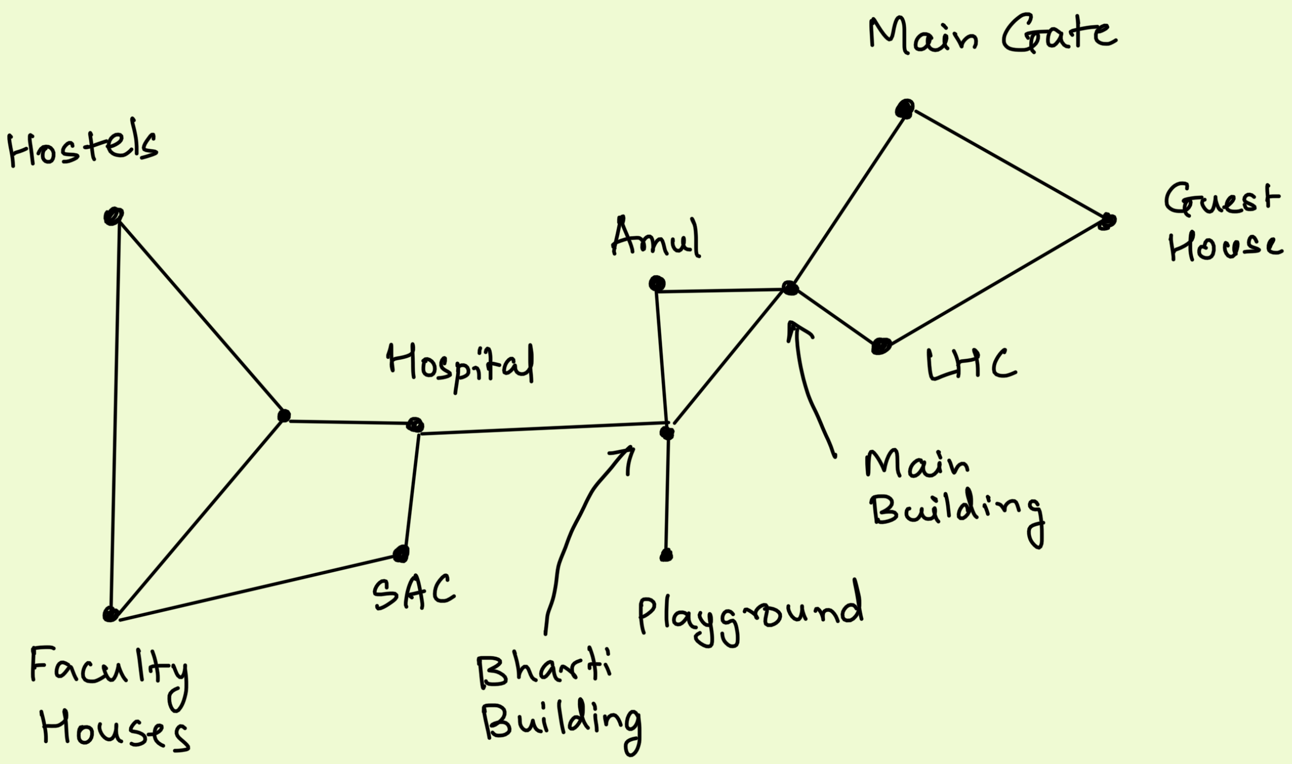
COL 106 Lecture 28

Topic : Graphs – Terminology

Motivating Example – IITD campus layout



Mathematical definition of a graph

A graph is a pair (V, E), where

- V is a finite set (a.k.a. set of vertices)
- E is a set of 2-subsets of V (a.k.a. set of edges)

Vertex $v \in V$ is an endpoint of edge $e \in E$ and e is said to be incident on v if $v \in e$.

Degree of a vertex is the number of edges incident on it.

Handshake lemma: The sum of the degrees of all vertices of a graph is twice the number of edges.

Proof: Count elements of the following set in two ways:

$$I = \{ (v, e) \mid v \in V,\ e \in E,\ e \text{ is incident on } v \}$$

$$|I| = \sum_{e \in E} \left| \{ v \mid e \text{ is incident on } v \} \right| = 2 \cdot |E|.$$

$$|I| = \sum_{v \in V} \left| \{ e \mid e \text{ is incident on } v \} \right| = \sum_{v \in V} \text{degree of } v. \blacksquare$$

More definitions:

Empty graph: Graph whose edge set is empty

Complete graph: Graph whose edge set contains all 2-subsets of the vertex set. $\left(|E| = \binom{|V|}{2} = \frac{|V| \times (|V|-1)}{2}\right)$

Walk: Sequence of vertices $v_0, v_1, \ldots, v_m$ such that for each i, $\{v_{i-1}, v_i\}$ is an edge.

Path: Walk $v_0, v_1, \ldots, v_m$ such that $v_0 \ldots v_m$ are distinct vertices

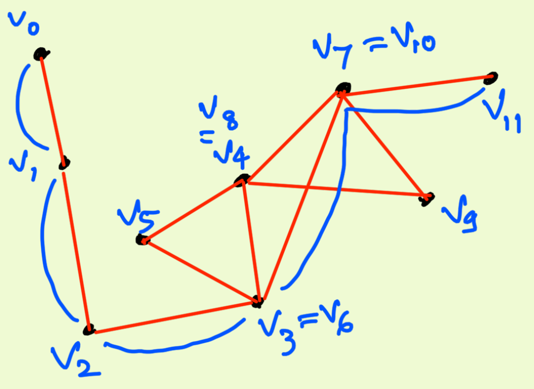
Claim: Given a graph G and its vertices u and v,
$\left\{\begin{matrix}\text{there is a walk} \\ \text{from } u \text{ to } v\end{matrix}\right\}$ if and only if $\left\{\begin{matrix}\text{there is a path} \\ \text{from } u \text{ to } v\end{matrix}\right\}$

Proof: "If": obvious because a path is a walk

"only if": Suppose there is a u to v walk.

Consider a shortest u to v walk. It must be a path.

(If not, can remove a segment between two visits to the same vertex and get a shorter walk.)



Claim: For every graph G and vertices x, y, z of G:

$$\left(\exists \text{ a } \begin{Bmatrix} \text{walk} \\ \text{path} \end{Bmatrix} \text{ from } x \text{ to } y \right.$$

AND

$$\left. \exists \text{ a } \begin{Bmatrix} \text{walk} \\ \text{path} \end{Bmatrix} \text{ from } y \text{ to } z \right)$$

IMPLIES

$$\exists \text{ a } \begin{Bmatrix} \text{walk} \\ \text{path} \end{Bmatrix} \text{ from } x \text{ to } z.$$

Proof: Concatenation of x to y walk and y to z walk is an x to z walk.