

CHEATSHEET FOR MATHEMATICAL FINANCE STUDENTS

VEGA INSTITUTE FOUNDATION

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The Foreword

Part 1. Probability Theory and Stochastic Processes (TBD)

Part 2. Stochastic Calculus

1. GIRSANOV THEOREM AND NOVIKOV CONDITION

Let us take a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$, and a Brownian motion W_t defined on it.

Theorem 1.1. (*Girsanov*) Let μ_t be a predictable process such that $\int_0^T \mu_t^2 dt < \infty$ almost surely. Define a process

$$(1.1) \quad Z_t = \exp \left(- \int_0^t \mu_s dW_s - \frac{1}{2} \int_0^t \mu_s^2 ds \right)$$

If Z_t is a martingale, then

$$(1.2) \quad \tilde{W}_t = W_t + \int_0^t \mu_s ds$$

is a Brownian motion w.r.t. measure Q such that $dQ = Z_t dP$.

Theorem 1.2. (*Girsanov*) Let W_t be a d -dimensional Brownian motion with non-correlated components, μ_t be a d -dimensional predictable process such that $\int_0^T \|\mu_t\|^2 dt < \infty$ almost surely. Define a process

$$(1.3) \quad Z_t = \exp \left(- \int_0^t \mu_s \cdot dW_s - \frac{1}{2} \int_0^t \|\mu_s\|^2 ds \right)$$

If Z_t is a martingale, then

$$(1.4) \quad \tilde{W}_t = W_t + \int_0^t \mu_s ds$$

is a Brownian motion with non-correlated components w.r.t. measure Q such that $dQ = Z_t dP$.

When Z_t is a martingale? One of the easiest ways to answer this question is to remember the

Theorem 1.3. (*Novikov condition*) Let

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T \|\mu_t\|^2 dt \right) \right] < \infty.$$

Then

$$(1.5) \quad Z_t = \exp \left(- \int_0^t \mu_s \cdot dW_s - \frac{1}{2} \int_0^t \|\mu_s\|^2 ds \right)$$

is a martingale.

Theorem 1.4. If $Q \sim P$, $dQ = Z_t dP$, then for any \mathcal{F}_T -measurable r.v. X

$$(1.6) \quad \mathbb{E}^Q [X | \mathcal{F}_t] = \frac{1}{Z_t} \mathbb{E}^P [Z_T X | \mathcal{F}_t]$$

2. WEAK AND STRONG SOLUTIONS OF A SDE

Assume a one-dimensional SDE

$$(2.1) \quad dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x$$

Definition 2.1. A *weak* solution of equation (2.1) is a pair of processes (X_t, W_t) on some probability space with filtration $(\mathcal{F}_t)_{t \geq 0}$ such that W_t is a Brownian motion w.r.t. \mathcal{F}_t and

$$(2.2) \quad X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,$$

and both integrals exist for all $t \leq T$

Definition 2.2. A *strong* solution of equation (2.1) is a weak solution (X_t, W_t) , where X_t is adapted to filtration generated by W_t .

Definition 2.3. Equation (2.1) has a *weak uniqueness* of a solution, if for any two weak solutions, X_t and \tilde{X}_t are equal in distribution.

Definition 2.4. Equation (2.1) has a *strong uniqueness* of a solution, if for any two strong solutions, X_t and \tilde{X}_t are equal a.s.

Theorem 2.1. (*Itô*) Let there exist $C > 0$ such that

$$(2.3) \quad |b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| < C|x - y|$$

$$(2.4) \quad |b(t, x)| + |\sigma(t, x)| < C(1 + |x|)$$

Then there exists a strong unique solution for the equation (2.1).

Theorem 2.2. (*Zvonkin*) Let (2.1) satisfy the following conditions:

- (1) b is bounded
- (2) σ is continuous
- (3) $|\sigma(t, x) - \sigma(t, y)| < C\sqrt{|x - y|}$, $|\sigma(t, x)| \geq \epsilon > 0$

Then there exists a strong unique solution for the equation (2.1).

Theorem 2.3. (*Skorokhod*) Let $b(t, x)$, $\sigma(t, x)$ be bounded and continuous. Then there exists a weak unique solution for the equation (2.1).

Theorem 2.4. (*Struk, Varadan*) Let (2.1) satisfy the following conditions:

- (1) b is bounded
- (2) σ is continuous
- (3) $\sigma(t, x) \neq 0$

Then there exists a weak unique solution for the equation (2.1).

