Student Research Group Report

Monte-Carlo Methods for the Heston Model

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Introduction

One of the first diffusion-based models in mathematical finance was introduced in 1973 in the paper by Fisher Black and Myron Sholes [BS73]. However, the model was not very realistic, as it did not take into account the variability of the volatility process, which was proven not to be a constant in the real stock market. The implied volatility of the stock options was not the same for different maturities and strikes.

Later, the class of so-called local volatility models was developed (Dupire et. al.). They fixed the problem of the spot implied volatility: now we could get a perfect fin into the spot prices of the options. However, the local volatility models give us the wrong dynamics, which is crucial to valuate the price of different derivatives.

In 1993, Steven Heston introduced a new diffusion-based model [Hes93], but he made a vital assumption: the variance process is not a constant, not a determenistic function of time and stock price, but follows a diffusion process, called the Cox-Ingersol-Ross (CIR) process. The stochastic volatility models cannot be perfectly calibrated to fit the volatility smile, but they give us a realistic dynamics of the implied volatility surface.

In this paper we revise the Heston model and its most popular simulation methods. We remind the reader of some basic facts abou the Monte-Carlo methods in finance. We also study the empirical speed of convergence of the simulation methods and the accuracy of the option greeks. Futhermore, we implement a multi-threaded versions of the desired simulation teqniques and optimize them for the best possible performance in **Python**.

We provide the reader with the code for the simulation methods and the greeks computation for the results to be reproductible.

Part I

Monte-Carlo Methods for the Heston Model: A Theoretical Review

Chapter 1

A review of the original Heston model

Sources: [Hes93], [Gat12], [Zhi22]

1.1 Basic facts

Assume that the spot asset's price S at time t follows the diffusion (1.1) – (1.2):

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dZ_1(t), \tag{1.1}$$

$$dv(t) = \left(\delta^2 - 2\beta v(t)\right) dt + 2\delta\sqrt{v(t)}dZ_2(t), \tag{1.2}$$

where Z_1, Z_2 are the correlated Wiener processes with $dZ_1 dZ_2 = \rho dt$.

1.2 PDEs

1.3 A closed-form solution for the European call option

Chapter 2

A review of the Monte-Carlo methods for diffusions

Sources: [Kol83], [Zhi22], [KK22], [KP92]

2.1 Randomness in Probability Theory

A. N. Kolmogorov in «On Logical Foundations of Probability Theory»: *In everyday language we call random these phenomena where we cannot find a regularity allowing us to predict precisely their results. Generally speaking there is no ground to believe that a random phenomenon should possess any definite probability. Therefore, we should have distinguished between randomness proper (as absence of any regularity) and stochastic randomness (which is the subject of the probability theory). Since randomness is defined as absence of regularity, we should primarily specify the concept of regularity. The natural means of such a specification is the theory of algorithms and recursive functions...*

2.2 Laws of large numbers and central limit theorems

Theorem 1 (Khinchin). *Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with $\mathbb{E}X_i = \mu$. Then*

$$\mathbb{P}\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = \mu. \tag{2.1}$$

Theorem 2 (Kolmogorov). *Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables. Then $\exists \mathbb{E}X_i = \mu$, if and only if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \overset{a.s.}{=} \mu. \tag{2.2}$$

Theorem 3 (Lindeberg-Lévy). *Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = = \mu$ and $\operatorname{var} X_i = \sigma^2$. Then as n approaches infinity, the random variables $\sqrt{n}(\bar{X}_n - \mu)$ converge in law to a normal distribution $\mathcal{N}(0, \sigma^2)$, i.e.*

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right). \tag{2.3}$$

2.3 The statistical foundations of the Monte-Carlo methods

Lemma 4. *Let $X_1, X_2, \dots, X_n$ be a series of independent and identically distributed random variables, and $h : \mathbb{R} \to \mathbb{R}$ be a borel function. Then $h(X_1), h(X_2), \dots, h(X_n)$ is a series of independent and identically distributed random variables.*

Thus, we could write an unbiased consistent estimator of $\mathbb{E}\left[h(X)\right]$ as follows:

$$\widehat{\mathbb{E}\left[h(X)\right]} = \frac{1}{n}\sum_{i=1}^{n} h(X_i). \tag{2.4}$$

Definition 1. *Monte Carlo simulation is a set of techniques that use pseudorandom number generators to solve problems that might be too complicated to be solved analytically. It is based on the central limit theorem.*

Asymptotic confidence interval for $\hat{\mu} = \widehat{\mathbb{E}\left[X\right]}$ at the confidence level α:

$$\mu \in \left(\hat{\mu} - z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}, \hat{\mu} + z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}\right). \tag{2.5}$$

That means that the estimation error is equal to $2z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}$.

