CPW expansions of Celestial Amplitudes

in which I try to evaluate:

$$\int_0^1 \frac{dz}{z^4}(1-z)^{a+b} F_h(z)\, F_{\tilde{h}}(z)$$

Alex Atanasov

Looking through Ana's notes:

$$C_4 = \langle \varphi_1(z_1,\bar{z}_1)\cdots\varphi_4(z_4,\bar{z}_4)\rangle$$

$$= \underbrace{\frac{1}{z_{12}^{h_1+h_2}\, z_{34}^{h_3+h_4}}}_{\text{standard form in literature.}} \left(\frac{z_{24}}{z_{14}}\right)^{h_{12}} \left(\frac{z_{14}}{z_{13}}\right)^{h_{34}} \times (\text{c.c.}) \times F(z,\bar{z})$$

differ by poly of z

Alt. $$C_4 = \prod_{i<j} z_{ij}^{h/3 - h_i - h_j} \times (\text{c.c.})\, f(z,\bar{z}) \qquad h = \sum h_i$$

$$f(z,\bar{z}) = z_{12}^{-h/3}\, z_{34}^{-h/3}\, z_{13}^{-h/3+h_1+h_3-h_3+h_4}\, z_{24}^{-h/3+h_2+h_4+h_1-h_2}\, z_{14}^{-h/3+h_1+h_4-h_1+h_2+h_3-h_4}\, z_{23}^{-h/3+h_2+h_3} \times \text{c.c.} \times F(z,\bar{z})$$

$$= \frac{z_{13}^{h_1+h_4}\, z_{24}^{h_1+h_4}\, z_{14}^{h_2+h_3}\, z_{23}^{h_2+h_3}}{(z_{12} z_{34} z_{13} z_{24} z_{14} z_{23})^{h/3}} \times \text{c.c.}$$

$$= \left(\frac{z_{13} z_{24}}{z_{14} z_{23}}\right)^{h_1+h_4} \frac{z_{14}^{2h/3}\, z_{23}^{2h/3}}{(z_{12} z_{34} z_{13} z_{24})^{h/3}} \times \text{c.c.} \times F(z,\bar{z})$$

$$= (1-z)^{-h_1+h_4} (1-z)^{2h/3} z^{-h/3}\, F(z,\bar{z})$$

$$= z^{-h/3} (1-z)^{2h/3 - h_1 - h_4} \times (\text{c.c.})\, F(z,\bar{z})$$

$F(z,\bar{z})$ is more symm.

Translation invariance $\Rightarrow f$ depends only on sums of external dims

Block decomp:

$$C_4 = \sum_{\substack{h,\bar h \ sl_2(\mathbb{R}) \\ \text{prim + desc}}} \langle \varphi_1(z_1,\bar z_1)\varphi_2(z_2,\bar z_2)\mathcal{O}_{h,\bar h}\rangle \langle \tilde{\mathcal{O}}^{\dagger}_{h,\bar h}\ \varphi_3(z_3,\bar z_3)\varphi_4(z_4,\bar z_4)\rangle$$

$$(M_1+M_2)^{AB}(M_1+M_2)_{AB}\, C_4$$

$$= \sum_{h,\bar h} C^{(2)}_{h,\bar h} \langle \varphi_1(z_1,\bar z_1)\varphi_2(z_2,\bar z_2)\mathcal{O}_{h,\bar h}\rangle \langle \tilde{\mathcal{O}}^{\dagger}_{h,\bar h}\ \varphi_3(z_3,\bar z_3)\varphi_4(z_4,\bar z_4)\rangle$$

with $C_{h,\bar h} = \Delta(\Delta-d) + J(J+d-2)$
$\qquad\qquad = \Delta(\Delta-2) + J^2$

note $\frac{1}{2}M^{AB}M_{AB} = M^{0i}M_{0i} + M^{1i}M_{1i} + \frac{1}{2}M^{ij}M_{ij} + M^{01}M_{01}$ $\quad i=2,3$

$M_{0i} = \frac{P_i+K_i}{2}$

$M_{1i} = \frac{P_i-K_i}{2}$

$M_{ij} = J_{ij}$

$M_{01} = D$

$$= \frac{1}{2}\frac{1}{2}\left[-(P+K)^2 + (P-K)^2\right] + \frac{1}{2}J_{ij}J_{ji} + D^2$$

From lowering 0

$$= -P\cdot K + \frac{1}{2}J_{ij}J_{ji} + D^2$$
$$+ \frac{1}{2}[P_i, K_i]$$

$$= -P\cdot K + \frac{1}{2}J_{ij}J_{ji} + D(D-2)$$

equiv: $\frac{1}{2}M^{AB}M_{AB} = L_0^2 - \frac{1}{2}(L_{-1}L_1 + L_1L_{-1}) - \frac{1}{2}(\bar L_{-1}\bar L_1 - \bar L_1\bar L_{-1})$

Take $L_{-1} = \partial_z$

$L_0 = z\partial_z + h$

$L_1 = z^2\partial_z + 2zh$

$L_0^2 = \partial_z^2$

$L_1 L_{-1}$

Come back to this!

Let's try Aryn's way

$z^3 (1-z)^\beta$ $\qquad\qquad \beta = -a-b$

$$\int_0^1 z^{-1+h+\tilde{h}+s+\tilde{s}} (1-z)^\beta = \frac{\Gamma(1+\beta)\,\Gamma(\Delta+s+\tilde{s})}{\Gamma(1+\beta+\Delta+s+\tilde{s})}$$

$$\frac{\Gamma(-s)\,\Gamma(s+h+a)\,\Gamma(s+h+b)}{\Gamma(s+2h)} \; \frac{\Gamma(h+\tilde{h}+s+\tilde{s})}{\Gamma(1+\beta+\Delta+s+\tilde{s})}$$

$${}_3F_2\left(\begin{matrix} h-a & h-b & \Delta+\tilde{s} \\ & 2h & 1+\beta+\Delta+\tilde{s} \end{matrix} ; 1\right)$$

Saalschutz

$$\frac{\Gamma(1+\beta+\Delta+\tilde{s})\,\Gamma(1-h-b)\,\Gamma(1-h-a)\,\Gamma(1-J+\tilde{s})\,\Gamma(\tilde{s},\,\hat{h}-a,\,\hat{h}-b)}{\Gamma(1-2h)\,\Gamma(1+\beta+\hat{h}+\tilde{s}+a)\,\Gamma(1+\beta+\hat{h}+\tilde{s}+b)}$$

$$\sim \; {}_4F_3\left(\begin{matrix} 1+\beta+\Delta & 1-J & \hat{h}-a & \hat{h}-b \\ 1+\beta+\hat{h}+a & 1+\beta+\hat{h}+b & 2\tilde{h} & \end{matrix}\right)$$

My way:

$$z^3(1-z)^{\beta}$$

$$\int_0^1 z^{-1+h+\tilde h}(1-z)^{\beta+s+\tilde s} = \frac{\Gamma(\Delta)\,\Gamma(1+\beta+s+\tilde s)}{\Gamma(1+\Delta+\beta+s+\tilde s)}$$

$$\frac{\Gamma(-s)\,\Gamma(h-a+s)\,\Gamma(h-b+s)\,\Gamma(a+b-s)\,\Gamma(1+\beta+s+\tilde s)}{\Gamma(1+\Delta+\beta+s+\tilde s)}$$

$$\rightsquigarrow {}_3F_2\left(\begin{matrix} h-a & h-b & 1+\beta+s+\tilde s \\ 1-a-b & & 1+\Delta+\beta+s+\tilde s \end{matrix}\right)$$

need $a+b$ integral I think

$$\rightsquigarrow \frac{\Gamma(h-a)\,\Gamma(h-b)\,\Gamma(1+\beta+\tilde s)\,\Gamma(h+a)\,\Gamma(h+b)\,\Gamma(1+\beta+a+b+s+\tilde s)}{\Gamma(1+\tilde h+a+\cancel{\beta+\tilde s})\,\Gamma(1+\tilde h+b+\tilde s)\,\Gamma(\Delta)}$$

$$f(z,\bar z) = z^{-h/3}(1-z)^{2h/3 - h_1 - h_4} \times (c.c.)\; F(z,\bar z) \qquad h = \sum h_i = 2$$

$$\Rightarrow z^{2/3}(1-z)^{-4/3+h_1+h_4} \times c.c.\; f(z,\bar z) = z^{4/3} z^{5/3} (1-z)^{-8/3-1/3+\Delta_1+\Delta_4}$$

$$z^{5/3}(1-z)^{-1/3}\,\delta(z-\bar z) \qquad = \quad z^{3}(1-z)^{\overbrace{-3+\Delta_1+\Delta_4}^{2+i\lambda_1+i\lambda_4}}\,\delta(z-\bar z)$$

