Exercise Medical Robotics CS4270 – Exercise Sheet 8

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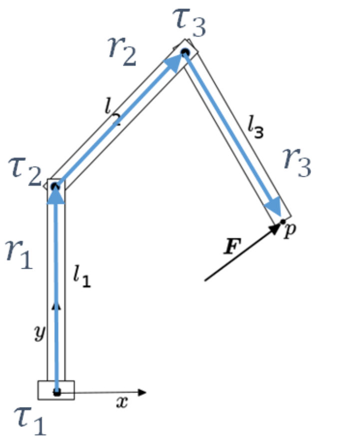
Motion Replication

SOLUTIONS

1 Motion Replication

a) The position vectors r of this manipulator are:



$$r_1 = l_1 \begin{pmatrix} \cos(\Theta_1) \\ \sin(\Theta_1) \\ 0 \end{pmatrix} = l_1 \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix}$$

$$r_2 = l_2 \begin{pmatrix} \cos(\Theta_1 + \Theta_2) \\ \sin(\Theta_1 + \Theta_2) \\ 0 \end{pmatrix} = l_2 \begin{pmatrix} c_{12} \\ s_{12} \\ 0 \end{pmatrix}$$

$$r_3 = l_3 \begin{pmatrix} \cos(\Theta_1 + \Theta_2 + \Theta_3) \\ \sin(\Theta_1 + \Theta_2 + \Theta_3) \\ 0 \end{pmatrix} = l_3 \begin{pmatrix} c_{123} \\ s_{123} \\ 0 \end{pmatrix}$$

Hints: 1) Because it is planar, all r_z and f_z components are 0.

2) In the case of $l_1 = l_2 = l_3$ these factors can be omitted.

The torque vectors t and torque magnitudes τ results:

Remember $t = r \times f,\ \tau = |t| = \sqrt{{t_x}^2 + {t_y}^2 + {t_z}^2}$

$$t_3 = r_3 \times f = l_3 \begin{pmatrix} c_{123} \\ s_{123} \\ 0 \end{pmatrix} \times \begin{pmatrix} f_x \\ f_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (l_3\, c_{123})\, f_y - (l_3\, s_{123})\, f_x \end{pmatrix}$$

$$\boldsymbol{\tau_3} = |t_3| = (l_3\, c_{123})\, f_y - (l_3\, s_{123})\, f_x$$

$$t_2 = (r_2 + r_3) \times f = \left(l_2 \begin{pmatrix} c_{12} \\ s_{12} \\ 0 \end{pmatrix} + l_3 \begin{pmatrix} c_{123} \\ s_{123} \\ 0 \end{pmatrix} \right) \times \begin{pmatrix} f_x \\ f_y \\ 0 \end{pmatrix} = \begin{pmatrix} l_2\, c_{12} + l_3\, c_{123} \\ l_2\, s_{12} + l_3\, s_{123} \\ 0 \end{pmatrix} \times \begin{pmatrix} f_x \\ f_y \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ (l_2\, c_{12} + l_3\, c_{123})\, f_y - (l_2\, s_{12} + l_3\, s_{123})\, f_x \end{pmatrix}$$

$$\boldsymbol{\tau_2} = |t_2| = (l_2\, c_{12} + l_3\, c_{123})\, f_y - (l_2\, s_{12} + l_3\, s_{123})\, f_x$$

$$t_1 = (r_1 + r_2 + r_3) \times f = \left(l_1 \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix} + l_2 \begin{pmatrix} c_{12} \\ s_{12} \\ 0 \end{pmatrix} + l_3 \begin{pmatrix} c_{123} \\ s_{123} \\ 0 \end{pmatrix} \right) \times \begin{pmatrix} f_x \\ f_y \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} l_1\, c_1 + l_2\, c_{12} + l_3\, c_{123} \\ l_1\, s_1 + l_2\, s_{12} + l_3\, s_{123} \\ 0 \end{pmatrix} \times \begin{pmatrix} f_x \\ f_y \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ (l_1\, c_1 + l_2\, c_{12} + l_3\, c_{123})\, f_y - (l_1\, s_1 + l_2\, s_{12} + l_3\, s_{123})\, f_x \end{pmatrix}$$

$$\boldsymbol{\tau_1} = |t_1| = (l_1\, c_1 + l_2\, c_{12} + l_3\, c_{123})\, f_y - (l_1\, s_1 + l_2\, s_{12} + l_3\, s_{123})\, f_x$$

b) Matrix expression A oft the tortque magnitudes vector τ: (see lecture, extend Formula 9.11)

