Sine - coupling. Eigenvalues

21.08.2019

$$\dot\varphi_j = \omega_0\left(1+\lambda_s \sin(\varphi_j-\varphi_{j-1}) + \lambda_s \sin(\varphi_j - \varphi_{j+1})\right)$$

$*$ $\varphi_j|_{t=0} = \underbrace{-\frac{2\pi k}{N} j}_{\text{k-twist}} + \underbrace{\varepsilon \sin\left(\frac{2\pi m}{N} j\right)}_{\text{perturbation with sine of m-twist}}$ (the same holds for cos)

$\Downarrow$

$$\dot\varphi_j|_{t=0} = \omega_0\left(1+\lambda_s \sin\delta_j - \lambda_s \sin(\delta_{j+1})\right) =$$

$\delta_j := \varphi_j - \varphi_{j-1}$

$$\sin\delta_j = \sin\left(-\frac{2\pi k}{N} + \varepsilon\sin\left(\frac{2\pi m}{N}j\right) - \varepsilon \sin\left(\frac{2\pi m}{N}(j-1)\right)\right) =$$

$$= \sin\left(-\frac{2\pi k}{N}\right)\cdot \underbrace{\cos(\varepsilon[..])}_{1+O(\varepsilon^2)} + \cos\frac{2\pi k}{N} \underbrace{\sin(\varepsilon[..])}_{\varepsilon[..]+O(\varepsilon^3)} =$$

$$= -\sin\left(\frac{2\pi k}{N}\right) + \cos\frac{2\pi k}{N}\cdot\varepsilon\left(\sin\left(\frac{2\pi m}{N}j\right) - \sin\left(\frac{2\pi m}{N}(j-1)\right)\right) + O(\varepsilon^2) =$$

trig expand

$$= -\sin\left(\frac{2\pi k}{N}\right) + \varepsilon\cos\frac{2\pi k}{N}\left(\sin\frac{2\pi m}{N}j\cdot\left(1-\cos\frac{2\pi m}{N}\right) + \cos\frac{2\pi m}{N}j\cdot\sin\frac{2\pi m}{N}\right) + O(\varepsilon^2)$$

$$\sin\delta_{j+1} = -\sin\left(\frac{2\pi k}{N}\right) + \varepsilon\cos\frac{2\pi k}{N}\left(\sin\frac{2\pi m}{N}(j+1) - \sin\left(\frac{2\pi m}{N}j\right)\right) + O(\varepsilon^2) =$$

$$= -\sin\frac{2\pi k}{N} + \varepsilon\cos\frac{2\pi k}{N}\left(-\sin\left(\frac{2\pi m}{N}j\right)\left(1-\cos\frac{2\pi m}{N}\right) + \cos\frac{2\pi m}{N}j\cdot \sin\left(\frac{2\pi m}{N}\right)\right)$$

$$= \omega_0\left(1+2\lambda_s\varepsilon\cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right)\sin\left(\frac{2\pi m}{N}j\right)\right) + O(\varepsilon^2)$$

$\cancel{\tilde\lambda := 2\lambda_s\cos}$ $\boxed{A := 2\omega_0\lambda_s\cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right)}$

$\Rightarrow$ ~~If~~ $\cancel{\varphi_j}$ - If $\varphi_j = -\frac{2\pi k}{N}j + \varepsilon\sin\left(\frac{2\pi m}{N}j\right)$ (+ const independent on j)

then $\dot\varphi_j = \omega_0 + \varepsilon A \sin\left(\frac{2\pi m}{N}j\right) + O(\varepsilon^2)$ ~~$\forall\varepsilon$ suffice~~

$*$ $\varphi_j(t) = -\frac{2\pi k}{N}j + \omega_0 t + \alpha(t)\sin\left(\frac{2\pi m}{N}j\right)$

$$\dot\varphi_j = \omega_0 + \alpha(t)A\sin\left(\frac{2\pi m}{N}j\right) = \omega_0 + \dot\alpha(t)\sin\left(\frac{2\pi m}{N}j\right) + O(\alpha^2)$$

$\dot\alpha = A\alpha + O(\alpha^2)$

$\alpha|_{t=0} = \varepsilon$

$\Longrightarrow$ $\alpha(t) = \varepsilon e^{At} + O(\varepsilon^2)$; $t = O\left(\frac{1}{A}\right)$

$t = O\left(\frac{1}{\omega_0}\right)$ $\boxed{\dot\alpha \sim \varepsilon(1+\lambda)}$

