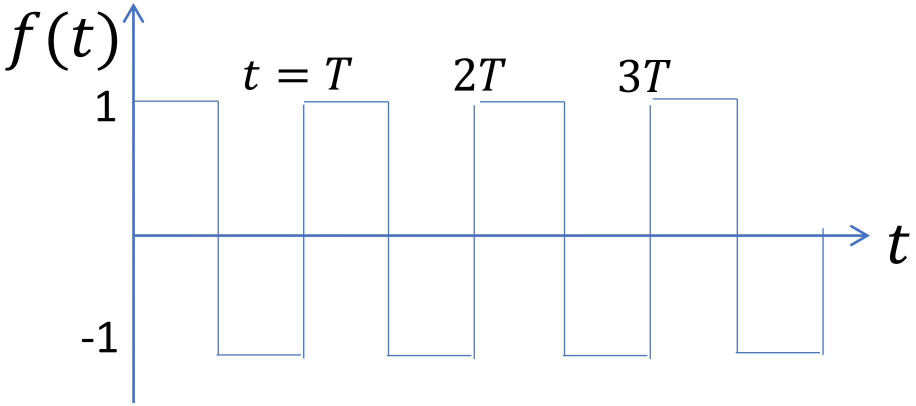
Similarly, it can be shown that

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt \qquad \text{for } n = 1, 2 \ \ldots \ldots$$

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

Periodic Signals and Fourier Series: Example

Imagine using a function generator to generate a square wave voltage signal to drive an electromechanical transducer. The force generated by the transducer would be of the form:



$$f(t) = \begin{cases} 1 & for\ 0 < t \leq T/2 \\ -1 & for\ T/2 < t \leq T \end{cases}$$

Fourier coefficients are given by:

$$a_m = \frac{2}{T}\int_0^T F(t)\cos(m\omega t)\, dt \quad \text{for } m = 0, 1, 2 \ \ldots\ldots$$

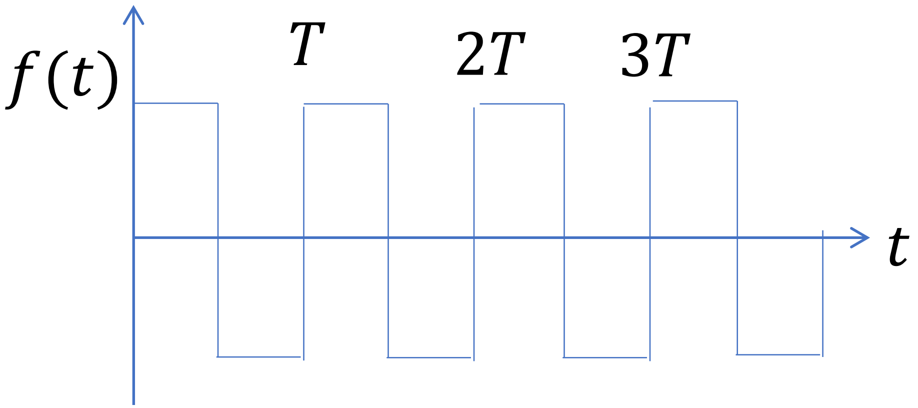
Substituting $F(t)$:

$$a_m = \frac{2}{T}\int_0^{T/2} \cos(m\omega t)\, dt - \frac{2}{T}\int_{\frac{T}{2}}^{T} \cos(m\omega t)\, dt$$

Recall: $T\omega = 2\pi$

Periodic Signals and Fourier Series: Example

$$f(t) = \begin{cases} 1 & for\ 0 < t < t/2 \\ -1 & for\ T/2 < t < T \end{cases}$$



$$a_m = \frac{2}{T}\int_0^T F(t)\cos(m\omega t)\,dt \quad \text{for } m = 0, 1, 2 \ldots\ldots$$

