**The Space $T_s^r$**

A tensor of type (r.s) or of order (r+s) is an element of the product space.

$$T_s^r = \underbrace{T \otimes \ldots\ldots..\otimes T}_{\text{r times}} \otimes \underbrace{T^* \ldots\ldots\ldots \otimes T^*}_{\text{s times}}$$

In particular T = $T^i$ and $T^*$= $T_1$

A member of $T^r$ a contravariant tensor of order r, a member of $T_s$, is a covariant tensor of order s, while a member of $T_s^r$ is a mixed tensor of order (r + s). A member of $T^r$ is also referred to as a tensor of type (r, 0). a member of $T_s$. as a tensor of type (0, s) and a member of $T_s^r$ a tensor of type (r, s), The contravariant and covariant vectors as tensors of types (1, 0) and (0,1) respectively and the scalars may be included as a tensor of type (0, 0).

Let $\{\underline{e}_a\}$ be the basis of T which induces naturally $\{\tilde{e}^a\}$ the dual basis of $T$ i.e. $T^*$and $\{\tilde{e}^a\}$ induce the basis set $\{\underline{g}_a\}$ of T which a dual to $T^*$. Since the dimension of each are equal i, e, n. Define $n^{r+s}$ functions

$$\underline{e}_{a_1\ldots\ldots\ldots a_r}^{b_1\ldots\ldots\ldots b_s} : \quad \underbrace{T^* \times \ldots\ldots\ldots \times T^*}_{\text{r times}} \times \underbrace{T \times \ldots\ldots \times T}_{\text{s times}} \to R \ by$$

$$\underline{e}_{a_1\ldots\ldots a_r}^{b_1\ldots\ldots b_s}\left(\tilde{\lambda}^1 \ldots\ldots \tilde{\lambda}^r, \underline{\mu}_{1,\ldots\ldots,}\underline{\mu}_s\right)\lambda_{a_1}^1 \ldots\ldots.\lambda_{a_r}^r \ldots\ldots.\mu_1^{b_1} \ldots\ldots\mu_s^{b_s}.$$

Here we lebels the vectors of $T^*$ by superscript and that of the vectors of T by subscript and their respective components superscript and subscript. In particular

$$\underline{e}_{a_1\ldots\ldots ar}^{b_1\ldots\ldots bs}\left(\tilde{e}^{c_1} \ldots\ldots \tilde{e}^{c_r}, \underline{g}_{d_1}, \ldots..\underline{g}_{d_s}\right) = \delta_{a_1}^{c_1} \ldots\ldots \delta_{a_r}^{c_r}\delta_{d_1}^{b_1} \ldots\ldots.\delta_{d_s}^{b_s}.$$

To show $\underline{e}_{a_1\ldots\ldots ar}^{b_1\ldots\ldots bs}$ the basis of $T_s^r$ we have to show that they spans $T_s^r$ that they are independent.

For any $\underline{\tau}\left(\tilde{e}^{a_1}, \ldots\ldots, \tilde{e}^{a_r}, \underline{g}_{b_1}, \ldots\ldots, \underline{g}_{b_s}\right) = \tau_{b_1\ldots\ldots b_s}^{a_1\ldots\ldots a_r}$

Then $\underline{\tau}\left(\tilde{\lambda}^1 \ldots\ldots \tilde{\lambda}^r, \underline{\mu}_{1,\ldots\ldots,}\underline{\mu}_s\right) = \underline{\tau}(\lambda_{a_1}^1\tilde{e}^{a_1}, \ldots\ldots.\lambda_{a_r}^r\tilde{e}^{a_r}.\mu_1^{b_1}\underline{g}_{b_1} \ldots\ldots\mu_s^{b_s}\underline{g}_{b_s})$

$$= \lambda_{a_1}^1 \ldots\ldots.\lambda_{a_r}^r.\mu_1^{b_1} \ldots\ldots\mu_s^{b_s}\underline{\tau}\left(\tilde{e}^{a_1}, \ldots\ldots, \tilde{e}^{a_r}, \underline{g}_{b_1}, \ldots\ldots, \underline{g}_{b_s}\right)$$

$$= \tau_{b_1\ldots\ldots b_s}^{a_1\ldots\ldots a_r}\lambda_{a_1}^1 \ldots\ldots.\lambda_{a_r}^r\mu_1^{b_1} \ldots\ldots\mu_s^{b_s} = \tau_{b_1\ldots\ldots b_s}^{a_1\ldots\ldots a_r}$$

$$= \tau_{b_1\ldots\ldots b_s}^{a_1\ldots\ldots a_r}\underline{e}_{a_1\ldots\ldots a_r}^{b_1\ldots\ldots b_s}\left(\tilde{\lambda}^1 \ldots\ldots \tilde{\lambda}^r, \underline{\mu}_{1,\ldots\ldots,}\underline{\mu}_s\right)$$

so for any $\underline{\tau}\epsilon T_s^r$ we have $\tau_{b_1\ldots\ldots b_s}^{a_1\ldots\ldots a_r}\underline{e}_{a_1\ldots\ldots a_r}^{b_1\ldots\ldots b_s}$, showing that the $\{\underline{e}_{a_1\ldots\ldots a_r}^{b_1\ldots\ldots b_s}\}$ spans$T_s^r$. More over, $\{\underline{e}_{a_1\ldots\ldots a_r}^{b_1\ldots\ldots b_s}\}$ is an independent set, for if