$a = h_{21} = i(\lambda_2-\lambda_1)/2$

$b = h_{34} = i(\lambda_3-\lambda_4)/2$

$\Rightarrow -a-b = i\lambda_1 + i\lambda_4$

$$= z^3(1-z)^{-1-a-b} = A(1^-,2^-,3^+,4^+)$$

$$A(1^-,2^-,3,4) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$A(1,2,4,3) = \frac{\langle 23\rangle\langle 41\rangle}{\langle 24\rangle\langle 31\rangle} A = -(1-z)A$$

$$A(1,3,2,4) = \langle 13\rangle$$

$$A(1,4,2,3) = -zA$$

Alt:

$$\tilde A^{11}$$

$$\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} \quad \text{vs} \quad \frac{\langle 12\rangle^4}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle}$$

$$x = \frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} = -z$$

$$\Rightarrow \tilde A = 2A\left[f^{12c}f^{c34} + (-z)\, f^{13c}f^{c24}\right]$$

$$\Rightarrow \tilde A = 2A \qquad\qquad f^{12c}f^{c34} + f^{14c}f^{c23}$$

$$\Rightarrow \tilde A = 2\,\tilde A(1^-,2^-,3^+,4^+)\left[f^{a_1a_2c}f^{ca_3a_4}(1-z) + f^{a_1a_4c}f^{ca_2a_3}\,z\right]$$

$$= 2\left[z^3(1-z)^{-a-b} + z^4(1-z)^{1-a-b}\right]$$

Anil's way using unjustified Saalschutz

$$\Rightarrow \int_0^1 dz\, z^{-1+h+\hat{h}+s+\tilde{s}}(1-z)^{a+b} \rightarrow \frac{\Gamma(\Delta+s+\tilde{s})\Gamma(1+a+b)}{\Gamma(1+\Delta+a+b+s+\tilde{s})}$$

$$\mathcal{N}\,\Gamma(1+a+b)\int ds\, d\tilde{s}\, \frac{\Gamma(-s)\Gamma(h+a+s)\Gamma(h+b+s)\,\Gamma(\Delta+s+\tilde{s})}{\Gamma(2h+s)\,\Gamma(1+\Delta+a+b+s+\tilde{s})} \times \hat{h}$$

$$\mathcal{N}\,\Gamma\left(\begin{matrix}1+a+b & \cancel{h+a} & \cancel{h+b}\\ & \cancel{2h} & \end{matrix}\right)\int d\tilde{s}\, \frac{\Gamma(\Delta+\tilde{s})}{\Gamma(1+\Delta+a+b+\tilde{s})}\; {}_3F_2\left(\begin{matrix}h+a & h+b & \Delta+\tilde{s}\\ & 2h & 1+\Delta+a+b+\tilde{s}\end{matrix};1\right)$$

$$\times \frac{\Gamma(-\tilde{s})\,\Gamma(\hat{h}+a+\tilde{s})\Gamma(\hat{h}+b+\tilde{s})}{\Gamma(2\hat{h}+\tilde{s})}$$

$${}_3F_2\left(\begin{matrix}a\, b\, c\\ e\; f\end{matrix};1\right) = \frac{\Gamma(e)\Gamma(1+a-f)\Gamma(1+b-f)\Gamma(1+c-f)}{\Gamma(1-f)\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)}$$

$$= \frac{\Gamma(1+\Delta+a+b+\tilde{s})\Gamma(1+a-h)\Gamma(1+b-h)\,\Gamma(1-J+\tilde{s})}{\Gamma(1-2h)\,\Gamma(1+\hat{h}+b+\tilde{s})\,\Gamma(1+\hat{h}+a+\tilde{s})\,\Gamma(1+a+b)}$$

$$\Rightarrow \mathcal{N}\,\frac{\Gamma(1+a-h)\Gamma(1+b-h)}{\Gamma(1-2h)}\int d\tilde{s}\, \frac{\Gamma(-\tilde{s})\,\Gamma(\Delta+\tilde{s})\,\Gamma(\hat{h}+a+\tilde{s})\Gamma(\hat{h}+b+\tilde{s})\Gamma(1-J+\tilde{s})}{\Gamma(2\hat{h}+\tilde{s})\,\Gamma(1+\hat{h}+b+\tilde{s})\,\Gamma(1+\hat{h}+a+\tilde{s})}$$

$$\Rightarrow \mathcal{N}\,\Gamma\left(\begin{matrix}1+a-h & 1+b-h & \cancel{\hat{h}+a} & \cancel{\hat{h}+b} & \Delta & 1-J\\ & 1-2h & \cancel{2\hat{h}} & & 1+\hat{h}+b & 1+\hat{h}+a\end{matrix}\right)$$

$$\times\; {}_4F_3\left(\begin{matrix}\Delta & 1-J & \hat{h}+a & \hat{h}+b\\ 1+\hat{h}+b & 1+\hat{h}+a & 2\hat{h} & \end{matrix}\right)$$

$$(-1)^s \frac{\Gamma(-s)\,\Gamma(h+a+s)\,\Gamma(h+b+s)\,\Gamma(\Delta+s+\tilde{s})}{\Gamma(2h+s)\,\Gamma(1+\Delta+a+b+s+\tilde{s})}$$

$$= \frac{\Gamma(-s)\sin(\pi(1+\Delta+a+b+s+\tilde{s}))}{\pi} \frac{\Gamma(h+a+s)\,\Gamma(h+b+s)\,\Gamma(\Delta+s+\tilde{s})\,\Gamma(-\Delta-a-b-s-\tilde{s})}{\Gamma(2h+s)}$$

$$= \frac{1}{\pi} \frac{\cancel{\Gamma(h+a)}\,\cancel{\Gamma(h+b)}\,\Gamma(\Delta+\tilde{s})\,\Gamma(-\tilde{h}-a-\tilde{s})\,\Gamma(-\tilde{h}-b-\tilde{s})}{\cancel{\Gamma(2h)}}$$

$$\Gamma(-\tilde{s}) \frac{\Gamma(\Delta+\tilde{s})\,\Gamma(-\tilde{h}-a-\tilde{s})\,\Gamma(-\tilde{h}-b-\tilde{s})\,\Gamma(\tilde{h}+a+s)\,\Gamma(\tilde{h}+b+s)}{\Gamma(2h+s)}$$

$\sin(h+a+s)$

$\sin(h+b+s)$

Integral vl wont:

$$\int_0^1 z^{-1+h+\tilde{h}}(1-z)^{a+b} \; {}_2F_1\left(\begin{matrix} h+a & h+b \\ 2h \end{matrix}; z\right) {}_2F_1\left(\begin{matrix} \tilde{h}+a & \tilde{h}+b \\ 2\tilde{h} \end{matrix}; z\right) + (\text{Shadow})$$

take $a=b=0$

$$\int ds\, \Gamma(-s) \frac{\Gamma(h+s)\Gamma(h+s)}{\Gamma(2h+s)} \frac{\Gamma(\Delta+s+\tilde{s})\Gamma(1)}{\Gamma(1+\Delta+s+\tilde{s})}$$

$$\int d\tilde{s}\, \Gamma(-\tilde{s}) \frac{\Gamma(\tilde{h}+\tilde{s})\Gamma(\tilde{h}+\tilde{s})}{\Gamma(2\tilde{h}+\tilde{s})} \; {}_3F_2\left(\begin{matrix} \Delta+\tilde{s} & h & h \\ 1+\Delta+\tilde{s} & 2h \end{matrix}; 1\right) \Gamma(\Delta+\tilde{s})$$

singular at $\Delta \in -1-\mathbb{N}-\tilde{s}$
$2h \in -\mathbb{N}$

$$\frac{\Gamma(\tilde{h}+a+s)\Gamma(\tilde{h}+b+s)}{\Gamma(2\tilde{h}+s)} \Gamma(h+a+s)\Gamma(h+b+s)\Gamma(-a-b-s) \int_0^1 z^{-1+h+\tilde{h}+\tilde{s}} (1-z)^s$$

mixed wont work

$$\frac{\Gamma(\Delta+\tilde{s})\Gamma(1+s)}{\Gamma(\Delta+s+\tilde{s}+1)}$$

Back to my way:

$$\int \underset{(-1)^{s+\tilde s}}{\Gamma(-s)\Gamma(-\tilde s)}\; \underset{\times \tilde s,\ \tilde h}{\Gamma(h+a+s)}\Gamma(h+b+s)\Gamma(-a-b-s)\; \underbrace{\int_0^1 dz\, z^{-1+\Delta}(1-z)^{a+b+s+\tilde s}}$$

$$\int \underset{\times \tilde s\ \tilde h}{\Gamma(-\tilde s)}\; \Gamma(h+a+s)\Gamma(h+b+s)\Gamma(-a-b-s)\frac{\Gamma(\Delta)\,\Gamma(1+a+b+s+\tilde s)}{\Gamma(\Delta+a+b+s+\tilde s)}$$

$$= {}_3F_2\left(\begin{matrix} h+a & h+b & 1+a+b+\tilde s \\ 1+a+b & \Delta+a+b+\tilde s & \end{matrix} ; 1\right)\frac{\Gamma(1+a+b+\tilde s)}{\Gamma(\Delta+a+b+\tilde s)}$$

seems like I can't get past this...

$$\int ds\, \Gamma(-s)\,\frac{\Gamma(h+s)\Gamma(h+s)}{\Gamma(2h+s)}\,\frac{\Gamma(\Delta+s+\tilde s)}{\Gamma(1+\Delta+s+\tilde s)}$$

massage to Barnes II

$$\int ds\, \Gamma(-s)\,\frac{\Gamma(h+s)\Gamma(h+s)}{\Gamma(2h+s)}\,\frac{\Gamma(\Delta+s+\tilde s)\Gamma(-\Delta-s-\tilde s)}{\pi}\,\frac{\sin[\pi(\Delta+s+\tilde s)]}{\sin(\pi\Delta)\cancel{(-1)^{s+\tilde s}}}$$

$$\frac{\Gamma(-s)\,\Gamma(h+s)\Gamma(h+s)\Gamma(\Delta+s+\tilde s)\Gamma(-\Delta-s-\tilde s)}{\Gamma(2h+s)}$$

$$= \frac{\cancel{\Gamma(h)}\cancel{\Gamma(h)}\,\Gamma(\Delta+\tilde s)}{\cancel{\Gamma(h)}\cancel{\Gamma(h)}}\;\Gamma(h-\Delta-\tilde s)\,\Gamma(h-\Delta+\tilde s)\,\Gamma(0)\,??$$

$$\Gamma(2h-\Delta-\tilde s)$$

$$\int ds \, \frac{\Gamma(-s)}{(-1)^s} \frac{\Gamma(h+s)\Gamma(h+s)}{\Gamma(2h+s)} \qquad \frac{\Gamma(\Delta+s+\tilde{s})\Gamma(-\Delta-s-\tilde{s})}{\pi}$$

$$\sim \Gamma(-s) \frac{\sin(2h)}{\pi} \, \Gamma(h+s)\Gamma(h+s)\Gamma(1-2h-s) \frac{\Gamma(\Delta+s+\tilde{s})}{\Gamma(1+\Delta+s+\tilde{s})}$$

actually nontriv unless pole sequences coincide!

