Contents

Notation

- $\mathbf{k}$ - wave vector
- $\mathbf{\Phi} = (\varphi_1, \varphi_2, .., \varphi_N)^T$ - phase vector
- $\mathbf{Q}(\mathbf{\Phi})$ - vector of active driving forces
- $\mathbf{\Gamma}$ - friction coefficients NxN-matrix
- $\mathbf{\Phi_k}$ - m-twist solution
- $\phi(\mathbf{\Phi})$ - global phase, according to one of the definitions. *Currently mean phase is the default choice for global phase.*
- $\mathcal{L} : H \to H$ - Poincare map
- $\mathbf{\Phi}^* \in H$ - fix point of the Poincare map
- $\mathbf{\Phi}^*_\mathbf{k}$ - fix point close to the m-twist solution
- $\mathbf{\Delta_0}$, $\mathbf{\Delta_1}$ - perturbation initial and after one cycle: $\mathcal{L}(\mathbf{\Phi}^* + \mathbf{\Delta_0}) = \mathbf{\Phi}^* + \mathbf{\Delta_1}$
- $\mathbf{L} = \mathrm{D}\mathcal{L}(\mathbf{\Phi}^*)$ linearized Poincare map at a fixed point (matrix). In code 'Lmat'
- Λ_j - eigenvalues of $\mathbf{L}$, Floquet multipliers
- $? = \ln \mathbf{L}$ logarithm of the linearized Poincare map. In code 'Lmat_log'
- λ_j - eigenvalues of $\ln \mathbf{L}$, dimensionless Lyapunov exponents

Conflicts:

- Fixpoint notation and complex conjugation.
- Once δ and Δ are taken - how to denote a difference of some values?
- global phase and components of $\mathbf{\Phi}$

Geometry

Unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Position vectors

$$\mathbf{x}_{n,m} = n\,a\,\mathbf{e}_1 + m\,a\,\mathbf{e}_2$$

Honeycomb lattice

$$\mathcal{H} = \{\mathbf{x}_{n,m} | 2n + m \equiv 0 \text{ or } 2 \bmod 3\}$$

Triangular lattice

$$\mathcal{T} = \{\mathbf{x}_{n,m} | n, m \in \mathrm{Z}\}$$

m-twist solutions

Wave vector of metachronal wave consistent with periodic boundary condition

$$\mathbf{k} = \frac{a_1}{L_1} k_1 \, \mathbf{a}_1^* + \frac{a_2}{L_2} k_2 \, \mathbf{a}_2^*,$$

where $k_1, k_2 \in Z$ and $a_1 = a$, $a_2 = \sqrt{3}a/2$

Meta-chronal wave solutions:

Phase-space vector $\Phi(t) = (\varphi_1, \varphi_2, .., \varphi_N)^T$ with components

$$\varphi_j(t) = \varphi(t) - \mathbf{k} \cdot \mathbf{x}_j,$$

for some global phase $\varphi(t)$. We expect $\dot{\varphi} \approx \omega_0$, but only approximately, since the calibration of active driving forces does not take into account the (weak) hydrodynamic interactions.

Dynamic equation from force-balance equation

We start with

$$\mathbf{\Gamma}(\Phi)\dot{\Phi} = Q(\Phi),$$

or

$$\dot{\Phi} = \mathbf{\Gamma}^{-1}(\Phi)Q(\Phi)$$

This linear system can be solved directly in Python.

We calibrate dynamical simulations with pair-wise coupling functions from hydrodynamical simulations

$$\Gamma_{ij}(\Phi) = 0 \quad \text{if not neighbours}$$

$$\Gamma_{ij}(\Phi) = \Gamma_{12}^{(\mathrm{loc})}(\varphi_i, \varphi_j; a, \psi) \quad \text{if neighbours}$$

where the translation vector pointing from cilium i to cilium j reads

$$[n(j) - n(i)] \, \mathbf{e}_1 + [m(j) - m(i)] \, \mathbf{e}_2 = a \begin{pmatrix} \sin\psi \\ \cos\psi \end{pmatrix}.$$

Self-friction:

$$\Gamma_{ii}(\Phi) = \Gamma_{11}^{(\mathrm{loc})}(\varphi_i, \varphi_j; a, \psi)$$

Note, self-friction $\Gamma_{11}^{(\mathrm{loc})}(\varphi_i, \varphi_j; a, \psi)$ dependence on φ_j, a, ψ is so weak[1], that we can take $\Gamma_{11}^{(\mathrm{loc})}(\varphi_i, \varphi_j; a, \psi) = \Gamma_{11}^{(\mathrm{loc})}(\varphi_i)$

We therefore calibrate active driving forces as following

$$Q_i(\Phi) = \Gamma_{11}^{(\mathrm{loc})}(\varphi_i)\omega_0$$

[1] assuming a is big enough

Global Phase

Naive phase

For some index j, $\varphi = \varphi_j$

Circular average phase

We define order parameters for each metachronal wave vector $\mathbf{k}$

$$S_{\mathbf{k}} = \left| \frac{1}{N} \sum_j \exp\left[i(\varphi_j - \varphi_{\mathbf{k},j}) \right] \right|,$$

where $\varphi_{\mathbf{k},j}$ - j-th component of $\Phi_{\mathbf{k}}$, m-twist solution defined by vector $\mathbf{k}$.

Global phase:

For a fixed $\mathbf{k}$

$$\varphi(t; \mathbf{k}) = \arg\left(\frac{1}{N} \sum_j \exp\left[i(\varphi_j(t) - \varphi_{\mathbf{k},j}) \right] \right)$$

- Only has meaning when $S_{\mathbf{k}}$ is close to 1.

Mean phase

Define the global phase as algebraic mean of all phases

$$\varphi = \frac{1}{N} \sum_{j=1}^{N} \varphi_j$$

Pros - Additive: if $\mathbf{\Phi} = \mathbf{\Phi_0} + \mathbf{\Phi_1}$, then $\varphi = \varphi_0 + \varphi_1$ - As an implication, it is independent of which initial condition we consider. - Poincare surface will be a plane, defined by normal $(1, 1, 1, 1.., 1)$. [A language of quotient vector space could be useful; just keep this reference here for now http://mathworld.wolfram.com/QuotientVectorSpace.html] (quotient = частное)

Cons - φ_j can't jump from 2π to 0, otherwise the global phase will make a jump by $-\frac{2\pi}{N}$.