Definition 2.5. A SDE is called *homogeneous* if it has the form of

$$(2.5) \quad dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

Definition 2.6. For a homogeneous SDE we define the following functions:

$$(2.6) \quad \rho(x) := \exp\left(-\int \frac{2b(x)}{\sigma^2(x)}dx\right) \quad s(x) = \int \rho(x)dx$$

The function $s(x)$ is called a *scale*.

NB. Using the Itô's formula one can show that if $\forall x \ s(x) < \infty$, then for any weak solution X_t , $s(X_t)$ is a local martingale. Note that $\rho(x)$ is defined up to multiplication by a positive scalar, and $s(x)$ is defined up to an affine transformation.

Theorem 2.5. (*Engelbert, Schmidt*) Let (2.5) satisfy the following conditions:

- (1) $\sigma(t, x) \neq 0$
- (2) $\int_K \frac{1+|b(y)|}{\sigma^2(y)} dy < \infty$ for any compact set $K \subseteq \mathbb{R}$
- (3) one of the following conditions stand:

$$(2.7) \quad \int_{\mathbb{R}} \rho(x) dx < \infty \text{ or } \int_{\mathbb{R}} \rho(x) dx = \infty, \quad \int_{\mathbb{R}} \frac{|s(x)|}{\rho(x)\sigma^2(x)} dx = \infty$$

Then there exists a weak unique solution for the equation (2.5).

3. CLASSIFICATION OF SINGULAR POINTS OF A SDE

In this section we assume equation (2.5) with $\sigma(x) \neq 0 \ \forall x \in \mathbb{R}$.

Definition 3.1. A point $y \in \mathbb{R}$ is called a *singular point* of equation (2.5), if

$$(3.1) \quad \forall \epsilon > 0 \quad \int_{y-\epsilon}^{y+\epsilon} \frac{1+|b(x)|}{\sigma^2(x)} dx = \infty$$

Further we assume $x = 0$ to be the only singular point of a SDE (2.5). For some $a > 0$ define

$$(3.2) \quad \rho(x) := \exp\left(-\int_x^a \frac{2b(y)}{\sigma^2(y)} dy\right)$$

and

$$(3.3) \quad s(x) = \begin{cases} \int_0^x \rho(y) dy, & \text{if } \int_0^a \rho(y) dy < \infty \\ -\int_x^a \rho(y) dy, & \text{if } \int_0^a \rho(y) dy = \infty \end{cases}$$

Type	Conditions
0	$\frac{1+ b(y) }{\sigma^2(y)} dy < \infty$
1	$\int_0^a \rho(y) dy < \infty, \quad \int_0^a \frac{1+ b(y) }{\rho(y)\sigma^2(y)} dy = \infty, \quad \int_0^a \frac{1+ b(y) }{\rho(y)\sigma^2(y)} s(y) dy < \infty$
2	$\int_0^a \rho(y) dy < \infty, \quad \int_0^a \frac{1+ b(y) }{\rho(y)\sigma^2(y)} dy < \infty, \quad \int_0^a \frac{ b(y) }{\sigma^2(y)} dy = \infty$
3	$\int_0^a \rho(y) dy = \infty, \quad \int_0^a \frac{1+ b(y) }{\rho(y)\sigma^2(y)} s(y) dy < \infty$
4	$\int_0^a \rho(y) dy < \infty, \quad \int_0^a \frac{s(y)}{\rho(y)\sigma^2(y)} dy = \infty$
5	$\int_0^a \rho(y) dy = \infty, \quad \int_0^a \frac{1+ b(y) }{\rho(y)\sigma^2(y)} s(y) dy = \infty$
6	$\int_0^a \rho(y) dy < \infty, \quad \int_0^a \frac{1+ b(y) }{\rho(y)\sigma^2(y)} dy = \infty, \quad \int_0^a \frac{ s(y) }{\rho(y)\sigma^2(y)} dy < \infty$

TABLE 1. Classification of Finite Singular Points, Right Type

For some $a > 0, x \geq a$ define

$$(3.4) \quad \rho(x) := \exp\left(-\int_a^x \frac{2b(y)}{\sigma^2(y)} dy\right), \quad s(x) = -\int_x^\infty \rho(y) dy$$

