1 Linear Equations in Linear Algebra

INTRODUCTORY EXAMPLE

Linear Models in Economics and Engineering

It was late summer in 1949. Harvard Professor Wassily Leontief was carefully feeding the last of his punched cards into the university's Mark II computer. The cards contained information about the U.S. economy and represented a summary of more than 250,000 pieces of information produced by the U.S. Bureau of Labor Statistics after two years of intensive work. Leontief had divided the U.S. economy into 500 "sectors," such as the coal industry, the automotive industry, communications, and so on. For each sector, he had written a linear equation that described how the sector distributed its output to the other sectors of the economy. Because the Mark II, one of the largest computers of its day, could not handle the resulting system of 500 equations in 500 unknowns, Leontief had distilled the problem into a system of 42 equations in 42 unknowns.

Programming the Mark II computer for Leontief's 42 equations had required several months of effort, and he was anxious to see how long the computer would take to solve the problem. The Mark II hummed and blinked for 56 hours before finally producing a solution. We will discuss the nature of this solution in Sections 1.6 and 2.6.

Leontief, who was awarded the 1973 Nobel Prize in Economic Science, opened the door to a new era in mathematical modeling in economics. His efforts at Harvard in 1949 marked one of the first significant uses of computers to analyze what was then a large-scale mathematical model. Since that time, researchers in many other fields have employed computers to analyze mathematical models. Because of the massive amounts of data involved, the models are usually *linear*; that is, they are described by *systems of linear equations*.

The importance of linear algebra for applications has risen in direct proportion to the increase in computing power, with each new generation of hardware and software triggering a demand for even greater capabilities. Computer science is thus intricately linked with linear algebra through the explosive growth of parallel processing and large-scale computations.

Scientists and engineers now work on problems far more complex than even dreamed possible a few decades ago. Today, linear algebra has more potential value for students in many scientific and business fields than any other undergraduate mathematics subject! The material in this text provides the foundation for further work in many interesting areas. Here are a few possibilities; others will be described later.

- *Oil exploration*. When a ship searches for offshore oil deposits, its computers solve thousands of separate systems of linear equations *every day*.

The seismic data for the equations are obtained from underwater shock waves created by explosions from air guns. The waves bounce off subsurface rocks and are measured by geophones attached to mile-long cables behind the ship.

- *Linear programming*. Many important management decisions today are made on the basis of linear programming models that use hundreds of variables. The airline industry, for instance, employs linear programs that schedule flight crews, monitor the locations of aircraft, or plan the varied schedules of support services such as maintenance and terminal operations.
- *Electrical networks*. Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on linear algebra techniques and systems of linear equations.

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Systems of linear equations lie at the heart of linear algebra, and this chapter uses them to introduce some of the central concepts of linear algebra in a simple and concrete setting. Sections 1.1 and 1.2 present a systematic method for solving systems of linear equations. This algorithm will be used for computations throughout the text. Sections 1.3 and 1.4 show how a system of linear equations is equivalent to a *vector equation* and to a *matrix equation*. This equivalence will reduce problems involving linear combinations of vectors to questions about systems of linear equations. The fundamental concepts of spanning, linear independence, and linear transformations, studied in the second half of the chapter, will play an essential role throughout the text as we explore the beauty and power of linear algebra.

1.1 SYSTEMS OF LINEAR EQUATIONS

A **linear equation** in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where b and the **coefficients** $a_1, \dots, a_n$ are real or complex numbers, usually known in advance. The subscript n may be any positive integer. In textbook examples and exercises, n is normally between 2 and 5. In real-life problems, n might be 50 or 5000, or even larger.

The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2\left(\sqrt{6} - x_1\right) + x_3$$

are both linear because they can be rearranged algebraically as in equation (1):

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

The equations

$$4x_1 - 5x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

are not linear because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables—say, $x_1, \dots, x_n$. An example is

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad\;\; - 4x_3 &= -7 \end{aligned} \tag{2}$$

A **solution** of the system is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$, respectively. For instance, $(5, 6.5, 3)$ is a solution of system (2) because, when these values are substituted in (2) for x_1, x_2, x_3, respectively, the equations simplify to $8 = 8$ and $-7 = -7$.

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

The graphs of these equations are lines, which we denote by ℓ_1 and ℓ_2. A pair of numbers (x_1, x_2) satisfies *both* equations in the system if and only if the point (x_1, x_2) lies on both ℓ_1 and ℓ_2. In the system above, the solution is the single point $(3, 2)$, as you can easily verify. See Figure 1.

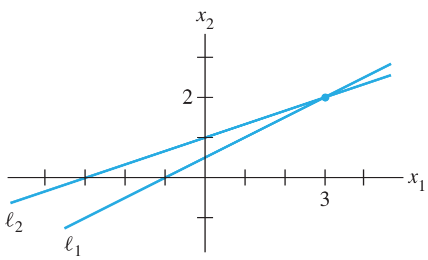
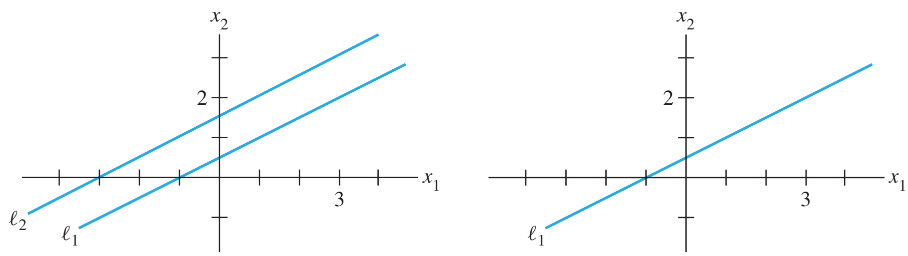


FIGURE 1 Exactly one solution.

Of course, two lines need not intersect in a single point—they could be parallel, or they could coincide and hence "intersect" at every point on the line. Figure 2 shows the graphs that correspond to the following systems:

$$\text{(a)} \quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$



(a) (b)

FIGURE 2 (a) No solution. (b) Infinitely many solutions.

Figures 1 and 2 illustrate the following general fact about linear systems, to be verified in Section 1.2.

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Matrix Notation

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned} \tag{3}$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

is called the **coefficient matrix** (or **matrix of coefficients**) of the system (3), and

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \tag{4}$$

is called the **augmented matrix** of the system. (The second row here contains a zero because the second equation could be written as $0 \cdot x_1 + 2x_2 - 8x_3 = 8$.) An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

The **size** of a matrix tells how many rows and columns it has. The augmented matrix (4) above has 3 rows and 4 columns and is called a 3×4 (read "3 by 4") matrix. If m and n are positive integers, an $\boldsymbol{m \times n}$ **matrix** is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.) Matrix notation will simplify the calculations in the examples that follow.

Solving a Linear System

This section and the next describe an algorithm, or a systematic procedure, for solving linear systems. The basic strategy is *to replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve.*

Roughly speaking, use the x_1 term in the first equation of a system to eliminate the x_1 terms in the other equations. Then use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until you finally obtain a very simple equivalent system of equations.

Three basic operations are used to simplify a linear system: Replace one equation by the sum of itself and a multiple of another equation, interchange two equations, and multiply all the terms in an equation by a nonzero constant. After the first example, you will see why these three operations do not change the solution set of the system.

EXAMPLE 1 Solve system (3).

SOLUTION The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Keep x_1 in the first equation and eliminate it from the other equations. To do so, add -5 times equation 1 to equation 3. After some practice, this type of calculation is usually performed mentally:

$$\begin{array}{r} -5 \cdot [\text{equation 1}] \\ +\,[\text{equation 3}] \\ \hline [\text{new equation 3}] \end{array} \qquad \begin{array}{r} -5x_1 + 10x_2 - 5x_3 = 0 \\ 5x_1 - 5x_3 = 10 \\ \hline 10x_2 - 10x_3 = 10 \end{array}$$

The result of this calculation is written in place of the original third equation:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Now, multiply equation 2 by $\frac{1}{2}$ in order to obtain 1 as the coefficient for x_2. (This calculation will simplify the arithmetic in the next step.)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Use the x_2 in equation 2 to eliminate the $10x_2$ in equation 3. The "mental" computation is

$$\begin{array}{r} -10 \cdot [\text{equation 2}] \\ +\,[\text{equation 3}] \\ \hline [\text{new equation 3}] \end{array} \qquad \begin{array}{r} -10x_2 + 40x_3 = -40 \\ 10x_2 - 10x_3 = 10 \\ \hline 30x_3 = -30 \end{array}$$

The result of this calculation is written in place of the previous third equation (row):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 30x_3 &= -30 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Now, multiply equation 3 by $\frac{1}{30}$ in order to obtain 1 as the coefficient for x_3. (This calculation will simplify the arithmetic in the next step.)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The new system has a *triangular* form (the intuitive term *triangular* will be replaced by a precise term in the next section):

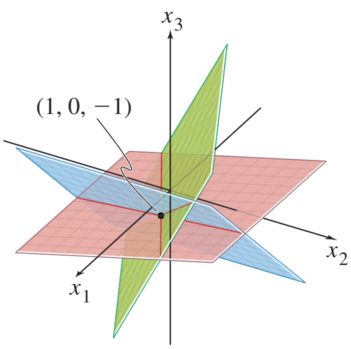
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Eventually, you want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 in equation 3 first, to eliminate the $-4x_3$ and $+x_3$ terms in equations 2 and 1. The two "mental" calculations are

$$\begin{array}{rr} 4\cdot[\text{equation 3}] & 4x_3 = -4 \\ +\,[\text{equation 2}] & x_2 - 4x_3 = 4 \\ \hline [\text{new equation 2}] & x_2 \phantom{{}-4x_3} = 0 \end{array} \qquad \begin{array}{rr} -1\cdot[\text{equation 3}] & -x_3 = 1 \\ +\,[\text{equation 1}] & x_1 - 2x_2 + x_3 = 0 \\ \hline [\text{new equation 1}] & x_1 - 2x_2 \phantom{{}+x_3} = 1 \end{array}$$

It is convenient to combine the results of these two operations:

$$\begin{aligned} x_1 - 2x_2 \phantom{{}+x_3} &= 1 \\ x_2 \phantom{{}+x_3} &= 0 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$



Each of the original equations determines a plane in three-dimensional space. The point $(1, 0, -1)$ lies in all three planes.

Now, having cleaned out the column above the x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Because of the previous work with x_3, there is now no arithmetic involving x_3 terms. Add 2 times equation 2 to equation 1 and obtain the system:

$$\begin{aligned} x_1 \phantom{{}+x_2+x_3} &= 1 \\ x_2 \phantom{{}+x_3} &= 0 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The work is essentially done. It shows that the only solution of the original system is $(1, 0, -1)$. However, since there are so many calculations involved, it is a good practice to check the work. To verify that $(1, 0, -1)$ *is* a solution, substitute these values into the left side of the original system, and compute:

$$\begin{aligned} 1(1) - 2(0) + 1(-1) &= 1 - 0 - 1 = 0 \\ 2(0) - 8(-1) &= \phantom{1-{}} 0 + 8 = 8 \\ 5(1) \phantom{{}-2(0)} - 5(-1) &= 5 \phantom{{}-0} + 5 = 10 \end{aligned}$$

The results agree with the right side of the original system, so $(1, 0, -1)$ is a solution of the system. ■

Example 1 illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix. The three basic operations listed earlier correspond to the following operations on the augmented matrix.