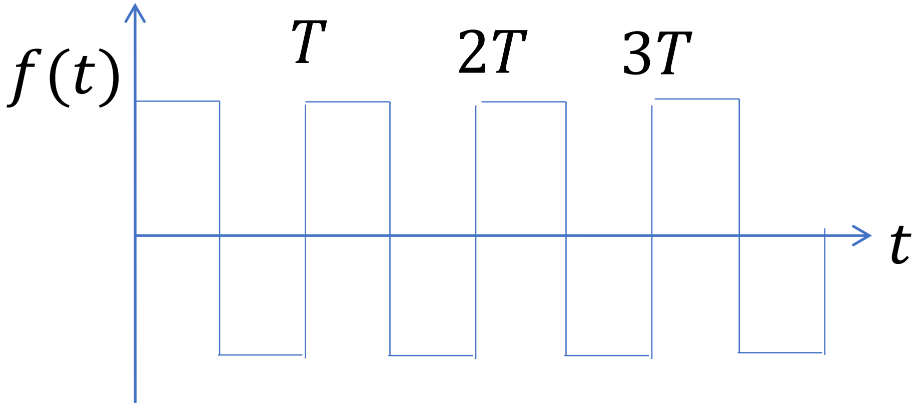
$$a_m = \frac{2}{T}\int_0^{T/2} \cos(m\omega t)\,dt - \frac{2}{T}\int_{\frac{T}{2}}^{T} \cos(m\omega t)\,dt$$

Recall: $T\omega = 2\pi$

$$a_m = \frac{2}{T(m\omega)}\left(\sin(m\omega t)\Big|_0^{\frac{T}{2}} - \sin(m\omega t)\Big|_{\frac{T}{2}}^{T}\right) = 0$$

Periodic Signals and Fourier Series: Example

$$f(t) = \begin{cases} 1 & for\; 0 < t < t/2 \\ -1 & for\; T/2 < t < T \end{cases}$$



$$b_n = \frac{2}{T}\int_0^T F(t)\sin(n\omega t)\,dt$$ for $n = 1, 2 \;\ldots\ldots$

$$b_n = \frac{2}{T}\int_0^{T/2} \sin(n\omega t)\,dt - \frac{2}{T}\int_{\frac{T}{2}}^{T} \sin(n\omega t)\,dt$$

$$= \frac{-2}{T(n\omega)}\left(\cos(n\omega t)\Big|_0^{\frac{T}{2}} - \cos(n\omega t)\Big|_{\frac{T}{2}}^{T}\right) = \frac{-2}{n\pi}(\cos(n\pi) - 1)$$

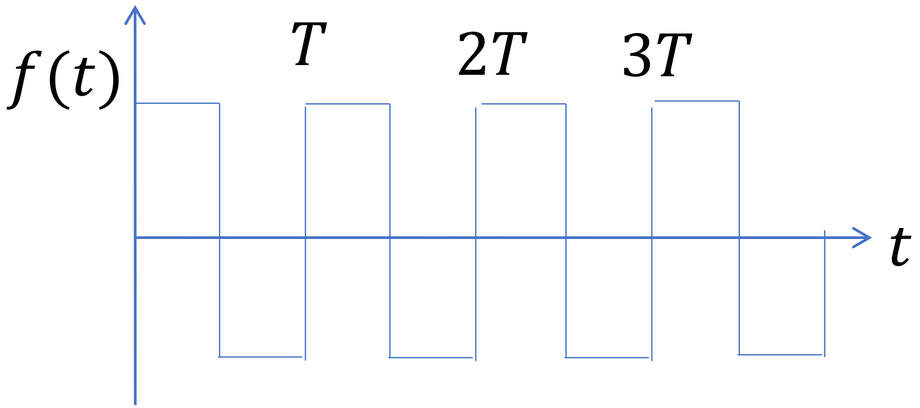
$$b_n = \begin{cases} 0 \;\; if\; n\; even \\ \dfrac{4}{n\pi}\;\; if\; n\; odd \end{cases}$$