2.4 Variance reduction methods for the Monte-Carlo simulations

Suppose we need to estimate a parameter $\theta = \mathbb{E}\left[Y\right]$. From the statistics course we know that $\bar{Y}$ is a consistent unbiased estimator of θ.

2.4.1 Control Variates

Suppose that we have another random variable Z that is correlated with Y and $\mathbb{E}\left[Z\right] = \mu$ is known. Then we could introduce the following estimator:

$$\hat{\theta}^b = \bar{Y} + b(\bar{Z} - \mu), \tag{2.6}$$

where b is a constant. Obviously, $\hat{\theta}^b$ is a consistent unbiased estimator of θ. How do we choose b? We need to minimize the variance of $\hat{\theta}^b$. A simple unconstrained optimization problem:

$$\operatorname{var}\hat{\theta}^b = \operatorname{var}\bar{Y} + b^2 \operatorname{var}\bar{Z} - 2b\operatorname{cov}[\bar{Y}, \bar{Z}] \to \min_b .$$

The solution is

$$b^* = \frac{\operatorname{cov}[Y, Z]}{\operatorname{var} Z}. \tag{2.7}$$

From this we may see that

$$\operatorname{var}\hat{\theta}^b = (1 - \rho^2)\operatorname{var}\bar{Y},$$

where $\rho = \frac{\operatorname{cov}[Y,Z]}{\sqrt{\operatorname{var} Y \operatorname{var} Z}}$ is the correlation coefficient between Y and Z. Thus, in order to reduce the variance of an estimator, we need to find a strongly correlated random variable Z. If we don't know the theoretical closed-form solution for the correlation coefficient, then we change the estimator to the following:

$$\hat{\theta}^b = \bar{Y} + \hat{b}_n(\bar{Z} - \mu), \qquad \hat{b}_n = \frac{\sum_{i=1}^n (Z_i - \bar{Z})(Y_i - \bar{Y})}{\sum_{i=1}^n (Z_i - \bar{Z})^2}.$$

Furthermore, all good properties of an estimator are preserved due to the convergence inheritance theorem.

Conclusion: with a b chosen near the optimal value (2.7), the variance reduction effect is strictly determined by the correlation coefficient ρ.

2.4.2 Antithetic Variates

Suppose that we have two correlated identically distributed samples Y^1 and Y^2: $\operatorname{cov}[Y_i^1, Y_j^2] = \delta_{ij} \operatorname{cov}[Y_i^1, Y_i^2]$. Then we could introduce the following estimator:

$$\hat{\theta}_{\text{AV}} = \frac{\bar{Y}^1 + \bar{Y}^2}{2}. \tag{2.8}$$

Again, we can see that this estimator is unbiased and consistent. The variance of this estimator is

$$\operatorname{var} \hat{\theta}_{\text{AV}} = \frac{1}{4} \operatorname{var}[\bar{Y}^1 + \bar{Y}^2] = \frac{1}{4} \operatorname{var}[\bar{Y}^1] + \frac{1}{4} \operatorname{var}[\bar{Y}^2] + \frac{1}{2} \operatorname{cov}[\bar{Y}^1, \bar{Y}^2].$$

Thus, the variance reduction effect takes place when $\rho < 0$. If the random variable is generated by the Smirnov's transform $Y^1 = g(U)$, then its antithetic variable is $Y^2 = g(1 - U)$, where U is a uniform over $[0, 1]$ random variable. The same could be generalized to the case of $Y = f(Z)$. If Z is symmetric, then we define $Y^1 = f(Z)$ and $Y^2 = f(-Z)$.

2.4.3 Importance Sampling

OTM options tend not to expire almost always or almost never. Thus, we need some ways to reduce the number of simulations for this kind of derivatives.