Poicnare map for an m-twist solution

Poincare map and limit cycle

We consider a Poincare section H defined by

$$H = \{\Phi : \varphi(\Phi) \equiv \varphi(\Phi_0) \mod 2\pi\}.$$

where φ denotes the global phase, Φ_0 is the phase vector at initial time t_0.

H is a $(N-1)$-dimensional hyper-surface in N-dimensional phase space. If φ is the mean phase, H is a hyperplane.

For each of m-twists solutions, we anticipate a corresponding limit cycle $C_{\mathbf{k}}$, piercing H close to Φ_k i.e.

$$C_{\mathbf{k}} \cap H_0 = \Phi_{\mathbf{k}} + \mathbf{E}^*,$$

with a small correction vector $\mathbf{E}^*$. The reason for the small correction $\mathbf{E}^*$ is that the calibration of active driving forces causes small phase-dependent variations of the instantaneous phase speed *(a cilium will speed up at one part of the beat cycle, and slow down at another; but then the phase difference will vary during the cycle, therefore we don't get a perfect m-twist, which has constant phase difference?).*

Therefore, the first step is to find $\Phi^*_{\mathbf{k}} = \Phi_{\mathbf{k}} + \mathbf{E}^*$.

We define the **Poincare map** $\mathcal{L} : H \to H$ as

$$\mathcal{L}(\Phi_0) = \Phi_1,$$

where $\Phi_0 = \Phi(t_0)$ - some initial phase[2], $\Phi_1 = \Phi(t_1)$ - where t_1 is the next time when our phase trajectory intersects with the Poincare section.

With this definition $\Phi^*_{\mathbf{k}}$ is a fixed point of Poincare map.

TODO: explain why it cannot be another limit cycle in Poincare plane.

TODO: do we expect no extra fixed points/ limit cycles?

Procedure to find the fixpoint Φ^*

By definition, the fixpoint is such a point that

$$\mathcal{L}(\Phi^*) = \Phi^*$$

We define

$$D(\Phi) = \mathcal{L}(\Phi) - \Phi$$

and

$$d(\Phi) = \|D(\Phi)\|^2$$

Function $d(\Phi)$ is the squared distance between a phase vector and its Poincare map image.

Properties of norm imply that

- $d(\Phi) \geq 0$

[2]Note that for any Φ_0 we can define a Poincare section as defined above.

- $d(\Phi) = 0 \iff \Phi = \Phi^*$ — a fixed point

Therefore, to find the fixpoint $\Phi^*_{\mathbf{k}}$, we numerically find a minimum of function d with initial guess at $\Phi_{\mathbf{k}}$.[3]

Linearized Poincare map

We use linear stability analysis to study stability of limit cycles. Consider a fixpoint of Poincare map Φ^* (corresponds to a limit cycle) and apply a small perturbation Δ_0.

$$\mathcal{L}(\mathbf{\Phi}^* + \Delta_0) = \mathbf{\Phi}^* + \Delta_1$$

Power expansion in vicinity of the fixpoint yields

$$\mathcal{L}(\mathbf{\Phi}^* + \Delta_0) = \mathbf{\Phi}^* + \mathrm{D}\mathcal{L}(\mathbf{\Phi}^*)\Delta_0 + \mathcal{O}(\| \Delta_0\|^2)$$

where $\mathrm{D}\mathcal{L}(\mathbf{\Phi}^*)$ represents the linear contribution (Jacobian matrix), also known as linearized Poincare map [reference].

Further let's use short notation $\mathbf{L} = \mathrm{D}\mathcal{L}(\mathbf{\Phi}^*)$. In code 'Lmat'

TODO: procedure to find L

TODO:

$.. = \ln \mathbf{L}$

- $\ln \mathbf{L}$ - is not a symmetrical matrix. - $\ln \mathbf{L}$ is close to $\mathbf{L} - \mathbf{I}$, but numerically I found that real part of eigenvalue can differ up to 20

Linearized Poincare map: another vision

- Since Poincare section is N-1 dimensional hypersurface, the linearized map is represented by N-1 to N-1 matrix. - We construct it by perturbing the fixpoint with N-1 perturbations. - Those are such perturbations, that the perturbed state still lies in Poincare section. - To work with N to N matrix we add Nth perturbation $\Delta_0 = (1, ...1)$ - normal to the Poincare plane and declare that $\Delta_1 = \Delta_0$.

That will add another eigenvalue $\lambda = 1$ and other eigenvalues and eigenvectors will remain unchanged.

[3] Used library function 'scipy.optimize.minimize' with method 'BFGS'.

Numerical inaccuracy discussion

Inaccuracy in fixpoint

Fixpoint

Ben suggested a different notation:

- *difference between real fixpoint and approximate:* $\widetilde{\Delta}_0$
- *After one cycle distance to the real fixpoint:* $\widetilde{\Delta}_1$
- *Then always use* Δ_0, Δ_1 *like before*

Downsides:

- *New* $\widetilde{\Delta}_1$ *is not* Δ_**, but we can only compute latter.*
- *In code we subtract numerical fixpoint, but we should subtract its Poincare map image.*

We find fixpoint by optimization procedure [defined above] up to a tolerance, specified in code. Two tolerances are involved: (i) solver tolerance and (ii) minimizer tolerance. After some minimal testing, those were taken to be equal and further in this section both are referred simply as tolerance.