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Row operations can be applied to any matrix, not merely to one that arises as the augmented matrix of a linear system. Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

It is important to note that row operations are *reversible*. If two rows are interchanged, they can be returned to their original positions by another interchange. If a

[1] A common paraphrase of row replacement is "Add to one row a multiple of another row."

row is scaled by a nonzero constant c, then multiplying the new row by $1/c$ produces the original row. Finally, consider a replacement operation involving two rows—say, rows 1 and 2—and suppose that c times row 1 is added to row 2 to produce a new row 2. To "reverse" this operation, add $-c$ times row 1 to (new) row 2 and obtain the original row 2. See Exercises 29–32 at the end of this section.

At the moment, we are interested in row operations on the augmented matrix of a system of linear equations. Suppose a system is changed to a new one via row operations. By considering each type of row operation, you can see that any solution of the original system remains a solution of the new system. Conversely, since the original system can be produced via row operations on the new system, each solution of the new system is also a solution of the original system. This discussion justifies the following statement.

> If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Though Example 1 is lengthy, you will find that after some practice, the calculations go quickly. Row operations in the text and exercises will usually be extremely easy to perform, allowing you to focus on the underlying concepts. Still, you must learn to perform row operations accurately because they will be used throughout the text.

The rest of this section shows how to use row operations to determine the size of a solution set, without completely solving the linear system.

Existence and Uniqueness Questions

Section 1.2 will show why a solution set for a linear system contains either no solutions, one solution, or infinitely many solutions. Answers to the following two questions will determine the nature of the solution set for a linear system.

To determine which possibility is true for a particular system, we ask two questions.

> **TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM**
>
> **1.** Is the system consistent; that is, does at least one solution *exist*?
>
> **2.** If a solution exists, is it the *only* one; that is, is the solution *unique*?

These two questions will appear throughout the text, in many different guises. This section and the next will show how to answer these questions via row operations on the augmented matrix.

EXAMPLE 2 Determine if the following system is consistent:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

SOLUTION This is the system from Example 1. Suppose that we have performed the row operations necessary to obtain the triangular form

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

At this point, we know x_3. Were we to substitute the value of x_3 into equation 2, we could compute x_2 and hence could determine x_1 from equation 1. So a solution exists; the system is consistent. (In fact, x_2 is uniquely determined by equation 2 since x_3 has only one possible value, and x_1 is therefore uniquely determined by equation 1. So the solution is unique.) ■

EXAMPLE 3 Determine if the following system is consistent:

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned} \tag{5}$$

SOLUTION The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

To obtain an x_1 in the first equation, interchange rows 1 and 2:

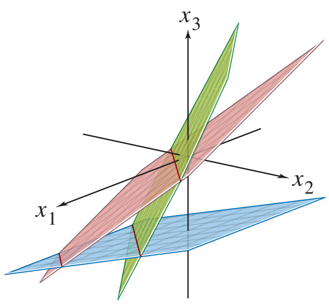
$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

To eliminate the $4x_1$ term in the third equation, add -2 times row 1 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix} \tag{6}$$

Next, use the x_2 term in the second equation to eliminate the $-2x_2$ term from the third equation. Add 2 times row 2 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix} \tag{7}$$



The system is inconsistent because there is no point that lies on all three planes.

The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation:

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 15 \end{aligned} \tag{8}$$

The equation $0 = 15$ is a short form of $0x_1 + 0x_2 + 0x_3 = 15$. This system in triangular form obviously has a built-in contradiction. There are no values of x_1, x_2, x_3 that satisfy (8) because the equation $0 = 15$ is never true. Since (8) and (5) have the same solution set, the original system is inconsistent (i.e., has no solution). ■

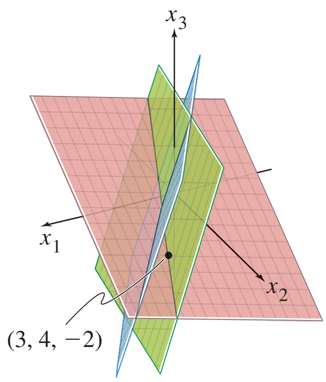
Pay close attention to the augmented matrix in (7). Its last row is typical of an inconsistent system in triangular form.

step now is to add 2 times equation 4 to equation 1. (After that, move to equation 3, multiply it by 1/2, and then use the equation to eliminate the x_3 terms above it.)

2. The system corresponding to the augmented matrix is

$$\begin{aligned} x_1 + 5x_2 + 2x_3 &= -6 \\ 4x_2 - 7x_3 &= 2 \\ 5x_3 &= 0 \end{aligned}$$

The third equation makes $x_3 = 0$, which is certainly an allowable value for x_3. After eliminating the x_3 terms in equations 1 and 2, you could go on to solve for unique values for x_2 and x_1. Hence a solution exists, and it is unique. Contrast this situation with that in Example 3.



Since $(3, 4, -2)$ satisfies the first two equations, it is on the line of the intersection of the first two planes. Since $(3, 4, -2)$ does not satisfy all three equations, it does not lie on all three planes.

3. It is easy to check if a specific list of numbers is a solution. Set $x_1 = 3$, $x_2 = 4$, and $x_3 = -2$, and find that

$$\begin{aligned} 5(3) - (4) + 2(-2) &= 15 - 4 - 4 = 7 \\ -2(3) + 6(4) + 9(-2) &= -6 + 24 - 18 = 0 \\ -7(3) + 5(4) - 3(-2) &= -21 + 20 + 6 = 5 \end{aligned}$$

Although the first two equations are satisfied, the third is not, so $(3, 4, -2)$ is not a solution of the system. Notice the use of parentheses when making the substitutions. They are strongly recommended as a guard against arithmetic errors.

4. When the second equation is replaced by its sum with 3 times the first equation, the system becomes

$$\begin{aligned} 2x_1 - x_2 &= h \\ 0 &= k + 3h \end{aligned}$$

If $k + 3h$ is nonzero, the system has no solution. The system is consistent for any values of h and k that make $k + 3h = 0$.

1.2 ROW REDUCTION AND ECHELON FORMS

This section refines the method of Section 1.1 into a row reduction algorithm that will enable us to analyze any system of linear equations.[1] By using only the first part of the algorithm, we will be able to answer the fundamental existence and uniqueness questions posed in Section 1.1.

The algorithm applies to any matrix, whether or not the matrix is viewed as an augmented matrix for a linear system. So the first part of this section concerns an arbitrary rectangular matrix and begins by introducing two important classes of matrices that include the "triangular" matrices of Section 1.1. In the definitions that follow, a *nonzero* row or column in a matrix means a row or column that contains at least one nonzero entry; a **leading entry** of a row refers to the leftmost nonzero entry (in a nonzero row).

[1] The algorithm here is a variant of what is commonly called *Gaussian elimination*. A similar elimination method for linear systems was used by Chinese mathematicians in about 250 B.C. The process was unknown in Western culture until the nineteenth century, when a famous German mathematician, Carl Friedrich Gauss, discovered it. A German engineer, Wilhelm Jordan, popularized the algorithm in an 1888 text on geodesy.

DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form). Property 2 says that the leading entries form an *echelon* ("steplike") pattern that moves down and to the right through the matrix. Property 3 is a simple consequence of property 2, but we include it for emphasis.

The "triangular" matrices of Section 1.1, such as

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

are in echelon form. In fact, the second matrix is in reduced echelon form. Here are additional examples.

EXAMPLE 1 The following matrices are in echelon form. The leading entries (■) may have any nonzero value; the starred entries ($*$) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

■

Any nonzero matrix may be **row reduced** (that is, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique. The following theorem is proved in Appendix A at the end of the text.

THEOREM 1

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

If a matrix A is row equivalent to an echelon matrix U, we call U **an echelon form** (or row echelon form) **of** A; if U is in reduced echelon form, we call U **the reduced echelon form of** A. [Most matrix programs and calculators with matrix capabilities use the abbreviation RREF for reduced (row) echelon form. Some use REF for (row) echelon form.]

Pivot Positions

When row operations on a matrix produce an echelon form, further row operations to obtain the reduced echelon form do not change the positions of the leading entries. Since the reduced echelon form is unique, *the leading entries are always in the same positions in any echelon form obtained from a given matrix*. These leading entries correspond to leading 1's in the reduced echelon form.

DEFINITION

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

In Example 1, the squares (■) identify the pivot positions. Many fundamental concepts in the first four chapters will be connected in one way or another with pivot positions in a matrix.

EXAMPLE 2 Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

SOLUTION Use the same basic strategy as in Section 1.1. The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position. A good choice is to interchange rows 1 and 4 (because the mental computations in the next step will not involve fractions).

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Pivot column} \end{array}$$

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain matrix (1) below. The pivot position in the second row must be as far left as possible—namely, in the second column. Choose the 2 in this position as the next pivot.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Next pivot column} \end{array} \tag{1}$$

Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \tag{2}$$

The matrix in (2) is different from any encountered in Section 1.1. There is no way to create a leading entry in column 3! (We can't use row 1 or 2 because doing so would destroy the echelon arrangement of the leading entries already produced.) However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{General form: } \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

Pivot positions

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \tag{3}$$

Pivot columns ■

A **pivot**, as illustrated in Example 2, is a nonzero number in a pivot position that is used as needed to create zeros via row operations. The pivots in Example 2 were 1, 2, and -5. Notice that these numbers are not the same as the actual elements of A in the highlighted pivot positions shown in (3).

With Example 2 as a guide, we are ready to describe an efficient procedure for transforming a matrix into an echelon or reduced echelon matrix. Careful study and mastery of this procedure now will pay rich dividends later in the course.

The Row Reduction Algorithm

The algorithm that follows consists of four steps, and it produces a matrix in echelon form. A fifth step produces a matrix in reduced echelon form. We illustrate the algorithm by an example.

EXAMPLE 3 Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

SOLUTION

STEP 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\begin{array}{c}\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \\ \text{Pivot column} \end{array}$$

STEP 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\begin{array}{c}\text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}\end{array}$$

STEP 3

Use row replacement operations to create zeros in all positions below the pivot.

As a preliminary step, we could divide the top row by the pivot, 3. But with two 3's in column 1, it is just as easy to add -1 times row 1 to row 2.

$$\begin{array}{c}\text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}\end{array}$$

STEP 4

Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the "top" entry in that column.

$$\begin{array}{c}\text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \\ \text{New pivot column}\end{array}$$

For step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the "top" row to the row below. This produces

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix having only one row:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot

Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. If we want the reduced echelon form, we perform one more step.

> **STEP 5**
>
> Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

$$\begin{array}{cl} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} & \begin{array}{l} \leftarrow \text{Row } 1 + (-6) \cdot \text{row } 3 \\ \leftarrow \text{Row } 2 + (-2) \cdot \text{row } 3 \\ \end{array} \end{array}$$

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{array}{cl} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} & \begin{array}{l} \\ \leftarrow \text{Row scaled by } \frac{1}{2} \\ \end{array} \end{array}$$

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{array}{cl} \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} & \begin{array}{l} \leftarrow \text{Row } 1 + (9) \cdot \text{row } 2 \\ \\ \end{array} \end{array}$$

Finally, scale row 1, dividing by the pivot, 3.

$$\begin{array}{cl} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} & \begin{array}{l} \leftarrow \text{Row scaled by } \frac{1}{3} \\ \\ \end{array} \end{array}$$

This is the reduced echelon form of the original matrix. ■

The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

> **NUMERICAL NOTE**
>
> In step 2 above, a computer program usually selects as a pivot the entry in a column having the largest absolute value. This strategy, called **partial pivoting**, is used because it reduces roundoff errors in the calculations.