$$\frac{\Gamma(h)\Gamma(h)\Gamma(1-h)\Gamma(1-h)\Gamma(\Delta+\tilde{s})\Gamma(\Delta+\tilde{s}+1-2h)}{\Gamma(1+\Delta+\tilde{s}-h)\Gamma(1+\Delta+\tilde{s}-h)\Gamma(1)}$$

not 0 or ∞!

$$\Gamma(h)^2\Gamma(1-h)^2 \int \frac{\Gamma(\hat{h}+s)\Gamma(\tilde{h}+s)}{\Gamma(2\hat{h}+s)} \frac{\Gamma(\Delta+\tilde{s})\Gamma(\Delta+\tilde{s}+1-2h)}{\Gamma(1+\Delta+\tilde{s}-h)^2}$$

$$\frac{\Gamma(1-J)\Gamma(\Delta)\Gamma(h)^2\Gamma(1-h)^2}{\Gamma(1+\tilde{h})^2\,\Gamma(2\hat{h})} \qquad {}_4F_3\left(\begin{matrix} \hat{h} & \tilde{h} & h+\hat{h} & 1-h+\hat{h} \\ 2\hat{h} & & 1+\hat{h} & 1+\tilde{h} \end{matrix} ; 1\right)$$

balanced

$$h = \frac{\Delta+J}{2}$$

$$2h = \Delta + J \leftarrow \text{integer}$$

Closing on the left:

$$\frac{\Gamma(h+a+s)\Gamma(\tilde{h}+b+s)}{\Gamma(2h+s)}\int_0^1 z^{-1+\Delta+s+\tilde{s}}(1-z)^{a+b}\,dz = \frac{\Gamma(\Delta+s+\tilde{s})\Gamma(1+a+b)}{\Gamma(2h+s)\Gamma(\Delta+\tilde{s}+a+b+1)}$$

$$\int d\tilde{s}\,(-1)^{\tilde{s}}\Gamma(-\tilde{s})\,\frac{\Gamma(\tilde{h}+a+\tilde{s})\Gamma(\tilde{h}+b+\tilde{s})}{\Gamma(2\tilde{h}+\tilde{s})}\;{}_3F_2\left(\begin{matrix}\Delta+\tilde{s} & h+a & h+b\\ \Delta+\tilde{s}+a+b+1 & 2h & \end{matrix}\right)\frac{\Gamma(\Delta+\tilde{s})}{\Gamma(\Delta+\tilde{s}+a+b+1)}$$

What is the pole structure?

$(\Delta+\tilde{s}+1)(2h)$

$(\Delta+\tilde{s})h+(\Delta+\tilde{s})h+n^2$

$\Rightarrow$ no poles from ${}_3F_2$.

$\tilde{s}=-n$

Poles @ $\tilde{s}=-\Delta-n$

$\tilde{s}=-\tilde{h}-a-n$

$\tilde{s}=-\tilde{h}-b-n$

Saalschutz ✓

$$\frac{(h-a)_n(h-b)_n}{(2h)_n(-a-b)_n}$$

$$\delta_1\sum_n\frac{1}{n!}\frac{\Gamma(\Delta+n)\Gamma(-h+a-n)\Gamma(-h-b-n)}{\Gamma(\underbrace{\tilde{h}-h}_{-J}-n)\Gamma(1+a+b-n)}$$

Agrees w/ Ana

$\Gamma(\Delta+n)\Gamma(1+J+n)$

Ok

$$\sim {}_4F_3\left(\begin{matrix}h-a & h-b & \Delta & 1-J\\ 1+h-a & 1+h-b & 2h & \end{matrix}\right)$$

What goals can I set?

- Understand Volovich's Partial wave calc'n in full
- Confirm the left/right integral expression of ${}_pF_q$
  maybe just works for ${}_pF_p$ or ${}_{p+1}F_p$
- Argue for left/right closure
- Understand inversion prescription
  $\rightsquigarrow \sum_J \int dA$ or $\iint dh\, d\bar{h}$
- Try to get B-function coeffs out of this @ $J = \pm 1$
- In particular, try to argue that the pole sequences $-h-a-n$, $-h-b-n$ don't contrib.
- What if we evaluated everything through integral expressions alone?

$$\int d^2z\, d^2w \quad \frac{z^{-1}(1-z)^{a+b}\,\delta(z-\bar{z})}{}$$

CPWs in Volovich

$$F_+ = z^{-b} \, {}_2F_1\left(\begin{matrix} h+b & 1-h+b \\ 1-a+b & \end{matrix} ; \frac{1}{z}\right) \times c.c.$$

$$F_- = a \leftrightarrow b$$

consider Dolan + Osborne : Further Math. results

A.11 $\quad x' = \dfrac{x}{x-1}$

$$1-(1-x)' = 1 + \frac{1-x}{x} = 1/x$$

Take $x \to 1-x$

Doing Ana's Calc'n:

$$\int_0^1 \frac{dz\,d\bar z}{(z\bar z)^2}\,\delta(iz-i\bar z)(1-\bar z)^{a+b}\underbrace{z^3(1-z)}_{A}{}^{-a-b}\,\Psi_{h,\tilde h}(z,\bar z) \qquad (\tilde a=\tilde b)$$

$$=\int_0^1 dz\, z^{-1}(1-z)^{-a-b} z^{h+\tilde h}\,{}_2F_1\left(\begin{matrix}h+a \quad h+b\\ 2h\end{matrix};z\right){}_2F_1\left(\begin{matrix}\tilde h+a \quad \tilde h+b\\ 2\tilde h\end{matrix};z\right)$$

$$=\int \frac{ds}{2\pi i}\frac{d\tilde s}{2\pi i}(-1)^s\,\Gamma(-s)\,\frac{\Gamma(h+a+s)\,\Gamma(h+b+s)}{\Gamma(2h+s)}\times\left(\begin{matrix}h\to\tilde h\\ s\to\tilde s\end{matrix}\right)$$

$$\times\int_0^1 dz\, z^{-1+h+\tilde h+s+\tilde s}(1-z)^{-a-b}$$

$$=\int \frac{ds}{2\pi i}\frac{d\tilde s}{2\pi i}(-1)^s\,\Gamma(-s)\,\frac{\Gamma(h+a+s)\,\Gamma(h+b+s)}{\Gamma(2h+s)}\times(h\to\tilde h)$$

$$\times\frac{\Gamma(\underbrace{h+\tilde h}_{\Delta}+s+\tilde s)\,\Gamma(1-a-b)}{\Gamma(1+\underbrace{h+\tilde h}_{\Delta}+s+\tilde s-a-b)}$$

Now do s integral, yielding ${}_3F_2$

(next page)

$$\int d\tilde{s}\,(-1)^{\tilde{s}}\,\Gamma(-\tilde{s})\,\frac{\Gamma(\tilde{h}+a+\tilde{s})\Gamma(\tilde{h}+b+\tilde{s})\Gamma(\Delta+\tilde{s})}{\Gamma(2\tilde{h}+\tilde{s})\Gamma(1+\Delta+\tilde{s}+a+b)} \times \Gamma(1+a+b)$$

$$\times\; {}_3F_2\left(\begin{matrix}\Delta+\tilde{s} & h+a & h+b \\ 1+\Delta+a+b+\tilde{s} & 2h & \end{matrix};1\right)$$

poles @ $s=n,\ s=-\Delta-n,\ s=-b-n,\ s=-a-n$

Consider just $s=-n$

$$\Rightarrow \sum_n (-1)^{-\Delta}\frac{\Gamma(\Delta+n)}{n!}\frac{\Gamma(-h+a-n)\Gamma(-h+b-n)}{\Gamma(-J-n)\Gamma(1+a+b-n)} \times \underbrace{{}_3F_2\ldots}_{\text{Saalschutz}} \quad \Gamma(1+a+b)$$

$${}_3F_2\left(\begin{matrix}-n & h+a & h+b \\ 1+a+b-n & 2h & \end{matrix}\right) = \frac{\Gamma(1+a+b-n)\Gamma(1-h+a)\Gamma(1-h+b)\Gamma(1-2h-n)}{\Gamma(1-2h)\Gamma(1-h+b-n)\Gamma(1-h+a-n)\Gamma(1+a+b)}$$

$$\sum_n (-1)^{-\Delta}\frac{\Gamma(\Delta+n)}{n!}\frac{\Gamma(-h+a-n)\Gamma(-h+b-n)\Gamma(1-2h-n)\Gamma(1-h+a)\Gamma(1-h+b)}{\Gamma(-J-n)\Gamma(1-2h)\Gamma(1-h+b-n)\Gamma(1-h+a-n)}$$

