Exercise Medical Robotics CS4270 – Exercise Sheet 4

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Mutual Information and Registration

SOLUTIONS

Please submit your solutions before next Monday at 14:15. The names of all group members must be included in the solution sheets/files. Handwritten solutions can be either submitted in the Institute for Robotics and Cognitive Systems (postbox in front of room 98) or scanned and uploaded in Moodle. MATLAB codes must be properly and briefly commented and uploaded in Moodle.

1 Mutual Information and Registration (64 Points)

Mutual information is a measure that can be used to register two images with each other. It was shown in the lecture that there is a close link between the entropy and mutual information. The mutual information of two images A and B is given by

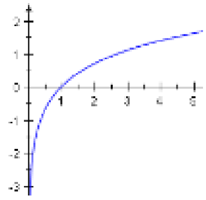
$$I(A,B) = \sum_{q_1 \in \Omega} \sum_{q_2 \in \Omega} p_{AB}\big(A(q_1), B(q_2)\big) \times \log\left(\frac{p_{AB}\big(A(q_1), B(q_2)\big)}{p_A\big(A(q_1)\big) \times p_B\big(B(q_2)\big)}\right)$$

where q_1 and q_2 denote a pixel in the overlapping region of the two images, and $A(q)$ denotes the intensity of pixel q in image A. An instance of A is also denoted by $A(q) = a$. The joint probability distribution is denoted by p_{AB} and the two marginal distributions are denoted by p_A and p_B, respectively.

This exercise will revisit some of the information theoretical background before addressing a simple practical example.

(a) Show that the entropy of any random variable A is always positive. (4P)

$$H(A) = -\sum_{q \in A} p(a) \log\big(p(a)\big)$$



Since $a > 0, \quad 0 \leq p(a) \leq 1, \quad \log\big(p(a)\big) \leq 0,$

the term $\quad p(a)\log\big(p(a)\big) \leq 0 \quad$ applies. $\quad$ Therefore $\quad H(A) \geq 0$

(b) The entropy is maximal for uniform probability distributions. Given an image consisting of N-bit gray values derive the upper entropy bound for such an image. (6P)

With 2^N possibility, probability $p(a) = \frac{1}{2^N}$ is uniform.

$$H(A) = -\sum_{a \in A}^{2^N} \frac{1}{2^N} \log_2\left(\frac{1}{2^N}\right) = -\sum_{a \in A}^{2^N} \frac{1}{2^N}(\log_2 1 - N \log_2 2) = -\sum_{a \in A}^{2^N} \frac{1}{2^N} * -N = \frac{2^N}{2^N} N = N$$

Whereas N is the upper bound.

(c) The mutual information of two images A and B can be decomposed into the entropy of both images and the joint entropy. Derive this decomposition. (8P)

$$MI(A,B) = \sum_{a\in\Omega}\sum_{b\in\Omega} p(a,b)\log\left(\frac{p(a,b)}{p(a)p(b)}\right)$$

$$= \sum_{a\in\Omega}\sum_{b\in\Omega} p(a,b)\log\big(p(a,b)\big) - \sum_{a\in\Omega} p(a)\log\big(p(a)\big) - \sum_{b\in\Omega} p(b)\log\big(p(b)\big)$$

$$= -H(A,B) + H(A) + H(B)$$

(d) Assume two extreme cases:

i. two random variables are entirely independent, and

$p(a,b) = p(a)p(b)$

$$I(A,B) = \sum_{a\in\Omega}\sum_{b\in\Omega} p(a,b)\log\left(\frac{p(a)p(b)}{p(a)p(b)}\right) = \sum_{a\in\Omega}\sum_{b\in\Omega} p(a,b)\log(1) = 0$$

ii. two random variables are entirely dependent (there exists an unambiguous function $f(a) = b; \forall a \in A \ \land \ \forall b \in B$).

From Bayes rule: $p(a,b) = p(a|b)p(b)$

with $$p(a|b) = \begin{cases} 1, & b = f(a) \\ 0, & otherwise \end{cases}$$

and $$p(a,b) = \begin{cases} p(a), & b = f(a) \\ 0, & otherwise \end{cases}$$

$$I(A,B) = \sum_{a\in\Omega}\sum_{b\in\Omega} p(a,b)\log\left(\frac{p(a|b)\,p(b)}{p(a)\ \ p(b)}\right) = \sum_{a\in\Omega}\sum_{b\in\Omega} p(a)\log\left(\frac{p(a,b)}{p(a)}\right)$$

$$= -\sum_{a\in\Omega} p(a)\log\big(p(a)\big) \ = \ H(A) \ = \ H(B)$$

The upper bound of mutual information is given by the image entropy.

