Disk Based Data Structures

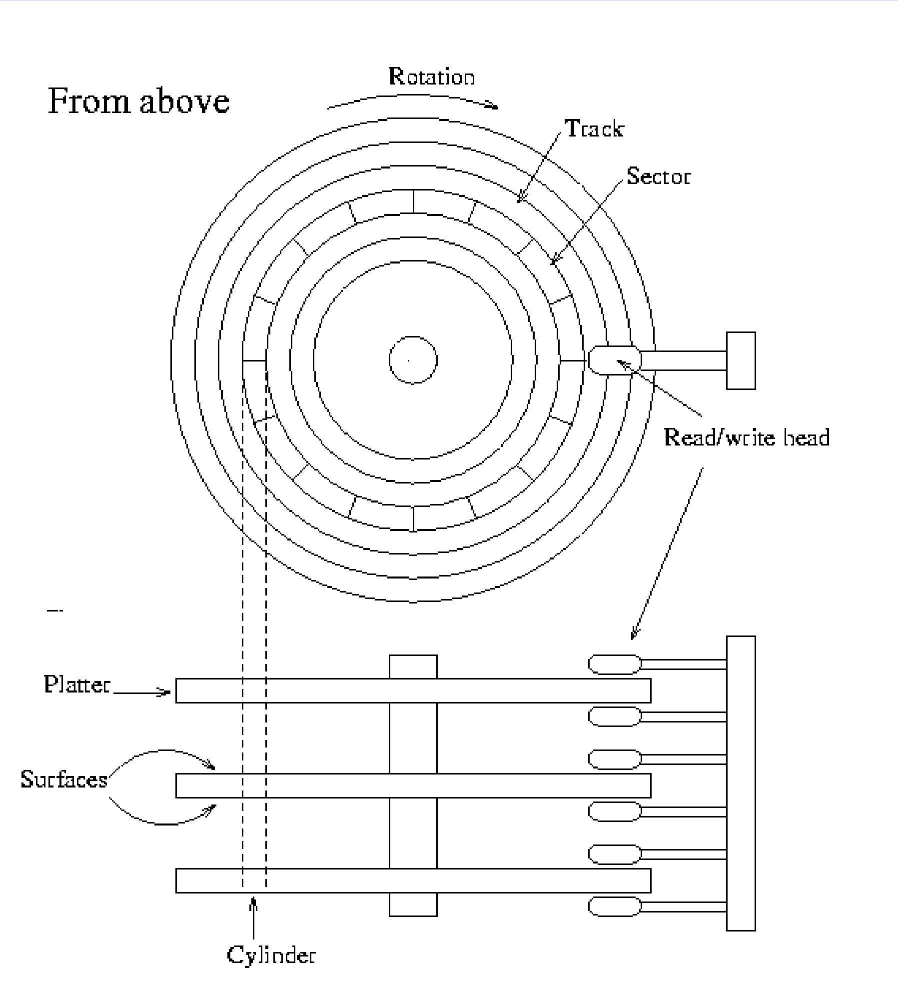
- So far search trees were limited to main memory structures

Where data stored on RAM

  - Assumption: the dataset organized in a search tree fits in main memory (including the tree overhead)
- Counter-example: transaction data of a bank > 1 GB per day
  - use secondary storage media (punch cards, hard disks, magnetic tapes, etc.)
- Consequence: make a search tree structure secondary-storage-enabled

Hard Disks

- **Large amounts of storage, but slow access!**
- Identifying a page takes a long time (seek time plus rotational delay – 5-10ms), reading it is fast
  - It pays off to read or write data in **pages** (or blocks) of 2-16 Kb in size.



Algorithm analysis

- The running time of disk-based algorithms is measured in terms of
  - computing time (CPU)
  - number of disk accesses
    - sequential reads
    - random reads

Accessing data from main memory takes very little time but disc takes time

Our BTree should sit some part in main and some on disc

Here most of our pointers are pointing to disc

- Regular main-memory algorithms that work one data element at a time can not be "ported" to secondary storage in a straight-forward way

Principles

- Pointers in data structures are no longer addresses in main memory but locations in files
- If *x* is a pointer to an object
  - if *x* is in main memory *key*[*x*] refers to it
  - otherwise DiskRead(*x*) reads the object from disk into main memory (DiskWrite(*x*) – writes it back to disk)

Principles (2)

- A typical working pattern

```
…
x ← a pointer to some object
DiskRead(x)
operations that access and/or modify x
DiskWrite(x) //omitted if nothing changed
other operations, only access no modify
…
```

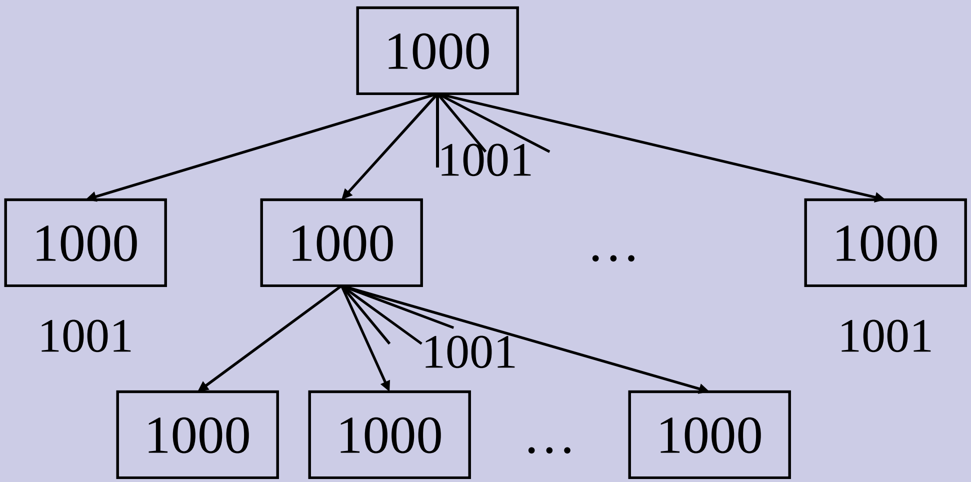
- Operations:
  - DiskRead(x:pointer_to_a_node)
  - DiskWrite(x:pointer_to_a_node)
  - AllocateNode():pointer_to_a_node

