|Sine - coupling. Eigenvalues|

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$$\dot\varphi_j = \omega_0\left(1+\lambda_s \sin(\varphi_j-\varphi_{j-1}) + \lambda_s \sin(\varphi_j-\varphi_{j+1})\right)$$

* $\varphi_j|_{t=0} = \underbrace{-\frac{2\pi k}{N} j}_{\text{k-twist}} + \underbrace{\varepsilon \sin\left(\frac{2\pi m}{N} j\right)}_{\text{perturbation with sine of m-twist}}$ (the same holds for cos)

$\Downarrow$

$$\dot\varphi_j|_{t=0} = \omega_0\left(1+\lambda_s \sin\delta_j - \lambda_s \sin(\delta_{j+1})\right) \;=$$

$\delta_j := \varphi_j - \varphi_{j-1}$

$$\sin\delta_j = \sin\left(-\frac{2\pi k}{N} + \varepsilon \sin\left(\frac{2\pi m}{N} j\right) - \varepsilon \sin\left(\frac{2\pi m}{N}(j-1)\right)\right) =$$

$$= \sin\left(-\frac{2\pi k}{N}\right)\cdot \underbrace{\cos(\varepsilon[..])}_{1+O(\varepsilon^2)} + \cos\frac{2\pi k}{N}\underbrace{\sin(\varepsilon[..])}_{\varepsilon[..]+O(\varepsilon^3)} =$$

$$= -\sin\left(\frac{2\pi k}{N}\right) + \cos\frac{2\pi k}{N}\cdot \varepsilon\left(\sin\left(\frac{2\pi m}{N} j\right) - \sin\left(\frac{2\pi m}{N}(j-1)\right)\right) + O(\varepsilon^2) =$$

$$= -\sin\left(\frac{2\pi k}{N}\right) + \varepsilon\cos\frac{2\pi k}{N}\left(\sin\frac{2\pi m}{N} j\cdot\left(1-\cos\frac{2\pi m}{N}\right) + \cos\frac{2\pi m}{N} j\cdot \sin\frac{2\pi m}{N}\right)$$

trig expand

$$\sin\delta_{j+1} = -\sin\left(\frac{2\pi k}{N}\right) + \varepsilon\cos\frac{2\pi k}{N}\left(\sin\frac{2\pi m}{N}(j+1) - \sin\left(\frac{2\pi m}{N} j\right)\right) + O(\varepsilon^2) =$$

$$= -\sin\frac{2\pi k}{N} + \varepsilon\cos\frac{2\pi k}{N}\left(-\sin\left(\frac{2\pi m}{N} j\right)\left(1-\cos\frac{2\pi m}{N}\right) + \cos\frac{2\pi m}{N} j\cdot \sin\left(\frac{2\pi m}{N}\right)\right)$$

$$= \omega_0\left(1+2\lambda_s\varepsilon\cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right)\sin\left(\frac{2\pi m}{N} j\right)\right) + O(\lambda_s\varepsilon^2\omega_0)$$

~~$\tilde\lambda := 2\lambda_s \cos$~~ $\boxed{A := 2\omega_0\lambda_s\cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right)}$

$\Rightarrow$ ~~If φ_j~~ - If $\varphi_j = -\frac{2\pi k}{N} j + \varepsilon\sin\left(\frac{2\pi m}{N} j\right)$ (+ const independent on j)

then $\dot\varphi_j = \omega_0 + \varepsilon A\sin\left(\frac{2\pi m}{N} j\right) + O(\lambda_s\varepsilon^2\omega_0)$ ~~$\forall\varepsilon$ suffice~~

* $\varphi_j(t) = -\frac{2\pi k}{N} j + \omega_0 t + \alpha(t)\sin\left(\frac{2\pi m}{N} j\right)$

$$\dot\varphi_j = \omega_0 + \alpha(t) A\sin\left(\frac{2\pi m}{N} j\right) = \omega_0 + \dot\alpha(t)\sin\left(\frac{2\pi m}{N} j\right) + O(\alpha^2)$$

dimensions

$\dot\alpha = A\alpha + O(\alpha^2)$

$\alpha|_{t=0} = \varepsilon$

$\Longrightarrow$ $\alpha(t) = \varepsilon e^{At} + O(\varepsilon^2)$; ~~$t = O(\frac{1}{A})$~~

$t = O(\frac{1}{\omega_0})$ | $\alpha \sim \varepsilon(1+\lambda)$

T: mean phase $= 2\pi$:

$$T=\frac{2\pi}{\omega_0} \Rightarrow \alpha(T) = e^{A\frac{2\pi}{\omega_0}} = e^{\overbrace{4\pi\lambda_s \cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right)}^{\hat\lambda}}$$

$$\varphi_j(T)-\varphi_j(0) = 2\pi + \underbrace{\left(e^{\hat\lambda}-1\right)}_{\lambda}\cdot\varepsilon\sin\left(\frac{2\pi m}{N}j\right) + O\Big(\underbrace{\lambda\alpha^2}_{\varepsilon^2 e^{2\hat\lambda}}\Big)$$

