

4 Formal Logic Cheatsheet

4.1 Propositional Logic

- * **Proposition**  is a statement which can be either true or false. Truth-bearer
- * **Alphabet**  of propositional logic consists of (1) atomic symbols and (2) operator symbols.
- * **Atomic formula**  (**atom**) is an irreducible formula without logical connectives.
 - **Propositional variables:** A, B, C, \dots, Z . With indices, if needed: $A_1, A_2, \dots, Z_1, Z_2, \dots$
 - **Logical constants:** \top for always true proposition (*tautology*), \perp for always false proposition (*contradiction*).
- * **Logical connectives**  (**operators**):

Type	Natural meaning	Symbolization
 Negation	It is not the case that \mathcal{P} . It is false that \mathcal{P} . It is not true that \mathcal{P} .	$\neg\mathcal{P}$
 Conjunction	Both \mathcal{P} and \mathcal{Q} . \mathcal{P} but \mathcal{Q} . \mathcal{P} , although \mathcal{Q} .	$\mathcal{P} \wedge \mathcal{Q}$
 Disjunction	Either \mathcal{P} or \mathcal{Q} (or both). \mathcal{P} unless \mathcal{Q} .	$\mathcal{P} \vee \mathcal{Q}$
 Exclusive or (Xor)	Either \mathcal{P} or \mathcal{Q} (but not both). \mathcal{P} xor \mathcal{Q} .	$\mathcal{P} \oplus \mathcal{Q}$
 Implication (Conditional)	If \mathcal{P} , then \mathcal{Q} . \mathcal{P} only if \mathcal{Q} . \mathcal{Q} if \mathcal{P} .	$\mathcal{P} \rightarrow \mathcal{Q}$
 Biconditional	\mathcal{P} , if and only if \mathcal{Q} . \mathcal{P} iff \mathcal{Q} . \mathcal{P} just in case \mathcal{Q} .	$\mathcal{P} \leftrightarrow \mathcal{Q}$

- * **Sentence** of propositional logic is defined inductively: Well-formed formula (WFF)

 1. Every propositional variable/constant is a sentence.
 2. If \mathcal{A} is a sentence, then $\neg\mathcal{A}$ is a sentence.
 3. If \mathcal{A} and \mathcal{B} are sentences, then $(\mathcal{A} \wedge \mathcal{B})$, $(\mathcal{A} \vee \mathcal{B})$, $(\mathcal{A} \rightarrow \mathcal{B})$, $(\mathcal{A} \leftrightarrow \mathcal{B})$ are sentences.
 4. Nothing else is a sentence.

- * Well-formed formulae grammar:

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<sentence> ::= <constant>
  | <variable>
  |  $\neg$  <sentence>
  | '(' <sentence> <binop> <sentence> ')'
  
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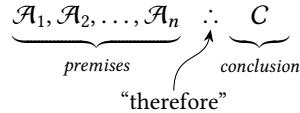
$\langle \text{constant} \rangle ::= \top \mid \perp$

$\langle \text{variable} \rangle ::= A \mid \dots \mid Z \mid A_1 \mid \dots \mid Z_n$

$\langle \text{binop} \rangle ::= \wedge \mid \vee \mid \oplus \mid \rightarrow \mid \leftarrow \mid \leftrightarrow$

Backus-Naur form (BNF)

- * **Literal**  is a propositional variable or its negation: $\mathcal{L}_i = X_i$ (*positive literal*), $\mathcal{L}_j = \neg X_j$ (*negative literal*).
- * **Argument**  is a set of logical statements, called *premises*, intended to support or infer a claim (*conclusion*):



- * An argument is **valid** if whenever all the premises are true, the conclusion is also true. Validity
- * An argument is **invalid** if there is a case (*a counterexample*) when all the premises are true, but the conclusion is false.