Periodic Signals and Fourier Series: Example

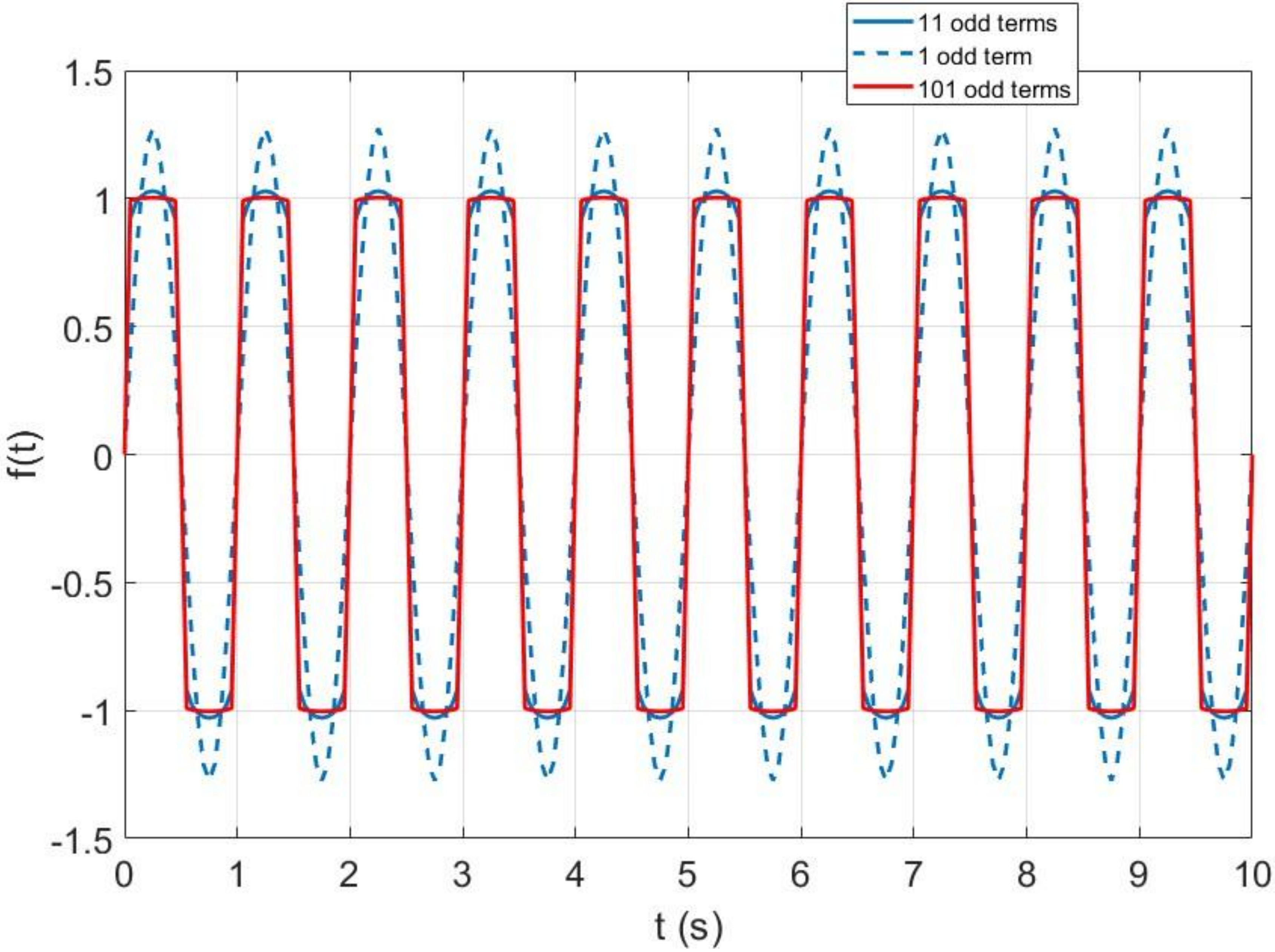
$$f(t) = \begin{cases} 1 & for\ 0 < t < T/2 \\ -1 & for\ T/2 < t < T \end{cases}$$