Suppose we found numerically a fixpoint $\widetilde{\Phi}^*$, but it is not perfect and $\mathcal{L}(\widetilde{\Phi^*}) \neq \widetilde{\Phi^*}$:

$$\mathcal{L}(\widetilde{\Phi^*}) = \widetilde{\Phi^*} + \Delta_*$$

This means, that real fixpoint is somewhere else:

$$\widetilde{\Phi^*} = \Phi^* + \widetilde{\Delta}$$

If we consider linear approximation in vicinity of Φ^*

$$\mathcal{L}(\widetilde{\Phi^*}) = \Phi^* + \mathbf{L}\widetilde{\Delta}$$

$$\widetilde{\Phi^*} + \Delta_* = \Phi^* + \mathbf{L}\widetilde{\Delta}$$

$$\Phi^* + \widetilde{\Delta} + \Delta_* = \Phi^* + \mathbf{L}\widetilde{\Delta}$$

Therefore

$$\Delta_* = (\mathbf{L} - \mathbf{I})\widetilde{\Delta} + \mathcal{O}(\|\widetilde{\Delta}\|^2) \tag{1}$$

Note

- We don't know real $\mathbf{L}$ (only approximate at $\widetilde{\Phi^*}$)
- We don't know distance to the real fixpoint $\widetilde{\Delta}$.
- We can, however, calculate Δ_*.

Let's denote $\varepsilon_F = \|\Delta_*\| = \|\mathcal{L}(\widetilde{\Phi^*}) - \widetilde{\Phi^*}\|$ and $\delta_F = \|\widetilde{\Delta}\|$. Then

$$\|\Delta_*\| = \|(\mathbf{L} - \mathbf{I})\widetilde{\Delta}\|$$

$$\min|\lambda|\delta_F \leq \varepsilon_F \leq \max|\lambda|\delta_F$$

$$\varepsilon_F \sim |\lambda|\delta_F$$

Perturbed state

Let's consider a perturbed (approximate) fixpoint. Fixpoint error will add up to the perturbation:

$$\mathcal{L}(\widetilde{\Phi}^* + \Delta_0) = \mathcal{L}(\Phi^* + \widetilde{\Delta} + \Delta_0) = \Phi^* + \mathbf{L}(\widetilde{\Delta} + \Delta_0)$$

The left side is equal to the initial state plus deviation

$$\mathcal{L}(\widetilde{\Phi^*} + \Delta_0) = \widetilde{\Phi^*} + \Delta_1 = \Phi^* + \widetilde{\Delta} + \Delta_1$$

And combining these two equalities

$$\Delta_1 = \mathbf{L}\Delta_0 + (\mathbf{L} - \mathbf{I})\widetilde{\Delta}$$

$$\Delta_1 - \Delta_0 = (\mathbf{L} - \mathbf{I})\Delta_0 + (\mathbf{L} - \mathbf{I})\widetilde{\Delta}$$

$$\Delta_1 - \Delta_0 = (\mathbf{L} - \mathbf{I})\Delta_0 + \Delta_*$$

Now let's estimate norms

$$\|\Delta_1 - \Delta_0\| \sim |\lambda|(\delta_0 + \delta_F) \sim |\lambda|\delta_0 + \varepsilon_F$$

$$\frac{\|\Delta_*\|}{\|\Delta_1 - \Delta_0\|} \sim \frac{\varepsilon_F}{|\lambda|\delta_0 + \varepsilon_F}$$

If $|\lambda|\delta_0 >> \varepsilon_F$:

$$\frac{\|\Delta_*\|}{\|\Delta_1 - \Delta_0\|} \sim \frac{\varepsilon_F}{|\lambda|\delta_0} \sim \frac{\delta_F}{\delta_0}$$

Conclusions - We want to make sure that $\frac{\|\Delta_*\|}{\|\Delta_1 - \Delta_0\|}$ is small. Otherwise, $\Delta_1 - \Delta_0$ is significantly affected by Δ_* contribution, and we will have big error when we calculate matrix $\mathbf{L}$. TODO: quantify

- $\Delta_1 - \Delta_0$ has a constant contribution equal to Δ_*. - Note that Δ_0, Δ_1 - simulation input and outputs, so that's something we can directly obtain. - Δ_* - is something that we can measure, and can control indirectly, by tuning algorithm tolerance. - Therefore, we must make sure that - Test at $N = 6$ showed that this ratio is around or below 10^{-2} if fixpoint tolerance is 10^{-8}.

- Also, just for fun, let's note

$$\mathcal{L}(\widetilde{\Phi^*} + \Delta_0) - \mathcal{L}(\Phi^* + \Delta_0) = D\mathcal{L}|_{\widetilde{\Phi^*}+\Delta_0}\widetilde{\Delta} + \mathcal{O}(\delta_F^2)$$

Linearized map

Expand Poincare map $\mathcal{L}$ in vicinity of Φ^*

$$\mathcal{L}(\widetilde{\Phi^*} + \Delta_0) = \Phi^* + \mathbf{L}(\widetilde{\Delta} + \Delta_0) + \mathcal{O}((\delta_F + \delta_0)^2)$$

Do the same in vicinity of $\widetilde{\Phi^*}$

$$\mathcal{L}(\widetilde{\Phi^*} + \Delta_0) = \mathcal{L}(\widetilde{\Phi^*}) + \widetilde{\mathbf{L}}\Delta_0 + \mathcal{O}(\delta_0^2) =$$

$$= \Phi^* + \mathbf{L}\widetilde{\Delta} + \mathcal{O}(\delta_F^2) + \widetilde{\mathbf{L}}\Delta_0 + \mathcal{O}(\delta_0^2)$$

Combining those two

$$\mathbf{L}(\widetilde{\Delta} + \Delta_0) + \mathcal{O}((\delta_F + \delta_0)^2) = \mathbf{L}\widetilde{\Delta} + \mathcal{O}(\delta_F^2) + \widetilde{\mathbf{L}}\Delta_0 + \mathcal{O}(\delta_0^2)$$

$$\mathbf{L}\widetilde{\Delta} = \widetilde{\mathbf{L}}\Delta_0 + \mathcal{O}((\delta_F + \delta_0)^2)$$

$$(\mathbf{L} - \widetilde{\mathbf{L}})\Delta_0 = a_{10}\delta_0 + a_{01}\delta_F + a_{20}\delta_0^2 + a_{02}\delta_F^2 + a_{11}\delta_0\delta_F + ...$$

The left side scales linearly with δ_0, since it must hold for any δ_0 (sufficiently small), only terms which scale linearly with δ_0 can exist.

$$(\mathbf{L} - \widetilde{\mathbf{L}})\Delta_0 = \delta_0 \mathcal{O}(\delta_F)$$

