Exercise Medical Robotics CS4270 – Exercise Sheet 5

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Treatment Planning

SOLUTIONS

Please submit your solutions before next Monday at 14:15. The names of all group members must be included in the solution sheets/files. Handwritten solutions can be either submitted in the Institute for Robotics and Cognitive Systems (postbox in front of room 98) or scanned and uploaded in Moodle. MATLAB codes must be properly and briefly commented and uploaded in Moodle.

1 Treatment Planning (20 Points)

1.1 Graphic Solution of LP Problems

Given the Linear Programming problem, maximize

$$5x_1 + x_2 \tag{1}$$

subject to

(I) $x_1 + x_2 \leq 10$ (2)

(II) $x_1 \geq 5$ (3)

(III) $x_2 \geq 4$ (4)

Determine a solution by computing all intersection points of bounding lines and inserting them into objective function. Illustrate those points with the feasible region. (4P)

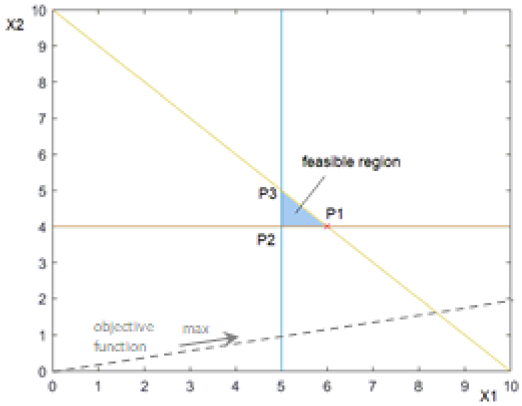
$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ -4 \end{pmatrix}$$

Intersections are at $p_1 = (6,4)$, $p_2 = (5,4)$ and $p_3 = (5,5)$

By **maximizing** the term $z = 5x_1 + x_2$

$z_1 = 34$, $z_2 = 29$ and $z_3 = 30$ is at its maximum at $x_1 = 6$ and $x_2 = 4$

Note: (I) and (III) are the **carrier lines** of the optimal solution point.



1.2 Dual of a Linear Program

Suppose we are given a linear program in matrix form as

$$(\max cx \;\; s.t. \;\; Ax \leq b) \tag{5}$$

The dual of an LP problem is also an LP problem. Here the roles of m and n, and the roles of the vectors b and c are exchanged. The dual LP is obtained by introducing new variables $(\alpha_1, \dots, \alpha_m)$:

$$(\min b\alpha \;\; s.t. \;\; A^T\alpha \geq c) \tag{6}$$

Here $A^T\alpha$ denotes the transpose of the matrix A, and the vector α denotes the vector of the variables α_i, i.e. $\alpha = (\alpha_i, \dots, \alpha_m)$. The variables $(\alpha_i, \dots, \alpha_m)$ are called dual variables. To differentiate between the original version of the LP problem, as in equation 5, and the dual in equation 6, the original version is also called the primal linear program.

(a) Set up the dual of the linear program in exercise 1.1. Illustrate the feasible region of this dual system, and find a solution vector of the dual LP by evaluating the objective function at the intersection points defined by the (dual) constraints. (6P)

Primal LP Problem: max. $5x_1 + x_2$

subject to

$1x_1 \;\; + 1x_2 \leq \;\; 10$ (I)

$-1x_1 + 0x_2 \leq -5$ (II)

$0x_1 \;\;\; -1x_2 \leq -4$ (III)

Dual LP Problem: **<u>minimize</u>** the term $10\alpha_1 - 5\alpha_2 - 4\alpha_3$

subject to

$1\alpha_1 - 1\alpha_2 + 0\alpha_2 \geq 5$ and

$1\alpha_1 + 0\alpha_2 - 1\alpha_3 \geq 1$

whereas

$\alpha_1 \geq 0$

$\alpha_2 \geq 0$

$\alpha_3 \geq 0$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

(I) (II) (III)

$1\alpha_1 - 1\alpha_2 + 0\alpha_2 = 5$

$1\alpha_1 + 0\alpha_2 - 1\alpha_3 = 1$

(II) is not a carrier line → $\alpha_2 = 0$

$\alpha_1 = 5$, and

$\alpha_3 = 4$

$z = 10 * \; 5 - 5 * 0 - 4 * 4 = 34$

(b) Let x_0 be a solution vector of a linear program. Then a solution vector y_0 of the dual linear program can also be found by the equilibrium theorem of linear programming. This theorem relates inequalities of the primal problem to variables y_i of the dual problem. Notice that the number of inequalities of the primal is m, and the number of variables of the dual is also m. Specifically, from the above graphical solutions, you will find that some of the inequalities are the *carrier lines* of the solution point.

I.e. the solution point x_0 is the intersection point of two lines. These two lines will be called the *carrier lines* of the solution point. According to the equilibrium theorem, the carrier lines correspond exactly (in terms of their indexes) to the dual variables having *non-zero* values in the solution y_0. This theorem holds both ways.

From this information alone, determine the solution of the dual for the LP problem in exercise 1.1, given the solution x_0 for the primal problem. (10P)

From $5x_1 + x_2 = 10\alpha_1 - 5\alpha_2 - 4\alpha_3 = 34$

$5x_1 + x_2 = 34$

$x_2 = 34 - 5x_1$

and from (I): $x_1 + x_2 = 10$

$x_1 + 34 - 5x_1 = 10$

$-4x_1 = -24$

$x_1 = 6$ and $x_2 = 4$

Or

from (III): $0x_1 + 1x_2 = 4$

$x_2 = 4$

and from (I): $x_1 + x_2 = 10$

$x_1 + 4 = 10$

$x_1 = 6$ and $x_2 = 4$

But not from (II) because this is not a carrier line!

The inequality constraints of the carrier lines becomes equalities by changing the "≤" or "≥" to "=".