Fourier Series

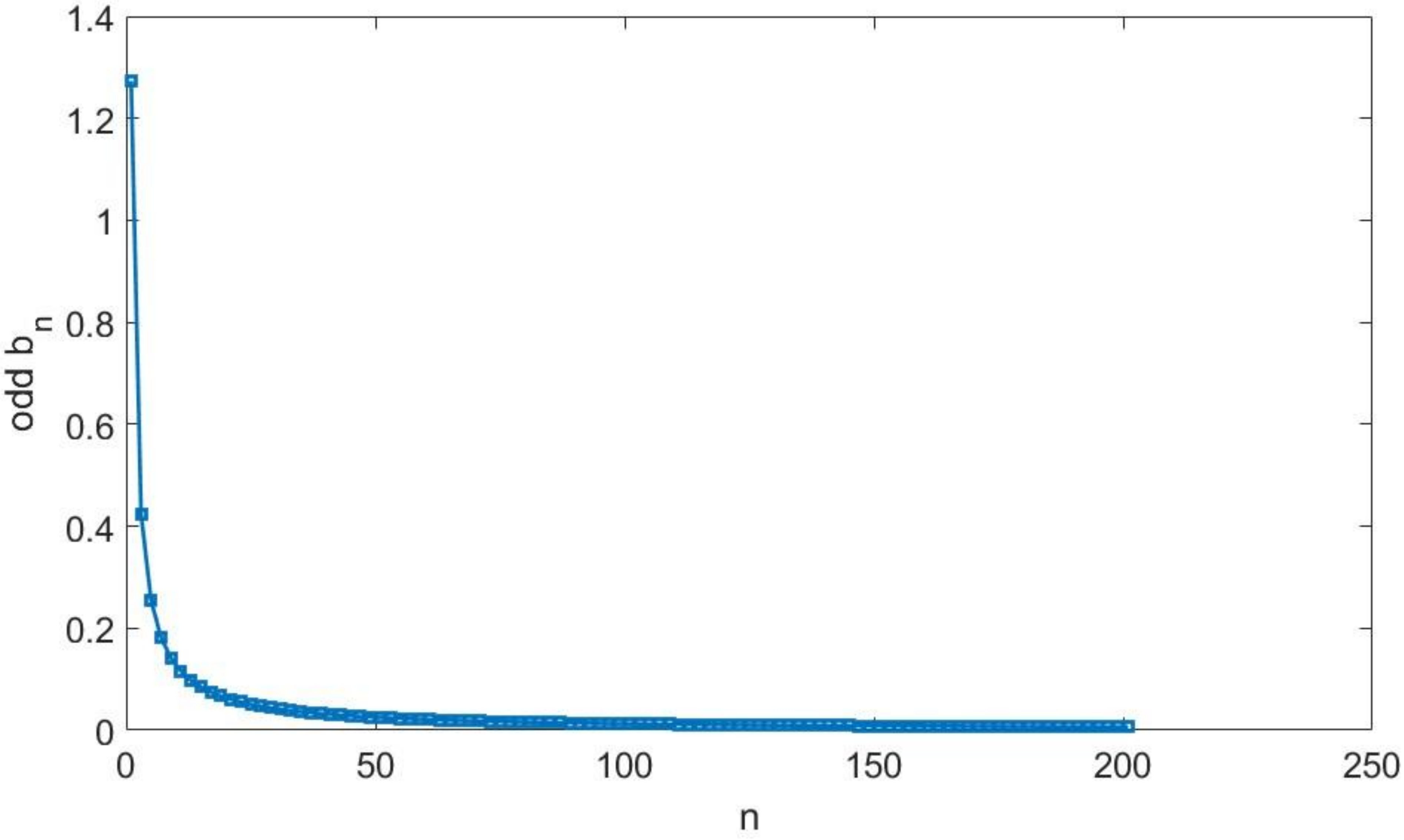
$$f(t) = \sum_{n=1,3,..(odd)}^{\infty} \frac{4}{n\pi} \sin(n\omega t)$$