Binary-trees vs. B-trees

- Size of B-tree nodes is determined by the page size. One page – one node.

unit transferred from disc to main memory

- A B-tree of height 2 may contain > 1 billion keys!
- Heights of Binary-tree and B-tree are logarithmic
  - B-tree: logarithm of base, e.g., 1000
  - Binary-tree: logarithm of base 2



1 node
1000 keys

1001 nodes,
1,001,000 keys

1,002,001 nodes,
1,002,001,000 keys

B-tree Definitions

- Node x has fields
  - $n[x]$: the number of keys of that the node
  - $key_1[x] \leq \ldots \leq key_{n[x]}[x]$: the keys in ascending order
  - leaf[x]: true if leaf node, false if internal node
  - if internal node, then $c_1[x], \ldots, c_{n[x]+1}[x]$: pointers to children
- Keys separate the ranges of keys in the sub-trees. If k_i is an arbitrary key in the subtree $c_i[x]$ then $k_i \leq key_i[x] \leq k_{i+1}$

B-tree Definitions (2)

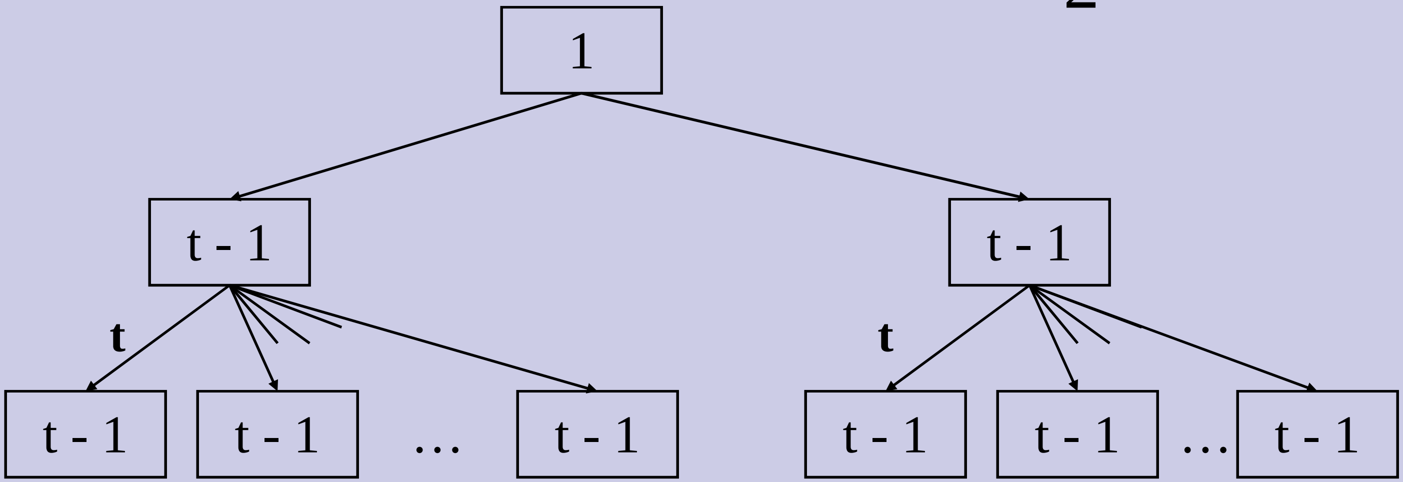
- Every leaf has the same depth
- In a B-tree of a **degree** *t* all nodes except the root node have between *t* and 2*t* children (i.e., between *t*–1 and 2*t*–1 keys).
- The root node has between 0 and 2*t* children (i.e., between 0 and 2*t*–1 keys)

Why between t and 2t?
If we had some fraction like t and 1.5 t then merge operations of t-1 and t-1 keys give 2(t-1) keys which is more than 1.5t
If we had t and 10t then lot of memory gets wasted as some may have t children, others 2t but every node has size 10t

Height of a B-tree

- B-tree *T* of height *h*, containing $n \geq 1$ keys and minimum degree $t \geq 2$, the following restriction on the height holds:

$$h \leq \log_t \frac{n+1}{2}$$



depth	#of nodes
0	1
1	2
2	2t

$$n \geq 1 + (t-1)\sum_{i=1}^{h} 2t^{i-1} = 2t^h - 1$$

B-tree Operations

- An implementation needs to suport the following B-tree operations
  - **Searching** (simple)
  - **Creating** an empty tree (trivial)
  - **Insertion** (complex)
  - **Deletion** (complex)

Searching

- Straightforward generalization of a binary tree search

```
BTreeSearch(x,k)
i ← 1
while i ≤ n[x] and k > key_i[x]
   i ← i+1
if i ≤ n[x] and k = key_i[x] then
   return(x,i)
if leaf[x] then
   return NIL
   else DiskRead(c_i[x])
        return BTtreeSearch(c_i[x],k)
```

Creating an Empty Tree

- Empty B-tree = create a root & write it to disk!

