Student Research Group ’Stochastic Volatility Models’

Methods of Simulation of the Heston Model: A Review

Artemy Sazonov, Danil Legenky, Kirill Korban

Lomonosov Moscow State Univesity, Faculty of Mechanics and Mathematics

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Heston Model Definition

Assume that the spot asset at time t follows the diffusion

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dZ_1(t), \tag{1}$$

$$dv(t) = \left(\delta^2 - 2\beta v(t)\right) dt + 2\delta\sqrt{v(t)}dZ_2(t), \tag{2}$$

where Z_1, Z_2 are the correlated Wiener processes with $dZ_1 dZ_2 = \rho dt$.

Table of Contents

Statistical Estimation

Let $X, X_1, X_2, \ldots, X_n$ be a series of independent and identically distributed random variables with a distribution function $F(x)$. Unbiased consistent estimators of $\mathbb{E}[X]$ and $\mathrm{var}[X]$ are:

- The sample mean is the average of the sample values:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{3}$$

- The sample variance is the average squared deviation from the mean:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{4}$$

Statistical Estimation

Lemma 1

Let $X_1, X_2, \ldots, X_n$ be a series of independent and identically distributed random variables, and $h : \mathbb{R} \to \mathbb{R}$ be a borel function. Then $h(X_1), h(X_2), \ldots, h(X_n)$ is a series of independent and identically distributed random variables.

Thus, we could write the unbiased consistent estimator of $\mathbb{E}\left[h(X)\right]$ as follows:

$$\widehat{\mathbb{E}\left[h(X)\right]} = \frac{1}{n}\sum_{i=1}^{n} h(X_i). \tag{5}$$

Random and Pseudo-Random Numbers

Definition 2

A random sequence of numbers is a sequence (in time or in space) with no discernable pattern.

Definition 3

Pseudorandom number generator is an algorithm that generates a sequence of numbers that has no internal pattern. However, the series requires a starting seed, and if the algorithm is started repeatedly with the same seed, it will go through precisely the same sequence of numbers.

Check out the lecture by A.N.Shiryaev (Probability Theory Department x Vega Institute Foundation Seminar, Oct 19) for more details.

Law of Large Numbers

Theorem 4 (Khinchin)

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with a distribution function $F(x)$ and $\mathbb{E}X_i = \mu$. Then

$$\underset{n\to\infty}{\mathbb{P}\text{-}\lim} \frac{1}{n} \sum_{i=1}^{n} X_i = \mu. \tag{6}$$

Theorem 5 (Kolmogorov)

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with a distribution function $F(x)$ and $\mathbb{E}X_i = \mu$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \overset{a.s.}{=} \mu. \tag{7}$$

Central Limit Theorem

Theorem 6 (Lindeberg-Lévy)

Let $X_1, \ldots, X_n$ *be a sequence of i.i.d. random variables with* $\mathbb{E}[X_i] = \mu$ *and* $\operatorname{var}[X_i] = \sigma^2$. *Then as n approaches infinity, the random variables* $\sqrt{n}(\bar{X}_n - \mu)$ *converge in law to a normal distribution* $\mathcal{N}(0, \sigma^2)$, *i.e.*

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right). \tag{8}$$

Table of Contents

Table of Contents

Table of Contents

Table of Contents

Table of Contents

Table of Contents

Conclusion

We introduced the three most common simulation methods for dynamics of the Heston stochastic volatility model:

1. Euler scheme;
2. Broadie-Kaya scheme;
3. Andersen scheme.