Solutions of Linear Systems

The row reduction algorithm leads directly to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 \qquad - 5x_3 &= 1 \\ x_2 + \quad x_3 &= 4 \\ 0 &= 0 \end{aligned} \tag{4}$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**.[2] The other variable, x_3, is called a **free variable**.

Whenever a system is consistent, as in (4), the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables. This operation is possible because the reduced echelon form places each basic variable in one and only one equation. In (4), solve the first equation for x_1 and the second for x_2. (Ignore the third equation; it offers no restriction on the variables.)

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases} \tag{5}$$

The statement "x_3 is free" means that you are free to choose any value for x_3. Once that is done, the formulas in (5) determine the values for x_1 and x_2. For instance, when $x_3 = 0$, the solution is $(1, 4, 0)$; when $x_3 = 1$, the solution is $(6, 3, 1)$. *Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3.*

EXAMPLE 4 Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

SOLUTION The matrix is in echelon form, but we want the reduced echelon form before solving for the basic variables. The row reduction is completed next. The symbol $\sim$ before a matrix indicates that the matrix is row equivalent to the preceding matrix.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

[2] Some texts use the term *leading variables* because they correspond to the columns containing leading entries.

There are five variables because the augmented matrix has six columns. The associated system now is

$$\begin{aligned} x_1 + 6x_2 \qquad + 3x_4 \qquad &= 0 \\ x_3 - 4x_4 \qquad &= 5 \\ x_5 &= 7 \end{aligned} \tag{6}$$

The pivot columns of the matrix are 1, 3, and 5, so the basic variables are x_1, x_3, and x_5. The remaining variables, x_2 and x_4, must be free. Solve for the basic variables to obtain the general solution:

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases} \tag{7}$$

Note that the value of x_5 is already fixed by the third equation in system (6). ■

Parametric Descriptions of Solution Sets

The descriptions in (5) and (7) are *parametric descriptions* of solution sets in which the free variables act as parameters. *Solving a system* amounts to finding a parametric description of the solution set or determining that the solution set is empty.

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions. For instance, in system (4), we may add 5 times equation 2 to equation 1 and obtain the equivalent system

$$\begin{aligned} x_1 + 5x_2 \qquad &= 21 \\ x_2 + x_3 &= 4 \end{aligned}$$

We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have an accurate description of the solution set. However, to be consistent, we make the (arbitrary) convention of always using the free variables as the parameters for describing a solution set. (The answer section at the end of the text also reflects this convention.)

Whenever a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has *no* parametric representation.

Back-Substitution

Consider the following system, whose augmented matrix is in echelon form but is *not* in reduced echelon form:

$$\begin{aligned} x_1 - 7x_2 + 2x_3 - 5x_4 + 8x_5 &= 10 \\ x_2 - 3x_3 + 3x_4 + x_5 &= -5 \\ x_4 - x_5 &= 4 \end{aligned}$$

A computer program would solve this system by back-substitution, rather than by computing the reduced echelon form. That is, the program would solve equation 3 for x_4 in terms of x_5 and substitute the expression for x_4 into equation 2, solve equation 2 for x_2, and then substitute the expressions for x_2 and x_4 into equation 1 and solve for x_1.

Our matrix format for the backward phase of row reduction, which produces the reduced echelon form, has the same number of arithmetic operations as back-substitution. But the discipline of the matrix format substantially reduces the likelihood of errors during hand computations. The best strategy is to use only the *reduced* echelon form to solve a system! The *Study Guide* that accompanies this text offers several helpful suggestions for performing row operations accurately and rapidly.

NUMERICAL NOTE

In general, the forward phase of row reduction takes much longer than the backward phase. An algorithm for solving a system is usually measured in flops (or floating point operations). A **flop** is one arithmetic operation $(+, -, *, /)$ on two real floating point numbers.[3] For an $n \times (n+1)$ matrix, the reduction to echelon form can take $2n^3/3 + n^2/2 - 7n/6$ flops (which is approximately $2n^3/3$ flops when n is moderately large—say, $n \geq 30$). In contrast, further reduction to reduced echelon form needs at most n^2 flops.

Existence and Uniqueness Questions

Although a nonreduced echelon form is a poor tool for solving a system, this form is just the right device for answering two fundamental questions posed in Section 1.1.

EXAMPLE 5 Determine the existence and uniqueness of the solutions to the system

$$
\begin{aligned}
3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\
3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\
3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15
\end{aligned}
$$

SOLUTION The augmented matrix of this system was row reduced in Example 3 to

$$
\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \tag{8}
$$

The basic variables are x_1, x_2, and x_5; the free variables are x_3 and x_4. There is no equation such as $0 = 1$ that would indicate an inconsistent system, so we could use back-substitution to find a solution. But the *existence* of a solution is already clear in (8). Also, the solution is *not unique* because there are free variables. Each different choice of x_3 and x_4 determines a different solution. Thus the system has infinitely many solutions. ■

When a system is in echelon form and contains no equation of the form $0 = b$, with b nonzero, every nonzero equation contains a basic variable with a nonzero coefficient. Either the basic variables are completely determined (with no free variables) or at least one of the basic variables may be expressed in terms of one or more free variables. In the former case, there is a unique solution; in the latter case, there are infinitely many solutions (one for each choice of values for the free variables).

These remarks justify the following theorem.

---

[3] Traditionally, a *flop* was only a multiplication or division, because addition and subtraction took much less time and could be ignored. The definition of *flop* given here is preferred now, as a result of advances in computer architecture. See Golub and Van Loan, *Matrix Computations*, 2nd ed. (Baltimore: The Johns Hopkins Press, 1989), pp. 19–20.

THEOREM 2

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[\,0 \quad \cdots \quad 0 \quad b\,] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

The following procedure outlines how to find and describe all solutions of a linear system.

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

PRACTICE PROBLEMS

1. Find the general solution of the linear system whose augmented matrix is

$$\begin{bmatrix} 1 & -3 & -5 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

2. Find the general solution of the system

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$$

3. Suppose a 4×7 coefficient matrix for a system of equations has 4 pivots. Is the system consistent? If the system is consistent, how many solutions are there?

1.2 EXERCISES

In Exercises 1 and 2, determine which matrices are in reduced echelon form and which others are only in echelon form.

1. a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

2. Row reduce the system's augmented matrix:

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

This echelon matrix shows that the system is *inconsistent*, because its rightmost column is a pivot column; the third row corresponds to the equation $0 = 5$. There is no need to perform any more row operations. Note that the presence of the free variables in this problem is irrelevant because the system is inconsistent.

3. Since the coefficient matrix has four pivots, there is a pivot in every row of the coefficient matrix. This means that when the coefficient matrix is row reduced, it will *not* have a row of zeros, thus the corresponding row reduced augmented matrix can never have a row of the form $[0 \;\; 0 \;\; \cdots \;\; 0 \;\; b]$, where b is a nonzero number. By Theorem 2, the system is consistent. Moreover, since there are seven columns in the coefficient matrix and only four pivot columns, there will be three free variables resulting in infinitely many solutions.

1.3 VECTOR EQUATIONS

Important properties of linear systems can be described with the concept and notation of vectors. This section connects equations involving vectors to ordinary systems of equations. The term *vector* appears in a variety of mathematical and physical contexts, which we will discuss in Chapter 4, "Vector Spaces." Until then, *vector* will mean an *ordered list of numbers*. This simple idea enables us to get to interesting and important applications as quickly as possible.

Vectors in $\mathbb{R}^2$

A matrix with only one column is called a **column vector**, or simply a **vector**. Examples of vectors with two entries are

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} .2 \\ .3 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where w_1 and w_2 are any real numbers. The set of all vectors with two entries is denoted by $\mathbb{R}^2$ (read "r-two"). The $\mathbb{R}$ stands for the real numbers that appear as entries in the vectors, and the exponent 2 indicates that each vector contains two entries.[1]

Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal. Thus $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ are *not* equal, because vectors in $\mathbb{R}^2$ are *ordered pairs* of real numbers.

[1] Most of the text concerns vectors and matrices that have only real entries. However, all definitions and theorems in Chapters 1–5, and in most of the rest of the text, remain valid if the entries are complex numbers. Complex vectors and matrices arise naturally, for example, in electrical engineering and physics.

Given two vectors **u** and **v** in $\mathbb{R}^2$, their **sum** is the vector $\mathbf{u}+\mathbf{v}$ obtained by adding corresponding entries of **u** and **v**. For example,

$$\begin{bmatrix}1\\-2\end{bmatrix}+\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}1+2\\-2+5\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$$

Given a vector **u** and a real number c, the **scalar multiple** of **u** by c is the vector $c\mathbf{u}$ obtained by multiplying each entry in **u** by c. For instance,

$$\text{if}\quad \mathbf{u}=\begin{bmatrix}3\\-1\end{bmatrix}\quad\text{and}\quad c=5,\qquad\text{then}\quad c\mathbf{u}=5\begin{bmatrix}3\\-1\end{bmatrix}=\begin{bmatrix}15\\-5\end{bmatrix}$$

The number c in $c\mathbf{u}$ is called a **scalar**; it is written in lightface type to distinguish it from the boldface vector **u**.

The operations of scalar multiplication and vector addition can be combined, as in the following example.

EXAMPLE 1 Given $\mathbf{u}=\begin{bmatrix}1\\-2\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}2\\-5\end{bmatrix}$, find $4\mathbf{u}$, $(-3)\mathbf{v}$, and $4\mathbf{u}+(-3)\mathbf{v}$.

SOLUTION

$$4\mathbf{u}=\begin{bmatrix}4\\-8\end{bmatrix},\qquad (-3)\mathbf{v}=\begin{bmatrix}-6\\15\end{bmatrix}$$

and

$$4\mathbf{u}+(-3)\mathbf{v}=\begin{bmatrix}4\\-8\end{bmatrix}+\begin{bmatrix}-6\\15\end{bmatrix}=\begin{bmatrix}-2\\7\end{bmatrix}$$ ■

Sometimes, for convenience (and also to save space), this text may write a column vector such as $\begin{bmatrix}3\\-1\end{bmatrix}$ in the form $(3,-1)$. In this case, the parentheses and the comma distinguish the vector $(3,-1)$ from the 1×2 row matrix $\begin{bmatrix}3 & -1\end{bmatrix}$, written with brackets and no comma. Thus

$$\begin{bmatrix}3\\-1\end{bmatrix}\neq\begin{bmatrix}3 & -1\end{bmatrix}$$

because the matrices have different shapes, even though they have the same entries.

Geometric Descriptions of $\mathbb{R}^2$

Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point* (a,b) *with the column vector* $\begin{bmatrix}a\\b\end{bmatrix}$. So we may regard $\mathbb{R}^2$ as the set of all points in the plane. See Figure 1.

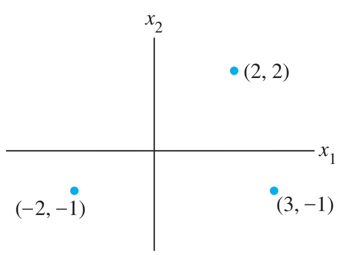


FIGURE 1 Vectors as points.

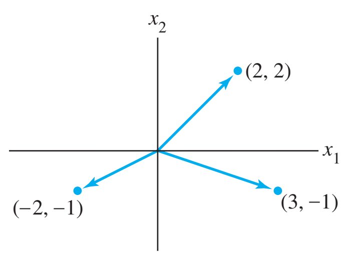


FIGURE 2 Vectors with arrows.