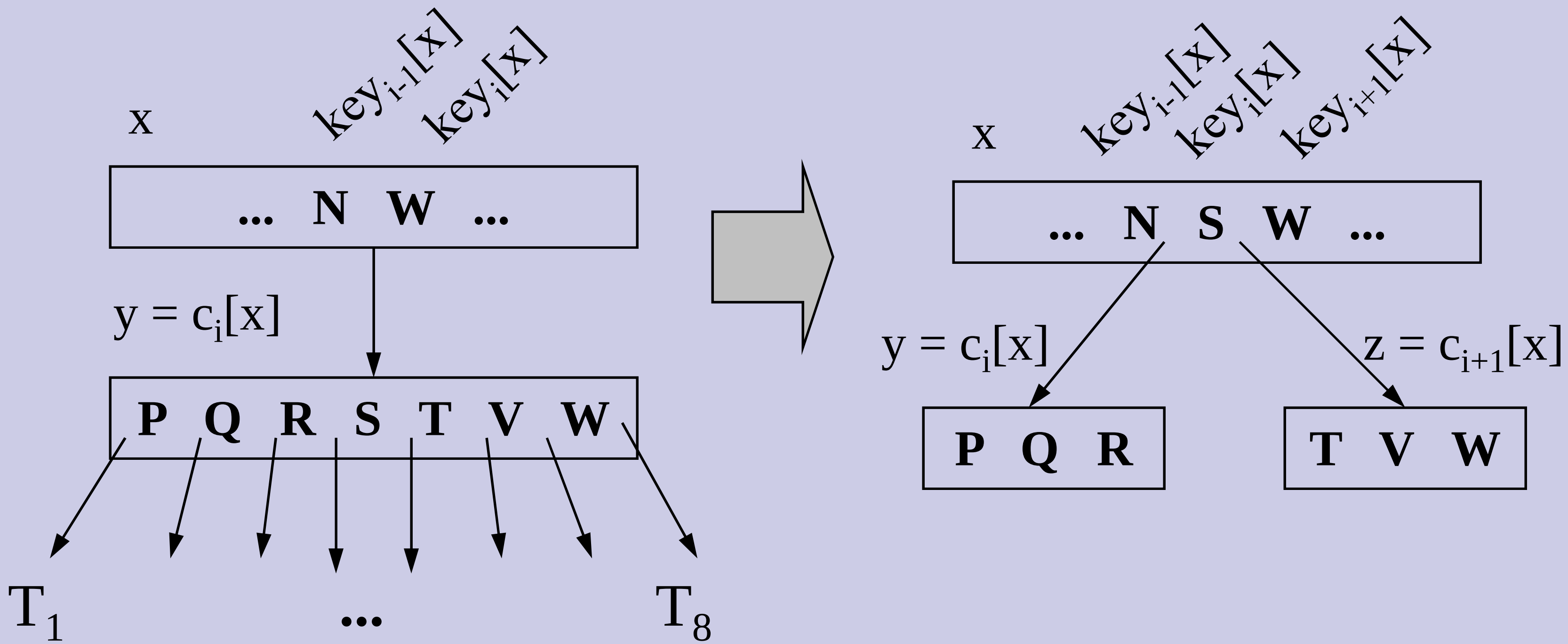
```
BTreeCreate(T)
x ← AllocateNode();
leaf[x] ← TRUE;
n[x] ← 0;
DiskWrite(x);
root[T] ← x
```

Splitting Nodes

- Nodes fill up and reach their maximum capacity $2t - 1$
- Before we can insert a new key, we have to “make room,” i.e., split nodes

Splitting Nodes (2)

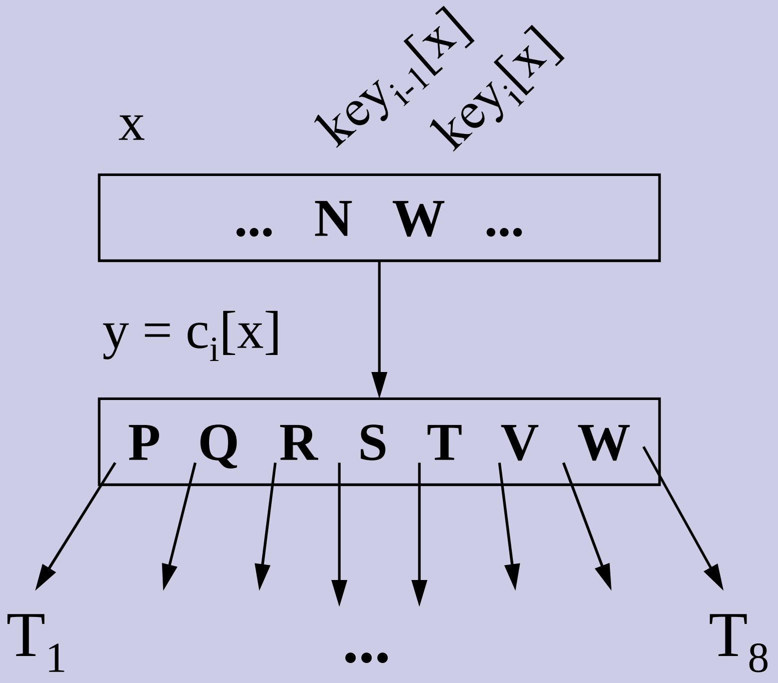
- Result: one key of *x* moves up to parent + 2 nodes with *t*-1 keys



Splitting Nodes (2)

```
BTreeSplitChild(x,i,y)
 z ← AllocateNode()
 leaf[z] ← leaf[y]
 n[z] ← t-1
 for j ← 1 to t-1
    key_j[z] ← key_{j+t}[y]
 if not leaf[y] then
    for j ← 1 to t
       c_j[z] ← c_{j+t}[y]
 n[y] ← t-1
 for j ← n[x]+1 downto i+1
    c_{j+1}[x] ← c_j[x]
 c_{i+1}[x] ←  z
 for j ← n[x] downto i
    key_{j+1}[x] ← key_j[x]
 key_i[x] ← key_t[y]
 n[x] ← n[x]+1
 DiskWrite(y)
 DiskWrite(z)
 DiskWrite(x)
```

x: parent node
y: node to be split and child of x
i: index in x
z: new node



Split: Running Time

- A local operation that does not traverse the tree
- $\Theta(t)$ CPU-time, since two loops run t times
- 3 I/Os

Inserting Keys

- Done recursively, by starting from the root and recursively traversing down the tree to the leaf level
- Before descending to a lower level in the tree, make sure that the node contains $< 2t - 1$ keys:
  - so that if we split a node in a lower level we will have space to include a new key

