Burger's equation solving by Rk4 method

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Burger's Equation

The one-dimensional Burger's equation is given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

Where:

- u is the velocity field.
- t is time.
- x is the spatial coordinate.
- ν is the kinematic viscosity.

1 Discretization in Time

The time derivative $\frac{\partial u}{\partial t}$ can be approximated using the forward difference:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

Where:

- u_i^n represents the velocity at spatial point i and time step n.
- Δt is the time step.

2 Discretization in Space

The first derivative $\frac{\partial u}{\partial x}$ is approximated as:

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

The second derivative $\frac{\partial^2 u}{\partial x^2}$ is approximated as:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Where:

- Δx is the spatial step.

These approximations are used in the discretized form of Burger's equation for numerical simulations.

3 Discretized Burger's Equation

Substitute the discretized derivatives into Burger's equation:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \left(2u_{i+1}^n - u_{i-1}^n\right) = \nu \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

Solving for u_i^{n+1}, we get:

$$u_i^{n+1} = u_i^n - u_i^n \frac{\Delta t}{2\Delta x} \left(2u_{i+1}^n - u_{i-1}^n\right) + \nu \frac{\Delta t}{\Delta x^2} \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

This discretized form is used in the RK4 method to numerically solve Burger's equation at each time step. The RK4 method provides a way to approximate the solution of ordinary differential equations (ODEs) and can be adapted to solve partial differential equations (PDEs) like Burger's equation when combined with finite difference methods for discretization.
We can write the equation in terms of ODE as:

$\frac{du_i^{n+1}}{dt} = -u_i^n \frac{1}{2\Delta x} \left(2u_{i+1}^n - u_{i-1}^n\right) + \nu \frac{1}{\Delta x^2} \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$

Reference: Click Here