CHEATSHEET FOR MATHEMATICAL FINANCE STUDENTS

VEGA INSTITUTE FOUNDATION

Contents

The Foreword

Part 1. Probability Theory and Stochastic Processes (TBD)

Part 2. Stochastic Calculus

1. Girsanov Theorem and Novikov Condition

Let us take a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^{T}, P\right)$, and a Brownian motion W_t defined on it.

Theorem 1.1. *(Girsanov)* Let μ_t be a predictable process such that $\int_0^T \mu_t^2 dt < \infty$ almost surely. Define a process

$$Z_t = exp\left(-\int_0^t \mu_s dW_s - \frac{1}{2}\int_0^t \mu_s^2 ds\right) \tag{1.1}$$

If Z_t is a martingale, then

$$\tilde{W}_t = W_t + \int_0^t \mu_s ds \tag{1.2}$$

is a Brownian motion w.r.t. measure Q such that $dQ = Z_t dP$.

Theorem 1.2. *(Girsanov)* Let W_t be a d-dimensional Brownian motion with non-correlated components, μ_t be a d-dimensional predictable process such that $\int_0^T \|\mu_t\|^2 dt < \infty$ almost surely. Define a process

$$Z_t = exp\left(-\int_0^t \mu_s \cdot dW_s - \frac{1}{2}\int_0^t \|\mu_s\|^2 ds\right) \tag{1.3}$$

If Z_t is a martingale, then

$$\tilde{W}_t = W_t + \int_0^t \mu_s ds \tag{1.4}$$

is a Brownian motion with non-correlated components w.r.t. measure Q such that $dQ = Z_t dP$.

When Z_t is a martingale? One of the easiest ways to answer this question is to remember the

Theorem 1.3. *(Novikov condition)* Let

$$\mathbb{E}\left[exp\left(\frac{1}{2}\int_0^T \|\mu_t\|^2 dt\right)\right] < \infty.$$

Then

$$Z_t = exp\left(-\int_0^t \mu_s \cdot dW_s - \frac{1}{2}\int_0^t \|\mu_s\|^2 ds\right) \tag{1.5}$$

is a martingale.

Theorem 1.4. If $Q \sim P$, $dQ = Z_t dP$, then for any $\mathcal{F}_T$-measurable r.v. X

$$\mathbb{E}^Q\left[X|\mathcal{F}_t\right] = \frac{1}{Z_t}\mathbb{E}^P\left[Z_T X|\mathcal{F}_t\right] \tag{1.6}$$

2. Weak and Strong Solutions of a SDE

Assume a one-dimensional SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad X_0 = x \tag{2.1}$$

Definition 2.1. A *weak* solution of equation (2.1) is a pair of processes (X_t, W_t) on some probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ such that W_t is a Brownian motion w.r.t. $\mathcal{F}_t$ and

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{2.2}$$

and both integrals exist for all $t \leq T$

Definition 2.2. A *strong* solution of equation (2.1) is a weak solution (X_t, W_t), where X_t is adapted to filtration generated by W_t.

Definition 2.3. Equation (2.1) has a *weak uniqueness* of a solution, if for any two weak solutions, X_t and $\tilde{X}_t$ are equal in distribution.

Definition 2.4. Equation (2.1) has a *strong uniqueness* of a solution, if for any two strong solutions, X_t and $\tilde{X}_t$ are equal a.s.

Theorem 2.1. *(Itô)* Let there exist $C > 0$ such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| < C|x - y| \tag{2.3}$$

$$|b(t, x)| + |\sigma(t, x)| < C(1 + |x|) \tag{2.4}$$

Then there exists a strong unique solution for the equation (2.1).

Theorem 2.2. *(Zvonkin)* Let (2.1) satisfy the following conditions:

(1) b is bounded
(2) σ is continuous
(3) $|\sigma(t, x) - \sigma(t, y)| < C\sqrt{|x - y|}$, $|\sigma(t, x)| \geq \epsilon > 0$

Then there exists a strong unique solution for the equation (2.1).

Theorem 2.3. *(Skorokhod)* Let $b(t, x)$, $\sigma(t, x)$ be bounded and continuous Then there exists a weak unique solution for the equation (2.1).

Theorem 2.4. *(Struk, Varadan)* Let (2.1) satisfy the following conditions:

(1) b is bounded
(2) σ is continuous
(3) $\sigma(t, x) \neq 0$

Then there exists a weak unique solution for the equation (2.1).