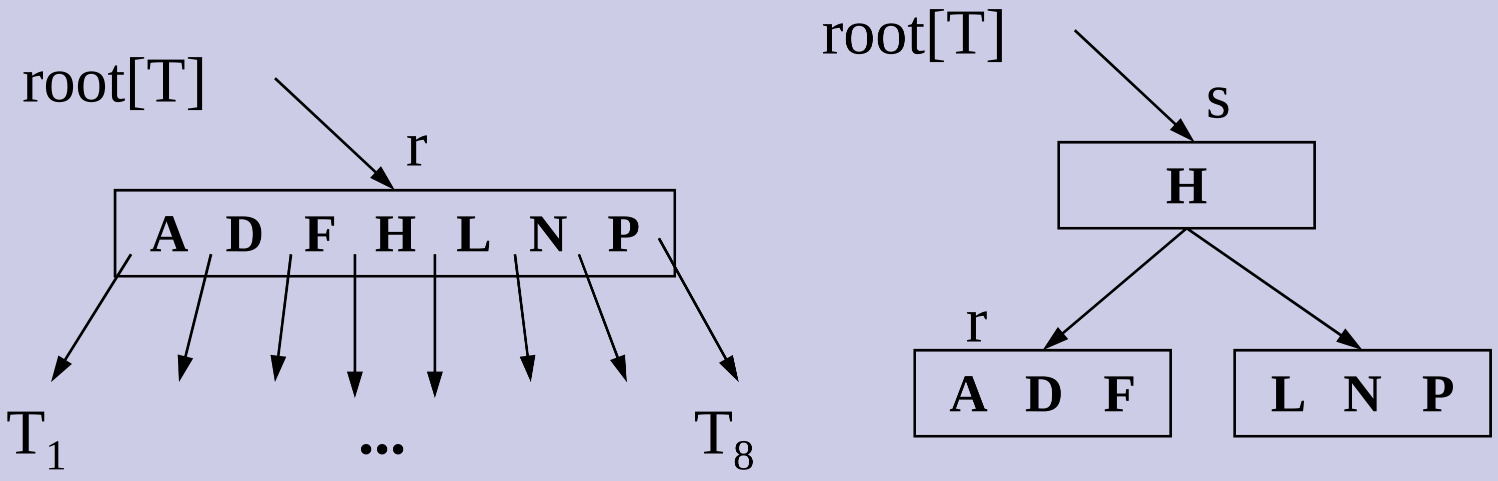
Inserting Keys (2)

- Special case: root is full (BtreeInsert)

```
BTreeInsert(T)
 r ← root[T]
 if n[r] = 2t – 1 then
    s ← AllocateNode()
    root[T] ← s
    leaf[s] ← FALSE
    n[s] ← 0
    c1[s] ← r
    BTreeSplitChild(s,1,r)
    BTreeInsertNonFull(s,k)
 else BTreeInsertNonFull(r,k)
```

Splitting the Root

- Splitting the root requires the creation of a new root



- The tree grows at the top instead of the bottom

Inserting Keys

- BtreeNonFull tries to insert a key *k* into a node *x*, which is **assumed to be non-full** when the procedure is called
- BTreeInsert and the recursion in BTreeInsertNonFull guarantee that this assumption is true!

Inserting Keys: Pseudo Code

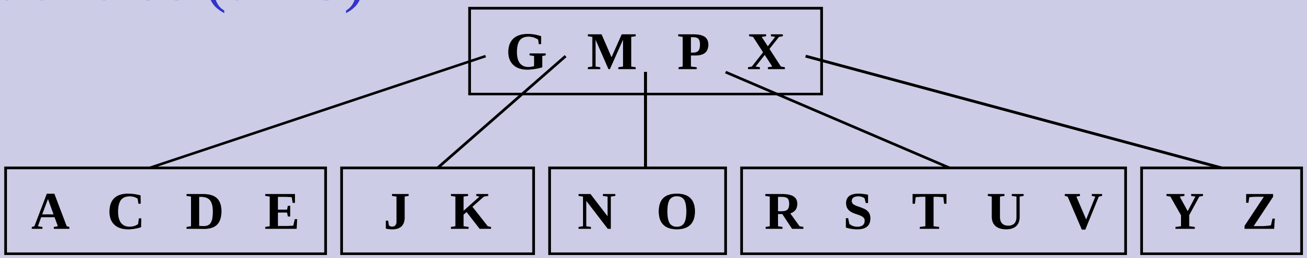
```
BTreeInsertNonFull(x,k)
i ← n[x]
if leaf[x] then
   while i ≥ 1 and k < key_i[x]
      key_i+1[x] ← key_i[x]
      i ← i - 1
   key_i+1[x] ← k
   n[x] ← n[x] + 1
   DiskWrite(x)
else while i ≥ 1 and k < key_i[x]
      i ← i - 1
   i ← i + 1
   DiskRead c_i[x]
   if n[c_i[x]] = 2t – 1 then
      BTreeSplitChild(x,i,c_i[x])
      if k > key_i[x] then
         i ← i + 1
   BTreeInsertNonFull(c_i[x],k)
```

leaf insertion

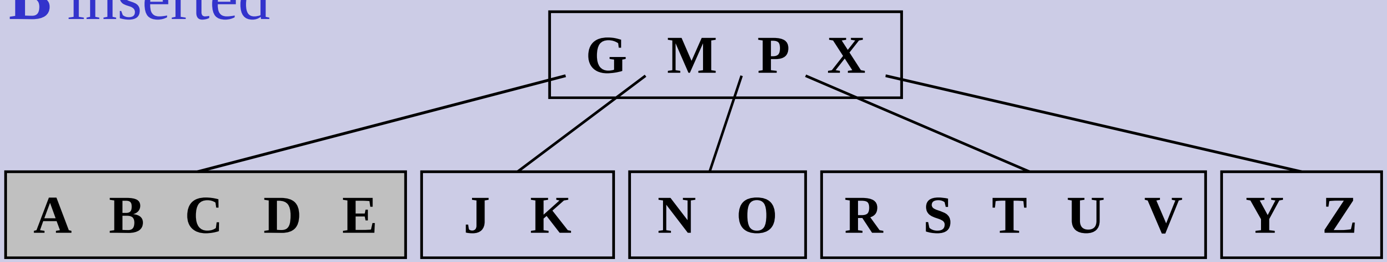
internal node: traversing tree

Insertion: Example

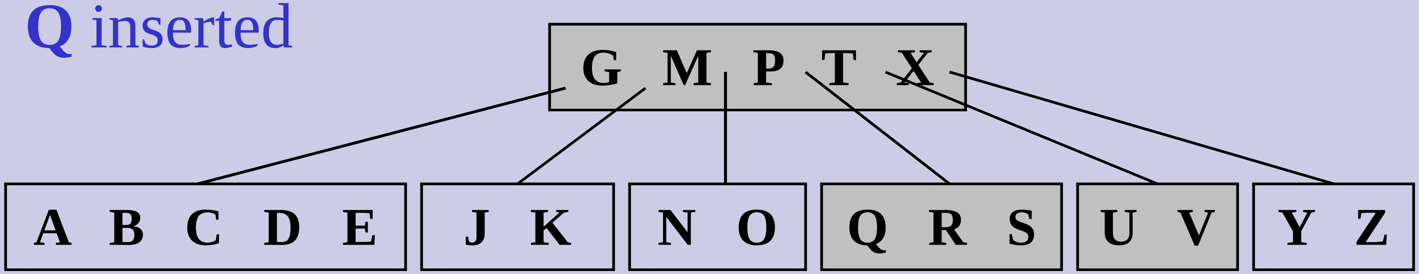
initial tree (t = 3)



