COL 106 Lecture 35

Topic : Trees

$G = (V, E)$: Connected graph

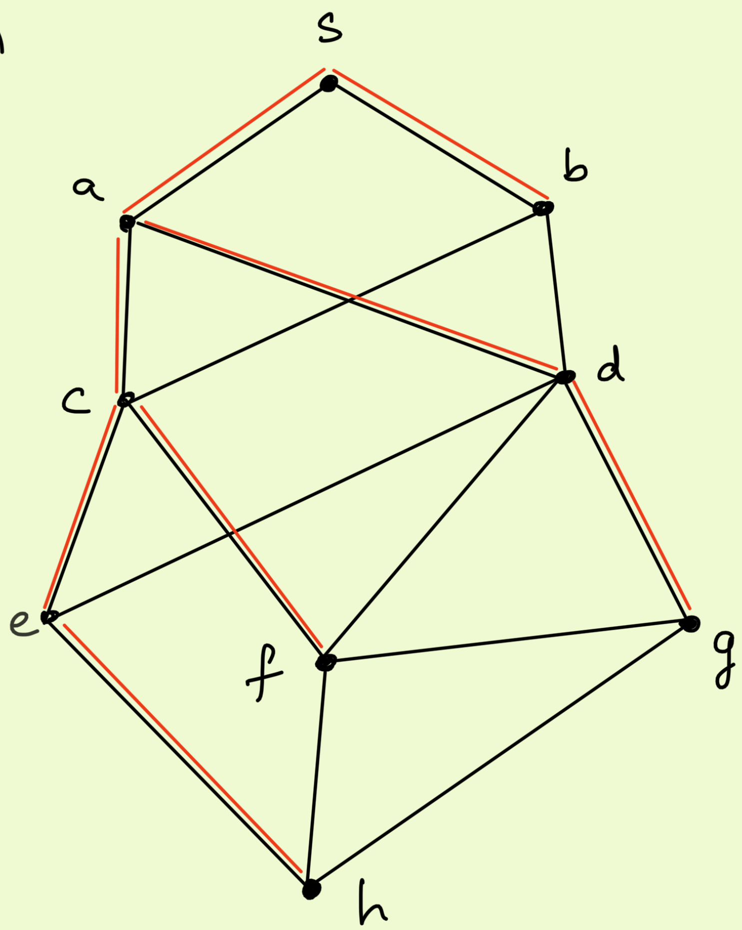
s : vertex of G.

Run BFT of G starting from s.

Consider graph $T = (V, E')$, where

$E' = \{ \{u, prev[u]\} \mid u \in V \}$

What kind of graph is T?



<u>Cycle</u>: Sequence of <u>distinct</u> vertices $v_0, v_1, \ldots, v_{\ell-1}$ $(\ell \geq 3)$ such that each $\{v_{i-1}, v_i\}$ is an edge, and $\{v_{\ell-1}, v_0\}$ is an edge

<u>Tree</u>: Connected graph that doesn't have a cycle.

Question: Suppose T is a tree with n vertices. What is the $\left\{\begin{matrix} \text{minimum} \\ \text{maximum} \end{matrix}\right\}$ possible # edges in T?

<u>Claim</u>: Let G be a graph with $n\,(>0)$ vertices and m edges. If any two of the following statements are true, then so is the remaining.

1. G is connected
2. G doesn't have a cycle
3. $m = n-1$.

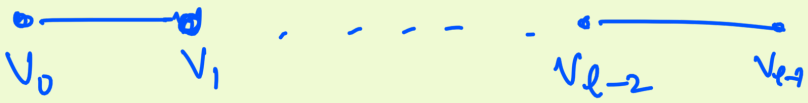
1. G is connected
AND
2. G doesn't have a cycle

$\Rightarrow$ 3. $m = n - 1$.

Proof: Induction on n; Base case of $n = 1$ is obvious.

Suppose claim true for $n-1$ vertex graphs.

G: n-vertex graph. Let $v_0 v_1 \cdots v_{\ell-1}$ be a longest path in G.



v_0 doesn't have a nbr in $\{v_2, \ldots, v_{\ell-1}\}$. $\longrightarrow$ otherwise there is a cycle,

v_0 cannot have a nbr outside this path $\rightarrow$ otherwise we get a longer path.

Remove vertex v_0 and edge $\{v_0 v_1\}$ to get graph G'.