T: mean phase $= 2\pi$:

$$T = \frac{2\pi}{\omega_0} \Rightarrow \alpha(T) = e^{A\frac{2\pi}{\omega_0}} = e^{\overbrace{4\pi\lambda_s \cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right)}^{\lambda}}$$

$$\varphi_j(T) - \varphi_j(0) = 2\pi + \underbrace{(e^{\lambda}-1)}_{\lambda} \cdot \varepsilon \sin\left(\frac{2\pi m}{N} j\right) + O(\underbrace{\alpha^2}_{\varepsilon^2 e^{2\lambda}})$$

[assuming $A = O(\omega_0)$]

$\lambda_s \ll 1 \Rightarrow$ ✓

$$\varphi_j(T) - \varphi_j(0) - 2\pi = \lambda \underbrace{\varepsilon \sin\frac{2\pi m}{N} j}_{\text{init perturbation}} + O(\varepsilon^2)$$

$$\boxed{\lambda = e^{4\pi\lambda_s \cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right)} - 1}$$

$$\boxed{\lambda = 4\pi\lambda_s \cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right) + O(\lambda_s^2)}$$

↑ e.value of $L - I$.

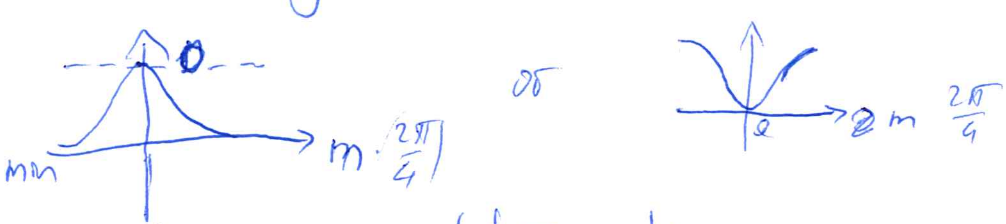
A - Lyapunov exponent.

Result (tested in simulation!)

(1) $\lambda_s < 0 \Rightarrow$ stable in vicinity of 2 twist: $|k| < \frac{N}{4}$

(2) $|k| = \frac{N}{4} \Rightarrow$ all eigenvalues are zero

(3) $\lambda(m)$:



(4) All eigenvalues are ~~st~~ real;
degeneracy $= 2$ except max and min e.v.

sin + cos eigenvalues 22.08.19

$$\dot\varphi_j = \omega_0 \big(1+\lambda_s \sin(\varphi_j-\varphi_{j-1}) + \lambda_s \sin(\varphi_j-\varphi_{j+1}) + \lambda_c \cos(\varphi_j-\varphi_{j-1}) + \lambda_c \cos(\varphi_j-\varphi_{j+1})\big)$$

Let $\varphi_j|_{t=0} = \underbrace{-\frac{2\pi k}{N} j}_{k\text{-twist}} + \underbrace{\varepsilon e^{i\frac{2\pi m}{N} j}}_{\text{perturbation}}$

$s := \sin$ $\quad \sin(\frac{2\pi m}{N} j) =: s_m^j$

$c := \cos$ $\quad \ldots\ c_m^j$

$e := \exp(i\cdot) = c + is$ $\quad \ldots\ e_m^j$

$\delta_j := \varphi_j - \varphi_{j-1}$

$$\dot\varphi_j = \omega_0(1+\lambda_s s(\delta_j) - \lambda_s s(\delta_{j+1}) + \lambda_c c(\delta_j) + \lambda_c c(\delta_{j+1}))$$

$$\delta_j|_{t=0} = \varphi_j - \varphi_{j-1} = -\frac{2\pi k}{N} + \varepsilon(e_m^j - e_m^{j-1})$$

Let $\delta_j = -\frac{2\pi k}{N} + \varepsilon \delta_{j1}$

Then expression from $\dot\varphi_j|_{t=0}$: $s(\delta_j) \cancel{- s(\delta_{j+1})} =$

$$= s(-\tfrac{2\pi k}{N}) c(\varepsilon \delta_{j1}) + c(\tfrac{2\pi k}{N}) s(\varepsilon\delta_{j1}) =$$

$$= -s(\tfrac{2\pi k}{N})(1+O(\varepsilon^2)) + c(\tfrac{2\pi k}{N})(\varepsilon\delta_{j1} + O(\varepsilon^3)) =$$

$$= -s_k^1 + \varepsilon c_k^1 \delta_{j1} + O(\varepsilon^2)$$

$$s(\delta_j) - s(\delta_{j+1}) = \varepsilon c_k^1 (\delta_{j1} - \delta_{j+1\,1}) + O(\varepsilon^2)$$

$$c(\delta_j) = c(-\tfrac{2\pi k}{N} + \varepsilon\delta_{j1}) = c(\tfrac{2\pi k}{N}) c(\varepsilon\delta_{j1}) - s(-\tfrac{2\pi k}{N}) s(\varepsilon\delta_{j1}) =$$

