Student Research Group ’Stochastic Volatility Models’, Project ’Heston−2’

Methods of Simulation of the Heston Model: A Review

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January 21, 2023

Heston Model Definition

Assume that the spot asset at time t follows the diffusion

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dZ_1(t), \tag{1}$$

$$dv(t) = \left(\delta^2 - 2\beta v(t)\right) dt + \sigma\sqrt{v(t)}dZ_2(t), \tag{2}$$

where Z_1, Z_2 are the correlated Wiener processes with $dZ_1 dZ_2 = \rho dt$.

Outline

Truncated Gaussian Scheme

Exact+Milstein Scheme

Conclusion

Truncated Gaussian Discretization Scheme

Idea

Andersen:

In this scheme the idea is to sample from a moment-matched Gaussian density where all probability mass below zero is inserted into a delta-function at the origin.

Same, but in the formular form:

$$\left(\hat{V}(t+\Delta)\middle|\, V(t)\right) = (\mu + \sigma Z)^{+}\,, \tag{3}$$

where Z is a standard normal random variable and μ and σ are the 'mean' and the 'standard deviation' of the desired distribution. We find μ and σ from the same old moment-matching techniques (see Slide ??).

Truncated Gaussian Discretization Scheme

Finding the constants

Proposition 1

Let $\phi(x)$ be a standart Gaussian density and define a function $r : \mathbb{R} \to \mathbb{R}$ by the following equation:

$$r(x)\phi(r(x)) + \Phi(r(x))(1 + r(x)^2) = (1 + x)\left(\phi(r(x)) + r(x)\Phi(r(x))\right)^2 . \tag{4}$$

Then the moment-matching parameters are

$$\mu = \frac{m}{\frac{\phi(r(\psi))}{r(\psi)} + \Phi(r(\psi))}, \tag{5}$$

$$\sigma = \frac{m}{\phi(r(\psi)) + r(\psi)\Phi(r(\psi))}. \tag{6}$$

Truncated Gaussian Discretization Scheme

Finding the numerical integration interval

Problem: no closed-form solution for $r(\psi)$.
Solution: numerical solution.
Problem: no known limits to use the numerical solution.
Solution:

$$m = \frac{\delta^2}{2\beta} + \left(\hat{V}(t) - \frac{\delta^2}{2\beta}\right) e^{-2\beta\Delta}, \tag{7}$$

$$s^2 = \frac{\hat{V}(t)\sigma^2 e^{-2\beta\Delta}}{2\beta}\left(1 - e^{-2\beta\Delta}\right) + \frac{\delta^2\sigma^2}{8\beta^2}\left(1 - e^{-2\beta\Delta}\right)^2. \tag{8}$$

Truncated Gaussian Discretization Scheme

Finding the numerical integration interval

$$\psi = \frac{s^2}{m^2} = \frac{\frac{\hat{V}(t)\sigma^2 e^{-2\beta\Delta}}{2\beta}\left(1 - e^{-2\beta\Delta}\right) + \frac{\delta^2\sigma^2}{8\beta^2}\left(1 - e^{-2\beta\Delta}\right)^2}{\left(\frac{\delta^2}{2\beta} + \left(\hat{V}(t) - \frac{\delta^2}{2\beta}\right) e^{-2\beta\Delta}\right)^2}. \quad (9)$$

Differentiating this expression with respect to $V(t)$ shows that $\frac{\partial\psi}{\partial V(t)} < 0$ for all $V(t) \geq 0$, such that the largest possible value for π is obtained for $V(t) = 0$, and the smallest possible value for$V(t) = \infty$. Inserting these values for $V(t)$ into (9) shows that $\psi \in (0, \frac{\beta^2\sigma^2}{2\delta^2})$.

Truncated Gaussian Discretization Scheme

Finding the numerical integration interval

As a final computational trick, note that once we have established the function r we can write

$$\mu = m \cdot f_\mu(\psi), \quad f_\mu(\psi) = \frac{r(\psi)}{\phi(r(\psi)) + r(\psi)\Phi(r(\psi))} \tag{10}$$

$$\sigma = s \cdot f_\sigma(\psi), \quad f_\sigma(\psi) = \frac{\psi^{-\frac{1}{2}}}{\phi(r(\psi)) + r(\psi)\Phi(r(\psi))} \tag{11}$$

The two functions $f_\mu(\psi)$ and $f_\sigma(\psi)$ are ultimately what we should cache on a computer onceand for all, on an equidistant grid for ψ large enough to span the domain.

Outline

E+M Idea

- The idea behind the scheme is to use the exact scheme for the underlying asset process and the Euler scheme for the volatility process.
- The motivation for this is the fact that the frequency of negative values of the variance process decreases for the Milstein scheme compared to Euler.
- Why do we use the B&K approach? It can be used for discretizations on a larger number of time instants when not trying to sample the integral, but using an approximation instead.

E+M Scheme

$$X_t = X_u + r(t-u) - \frac{1}{2}\int_u^t V_s ds + \int_u^t \sqrt{V_s} dW_s, \tag{12}$$

$$V_{i+1} = V_i + \kappa(\theta - V_i)h + \gamma\sqrt{V_i h}Z_2 + \frac{1}{4}\gamma^2 h(Z_2^2 - 1). \tag{13}$$

Outline