[assuming $A=O(\omega_0)$]
$\lambda_s \ll 1 \Rightarrow$ ✓

$$\varphi_j(T)-\varphi_j(0)-2\pi = \lambda\underbrace{\varepsilon\sin\frac{2\pi m}{N}j}_{\text{init perturbation}} + O(\lambda_s\cdot\varepsilon^2)$$ ← scaling verified by sim fig 14a

$$\boxed{\lambda = e^{4\pi\lambda_s\cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right)}-1}$$

$$\boxed{\lambda \approx 4\pi\lambda_s \cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right) + O(\lambda_s^2)}$$

↑ e.value of $L-I$.

A - Lyapunov exponent.

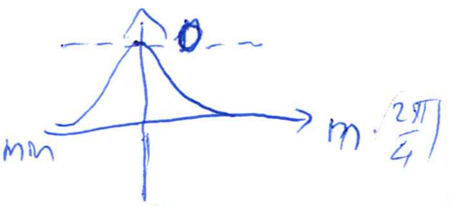
UPD 1: $m \mapsto -m$ to comply with my definition of m-twist

Result (tested in simulation)

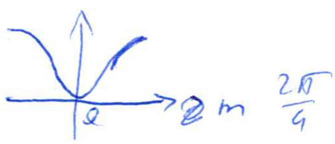
(1) $\lambda_s<0 \Rightarrow$ stable in vicinity of 0-twist: $|k|<\frac{N}{4}$

(2) $|k|=\frac{N}{4} \Rightarrow$ all eigenvalues are zero

(3) $\lambda(m)$:



or



(4) All eigenvalues are ~~std~~ real; degeneracy = 2 except max and min e.v.

sin + cos eigenvalues

$$\dot\varphi_j = \omega_0 \big(1 + \lambda_s \sin(\varphi_j - \varphi_{j-1}) + \lambda_s \sin(\varphi_j - \varphi_{j+1}) + \lambda_c \cos(\varphi_j - \varphi_{j-1}) + \lambda_c \cos(\varphi_j - \varphi_{j+1})\big)$$

Let $$\varphi_j\big|_{t=0} = \underbrace{-\frac{2\pi k}{N} j}_{k\text{-twist}} + \underbrace{\varepsilon e^{i\frac{2\pi m}{N} j}}_{\text{perturbation}}$$

UPD complex exp mimics m-twist | afterwards should [illegible]

$s := \sin$ $\quad \sin\left(\frac{2\pi m}{N} j\right) =: s_m^j$

$c := \cos$ $\quad \ldots \quad e_m^j$

$e := \exp(i\cdot) = c + is$ $\quad \ldots \quad c_m^j$

$\delta_j := \varphi_j - \varphi_{j-1}$

$$\dot\varphi_j = \omega_0\big(1 + \lambda_s s(\delta_j) - \lambda_s s(\delta_{j+1}) + \lambda_c c(\delta_j) + \lambda_c c(\delta_{j+1})\big)$$

$$\delta_j\big|_{t=0} = \varphi_j - \varphi_{j-1} = -\frac{2\pi k}{N} + \varepsilon\left(e_m^j - e_m^{j-1}\right)$$

Let $\delta_j = -\frac{2\pi k}{N} + \varepsilon \delta_{j1}$

Then (expression from $\dot\varphi_j|_{t=0}$): $s(\delta_j) - \cancel{s(\delta_{j+1})} =$

$$= s\left(-\tfrac{2\pi k}{N}\right) c(\varepsilon \delta_{j1}) + c\left(\tfrac{2\pi k}{N}\right) s(\varepsilon\delta_{j1}) =$$

$$= -s\left(\tfrac{2\pi k}{N}\right)\left(1 + O(\varepsilon^2)\right) + c\left(\tfrac{2\pi k}{N}\right)\left(\varepsilon \delta_{j1} + O(\varepsilon^3)\right) =$$

$$= -s_k^1 + \varepsilon c_k^1 \delta_{j1} + O(\varepsilon^2)$$

$$s(\delta_j) - s(\delta_{j+1}) = \varepsilon c_k^1 \left(\delta_{j1} - \delta_{j+1\,1}\right) + O(\varepsilon^2)$$

$$c(\delta_j) = c\left(-\tfrac{2\pi k}{N} + \varepsilon \delta_{j1}\right) = c\left(\tfrac{2\pi k}{N}\right) c(\varepsilon\delta_{j1}) - s\left(-\tfrac{2\pi k}{N}\right) s(\varepsilon \delta_{j1}) =$$

$$= c_k^1 + \varepsilon s_k^1 \delta_{j1} + O(\varepsilon^2)$$

$$c(\delta_j) + c(\delta_{j+1}) = 2c_k^1 + \varepsilon s_k^1 \left(\delta_{j1} + \delta_{j+1\,1}\right) + O(\varepsilon^2)$$

$$\boxed{\delta_{j1}} = e_m^j - e_m^{j-1} = c_m^j + i s_m^j - c_m^{j-1} - i s_m^{j-1} = c_m^j(1 - c_m^1) + (-s_m^1 s_m^j) + i s_m^j (1 - c_m^1) + i s_m^1 c_m^j = \boxed{(1 - c_m^1) e_m^j + i s_m^1 e_m^j}$$