4.2 Semantics of Propositional Logic

- * **Valuation** \square is any assignment of truth values to propositional variables. Interpretation
- * \mathcal{A} is a **tautology** \square (valid) iff it is true on *every* valuation. Might be symbolized as “ $\models \mathcal{A}$ ”.
- * \mathcal{A} is a **contradiction** \square iff it is false on *every* valuation. Might be symbolized as “ $\mathcal{A} \models$ ”.
- * \mathcal{A} is a **contingency** \square iff it is true on some valuation and false on another. In other words, a **contingent** proposition is neither a tautology nor a contradiction.
- * \mathcal{A} is **satisfiable** iff it is true on *some* valuation. Satisfiability
- * \mathcal{A} is **falsifiable** iff it is not valid, *i.e.* it is false on *some* valuation. Falsifiability
- * \mathcal{A} and \mathcal{B} are **equivalent** \square (symbolized as $\mathcal{A} \equiv \mathcal{B}$) iff, for every valuation, their truth values agree, *i.e.* there is no valuation in which they have opposite truth values. Equivalence check
- * $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are **consistent** (**jointly satisfiable**) iff there is *some* valuation which makes them all true. Sentences are **inconsistent** (**jointly unsatisfiable**) iff there is *no* valuation that makes them all true. Consistency
- * The sentences $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ **entail** the sentence C (symbolized as $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$) if there is no valuation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ true and C false. Semantic entailment
- * If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$, then the argument $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \therefore C$ is **valid**.

Validity check examples:

A	B	$A \rightarrow B$	$A \quad \vdots \quad B$	$\neg A \rightarrow \neg B$	$\vdots \quad B \rightarrow A$	$A \rightarrow B$	$B \quad \vdots \quad \neg(B \rightarrow A)$
0	0	1	0 · 0	1 1 1	✓ 1	1	0 · 0 1
0	1	1	0 · 1	1 0 0	· 0	1	1 ✓ 1 0
1	0	0	1 · 0	0 1 1	✓ 1	0	0 · 0 1
1	1	1	1 ✓ 1	0 1 0	✓ 1	1	1 1 ✗ 0 1

R	S	T	$R \vee S$	$S \vee T$	$\neg R \quad \vdots \quad S \wedge T$	$(R \wedge S) \rightarrow T$	$\vdots \quad R \rightarrow (S \rightarrow T)$
0	0	0	0	0	0 · 0	✓ 0	1
0	0	1	0	1	0 · 0	✓ 0	1
0	1	0	1	1	✗ 0	0	1
0	1	1	1	1	✓ 1	0	1
1	0	0	1	0	0 · 0	✓ 1	1
1	0	1	1	1	0 · 0	✓ 1	1
1	1	0	1	1	0 · 0	1 0	0
1	1	1	1	1	0 · 1	✓ 1	1

* **Soundness** \square : $\Gamma \vdash \mathcal{A} \rightarrow \Gamma \models \mathcal{A}$ “Every provable statement is in fact true”

* **Completeness**: \square $\Gamma \models \mathcal{A} \rightarrow \Gamma \vdash \mathcal{A}$ “Every true statement has a proof”