Then apply I.H. to G' to get

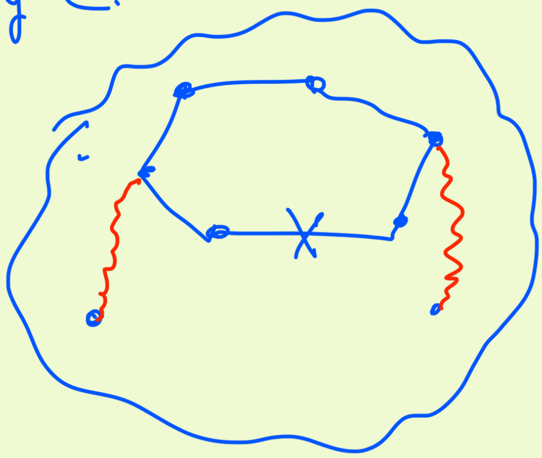
$$m - 1 = (n-1) - 1 \quad \therefore \quad m = n-1$$

1. G is connected
AND
3. $m = n - 1$.

$\Rightarrow$ 2. G doesn't have a cycle

Proof sketch: While G has a cycle, say C, remove any edge in C from G and destroy C.

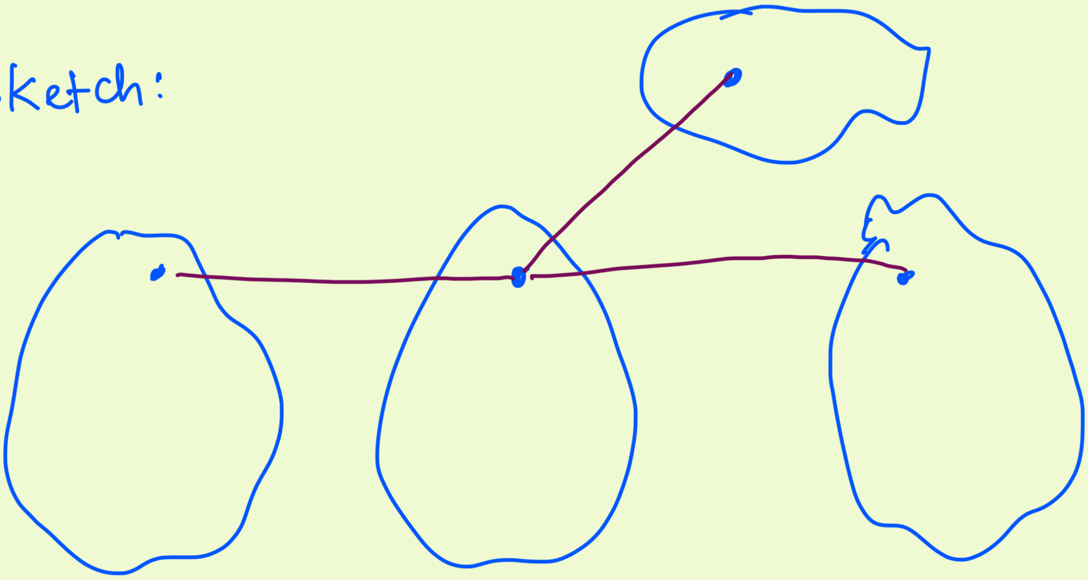
This maintains connectivity



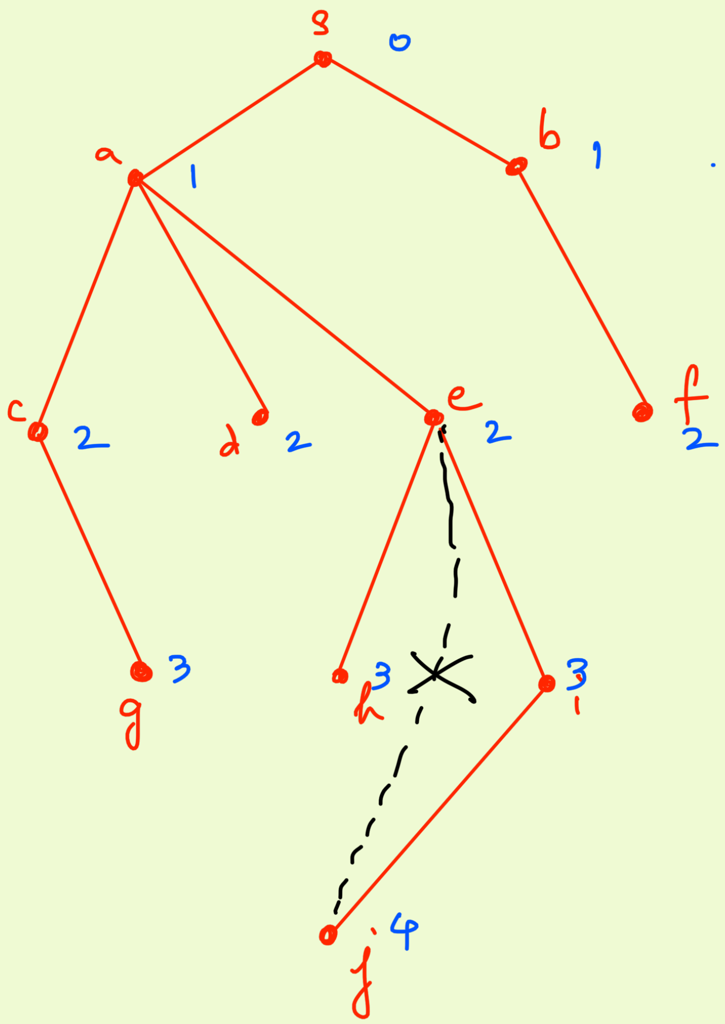
2. G doesn't have a cycle
AND
3. $m = n - 1$.