$$x_{b_1\ldots\ldots b_s}^{a_1\ldots\ldots a_r}\underline{e}_{a_1\ldots\ldots a_r}^{b_1\ldots\ldots b_s} = \underline{0}$$

$$\therefore\ x_{b_1\ldots\ldots b_s}^{a_1\ldots\ldots a_r}\underline{e}_{a_1\ldots\ldots a_r}^{b_1\ldots\ldots b_s}\left(\tilde{e}^{c_1}, \ldots\ldots, \tilde{e}^{c_r}, \underline{g}_{d_1}, \ldots\ldots, \underline{g}_{d_s}\right) = 0$$

$$or\ \ x_{b_1\ldots\ldots b_s}^{a_1\ldots\ldots a_r}\delta_{a_1}^{c_1} \ldots\ldots \delta_{a_r}^{c_r}\delta_{d_1}^{b_1} \ldots\ldots.\delta_{d_s}^{b_s} = 0$$

$or\ x^{c_1 \dots\dots c_r}_{d_1 \dots\dots d_s} = 0$

for all $c_1 \dots \dots\ c_r, d_1 \dots \dots\ d_s$ . On using equation ( Equation Hobay)

Thus we have seen that any $\underline{\tau} \epsilon T^r_s$ can be expressed as a linear combination of

$\underline{e}^{b_1 \dots\dots b_s}_{a_1 \dots\dots a_r}$ $and$ $\underline{e}^{b_1 \dots\dots b_s}_{a_1 \dots\dots a_r}$ are independent hence form the basis of the space $T^r_s$.

Now for the non-singular transformations of the bases

$$\underline{e}'_a = X^c_{a'} \underline{e}_c, \tilde{e}'^a = X^{a'}_d \tilde{e}^d \cdot X^c_{a'} X^{a'}_d = \delta^c_d \text{ ( Equation Hobay)}$$

Change the components of the tensor $T^r_s$ according to the transforamtion law

$$\tau'^{a_1 \dots\dots a_r}_{b_1 \dots\dots b_s} = X^{a'_1}_{c_1} \cdots\cdots X^{a'_r}_{c_r} X^{d_1}_{b'_1} \cdots\cdots X^{d_s}_{b'_s} \tau^{c_1 \dots\dots c_r}_{d_1 \dots\dots d_s} \text{ ( Equation Hobay)}$$

Also the basis of $T^r_s$ transform like

$$\underline{e}'^{b_1 \dots\dots b_s}_{a_1 \dots\dots a_r} = X^{c_1}_{a'_1} \cdots\cdots X^{c_r}_{a'_r} X^{b'_1}_{d_1} \cdots\cdots\cdots X^{b'_s}_{d_s} \underline{e}^{d_1 \dots\dots d_s}_{c_1 \dots\dots c_r} \text{ ( Equation Hobay)}$$

A useful theorem for determining tensor character is the so called quotient theorem. If the result of taking the product (outer or inner) of a given set of elements with a tensor of any specified type of arbitrary components is known to be a tensor, then the given set of elements are also components of a tensor, we consider an example, suppose $\tau^a_{bc}$ are a set of $n^3$ quantities. let $\lambda^f_g$ be a mixed tensor of rank two whose components can be choosen arbitrarily and suppose it is given that the inner product

$$\tau^a_{bc}\ \lambda^c_g = \mu^a_{bg} \text{ ( Equation Hobay)}$$

is a tensor for all such $\lambda^f_g$ . For transformed basis, we get

$$\tau^{*a}_{\ bc} \lambda'^c_g = \mu'^a_{bg} \text{ ( Equation Hobay)}$$

where $\tau^{*a}_{\ bc}$ are transformed components, Let $\tau'^a_{bc}$ be defined for the same transformed basis by the following relation,

$$\tau'^a_{bc} = X^{a'}_d X^i_{b'} X^h_{c'} \tau^d_{ih} \text{ ( Equation Hobay)}$$

Since this is a tensor transformation law, we know the elements, so defined will satisfy.

$$\tau'^a_{bc} \lambda'^c_g = \mu'^a_{bg}$$

Subtracting (1,4,9) from (1.4,7) we get

$$(\tau^{*a}_{\ bc} - \tau'^a_{bc}) \lambda'^c_g = 0$$

Since $\lambda^f_g$ has arbitrary components its components in transform basis are also arbitrary. $\lambda'^c_g$ can be assume any convenient values. Thus taking

$$\left.\begin{aligned} \lambda'^{c}_{g} &= 1 \qquad \text{when} \quad c = g \\ &= 0 \qquad \text{otherwise} \end{aligned}\right\} \text{yields}$$

$$\tau^{*a}_{\ bd} - \tau'^{a}_{bd} = 0$$

$$\tau^{*a}_{\ bd} = \tau'^{a}_{bd}$$

This being true for j = 1,2, ............, n. we have quite generally

$$\tau^{*a}_{\ bc} = \tau'^{a}_{bc}$$

We are now interested to introduce the special type of tensor, Consider the tensor p of type (1,1) defined by specifying its components relative to a given basis $\{\underline{e}_a\}$ i to be the Kronecker delta, i.e. $p^a_b = \delta^a_b$ . Relative to another basis $\{\underline{e}'_a\}$ its components are

$$p'^{a}_{b} = X^{a'}_{c} X^{d}_{b'} \delta^{c}_{d} = X^{a'}_{c} X^{c}_{b'} = \delta^{a}_{b}$$

Thus P has the same components relative to any basis. This special tensor is called the Kronecker tensor and it is customary to use $\delta^a_b$ rather than $p^a_b$ for its components.