$$\frac{\Gamma(1-x)}{\Gamma(1-x-n)} = \frac{\Gamma(x+n)}{\Gamma(x)}\overbrace{\frac{\sin(x-n)}{\sin x}}^{(-1)^n} = (-1)^n (x)_n$$

$$= \Gamma(a-h)\Gamma(b-h)\sum_n (-1)^{-\Delta}\frac{\Gamma(\Delta+n)}{n!}\frac{1}{\Gamma(-J-n)(2h)_n}\frac{(h-a)_n(h-b)_n}{(1+h-a)_n(1+h-b)_n}$$

$$= (-1)^{-\Delta}\frac{\Gamma(\Delta)\Gamma(a-h)\Gamma(b-h)}{\Gamma(-J)}\,{}_4F_3\left(\begin{matrix}\Delta & 1+J & h-a & h-b \\ 2h & & h-a+1 & h-b+1\end{matrix}\right)$$

$$\Rightarrow (-1)^{-\Delta}\frac{\Gamma(\Delta)}{\Gamma(-J)}\,\frac{\Gamma(2\tilde{h})\Gamma(a-h)\Gamma(b-h)}{\Gamma(\tilde{h}+a)\Gamma(\tilde{h}+b)}\,{}_4F_3\left(\begin{matrix}\Delta & 1+J & h-a & h-b \\ 2h & h-a+1 & h-b+1 & \end{matrix}\right)$$

$$K_{\Delta,J} = (-1)^J\frac{\Gamma(2h-1)}{\Gamma(2-2\tilde{h})}\,\frac{\Gamma(1-\hat{h}-a)\Gamma(1-\hat{h}+a)}{\Gamma(h-a)\Gamma(h+a)}$$

$$\Rightarrow K_{\tilde{\Delta},J} = (-1)^J\frac{\Gamma(1-2\tilde{h})\Gamma(h-a)\Gamma(h+a)}{\Gamma(2h)\Gamma(1-\hat{h}+a)\Gamma(1-\hat{h}+a)}$$

I took a break
on June 10

K. Symanzik –

On Calc'ns in Conf Inv. Theories

Look at this integral:
(Euclidean metric)

$$I(x_1 \cdots x_n, h_1 \cdots h_n) = \frac{1}{\sqrt{\pi}^D} \int d^D u \prod_i \frac{\Gamma(\delta_i)}{[(u-x_i)^2]^{\delta_i}}$$

Notice the $\Gamma(\delta_i)$ factors he has

We need $\sum \delta = D$

Wicks thm?

Write this as $\frac{1}{\sqrt{\pi}^D} \int \prod_i d\alpha_i \, \alpha^{\delta_i - 1} \int du \, \exp\left[-\sum_i \alpha_i (u - x_i)^2\right]$

$$= \int \prod_i d\alpha_i \, \alpha_i^{\delta_i - 1} \frac{1}{(\sum \alpha_i)^{D/2}} \exp\left[- \frac{\sum_{i<j} \alpha_i \alpha_j (x_i - x_j)^2}{\sum_i \alpha_i} \right]$$ ✓

Consider more generally

$$\int \prod_i d\alpha_i \, \alpha_i^{\delta_i - 1} (\sum \kappa_i \alpha_i)^{-D/2} \exp\left[- \frac{\sum_{i<j} \alpha_i \alpha_j x_{ij}^2}{\sum_i \kappa_i \alpha_i} \right]$$

CPWs:

Appendix A of Dolan & Osborne - Further Mathematical...

Want to evaluate: $I = \frac{1}{\pi}\int d^2 z \prod_{i=1}^{n} \frac{1}{(z-z_i)^{h_i}} \frac{1}{(\bar{z}-\bar{z}_i)^{\bar{h}_i}}$

$\underbrace{\sum h_i = 2}_{\text{For } I \text{ to be invariant}}$, $\sum \bar{h}_i = 2$ $\underset{\text{in } z}{\sim}$ $\underbrace{h_i - \bar{h}_i \in \mathbb{Z}}_{\text{For } I \text{ to be single-valued}}$

avoidence of singularities requires $h_i + \bar{h}_i < 2 \quad \forall i$

$n=2$:

$$I_2 = \frac{1}{\pi}\int \frac{d^2 z}{z^{h_1}\bar{z}^{\bar{h}_1}(z-1)^{2-h_1}(\bar{z}-1)^{2-\bar{h}_1}}$$

First, $$\int_{-\infty}^{\infty} \frac{dz}{z^h (z-1)^{2-h}} = \underbrace{\int_{-\infty}^{0} \cdots}_{\frac{1}{1-h}} + \underbrace{\int_{0}^{\infty}}_{\frac{-1}{1-h}} = 0 \qquad /.\ \mathrm{Re}\, h < 1$$

```
In[34]:= Integrate[1/((z)^h (z - 1)^(2-h)), {z, -Infinity, 0}]
         Integrate[1/((z)^h (z - 1)^(2-h)), {z, 0, Infinity}, PrincipalValue -> True]
         Integrate[1/((z)^h (z - 1)^(2-h)), {z, 1, Infinity}]
Out[34]= ConditionalExpression[1/(1 - h), Re[h] < 1]
Out[35]= ConditionalExpression[1/(-1 + h), Re[h] < 1]
Out[36]= ConditionalExpression[1/(-1 + h), Re[h] > 1]
```

So if $z_1 \neq z_2$, 0

If $z_1 = z_2$, we instead get

$$\int \frac{d^2 z}{z^2 \bar{z}^2} = \infty$$

how do we get $\delta^2(z_1 - z_2)$ generally?

In general:

$$\underbrace{\left[\partial_z^2 + \partial_z \sum_{i=2}^{n} \frac{h_i}{z-z_i}\right]}_{\text{total derivative in } z} f_n(z) = \left[\partial_{z_1}^2 + \sum_{i=2}^{n} \frac{1}{z_{i1}}\left(h_i \partial_{z_i} - h_1 \partial_{z_1}\right)\right] f_n(z)$$

$$\Rightarrow \left[\partial_{z_1}^2 + \sum_{i=2}^{n} \frac{1}{z_{i1}}\left(h_i \partial_{z_i} - h_1 \partial_{z_1}\right)\right] I = 0 \qquad + \text{c.c}$$

can I interpret all this through surface op. deformation?

$n=3$: Conformal invariance demands:

$$I = K_{123}\, z_{12}^{h_3-1} z_{23}^{h_1-1} z_{31}^{h_2-1} \times c.c.$$

take $z_1=0$, $z_2=1$, $z_3\to\infty$

$$\frac{1}{\infty^{h_3}}\int_{-\infty}^{\infty} \frac{dz}{z^{h_1}(z-1)^{h_2}} = \frac{1}{\infty^{h_3}}$$

I don't know how to do this

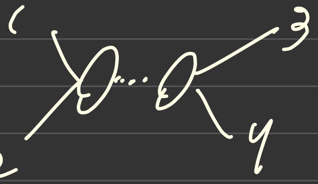
$n=4$ requires:

$$I = \frac{1}{\pi}\int d^2z \prod_{i=1}^{n} \frac{1}{(z-z_i)^{h_i}} \frac{1}{(\bar{z}-\bar{z}_i)^{\bar{h}_i}}$$

Why this form?

$$= z_{12}^{h_3+h_4-1}\, z_{23}^{h_2+h_3-1}\, z_{31}^{h_1-1}\, z_{24}^{-h_4} \times c.c. \quad I$$

For CPW we have:



$$= \int d^dx \langle O(x_1) O(x_2) O(x)\rangle \langle O(x_3) O(x_4) \tilde{O}(x)\rangle$$

$$= \frac{1}{z_{12}^{\Delta_1+\Delta_2-\Delta}\, z_{34}^{\Delta_3+\Delta_4-\Delta}} \int \frac{\cdots}{(z-z_1)^{\Delta+\Delta_1-\Delta_2}}$$

$$\begin{aligned}
h_1 &= h + h_1 - h_2 = h - a \\
h_2 &= h + h_2 - h_1 = h + a \\
h_3 &= 1 - h + h_3 - h_4 = 1 - h - b \\
h_4 &= 1 - h + h_4 - h_3 = 1 - h + b
\end{aligned}$$

$$\Rightarrow \tilde{I} = K_4\, F(1-h-a,\, 1-h+b,\, 2-2h;\, z) \times c.c.$$

$$+ \tilde{K}_4\, (-1)^{h_1+h_4-\bar{h}_1-\bar{h}_4}\, z^{2h-1} F(h+b,\, h-a,\, 2h;\, z) \times c.c.$$

$$z_{12}^{1-2h}\, z_{23}^{h_{12}-h_{34}}\, z_{31}^{-1-h-a}\, z_{24}^{-1+h-b}$$

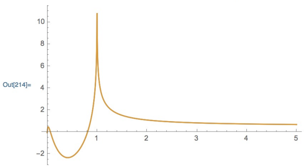
I just want Valovich's result.

