Student Research Group Report

Monte-Carlo Methods for the Heston Model

Artemy Sazonov, Danil Legenky, Kirill Korban

Contents

List of Figures

List of Tables

Introduction

One of the first diffusion-based models in mathematical finance was introduced in 1973 in the paper by Fisher Black and Myron Sholes [BS73]. However, the model was not very realistic, as it did not take into account the variability of the volatility process, which was proven not to be a constant in the real stock market. The implied volatility of the stock options was not the same for different maturities and strikes.

Later, the class of so-called local volatility models was developed (Dupire et. al.). They fixed the problem of the spot implied volatility: now we could get a perfect fin into the spot prices of the options. However, the local volatility models give us the wrong dynamics, which is crucial to valuate the price of different derivatives.

In 1993, Steven Heston introduced a new diffusion-based model [Hes93], but he made a vital assumption: the variance process is not a constant, not a determenistic function of time and stock price, but follows a diffusion process, called the Cox-Ingersol-Ross (CIR) process. The stochastic volatility models cannot be perfectly calibrated to fit the volatility smile, but they give us a realistic dynamics of the implied volatility surface.

In this paper we revise the Heston model and its most popular simulation methods. We remind the reader of some basic facts abou the Monte-Carlo methods in finance. We also study the empirical speed of convergence of the simulation methods and the accuracy of the option greeks. Futhermore, we implement a multi-threaded versions of the desired simulation teqniques and optimize them for the best possible performance in **Python**.

We provide the reader with the code for the simulation methods and the greeks computation for the results to be reproductible.

Part I

Monte-Carlo Methods for the Heston Model: A Theoretical Review

Chapter 1

A review of the original Heston model

1.1 Basic facts

We shall use the following resources in this chapter: [Hes93] and [Gat12]. Assume that the spot asset's price S at time t follows the diffusion (1.1) – (1.2):

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dZ_1(t), \tag{1.1}$$

$$dv(t) = \left(\delta^2 - 2\beta v(t)\right) dt + 2\delta\sqrt{v(t)}dZ_2(t), \tag{1.2}$$

where Z_1, Z_2 are the correlated Wiener processes with $dZ_1 dZ_2 = \rho dt$.

1.2 PDEs

1.3 A closed-form solution for the European call option

Chapter 2

A review of the Monte-Carlo methods for diffusions

2.1 Randomness in Probability Theory

A. N. Kolmogorov in «On Logical Foundations of Probability Theory»:

In everyday language we call random these phenomena where we cannot find a regularity allowing us to predict precisely their results. Generally speaking there is no ground to believe that a random phenomenon should possess any definite probability. Therefore, we should have distinguished between randomness proper (as absence of any regularity) and stochastic randomness (which is the subject of the probability theory). Since randomness is defined as absence of regularity, we should primarily specify the concept of regularity. The natural means of such a specification is the theory of algorithms and recursive functions...

Check out the lecture by A. N. Shiryaev for more details.

2.2 Laws of Large Numbers and Central Limit Theorem

Theorem 1 (Khinchin). *Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with $\mathbb{E}X_i = \mu$. Then*

$$\mathbb{P}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = \mu. \tag{2.1}$$

Theorem 2 (Kolmogorov). *Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables. Then $\exists \mathbb{E}X_i = \mu$, if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \overset{a.s.}{=} \mu. \tag{2.2}$$

Theorem 3 (Lindeberg-Lévy). *Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $\operatorname{var}[X_i] = \sigma^2$. Then as n approaches infinity, the random variables $\sqrt{n}(\bar{X}_n - \mu)$ converge in law to a normal distribution $\mathcal{N}(0, \sigma^2)$, i.e.*

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right). \tag{2.3}$$

Law of large numbers is an informal corollary of the central limit theorem.

2.3 The Statistical Foundations of the Monte-Carlo Methods

Lemma 4. *Let $X_1, X_2, \dots, X_n$ be a series of independent and identically distributed random variables, and $h : \mathbb{R} \to \mathbb{R}$ be a borel function. Then $h(X_1), h(X_2), \dots, h(X_n)$ is a series of independent and identically distributed random variables.*

Thus, we could write an unbiased consistent estimator of $\mathbb{E}\left[h(X)\right]$ as follows:

$$\widehat{\mathbb{E}\left[h(X)\right]} = \frac{1}{n}\sum_{i=1}^{n} h(X_i). \tag{2.4}$$

Definition 1. *Monte Carlo simulation is a set of techniques that use pseudorandom number generators to solve problems that might be too complicated to be solved analytically. It is based on the central limit theorem.*

Asymptotic confidence interval for $\hat{\mu} = \widehat{\mathbb{E}\left[X\right]}$ at the confidence level α:

$$\mu \in \left(\hat{\mu} - z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}, \hat{\mu} + z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}\right). \tag{2.5}$$

That means that the estimation error is equal to $2z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}$.

2.4 General Monte-Carlo Methods for Gaussian Diffusions

2.5 The Main Three Methods

2.5.1 Euler-Maruyama

Forward Euler Scheme for ODEs

Suppose that we have an ODE of the form

$$dX(t) = f(X(t), t)dt, \quad X(0) = X_0. \tag{2.6}$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_n, X_n)h_n, \tag{2.7}$$

where $t_n = \sum_{k=1}^{n} h_n, t_0 = 0$ is a grid.