The geometric visualization of a vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is often aided by including an arrow (directed line segment) from the origin (0, 0) to the point (3, −1), as in Figure 2. In this case, the individual points along the arrow itself have no special significance.[2]

The sum of two vectors has a useful geometric representation. The following rule can be verified by analytic geometry.

Parallelogram Rule for Addition

If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, **0**, and **v**. See Figure 3.

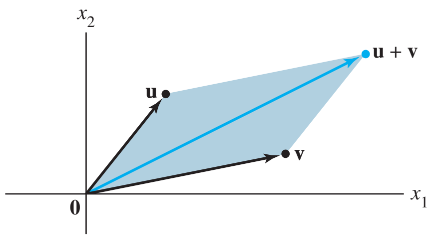


FIGURE 3 The parallelogram rule.

EXAMPLE 2 The vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are displayed in Figure 4. ■

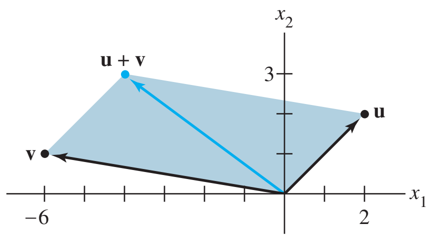


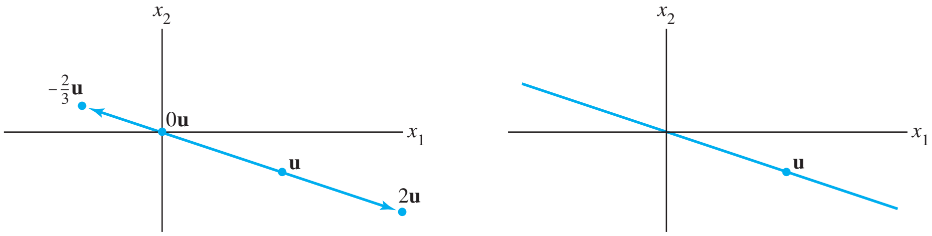
FIGURE 4

The next example illustrates the fact that the set of all scalar multiples of one fixed nonzero vector is a line through the origin, (0, 0).

EXAMPLE 3 Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Display the vectors **u**, 2**u**, and $-\frac{2}{3}\mathbf{u}$ on a graph.

SOLUTION See Figure 5, where **u**, $2\mathbf{u} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $-\frac{2}{3}\mathbf{u} = \begin{bmatrix} -2 \\ 2/3 \end{bmatrix}$ are displayed. The arrow for 2**u** is twice as long as the arrow for **u**, and the arrows point in the same direction. The arrow for $-\frac{2}{3}\mathbf{u}$ is two-thirds the length of the arrow for **u**, and the arrows point in opposite directions. In general, the length of the arrow for $c\mathbf{u}$ is $|c|$ times the length of the arrow for **u**. [Recall that the length of the line segment from (0, 0) to (a, b) is $\sqrt{a^2 + b^2}$. We shall discuss this further in Chapter 6.]

[2] In physics, arrows can represent forces and usually are free to move about in space. This interpretation of vectors will be discussed in Section 4.1.



Typical multiples of **u** The set of all multiples of **u**

FIGURE 5

Vectors in $\mathbb{R}^3$

Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries. They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin sometimes included for visual clarity. The vectors $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $2\mathbf{a}$ are displayed in Figure 6.

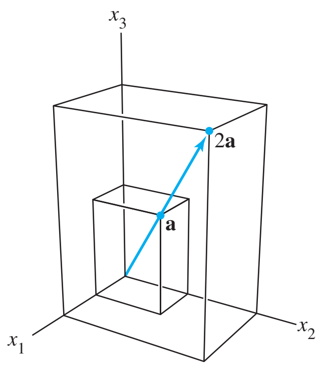


FIGURE 6
Scalar multiples.

Vectors in $\mathbb{R}^n$

If n is a positive integer, $\mathbb{R}^n$ (read "r-n") denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$. (The number of entries in $\mathbf{0}$ will be clear from the context.)

Equality of vectors in $\mathbb{R}^n$ and the operations of scalar multiplication and vector addition in $\mathbb{R}^n$ are defined entry by entry just as in $\mathbb{R}^2$. These operations on vectors have the following properties, which can be verified directly from the corresponding properties for real numbers. See Practice Problem 1 and Exercises 33 and 34 at the end of this section.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

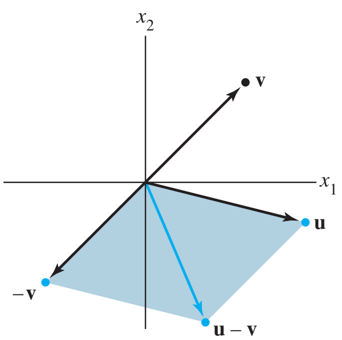


FIGURE 7
Vector subtraction.

For simplicity of notation, a vector such as $\mathbf{u} + (-1)\mathbf{v}$ is often written as $\mathbf{u} - \mathbf{v}$. Figure 7 shows $\mathbf{u} - \mathbf{v}$ as the sum of $\mathbf{u}$ and $-\mathbf{v}$.

Linear Combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$. Property (ii) above permits us to omit parentheses when forming such a linear combination. The weights in a linear combination can be any real numbers, including zero. For example, some linear combinations of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are

$$\sqrt{3}\,\mathbf{v}_1 + \mathbf{v}_2, \quad \tfrac{1}{2}\mathbf{v}_1 \;(= \tfrac{1}{2}\mathbf{v}_1 + 0\mathbf{v}_2), \quad \text{and} \quad \mathbf{0}\;(= 0\mathbf{v}_1 + 0\mathbf{v}_2)$$

EXAMPLE 4 Figure 8 identifies selected linear combinations of $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. (Note that sets of parallel grid lines are drawn through integer multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$.) Estimate the linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ that generate the vectors $\mathbf{u}$ and $\mathbf{w}$.

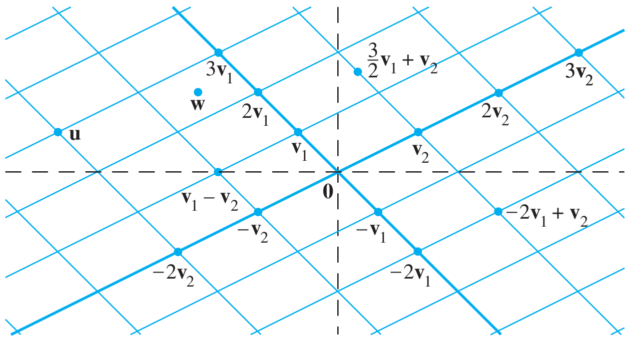


FIGURE 8 Linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.

SOLUTION The parallelogram rule shows that $\mathbf{u}$ is the sum of $3\mathbf{v}_1$ and $-2\mathbf{v}_2$; that is,

$$\mathbf{u} = 3\mathbf{v}_1 - 2\mathbf{v}_2$$

This expression for $\mathbf{u}$ can be interpreted as instructions for traveling from the origin to $\mathbf{u}$ along two straight paths. First, travel 3 units in the $\mathbf{v}_1$ direction to $3\mathbf{v}_1$, and then travel -2 units in the $\mathbf{v}_2$ direction (parallel to the line through $\mathbf{v}_2$ and $\mathbf{0}$). Next, although the vector $\mathbf{w}$ is not on a grid line, $\mathbf{w}$ appears to be about halfway between two pairs of grid lines, at the vertex of a parallelogram determined by $(5/2)\mathbf{v}_1$ and $(-1/2)\mathbf{v}_2$. (See Figure 9.) Thus a reasonable estimate for $\mathbf{w}$ is

$$\mathbf{w} = \tfrac{5}{2}\mathbf{v}_1 - \tfrac{1}{2}\mathbf{v}_2$$

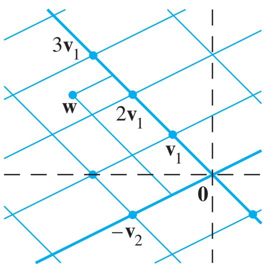


FIGURE 9

The next example connects a problem about linear combinations to the fundamental existence question studied in Sections 1.1 and 1.2.

EXAMPLE 5 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether $\mathbf{b}$ can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \tag{1}$$

If vector equation (1) has a solution, find it.

SOLUTION Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \underset{\mathbf{a}_1}{\underset{\uparrow}{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}}} + x_2 \underset{\mathbf{a}_2}{\underset{\uparrow}{\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}}} = \underset{\mathbf{b}}{\underset{\uparrow}{\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}}}$$

which is the same as

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \tag{2}$$

The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is, x_1 and x_2 make the vector equation (1) true if and only if x_1 and x_2 satisfy the system

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \tag{3}$$

To solve this system, row reduce the augmented matrix of the system as follows:[3]

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \qquad \blacksquare$$

Observe in Example 5 that the original vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix that we row reduced:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{array} \end{array}$$

For brevity, write this matrix in a way that identifies its columns—namely,

$$[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}\,] \tag{4}$$

It is clear how to write this augmented matrix immediately from vector equation (1), without going through the intermediate steps of Example 5. Take the vectors in the order in which they appear in (1) and put them into the columns of a matrix as in (4).

The discussion above is easily modified to establish the following fundamental fact.

[3] The symbol $\sim$ between matrices denotes row equivalence (Section 1.2).

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{5}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors.

DEFINITION

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

Asking whether a vector $\mathbf{b}$ is in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix $[\,\mathbf{v}_1 \;\; \cdots \;\; \mathbf{v}_p \;\; \mathbf{b}\,]$ has a solution.

Note that Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$ (for example), since $c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$. In particular, the zero vector must be in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

A Geometric Description of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u}, \mathbf{v}\}$

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Then Span $\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$, which is the set of points on the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$. See Figure 10.

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}$ not a multiple of $\mathbf{u}$, then Span $\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$. In particular, Span $\{\mathbf{u}, \mathbf{v}\}$ contains the line in $\mathbb{R}^3$ through $\mathbf{u}$ and $\mathbf{0}$ and the line through $\mathbf{v}$ and $\mathbf{0}$. See Figure 11.

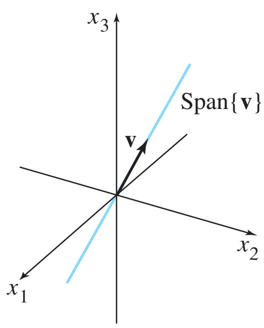


FIGURE 10 Span $\{\mathbf{v}\}$ as a line through the origin.

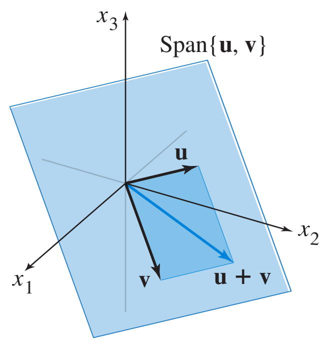


FIGURE 11 Span $\{\mathbf{u}, \mathbf{v}\}$ as a plane through the origin.

EXAMPLE 6 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Then Span $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\mathbf{b}$ in that plane?