$$= c_k^1 + \varepsilon s_k^1 \delta_{j1} + O(\varepsilon^2)$$

$$c(\delta_j) + c(\delta_{j+1}) = 2c_k^1 + \varepsilon s_k^1(\delta_{j1} + \delta_{j+1\,1}) + O(\varepsilon^2)$$

$$\boxed{\delta_{j1}} = e_m^j - e_m^{j-1} = c_m^j + i s_m^j - c_m^{j-1} - i s_m^{j-1} = c_m^j(1 - c_m^1) +$$

$$+ (-s_m^1 s_m^j) + i s_m^j (1 - c_m^1) + i s_m^1 c_m^j = \boxed{(1-c_m^1) e_m^j + i s_m^1 e_m^j}$$

$c_m^{j-1} = c_m^1 c_m^j + s_m^1 s_m^j$

$s_m^{j-1} = c_m^1 s_m^j - s_m^1 c_m^j$

$$\delta_{j+1\,1} = e_m^{j+1} - e_m^j = c_m^{j+1} + i s_m^{j+1} - c_m^j - i s_m^j =$$

$$= c_m^j(-1 + c_m^1) - s_m^1 s_m^j + i s_m^j(-1 + c_m^j) + i c_m^1 s_m^j = -(1-c_m^1) e_m^j + i s_m^1 e_m^j$$

$c_m^{j+1} = c_m^1 c_m^j - s_m^1 s_m^j$

$s_m^{j+1} = c_m^1 s_m^j + s_m^1 c_m^j$

$$\delta_{j1}^{*} - \delta_{j+1,1} = 2(1 - c_m^1)\, e_m^j$$

$$\delta_{j1} + \delta_{j+1} = 2 i s_m^1\, e_m^j$$

$$\Rightarrow \boxed{\dot{\varphi}_j} = \omega_0\left(1 + \lambda_s\left(s(\delta_j) - s(\delta_{j+1})\right) + \lambda_c\left(c(\delta_j) + c(\delta_{j+1})\right)\right) =$$

$$= \omega_0\Big(1 + \lambda_s \varepsilon c_k^1 (\delta_{j1} - \delta_{j+1,1}) + \lambda_c s_k^1 \varepsilon (\delta_{j1} + \delta_{j+1,1})\Big) + \underbrace{+ 2\lambda_c c_k^1}\; + O(\varepsilon^2 \lambda_s \omega_0) + O(\varepsilon^2 \lambda_c \cdot \omega_0) \Leftrightarrow$$

$$\Leftrightarrow \boxed{\omega_0\left(1 + 2\lambda_c c_k^1 \varepsilon + \begin{pmatrix} 2\lambda_s c_k^1 (1 - c_m^1) \\ + 2i\lambda_c s_k^1 s_m^1 \end{pmatrix} \varepsilon e_m^j\right) + O(\varepsilon^2 \lambda_0 \cdot \omega_0)}$$

[assume $\lambda_s, \lambda_c = O(\lambda_0)$
e.g. $\lambda_0 = \max(|\lambda_s|, |\lambda_c|)$]

$\omega = \left(1 + 2\lambda_c \varepsilon \cos\left(\frac{2\pi k}{N}\right)\right)\omega_0$ – Boosted frequency
effective

$A = \Big(2\lambda_s \cos\left(\frac{2\pi k}{N}\right)\left(1 - \cos\frac{2\pi m}{N}\right) + 2i\lambda_c \sin\left(\frac{2\pi k}{N}\right) \sin\left(\frac{2\pi m}{N}\right)\Big)\omega_0$ – Lyapunov exponent

$T = \frac{2\pi}{\omega}$: $\quad \lambda = e^{AT} - 1 = \exp\left(\dfrac{4\pi\left(\lambda_s \cos\left(\frac{2\pi k}{N}\right)\left(1 - \cos\frac{2\pi k}{N}\right) + i\lambda_c \sin\left(\frac{2\pi k}{N}\right)\sin\left(\frac{2\pi m}{N}\right)\right)}{1 + 2\lambda_c \cos\left(\frac{2\pi k}{N}\right)\varepsilon}\right) - 1 =$

(L-T eigenvalue)

depends on ε!

$$= \underbrace{4\pi\left(\lambda_s \cos\frac{2\pi k}{N}\left(1 - \cos\frac{2\pi m}{N}\right) + i\lambda_c \sin\left(\frac{2\pi k}{N}\right)\sin\left(\frac{2\pi m}{N}\right)\right)}_{\tilde{\lambda}} + O(\lambda_0^2):$$

OR

$$\lambda = \exp(\underset{A \cdot T_0}{\tilde{\lambda}}) - 1 + O(\lambda_0^2 \varepsilon)$$