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} (l_1\,c_1 + l_2\,c_{12} + l_3\,c_{123})\,\boldsymbol{f_y} - (l_1\,s_1 + l_2\,s_{12} + l_3\,s_{123})\boldsymbol{f_x} \\ (l_2\,c_{12} + \,l_3\,c_{123})\,\boldsymbol{f_y} - (l_2\,s_{12} + \,l_3\,s_{123})\boldsymbol{f_x} \\ (l_3\,c_{123})\,\boldsymbol{f_y} - (l_3\,s_{123})\boldsymbol{f_x} \end{pmatrix}$$

$$= \begin{pmatrix} -l_1\,s_1 - l_2\,s_{12} - l_3\,s_{123} & l_1\,c_1 + l_2\,c_{12} + l_3\,c_{123} \\ -l_2\,s_{12} - \,l_3\,s_{123} & l_2\,c_{12} + \,l_3\,c_{123} \\ -l_3\,s_{123} & l_3\,c_{123} \end{pmatrix} \begin{pmatrix} \boldsymbol{f_x} \\ \boldsymbol{f_y} \end{pmatrix}$$

Derive A from schema $\tau = A\,f_B$, $f_B = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$ (see lecture, extend Formula 9.13)

$$\boldsymbol{A} = \begin{pmatrix} -l_1\,s_1 - l_2\,s_{12} - l_3\,s_{123} & l_1\,c_1 + l_2\,c_{12} + l_3\,c_{123} & 0 \\ -l_2\,s_{12} - \,l_3\,s_{123} & l_2\,c_{12} + \,l_3\,c_{123} & 0 \\ -l_3\,s_{123} & l_3\,c_{123} & 0 \end{pmatrix}$$

c) Showing that A is the transpose J^T of the Jacobian J

Calculate the origins p_{i-1} of the coordinate systems at the joints i:

$$\boldsymbol{p_0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{p_1} = p_0 + r_1 = l_1 \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix}, \quad \boldsymbol{p_2} = p_1 + r_2 = l_1 \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix} + l_2 \begin{pmatrix} c_{12} \\ s_{12} \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ 0 \end{pmatrix},$$

$$\boldsymbol{p_3} = p_2 + r_3 = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ 0 \end{pmatrix} + l_3 \begin{pmatrix} c_{123} \\ s_{123} \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{pmatrix},$$

$$\boldsymbol{p_3} - \boldsymbol{p_0} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{pmatrix}, \quad \boldsymbol{p_3} - \boldsymbol{p_1} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{pmatrix} - l_1 \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix} = \begin{pmatrix} l_2 c_{12} + l_3 c_{123} \\ l_2 s_{12} + l_3 s_{123} \\ 0 \end{pmatrix},$$

$$\boldsymbol{p_3} - \boldsymbol{p_2} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{pmatrix} - \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ 0 \end{pmatrix} = \begin{pmatrix} l_3 c_{123} \\ l_3 s_{123} \\ 0 \end{pmatrix}$$

Calculate the lines of the Jacobian:

$$\boldsymbol{z_0} \times (\boldsymbol{p_3} - \boldsymbol{p_0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{pmatrix} = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ 0 \end{pmatrix}$$

$$\boldsymbol{z_1} \times (\boldsymbol{p_3} - \boldsymbol{p_1}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} l_2 c_{12} + l_3 c_{123} \\ l_2 s_{12} + l_3 s_{123} \\ 0 \end{pmatrix} = \begin{pmatrix} -l_2 s_{12} - l_3 s_{123} \\ l_2 c_{12} + l_3 c_{123} \\ 0 \end{pmatrix}$$

$$\boldsymbol{z_2} \times (\boldsymbol{p_3} - \boldsymbol{p_2}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} l_3 c_{123} \\ l_3 s_{123} \\ 0 \end{pmatrix} = \begin{pmatrix} -l_3 s_{123} \\ l_3 c_{123} \\ 0 \end{pmatrix}$$

Creating the Jacobian J_v^T: (see lecture, Formula 9.32)

$$\boldsymbol{J_v^T} = \begin{pmatrix} z_0 \times (p_3 - p_0)^T \\ z_1 \times (p_3 - p_1)^T \\ z_2 \times (p_3 - p_2)^T \end{pmatrix} = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & l_1 c_1 + l_2 c_{12} + l_3 c_{123} & 0 \\ -l_2 s_{12} - l_3 s_{123} & l_2 c_{12} + l_3 c_{123} & 0 \\ -l_3 s_{123} & l_3 c_{123} & 0 \end{pmatrix}$$

$= \boldsymbol{A}$ (see above) **Proof!**