SOLUTION Does the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ have a solution? To answer this, row reduce the augmented matrix $[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}\,]$:

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The third equation is $0 = -2$, which shows that the system has no solution. The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ has no solution, and so $\mathbf{b}$ is *not* in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$. ■

Linear Combinations in Applications

The final example shows how scalar multiples and linear combinations can arise when a quantity such as "cost" is broken down into several categories. The basic principle for the example concerns the cost of producing several units of an item when the cost per unit is known:

$$\begin{Bmatrix} \text{number} \\ \text{of units} \end{Bmatrix} \cdot \begin{Bmatrix} \text{cost} \\ \text{per unit} \end{Bmatrix} = \begin{Bmatrix} \text{total} \\ \text{cost} \end{Bmatrix}$$

EXAMPLE 7 A company manufactures two products. For \$1.00 worth of product B, the company spends \$.45 on materials, \$.25 on labor, and \$.15 on overhead. For \$1.00 worth of product C, the company spends \$.40 on materials, \$.30 on labor, and \$.15 on overhead. Let

$$\mathbf{b} = \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} .40 \\ .30 \\ .15 \end{bmatrix}$$

Then $\mathbf{b}$ and $\mathbf{c}$ represent the "costs per dollar of income" for the two products.

a. What economic interpretation can be given to the vector $100\mathbf{b}$?

b. Suppose the company wishes to manufacture x_1 dollars worth of product B and x_2 dollars worth of product C. Give a vector that describes the various costs the company will have (for materials, labor, and overhead).

SOLUTION

a. Compute

$$100\mathbf{b} = 100\begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} = \begin{bmatrix} 45 \\ 25 \\ 15 \end{bmatrix}$$

The vector $100\mathbf{b}$ lists the various costs for producing \$100 worth of product B—namely, \$45 for materials, \$25 for labor, and \$15 for overhead.

b. The costs of manufacturing x_1 dollars worth of B are given by the vector $x_1\mathbf{b}$, and the costs of manufacturing x_2 dollars worth of C are given by $x_2\mathbf{c}$. Hence the total costs for both products are given by the vector $x_1\mathbf{b} + x_2\mathbf{c}$. ■

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

The system is consistent if and only if there is no pivot in the fourth column. That is, $h - 5$ must be 0. So $\mathbf{y}$ is in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h = 5$.

Remember: The presence of a free variable in a system does not guarantee that the system is consistent.

3. Since the vectors $\mathbf{u}$ and $\mathbf{v}$ are in Span $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, there exist scalars c_1, c_2, c_3 and d_1, d_2, d_3 such that

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \quad \text{and} \quad \mathbf{v} = d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3.$$

Notice

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3 \\ &= (c_1 + d_1)\mathbf{w}_1 + (c_2 + d_2)\mathbf{w}_2 + (c_3 + d_3)\mathbf{w}_3 \end{aligned}$$

Since $c_1 + d_1, c_2 + d_2$, and $c_3 + d_3$ are also scalars, the vector $\mathbf{u} + \mathbf{v}$ is in Span $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

1.4 THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector. The following definition permits us to rephrase some of the concepts of Section 1.3 in new ways.

DEFINITION

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

EXAMPLE 1

a. $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

b. $$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$$ ■

EXAMPLE 2 For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.

SOLUTION Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix A and place the weights $3, -5$, and 7 into a vector $\mathbf{x}$. That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x}$$

Section 1.3 showed how to write a system of linear equations as a vector equation involving a linear combination of vectors. For example, the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \tag{1}$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{2}$$

As in Example 2, the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{3}$$

Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**, to distinguish it from a vector equation such as is shown in (2).

Notice how the matrix in (3) is just the matrix of coefficients of the system (1). Similar calculations show that any system of linear equations, or any vector equation such as (2), can be written as an equivalent matrix equation in the form $A\mathbf{x} = \mathbf{b}$. This simple observation will be used repeatedly throughout the text.

Here is the formal result.

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{6}$$

Theorem 3 provides a powerful tool for gaining insight into problems in linear algebra, because a system of linear equations may now be viewed in three different but equivalent ways: as a matrix equation, as a vector equation, or as a system of linear equations. Whenever you construct a mathematical model of a problem in real life, you are free to choose whichever viewpoint is most natural. Then you may switch from one formulation of a problem to another whenever it is convenient. In any case, the matrix equation (4), the vector equation (5), and the system of equations are all solved in the same way—by row reducing the augmented matrix (6). Other methods of solution will be discussed later.

Existence of Solutions

The definition of $A\mathbf{x}$ leads directly to the following useful fact.

> The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Section 1.3 considered the existence question, "Is $\mathbf{b}$ in Span $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$?" Equivalently, "Is $A\mathbf{x} = \mathbf{b}$ consistent?" A harder existence problem is to determine whether the equation $A\mathbf{x} = \mathbf{b}$ is consistent *for all* possible $\mathbf{b}$.

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

SOLUTION Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is *not* consistent for every $\mathbf{b}$ because some choices of $\mathbf{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero. ■

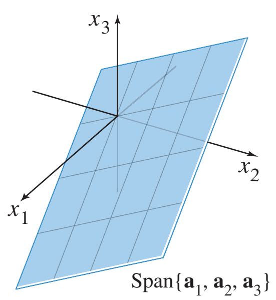


FIGURE 1
The columns of $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3\,]$ span a plane through $\mathbf{0}$.

The reduced matrix in Example 3 provides a description of all $\mathbf{b}$ for which the equation $A\mathbf{x} = \mathbf{b}$ *is* consistent: The entries in $\mathbf{b}$ must satisfy

$$b_1 - \tfrac{1}{2}b_2 + b_3 = 0$$

This is the equation of a plane through the origin in $\mathbb{R}^3$. The plane is the set of all linear combinations of the three columns of A. See Figure 1.

The equation $A\mathbf{x} = \mathbf{b}$ in Example 3 fails to be consistent for all $\mathbf{b}$ because the echelon form of A has a row of zeros. If A had a pivot in all three rows, we would not care about the calculations in the augmented column because in this case an echelon form of the augmented matrix could not have a row such as $[\,0 \quad 0 \quad 0 \quad 1\,]$.

In the next theorem, the sentence "The columns of A span $\mathbb{R}^m$" means that *every* $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A. In general, a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ **spans** (or **generates**) $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$—that is, if Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = \mathbb{R}^m$.

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

Theorem 4 is one of the most useful theorems in this chapter. Statements (a), (b), and (c) are equivalent because of the definition of $A\mathbf{x}$ and what it means for a set of vectors to span $\mathbb{R}^m$. The discussion after Example 3 suggests why (a) and (d) are equivalent; a proof is given at the end of the section. The exercises will provide examples of how Theorem 4 is used.

Warning: Theorem 4 is about a *coefficient matrix*, not an augmented matrix. If an augmented matrix $[\,A \quad \mathbf{b}\,]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ may or may not be consistent.

Computation of $A\mathbf{x}$

The calculations in Example 1 were based on the definition of the product of a matrix A and a vector $\mathbf{x}$. The following simple example will lead to a more efficient method for calculating the entries in $A\mathbf{x}$ when working problems by hand.

EXAMPLE 4 Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

SOLUTION From the definition,

$$\begin{aligned} \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= x_1\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix} \end{aligned} \tag{7}$$

The first entry in the product $A\mathbf{x}$ is a sum of products (sometimes called a *dot product*), using the first row of A and the entries in $\mathbf{x}$. That is,

$$\begin{bmatrix} 2 & 3 & 4 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ \\ \\ \end{bmatrix}$$

This matrix shows how to compute the first entry in $A\mathbf{x}$ directly, without writing down all the calculations shown in (7). Similarly, the second entry in $A\mathbf{x}$ can be calculated at once by multiplying the entries in the second row of A by the corresponding entries in $\mathbf{x}$ and then summing the resulting products:

$$\begin{bmatrix} & & \\ -1 & 5 & -3 \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ -x_1 + 5x_2 - 3x_3 \\ \\ \end{bmatrix}$$

Likewise, the third entry in $A\mathbf{x}$ can be calculated from the third row of A and the entries in $\mathbf{x}$. ■

Row–Vector Rule for Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

EXAMPLE 5

a. $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 3 + (-1) \cdot 7 \\ 0 \cdot 4 + (-5) \cdot 3 + 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

b. $$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \cdot 4 + (-3) \cdot 7 \\ 8 \cdot 4 + 0 \cdot 7 \\ (-5) \cdot 4 + 2 \cdot 7 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$$

c. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \cdot r + 0 \cdot s + 0 \cdot t \\ 0 \cdot r + 1 \cdot s + 0 \cdot t \\ 0 \cdot r + 0 \cdot s + 1 \cdot t \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$ ■

By definition, the matrix in Example 5(c) with 1's on the diagonal and 0's elsewhere is called an **identity matrix** and is denoted by I. The calculation in part (c) shows that $I\mathbf{x} = \mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^3$. There is an analogous $n \times n$ identity matrix, sometimes written as I_n. As in part (c), $I_n\mathbf{x} = \mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^n$.

Properties of the Matrix–Vector Product $A\mathbf{x}$

The facts in the next theorem are important and will be used throughout the text. The proof relies on the definition of $A\mathbf{x}$ and the algebraic properties of $\mathbb{R}^n$.

THEOREM 5

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

PROOF For simplicity, take $n = 3$, $A = [\,\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3\,]$, and $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$. (The proof of the general case is similar.) For $i = 1, 2, 3$, let u_i and v_i be the ith entries in $\mathbf{u}$ and $\mathbf{v}$, respectively. To prove statement (a), compute $A(\mathbf{u} + \mathbf{v})$ as a linear combination of the columns of A using the entries in $\mathbf{u} + \mathbf{v}$ as weights.

$$A(\mathbf{u} + \mathbf{v}) = [\,\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3\,]\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

Entries in $\mathbf{u} + \mathbf{v}$

$$= (u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3$$

Columns of A

$$\begin{aligned} &= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3) \\ &= A\mathbf{u} + A\mathbf{v} \end{aligned}$$

To prove statement (b), compute $A(c\mathbf{u})$ as a linear combination of the columns of A using the entries in $c\mathbf{u}$ as weights.

$$\begin{aligned} A(c\mathbf{u}) = [\,\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3\,]\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} &= (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + (cu_3)\mathbf{a}_3 \\ &= c(u_1\mathbf{a}_1) + c(u_2\mathbf{a}_2) + c(u_3\mathbf{a}_3) \\ &= c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) \\ &= c(A\mathbf{u}) \end{aligned}$$ ■

NUMERICAL NOTE

To optimize a computer algorithm to compute $A\mathbf{x}$, the sequence of calculations should involve data stored in contiguous memory locations. The most widely used professional algorithms for matrix computations are written in Fortran, a language that stores a matrix as a set of columns. Such algorithms compute $A\mathbf{x}$ as a linear combination of the columns of A. In contrast, if a program is written in the popular language C, which stores matrices by rows, $A\mathbf{x}$ should be computed via the alternative rule that uses the rows of A.

PROOF OF THEOREM 4 As was pointed out after Theorem 4, statements (a), (b), and (c) are logically equivalent. So, it suffices to show (for an arbitrary matrix A) that (a) and (d) are either both true or both false. This will tie all four statements together.