Periodic Signals and Fourier Series: Example



Periodic Signals and Fourier Series: Example



Periodic Signals and Fourier Series: exponential representation

Since sine and cosine can be written in terms of exponentials, we can also rewrite the Fourier Series as:

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \left(\frac{e^{jm\omega t} + e^{-jm\omega t}}{2}\right) + \sum_{n=1}^{\infty} b_n \left(\frac{e^{jn\omega t} - e^{jn\omega t}}{2}\right)$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

where the coefficients c_k are a linear combination of coefficients a_m and b_n, and $\omega = \frac{2\pi}{T}$ with T being the time-period.

Similar to the method used to determine a_m and b_n, the coefficients c_k can be derived as

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jk\omega t}\, dt$$

Periodic Signals and Fourier Series: exponential representation

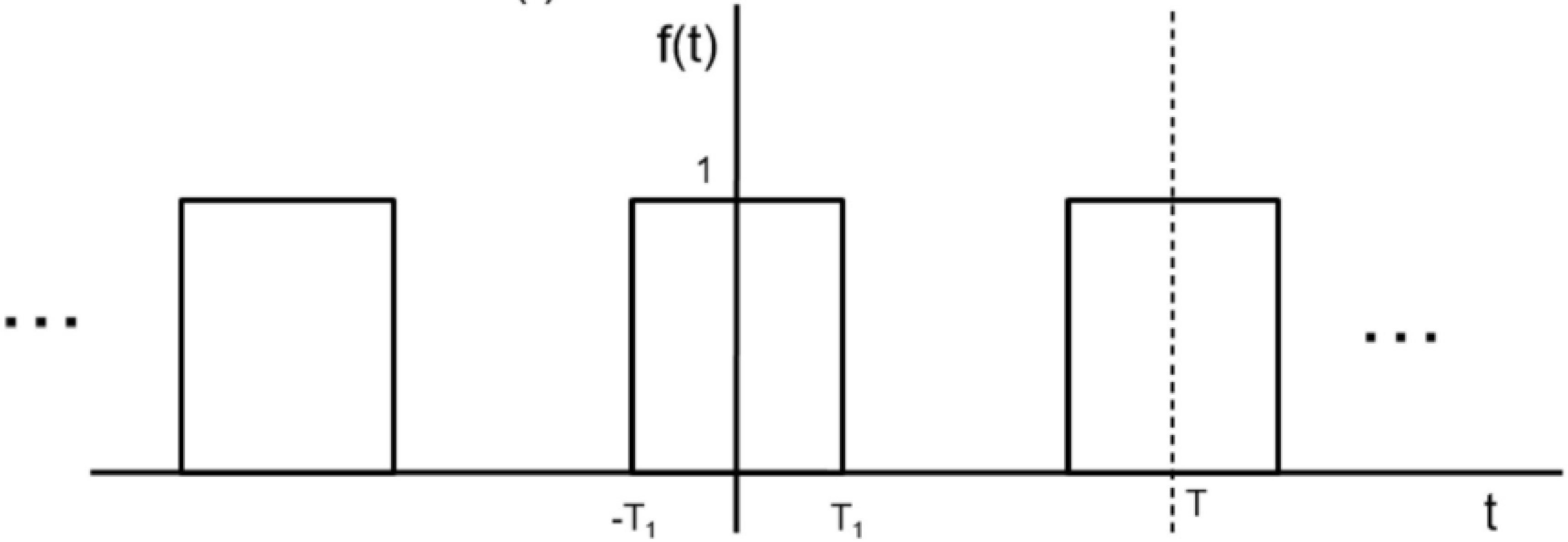
$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

- The coefficients c_k are called the Fourier Series Coefficients of $f(t)$
- c_k vs k, or c_k vs. $k\omega$ represents the frequency-domain representation of the periodic signal $f(t)$.