2.5 Monte-Carlo methods for Gaussian diffusions

2.5.1 Euler-Maruyama Scheme

Forward Euler Scheme for ODEs

Suppose that we have an ODE of the form

$$dX(t) = f(X(t), t)dt, \quad X(0) = X_0. \tag{2.9}$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_n, X_n)h_n, \tag{2.10}$$

where $t_n = \sum_{k=1}^n h_n, t_0 = 0$ is a grid.

Backward Euler Scheme for ODEs

Suppose that we have an ODE of the form

$$dX(t) = f(X(t), t)dt, \quad X(0) = X_0. \tag{2.11}$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_{n+1}, X_{n+1})h_n, \tag{2.12}$$

where $t_n = \sum_{k=1}^n h_n, t_0 = 0$ is a grid.

Euler-Maruyama Scheme for SDEs

Suppose we have a diffusion of the form

$$dX(t) = f(X(t), t)dt + \sigma(X(t), t)dW(t), \quad X_0 = X_0.$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_n, X_n)h_n + \sigma(t_n, X_n)\sqrt{h_n}Z_n, \tag{2.13}$$

where $(Z_n)_{n=1,2,\dots}$ is a sample of standard normal random variables, and $t_n = \sum_{k=1}^n h_n, t_0 = 0$ is a grid. The same method could be generalized for the two-factor Gaussian diffusions. Further we assume that $(t_i)_{i=0,1,\dots}$ is a uniform grid with $t_i = ih$.

Definition 2. *Let $\hat{X}^n(t)$ be a piecewise mesh approximation of an SDE solution $X(t)$ (we assume that there exists a unique strong solution). Then a scheme is said to have a strong convergence of order p if*

$$\mathbb{E}\left[\left|\hat{X}^n(T) - X(T)\right|\right] \le Ch^p, \quad n \to \infty. \tag{2.14}$$

A scheme is said to have a weak convergence of order p if for any polynomial $f : \mathbb{R} \to \mathbb{R}$ we have

$$\left|\mathbb{E}\left[f(\hat{X}^n(T))\right] - \mathbb{E}\left[f(X(T))\right]\right| \le Ch^p, \quad n \to \infty. \tag{2.15}$$

Theorem 5. *Under some technical assumptions the Euler-Maruyama scheme (2.13) has a strong convergence of order $1/2$ and a weak convergence of order 1.*

Proof. TODO. WRITE TECHNICAL ASSUMPTIONS □

Remark. *Weak convergence of the scheme guarantees that the approximation of the expectation is correct only at a given time, not in the whole time interval. I.e. a European call price may converge with a weak convergence rate 1, but the price of an Asian call option may not converge with a given weak convergence rate.*

2.5.2 Milstein Scheme

2.5.3 Stochastic Runge-Kutta Scheme

Chapter 3

Methods of simulation of the Heston stochastic volatility model

3.1 Euler Scheme

Suppose we have the Heston model (1.1) – (1.2). Then it could be numerically solved by the following finite difference scheme:

$$S_{n+1} = S_n + \mu S_n h_n + \sqrt{v_n} S_n \sqrt{h_n} Z_{1,n}, \tag{3.1}$$

$$v_{n+1} = v_n + \left(\delta^2 - 2\beta v_n\right) h_n + \sigma \sqrt{v_n} \sqrt{h_n} Z_{2,n}, \tag{3.2}$$

where $(Z_{1,n})_{n=1,2,\ldots}$ and $(Z_{2,n})_{n=1,2,\ldots}$ are the ρ-correlated samples of standard normal random variables, and $t_n = \sum_{k=1}^{n} h_n$ is a mesh grid. But we have a problem: during simulation of the Heston model using Euler method S_{t_n} and v_{t_n} could be negative. How do we deal with this inconvenience? Let us introduce the log-prices

$$X(t) := \log \frac{S(t)}{S(0)}. \tag{3.3}$$

We take the positive part of the variance:

$$X_{n+1} = X_n + (\mu - 0.5 v_n^+) h_n + \sqrt{v_n^+} X_n \sqrt{h_n} Z_{1,n}, \tag{3.4}$$

$$v_{n+1} = v_n + \left(\delta^2 - 2\beta v_n^+\right) h_n + \sigma \sqrt{v_n^+} \sqrt{h_n} Z_{2,n}, \tag{3.5}$$

and then we take the exponential of the log-prices:

$$S_n = S_0 e^{X_n}. \tag{3.6}$$

However, the scheme is not accurate, since we ignore the $dZ_i dZ_j$ terms in the Itô-Taylor series expansion.