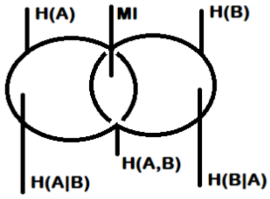
Derive the mutual information for these two cases. Considering your previous results, what is the upper bound for the mutual information of an N-bit image? (16P)

(e) Figure 1 illustrates two binary image pairs as they were considered in the lecture. Calculate the mutual information (pen and paper) for these two simple cases. (8P)
Hint: You may wish to use your results from the previous exercise.

(i) Independent (b: black, w: white)

$p_A(b) = 0.5,\ p_A(w) = 0.5,\ p_B(b) = 0,\ p_B(w) = 1$

$$I(A,B) = 0.5\log\left(\frac{0.5}{0.5\times 1}\right) = 0$$



(ii) Dependent (b: black, w: white)

$p_A(b) = 0.5,\ p_A(w) = 0.5,\ p_B(b) = 0.25,\ p_B(w) = 0.75,\ p_{A,B}(b,b) = 0.25,\ p_{A,B}(w,w) = 0.5,$
$p_{A,B}(b,w) = 0.25,\ p_{A,B}(w,b) = 0$

$$I(A,B) = 0.25\log\left(\frac{0.25}{0.5\times 0.25}\right) + 0.5\log\left(\frac{0.25}{0.5\times 0.75}\right) + 0.25\log\left(\frac{0.25}{0.5\times 0.75}\right) = 0.311$$

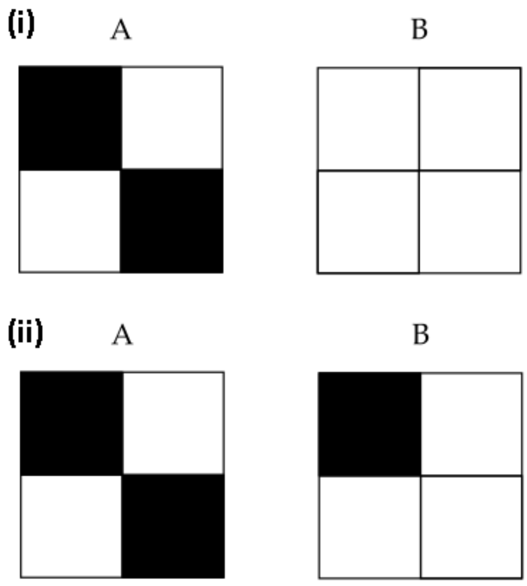


Figure 1: Two binary image pairs to be registered via mutual information

(f) Write a MATLAB function *reg MI()*, which takes two images A and B of arbitrary (yet known) gray value depth and outputs the mutual information for this image pair. (12P)

See attached code *reg_MI.m* and related files.

(g) A simple way of registering two images is represented by template matching. This approach shifts a smaller image (called template) across a larger one and computes a matching measure for each position. By searching the extreme value among various positions the registration between A and B is obtained. The algorithm requires the same sampling resolution in both images.

Implement a Matlab script that registers two images by considering only translational offsets in two dimensions. You can assume that image B (the template) is always smaller in size (given by the pixel dimensions) compared to image A. Use your function *reg MI()* as a matching measure. Test your script with the 2D CT and MR slices available from Moodle (take the CT as template). The sampling resolution is the same for both images. Plot the mutual information for pixel-wise shifts in both directions and give the maximum value as well as the corresponding translational shifts. Finally generate a registered image of MR slice size, where you place the CT image at the identified position. (12P)

See attached code *MI_solution.m* and related files. Translation = (35, 101)

(h) What problem occurs when incorporating rotations into this algorithm? Suggest a solution to resolve it. (4P)

Grids do not coincide anymore and therefore corresponding pixels would not align, interpolation or rounding of these pixels would be required.

(i) What is the advantage of using mutual information instead of SSD for registering between different imaging modalities? (2P)

MI is not sensitive towards intensity and is therefore imaging modality-independent.