First note $\underbrace{\frac{2\pi}{\sin \pi h}\, {}_2F_1\left(\begin{matrix}1-h & h\\ & 1\end{matrix}; \frac{1}{z}\right)}_{\text{Re @}} = \underbrace{\frac{\Gamma(h)^2}{\Gamma(2h)}\, {}_2F_1\left(\begin{matrix}h & h\\ & 2h\end{matrix}; z\right) + h \to 1-h}_{\text{Re @}}$

```
In[214]:= Plot[
  {2 Pi/Cosh[Pi I (h - 1/2)] Re[Hypergeometric2F1[1 - h, h, 1, 1/z]],
   Re[(Gamma[h]^2/Gamma[2 h] z^h Hypergeometric2F1[h, h, 2 h, z] +
      Gamma[1 - h]^2/Gamma[2 - 2 h] z^(1-h) Hypergeometric2F1[1 - h, 1 - h, 2 - 2 h, z])]} /. h → 1/2 + I //
  Evaluate, {z, 0, 5}, PlotRange → All]
```

sin πh

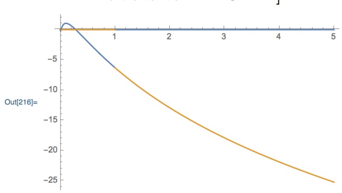
Out[214]=



As we know from experience, shouldnt expect Im to match:

```
In[216]:= Plot[
  {2 Pi/Cosh[Pi I (h - 1/2)] Im[Hypergeometric2F1[1 - h, h, 1, 1/z]],
   Im[(Gamma[h]^2/Gamma[2 h] z^h Hypergeometric2F1[h, h, 2 h, z] +
      Gamma[1 - h]^2/Gamma[2 - 2 h] z^(1-h) Hypergeometric2F1[1 - h, 1 - h, 2 - 2 h, z])]} /. h → 1/2 + I //
  Evaluate, {z, 0, 5}, PlotRange → All]
```

Out[216]=



I think the general prescription may be like $\frac{f(+i\epsilon)+f(-i\epsilon)}{2}$

Should we really expect these s-channel waves to be ok for $z>1$?

⤳ Understand what happens when representing restricted function in a complete basis

Taking external dimensions distinct

$${}_2F_1\left({a\ b \atop c}; x\right) = (1-x)^{-b} F\left({c-a\ \ b \atop c}; \frac{x}{x-1}\right)$$

$${}_2F_1\left({a\ b \atop c}; x\right) = (1-x)^{-a} F\left({a\ \ c-b \atop c}; \frac{x}{x-1}\right)$$

$$\Rightarrow z^h\, {}_2F_1\left({h+a\ \ h+b \atop 2h}; z\right) = z^h (1-z)^{-h-a}\, {}_2F_1\left({h+a\ \ h-b \atop 2h}; \frac{z}{z-1}\right)$$

$$= z^h (1-z)^{-h-b}\, {}_2F_1\left({h-a\ \ h+b \atop 2h}; \frac{z}{z-1}\right)$$

From Kummer:

$$w_1 = F\left({a\ b \atop c}; z\right)$$

$$w_5 = (-1)^a z^{-a} F\left({a\ \ a-c+1 \atop a-b+1}; 1/z\right)$$

$$w_6 = (-1)^b z^{-b} F\left({b\ \ b-c+1 \atop b-a+1}; 1/z\right)$$

Mathematica doesn't like this...

$$w_1 = \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)} w_5 + \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)} w_6$$

Yes it does!

$$\Rightarrow z^h F\left({h+a\ \ h+b \atop 2h}; z\right) = \frac{\Gamma(2h)}{\Gamma(h+b)}\frac{\Gamma(b-a)}{\Gamma(h-a)} (-1)^{h+a} z^{\cancel{-h}-a} F\left({h+a\ \ 1-h+a \atop 1+a-b}; \frac{1}{z}\right)$$

$$+ \frac{\Gamma(2h)}{\Gamma(h+a)}\frac{\Gamma(a-b)}{\Gamma(h-b)} (-1)^{h+b} z^{\cancel{-h}-b} F\left({h+b\ \ 1-h+b \atop 1-a+b}; \frac{1}{z}\right)$$

Starting to look like Volovich

From Osborn 2012

$$K_{h,\bar h} = \frac{\Gamma(2-2\bar h)}{\Gamma(2h-1)} \Rightarrow K_{1-h,1-\bar h} = \frac{\Gamma(2\bar h)}{\Gamma(1-2h)} = (-1)^{2(\bar h-h)}\frac{\Gamma(2h)}{\Gamma(1-2\bar h)}$$

$$\underset{\text{“}\Psi\text{”}}{\underset{\uparrow}{I}} = \frac{\Gamma(1-2\bar h)}{\Gamma(2h)}\,\frac{\Gamma(\bar h-a)\Gamma(\bar h+b)}{\Gamma(1-h+a)\Gamma(1-h-b)}\, z^h \bar z^{\bar h}\, {}_2F_1\left(\begin{matrix}h+a & h+b\\ 2h\end{matrix}; z\right){}_2F_1\left(\begin{matrix}\bar h+a & \bar h+b\\ 2\bar h\end{matrix}; \bar z\right)$$

$$+\frac{\Gamma(2h-1)}{\Gamma(2-2\bar h)}\,\frac{\Gamma(1-\bar h-a)\Gamma(1-\bar h+b)}{\Gamma(h+a)\Gamma(h-b)}\, z^{1-h}\bar z^{1-\bar h}\,{}_2F_1\left(\begin{matrix}1-h+a & 1-h+b\\ 2-2h\end{matrix}; z\right){}_2F_1\left(\begin{matrix}1-\bar h+a & 1-\bar h+b\\ 2-2\bar h\end{matrix}; \bar z\right)$$

Just look @ First term. Expanding gives 4 terms

$$\left[\frac{\Gamma(2h)\Gamma(b-a)}{\Gamma(h+b)\Gamma(h-a)}(-1)^{h+a} z^{-a} F\left(\begin{matrix}h+a & 1-h+a\\ 1+a-b\end{matrix};\frac1z\right) + \frac{\Gamma(2h)\Gamma(a-b)}{\Gamma(h+a)\Gamma(h-b)}(-1)^{h+b}z^{-b}F\left(\begin{matrix}h+b & 1-h+b\\ 1-a+b\end{matrix};\frac1z\right)\right]$$

$$\times\left[\begin{matrix}h\to\bar h\\ z\to\bar z\end{matrix}\right]$$

1) $h\bar h$ $\quad (-1)^{\Delta+2a}\;\dfrac{\cancel{\Gamma(2h)}\Gamma(b-a)\Gamma(\bar b-\bar a)}{\Gamma(h+b)\Gamma(h-a)\cancel{\Gamma(\bar h+\bar b)}\cancel{\Gamma(\bar h-\bar a)}}\,\Gamma(2\bar h)\times\dfrac{\Gamma(1-2\bar h)}{\cancel{\Gamma(2h)}}\,\dfrac{\cancel{\Gamma(\bar h-a)}\cancel{\Gamma(\bar h+b)}}{\Gamma(1-h+a)\Gamma(1-h-b)}$

NO $J+a-\bar a$
b/c $\bar z^h F(\cdot;\cdot;\bar z)$
identity is
conjugated

$$= (-1)^{J+a-\bar a}\,\frac{\sin\pi(h+b)\,\sin\pi(h-a)\;\Gamma(b-a)\Gamma(\bar b-\bar a)}{\pi\,\sin 2\pi\bar h} \qquad \times \boxed{\checkmark}$$

$+\; h\to 1-h$
$\quad\;\bar h\to 1-\bar h$

$$\to (-1)^{J+a-\bar a}\,\frac{\sin\pi(b-a)}{\pi}\,\Gamma(b-a)\Gamma(\bar b-\bar a)$$

```
In[354]:= (Sin[Pi (h + b)] Sin[Pi (h - a)] - Sin[Pi (1 - h + b)] Sin[Pi (1 - h - a)]) / (Pi Sin[2 Pi h]) // FullSimplify
Out[354]= -Sin[(a - b) π]/π
```

)

$$= (-1)^{J+a-\bar a}\,\frac{\Gamma(\bar b-\bar a)}{\Gamma(1-b+a)}$$

$$= (-1)^{J+a-\bar a}\,\frac{\sin\pi(b-a)}{\sin\pi(\bar b-\bar a)}\,\frac{\Gamma(b-a)}{\Gamma(1-\bar b+\bar a)}$$

$$\overset{?}{=} (-1)^{J+b-\bar b}\,\frac{\Gamma(b-a)}{\Gamma(1-\bar b+\bar a)}$$

$$\underset{\text{“}\Psi\text{”}}{\underset{\uparrow}{I}} = \frac{\Gamma(1-2\tilde{h})}{\Gamma(2h)} \frac{\Gamma(\tilde{h}-a)\Gamma(\tilde{h}+b)}{\Gamma(1-h+a)\Gamma(1-h-b)} z^{h}\bar{z}^{\tilde{h}} {}_2F_1\left(\begin{matrix} h+a & h+b \\ 2h \end{matrix}; z\right) {}_2F_1\left(\begin{matrix} \tilde{h}+a & \tilde{h}+b \\ 2\tilde{h} \end{matrix}; \bar{z}\right)$$

$$+ \frac{\Gamma(2h-1)}{\Gamma(2-2\tilde{h})} \frac{\Gamma(1-\tilde{h}-a)\Gamma(1-\tilde{h}+b)}{\Gamma(h+a)\Gamma(h-b)} z^{1-h}\bar{z}^{1-\tilde{h}} {}_2F_1\left(\begin{matrix} 1-h+a & 1-h+b \\ 2-2h \end{matrix}; z\right) {}_2F_1\left(\begin{matrix} 1-\tilde{h}+a & 1-\tilde{h}+b \\ 2-2\tilde{h} \end{matrix}; \bar{z}\right)$$