3.2 Andersen Scheme

Motivation for these schemes is the following two facts:

- Euler scheme is not very accurate, but fast and easy to implement;
- Broadie-Kaya scheme is more accurate, but significantly slower and way more complicated.

3.2.1 Quadratic-Exponential Discretization Scheme

We denote

$$m = \mathbb{E}\left[\hat{V}(t+\Delta)\middle|\hat{V}(t)\right], \tag{3.7}$$

$$s^2 = \mathbb{E}\left[\left(\hat{V}(t+\Delta)-m\right)^2\middle|\hat{V}(t)\right], \tag{3.8}$$

$$\psi = \frac{s^2}{m^2}. \tag{3.9}$$

Andersen proposes an approximation based on moment-matching techniques. His goal is then to speed up the first step of Broadie and Kaya's method. He observes that the conditional distribution of $\hat{V}(t+\Delta)$ given $\hat{V}(t)$ visually difers when $\hat{V}(t)$ is small or large (in the variation coefficient sense). The scheme is constructed from the following two subschemes:

1. Quadratic sampling scheme ($\psi \leq 2$);
2. Exponential sampling scheme ($\psi \geq 1$).

Fortunately, these two intervals cover the whole positive real line. Furthermore, these two schemes could be applied at the same time when $\psi \in [1, 2]$. This implies that there exists a critical value $\psi_{\text{crit}} \in [1, 2]$, which could be an indicator of which scheme is more applicable at the given value of ψ. Let us show you this.

Quadratic Sampling Scheme

For large enough $\hat{V}(t)$ (in the CV-sense) we can approximate the distribution of $\hat{V}(t+\Delta)$ by the scaled non-central chi-squared distribution with 1 degree of freedom:

$$\text{Law}\left(\hat{V}(t+\Delta)\middle|\hat{V}(t)\right) = a(\Delta, \hat{V}(t), VP)\chi_1'^2(b(\Delta, \hat{V}(t), VP)), \tag{3.10}$$

where VP is the vector of parameters of the CIR variance.

Lemma 6. *We have*

$$b^2 = \frac{2}{\psi} - 1 + \sqrt{\frac{2}{\psi}\left(\frac{2}{\psi}-1\right)}, \tag{3.11}$$

$$a = \frac{m}{1+b^2}. \tag{3.12}$$

Proof. Plain equating of the theoretical and real moments.

□

Remark. *The above lemma is not valid for $\psi \geq 2$.*

Therefore, if $\hat{V}(t)$ is close to zero, then we have a problem in finding such $a = a(\Delta, \hat{V}(t), VP)$ and $b = b(\Delta, \hat{V}(t), VP)$ such that the moments of the desired conditional distribution could be properly matched.

Exponential Sampling Scheme

Therefore, we approximate the desired distribution with the following method. Let ξ and η be independent random variables and $\xi \sim Be(1-p)$, $\eta \sim Exp(\beta)$ for some $p \in (0,1)$ and $\beta > 0$. Then we have (given $\hat{V}(t)$)

$$\hat{V}(t+\Delta) = \xi \cdot \eta, \tag{3.13}$$

what gives us the following distribution density:

$$p_{\hat{V}(t+\Delta)|\hat{V}(t)} = p \cdot \delta(x) + (1-p) \cdot \beta e^{-\beta x}, \tag{3.14}$$

where $\delta(x)$ is a standart delta function and for some β and p. Sampling ξ and η: Smirnov's transform. Or we can use the Smirnov transform with the cdf of the desired distribution.

Lemma 7. *We have*

$$p = \frac{\psi - 1}{\psi + 1}, \qquad \beta = \frac{1-p}{m} = \frac{2}{m(\psi+1)}. \tag{3.15}$$

Proof. By direct integration of the given densities we get the following:

$$\frac{1-p}{\beta} = m, \qquad \frac{1-p^2}{\beta^2} = s^2. \tag{3.16}$$

□

Remark. *The above lemma is not valid for $\psi \leq 1$.*

3.2.2 Truncated Gaussian Discretization Scheme

The main idea of the method: in this scheme the idea is to sample from a moment-matched Gaussian density where all probability mass below zero is inserted into a delta-function at the origin. Formalization of the idea:

$$\left(\hat{V}(t+\Delta)\middle| V(t)\right) = (\mu + \sigma Z)^+, \tag{3.17}$$

where Z is a standard normal random variable and μ and σ are the 'mean' and the 'standard deviation' of the desired distribution. We find μ and σ from the moment-matching techniques (see the previous method, equations (3.7) – (3.9)).