B inserted



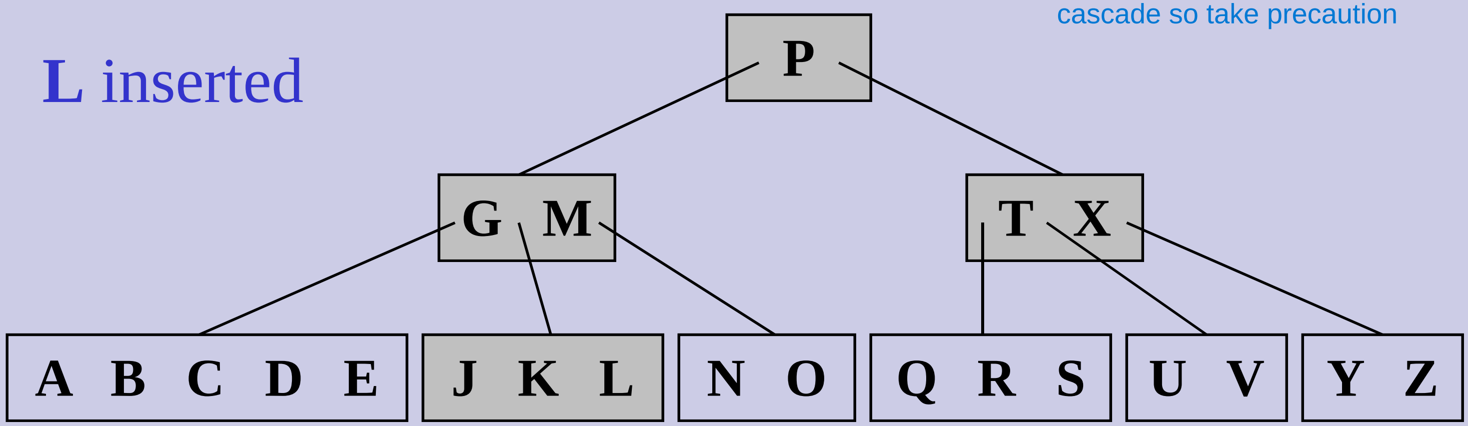
Q inserted



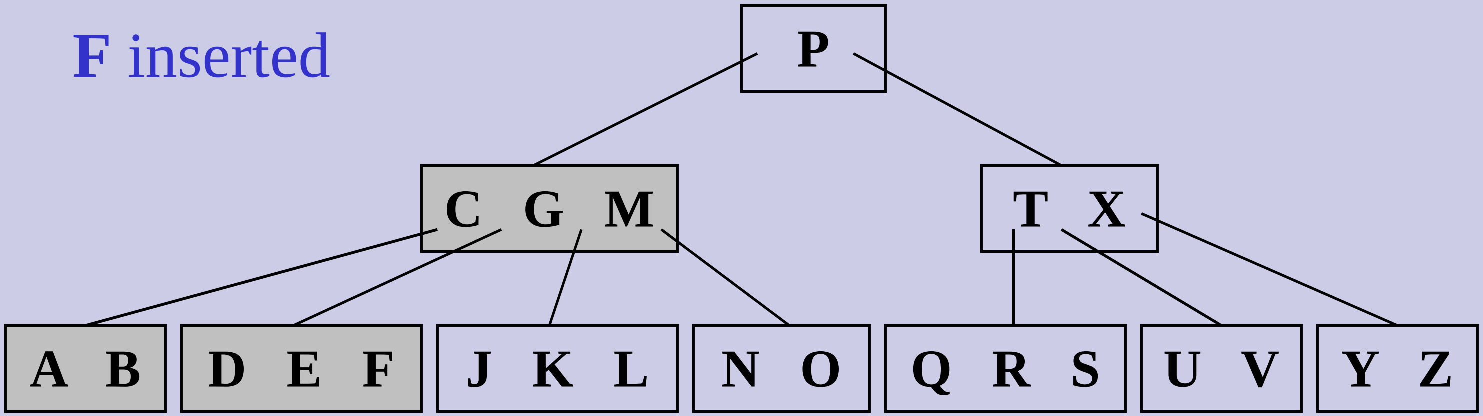
Insertion: Example (2)