Just look @ First term. Expanding gives 4 terms
from add shadow

$$\left[\frac{\Gamma(2h)}{\Gamma(h+b)} \frac{\Gamma(b-a)}{\Gamma(h-a)} (-1)^{h+a} z^{-a} F\left(\begin{matrix} h+a & 1-h+a \\ 1+a-b \end{matrix}; \frac{1}{z}\right) + \frac{\Gamma(2h)}{\Gamma(h+a)} \frac{\Gamma(a-b)}{\Gamma(h-b)} (-1)^{h+b} z^{-b} F\left(\begin{matrix} h+b & 1-h+b \\ 1-a-b \end{matrix}; \frac{1}{z}\right)\right]$$

$$\times \begin{bmatrix} h \to \tilde{h} \\ z \to \bar{z} \end{bmatrix}$$

2) $$\frac{\Gamma(1-2\tilde{h})}{\cancel{\Gamma(2h)}} \frac{\Gamma(\tilde{h}-a)\Gamma(\tilde{h}+b)}{\Gamma(1-h+a)\Gamma(1-h-b)} \frac{\cancel{\Gamma(2h)}\Gamma(a-b)\Gamma(2\tilde{h})\Gamma(\bar{a}-\bar{b})}{\Gamma(h+a)\Gamma(h-b)\Gamma(\tilde{h}+a)\Gamma(\tilde{h}-b)} \times (-1)^{J+b-\bar{b}}$$

$$= (-1)^{J+b-\bar{b}} \underbrace{\frac{\pi}{\sin 2\pi\tilde{h}} \frac{\Gamma(\tilde{h}-a)\Gamma(\tilde{h}+b)}{\Gamma(\tilde{h}+a)\Gamma(\tilde{h}-b)}}_{} \frac{\Gamma(a-b)\Gamma(\bar{a}-\bar{b})}{\Gamma(1-h+a)\Gamma(1-h-b)\Gamma(h+a)\Gamma(h-b)}$$

$\uparrow$ inv under $h \to 1-h$

$$\underbrace{\frac{\pi}{\sin 2\pi\tilde{h}}\left[1 - \frac{\sin\pi(\tilde{h}-a)\sin\pi(\tilde{h}+b)}{\sin\pi(\tilde{h}+a)\sin\pi(\tilde{h}-b)}\right] \frac{\Gamma(\tilde{h}-a)\Gamma(\tilde{h}+b)}{\Gamma(\tilde{h}+a)\Gamma(\tilde{h}-b)}}_{}$$

$$\frac{\pi \sin(a-b)}{\sin\pi(\tilde{h}+a)\sin\pi(\tilde{h}-b)} \frac{\Gamma(\tilde{h}-a)\Gamma(\tilde{h}+b)}{\Gamma(\tilde{h}+a)\Gamma(\tilde{h}-b)} = \frac{\sin\pi(a-b)}{\pi} \Gamma(\tilde{h}-a)\Gamma(\tilde{h}+b)\Gamma(1-\tilde{h}-a)\Gamma(1-\tilde{h}+b)$$

$$= (-1)^{J+b-\bar{b}} \frac{\Gamma(\bar{a}-\bar{b})}{\Gamma(1-a+b)} \frac{\Gamma(\tilde{h}-a)\Gamma(\tilde{h}+b)\Gamma(1-\tilde{h}-a)\Gamma(1-\tilde{h}+b)}{\Gamma(1-h+a)\Gamma(1-h-b)\Gamma(h+a)\Gamma(h-b)}$$

$$= (-1)^{J+a-\bar{a}} \frac{\Gamma(a-b)\Gamma(\tilde{h}-a)\Gamma(\tilde{h}+b)\Gamma(1-\tilde{h}-a)\Gamma(1-\tilde{h}+b)}{\Gamma(1-\bar{a}+\bar{b})\Gamma(1-h+a)\Gamma(h+a)\Gamma(1-h-b)\Gamma(h-b)}$$

$$= (-1)^{J+a-\bar{a}} \frac{\underline{\Gamma(\bar{h}-a)}\,\Gamma(\bar{h}+b)\,\Gamma(1-\bar{h}-a)\,\underline{\Gamma(1-\bar{h}+b)}}{\underline{\Gamma(1-h+a)}\,\Gamma(h+a)\,\Gamma(1-h-b)\,\underline{\Gamma(h-b)}} \times \frac{\Gamma(a-b)}{\Gamma(1-\bar{a}+b)}$$

$$= (-1)^{J+a-\bar{a}} \frac{\sin(h+a)\sin(h+b)}{\sin(\bar{h}+b)\sin(\bar{h}+a)} \times \boxed{}$$

$\bar{h}+J \qquad \bar{h}+J$

$\leftarrow$ 1? assuming integer spin

yes!

$$= (-1)^{-J+a-\bar{a}} \times 1 \times \boxed{} \quad \checkmark$$

(Fermions would suck...)

Lastly

Show 3), 4) Vanish

$$\left[\frac{\Gamma(2h)\,\Gamma(b-a)}{\Gamma(h+b)\,\Gamma(h-a)} (-1)^{h+a} z^{-a} F\left(\begin{matrix} h+a & 1-h+a \\ 1+a-b \end{matrix}; \frac{1}{z}\right) + \frac{\Gamma(2h)}{\Gamma(h+a)} \frac{\Gamma(a-b)}{\Gamma(h-b)} (-1)^{h+b} z^{-b} F\left(\begin{matrix} h+b & 1-h+b \\ 1-a-b \end{matrix}; \frac{1}{z}\right)\right]$$

$$\times \begin{bmatrix} h \to \bar{h} \\ z \to \bar{z} \end{bmatrix}$$

$\boxed{}$ = shadow-inv.

$$\Rightarrow \frac{\cancel{\Gamma(1-2h)}}{\cancel{\Gamma(2h)}} \underbrace{\frac{\Gamma(\bar{h}-\bar{a})\Gamma(\bar{h}+\bar{b})}{\cancel{\Gamma(1-h+a)}\,\cancel{\Gamma(1-h-b)}}}$$

$$\frac{\sin\pi(h-a)\sin\pi(h+b)}{\sin 2\pi h} \times \frac{\cancel{\Gamma(2h)}\,\cancel{\Gamma(2\bar{h})}\,\Gamma(b-a)\,\Gamma(\bar{a}-\bar{b})}{\cancel{\Gamma(h+b)}\,\cancel{\Gamma(h-a)}\,\Gamma(\bar{h}+\bar{a})\,\Gamma(\bar{h}-\bar{b})} \underbrace{(-1)^{J+a-\bar{b}} z^{-a-\bar{b}} F\left(\begin{matrix} h+a & 1-h+a \\ 1+a-b \end{matrix}; \frac{1}{z}\right) F\left(\begin{matrix} \bar{h}+\bar{b} & 1-\bar{h}+\bar{b} \\ 1-\bar{a}+\bar{b} \end{matrix}; \frac{1}{\bar{z}}\right)}$$

$$+ \begin{matrix} h \to 1-h \\ \bar{h} \to 1-\bar{h} \end{matrix}$$

shadow-inv

$$\to \underbrace{\frac{1}{\pi \sin 2\pi\bar{h}}}_{\text{shadow odd}} \left(\overbrace{\frac{\sin\pi(h-a)\sin\pi(h+b)}{\Gamma(\bar{h}+\bar{a})\Gamma(1-\bar{h}+\bar{a})\Gamma(\bar{h}-\bar{b})\Gamma(1-\bar{h}-\bar{b})} \times \sin\pi(\bar{h}-a)\,\sin\pi(\bar{h}+b)}^{\text{shadow-inv}} \right) \times \boxed{}$$

$\to \int = 0$

Talk w/ Ana + Walker 6/16

- Revisit small z arg for $\langle \Psi_i, \Psi_j \rangle$ orthogonal proof

- Find a proof that Ψ are orthogonal w.r.t. $\int dx$

✓ - Show Ana + Walker pictures of the two rep'ns of waves.

- Think abt. Shu-Heng's example.
  Don't you pick up poles w/ B^2 values after deforming

I don't think so...

- Antipodal matching may matter?

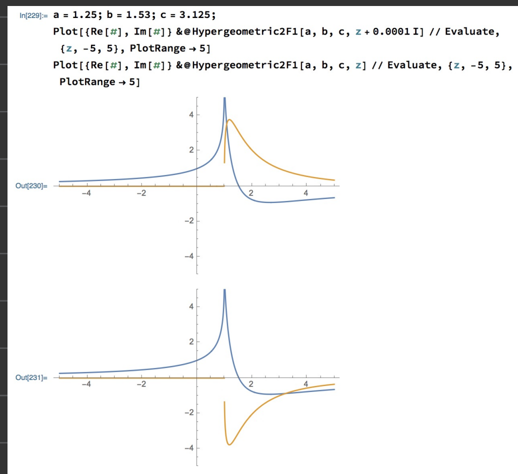
- Think about 1D CPWs

- Revisit Shu-(

- Show Andre Walker pictures of the two rep'ns of waves.

First off, it is worth understanding $_2F_1$ in full!