4.3 Natural Deduction Rules

Reiteration $\begin{array}{c c} m & \mathcal{A} \\ \hline \therefore & \mathcal{A} \quad R\ m \end{array}$	Explosion $\begin{array}{c c} m & \perp \\ \hline \therefore & \mathcal{A} \quad X\ m \end{array}$	Conditional $\begin{array}{c c} i & \mathcal{A} \\ j & \hline \mathcal{B} \\ \hline \therefore & \mathcal{A} \rightarrow \mathcal{B} \quad \rightarrow I\ i-j \end{array}$
Modus ponens $\begin{array}{c c} i & \mathcal{A} \rightarrow \mathcal{B} \\ j & \mathcal{A} \\ \hline \therefore & \mathcal{B} \quad MP\ i, j \end{array}$	Conjunction $\begin{array}{c c} i & \mathcal{A} \\ j & \mathcal{B} \\ \hline \therefore & \mathcal{A} \wedge \mathcal{B} \quad \wedge I\ i, j \end{array}$	Contraposition $\begin{array}{c c} m & \mathcal{A} \rightarrow \mathcal{B} \\ \hline \therefore & \neg \mathcal{B} \rightarrow \neg \mathcal{A} \quad Contra\ m \end{array}$
Modus tollens $\begin{array}{c c} i & \mathcal{A} \rightarrow \mathcal{B} \\ j & \neg \mathcal{B} \\ \hline \therefore & \neg \mathcal{A} \quad MT\ i, j \end{array}$	$\begin{array}{c c} m & \mathcal{A} \wedge \mathcal{B} \\ \hline \therefore & \mathcal{A} \quad \wedge E\ m \\ \therefore & \mathcal{B} \quad \wedge E\ m \end{array}$	Biconditional $\begin{array}{c c} i & \mathcal{A} \\ j & \hline \mathcal{B} \\ k & \mathcal{B} \\ l & \hline \mathcal{A} \\ \hline \therefore & \mathcal{A} \leftrightarrow \mathcal{B} \quad \leftrightarrow I\ i-j, k-l \end{array}$
Negation $\begin{array}{c c} i & \neg \mathcal{A} \\ j & \mathcal{A} \\ \hline \therefore & \perp \quad \neg E\ i, j \end{array}$	Disjunction $\begin{array}{c c} m & \mathcal{A} \\ \hline \therefore & \mathcal{A} \vee \mathcal{B} \quad \vee I\ m \end{array}$	$\begin{array}{c c} i & \mathcal{A} \leftrightarrow \mathcal{B} \\ j & \mathcal{A} \\ \hline \therefore & \mathcal{B} \quad \leftrightarrow E\ i, j \end{array}$
$\begin{array}{c c} i & \mathcal{A} \\ j & \hline \perp \\ \hline \therefore & \neg \mathcal{A} \quad \neg I\ i-j \end{array}$	$\begin{array}{c c} m & \mathcal{A} \\ \hline \therefore & \mathcal{B} \vee \mathcal{A} \quad \vee I\ m \end{array}$	$\begin{array}{c c} i & \mathcal{A} \leftrightarrow \mathcal{B} \\ j & \mathcal{B} \\ \hline \therefore & \mathcal{A} \quad \leftrightarrow E\ i, j \end{array}$
Indirect proof $\begin{array}{c c} i & \neg \mathcal{A} \\ j & \hline \perp \\ \hline \therefore & \mathcal{A} \quad IP\ i-j \end{array}$	$\begin{array}{c c} m & \mathcal{A} \vee \mathcal{B} \\ i & \mathcal{A} \\ j & \hline C \\ k & \mathcal{B} \\ l & \hline C \\ \hline \therefore & C \quad \vee E\ m, i-j, k-l \end{array}$	De Morgan Rules $\begin{array}{c c} m & \neg(\mathcal{A} \vee \mathcal{B}) \\ \hline \therefore & \neg \mathcal{A} \wedge \neg \mathcal{B} \quad DeM\ m \end{array}$
Double negation $\begin{array}{c c} m & \neg \neg \mathcal{A} \\ \hline \therefore & \mathcal{A} \quad \neg \neg E\ m \end{array}$	Disjunctive syllogism $\begin{array}{c c} i & \mathcal{A} \vee \mathcal{B} \\ j & \neg \mathcal{A} \\ \hline \therefore & \mathcal{B} \quad DS\ i, j \end{array}$	$\begin{array}{c c} m & \neg(\mathcal{A} \wedge \mathcal{B}) \\ \hline \therefore & \neg \mathcal{A} \vee \neg \mathcal{B} \quad DeM\ m \end{array}$
$\begin{array}{c c} i & \mathcal{A} \\ j & \hline \mathcal{B} \\ k & \neg \mathcal{A} \\ l & \hline \mathcal{B} \\ \hline \therefore & \mathcal{B} \quad LEM\ i-j, k-l \end{array}$	$\begin{array}{c c} i & \mathcal{A} \vee \mathcal{B} \\ j & \neg \mathcal{B} \\ \hline \therefore & \mathcal{A} \quad DS\ i, j \end{array}$	$\begin{array}{c c} m & \neg(\mathcal{A} \wedge \mathcal{B}) \\ \hline \therefore & \neg \mathcal{A} \vee \neg \mathcal{B} \quad DeM\ m \end{array}$
Law of excluded middle $\begin{array}{c c} i & \mathcal{A} \\ j & \hline \mathcal{B} \\ k & \neg \mathcal{A} \\ l & \hline \mathcal{B} \\ \hline \therefore & \mathcal{B} \quad LEM\ i-j, k-l \end{array}$	Hypothetical syllogism $\begin{array}{c c} i & \mathcal{A} \rightarrow \mathcal{B} \\ j & \mathcal{B} \rightarrow C \\ \hline \therefore & \mathcal{A} \rightarrow C \quad HS\ i, j \end{array}$	$\begin{array}{c c} m & \neg \mathcal{A} \vee \neg \mathcal{B} \\ \hline \therefore & \neg(\mathcal{A} \wedge \mathcal{B}) \quad DeM\ m \end{array}$

Green: basic rules.

Orange: derived rules.

More rules can be found in the “forall x: Calgary” book (p. 406).