preemptive split- maybe L could cause a cascade so take precaution

L inserted



F inserted



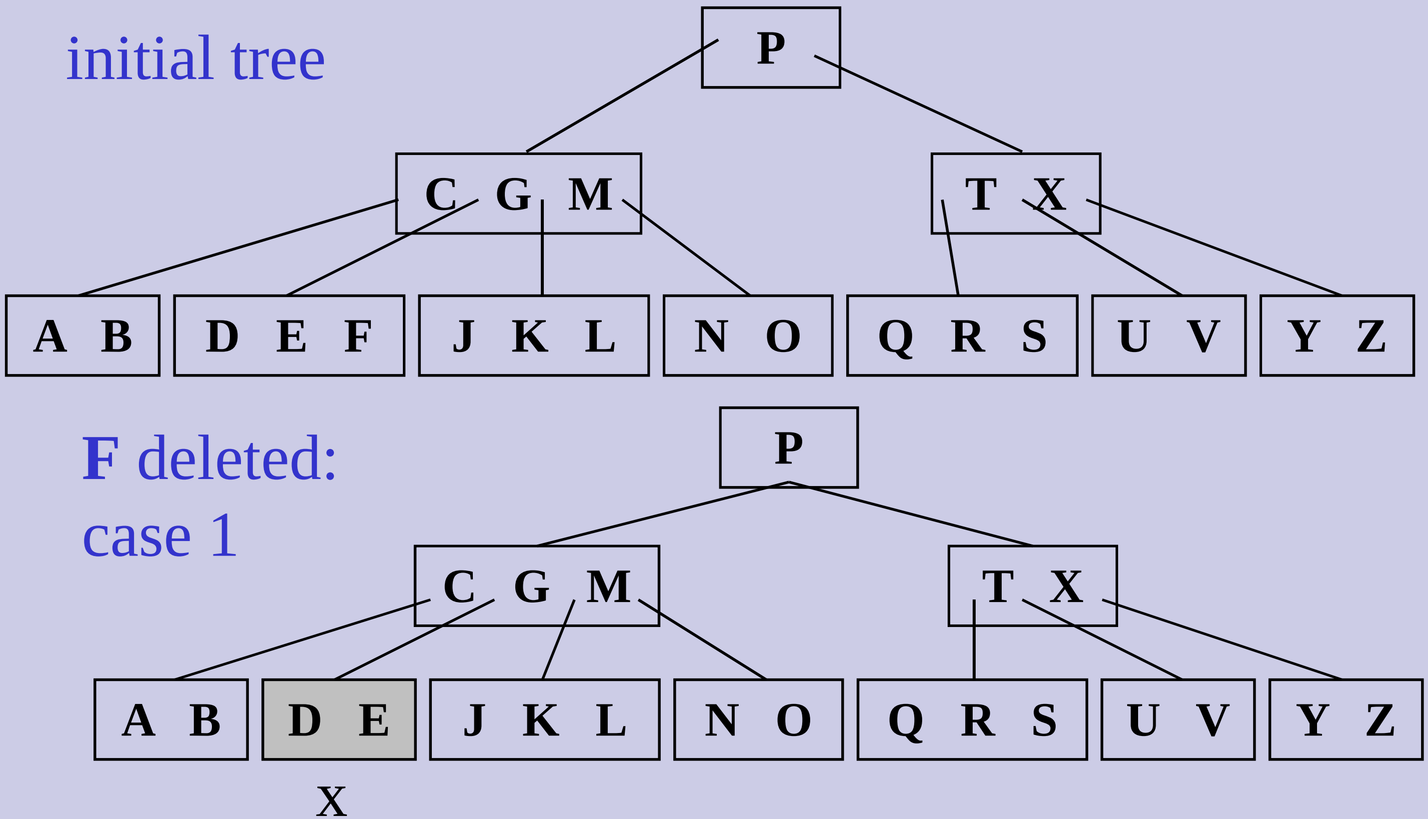
Insertion: Running Time

- Disk I/O: $O(h)$, since only $O(1)$ disk accesses are performed during recursive calls of BTreeInsertNonFull
- CPU: $O(th) = O(t \log_t n)$
- At any given time there are $O(1)$ number of disk pages in main memory

Deleting Keys

- Done recursively, by starting from the root and recursively traversing down the tree to the leaf level
- Before descending to a lower level in the tree, make sure that the node contains $\geq t$ keys (cf. insertion $< 2t - 1$ keys)
- BtreeDelete distinguishes three different stages/scenarios for deletion
  - Case 1: key *k* found in leaf node
  - Case 2: key *k* found in internal node
  - Case 3: key *k* suspected in lower level node

Deleting Keys (2)

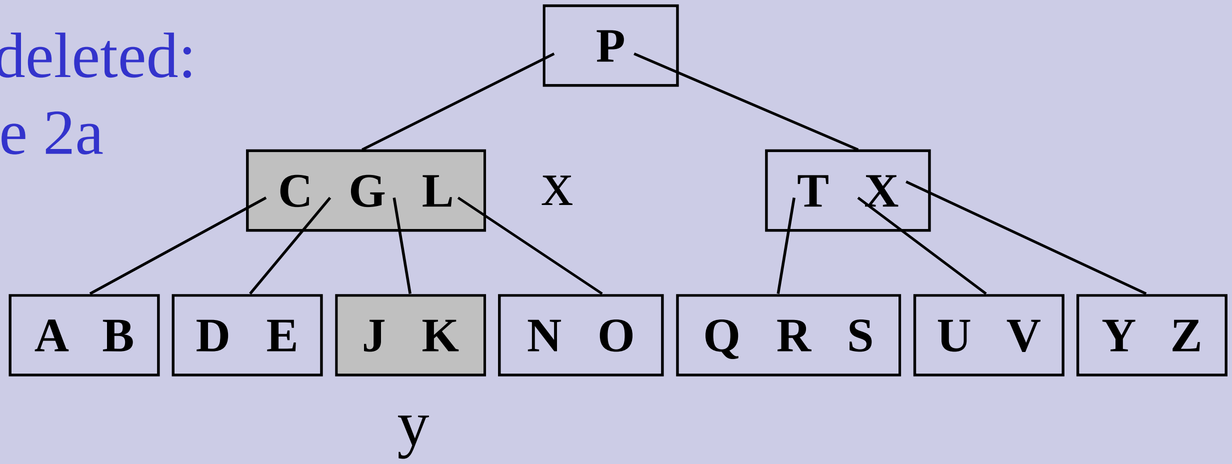


- Case 1: If the key *k* is in node *x*, and *x* is a leaf, delete *k* from *x*

Deleting Keys (3)

- Case 2: If the key *k* is in node *x*, and *x* is not a leaf, delete *k* from *x*
  - a) If the child *y* that precedes *k* in node *x* has at least *t* keys, then find the predecessor *k*' of *k* in the sub-tree rooted at *y*. Recursively delete *k*', and replace *k* with *k*' in *x*.
  - b) Symmetrically for successor node *z*