Plot $_2F_1(\cdot\,;\cdot\,;z)$ @ random $\cdot\,;\cdot$:



Goes as $1 + O(z)$ near $z=0$

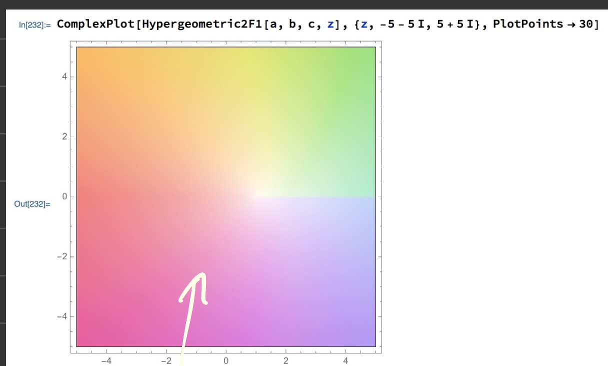
Goes as $(1-z)^{c-a-b}$ near $z=1$

not necessarily a pole

For $z>1$ has branch cut. Mathematica evaluates $z-i\epsilon$

$z+i\epsilon$ is − im part of $z-i\epsilon$

Complex plot:



Smooth everywhere away from $z \geq 1$

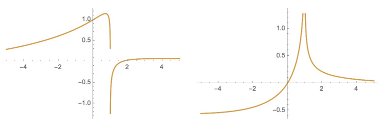
From Kummer:

$$w_1 = F\left({a\ b \atop c}; z\right) \qquad w_5 = (-1)^a z^{-a} F\left({a\quad a-c+1 \atop a-b+1}; 1/z\right)$$
$$w_6 = (-1)^b z^{-b} F\left({b\quad b-c+1 \atop b-a+1}; 1/z\right)$$

$$w_1 = \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}\, w_5 + \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}\, w_6$$

```
a = (1/2 + I + 0.124); b = (1/2 + I - 0.124); c = (1 + 2 I);
GraphicsRow[
 {Plot[
   Re /@ {Hypergeometric2F1[a, b, c, x + 0.0000001 I],
      (Gamma[c] Gamma[b - a])/(Gamma[b] Gamma[c - a]) Exp[Pi I a] x^-a Hypergeometric2F1[a, a - c + 1, 1 + a - b, 1/x - 0.00001 I] +
      (Gamma[c] Gamma[a - b])/(Gamma[a] Gamma[c - b]) Exp[Pi I b] x^-b Hypergeometric2F1[b, b - c + 1, 1 + b - a, 1/x - 0.00001 I]} // Evaluate,
   {x, -5, 5}],
  Plot[
   Im /@ {Hypergeometric2F1[a, b, c, x + 0.0000001 I],
      (Gamma[c] Gamma[b - a])/(Gamma[b] Gamma[c - a]) Exp[Pi I a] x^-a Hypergeometric2F1[a, a - c + 1, 1 + a - b, 1/x - 0.00001 I] +
      (Gamma[c] Gamma[a - b])/(Gamma[a] Gamma[c - b]) Exp[Pi I b] x^-b Hypergeometric2F1[b, b - c + 1, 1 + b - a, 1/x - 0.00001 I]} // Evaluate,
   {x, -5, 5}]
 }]
```

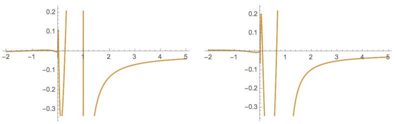
Out[535]=



$$\Rightarrow z^h F\left({h+a\quad h+b \atop 2h}; z\right) = \frac{\Gamma(2h)\,\Gamma(b-a)}{\Gamma(h+b)\,\Gamma(h-a)} (-1)^{h+a} z^{\cancel{h}-a} F\left({h+a\quad 1-h+a \atop 1+a-b}; \frac{1}{z}\right)$$
$$+ \frac{\Gamma(2h)\,\Gamma(a-b)}{\Gamma(h+a)\,\Gamma(h-b)} (-1)^{h+b} z^{\cancel{h}-b} F\left({h+b\quad 1-h+b \atop 1-a+b}; \frac{1}{z}\right)$$

```
In[554]:= a = (0.124); b = (0.524); h = 1/2 + 1.4 I;
GraphicsRow[
 {Plot[
   Re /@ {z^h Hypergeometric2F1[h + a, h + b, 2 h, z + 0.000001 I],
      (Gamma[2 h] Gamma[b - a])/(Gamma[h + b] Gamma[h - a]) Exp[Pi I (h + a)] z^-a Hypergeometric2F1[h + a, 1 - h + a, 1 + a - b, 1/z - 0.000001 I] +
      (Gamma[2 h] Gamma[a - b])/(Gamma[h + a] Gamma[h - b]) Exp[Pi I (h + b)] z^-b Hypergeometric2F1[h + b, 1 - h + b, 1 + b - a, 1/z - 0.000001 I]} // Evaluate,
   {z, -2, 5}],
  Plot[
   Im /@ {z^h Hypergeometric2F1[h + a, h + b, 2 h, z + 0.000001 I],
      (Gamma[2 h] Gamma[b - a])/(Gamma[h + b] Gamma[h - a]) Exp[Pi I (h + a)] z^-a Hypergeometric2F1[h + a, 1 - h + a, 1 + a - b, 1/z - 0.000001 I] +
      (Gamma[2 h] Gamma[a - b])/(Gamma[h + a] Gamma[h - b]) Exp[Pi I (h + b)] z^-b Hypergeometric2F1[h + b, 1 - h + b, 1 + b - a, 1/z - 0.000001 I]} // Evaluate,
   {z, -2, 5}]
 }]
```

Out[555]=



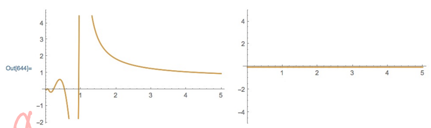
good on both counts

Final Step:

the trick is to be careful w/ $i\epsilon$ perscriptions: $z \to z + i\epsilon$ in Mathematica

$z \to z - i\epsilon$

```
In[643]:= a = (0.124); b = (0.524); h = 1/2 + 1.4 I; hb = h;
GraphicsRow[{Plot[Re /@ {(Gamma[1 - 2 h]/Gamma[2 hb]) (Gamma[hb - a] Gamma[hb + b])/(Gamma[1 - h + a] Gamma[1 - h - b]) (z + I ϵ)^h Hypergeometric2F1[h + a, h + b, 2 h, z + I ϵ] (z - I ϵ)^hb Hypergeometric2F1[hb + a, hb + b, 2 hb, z - I ϵ]
    + (Gamma[2 h - 1]/Gamma[2 - 2 hb]) (Gamma[1 - hb - a] Gamma[1 - hb + b])/(Gamma[h + a] Gamma[h - b]) (z + I ϵ)^(1-h) Hypergeometric2F1[1 - h + a, 1 - h + b, 2 - 2 h, z + I ϵ] (z - I ϵ)^(1-hb) Hypergeometric2F1[1 - hb + a, 1 - hb + b, 2 - 2 hb, z - I ϵ],
    (Gamma[a - b]/Gamma[1 + b - a]) (Gamma[1 - h - a] Gamma[h + b] Gamma[hb - a] Gamma[1 - hb + b])/(Gamma[1 - h + a] Gamma[h - b] Gamma[hb + a] Gamma[1 - hb - b]) (z + I ϵ)^-b Hypergeometric2F1[h + b, 1 - h + b, 1 + b - a, 1/z - I ϵ] (z - I ϵ)^-b Hypergeometric2F1[hb + b, 1 - hb + b, 1 + b - a, 1/z + I ϵ]
    + (Gamma[b - a]/Gamma[1 + a - b]) (z + I ϵ)^-a Hypergeometric2F1[h + a, 1 - h + a, 1 + a - b, 1/z - I ϵ] (z - I ϵ)^-a Hypergeometric2F1[hb + a, 1 - hb + a, 1 + a - b, 1/z + I ϵ]} // Evaluate, {z, 0, 5}],
  Plot[Im /@ {(Gamma[1 - 2 h]/Gamma[2 hb]) (Gamma[hb - a] Gamma[hb + b])/(Gamma[1 - h + a] Gamma[1 - h - b]) (z + I ϵ)^h Hypergeometric2F1[h + a, h + b, 2 h, z + I ϵ] (z - I ϵ)^hb Hypergeometric2F1[hb + a, hb + b, 2 hb, z - I ϵ]
    + (Gamma[2 h - 1]/Gamma[2 - 2 hb]) (Gamma[1 - hb - a] Gamma[1 - hb + b])/(Gamma[h + a] Gamma[h - b]) (z + I ϵ)^(1-h) Hypergeometric2F1[1 - h + a, 1 - h + b, 2 - 2 h, z + I ϵ] (z - I ϵ)^(1-hb) Hypergeometric2F1[1 - hb + a, 1 - hb + b, 2 - 2 hb, z - I ϵ],
    (Gamma[a - b]/Gamma[1 + b - a]) (Gamma[1 - h - a] Gamma[h + b] Gamma[hb - a] Gamma[1 - hb + b])/(Gamma[1 - h + a] Gamma[h - b] Gamma[hb + a] Gamma[1 - hb - b]) (z + I ϵ)^-b Hypergeometric2F1[h + b, 1 - h + b, 1 + b - a, 1/z - I ϵ] (z - I ϵ)^-b Hypergeometric2F1[hb + b, 1 - hb + b, 1 + b - a, 1/z + I ϵ]
    + (Gamma[b - a]/Gamma[1 + a - b]) (z + I ϵ)^-a Hypergeometric2F1[h + a, 1 - h + a, 1 + a - b, 1/z - I ϵ] (z - I ϵ)^-a Hypergeometric2F1[hb + a, 1 - hb + a, 1 + a - b, 1/z + I ϵ]} // Evaluate, {z, 0, 5}, PlotRange -> 5]}]
```



Everything seems to be confirmed.

