Student Research Group 'Stochastic Volatility Models', Project 'Heston–2'

Monte-Carlo Simulation: Speed and Error Control

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Outline

Monte-Carlo Error Control

Monte-Carlo Speed Control

Conclusion

Central Limit Theorem

Theorem 1 (Lindeberg-Lévy)

Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $\operatorname{var}[X_i] = \sigma^2$. Then as n approaches infinity, the random variables $\sqrt{n}(\bar{X}_n - \mu)$ converge in law to a normal distribution $\mathcal{N}(0, \sigma^2)$, i.e.

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right). \tag{1}$$

Monte Carlo Simulation

Statistical Estimation

Lemma 2

Let $X_1, X_2, \ldots, X_n$ be a series of independent and identically distributed random variables, and $h : \mathbb{R} \to \mathbb{R}$ be a borel function. Then $h(X_1), h(X_2), \ldots, h(X_n)$ is a series of independent and identically distributed random variables.

Thus, we could write an unbiased consistent estimator of $\mathbb{E}[h(X)]$ as follows:

$$\widehat{\mathbb{E}[h(X)]} = \frac{1}{n} \sum_{i=1}^{n} h(X_i). \tag{2}$$

Monte Carlo Simulation

Local Truncation Error

Asymptotic confidence interval for $\hat{\mu} = \widehat{\mathbb{E}[X]}$ at the confidence level α:

$$\mu \in \left(\hat{\mu} - z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}, \hat{\mu} + z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}} \right). \tag{3}$$

That means that the estimation error is equal to $2z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}$.

Euler-Maruyama Discretization Scheme for SDEs

Strong and weak convergence as a global truncation error analogue

Definition 3

Let $\hat{X}^n(t)$ be a piecewise mesh approximation of an SDE solution $X(t)$ (we assume that there exists a unique strong solution). Then a scheme is said to have a strong convergence of order p if

$$\mathbb{E}\left[\left|\hat{X}^n(T) - X(T)\right|\right] \leq Ch^p, \quad n \to \infty. \tag{4}$$

A scheme is said to have a weak convergence of order p if for any polynomial $f : \mathbb{R} \to \mathbb{R}$ we have

$$\left|\mathbb{E}\left[f(\hat{X}^n(T))\right] - \mathbb{E}\left[f(X(T))\right]\right| \leq Ch^p, \quad n \to \infty. \tag{5}$$

Euler-Maruyama Discretization Scheme for SDEs

Strong and weak convergence as a global truncation error analogue

Theorem 4

Under some technical assumptions the Euler-Maruyama Discretization scheme (??) has a strong convergence of order $1/2$ and a weak convergence of order 1.

NB

Since our goal is to approximate $\mathbb{E}\left[h(X)\right]$ with a given accuracy and the least possible number of simulations, we need to compare the weak convergence rate between the methods. But this will work only for the derivatives of the European type, since we cannot guarantee the convergence of the simulations for the times $t < T$.

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Control Variate Method

Motivation

- Let us consider the following random variables: X and Y.
- Let us suppose, that we somehow know $\mathbb{E}[X] = \mu_X$ and $\operatorname{cor}[X, Y] = \rho$.
- Our goal is to build a consistent estimator of $\mathbb{E}[Y] = \mu_Y$.
- Possible estimator is $\hat{\mu}_Y = \bar{Y}$. It is unbiased and consistent.
- Can we improve the speed of convergence of the estimator?

Control Variate Method

Solution to the problem

Let us consider the following estimator:

$$\hat{\mu}_Y^b = \bar{Y} + b\left(\bar{X} - \mu_X\right). \tag{6}$$

We can see that the estimator is consistent and unbiased. Let us minimise the variance of it using the following optimization problem:

$$\operatorname{var} \hat{\mu}_Y^b \to \min_{b \in \mathbb{R}}. \tag{7}$$

The solution is to use $b^* = \sqrt{\frac{\operatorname{var} Y}{\operatorname{var} X}}\rho$.

Other useful methods

- Antithetic variates: $\frac{Y_1+Y_2}{2}$, where Y_1 and Y_2 are the correlated series of iid random variables.
- Importance sampling: could be used to estimate the OTM options.

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To-dos

1. How do we approximate the log-prices?
2. Martingale correction in the Andersen schemes
3. Numerical stability of implied volatility calculations