And finally

$$\mathbf{L} - \widetilde{\mathbf{L}} = \mathcal{O}(\delta_F)$$

$\mathcal{L}(\widetilde{\Phi^*})$ *and* $\widetilde{\Phi^*}$ *are asymptotically at the same distance from* Φ^*.

$$\mathcal{L}(\widetilde{\Phi^*}) = \Phi^* + \mathbf{L}\widetilde{\Delta} + \mathcal{O}(\delta_F^2) \tag{2}$$

Or

$$\mathcal{L}(\widetilde{\Phi^*}) - \Phi^* = \mathcal{O}((1 + |\lambda|)\delta_F) = \mathcal{O}(|\lambda|) \tag{3}$$

Numerical estimations

If $N = 6$, fixpoint with $tol = 10^{-8}$:

$|\lambda| = 10^{-2}$ - 10^{-3}

$\varepsilon_F = 10^{-6}$ - 10^{-7}

Therefore $\delta_F \sim 10^{-4}$

If $\delta_0 = 10^{-3}$, $\frac{\delta_F}{\delta_0} = 10^{-1}$, and indeed, $\frac{\|\Delta_*\|}{\|\Delta_1 - \Delta_0\|}$ lie in range $2 * 10^{-2} - 10^{-3}$

Simulations

Eigenvectors

- Stored as **columns** of NxN array *evecs*
- Normalized
- Not orthogonal in general, but a lot of them *are* orthogonal
- Most of eigenvectors are complex, they come in pairs: eigenvector and its complex-conjugates, eigenvalue and its complex conjugate [algebra]
- Perturbation made of eigenvector and its complex conjugate will develop after a cycle as predicted to linear theory up to precision of $10^{-4} - 10^{-5}$.

  That's not very small, but it remains as small if I increase δ_0 to 10^{-1}.
- Imaginary part of eigenvalue gives only a small contribution ($e^{0.02i} \approx 0.9998 + 0.02i$).
- Most of eigenvectors have their components lying on a circle.

Eigenvector decomposition

Let's consider an eigenvector Δ. We can decompose it into basis of complex exponents of m-twists (coefficients are essentially multidimensional discrete Fourier transform output):

$$\Delta = \sum_{\mathbf{k}} d_k e^{i\Phi_{\mathbf{k}}}.$$

Observed (1D carpet,*try09v2*, *try12v2*) that in fact only one component gives major contribution $\mathbf{k} = \mathbf{k_0}$,

$$\Delta = d_{\mathbf{k_0}} e^{i\Phi_{\mathbf{k_0}}} + \mathbf{R_1},$$

where $\mathbf{R_1} = \sum_{\mathbf{k} \neq \mathbf{k_0}} d_k e^{i\Phi_{\mathbf{k}}}$ - residual.

In most of the cases $\|\mathbf{R_2}\| < 0.04$. It is observed [*try12v2*, ev vs k], that if eigenvalues are small by absolute magnitude ($\max|\lambda| \sim 10^{-3}$), residual can go up: $\|\mathbf{R_2}\|$ up to 0.5. Taking just one additional term would lower the residual back to 0.04. Either way, correctness of numerics in case of very small eigenvalues is under question, further let's neglect this case (which also happens usually only on the boundary between stable and unstable regions).

Representation of the eigenvector in the form

$$\Delta \approx d_{\mathbf{k_0}} e^{i\Phi_{\mathbf{k_0}}}$$

means that eigenvector components all lie on a circle in the complex plane.

Purely real eigenvalues

If eigenvalue is real, then eigenvector components lie on the real axis line instead of a circle. It can probably decomposed into cos/sin of m-twists. [Under investigation: pure sine coupling 'try14']

Real perturbation out of complex eigenvectors

We can construct real perturbation based on complex eigenvectors:

$$\mathrm{Re}(\Delta) = \frac{1}{2}(\Delta + \Delta^*) \approx \mathrm{Re}(d_{\mathbf{k_0}})\cos(\Phi_{\mathbf{k_0}}) - \mathrm{Im}(d_{\mathbf{k_0}})\sin(\Phi_{\mathbf{k_0}})$$

$$\mathrm{Im}(\Delta) = \frac{1}{2}(\Delta - \Delta^*) \approx \mathrm{Im}(d_{\mathbf{k_0}})\cos(\Phi_{\mathbf{k_0}}) + \mathrm{Re}(d_{\mathbf{k_0}})\sin(\Phi_{\mathbf{k_0}})$$

TODO: can I force coefficient d to be always real?

After one cycle:

$$\Delta_0 = \frac{1}{2}(\Delta + \Delta^*)$$

If $d_{\mathbf{k}_0}$ is real:

$$\Delta_0 = d_{\mathbf{k_0}}\cos(\Phi_{\mathbf{k_0}})$$

$$\Delta_1 - \Delta_0 = \lambda\Delta + \lambda^*\Delta^* = \mathrm{Re}(\lambda)d_{\mathbf{k_0}}\cos(\Phi_{\mathbf{k_0}}) - \mathrm{Im}(\lambda)d_{\mathbf{k_0}}\sin(\Phi_{\mathbf{k_0}})$$

Cilia carpet simulation

[*try11*]

Toy model: chain of oscillators with trigonometrical coupling

$$\dot{\varphi}_j = \omega_0 + \sum_{n,m}(s_{nm}\sin(m\phi_j - n\phi_{j-1}) + s_{nm}\sin(m\phi_j - n\phi_{j+1}) + \\ + c_{nm}\cos(m\phi_j - n\phi_{j-1}) + c_{nm}\cos(m\phi_j - n\phi_{j+1})), \quad j = 1..N$$

- If there no non-zero terms with $n \neq m$, pure m-twists will be limit cycles. Frequency will be modified by cosine terms.

Sine coupling

Jupyter notebook try14b

Formulation

Consider a chain of N oscillators with equal beating frequency ω_0, simple sinusoidal coupling and periodic boundary conditions (below we consider index j modulo N).

$$\dot{\varphi}_j = \omega_0(1 + \lambda_s \sin(\varphi_j - \varphi_{j-1}) + \lambda_s \sin(\varphi_j - \varphi_{j-1})), \quad j = 1..N \tag{4}$$

Limit cycles are perfect m-twists with some wave number k

$$\varphi_j(t) = \omega_0 t - \frac{2\pi k}{N} j, \quad j = 1..N$$

Indeed, if we substitute that into (4) we find that for $\forall j\ \dot{\varphi}_j = \omega_0$.