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jk\omega t}\, dt$$

Fourier Series (Example)

- Consider the function f(t) shown below



- The Fourier series coefficients are given by:

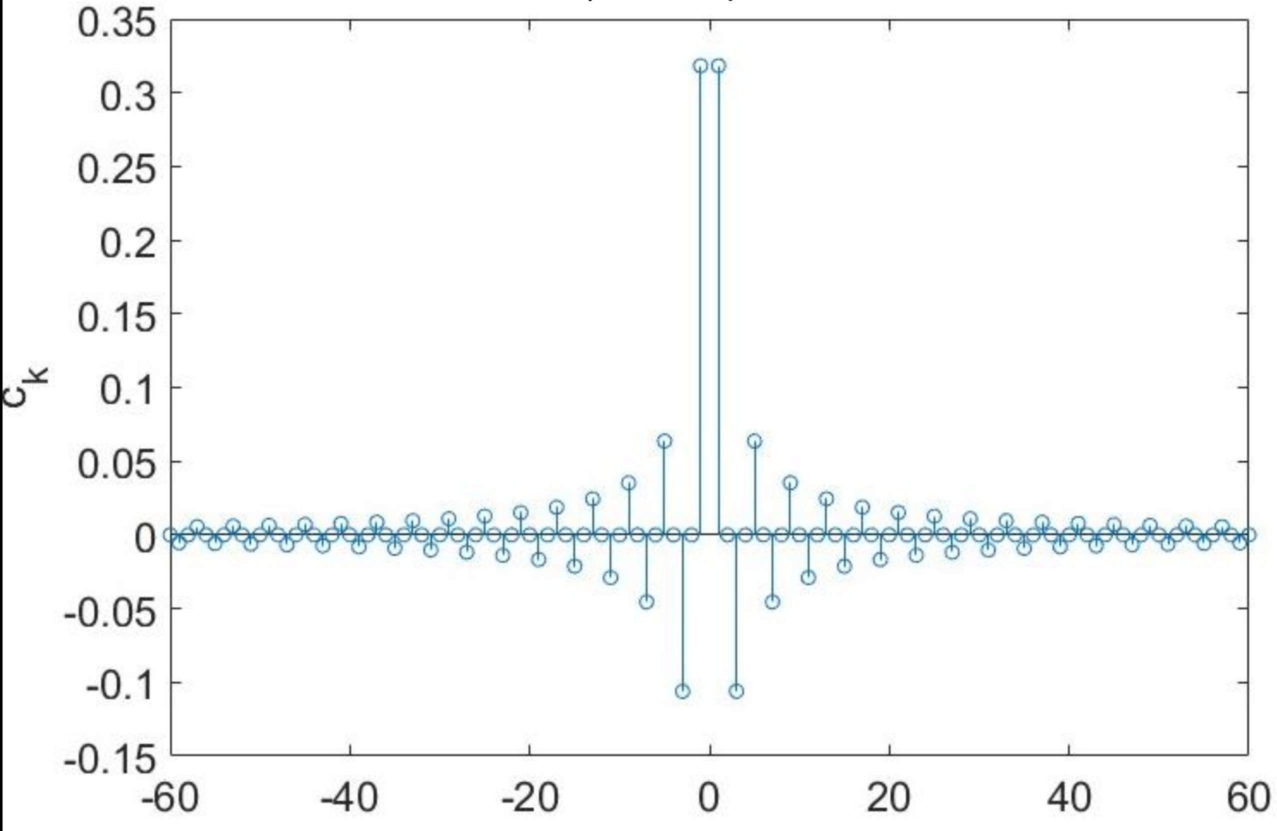
$$c_0 = \frac{2T_1}{T}$$

$$c_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = \frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{T(jk\omega_0)} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