Definition 2.5. A SDE is called *homogeneous* if it has the form of

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \tag{2.5}$$

Definition 2.6. For a homogeneous SDE we define the following functions:

$$\rho(x) := exp\left(-\int \frac{2b(x)}{\sigma^2(x)}dx\right) \qquad s(x) = \int \rho(x)dx \tag{2.6}$$

The function $s(x)$ is called a *scale*.

NB. Using the Itô's formula one can show that if $\forall x \;\; s(x) < \infty$, then for any weak solution X_t, $s(X_t)$ is a local martingale. Note that $\rho(x)$ is defined up to multiplication by a positive scalar, and $s(x)$ is defined up to an affine transformation.

Theorem 2.5. *(Engelbert, Schmidt)* Let (2.5) satisfy the following conditions:

(1) $\sigma(t,x) \neq 0$
(2) $\int_K \frac{1+|b(y)|}{\sigma^2(y)} dy < \infty$ for any compact set $K \subseteq \mathbb{R}$
(3) one of the following conditions stand:

$$\int_{\mathbb{R}} \rho(x)dx < \infty \text{ or } \int_{\mathbb{R}} \rho(x)dx = \infty, \quad \int_{\mathbb{R}} \frac{|s(x)|}{\rho(x)\sigma^2(x)}dx = \infty \tag{2.7}$$

Then there exists a weak unique solution for the equation (2.5).

3. Classification of Singular Points of a SDE

In this section we assume equation (2.5) with $\sigma(x) \neq 0 \;\; \forall x \in \mathbb{R}$.

Definition 3.1. A point $y \in \mathbb{R}$ is called a *singular* point of equation (2.5), if

$$\forall \epsilon > 0 \qquad \int_{y-\epsilon}^{y+\epsilon} \frac{1+|b(x)|}{\sigma^2(x)}dx = \infty \tag{3.1}$$

Further we assume $x = 0$ to be the only singular point of a SDE (2.5). For some $a > 0$ define

$$\rho(x) := exp\left(-\int_x^a \frac{2b(y)}{\sigma^2(y)}dy\right) \tag{3.2}$$

and

$$s(x) = \begin{cases} \int_0^x \rho(y)dy, \text{ if } \int_0^a \rho(y)dy < \infty \\ -\int_x^a \rho(y)dy, \text{ if } \int_0^a \rho(y)dy = \infty \end{cases} \tag{3.3}$$

Type	Conditions
0	$\frac{1+\lvert b(y)\rvert}{\sigma^2(y)}dy < \infty$
1	$\int_0^a \rho(y)dy < \infty, \quad \int_0^a \frac{1+\lvert b(y)\rvert}{\rho(y)\sigma^2(y)}dy = \infty, \quad \int_0^a \frac{1+\lvert b(y)\rvert}{\rho(y)\sigma^2(y)}s(y)dy < \infty$
2	$\int_0^a \rho(y)dy < \infty, \quad \int_0^a \frac{1+\lvert b(y)\rvert}{\rho(y)\sigma^2(y)}dy < \infty, \quad \int_0^a \frac{\lvert b(y)\rvert}{\sigma^2(y)}dy = \infty$
3	$\int_0^a \rho(y)dy = \infty, \quad \int_0^a \frac{1+\lvert b(y)\rvert}{\rho(y)\sigma^2(y)}\lvert s(y)\rvert dy < \infty$
4	$\int_0^a \rho(y)dy < \infty, \quad \int_0^a \frac{s(y)}{\rho(y)\sigma^2(y)}dy = \infty$
5	$\int_0^a \rho(y)dy = \infty, \quad \int_0^a \frac{1+\lvert b(y)\rvert}{\rho(y)\sigma^2(y)}\lvert s(y)\rvert dy = \infty$
6	$\int_0^a \rho(y)dy < \infty, \quad \int_0^a \frac{1+\lvert b(y)\rvert}{\rho(y)\sigma^2(y)}dy = \infty, \quad \int_0^a \frac{\lvert s(y)\rvert}{\rho(y)\sigma^2(y)}dy < \infty$

Table 1. Classification of Finite Singular Points, Right Type

For some $a > 0, x \geq a$ define

$$\rho(x) := exp\left(-\int_a^x \frac{2b(y)}{\sigma^2(y)}dy\right), \qquad s(x) = -\int_x^\infty \rho(y)dy \tag{3.4}$$

