Exercise Medical Robotics CS4270 – Exercise Sheet 9

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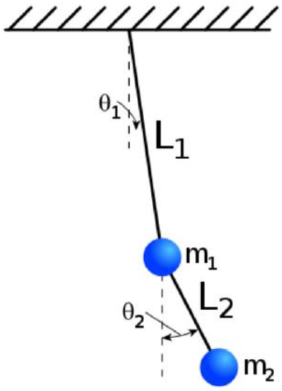
Dynamics

SOLUTIONS

1 Pendulum Motion

Refer to the attached MATLAB code $pendulum\_motion.m$.

2 Double Pendulum Motion



(a) Assume the bobs have mass m_1 and m_2, and the rods in this double pendulum are massless. Derive the kinetic $K(\theta, \dot{\theta})$ and potential $P(\theta)$ energy of the pendulum.

Potential energy

$P(\theta) = P_1(\theta) + P_2(\theta) = m_1 g h_1 + m_2 g h_2$

whereas

$h = -l\cos(\theta)$

therefore

$P(\theta) = -(m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2)$

Kinetic energy

$K(\theta, \dot{\theta}) = K_1(\theta, \dot{\theta}) + K_2(\theta, \dot{\theta}) = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

whereas

$v^2 = \dot{x}^2 + \dot{y}^2 = \left(l \sin(\theta)\dot{\theta}\right)^2 + \left(l\cos(\theta)\dot{\theta}\right)^2$

therefore

$K(\theta, \dot{\theta}) = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 \dot{\theta}_1 l_2 \dot{\theta}_2 (\cos(\theta_1 - \theta_2))\right)$

(b) As the Lagrange function defined as, set up the Lagrange equation and solve for the equation of motion for a double pendulum.

$$L(\theta,\dot{\theta}) = K(\theta,\dot{\theta}) - P(\theta)$$

from the solutions in part 2(a)

$$L(\theta,\dot{\theta}) = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1\dot{\theta}_1 l_2\dot{\theta}_2(\cos(\theta_1-\theta_2))\right) + (m_1+m_2)gl_1\cos(\theta_1) + m_2 g l_2\cos(\theta_2)$$

To find the angular acceleration $\ddot{\theta}$, we first calculate the torques using the Lagrange theorem

$$\tau_n = \frac{d}{dt}\frac{dL}{d\dot{\theta}_n} - \frac{dL}{d\theta_n}$$

$$\tau_1 = m_1 l_1^2\ddot{\theta}_1 + m_2 l_1^2\ddot{\theta}_1 + m_1 l_1 l_2\ddot{\theta}_2\cos(\theta_2-\theta_1) - m_2 l_1 l_2\dot{\theta}_2^2\sin(\theta_2-\theta_1) + (m_1+m_2)gl_1\sin(\theta_1)$$

$$\tau_2 = m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_2-\theta_1) + m_2 l_1 l_2\dot{\theta}_2^2\sin(\theta_2-\theta_1) + m_2 g l_2\sin(\theta_2)$$

we minimize the torque to zero to set the equations of angular acceleration and get

$$\ddot{\theta}_1 = \frac{-m_2 l_1 v_1^2\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) - g\sin(\theta_2)\cos(\theta_1-\theta_2) - m_2 l_2 v_2^2\sin(\theta_1-\theta_2) - (m_1+m_2)g\sin(\theta_1)}{l_1(m_1+m_2) - m_2 l_1\cos^2(\theta_1-\theta_2)}$$

$$\ddot{\theta}_2 = \frac{m_2 l_2 v_2^2\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) + (m_1+m_2)g\sin(\theta_1)\cos(\theta_1-\theta_2) + (m_1+m_2)l_1 v_1^2\sin(\theta_1-\theta_2) - (m_1+m_2)g\sin(\theta_2)}{l_2(m_1+m_2) - m_2 l_2\cos^2(\theta_1-\theta_2)}$$

(c) (Optional) Given mass $m_1 = m_2 = 0.5\ kg$ and rod $l_1 = l_2 = 0.5\ m$, visualize the motion of the double pendulum for time $t = 0$ to $10\ s$, initial $\theta_1 = -90°$ and $\theta_2 = 45°$.

Refer to the attached MATLAB code $double\_pendulum\_motion.m$.