Let U be an echelon form of A. Given $\mathbf{b}$ in $\mathbb{R}^m$, we can row reduce the augmented matrix $[\,A \quad \mathbf{b}\,]$ to an augmented matrix $[\,U \quad \mathbf{d}\,]$ for some $\mathbf{d}$ in $\mathbb{R}^m$:

$$[\,A \quad \mathbf{b}\,] \sim \cdots \sim [\,U \quad \mathbf{d}\,]$$

If statement (d) is true, then each row of U contains a pivot position and there can be no pivot in the augmented column. So $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$, and (a) is true. If (d) is false, the last row of U is all zeros. Let $\mathbf{d}$ be any vector with a 1 in its last entry. Then $[\,U \quad \mathbf{d}\,]$ represents an *inconsistent* system. Since row operations are reversible, $[\,U \quad \mathbf{d}\,]$ can be transformed into the form $[\,A \quad \mathbf{b}\,]$. The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false. ■

PRACTICE PROBLEMS

1. Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$. It can be shown that $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Use this fact to exhibit $\mathbf{b}$ as a specific linear combination of the columns of A.

2. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. Verify Theorem 5(a) in this case by computing $A(\mathbf{u} + \mathbf{v})$ and $A\mathbf{u} + A\mathbf{v}$.

3. Construct a 3×3 matrix A and vectors $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$ so that $A\mathbf{x} = \mathbf{b}$ has a solution, but $A\mathbf{x} = \mathbf{c}$ does not.

1.4 EXERCISES

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$

2. $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.

5. $\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

6. $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$

is equivalent to the vector equation

$$3\begin{bmatrix}1\\-3\\4\end{bmatrix} - 2\begin{bmatrix}5\\1\\-8\end{bmatrix} + 0\begin{bmatrix}-2\\9\\-1\end{bmatrix} - 4\begin{bmatrix}0\\-5\\7\end{bmatrix} = \begin{bmatrix}-7\\9\\0\end{bmatrix}$$

which expresses $\mathbf{b}$ as a linear combination of the columns of A.

2. $$\mathbf{u}+\mathbf{v} = \begin{bmatrix}4\\-1\end{bmatrix} + \begin{bmatrix}-3\\5\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$$

$$A(\mathbf{u}+\mathbf{v}) = \begin{bmatrix}2 & 5\\3 & 1\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}2+20\\3+4\end{bmatrix} = \begin{bmatrix}22\\7\end{bmatrix}$$

$$A\mathbf{u}+A\mathbf{v} = \begin{bmatrix}2 & 5\\3 & 1\end{bmatrix}\begin{bmatrix}4\\-1\end{bmatrix} + \begin{bmatrix}2 & 5\\3 & 1\end{bmatrix}\begin{bmatrix}-3\\5\end{bmatrix}$$

$$= \begin{bmatrix}3\\11\end{bmatrix} + \begin{bmatrix}19\\-4\end{bmatrix} = \begin{bmatrix}22\\7\end{bmatrix}$$

Remark: There are, in fact, infinitely many correct solutions to Practice Problem 3. When creating matrices to satisfy specified criteria, it is often useful to create matrices that are straightforward, such as those already in reduced echelon form. Here is one possible solution:

3. Let

$$A = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}, \mathbf{b} = \begin{bmatrix}3\\2\\0\end{bmatrix}, \text{and } \mathbf{c} = \begin{bmatrix}3\\2\\1\end{bmatrix}.$$

Notice the reduced echelon form of the augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix}1 & 0 & 1 & 3\\0 & 1 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix},$$

which corresponds to a consistent system, and hence $A\mathbf{x} = \mathbf{b}$ has solutions. The reduced echelon form of the augmented matrix corresponding to $A\mathbf{x} = \mathbf{c}$ is

$$\begin{bmatrix}1 & 0 & 1 & 3\\0 & 1 & 1 & 2\\0 & 0 & 0 & 1\end{bmatrix},$$

which corresponds to an inconsistent system, and hence $A\mathbf{x} = \mathbf{c}$ does not have any solutions.

1.5 SOLUTION SETS OF LINEAR SYSTEMS

Solution sets of linear systems are important objects of study in linear algebra. They will appear later in several different contexts. This section uses vector notation to give explicit and geometric descriptions of such solution sets.

Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the **trivial solution**. For a given equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$. The Existence and Uniqueness Theorem in Section 1.2 (Theorem 2) leads immediately to the following fact.

> The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

EXAMPLE 1 Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

SOLUTION Let A be the matrix of coefficients of the system and row reduce the augmented matrix $[\,A \quad \mathbf{0}\,]$ to echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is a free variable, $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions (one for each choice of x_3). To describe the solution set, continue the row reduction of $[\,A \quad \mathbf{0}\,]$ to *reduced* echelon form:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 &= 0 \\ x_2 \qquad\quad &= 0 \\ 0 &= 0 \end{aligned}$$

Solve for the basic variables x_1 and x_2 and obtain $x_1 = \frac{4}{3}x_3$, $x_2 = 0$, with x_3 free. As a vector, the general solution of $A\mathbf{x} = \mathbf{0}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v}, \quad \text{where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

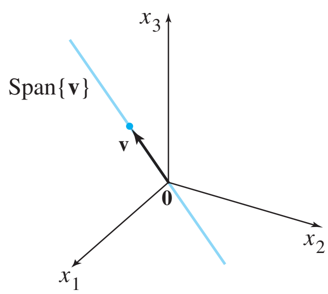


FIGURE 1

Here x_3 is factored out of the expression for the general solution vector. This shows that every solution of $A\mathbf{x} = \mathbf{0}$ in this case is a scalar multiple of $\mathbf{v}$. The trivial solution is obtained by choosing $x_3 = 0$. Geometrically, the solution set is a line through $\mathbf{0}$ in $\mathbb{R}^3$. See Figure 1. ■

Notice that a nontrivial solution $\mathbf{x}$ can have some zero entries so long as not all of its entries are zero.

EXAMPLE 2 A single linear equation can be treated as a very simple system of equations. Describe all solutions of the homogeneous "system"

$$10x_1 - 3x_2 - 2x_3 = 0 \tag{1}$$

SOLUTION There is no need for matrix notation. Solve for the basic variable x_1 in terms of the free variables. The general solution is $x_1 = .3x_2 + .2x_3$, with x_2 and x_3

free. As a vector, the general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \underbrace{\begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_3 \underbrace{\begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}} \quad (\text{with } x_2, x_3 \text{ free}) \tag{2}$$

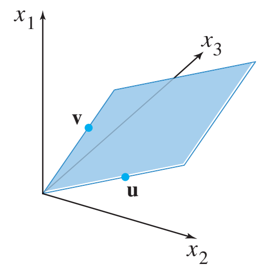


FIGURE 2

This calculation shows that every solution of (1) is a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, shown in (2). That is, the solution set is Span $\{\mathbf{u}, \mathbf{v}\}$. Since neither $\mathbf{u}$ nor $\mathbf{v}$ is a scalar multiple of the other, the solution set is a plane through the origin. See Figure 2. ■

Examples 1 and 2, along with the exercises, illustrate the fact that the solution set of a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can always be expressed explicitly as Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ for suitable vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$. If the only solution is the zero vector, then the solution set is Span $\{\mathbf{0}\}$. If the equation $A\mathbf{x} = \mathbf{0}$ has only one free variable, the solution set is a line through the origin, as in Figure 1. A plane through the origin, as in Figure 2, provides a good mental image for the solution set of $A\mathbf{x} = \mathbf{0}$ when there are two or more free variables. Note, however, that a similar figure can be used to visualize Span $\{\mathbf{u}, \mathbf{v}\}$ even when $\mathbf{u}$ and $\mathbf{v}$ do not arise as solutions of $A\mathbf{x} = \mathbf{0}$. See Figure 11 in Section 1.3.

Parametric Vector Form

The original equation (1) for the plane in Example 2 is an *implicit* description of the plane. Solving this equation amounts to finding an *explicit* description of the plane as the set spanned by $\mathbf{u}$ and $\mathbf{v}$. Equation (2) is called a **parametric vector equation** of the plane. Sometimes such an equation is written as

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \quad (s, t \text{ in } \mathbb{R})$$

to emphasize that the parameters vary over all real numbers. In Example 1, the equation $\mathbf{x} = x_3\mathbf{v}$ (with x_3 free), or $\mathbf{x} = t\mathbf{v}$ (with t in $\mathbb{R}$), is a parametric vector equation of a line. Whenever a solution set is described explicitly with vectors as in Examples 1 and 2, we say that the solution is in **parametric vector form**.

Solutions of Nonhomogeneous Systems

When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

EXAMPLE 3 Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

SOLUTION Here A is the matrix of coefficients from Example 1. Row operations on $[\,A \quad \mathbf{b}\,]$ produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 &= -1 \\ x_2 \qquad\quad &= 2 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free. As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underset{\substack{\uparrow \\ \mathbf{p}}}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + x_3 \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}$$

The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \tag{3}$$

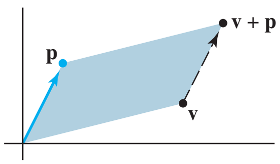


FIGURE 3
Adding $\mathbf{p}$ to $\mathbf{v}$ translates $\mathbf{v}$ to $\mathbf{v} + \mathbf{p}$.

describes the solution set of $A\mathbf{x} = \mathbf{b}$ in parametric vector form. Recall from Example 1 that the solution set of $A\mathbf{x} = \mathbf{0}$ has the parametric vector equation

$$\mathbf{x} = t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \tag{4}$$

[with the same $\mathbf{v}$ that appears in (3)]. Thus the solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by adding the vector $\mathbf{p}$ to the solutions of $A\mathbf{x} = \mathbf{0}$. The vector $\mathbf{p}$ itself is just one particular solution of $A\mathbf{x} = \mathbf{b}$ [corresponding to $t = 0$ in (3)]. ■

To describe the solution set of $A\mathbf{x} = \mathbf{b}$ geometrically, we can think of vector addition as a *translation*. Given $\mathbf{v}$ and $\mathbf{p}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding $\mathbf{p}$ to $\mathbf{v}$ is to *move* $\mathbf{v}$ in a direction parallel to the line through $\mathbf{p}$ and $\mathbf{0}$. We say that $\mathbf{v}$ is **translated by p** to $\mathbf{v} + \mathbf{p}$. See Figure 3. If each point on a line L in $\mathbb{R}^2$ or $\mathbb{R}^3$ is translated by a vector $\mathbf{p}$, the result is a line parallel to L. See Figure 4.

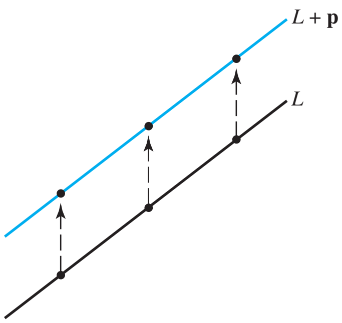


FIGURE 4
Translated line.

Suppose L is the line through $\mathbf{0}$ and $\mathbf{v}$, described by equation (4). Adding $\mathbf{p}$ to each point on L produces the translated line described by equation (3). Note that $\mathbf{p}$ is on the line in equation (3). We call (3) **the equation of the line through p parallel to v**. Thus *the solution set of* $A\mathbf{x} = \mathbf{b}$ *is a line through* $\mathbf{p}$ *parallel to the solution set of* $A\mathbf{x} = \mathbf{0}$. Figure 5 illustrates this case.

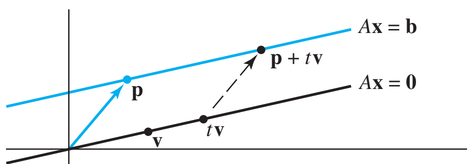


FIGURE 5 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

The relation between the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ shown in Figure 5 generalizes to any *consistent* equation $A\mathbf{x} = \mathbf{b}$, although the solution set will be larger than a line when there are several free variables. The following theorem gives the precise statement. See Exercise 25 for a proof.