From our simulations we guess that eigenvectors will be a sine or cosine of an m-twist with some wave number m.

To prove it, let's perturb the limit cycle and find how this perturbation will develop in time

$$\varphi_j\big|_{t=0} = -\frac{2\pi k}{N} j + \varepsilon \exp(-i\frac{2\pi m}{N} j), \quad j = 1..N$$

Note that calculations don't change if we perturb with only cos *or* sin *of an m-twist. General form is necessary when we add cosine coupling terms.*

Derivation

Substitute that into (4) - see scanned pdf notes *[and see appendix; references via latex gives only an empty link]*

$$\dot{\varphi}_j\big|_{t=0} = \omega_0 \left(1 + 2\lambda_s \cos\frac{2\pi k}{N} \left(1 - \cos\frac{2\pi m}{N}\right) \varepsilon \exp(-i\frac{2\pi m}{N} j)\right) + \mathcal{O}(\lambda_s \varepsilon^2 \omega_0) \tag{5}$$

Let's rewrite it in an abstract form

$$\varphi_j = -\frac{2\pi k}{N} j + \varepsilon v_j \implies \dot{\varphi}_j = \omega_0 + A\varepsilon v_j + \mathcal{O}(\lambda_s \varepsilon^2 \omega_0) \tag{6}$$

where $v_j = \exp(-i\frac{2\pi m}{N} j)$, A - a constant. Note that this relation doesn't change if we add any constant (independent of j) to φ_j. The relation holds for any small ε, so let's modify ε by time-dependent factor $\alpha(t)$, assuming $\alpha(t) = \mathcal{O}(1)$

$$\varphi_j(t) = -\frac{2\pi k}{N} j + \omega_0 t + \alpha(t)\varepsilon v_j, \quad \alpha(0) = 1 \tag{7}$$

Then the derivative relation (6) becomes

$$\dot{\varphi}_j = \omega_0 + A\varepsilon\alpha(t)v_j + \mathcal{O}(\lambda_s \varepsilon^2 \omega_0) \tag{8}$$

Substituting (7) to the left-side and omitting higher order terms on the right side, we obtain an ODE for $\alpha(t)$

$$\omega_0 + \dot{\alpha}\varepsilon v_j = \omega_0 + A\varepsilon\alpha(t)v_j \tag{9}$$

$$\dot{\alpha} = A\alpha(t), \quad \alpha(0) = 1 \tag{10}$$

The solution is an exponential function

$$\alpha(t) = e^{At} \tag{11}$$

and

$$\varphi_j(t) = -\frac{2\pi k}{N} j + \omega_0 t + e^{At}\varepsilon v_j. \quad t = \mathcal{O}(\mathrm{Re}(A)^{-1}) \tag{12}$$

Therefore we found how phases evolve in time after a perturbation in a vicinity of a limit cycle, and showed that v_j are indeed components of an eigenvector. In case of $\mathrm{Re}(A) > 0$ this solution is valid only on time scales of $\mathcal{O}(\mathrm{Re}(A)^{-1})$, otherwise $\alpha = \mathcal{O}(1)$ is not satisfied.

After one oscillation cycle at time T perturbation strength changes by factor e^{AT}, therefore

$$\varphi_j(T) - \varphi_j(0) = (e^{AT} - 1)v_j \tag{13}$$

This mean that linearized Poincare map eigenvalue *(Floquet multiplier)*

$$\Lambda = e^{AT}$$

Eigenvalue of the logarithm of linearized Poincare map *(dimensionless form of Lyapunov eigenvalue)*

$$\lambda = \ln \Lambda = AT$$

This relation with A is expected, as the coupling strength doesn't depend on phase (only on phase difference).

Results

The perturbation grows or decays with exponential rate A (Lyapunov exponent)

$$A = 2\lambda_s \omega_0 \cos\frac{2\pi k}{N}\left(1 - \cos\frac{2\pi m}{N}\right)$$

Since eigenvalues are real, we can construct purely real eigenvectors. These are $\varepsilon \sin\frac{2\pi m}{N} j$ and $\varepsilon \cos\frac{2\pi m}{N} j$: N linear-independent eigenvectors.

Oscillation frequency is unmodified in the presence of coupling. Period $T = \frac{2\pi}{\omega_0}$.

Poincare map eigenvalue

$$\lambda = 4\pi\lambda_s \cos\frac{2\pi k}{N}\left(1 - \cos\frac{2\pi m}{N}\right)$$

Floquet multiplier [4]

$$\Lambda = \exp\left(4\pi\lambda_s \cos\frac{2\pi k}{N}\left(1 - \cos\frac{2\pi m}{N}\right)\right)$$

Conclusions. Visualizations

- Fixed points (limit cycles) are pure m-twists.
- All eigenvalues (and eigenvectors) are real.
- Frequency is unchanged
- All eigenvalues are degenerate with degeneracy = 2, except for $m = 0$, $m = N/2$ (if N is even).
- At any fixed point, all eigenvalues are either non-negative, or non-positive.
- Eigenvalues at $k = \pm\frac{N}{4}$ are all zero.
- Checked the difference between simulation and analytical calculations: only higher order terms remain, as predicted by analysis.

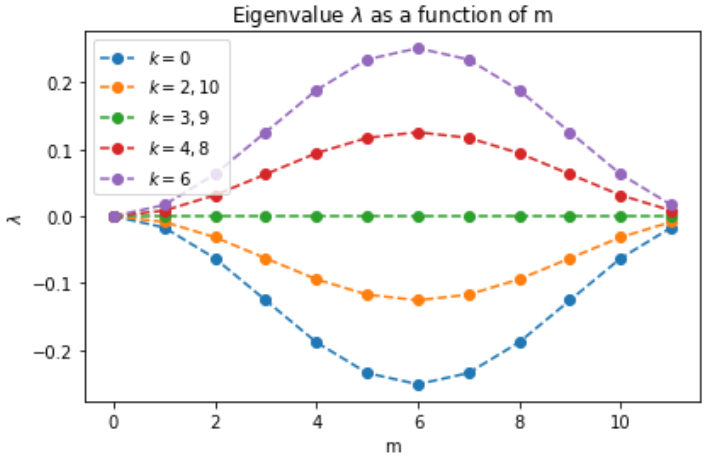
The perturbation decays if $\mathrm{Re}(A)$ is negative. The limit cycle is stable if for any perturbation A, or equivalently λ are negative (more precisely - non-positive, since we always have a neutral perturbation).

