Student Research Group ’Stochastic Volatility Models’

Methods of Simulation of the Heston Model: A Review

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Heston Model Definition

Assume that the spot asset at time t follows the diffusion

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dZ_1(t), \tag{1}$$

$$dv(t) = \left(\delta^2 - 2\beta v(t)\right) dt + 2\delta\sqrt{v(t)}dZ_2(t), \tag{2}$$

where Z_1, Z_2 are the correlated Wiener processes with $dZ_1 dZ_2 = \rho dt$.

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Statistical Estimation

Let $X, X_1, X_2, \ldots, X_n$ be a series of independent and identically distributed random variables with a distribution function $F(x)$. Unbiased consistent estimators of $\mathbb{E}[X]$ and $\mathrm{var}[X]$ are:

- The sample mean is the average of the sample values:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{3}$$

- The sample variance is the corrected by the factor of $\frac{n}{n-1}$ average squared deviation from the mean:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \tag{4}$$

Statistical Estimation

Lemma 1

Let $X_1, X_2, \ldots, X_n$ be a series of independent and identically distributed random variables, and $h : \mathbb{R} \to \mathbb{R}$ be a borel function. Then $h(X_1), h(X_2), \ldots, h(X_n)$ is a series of independent and identically distributed random variables.

Thus, we could write the unbiased consistent estimator of $\mathbb{E}\left[h(X)\right]$ as follows:

$$\widehat{\mathbb{E}\left[h(X)\right]} = \frac{1}{n}\sum_{i=1}^{n} h(X_i). \tag{5}$$

Random and Pseudo-Random Numbers

Definition 2

A random sequence of numbers is a sequence (in time or in space) with no discernable pattern.

Definition 3

Pseudorandom number generator is an algorithm that generates a sequence of numbers that has no internal pattern. However, the series requires a starting seed, and if the algorithm is started repeatedly with the same seed, it will go through precisely the same sequence of numbers.

Check out the lecture by A.N.Shiryaev (Probability Theory Department $\times$ Vega Institute Foundation Seminar, Oct 19) for more details.

Law of Large Numbers

Theorem 4 (Khinchin)

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with $\mathbb{E}X_i = \mu$. Then

$$\mathbb{P}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = \mu. \tag{6}$$

Theorem 5 (Kolmogorov)

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables. Then $\exists \mathbb{E}X_i = \mu$, if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \stackrel{a.s.}{=} \mu. \tag{7}$$

Central Limit Theorem

Theorem 6 (Lindeberg-Lévy)

Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $\operatorname{var}[X_i] = \sigma^2$. Then as n approaches infinity, the random variables $\sqrt{n}(\bar{X}_n - \mu)$ converge in law to a normal distribution $\mathcal{N}(0, \sigma^2)$, i.e.

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right). \tag{8}$$

NB

Law of large numbers is a corollarary of the central limit theorem.

Monte Carlo Simulation

Definition 7

Monte Carlo simulation is a set of techniques that use pseudo-random number generators to solve problems that might be too complicated to be solved analytically. It is based on the central limit theorem.

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Forward Euler Scheme for ODEs

Definition

Suppose that we have an ODE of the form

$$dX(t) = f(X(t), t)dt, \quad X(0) = X_0. \tag{9}$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_n, X_n)h_n, \tag{10}$$

where $t_n = \sum_{k=0}^{n} h_n$ is a mesh grid.

Forward Euler Scheme for ODEs

Global truncation error

Lemma 8

Let the solution of (9) have an M-bounded second derivative and let f be L-Lipshitz continious in its second argument. Then the global truncation error of the mesh solution (10) is

$$|X(T) - X_N| \leq \frac{hM}{2L}\left(e^{LT} - 1\right) = O(h). \tag{11}$$

Euler-Maruyama Scheme for SDEs

Definition

Suppose we have a diffusion of the form

$$dX(t) = f(X(t), t)dt + \sigma(X(t), t)dW(t), \quad X_0 = X_0.$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_n, X_n)h_n + \sigma(t_n, X_n)\sqrt{h_n}Z_n, \tag{12}$$

where $(Z_n)_{n=1,2,\ldots}$ is a sample of standard normal random variables, and $t_n = \sum_{k=0}^{n} h_n$ is a mesh grid. The same method could be generalized for the two-factor Gaussian diffusions.

Euler-Maruyama Scheme for SDEs TBD

Global truncation error

Let $\hat{X}^n(t)$ be a piecewise mesh approximation of an SDE solution $X(t)$ (we assume that there exists a unique strong solution).

Euler-Maruyama Scheme for SDEs TBD

Global truncation error

Euler Scheme for the Heston Model

Classical Euler-Maruyama Scheme

Suppose we have the Heston model (1) – (2). Then it could be numerically solved by the following finite difference scheme:

$$S_{n+1} = S_n + \mu S_n h_n + \sqrt{v_n} S_n \sqrt{h_n} Z_{1,n}, \tag{13}$$

$$v_{n+1} = v_n + \left(\delta^2 - 2\beta v_n\right) h_n + 2\delta \sqrt{v_n} \sqrt{h_n} Z_{2,n}, \tag{14}$$

where $(Z_{1,n})_{n=1,2,\ldots}$ and $(Z_{2,n})_{n=1,2,\ldots}$ are ρ-correlated samples of standard normal random variables, and $t_n = \sum_{k=0}^{n} h_n$ is a mesh grid.

Euler Scheme for the Heston Model

Modified Euler-Maruyama Scheme

But we have a problem: during simulation of the Heston model using Euler method S_{t_n} and v_{t_n} could be negative. How do we deal with this inconvenience? Let us introduce the log-prices

$$X(t) := \log \frac{S(t)}{S(0)}. \tag{15}$$

Euler Scheme for the Heston Model

Modified Euler-Maruyama Scheme

There are two options:

1. Positive part of variance:

$$X_{n+1} = X_n + (\mu - 0.5 v_n^+) h_n + \sqrt{v_n^+} X_n \sqrt{h_n} Z_{1,n}, \tag{16}$$

$$v_{n+1} = v_n + \left(\delta^2 - 2\beta v_n^+\right) h_n + 2\delta \sqrt{v_n^+} \sqrt{h_n} Z_{2,n}, \tag{17}$$

2. Absolute value of variance:

$$X_{n+1} = X_n + (\mu - 0.5 |v_n|) h_n + \sqrt{|v_n|} X_n \sqrt{h_n} Z_{1,n}, \tag{18}$$

$$v_{n+1} = |v_n| + \left(\delta^2 - 2\beta |v_n|\right) h_n + 2\delta \sqrt{|v_n|} \sqrt{h_n} Z_{2,n}, \tag{19}$$

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Conclusion

We introduced the three most common simulation methods for dynamics of the two-factor Gaussian diffusion model:

1. Euler scheme;
2. Broadie-Kaya scheme;
3. Andersen scheme.

Using these methods we simulated the dynamics of the Heston model and computed the price of the European call options with different strikes and maturities.