



Student Research Group 'Stochastic Volatility Models', Project 'Heston-2'

# **Methods of Simulation of the Heston Model: A Review**

Artemy Sazonov, Danil Legenky, Kirill Korban

Lomonosov Moscow State University, Faculty of Mechanics and Mathematics

November 5, 2022



## Heston Model Definition

Assume that the spot asset at time  $t$  follows the diffusion

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dZ_1(t), \quad (1)$$

$$dv(t) = \left( \delta^2 - 2\beta v(t) \right) dt + \sigma \sqrt{v(t)}dZ_2(t), \quad (2)$$

where  $Z_1, Z_2$  are the correlated Wiener processes with  $dZ_1dZ_2 = \rho dt$ .



# Table of Contents

Introduction to Monte-Carlo Methods

Euler Simulation Method

Broadie-Kaya Simulation Method

Andersen Simulation Method

Computation Examples

Greeks Computation



## Random and Pseudo-Random Numbers

### A. N. Kolmogorov – «On Logical Foundations of Probability Theory»

*In everyday language we call random these phenomena where we cannot find a regularity allowing us to predict precisely their results. Generally speaking there is no ground to believe that a random phenomenon should possess any definite probability. Therefore, we should have distinguished between randomness proper (as absence of any regularity) and stochastic randomness (which is the subject of the probability theory).*

...

*Since randomness is defined as absence of regularity, we should primarily specify the concept of regularity. The natural means of such a specification is the theory of algorithms and recursive functions...*

Check out the [lecture by A. N. Shiryaev](#) for more details.



# Random and Pseudo-Random Numbers

Collins Dictionary Definitions

## Definition 1 (Random numbers)

a sequence of numbers that do not form any progression, used to facilitate unbiased sampling of a population.

## Definition 2 (Pseudorandom numbers)

a sequence of numbers that satisfies statistical tests for randomness but is produced by a reproducible mathematical procedure.



# Law of Large Numbers

## Theorem 3 (Khinchin)

Let  $X_1, X_2, \dots, X_n$  be a sequence of independent and identically distributed random variables with  $\mathbb{E}X_i = \mu$ . Then

$$\mathbb{P}\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i = \mu. \quad (3)$$

## Theorem 4 (Kolmogorov)

Let  $X_1, X_2, \dots, X_n$  be a sequence of independent and identically distributed random variables. Then  $\exists \mathbb{E}X_i = \mu$ , if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i \stackrel{a.s.}{=} \mu. \quad (4)$$



# Central Limit Theorem

## Theorem 5 (Lindeberg-Lévy)

Let  $X_1, \dots, X_n$  be a sequence of i.i.d. random variables with  $\mathbb{E}[X_i] = \mu$  and  $\text{var}[X_i] = \sigma^2$ . Then as  $n$  approaches infinity, the random variables  $\sqrt{n}(\bar{X}_n - \mu)$  converge in law to a normal distribution  $\mathcal{N}(0, \sigma^2)$ , i.e.

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2). \quad (5)$$

## NB

Law of large numbers is an informal corollary of the central limit theorem.



# Monte Carlo Simulation

## Statistical Estimation

### Lemma 6

*Let  $X_1, X_2, \dots, X_n$  be a series of independent and identically distributed random variables, and  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a borel function. Then  $h(X_1), h(X_2), \dots, h(X_n)$  is a series of independent and identically distributed random variables.*

Thus, we could write an unbiased consistent estimator of  $\mathbb{E}[h(X)]$  as follows:

$$\widehat{\mathbb{E}[h(X)]} = \frac{1}{n} \sum_{i=1}^n h(X_i). \quad (6)$$





# Monte Carlo Simulation

## Local Truncation Error

### Definition 7

Monte Carlo simulation is a set of techniques that use pseudorandom number generators to solve problems that might be too complicated to be solved analytically. It is based on the central limit theorem.

Asymptotic confidence interval for  $\hat{\mu} = \widehat{\mathbb{E}[X]}$  at the confidence level  $\alpha$ :

$$\mu \in \left( \hat{\mu} - z_{\alpha/2} \sqrt{\frac{\sigma^2}{n}}, \hat{\mu} + z_{\alpha/2} \sqrt{\frac{\sigma^2}{n}} \right). \quad (7)$$

That means that  $LTE = 2z_{\alpha/2} \sqrt{\frac{\sigma^2}{n}}$ .



# Table of Contents

Introduction to Monte-Carlo Methods

Euler Simulation Method

Broadie-Kaya Simulation Method

Andersen Simulation Method

Computation Examples

Greeks Computation



# Forward Euler Scheme for ODEs

## Definition

Suppose that we have an ODE of the form

$$dX(t) = f(X(t), t)dt, \quad X(0) = X_0. \quad (8)$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_n, X_n)h_n, \quad (9)$$

where  $t_n = \sum_{k=0}^n h_k$  is a grid.



