Problem sheet on Asymptotic Analysis

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Exercises marked with a $*$ are not important from the exam point of view. All exercises are taken from or based on something in Goodrich and Tamassia's book.

Exercise 1. *Let $d(n), e(n), f(n)$ and $g(n)$ be functions mapping non-negative integers to non-negative reals. Prove that*

1. *If $d(n)$ is $O(f(n))$ then $ad(n)$ is $O(f(n))$ for any constant $a > 0$.*
2. *If $d(n)$ is $O(f(n))$ and $e(n)$ is $O(g(n))$ then $d(n) + e(n)$ is $O(f(n) + g(n))$.*
3. *If $d(n)$ is $O(f(n))$ and $e(n)$ is $O(g(n))$ then $d(n)e(n)$ is $O(f(n)g(n))$.*
4. *If $d(n)$ is $O(f(n))$ and $f(n)$ is $O(g(n))$ then $d(n)$ is $O(g(n))$.*

To give you an idea of how solutions to these problems are to be written, here is the solution to part 1 of Exercise 1.

Soln 1.1. Since $d(n)$ is $O(f(n))$, there exists real $c > 0$ and integer $n_0 > 0$ such that

$$d(n) \leq cf(n) \tag{1}$$

for all $n > n_0$.

Given any constant $a > 0$, multiplying (1) on both sides with a gives us that

$$ad(n) \leq acf(n),$$

for all $n \geq n_0$.

Hence if we set $c' = ac$ and $n_0' = n_0$ then we have found a c' and n_0' such that for all $n \geq n_0', ad(n) \leq c'f(n)$. Hence proved.

Exercise 2. *By finding the appropriate c, n_0 in each case, show that*

1. *If $f(n) = \sum_{i=0}^{d} a_i n^i, a_d > 0$ is a polynomial of degree d then $f(n)$ is $O(n^d)$.*
2. *n^x is $O(a^n)$ for any constant x (i.e. x does not change with n) and any $a > 1$.*
3. *$\log n^x$ is $O(\log n)$ for any constant $x > 0$.*
4. *$(\log n)^x$ is $O(n^y)$ for any constants $x, y > 0$.*

Exercise 3 (*). *In Goodrich and Tamassia's book it says that $f(n)$ is $o(g(n))$ if for every $c > 0$ and some $n_0 > 0$ $f(n) \leq c(g(n))$ for all $n > n_0$.*

1. *In class we defined this slightly differently: we said $f(n)$ is $o(g(n))$ if for every $c > 0$ and every $n_0 > 0$, there is an $n > n_0$ such that $f(n) \leq c(g(n))$ for all $n > n_0$. Is this equivalent to the definition given in the book?*

2. *Prove that $f(n)$ is $o(g(n)$ (as defined in the book) if and only if*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0,$$

provided the limit exists.

Exercise 4. *Order the following functions according to O notation, i.e., if $f(n)$ is $O(g(n))$ then $f(n)$ should come before $g(n)$. Also underline groups of functions that are θ of each other.*

1. $6n\log n; 2^{100}; \log\log n; \log^2 n; 2^{\log n}$.

2. $4^n; n^3; n^2\log n; 4^{\log n}; \sqrt{\log n}$.

Exercise 5. *Suppose you have an $n\times n$ 2-dimensional array A such that each row of the matrix consists of some number of 0s followed by some number of 1s. Describe a method for finding the row with the maximum number of 1s in it. What is the running time of your method? Is it $o(n^2)$? Is it possible to do this in $o(n^2)$ time? Is it possible to do it in $o(n)$ time?*