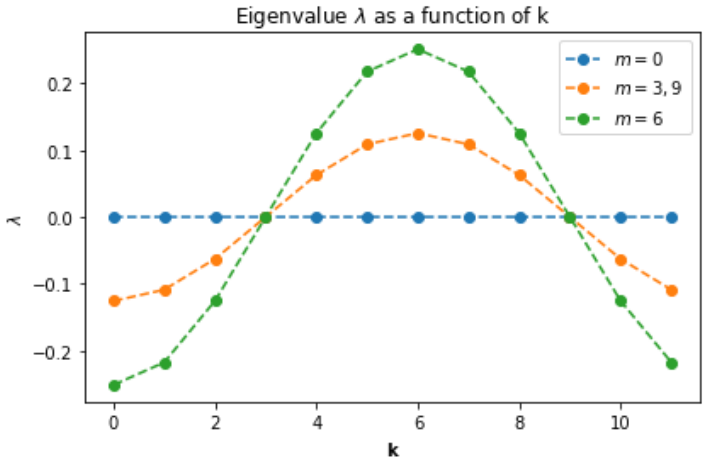
Visualize eigenvalues at $N = 12$, $\lambda_s = -0.01$.

[4] In case of sufficiently weak coupling we can write

$$\Lambda = 4\pi\lambda_s \cos\frac{2\pi k}{N}\left(1 - \cos\frac{2\pi m}{N}\right) - 1 + \mathcal{O}(\lambda_s^2),$$

but in practice to be safe we should stick to a general formula, as the difference is notable already at $\lambda_s = 0.01$.


Eigenvalue λ as a function of m
$k = 0$
$k = 2, 10$
$k = 3, 9$
$k = 4, 8$
$k = 6$
0.2
0.1
0.0
−0.1
−0.2
λ
0
2
4
6
8
10
m


Eigenvalue λ as a function of k
$m = 0$
$m = 3, 9$
$m = 6$
0.2
0.1
0.0
−0.1
−0.2
λ
0
2
4
6
8
10
k

Sine plus cosine coupling

[try14c]

Formulation

$$\dot{\varphi}_j = \omega_0 \left(1 + \lambda_s \sin(\varphi_j - \varphi_{j-1}) + \lambda_s \sin(\varphi_j - \varphi_{j-1}) + \lambda_c \cos(\varphi_j - \varphi_{j-1}) + \lambda_c \cos(\varphi_j - \varphi_{j-1})\right) \tag{14}$$

Let's consider a complex perturbation

$$\varphi_j\big|_{t=0} = -\frac{2\pi k}{N} j + \varepsilon \exp(-i\frac{2\pi m}{N} j)$$

Derivation

Again substitute into derivative [see pdf notes 2019-08] and end up with

$$\dot{\varphi}_j = \omega + A\varepsilon \exp(-i\frac{2\pi m}{N} j) + \mathcal{O}(\lambda_0 \varepsilon^2 \omega_0),$$

where

$\lambda_0 = \mathcal{O}(\lambda_s, \lambda_c)$, e.g. $\lambda_0 = (\lambda_s^2 + \lambda_c^2)^{\frac{1}{2}}$ - measure of coupling strength,

Results

ω - effective frequency[5]

$$\omega = \omega_0 (1 + \lambda_c \cos\frac{2\pi k}{N}) + \mathcal{O}(\lambda_0 \varepsilon^2 \omega_0),$$

A - perturbation change rate — Lyapunov exponent

$$A = 2\lambda_s \omega_0 \cos\frac{2\pi k}{N} \left(1 - \cos\frac{2\pi m}{N}\right) - 2i\lambda_c \omega_0 \sin\frac{2\pi k}{N} \sin\frac{2\pi m}{N},$$

Eigenvalue of logarithm of linearized Poincare map *(dimensionless Lyapunov exponent)* $\lambda = AT$, where $T = \frac{2\pi}{\omega}$

$$\lambda = \frac{4\pi}{1 + \lambda_c \cos\frac{2\pi k}{N}} \left(\lambda_s \cos\frac{2\pi k}{N} \left(1 - \cos\frac{2\pi m}{N}\right) - i\lambda_c \sin\frac{2\pi k}{N} \sin\frac{2\pi m}{N}\right)$$

Floquet multiplier

$$\Lambda = e^{\lambda}$$

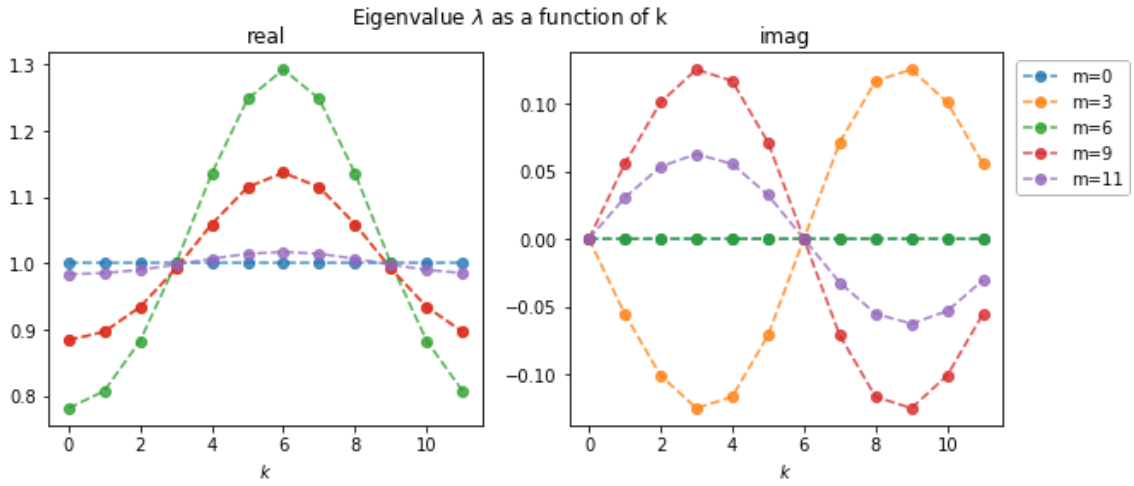
[5] Higher order terms scaling as ε^2 confirmed in simulation

