Exercise Medical Robotics CS4270 – Exercise Sheet 6

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Motion Correlation

SOLUTIONS

Please submit your solutions before next Monday at 14:15. The names of all group members must be included in the solution sheets/files. Handwritten solutions can be either submitted in the Institute for Robotics and Cognitive Systems (postbox in front of room 98) or scanned and uploaded in Moodle. MATLAB codes must be properly and briefly commented and uploaded in Moodle.

1 Motion Correlation I (10 Points)

Suppose we are given a set of data points $\{(1,9),(2,12),(3,33),(4,39),(5,50)\}$. To determine the motion correlation of this points, we may find the line that best fits the data. Thus, our goal is to fit the line $y = mx + b$ to the data. Then, we can simply setup a linear system as:

$$\begin{aligned} b + 1m &= 9 \\ b + 2m &= 21 \\ b + 3m &= 33 \\ b + 4m &= 39 \\ b + 5m &= 50 \end{aligned} \tag{1}$$

We can see that the system is over-determined. An approach to solve this system is to minimize the expression:

$$(b + 1m - 9)^2 + \cdots + (b + 5m - 50)^2 \tag{2}$$

(a) Find the differentiation of equation (2) with respect to b and m, and determine the solution of the linear system (1). (5P)

$$f = (b + m - 9)^2 + (b + 2m - 21)^2 + (b + 3m - 33)^2 + (b + 4m - 39)^2 + (b + 5m - 50)^2$$

$$\frac{df}{dm} = 2(b + m - 9) + 4(b + 2m - 21) + 6(b + 3m - 33) + 8(b + 4m - 39) + 10(b + 5m - 50)$$

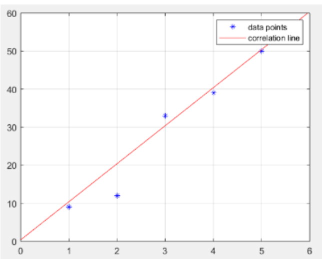
$$\frac{df}{dm} = 30b + 110m - 1112 = 15b + 55m - 556$$

$$\frac{df}{db} = 2(b + m - 9) + 2(b + 2m - 21) + 2(b + 3m - 33) + 2(b + 4m - 39) + 2(b + 5m - 50)$$

$$\frac{df}{db} = 10b + 30m - 304 = 5b + 15m - 152$$

Solving by minimizing the two derivatives to zero

$b = 0.4$ and $m = 10$



(b) We can rewrite equation 1 into a matrix form:

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix}, w = \begin{pmatrix} b \\ m \end{pmatrix}, y = \begin{pmatrix} 9 \\ 21 \\ 33 \\ 39 \\ 50 \end{pmatrix} \tag{3}$$

Then we will have a system, called normal equation, as:

$$B^T B w = B^T y \tag{4}$$

Solve the normal equation (4) by using Gaussian elimination and show that the least squares solution obtained by the approach in question 1(a) is the same as the solution obtained from the normal equation. (5P)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 9 \\ 21 \\ 33 \\ 39 \\ 50 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 15 \\ 15 & 55 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 152 \\ 556 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} 0.4 \\ 10 \end{pmatrix} \#$$

Hint: Matlab function mldivide “\” to solve linear equation systems w = B' * B \ B' * y

2 Motion Correlation II (12 Points)

In radiation therapy internal motion, e.g. obtained via implanted gold markers, needs to be linked to external motion, e.g. measured with IR LEDs attached to the patient's chest. In the simplest case this could be done using least squares regression, which fits a hyperplane $f(x)$ to the data $\{x_i, y_i\}_{i=1\ldots N}$ by minimizing the sum of the squared fitting errors e_i^2.

$$\min(L(w)) = \min\left(\sum_{i=1}^{N} e_i^2\right) = \min\left(\sum_{i=1}^{N} (f(x_i) - y_i)^2\right)$$

with the joint parameter vector $w = [w_0; b] \in \mathbb{R}^{(M+1)\times 1}$ containing slope w_0 and offset b in a M-dimensional space (to allow for b in the vector we concatenate a one to each data vector resulting in $\mathbb{R}^{(M)} \to \mathbb{R}^{(M+1)}$ for x). The hyperplane is given by $f(x) = w^T x$.