Type	Conditions
A	
B	
C	

TABLE 2. Classification of $+\infty$

4. PDES AND SDEs

Theorem 4.1. (*Feynman-Kac*) Consider the following PDE

$$(4.1) \quad \frac{\partial u}{\partial t}(x, t) + \mu(x, t) \frac{\partial u}{\partial x}(x, t) + \frac{1}{2} \sigma^2(x, t) \frac{\partial^2 u}{\partial x^2}(x, t) - V(x, t)u(x, t) + f(x, t) = 0,$$

defined for all $x \in \mathbb{R}$ and $t \in [0, T]$, subject to the terminal condition $u(x, T) = \psi(x)$. Then the solution can be written as a conditional expectation

$$u(x, t) = \mathbb{E}^Q \left[\int_t^T e^{-\int_t^r V(X_\tau, \tau) d\tau} f(X_r, r) dr + e^{-\int_t^T V(X_\tau, \tau) d\tau} \psi(X_T) \middle| X_t = x \right]$$

under the probability measure Q such that X is an Itô process driven by the equation

$$(4.2) \quad dX = \mu(X, t)dt + \sigma(X, t)dW^Q,$$

with $W^Q(t)$ is a Wiener process under Q , and the initial condition for X_t is $X_t = x$.

Part 3. Mathematical Finance

5. BASIC DEFINITIONS

Let us take a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, and a d -dimensional Brownian motion W_t . There is a one riskless asset and n risky assets in the market with prices satisfying the following SDEs

$$(5.1) \quad dB_t = r_t B_t dt, \quad B_0 = 1$$

$$(5.2) \quad dS_t^i = \mu_t^i dt + \sigma_t^i \cdot dW_t,$$

where $r_t \in [-a, \infty)$, $\mu_t \in \mathbb{R}^n$, $\sigma_t \in \mathbb{R}^{n \times d}$ are good enough predictable processes, $\sigma_t^i \cdot dW_t := \sum_{j=1}^d \sigma_t^{ij} dW_t^j$.

Definition 5.1. A *complete market* is a market with two conditions:

- (1) Negligible transaction costs and therefore also perfect information,
- (2) There is a price for every asset in every possible state of the world (all bounded measurable liabilities are replicable).

Definition 5.2. A *strategy* is a predictable process $\pi_t = (G_t, H_t^1, \dots, H_t^n)$ such that there exist the following Itô integrals:

$$(5.3) \quad \int_0^t G_u dB_u, \quad \int_0^t H_u^i dS_u^i, \quad t \in [0, T]$$

A *self-financing* strategy is a trading strategy which requires no extra cost during the trading except for the initial capital, i.e. $dV_t^\pi = G_t dB_t + H_t \cdot dS_t$. An *acceptable* strategy is a strategy which has a lower bound ($V_t^\pi \geq -c$ a.s. $\forall t > 0$).

NB. Further we assume that all strategies are self-financing and acceptable.

Definition 5.3. A *price* of a portfolio is a process $V_t^\pi := G_t B_t + H_t \cdot S_t$.

Definition 5.4. *Absence of arbitrage* (AoA, NA) means that there exists no strategy π_t such that

- (1) $V_0^\pi = 0$
- (2) $V_T^\pi \geq 0$
- (3) $P(V_T^\pi > 0) > 0$

Definition 5.5. *Equivalent martingale measure* (EMM) is a probability measure $Q \sim P$ such that the discounted prices $S_t^{*,i} := \frac{S_t^i}{B_t}$ are martingales w.r.t. Q .

6. FUNDAMENTAL THEORETICAL RESULTS IN MATHEMATICAL FINANCE

Theorem 6.1. (*First Fundamental Theorem of Mathematical Finance*) There is no arbitrage in the market if and only if the EMM exists and is unique.

Theorem 6.2. A *fair price* of a replicable liability X is equal to

$$(6.1) \quad V_t(X) = B_t \mathbb{E}^Q \left[\frac{X}{B_T} \middle| \mathcal{F}_t \right],$$

where Q is an EMM.

Theorem 6.3. A *fair price interval* of a non-replicable liability X is

$$(6.2) \quad \left(B_t \inf_Q \mathbb{E}^Q \left[\frac{X}{B_T} \middle| \mathcal{F}_t \right], B_t \sup_Q \mathbb{E}^Q \left[\frac{X}{B_T} \middle| \mathcal{F}_t \right] \right),$$

where Q is an EMM.

Theorem 6.4. (*Second Fundamental Theorem of Mathematical Finance*) An arbitrageless market is complete, i.e. any bounded liability is replicable if, and only if an EMM is unique.

Part 4. Useful Books, Links, and Other Materials