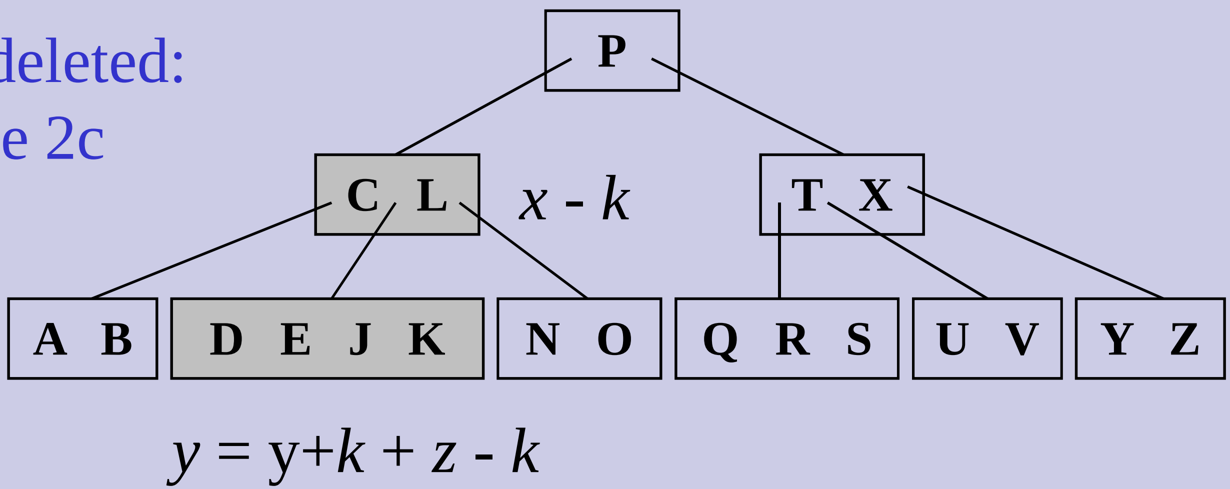
M deleted:
case 2a



Deleting Keys (4)

- If both *y* and *z* have only *t* –1 keys, ***merge*** *k* with the contents of *z* into *y*, so that *x* loses both *k* and the pointers to *z*, and *y* now contains 2*t* – 1 keys. Free *z* and recursively delete *k* from *y*.

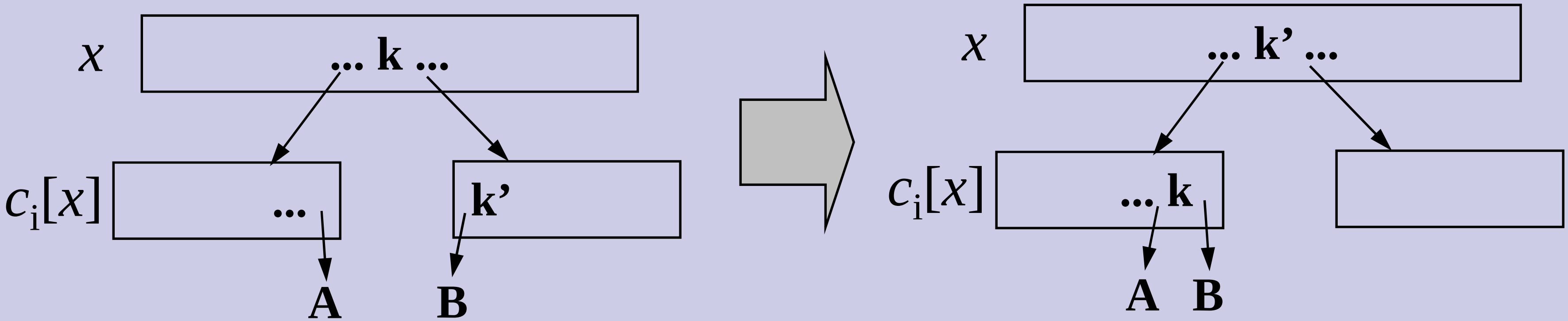
G deleted:
case 2c



Deleting Keys - Distribution

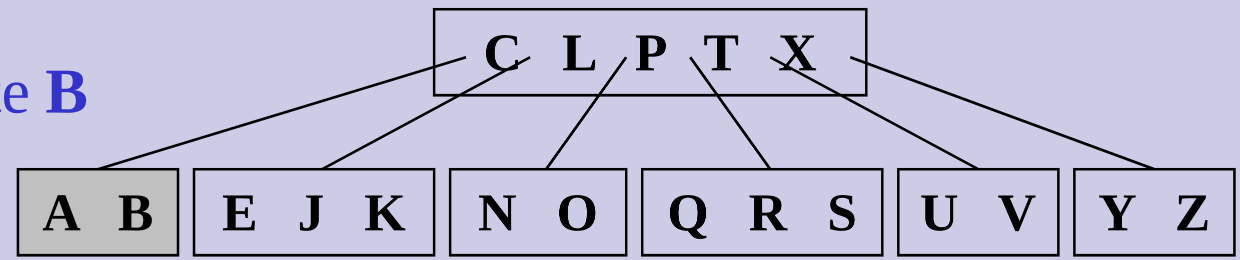
- Descending down the tree: if *k* not found in current node *x*, find the sub-tree $c_i[x]$ that has to contain k.
- If $c_i[x]$ has only $t - 1$ keys take action to ensure that we descent to a node of size at least *t.*
- We can encounter two cases.
  - If $c_i[x]$ has only t-1 keys, but a sibling with at least *t* keys, give $c_i[x]$ an extra key by moving a key from *x* to $c_i[x]$, moving a key from $c_i[x]$'s immediate left and right sibling up into *x*, and moving the appropriate child from the sibling into $c_i[x]$ - ***distribution***

Deleting Keys – Distribution(2)