(a) By exploiting the derivative of the loss function L, derive the normal equation, which analytically returns the optimal parameter vector for the least squares problem
Hint: Replace the sum above by introducing a data matrix $X \in \mathbb{R}^{(M+1)\times N}$ and $y \in \mathbb{R}^{1\times N}$. (5P)

Refer to matrix derivatives for scalar-by-vector-identities

Condition	Expression	Numerator layout, i.e. by $\mathbf{x}^\top$; result is row vector	Denominator layout, i.e. by $\mathbf{x}$; result is column vector
$\mathbf{u} = \mathbf{u}(\mathbf{x})$, $\mathbf{v} = \mathbf{v}(\mathbf{x})$	$\frac{\partial(\mathbf{u}\cdot\mathbf{v})}{\partial \mathbf{x}} = \frac{\partial \mathbf{u}^\top \mathbf{v}}{\partial \mathbf{x}} =$	$\mathbf{u}^\top \frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \mathbf{v}^\top \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \frac{\partial \mathbf{v}}{\partial \mathbf{x}}$ in numerator layout	$\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\mathbf{v} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{u}$ $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \frac{\partial \mathbf{v}}{\partial \mathbf{x}}$ in denominator layout

(Wikipedia: Matrix calculus)

Loss function $L(w) = \sum_{i=1}^{N}(f(x_i) - y_i)^2 = \sum_{i=1}^{N}(w^T x_i - y_i)^2 = (\underbrace{w^T x - y}_{u})(\underbrace{w^T x - y}_{v})^T$

while the term $(w^T x - y)$ and its transpose are scalar, that means they are equal to each other,

therefore $\frac{dL}{dw} = uv^T + vu^T$ with $L \in \mathbb{R} \triangleq Scalar \ (L = L^T)$.

$$\frac{dL}{dw} = x(w^T x - y)^T + x^T(w^T x - y)$$

$$= 2x(w^T x - y)^T = 2x^T(w^T x - y)$$

$$\frac{dL}{dw} = 2x^T(w^T x - y) = 0$$ set derivative to zero to get extremum value

$$L = w^T x x^T w - y x^T w - w^T x y^T + y y^T = w^T x x^T w - 2w^T x y^T + y y^T$$

$$\frac{dL}{dw} = w^T x x^T + w^T x^T x - 2y x^T = 2w^T x x^T - 2y x^T = 0$$

so

$$w^T = y x^T (x x^T)^{-1}$$

therefore

$w = (x x^T)^{-1} \ x y^T$ is the normal equation

(b) Real chest movement exhibits hysteresis as shown in Figure 1. This means there is different and nonlinear behavior for the inhale and exhale phase. Describe how you would use the least squares problem and the normal equation to fit a polynomial of order d into some data (instead of a linear hyperplane). Write a MATLAB script which fits

i. a linear function

ii. a second order polynomial, and

iii. third order polynomial

to the inhale phase of the data contained in '$respiration.mat$'. The fitted polynomial shall contain all powers of x below the order value $d \in \{1,2,3\}$. Plot each result and state the remaining RMS error. Comment on your results. (7P)

See attached code *script_correlation.m* and related files.

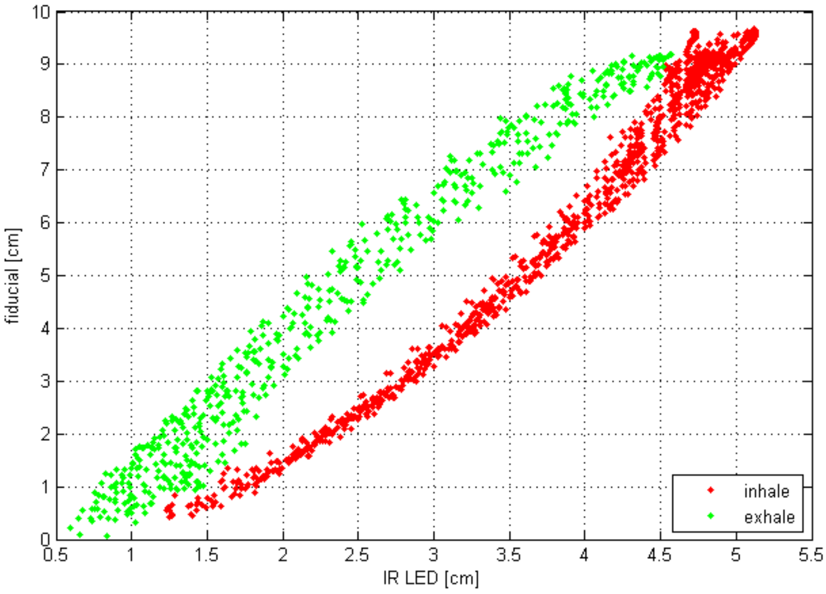


Figure 1: Respiratory hysteresis curve. The red part correlates LEDs and fiducials for the inhale and the green part for the exhale phase.