$c_m^{j-1} = c_m^1 c_m^j + s_m^1 s_m^j$

$s_m^{j-1} = c_m s_m^j - s_m^1 c_m^j$

$$\delta_{j+1\,1} = e_m^{j+1} - e_m^j = c_m^{j+1} + i s_m^{j+1} - c_m^j - i s_m^j =$$

$$= c_m^j(-1 + c_m^1) - s_m^1 s_m^j + i s_m^j(-1 + c_m^1) + i c_m^j s_m^1 = -(1 - c_m^1) e_m^j + i s_m^1 e_m^j$$

$c_m^{j+1} = c_m^1 c_m^j - s_m^1 s_m^j$

$s_m^{j+1} = c_m^1 s_m^j + s_m^1 c_m^j$

$$\delta_{j1}^{*} - \delta_{j+1,1} = 2(1-c_m^1)\, e_m^j$$

$$\delta_{j1} + \delta_{j+1} = 2 i s_m^1\, e_m^j$$

$$\Rightarrow \boxed{\dot{\varphi}_j} = \omega_0 \left(1 + \lambda_s \left(s(\delta_j) - s(\delta_{j+1})\right) + \lambda_c \left(c(\delta_j) + c(\delta_{j+1})\right)\right) =$$

$$= \omega_0 \left(1 + \lambda_s \varepsilon c_k^1 (\delta_{j1} - \delta_{j+1,1}) + \lambda_c s_k^1 \varepsilon (\delta_{j1} + \delta_{j+1,1})\right) + 2\lambda_c c_k^1 + O(\varepsilon_k^2) + O(\varepsilon^2 \lambda_c)(\cdot \omega_0) =$$

$$\Leftrightarrow \omega_0 \left(1 + 2\lambda_c c_k^1 + \begin{pmatrix} +2\lambda_s c_k^1 (1-c_m^1) \\ +2i\lambda_c s_k^1 s_m^1 \end{pmatrix} \varepsilon e_m^j \right) + O(\varepsilon^2 \lambda_0 \cdot \omega_0)$$

− in sim!

[assume $\lambda_s, \lambda_c = O(\lambda_0)$
e.g. $\lambda_0 = \max(|\lambda_s|, |\lambda_c|)$]

$\omega = \left(1 + 2\lambda_c \cos\left(\frac{2\pi k}{N}\right)\right)\omega_0$ – boosted frequency (effective)

$A = \left(2\lambda_s \cos\left(\frac{2\pi k}{N}\right)\left(1 - \cos\frac{2\pi m}{N}\right) + 2i\lambda_c \sin\left(\frac{2\pi k}{N}\right)\sin\left(\frac{2\pi m}{N}\right)\right)\omega_0$ – Lyapunov exponent

− in sim!

$T = \frac{2\pi}{\omega}$: $\lambda = e^{AT} - 1 = \exp\left(\frac{4\pi\left(\lambda_s \cos\left(\frac{2\pi k}{N}\right)\left(1-\cos\frac{2\pi k}{N}\right) + i\lambda_c \sin\left(\frac{2\pi k}{N}\right)\sin\left(\frac{2\pi m}{N}\right)\right)}{1 + 2\lambda_c \cos\left(\frac{2\pi k}{N}\right)}\right) - 1 =$

(L-T eigenvalue)

$$= 4\pi\left(\lambda_s \cos\frac{2\pi k}{N}\left(1 - \cos\frac{2\pi m}{N}\right) + i\lambda_c \sin\left(\frac{2\pi k}{N}\right)\sin\left(\frac{2\pi m}{N}\right)\right) + O(\lambda_0^2) \quad \text{OR} \quad \lambda = \exp(\tilde{\lambda}) - 1 \cdot O(\lambda_0^2 \varepsilon), \; \tilde{\lambda} = A \cdot T_0$$

$$\approx 4\pi\left(\lambda_s \cos\frac{2\pi k}{N}\left(1-\cos\frac{2\pi m}{N}\right) + i\lambda_c \sin\left(\frac{2\pi k}{N}\right)\sin\left(\frac{2\pi m}{N}\right)\right) + O(\lambda_0^2)$$

− in sim!

UPD: $m \to -m$ to comply with my definition of an m-twist

$$\Rightarrow A = \omega_0 \left(2\lambda_s c_k^1 (1 - c_m^1) - 2i s_k^1 s_m^1\right)$$