# Forward Euler Scheme for ODEs

Global truncation error

## Lemma 8

*Let the solution of (8) have an  $M$ -bounded second derivative and let  $f$  be  $L$ -Lipshitz continuous in its second argument. Then the global truncation error of the mesh solution (9) is*

$$|X(T) - X_N| \leq \frac{hM}{2L} \left( e^{LT} - 1 \right) = O(h). \quad (10)$$



# Euler-Maruyama Scheme for SDEs

## Definition

Suppose we have a diffusion of the form

$$dX(t) = f(X(t), t)dt + \sigma(X(t), t)dW(t), \quad X_0 = X_0.$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_n, X_n)h_n + \sigma(t_n, X_n)\sqrt{h_n}Z_n, \quad (11)$$

where  $(Z_n)_{n=1,2,\dots}$  is a sample of standard normal random variables, and  $t_n = \sum_{k=1}^n h_n, t_0 = 0$  is a grid. The same method could be generalized for the two-factor Gaussian diffusions. Further we assume that  $(t_i)_{i=0,1,\dots}$  is a uniform grid with  $t_i = ih$ .



# Euler-Maruyama Scheme for SDEs

Strong and weak convergence as a global truncation error analogue

## Definition 9

Let  $\hat{X}^n(t)$  be a piecewise mesh approximation of an SDE solution  $X(t)$  (we assume that there exists a unique strong solution). Then a scheme is said to have a strong convergence of order  $p$  if

$$\mathbb{E} \left[ \left| \hat{X}^n(T) - X(T) \right| \right] \leq Ch^p, \quad n \rightarrow \infty. \quad (12)$$

A scheme is said to have a weak convergence of order  $p$  if for any polynomial  $f : \mathbb{R} \rightarrow \mathbb{R}$  we have

$$\left| \mathbb{E} \left[ f(\hat{X}^n(T)) \right] - \mathbb{E} [f(X(T))] \right| \leq Ch^p, \quad n \rightarrow \infty. \quad (13)$$



# Euler-Maruyama Scheme for SDEs

Strong and weak convergence as a global truncation error analogue

## Theorem 10

*Under some technical assumptions the Euler-Maruyama scheme (11) has a strong convergence of order  $1/2$  and a weak convergence of order 1.*

## NB

*Since our goal is to approximate  $\mathbb{E}[h(X)]$  with a given accuracy and the least possible number of simulations, we need to compare the weak convergence rate between the methods.*



# Euler Scheme for the Heston Model

## Classical Euler-Maruyama Scheme

Suppose we have the Heston model (1) – (2). Then it could be numerically solved by the following finite difference scheme:

$$S_{n+1} = S_n + \mu S_n h_n + \sqrt{v_n} S_n \sqrt{h_n} Z_{1,n}, \quad (14)$$

$$v_{n+1} = v_n + \left( \delta^2 - 2\beta v_n \right) h_n + \sigma \sqrt{v_n} \sqrt{h_n} Z_{2,n}, \quad (15)$$

where  $(Z_{1,n})_{n=1,2,\dots}$  and  $(Z_{2,n})_{n=1,2,\dots}$  are  $\rho$ -correlated samples of standard normal random variables, and  $t_n = \sum_{k=0}^n h_k$  is a mesh grid.





# Euler Scheme for the Heston Model

Modified Euler-Maruyama Scheme

But we have a problem: during simulation of the Heston model using Euler method  $S_{t_n}$  and  $v_{t_n}$  could be negative. How do we deal with this inconvenience? Let us introduce the log-prices

$$X(t) := \log \frac{S(t)}{S(0)}. \quad (16)$$



# Euler Scheme for the Heston Model

## Modified Euler-Maruyama Scheme

We take the positive part of the variance:

$$X_{n+1} = X_n + (\mu - 0.5v_n^+)h_n + \sqrt{v_n^+}X_n\sqrt{h_n}Z_{1,n}, \quad (17)$$

$$v_{n+1} = v_n + \left(\delta^2 - 2\beta v_n^+\right)h_n + \sigma\sqrt{v_n^+}\sqrt{h_n}Z_{2,n}, \quad (18)$$

and then we take the exponential of the log-prices:

$$S_n = S_0 e^{X_n}. \quad (19)$$

However, the scheme is not accurate, since we ignore the  $dZ_i dZ_j$  terms in the Itô-Taylor expansion.



# Table of Contents

Introduction to Monte-Carlo Methods

Euler Simulation Method

**Broadie-Kaya Simulation Method**

Andersen Simulation Method

Computation Examples

Greeks Computation





# Table of Contents

Introduction to Monte-Carlo Methods

Euler Simulation Method

Broadie-Kaya Simulation Method

Andersen Simulation Method

Computation Examples

Greeks Computation





# Table of Contents

Introduction to Monte-Carlo Methods

Euler Simulation Method

Broadie-Kaya Simulation Method

Andersen Simulation Method

Computation Examples

Greeks Computation







# Table of Contents

Introduction to Monte-Carlo Methods

Euler Simulation Method

Broadie-Kaya Simulation Method

Andersen Simulation Method

Computation Examples

Greeks Computation





# Table of Contents

Introduction to Monte-Carlo Methods

Euler Simulation Method

Broadie-Kaya Simulation Method

Andersen Simulation Method

Computation Examples

Greeks Computation



## Conclusion

We introduced the three most common simulation methods for dynamics of the two-factor Gaussian diffusion model:

1. Euler scheme;
2. Broadie-Kaya scheme;
3. Andersen scheme.

Using these methods we simulated the dynamics of the Heston model and computed the price of the European call options with different strikes and maturities.