Lemma 8. *Let $\phi(x)$ be a standart Gaussian density and define a function $r : \mathbb{R} \to \mathbb{R}$ by the following equation:*

$$r(x)\phi(r(x)) + \Phi(r(x))(1 + r(x)^2) = (1+x)\left(\phi(r(x)) + r(x)\Phi(r(x))\right)^2. \tag{3.18}$$

Then the moment-matching parameters are

$$\mu = \frac{m}{\frac{\phi(r(\psi))}{r(\psi)} + \Phi(r(\psi))}, \tag{3.19}$$

$$\sigma = \frac{m}{\phi(r(\psi)) + r(\psi)\Phi(r(\psi))}. \tag{3.20}$$

Proof. PROOF HERE □

Problem: no closed-form solution for $r(\psi)$.
Solution: numerical solution.
Problem: no known limits to use the numerical solution.
Solution:

$$m = \frac{\delta^2}{2\beta} + \left(\hat{V}(t) - \frac{\delta^2}{2\beta}\right) e^{-2\beta\Delta}, \tag{3.21}$$

$$s^2 = \frac{\hat{V}(t)\sigma^2 e^{-2\beta\Delta}}{2\beta}\left(1 - e^{-2\beta\Delta}\right) + \frac{\delta^2\sigma^2}{8\beta^2}\left(1 - e^{-2\beta\Delta}\right)^2. \tag{3.22}$$

Then we analyze ψ wrt $\hat{V}(t)$ and obtain a finite interval as a domain for $r(\psi)$.

Proof. PROOF HERE. Redo as a lemma

□

3.3 Broadie-Kaya Scheme

It follows from Heston model that for $t > u$

$$S_t = S_u e^{\left(r(t-u) - \frac{1}{2}\int_u^t v_s\,ds + \rho\int_u^t \sqrt{v_s}\,dZ_1(s) + (1-\rho)\int_u^t \sqrt{v_s}\,dZ_2(s)\right)}, \tag{3.23}$$

$$v_t = v_u + \kappa\theta(t-u) - \kappa\int_u^t v_s\,ds + \sigma\int_u^t \sqrt{v_s}\,dZ_2(s), \tag{3.24}$$

Exact simulation algorithm for the Heston model:

1. Generate a sample from the distribution of v_t given v_u;
2. Generate a sample from the distribution of $\int_u^t V_s ds$ given v_t and v_u;
3. Recover $\int_u^t \sqrt{v_s} dZ_1(s)$ given v_t, v_u, and $\int_t^u v_s ds$;
4. Generate a sample from the distribution of S_t given $\int_u^t \sqrt{v_s} dZ_1(s)$, $\int_u^t \sqrt{v_s} dZ_2(s)$, $\int_u^t v_s ds$.

Step 1: Generate a sample from the distribution of v_t given v_u

As shown in [CIR85] the distribution of v_t given v_u for some $u < t$ is, up to a scale factor, a noncentral chi-squared distribution. The transition law of v_t can be expressed as:

$$v_t = \frac{\sigma^2(1 - e^{-\kappa(t-u)})}{4\kappa}\chi_d'^2\left(\frac{4\kappa e^{-\kappa(t-u)}}{\sigma^2(1 - e^{-\kappa(t-u)})}v_u\right), \quad t > u, \tag{3.25}$$

where $\chi_d'^2(\lambda)$ denotes the noncentral chi-squared random variable with d degrees of freedom, and noncentrality parameter λ, and

$$d = \frac{4\theta\kappa}{\sigma^2}. \tag{3.26}$$

Thus, we can sample from the distribution of v_t exactly, provided that we can sample from the noncentral chisquared distribution. [JKB94] show that for $d > 1$, the following representation is valid:

$$\chi_d'^2(\lambda) = \chi_1'^2(\lambda) + \chi_{d-1}'^2 = N(\lambda, 1)^2 + \chi_{d-1}^2. \tag{3.27}$$

Therefore, when $d > 1$, sampling from a noncentral chi-squared distribution is reduced to sampling from an ordinary chi-squared and an independent normal. When $d < 1$ we can use the the fact that

$$\chi_d'^2(\lambda) \sim \chi^2_{d+2N}, \tag{3.28}$$

where N is a Poisson random variable with mean $\frac{\lambda}{2}$.