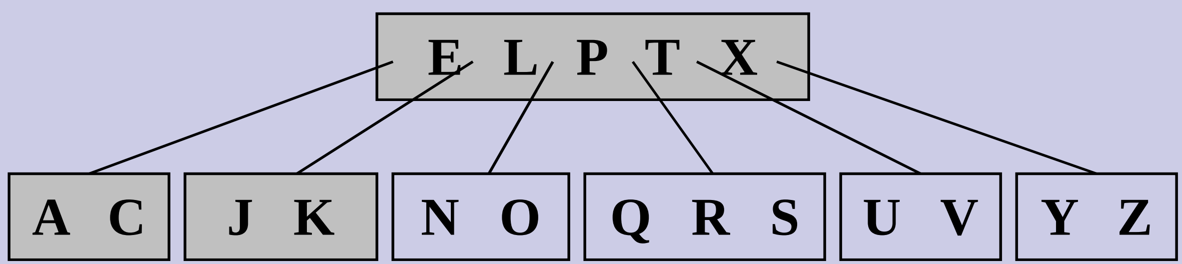
delete **B**

$c_i[x]$



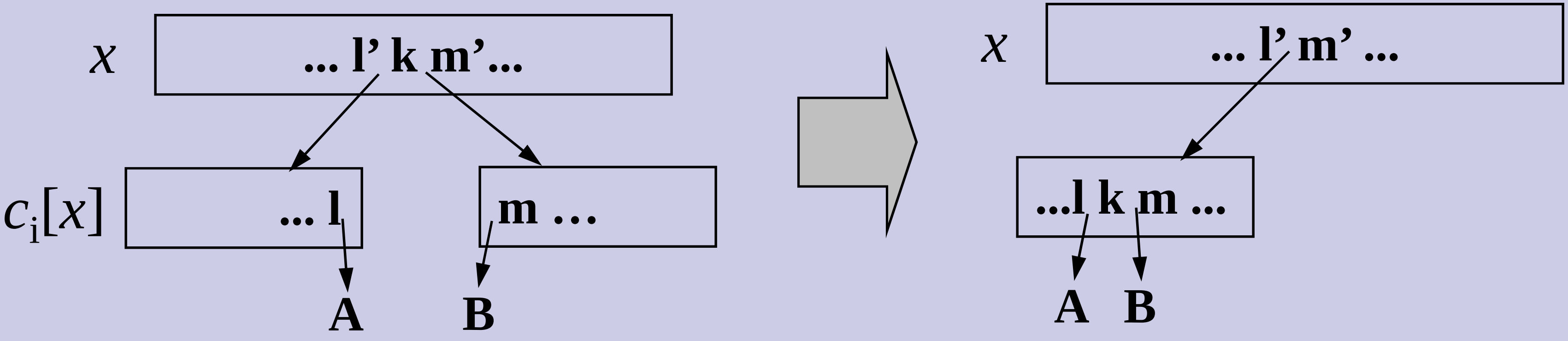
sibling

B deleted:



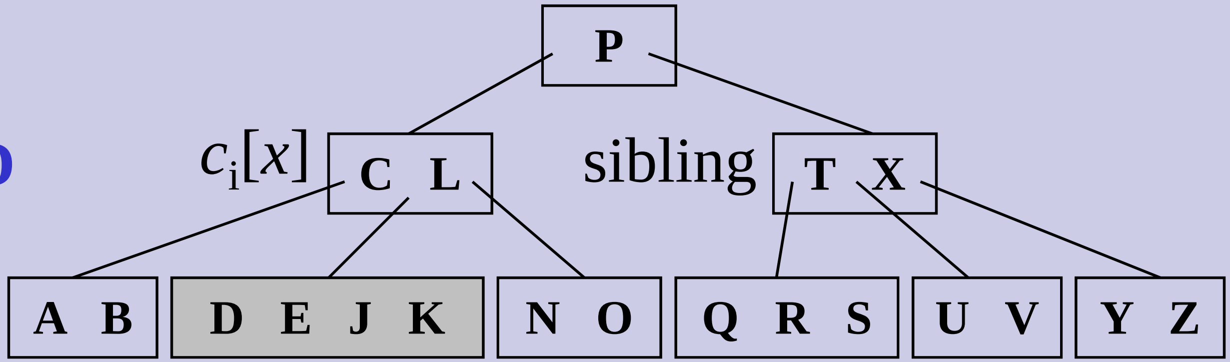
Deleting Keys - Merging

- If $c_i[x]$ and both of $c_i[x]$'s siblings have t – 1 keys, ***merge*** c_i with one sibling, which involves moving a key from *x* down into the new merged node to become the median key for that node

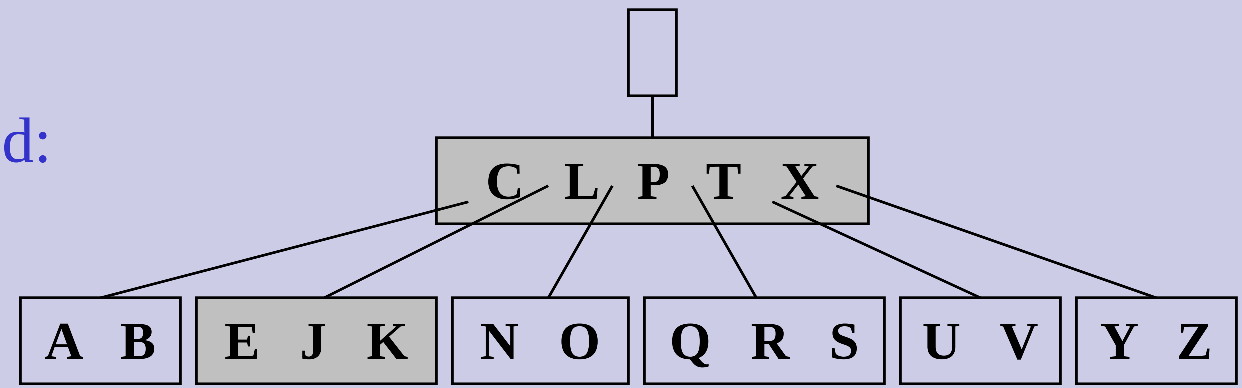


Deleting Keys – Merging (2)

delete **D**



D deleted:



tree shrinks in height

Deletion: Running Time

- Most of the keys are in the leaf, thus deletion most often occurs there!
- In this case deletion happens in one downward pass to the leaf level of the tree
- Deletion from an internal node might require “backing up” (case 2)
- Disk I/O: O(h), since only O(1) disk operations are produced during recursive calls
- CPU: $O(th) = O(t \log_t n)$

Two-pass Operations

- Simpler, practical versions of algorithms use two passes (down and up the tree):
  - *Down* – Find the node where deletion or insertion should occur
  - *Up* – If needed, split, merge, or distribute; propagate splits, merges, or distributes up the tree
- To avoid reading the same nodes twice, use a buffer of nodes

If insertion and deletion are interspersed, then two pass method is better