Fourier Coefficients vs. frequency (example)

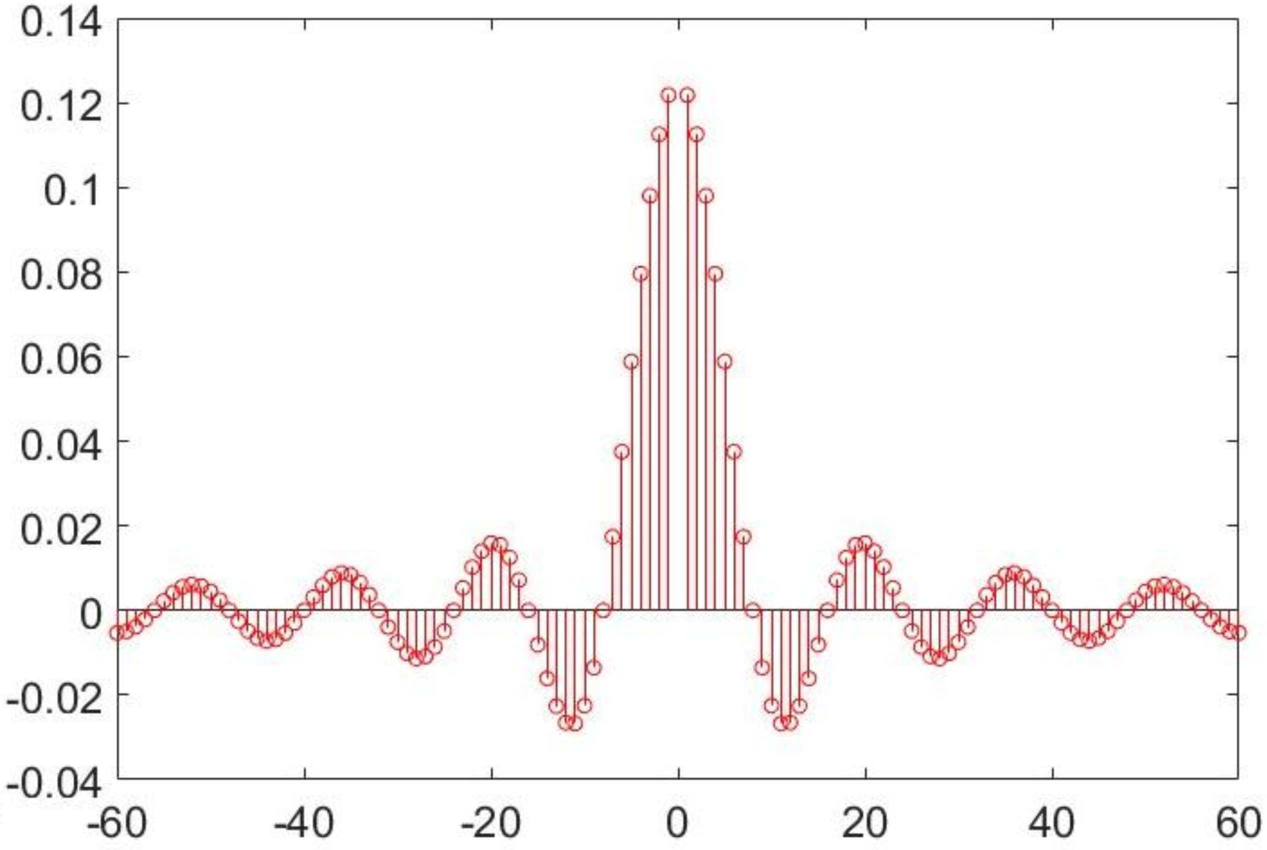
As T_1 decreases for the same period T, higher frequency components also become relevant

T=1 s, T1 = 1/4 s



$k\omega$

T=1 s, T1 = 1/16 s



$k\omega$