Step 2: Generate a sample from the distribution of $\int_u^t V_s ds$ given v_t and v_u

The folowing formula can be derived. The derivation could be found in in the original paper. DERIVE HERE

$$\phi(a) = \mathbb{E}\left[\exp\left(ia\int_u^t V_s ds\right)\middle| v_u, v_t\right] = \frac{\gamma(a)e^{-(1/2)(\gamma(a)-\kappa)(t-u)}}{\kappa(1-e^{-\gamma(a)(t-u)})}$$
$$\exp\left(\frac{v_u+v_t}{\sigma^2}\left[\frac{\kappa(1+e^{-\kappa(t-u)})}{1-e^{-\kappa(1-u)}}\right]\right)\frac{I_{0.5d-1}\left(\sqrt{v_u v_t}\frac{4\gamma(a)e^{-0.5\gamma(a)(t-u)}}{\sigma^2(1-e^{-\kappa(a)(t-u)})}\right)}{I_{0.5d-1}\left(\sqrt{v_u v_t}\frac{4\kappa e^{-0.5\kappa(t-u)}}{\sigma^2(1-e^{-\kappa(t-u)})}\right)}, \tag{3.29}$$

where $\gamma(a) = \sqrt{\kappa^2 - 2\sigma ia}$ and $I_{0.5d-1}$ is a modified Bessel function of the first kind.

Let $V(u,t)$ denote the random variable that has the conditional distribution of the integral $\int_u^t V_s ds$ given v_u and v_t. Then we need to invert the characteristic function to get the cumulative distribution function

$$F(x) = \mathbb{P}(V(u,t) \le x) = E\left[e^{iaV(u,t)}\middle| v_u, v_t\right] =$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin ux}{u}\Phi(u)du = \frac{2}{\pi}\int_0^{\infty}\frac{\sin ux}{u}\Phi(u)du. \tag{3.30}$$

To calculate the integral the trapezoidal rule is being used:

$$\mathbb{P}(V(u,t) \le X) = \frac{hx}{\pi} + \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{\sin hjx}{j}\Re[\Phi(hj)] - e_d(h), \tag{3.31}$$

where h is a grid scale and $e_d(h)$ is the discretization error e_d. It can be bounded above by using a Poisson summation formula:

$$0 \le e_d(h) = \sum_{k=1}^{\infty}\left[F\left(\frac{2k\pi}{h}+x\right) - F\left(\frac{2k\pi}{h}-x\right)\right] \le 1 - F\left(\frac{2\pi}{h}-x\right). \tag{3.32}$$

If we want to achieve a discretization error α, then the step size should be

$$h = 2\frac{2\pi}{x+u_\alpha} \ge \frac{\pi}{u_\alpha}, \tag{3.33}$$

where $1 - F(u_\alpha) = \alpha$ and $0 \le x \le u_\alpha$. To be able to calculate $P(V(u,t) < x)$ using (3.31), we need to determine a point at which the summation can be terminated. Let N represent the last term to be calculated so that the approximation becomes

$$F(x) = \mathbb{P}(V(u,t) \le X) = \frac{hx}{\pi} + \frac{2}{\pi}\sum_{j=1}^{N}\frac{\sin hjx}{j}\Re[\Phi(hj)] - e_d(h) - e_T(N). \tag{3.34}$$

Because $|\sin ux| \leq 1$, the integrand in (3.32) is bounded by

$$\frac{2|\Re[\Phi(u)]|}{\pi u} \leq \frac{2|\Phi(u)|}{\pi u}. \tag{3.35}$$

To simulate the value of the integral, the Smirnov's transform method is used. We generate a uniform random variable U and then find the value of x for which

$$\mathbb{P}(V(u,t) \leq x) = U. \tag{3.36}$$

Step 3: Generate a sample from the distribution of $V(u,t)$ given v_u and v_t

The following formula can be used to calculate $\int_u^t \sqrt{v_s} dZ_1(s)$, as we already generated samples for $v_t, v_u, V(u,t)$

$$\int_u^t \sqrt{v_s} dZ_1(s) = \frac{1}{\sigma}(v_t v_u) - \kappa\theta(t-u) + V(u,t). \tag{3.37}$$