$\Rightarrow$ 1. G is connected

Proof sketch:



Add edges to connect the graph while maintaining acyclicity.



Claim (*): If G: connected graph
T is a BFT tree of G.
Then every edge of G is between vertices that lie in the same or adjacent levels of T

"Triangle inequality" of graphs: For any three vertices a, b, c:

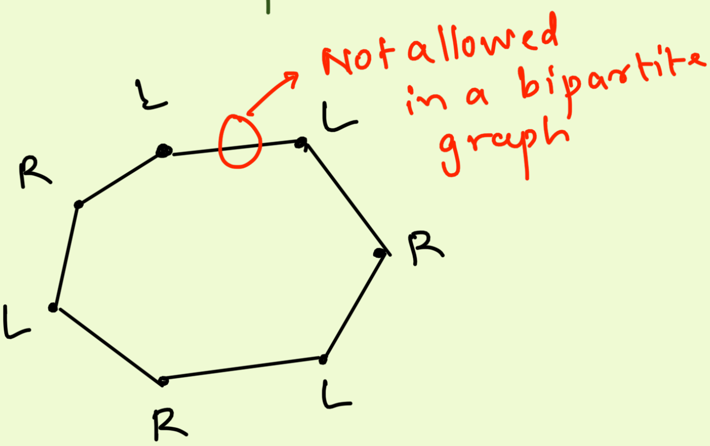
$$\text{dist}(a,b) + \text{dist}(b,c) \geq \text{dist}(a,c)$$

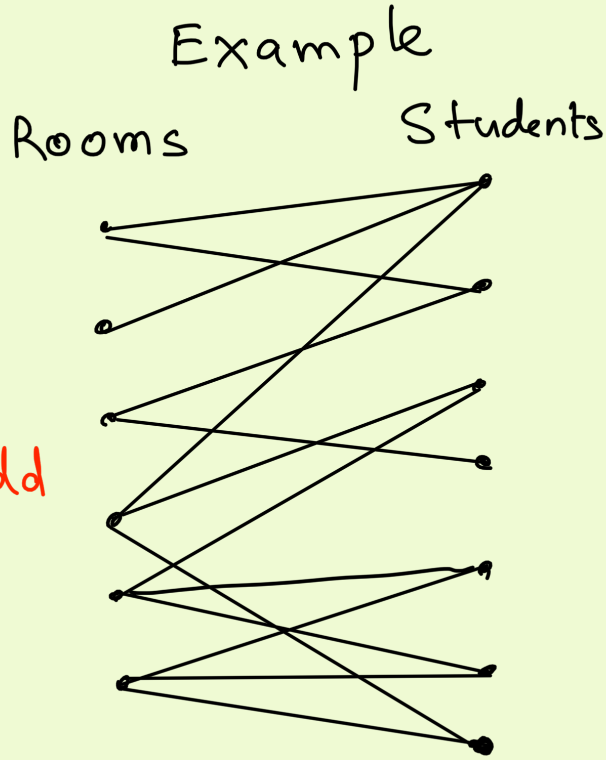
Proof: Concatenate shortest $a-b$ path with shortest $b-c$ path to get an $a-c$ walk.

Proof of Claim (*): If $\{x,y\}$ is an edge of G, then

$$1 = \text{dist}(x,y) \underset{\substack{\uparrow \\ \text{triangle inequality}}}{\geq} |\text{dist}(s,x) - \text{dist}(s,y)|$$

Definition: A graph $G = (V, E)$ is bipartite if V can be partitioned into L, R such that every edge of G has one endpoint in L and one endpoint in R.





Observation: A graph having an odd length cycle is not bipartite.

Question: Is the converse true?