THEOREM 6

> Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem 6 says that if $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$ for the translation. Figure 6 illustrates the case in which there are two free variables. Even when $n > 3$, our mental image of the solution set of a consistent system $A\mathbf{x} = \mathbf{b}$ (with $\mathbf{b} \neq \mathbf{0}$) is either a single nonzero point or a line or plane not passing through the origin.

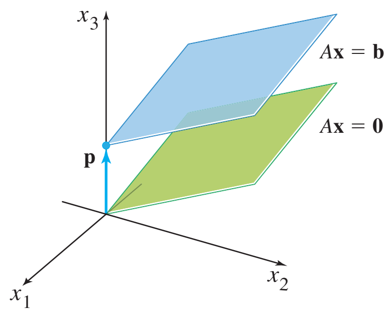


FIGURE 6 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

Warning: Theorem 6 and Figure 6 apply only to an equation $A\mathbf{x} = \mathbf{b}$ that has at least one nonzero solution $\mathbf{p}$. When $A\mathbf{x} = \mathbf{b}$ has no solution, the solution set is empty.

The following algorithm outlines the calculations shown in Examples 1, 2, and 3.

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

PRACTICE PROBLEMS

1. Each of the following equations determines a plane in $\mathbb{R}^3$. Do the two planes intersect? If so, describe their intersection.

$$\begin{aligned} x_1 + 4x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + 8x_3 &= 9 \end{aligned}$$

2. Write the general solution of $10x_1 - 3x_2 - 2x_3 = 7$ in parametric vector form, and relate the solution set to the one found in Example 2.
3. Prove the first part of Theorem 6: Suppose that $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$, so that $A\mathbf{p} = \mathbf{b}$. Let $\mathbf{v}_h$ be any solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, and let $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$. Show that $\mathbf{w}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

2. The augmented matrix $\begin{bmatrix} 10 & -3 & -2 & 7 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & -.3 & -.2 & .7 \end{bmatrix}$, and the general solution is $x_1 = .7 + .3x_2 + .2x_3$, with x_2 and x_3 free. That is,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 + .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix}$$
$$= \mathbf{p} + x_2\mathbf{u} + x_3\mathbf{v}$$

The solution set of the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is the translated plane $\mathbf{p} + \text{Span}\,\{\mathbf{u}, \mathbf{v}\}$, which passes through $\mathbf{p}$ and is parallel to the solution set of the homogeneous equation in Example 2.

3. Using Theorem 5 from Section 1.4, notice

$$A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h = \mathbf{b} + \mathbf{0} = \mathbf{b},$$

hence $\mathbf{p} + \mathbf{v}_h$ is a solution to $A\mathbf{x} = \mathbf{b}$.

1.6 APPLICATIONS OF LINEAR SYSTEMS

You might expect that a real-life problem involving linear algebra would have only one solution, or perhaps no solution. The purpose of this section is to show how linear systems with many solutions can arise naturally. The applications here come from economics, chemistry, and network flow.

A Homogeneous System in Economics

WEB

The system of 500 equations in 500 variables, mentioned in this chapter's introduction, is now known as a Leontief "input–output" (or "production") model.[1] Section 2.6 will examine this model in more detail, when more theory and better notation are available. For now, we look at a simpler "exchange model," also due to Leontief.

Suppose a nation's economy is divided into many sectors, such as various manufacturing, communication, entertainment, and service industries. Suppose that for each sector we know its total output for one year and we know exactly how this output is divided or "exchanged" among the other sectors of the economy. Let the total dollar value of a sector's output be called the **price** of that output. Leontief proved the following result.

> There exist *equilibrium prices* that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.

The following example shows how to find the equilibrium prices.

EXAMPLE 1 Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as shown in Table 1, where the entries in a column represent the fractional parts of a sector's total output.

The second column of Table 1, for instance, says that the total output of the Electric sector is divided as follows: 40% to Coal, 50% to Steel, and the remaining 10% to Electric. (Electric treats this 10% as an expense it incurs in order to operate its

[1] See Wassily W. Leontief, "Input–Output Economics," *Scientific American*, October 1951, pp. 15–21.

business.) Since all output must be taken into account, the decimal fractions in each column must sum to 1.

Denote the prices (i.e., dollar values) of the total annual outputs of the Coal, Electric, and Steel sectors by p_C, p_E, and p_S, respectively. If possible, find equilibrium prices that make each sector's income match its expenditures.

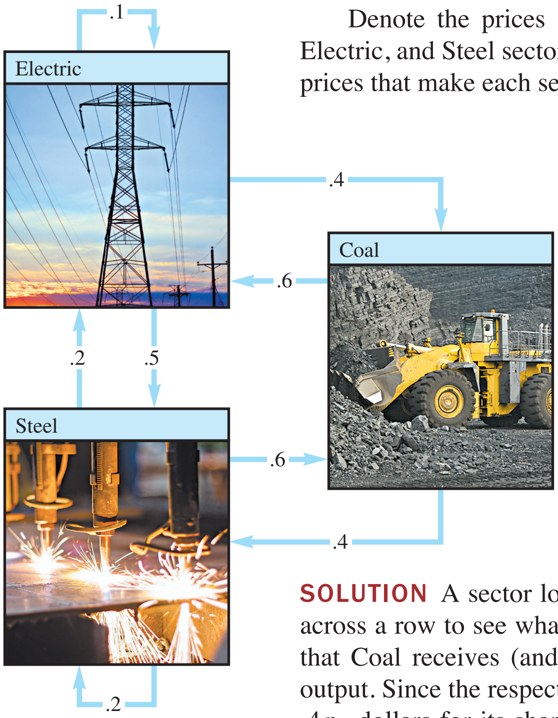


TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

SOLUTION A sector looks down a column to see where its output goes, and it looks across a row to see what it needs as inputs. For instance, the first row of Table 1 says that Coal receives (and pays for) 40% of the Electric output and 60% of the Steel output. Since the respective values of the total outputs are p_E and p_S, Coal must spend $.4p_E$ dollars for its share of Electric's output and $.6p_S$ for its share of Steel's output. Thus Coal's total expenses are $.4p_E + .6p_S$. To make Coal's income, p_C, equal to its expenses, we want

$$p_C = .4p_E + .6p_S \tag{1}$$

The second row of the exchange table shows that the Electric sector spends $.6p_C$ for coal, $.1p_E$ for electricity, and $.2p_S$ for steel. Hence the income/expense requirement for Electric is

$$p_E = .6p_C + .1p_E + .2p_S \tag{2}$$

Finally, the third row of the exchange table leads to the final requirement:

$$p_S = .4p_C + .5p_E + .2p_S \tag{3}$$

To solve the system of equations (1), (2), and (3), move all the unknowns to the left sides of the equations and combine like terms. [For instance, on the left side of (2), write $p_E - .1p_E$ as $.9p_E$.]

$$\begin{aligned} p_C - .4p_E - .6p_S &= 0 \\ -.6p_C + .9p_E - .2p_S &= 0 \\ -.4p_C - .5p_E + .8p_S &= 0 \end{aligned}$$

Row reduction is next. For simplicity here, decimals are rounded to two places.

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & -.66 & .56 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $p_C = .94p_S$, $p_E = .85p_S$, and p_S is free. The equilibrium price vector for the economy has the form

$$\mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

Any (nonnegative) choice for p_S results in a choice of equilibrium prices. For instance, if we take p_S to be 100 (or \$100 million), then $p_C = 94$ and $p_E = 85$. The incomes and expenditures of each sector will be equal if the output of Coal is priced at \$94 million, that of Electric at \$85 million, and that of Steel at \$100 million. ■

Balancing Chemical Equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O), according to an equation of the form

$$(x_1)C_3H_8 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O \tag{4}$$

To "balance" this equation, a chemist must find whole numbers $x_1, \dots, x_4$ such that the total numbers of carbon (C), hydrogen (H), and oxygen (O) atoms on the left match the corresponding numbers of atoms on the right (because atoms are neither destroyed nor created in the reaction).

A systematic method for balancing chemical equations is to set up a vector equation that describes the numbers of atoms of each type present in a reaction. Since equation (4) involves three types of atoms (carbon, hydrogen, and oxygen), construct a vector in $\mathbb{R}^3$ for each reactant and product in (4) that lists the numbers of "atoms per molecule," as follows:

$$C_3H_8: \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, \quad O_2: \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad CO_2: \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad H_2O: \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{Carbon} \\ \leftarrow \text{Hydrogen} \\ \leftarrow \text{Oxygen} \end{array}$$

To balance equation (4), the coefficients $x_1, \dots, x_4$ must satisfy

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

To solve, move all the terms to the left (changing the signs in the third and fourth vectors):

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reduction of the augmented matrix for this equation leads to the general solution

$$x_1 = \tfrac{1}{4}x_4, \;\; x_2 = \tfrac{5}{4}x_4, \;\; x_3 = \tfrac{3}{4}x_4, \;\; \text{with } x_4 \text{ free}$$

Since the coefficients in a chemical equation must be integers, take $x_4 = 4$, in which case $x_1 = 1$, $x_2 = 5$, and $x_3 = 3$. The balanced equation is

$$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$$

The equation would also be balanced if, for example, each coefficient were doubled. For most purposes, however, chemists prefer to use a balanced equation whose coefficients are the smallest possible whole numbers.

Network Flow

WEB

Systems of linear equations arise naturally when scientists, engineers, or economists study the flow of some quantity through a network. For instance, urban planners and traffic engineers monitor the pattern of traffic flow in a grid of city streets. Electrical engineers calculate current flow through electrical circuits. And economists analyze the distribution of products from manufacturers to consumers through a network of wholesalers and retailers. For many networks, the systems of equations involve hundreds or even thousands of variables and equations.

A *network* consists of a set of points called *junctions*, or *nodes*, with lines or arcs called *branches* connecting some or all of the junctions. The direction of flow in each branch is indicated, and the flow amount (or rate) is either shown or is denoted by a variable.

The basic assumption of network flow is that the total flow into the network equals the total flow out of the network and that the total flow into a junction equals the total flow out of the junction. For example, Figure 1 shows 30 units flowing into a junction through one branch, with x_1 and x_2 denoting the flows out of the junction through other branches. Since the flow is "conserved" at each junction, we must have $x_1 + x_2 = 30$. In a similar fashion, the flow at each junction is described by a linear equation. The problem of network analysis is to determine the flow in each branch when partial information (such as the flow into and out of the network) is known.

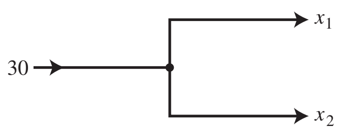


FIGURE 1
A junction, or node.

EXAMPLE 2 The network in Figure 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

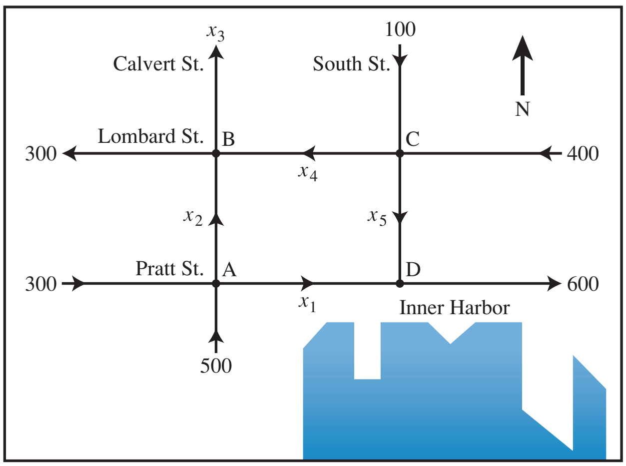


FIGURE 2 Baltimore streets.

