Student Research Group 'Stochastic Volatility Models'
# Methods of Simulation of the Heston Model: A Review

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## Heston Model Definition

Assume that the spot asset at time $t$ follows the diffusion

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dZ_1(t), \tag{1}$$

$$dv(t) = \left(\delta^2 - 2\beta v(t)\right)dt + 2\delta\sqrt{v(t)}dZ_2(t), \tag{2}$$

where $Z_1, Z_2$ are the correlated Wiener processes with $dZ_1 dZ_2 = \rho dt$.

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## Statistical Estimation

Let $X, X_1, X_2, \ldots, X_n$ be a series of independent and identically distributed random variables with a distribution function $F(x)$. Unbiased consistent estimators of $\mathbb{E}[X]$ and $\mathrm{var}[X]$ are:

- The sample mean is the average of the sample values:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{3}$$

- The sample variance is the corrected by the factor of $\frac{n}{n-1}$ average squared deviation from the mean:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \tag{4}$$

## Statistical Estimation

### Lemma 1

*Let $X_1, X_2, \ldots, X_n$ be a series of independent and identically distributed random variables, and $h : \mathbb{R} \to \mathbb{R}$ be a borel function. Then $h(X_1), h(X_2), \ldots, h(X_n)$ is a series of independent and identically distributed random variables.*

Thus, we could write the unbiased consistent estimator of $\mathbb{E}\left[h(X)\right]$ as follows:

$$\widehat{\mathbb{E}\left[h(X)\right]} = \frac{1}{n} \sum_{i=1}^{n} h(X_i). \tag{5}$$

# Random and Pseudo-Random Numbers

## Definition 2

A random sequence of numbers is a sequence (in time or in space) with no discernable pattern.

## Definition 3

Pseudorandom number generator is an algorithm that generates a sequence of numbers that has no internal pattern. However, the series requires a starting seed, and if the algorithm is started repeatedly with the same seed, it will go through precisely the same sequence of numbers.

Check out the lecture by A.N.Shiryaev (Probability Theory Department $\times$ Vega Institute Foundation Seminar, Oct 19) for more details.

## Law of Large Numbers

### Theorem 4 (Khinchin)

*Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with $\mathbb{E}X_i = \mu$. Then*

$$\underset{n \to \infty}{\mathbb{P}\text{-}\lim} \frac{1}{n} \sum_{i=1}^{n} X_i = \mu. \tag{6}$$

### Theorem 5 (Kolmogorov)

*Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables. Then $\exists \mathbb{E}X_i = \mu$, if and only if*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i \overset{a.s.}{=} \mu. \tag{7}$$

# Central Limit Theorem

## Theorem 6 (Lindeberg-Lévy)

*Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $\operatorname{var}[X_i] = \sigma^2$. Then as $n$ approaches infinity, the random variables $\sqrt{n}(\bar{X}_n - \mu)$ converge in law to a normal distribution $\mathcal{N}(0, \sigma^2)$, i.e.*

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right). \tag{8}$$

## NB

*Law of large numbers is a corollarary of the central limit theorem.*

# Monte Carlo Simulation

### Definition 7

Monte Carlo simulation is a set of techniques that use pseudo-random number generators to solve problems that might be too complicated to be solved analytically. It is based on the central limit theorem.

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Suppose that we have an ODE of the form

$$dX(t) = f(X(t), t)dt, \quad X(0) = X_0. \tag{9}$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_n, X_n)h_n, \tag{10}$$

where $t_n = \sum_{k=0}^{n} h_n$ is a mesh grid.

# Forward Euler Scheme for ODEs

Global truncation error

## Lemma 8

*Let the solution of* (9) *have an $M$-bounded second derivative and let $f$ be $L$-Lipshitz continious in its second argument. Then the global truncation error of the mesh solution* (10) *is*

$$|X(T) - X_N| \leq \frac{hM}{2L} \left( e^{LT} - 1 \right) = O(h).$$ (11)

## Euler-Maruyama Scheme for SDEs

Definition

Suppose we have a diffusion of the form

$$dX(t) = f(X(t), t)dt + \sigma(X(t), t)dW(t), \quad X_0 = X_0.$$

Then it could be numerically solved by the following finite difference scheme:

$$X_{n+1} = X_n + f(t_n, X_n)h_n + \sigma(t_n, X_n)\sqrt{h_n}Z_n, \tag{12}$$

where $(Z_n)_{n=1,2,...}$ is a sample of standard normal random variables, and $t_n = \sum_{k=0}^{n} h_n$ is a mesh grid. The same method could be generalized for the two-factor Gaussian diffusions.

Let $\hat{X}(t)^n$ be a mesh approximation of an SDE solution $X(t)$ (we assume that there exists a unique strong solution).

$$\hat{X}^n(t) \to X(t) \quad \text{in probability} \quad \text{as} \quad n \to \infty. \tag{13}$$

## Euler Scheme for the Heston Model

Classical Euler-Maruyama Scheme

Suppose we have the Heston model of the form

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dZ_1(t),$$
$$dv(t) = \left(\delta^2 - 2\beta v(t)\right)dt + 2\delta\sqrt{v(t)}dZ_2(t),$$

where $Z_1$, $Z_2$ are the correlated Wiener processes with $dZ_1 dZ_2 = \rho dt$. Then it could be numerically solved by the following finite difference scheme:

$$S_{n+1} = S_n + \mu S_n h_n + \sqrt{v_n}S_n\sqrt{h_n}Z_{1,n}, \tag{14}$$
$$v_{n+1} = v_n + \left(\delta^2 - 2\beta v_n\right)h_n + 2\delta\sqrt{v_n}\sqrt{h_n}Z_{2,n}, \tag{15}$$

where $(Z_{1,n})_{n=1,2,\ldots}$ and $(Z_{2,n})_{n=1,2,\ldots}$ are $\rho$-correlated samples of standard normal random variables, and $t_n = \sum_{k=0}^{n} h_n$ is a mesh grid.

But we have a problem: during simulation of the Heston model using Euler method $S_{t_n}$ and $v_{t_n}$ could be negative. How do we deal with this inconvinience? Let us introduce the log-prices

$$X(t) := \log \frac{S(t)}{S(0)}. \tag{16}$$

## Euler Scheme for the Heston Model

Modified Euler-Maruyama Scheme

There are two options:

1. Positive part of variance:

$$X_{n+1} = X_n + (\mu - 0.5v_n^+)h_n + \sqrt{v_n^+}X_n\sqrt{h_n}Z_{1,n}, \qquad (17)$$

$$v_{n+1} = v_n + \left(\delta^2 - 2\beta v_n^+\right)h_n + 2\delta\sqrt{v_n^+}\sqrt{h_n}Z_{2,n}, \qquad (18)$$

2. Absolute value of variance:

$$X_{n+1} = X_n + (\mu - 0.5|v_n|)h_n + \sqrt{|v_n|}X_n\sqrt{h_n}Z_{1,n}, \qquad (19)$$

$$v_{n+1} = |v_n| + \left(\delta^2 - 2\beta|v_n|\right)h_n + 2\delta\sqrt{|v_n|}\sqrt{h_n}Z_{2,n}, \qquad (20)$$

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# Conclusion

We introduced the three most common simulation methods for dynamics of the two-factor Gaussian diffusion model:

1. Euler scheme;
2. Broadie-Kaya scheme;
3. Andersen scheme.

Using these methods we simulated the dynamics of the Heston model and computed the price of the European call options with different strikes and maturities.