Step 4: Generate a sample from the distribution of $V(u,t)$ given v_u and v_t

Lastly, we need to bring everything together:

- $\int_u^t \sqrt{v_s} dZ_1(s)$ and $\int_u^t \sqrt{v_s} dZ_2(s)$ are already calculated;
- $V(u,t) = \int_u^t v_s ds$ is also calculated.

$$S_t = S_u \exp\left(r(t-u) - \frac{1}{2}V(u,t) + \rho \int_u^t \sqrt{v_s}\, dZ_1(s) + (1-\rho)\int_u^t \sqrt{v_s}\, dZ_2(s)\right) \tag{3.38}$$

Part II

Implementation Problems and Pricing Exotics

Chapter 4

Implementation of the Methods

4.1 General Problems and `mc_price` Function

The error of discretization consists of two parts: the discretization error itself (the one coming from the transition from an stochastic differential equation to the stochastic difference equation) and the Monte-Carlo error (see Section 2.3). We controlled the Monte-Carlo error with the following method:

Algorithm 1 Outer loop of the Monte-Carlo method (`mc_price`)

```
desired precision = 0.01 with confidence level 95%
prices = [ ]
while len(prices_confidence_interval) > desired precision and iter < MAX_ITER do
    paths = simulate(batch_size)
    prices ← payoff(paths)
    prices_confidence_interval = confidence_interval(prices)
end while
return mean(prices)
```

4.2 Euler Scheme

4.3 Andersen Scheme

4.4 Broadie-Kaya Scheme

Chapter 5

Comparison of the Methods

We shall compare the described methods for the European call option prices due to the fact that we have a closed-form solution for it.

5.1 Performance

5.2 Accuracy

Chapter 6

Pricing Exotics

Conclusion

Bibliography

[BS73] Fischer Black and Myron Sholes. “The Pricing of Options and Corporate Liabilities”. In: *Journal of Political Economy* 81.3 (1973), pp. 637–657.

[Kol83] Andrey Nikolaevich Kolmogorov. “On Logical Foundations of Probability Theory”. In: *Probability Theory and Mathematical Statistics*. Ed. by Jurii V. Prokhorov and Kiyosi Itô. Berlin, Heidelberg: Springer Berlin Heidelberg, 1983, pp. 1–5. ISBN: 978-3-540-38701-5.

[CIR85] John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross. “A Theory of the Term Structure of Interest Rates”. In: *Econometrica* 53.2 (1985), pp. 385–407. ISSN: 00129682, 14680262.

[KP92] Peter E. Kloeden and Eckhard Platen. *Numerical Solution of Stochastic Differential Equations*. Springer Berlin Heidelberg, 1992. ISBN: 978-3-662-12616-5. DOI: 10.1007/978-3-662-12616-5.

[Hes93] Steven L. Heston. “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options”. In: *Review of Financial Studies* 6.2 (1993), pp. 327–343.

[JKB94] Norman L. Johnson, Samuel Kotz, and Narayanaswamy Balakrishnan. *Continuous univariate distributions*. Vol. 2. Wiley, 1994. ISBN: 978-0-471-58494-0.

[Gat12] Jim Gatheral. *The Volatility Surface*. John Wiley & Sons, Ltd, 2012. Chap. 1-3, pp. 1–42. ISBN: 978-1-119-20207-3. DOI: 10.1002/9781119202073.

[KK22] Sergey Kobelkov and Yerkin Kitapbayev. *Numerical Methods in Finance I*. Vega Institute Foundation. 2022. URL: vega-institute.org/en/students/courses/chislennye-metody-v-finansakh-i/.

[Zhi22] Mikhail Zhitlukhin. *Stochastic Volatility Models*. Vega Institute Foundation. 2022. URL: vega-institute.org/en/students/courses/modeli-stokhasticheskoy-volatilnosti-/.