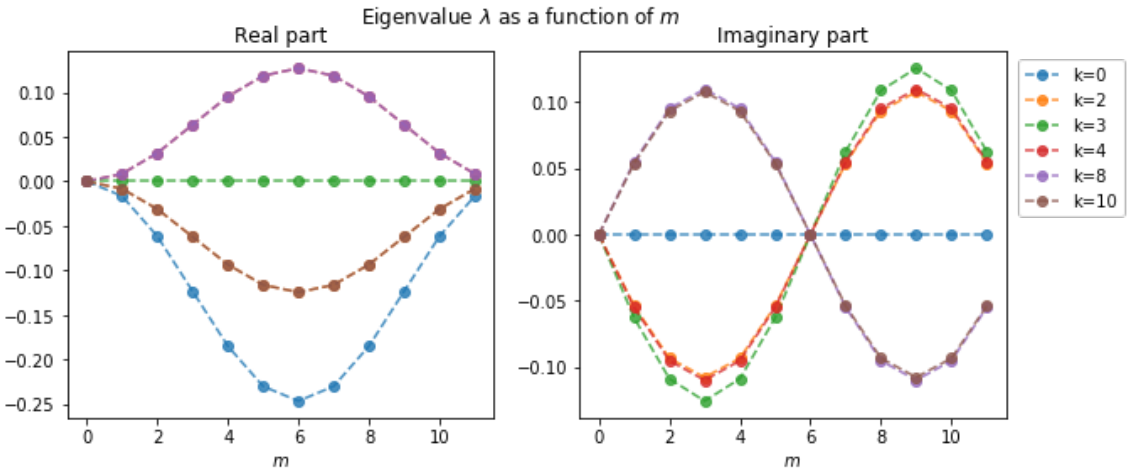
Conclusions. Visualizations

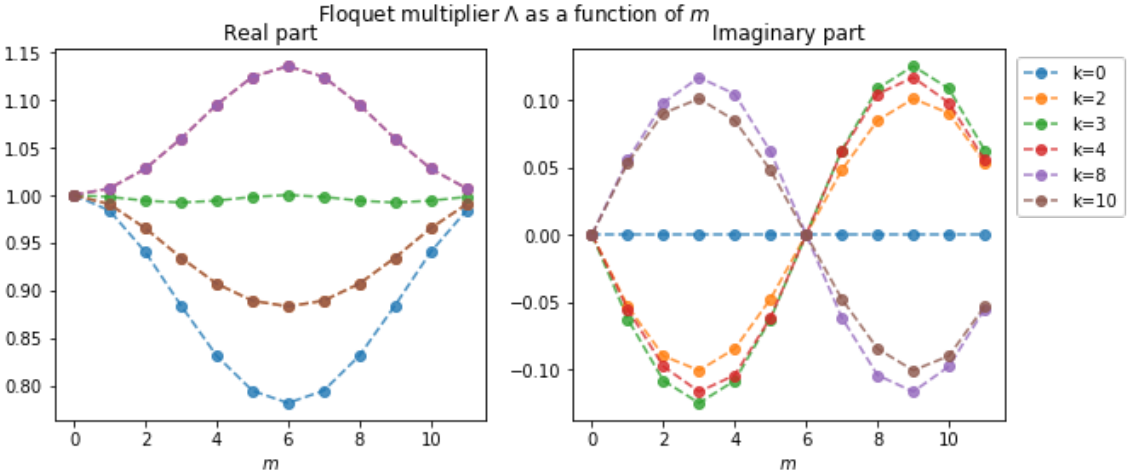
- Fixed points (limit cycles) are pure m-twists.
- Frequency depends on cosine coupling strength, and phase difference at the limit cycle.
- Now eigenvalues and eigenvectors are in general complex, degeneracy is broken (but not always: $k = \pm\frac{N}{4}$).
- Lyapunov exponent at $k = \pm\frac{N}{4}$ is purely imaginary, degeneracy $= 2$.
- At any fixed point, all Lyapunov exponents have either all non-negative, or all non-positive real parts.
- Contributions from sine and cosine coupling terms are additive in Lyapunov exponent, but not in dimensionless one, and especially not in Poincare map eigenvalue, unless λ_0 is sufficiently small.
- Checked the difference between simulation and analytical calculations: only higher order terms remain, as predicted by analysis.

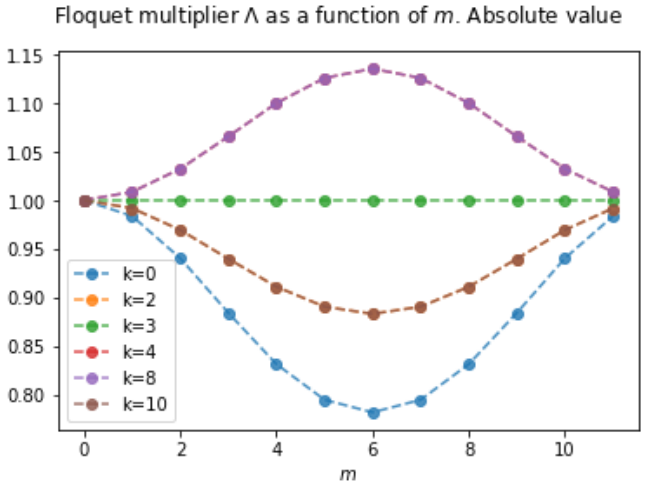
Visualize eigenvalues at $\lambda_s = -0.01$, $\lambda_c = 0.01$, $N = 12$.

Lyapunov/Poincare exponents λ and Floquet multipliers Λ change similarly with m, but not exactly in the same way. Lyapunov exponents which have the same imaginary part, but different real time evolve in time differently, and therefore Poincare map eigenvalues have slightly different imaginary parts.