SOLUTION Write equations that describe the flow, and then find the general solution of the system. Label the street intersections (junctions) and the unknown flows in the branches, as shown in Figure 2. At each intersection, set the flow in equal to the flow out.

Intersection	Flow in		Flow out
A	$300 + 500$	$=$	$x_1 + x_2$
B	$x_2 + x_4$	$=$	$300 + x_3$
C	$100 + 400$	$=$	$x_4 + x_5$
D	$x_1 + x_5$	$=$	600

Also, the total flow into the network $(500 + 300 + 100 + 400)$ equals the total flow out of the network $(300 + x_3 + 600)$, which simplifies to $x_3 = 400$. Combine this equation with a rearrangement of the first four equations to obtain the following system of equations:

$$\begin{aligned}
x_1 + x_2 \qquad\qquad\qquad &= 800 \\
x_2 - x_3 + x_4 \qquad &= 300 \\
x_4 + x_5 &= 500 \\
x_1 \qquad\qquad\qquad + x_5 &= 600 \\
x_3 \qquad\qquad &= 400
\end{aligned}$$

Row reduction of the associated augmented matrix leads to

$$\begin{aligned}
x_1 \qquad\qquad + x_5 &= 600 \\
x_2 \qquad - x_5 &= 200 \\
x_3 \qquad &= 400 \\
x_4 + x_5 &= 500
\end{aligned}$$

The general flow pattern for the network is described by

$$\begin{cases}
x_1 = 600 - x_5 \\
x_2 = 200 + x_5 \\
x_3 = 400 \\
x_4 = 500 - x_5 \\
x_5 \text{ is free}
\end{cases}$$

A negative flow in a network branch corresponds to flow in the direction opposite to that shown on the model. Since the streets in this problem are one-way, none of the variables here can be negative. This fact leads to certain limitations on the possible values of the variables. For instance, $x_5 \leq 500$ because x_4 cannot be negative. Other constraints on the variables are considered in Practice Problem 2. ■

PRACTICE PROBLEMS

1. Suppose an economy has three sectors: Agriculture, Mining, and Manufacturing. Agriculture sells 5% of its output to Mining and 30% to Manufacturing, and retains the rest. Mining sells 20% of its output to Agriculture and 70% to Manufacturing, and retains the rest. Manufacturing sells 20% of its output to Agriculture and 30% to Mining, and retains the rest. Determine the exchange table for this economy, where the columns describe how the output of each sector is exchanged among the three sectors.

2. Consider the network flow studied in Example 2. Determine the possible range of values of x_1 and x_2. [*Hint:* The example showed that $x_5 \leq 500$. What does this imply about x_1 and x_2? Also, use the fact that $x_5 \geq 0$.]

For instance, consider the equation

$$x_1\begin{bmatrix}1\\2\\3\end{bmatrix}+x_2\begin{bmatrix}4\\5\\6\end{bmatrix}+x_3\begin{bmatrix}2\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{1}$$

This equation has a trivial solution, of course, where $x_1 = x_2 = x_3 = 0$. As in Section 1.5, the main issue is whether the trivial solution is the *only one*.

DEFINITION

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{2}$$

Equation (2) is called a **linear dependence relation** among $\mathbf{v}_1, \dots, \mathbf{v}_p$ when the weights are not all zero. An indexed set is linearly dependent if and only if it is not linearly independent. For brevity, we may say that $\mathbf{v}_1, \dots, \mathbf{v}_p$ are linearly dependent when we mean that $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly dependent set. We use analogous terminology for linearly independent sets.

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$.

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

SOLUTION

a. We must determine if there is a nontrivial solution of equation (1) above. Row operations on the associated augmented matrix show that

$$\begin{bmatrix}1&4&2&0\\2&5&1&0\\3&6&0&0\end{bmatrix} \sim \begin{bmatrix}1&4&2&0\\0&-3&-3&0\\0&0&0&0\end{bmatrix}$$

Clearly, x_1 and x_2 are basic variables, and x_3 is free. Each nonzero value of x_3 determines a nontrivial solution of (1). Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent (and not linearly independent).

b. To find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix}1&0&-2&0\\0&1&1&0\\0&0&0&0\end{bmatrix} \qquad \begin{aligned} x_1 \quad\;\; - 2x_3 &= 0\\ x_2 + \;\; x_3 &= 0\\ 0 &= 0\end{aligned}$$

Thus $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free. Choose any nonzero value for x_3—say, $x_3 = 5$. Then $x_1 = 10$ and $x_2 = -5$. Substitute these values into equation (1) and obtain

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$$

This is one (out of infinitely many) possible linear dependence relations among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. ■

Linear Independence of Matrix Columns

Suppose that we begin with a matrix $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. Thus we have the following important fact.

> The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

EXAMPLE 2 Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

SOLUTION To study $A\mathbf{x} = \mathbf{0}$, row reduce the augmented matrix:

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

At this point, it is clear that there are three basic variables and no free variables. So the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and the columns of A are linearly independent. ■

Sets of One or Two Vectors

A set containing only one vector—say, $\mathbf{v}$—is linearly independent if and only if $\mathbf{v}$ is not the zero vector. This is because the vector equation $x_1\mathbf{v} = \mathbf{0}$ has only the trivial solution when $\mathbf{v} \neq \mathbf{0}$. The zero vector is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

The next example will explain the nature of a linearly dependent set of two vectors.

EXAMPLE 3 Determine if the following sets of vectors are linearly independent.

a. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ b. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

SOLUTION

a. Notice that $\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$, namely, $\mathbf{v}_2 = 2\mathbf{v}_1$. Hence $-2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$, which shows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent.

b. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are certainly *not* multiples of one another. Could they be linearly dependent? Suppose c and d satisfy

$$c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}$$

If $c \neq 0$, then we can solve for $\mathbf{v}_1$ in terms of $\mathbf{v}_2$, namely, $\mathbf{v}_1 = (-d/c)\mathbf{v}_2$. This result is impossible because $\mathbf{v}_1$ is *not* a multiple of $\mathbf{v}_2$. So c must be zero. Similarly, d must also be zero. Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set. ■

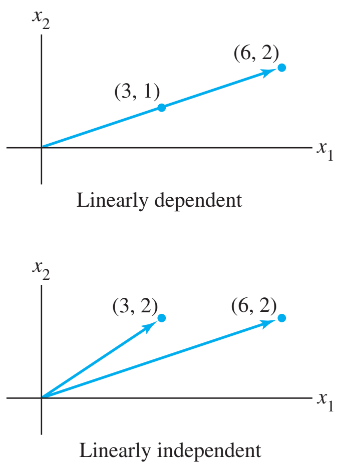


FIGURE 1

The arguments in Example 3 show that you can always decide *by inspection* when a set of two vectors is linearly dependent. Row operations are unnecessary. Simply check whether at least one of the vectors is a scalar times the other. (The test applies only to sets of *two* vectors.)

> A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

In geometric terms, two vectors are linearly dependent if and only if they lie on the same line through the origin. Figure 1 shows the vectors from Example 3.

Sets of Two or More Vectors

The proof of the next theorem is similar to the solution of Example 3. Details are given at the end of this section.

THEOREM 7 **Characterization of Linearly Dependent Sets**

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Warning: Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors. See Practice Problem 1(c).

EXAMPLE 4 Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by $\mathbf{u}$ and $\mathbf{v}$, and explain why a vector $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

SOLUTION The vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent because neither vector is a multiple of the other, and so they span a plane in $\mathbb{R}^3$. (See Section 1.3.) In fact, Span $\{\mathbf{u}, \mathbf{v}\}$ is the x_1x_2-plane (with $x_3 = 0$). If $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, by Theorem 7. Conversely, suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. By Theorem 7, some vector in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linear combination of the preceding vectors (since $\mathbf{u} \neq \mathbf{0}$). That vector must be $\mathbf{w}$, since $\mathbf{v}$ is not a multiple of $\mathbf{u}$. So $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$. See Figure 2. ■

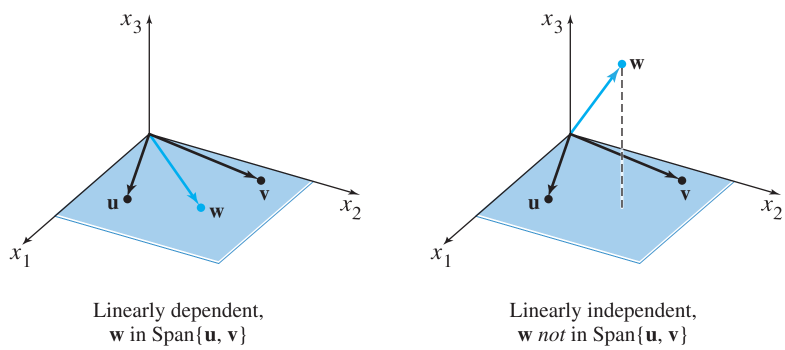


FIGURE 2 Linear dependence in $\mathbb{R}^3$.

Example 4 generalizes to any set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$ with $\mathbf{u}$ and $\mathbf{v}$ linearly independent. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ will be linearly dependent if and only if $\mathbf{w}$ is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

The next two theorems describe special cases in which the linear dependence of a set is automatic. Moreover, Theorem 8 will be a key result for work in later chapters.

THEOREM 8

> If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

FIGURE 3
If $p > n$, the columns are linearly dependent.

PROOF Let $A = [\,\mathbf{v}_1 \;\; \cdots \;\; \mathbf{v}_p\,]$. Then A is $n \times p$, and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of n equations in p unknowns. If $p > n$, there are more variables than equations, so there must be a free variable. Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and the columns of A are linearly dependent. See Figure 3 for a matrix version of this theorem. ■

Warning: Theorem 8 says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector.

EXAMPLE 5 The vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are linearly dependent by Theorem 8, because there are three vectors in the set and there are only two entries in each vector. Notice, however, that none of the vectors is a multiple of one of the other vectors. See Figure 4. ■

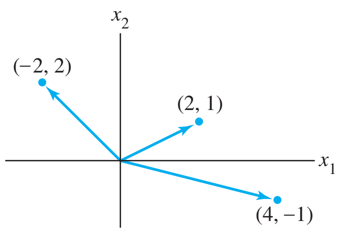


FIGURE 4
A linearly dependent set in $\mathbb{R}^2$.

THEOREM 9

> If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

PROOF By renumbering the vectors, we may suppose $\mathbf{v}_1 = \mathbf{0}$. Then the equation $1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p = \mathbf{0}$ shows that S is linearly dependent. ■

EXAMPLE 6 Determine by inspection if the given set is linearly dependent.

a. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ b. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$ c. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

SOLUTION

a. The set contains four vectors, each of which has only three entries. So the set is linearly dependent by Theorem 8.
b. Theorem 8 does not apply here because the number of vectors does not exceed the number of entries in each vector. Since the zero vector is in the set, the set is linearly dependent by Theorem 9.
c. Compare the corresponding entries of the two vectors. The second vector seems to be $-3/2$ times the first vector. This relation holds for the first three pairs of entries, but fails for the fourth pair. Thus neither of the vectors is a multiple of the other, and hence they are linearly independent. ■