Backward Euler Scheme for ODEs

Suppose that we have an ODE of the form

$$dX(t) = f(X(t), t)dt, \quad X(0) = X_0. \tag{2.8}$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_{n+1}, X_{n+1})h_n, \tag{2.9}$$

where $t_n = \sum_{k=1}^{n} h_n, t_0 = 0$ is a grid.

Euler-Maruyama Scheme for SDEs

Suppose we have a diffusion of the form

$$dX(t) = f(X(t), t)dt + \sigma(X(t), t)dW(t), \quad X_0 = X_0.$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_n, X_n)h_n + \sigma(t_n, X_n)\sqrt{h_n}Z_n, \tag{2.10}$$

where $(Z_n)_{n=1,2,\ldots}$ is a sample of standard normal random variables, and $t_n = \sum_{k=1}^{n} h_n, t_0 = 0$ is a grid. The same method could be generalized for the two-factor Gaussian diffusions. Further we assume that $(t_i)_{i=0,1,\ldots}$ is a uniform grid with $t_i = ih$.

Definition 2. *Let $\hat{X}^n(t)$ be a piecewise mesh approximation of an SDE solution $X(t)$ (we assume that there exists a unique strong solution). Then a scheme is said to have a strong convergence of order p if*

$$\mathbb{E}\left[\left|\hat{X}^n(T) - X(T)\right|\right] \leq Ch^p, \quad n \to \infty. \tag{2.11}$$

A scheme is said to have a weak convergence of order p if for any polynomial $f : \mathbb{R} \to \mathbb{R}$ we have

$$\left|\mathbb{E}\left[f(\hat{X}^n(T))\right] - \mathbb{E}\left[f(X(T))\right]\right| \leq Ch^p, \quad n \to \infty. \tag{2.12}$$

Theorem 5. *Under some technical assumptions the Euler-Maruyama scheme (2.10) has a strong convergence of order $1/2$ and a weak convergence of order 1.*

Since our goal is to approximate $\mathbb{E}\left[h(X)\right]$ with a given accuracy and the least possible number of simulations, we need to compare the weak convergence rate between the methods.

Chapter 3

The Three Methods of Simulation of the Heston Model

3.1 Euler Scheme

Suppose we have the Heston model (1.1) – (1.2). Then it could be numerically solved by the following finite difference scheme:

$$S_{n+1} = S_n + \mu S_n h_n + \sqrt{v_n} S_n \sqrt{h_n} Z_{1,n}, \tag{3.1}$$

$$v_{n+1} = v_n + \left(\delta^2 - 2\beta v_n\right) h_n + \sigma \sqrt{v_n} \sqrt{h_n} Z_{2,n}, \tag{3.2}$$

where $(Z_{1,n})_{n=1,2,\dots}$ and $(Z_{2,n})_{n=1,2,\dots}$ are ρ-correlated samples of standard normal random variables, and $t_n = \sum_{k=1}^{n} h_n$ is a mesh grid. But we have a problem: during simulation of the Heston model using Euler method S_{t_n} and v_{t_n} could be negative. How do we deal with this inconvenience? Let us introduce the log-prices

$$X(t) := \log \frac{S(t)}{S(0)}. \tag{3.3}$$

We take the positive part of the variance:

$$X_{n+1} = X_n + (\mu - 0.5 v_n^+) h_n + \sqrt{v_n^+} X_n \sqrt{h_n} Z_{1,n}, \tag{3.4}$$

$$v_{n+1} = v_n + \left(\delta^2 - 2\beta v_n^+\right) h_n + \sigma \sqrt{v_n^+} \sqrt{h_n} Z_{2,n}, \tag{3.5}$$

and then we take the exponential of the log-prices:

$$S_n = S_0 e^{X_n}. \tag{3.6}$$

However, the scheme is not accurate, since we ignore the $dZ_i dZ_j$ terms in the Itô-Taylor series approximation.

3.2 Broadie-Kaya Scheme

3.3 Andersen Scheme

Part II

Practical Problems and Pricing Exotics

Chapter 4

Implementation of the Methods

4.1 Euler Scheme

4.2 E+M Scheme

4.3 Broadie-Kaya Scheme

4.4 Andersen Scheme

Chapter 5

Comparison of the Methods

5.1 Performance

5.2 Accuracy

Chapter 6

Pricing Exotics

Conclusion

Bibliography

[BS73] Fischer Black and Myron Sholes. “The Pricing of Options and Corporate Liabilities”. In: *Journal of Political Economy* 81.3 (1973), pp. 637–657.

[Hes93] Steven L. Heston. “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options”. In: *Review of Financial Studies* 6.2 (1993), pp. 327–343.

[Gat12] Jim Gatheral. *The Volatility Surface*. John Wiley & Sons, Ltd, 2012. Chap. 1-3, pp. 1–42. ISBN: 9781119202073. DOI: https://doi.org/10.1002/9781119202073.