Eigenvalue λ as a function of m
Real part
Imaginary part
m
k=0
k=2
k=3
k=4
k=8
k=10


Floquet multiplier Λ as a function of m
Real part
Imaginary part
m
k=0
k=2
k=3
k=4
k=8
k=10


Floquet multiplier Λ as a function of m. Absolute value
m
k=0
k=2
k=3
k=4
k=8
k=10

Analytical derivation of eigenvalues for a chain of oscillators with sinusoidal and cosinusoidal coupling

Sin+cos eigenvalues

28.08.19

$$\dot{\varphi}_j = \omega_0 \left(1 + \lambda_s \sin(\varphi_j - \varphi_{j-1}) + \lambda_s \sin(\varphi_j - \varphi_{j+1}) + \lambda_c \cos(\varphi_j - \varphi_{j-1}) + \lambda_c \cos(\varphi_j - \varphi_{j+1})\right)$$

Let $\varphi_j|_{t=0} = \underbrace{-\frac{2\pi k}{N} j}_{\text{fixpoint. m-twisted, k}} + \underbrace{\varepsilon e^{-i\frac{2\pi m}{N} j}}_{\text{perturbation; complex exponent of on-twisted with wave number m}}$

$c_j := \cos$

$s_j := \sin \quad s_m^j := \sin\left(\frac{2\pi m}{N} j\right) \qquad \delta_j := \varphi_j - \varphi_{j-1}$

$e_j := c + is$

$$\delta_j|_{t=0} = \varphi_j - \varphi_{j-1} = \varepsilon\left(e_m^{-j} - e_m^{-(j-1)}\right) - \frac{2\pi k}{N}$$

Let $\delta_j = \frac{2\pi k}{N} + \varepsilon \delta_{j1}$

Then

From $\dot{\varphi}_j|_{t=0}$: $\dot{\delta}(\delta_j) = \dot{s}\left(-\frac{2\pi k}{N}\right) \cdot \varepsilon(\varepsilon\delta_{j1}) + \dot{c}\left(\frac{2\pi k}{N}\right) s(\varepsilon\delta_{j1}) =$

$$= -s_k' \cdot (1 + O(\varepsilon^2)) + c_k' \cdot (\varepsilon\delta_{j1} + O(\varepsilon^3)) =$$

$$= -s_k' + \varepsilon c_k' \delta_{j1}$$

$$\dot{\delta}(\delta_j) - \dot{\delta}(\delta_{j+1}) = \varepsilon c_k'(\delta_{j1} - \delta_{(j+1)1}) + O(\varepsilon^2)$$

$$c(\delta_j) = c\left(-\frac{2\pi k}{N} + \varepsilon\delta_{j1}\right) = c\left(\frac{2\pi k}{N}\right) c(\varepsilon\delta_{j1}) - s\left(-\frac{2\pi k}{N}\right) s(\varepsilon\delta_{j1}) =$$

$$= c_k' + \varepsilon s_k' \delta_{j1} + O(\varepsilon^2)$$

$$c(\delta_j) + c(\delta_{j+1}) = 2c_k' + \varepsilon s_k'(\delta_{j-1} + \delta_{j+1,1}) + O(\varepsilon^2)$$

$\delta_{j1} = e_m^{-j} - e_m^{-j+1} = c_m^j - is_m^j - c_m^{j-1} + s_m^{j-1} = c_m^j(1 - c_m') + (-s_m' s_m^j)$

$c_m^{j-1} = c_m' c_m^j + s_m' s_m^j$

$s_m^{j-1} = c_m' s_m^j - s_m' c_m^j$

$+ is_m^j(c_m' - 1) + (-i) s_m' c_m^j = (1 - c_m') e_m^{-j} - is_m' e_m^{-j}$

$\delta_{j+1,1} = e_m^{-j-1} - e_m^{-j} = -(1 - c_m') e_m^{-j} - is_m' e_m^{-j}$

$$\Rightarrow \delta_{j1} - \delta_{j+1,1} = 2(1 - c'_m) e^{ij}_m$$

$$\delta_{j1} + \delta_{j+1,1} = -2i s'_m e^{ij}_m$$

$$\dot{\varphi}_j = \omega_0 (1 + \lambda_s (S(\delta_j) - S(\delta_{j+1})) + \lambda_c (C(\delta_j) + C(\delta_{j+1}))$$

$$= \omega_0 (1 + \lambda_s \varepsilon c'_k (\delta_{j1} - \delta_{j+1,1}) + \lambda_c s'_k \varepsilon (\delta_{j1} + \delta_{j+1,1}) + 2\lambda_c c'_k) + O(\varepsilon^2 \lambda_0 \omega_0) =$$

$$= \omega_0 (1 + 2\lambda_c c'_k + \varepsilon e^{ij}_m (2\lambda_s c'_k (1 - c'_m) - 2i s'_k s'_m \lambda_c)) + O(\varepsilon^2 \lambda_0 \omega_0)$$

$$\omega = \left(1 + 2\lambda_c \cos \frac{2\pi k}{N}\right) \omega_0$$

$$A = \left(2\lambda_s \cos \frac{2\pi k}{N} \left(1 - \cos \frac{2\pi m}{N}\right) - 2i\lambda_c \sin \frac{2\pi k}{N} \sin \frac{2\pi m}{N}\right) \omega_0 \qquad \lambda_0 = (\lambda_c^2 + \lambda_s^2)^{1/2}$$

A: Lyapunov

$$\lambda = AT = A \frac{2\pi}{\omega} = \frac{4\pi \left(\lambda_s \cos \frac{2\pi k}{N} \left(1 - \cos \frac{2\pi m}{N}\right) - i\lambda_c \sin \frac{2\pi k}{N} \sin \frac{2\pi m}{N}\right)}{1 + 2\lambda_c \cos \frac{2\pi k}{N}}$$

λ: Lyapunov, dimensionless

$$\Lambda = e^{AT}$$

$\lambda_c = 0$: pure sine coupling

$\omega = \omega_0$

$$\lambda = 4\pi \lambda_s \cos \frac{2\pi k}{N} \left(1 - \cos \frac{2\pi m}{N}\right)$$

Real perturbations: $\sin \frac{2\pi m}{N} j$, $\cos \frac{2\pi m}{N} j$