Type	Conditions
A	
B	
C	

Table 2. Classification of $+\infty$

4. PDEs and SDEs

Theorem 4.1. *(Feynman-Kac)* Consider the following PDE

$$\frac{\partial u}{\partial t}(x,t) + \mu(x,t)\frac{\partial u}{\partial x}(x,t) + \tfrac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) - \\ - V(x,t)u(x,t) + f(x,t) = 0, \tag{4.1}$$

defined for all $x \in \mathbb{R}$ and $t \in [0,T]$, subject to the terminal condition $u(x,T) = \psi(x)$. Then the solution can be written as a conditional expectation

$$u(x,t) = \mathbb{E}^Q \left[\int_t^T e^{-\int_t^r V(X_\tau,\tau)d\tau} f(X_r, r)dr + e^{-\int_t^T V(X_\tau,\tau)d\tau}\psi(X_T) \middle| X_t = x \right]$$

under the probability measure Q such that X is an Itô process driven by the equation

$$dX = \mu(X,t)dt + \sigma(X,t)dW^Q, \tag{4.2}$$

with $W^Q(t)$ is a Wiener process under Q, and the initial condition for X_t is $X_t = x$.

Part 3. Mathematical Finance

5. Basic Definitions

Let us take a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, and a d-dimensional Brownian motion W_t. There is a one riskless asset and n risky assets in the market with prices satisfying the following SDEs

$$dB_t = r_t B_t dt, \qquad B_0 = 1 \tag{5.1}$$

$$dS_t^i = \mu_t^i dt + \sigma_t^i \cdot dW_t, \tag{5.2}$$

where $r_t \in [-a, \infty)$, $\mu_t \in \mathbb{R}^n$, $\sigma_t \in \mathbb{R}^{n \times d}$ are good enough predictable processes, $\sigma_t^i \cdot dW_t := \sum_{j=1}^d \sigma_t^{ij} dW_t^j$.

Definition 5.1. A *complete market* is a market with two conditions:

(1) Negligible transaction costs and therefore also perfect information,
(2) There is a price for every asset in every possible state of the world (all bounded measurable liabilities are replicable).

Definition 5.2. A *strategy* is a predictable process $\pi_t = \left(G_t, H_t^1, \dots, H_t^n\right)$ such that there exist the following Itô integrals:

$$\int_0^t G_u dB_u, \quad \int_0^t H_u^i dS_u^t, \quad t \in [0, T] \tag{5.3}$$

A *self-financing* strategy is a trading strategy which requires no extra cost during the trading except for the initial capital, i.e. $dV_t^\pi = G_t dB_t + H_t \cdot dS_t$. An *acceptable* strategy is a strategy which has a lower bound ($V_t^\pi \geq -c$ a.s. $\quad \forall t > 0$).

NB. Futher we assume that all strategies are self-financing and acceptable.

Definition 5.3. A *price* of a portfolio is a process $V_t^\pi := G_t B_t + H_t \cdot S_t$.

Definition 5.4. *Absence of arbitrage* (AoA, NA) means that there exists no strategy π_t such that

(1) $V_0^\pi = 0$
(2) $V_T^\pi \geq 0$
(3) $P(V_t^\pi > 0) > 0$

Definition 5.5. *Equivalent martingale measure* (EMM) is a probability measure $Q \sim P$ such that the discounted prices $S_t^{*,i} := \frac{S_t^i}{B_t}$ are martingales w.r.t. Q.

6. Fundamental Theoretical Results in Mathematical Finance

Theorem 6.1. *(First Fundamental Theorem of Mathematical Finance)* There is no arbitrage in the market if and only if the EMM exists and is unique.

Theorem 6.2. A *fair price* of a replicable liability X is equal to

$$V_t(X) = B_t \mathbb{E}^Q \left[\frac{X}{B_T} \middle| \mathcal{F}_t \right], \tag{6.1}$$

where Q is an EMM.

Theorem 6.3. A *fair price interval* of a non-replicable liability X is

$$\left(B_t \inf_Q \mathbb{E}^Q \left[\frac{X}{B_T} \middle| \mathcal{F}_t \right], B_t \sup_Q \mathbb{E}^Q \left[\frac{X}{B_T} \middle| \mathcal{F}_t \right] \right), \tag{6.2}$$

where Q is an EMM.

Theorem 6.4. *(Second Fundamental Theorem of Mathematical Finance)* An arbitrageless market is complete, i.e. any bounded liability is replicable if, and only if an EMM is unique.

Part 4. **Useful Books, Links, and Other Materials**