Also works for $z < 0$ ✓

Working through Ana's note

1D CPWs:

$$\Psi_h = z_{12}^{h_1+h_2} z_{34}^{h_3+h_4} \frac{z_{14}^{h_{12}} z_{13}^{h_{34}}}{z_{24}^{h_{12}} z_{14}^{h_{34}}} \int_{-\infty}^{\infty} dy \, \langle O_1 O_2 O_h(y) \rangle \langle \tilde{O}_h^{+}(y) O_3 O_4 \rangle$$

$$= \cancel{z_{12}^{h_1+h_2}} \cancel{z_{34}^{h_3+h_4}} \frac{z_{14}^{h_{12}} z_{13}^{h_{34}}}{z_{24}^{h_{12}} z_{14}^{h_{34}}} \int_{-\infty}^{\infty} \frac{dy}{z_{12}^{h_1+h_2-h} z_{1y}^{h_1+h-h_2} z_{2y}^{h_2+h_1-h} \times \begin{pmatrix} 1\to 3 \\ 2\to 4 \\ h \to 1-h \end{pmatrix}}$$

$$= \frac{z_{14}^{h_{12}} z_{13}^{h_{34}}}{z_{24}^{h_{12}} z_{14}^{h_{34}}} \int_{-\infty}^{\infty} dy \frac{z_{12}^{h} z_{34}^{1-h}}{z_{1y}^{h_1+h-h_2} z_{2y}^{h_2+h-h_1} z_{y3}^{1-h+h_3-h_4} z_{y4}^{1-h+h_4-h_3}}$$

Inner Product:

"stripped"

$$\langle \Psi_h, f \rangle = \frac{1}{\mathrm{Vol}(SL_2\mathbb{R})} \int_{\mathbb{R}} \frac{dz_1 dz_2 dz_3 dz_4}{z_{12}^2 z_{34}^2} \Psi_h(z_i) f^*(z_i)$$

i.e. shadowed + $h_{ij}^* = -h_{ij}$

Interested in $f = z^\alpha (1-z)^\beta$

cross ratio

Can use $SL_2(\mathbb{R})$ to fix 3 pts

$z_1 = 0$
$z_3 = 1$
$z_4 = \infty$

yields our usual $\int \frac{dz}{z} \int \frac{dy \, z^h}{y^{h_1+h-h_2} (z-y)^{h_2+h-h_1} (1-y)^{1-h+h_3-h_4}}$

Alt: $z_3 = 1$
$z_4 = 0$
$y \to \infty$

yields: $\int dz_1 dz_2 \frac{z_1^{h_{12}-h_{34}} (z_1-1)^{h_{34}}}{z_2^{h_{12}}} z_{12}^{h-2} f^*(z_1, z_2, 1, 0)$

Let's work w/

$z_3 = 1$
$z_4 = 0$
$y \to \infty$

yields: $\int dz_1 dz_2 \, \frac{z_1^{h_{12}-h_{34}} (z_1 - 1)^{h_{34}}}{z_2^{h_{12}}} z_{12}^{h-2} f^*(z_1, z_2, 1, 0)$

take $h_{12} = h_{34} = 0$ for simp.

$$f(z) = \left(\frac{z_{12}}{(z_1 - 1) z_2} \right)^{\alpha} \left(\frac{z_1 (z_2 - 1)}{(z_1 - 1) z_2} \right)^{\beta}$$

$$\langle \Psi_h, f \rangle = \int dz_1 dz_2 \, z_{12}^{h-2+\alpha^*} (z_2 (z_1 - 1))^{-\alpha^* - \beta^*} (z_1 (z_2 - 1))^{\beta^*}$$

we care abt $z \in [0, 1] \Rightarrow \Theta(z) \Theta(1 - z)$

Notes:

- Huge degeneracy @ a given Δ
  $\Rightarrow$ don't expect factorization
- CPWs using trivial 3-pt structure
- Systematically disprove optical thm.
- Factorization of MFT in 1D on [0,1]
  $\rightsquigarrow$ then write $\frac{z}{\sqrt{1-z}}$ in terms of wgt shifting
- Do Volovich in Aras method
  $z>1$, etc
- Just try the $J=1$ coeff in the old way, use your method if necessary!
  $\Downarrow$
  lowest wght state
  $\Rightarrow$ various raising ops can act on this
- things factor w/o $\delta(z-\bar{z})$ Maybe replace w/ $\sum_{n\in\mathbb{Z}}(z-\bar{z})^n$

this seems most promising

Todo over the Weeke

- Cheung Paper
- Ana + Monica OPE Paper
- Soft limits paper

δ-function rep'ns:

$$\delta(x-y) = \sum_{k=-\infty}^{\infty} e^{ik(x-y)}$$

x-y can differ by 2π

$x = -i\log z$, $y = -i\log \bar{z}$

$$\delta(-i\log z + i\log \bar{z}) = \sum_{k=-\infty}^{\infty} e^{i(-i\log z)k} e^{-i(-i\log \bar{z})}$$

ε prescriptions

$$= \sum_{k=-\infty}^{\infty} \left(\frac{z}{\bar{z}}\right)^k = z\,\delta(z-\bar{z})$$

$$\int \cancel{\frac{dz\, d\bar{z}}{z^2 \bar{z}^2}}\, z \ldots \left(\frac{z}{\bar{z}}\right)^k z^h {}_2F_1(\cdot\cdot ; z)\, \bar{z}^{\bar{h}}\, {}_2F_1(\cdot\cdot ; \bar{z})$$

factorizable

I worry about convergence..

→ analytically continue?

Dont forget we have those z^h, $\bar{z}^{\bar{h}}$ which are allowed to oscillate wildly!

Get an answer using that δ-rep'n

- Need to figure out how to split $z^3(1-z)^{-a-b}$

Should be independent, no?

$$\sum_{\lambda\in\mathbb{Z}} \frac{\Gamma(2h)}{\Gamma(h+a)\Gamma(h+b)} \int \frac{ds}{2\pi i} (-1)^s \Gamma(-s) \frac{\Gamma(h+a+s)\Gamma(h+b+s)}{\Gamma(2h+s)} \underbrace{\int_0^1 \frac{dz}{z^2} z^{1+h+\lambda+s}(1-z)^{\beta}}_{\frac{\Gamma(h+\lambda+s)\Gamma(1+\beta)}{\Gamma(1+h+\lambda+s+\beta)}} \times \text{c.c.}$$

$$= \sum_{\lambda\in\mathbb{Z}} \frac{\Gamma(h+\lambda)\Gamma(1+\beta)}{\Gamma(h+\lambda+\beta)}\, {}_3F_2\left(\begin{matrix} h+a & h+b & h+\lambda \\ 2h & 1+h+\lambda+\beta & \end{matrix}\right) \times \text{c.c.}$$

poles @ $h=-\lambda$ poles @ $\bar{h}=+\lambda$

(I may be off by a shift)

I was @ Cape Cod
w/ Nick + Christina
→ Didn't do research on
Tuesday 6/24

$$\frac{\Gamma(2+h+s)}{\Gamma(3+h+s)} \to \frac{\Gamma(2+h)}{\Gamma(3+h)}$$

$$\int \frac{d^2 z}{(z\bar{z})^2}\, z^{3+h}\, \bar{z}^{\bar{h}}\, {}_2F_1\left(\begin{matrix} h\ h \\ 2h \end{matrix}; z\right) {}_2F_1\left(\begin{matrix} \bar{h}\ \bar{h} \\ 2\bar{h} \end{matrix}; \bar{z}\right)$$

$$= \frac{\Gamma(2+h+\lambda)}{\Gamma(3+h+\lambda)}\, {}_3F_2\left(\begin{matrix} h\ h\ 2+h+\lambda \\ 2h\ \ 3+h+\lambda \end{matrix}; 1\right) \frac{\Gamma(-1+\bar{h}-\lambda)}{\Gamma(\bar{h}-\lambda)}\, {}_3F_2\left(\begin{matrix} \bar{h}\ \bar{h}\ -1+\bar{h}-\lambda \\ 2\bar{h}\ \ \bar{h}-\lambda \end{matrix}; 1\right)$$

$$\Gamma\ {}_3F_2\left(\begin{matrix} h\ h\ \ \tfrac{1}{2}+h+\lambda \\ 2h\ \ \tfrac{3}{2}+h+\lambda \end{matrix}; 1\right) {}_3F_2\left(\begin{matrix} \bar{h}\ \bar{h}\ \ \tfrac{1}{2}+\bar{h}-\lambda \\ 2\bar{h}\ \ \tfrac{3}{2}+\bar{h}-\lambda \end{matrix}; 1\right)$$

1) Be sure about pole structure

2) Reproduce Vdovich Poles

4) End goal: Blocks

2D ψ Norm, arbitrary Δ_i

From DSD & Witten "Lorenzian Inversion"

$$K\tilde{K} = \frac{\pi^2}{2^{2J}} \frac{1}{(\Delta+J-1)(1-\Delta+J)}$$

$$n_{\Delta,J} = \frac{\pi^3}{2^{2J}} \frac{1}{\underbrace{(\Delta+J-1)}_{2h-1}\underbrace{(1-\Delta+J)}_{1-2\bar{h}}} \times 2$$

Can I get Volovich from the 5-trick?

$$\int_1^\infty \frac{d^2 z}{(z-\bar{z})^2} \, z^{2+\lambda+h} (1-z)^{-a-b} \, \bar{z}^{-\lambda+h} \, {}_2F_1\left(\begin{matrix} h+a & h+b \\ & 2h \end{matrix} ; z\right) + \text{c.c.}$$

$$\rightarrow \quad z^{2+\lambda+h+s} (1-z)^{-a-b}$$

"Exactly!"
~Andy

G as partition function

Ψ_Δ as partition function for unitary principal series

Ψ is conf. block for unitary principal series

"$z=0$ corresp to $\mathcal{O}^- \mathcal{O}^-$"

- I need to confirm where $\mathcal{O}^-$, $\mathcal{O}^\epsilon$s are
- Understand principal series rep'ns

- Check Ana's calc'n
- Explain $\gamma_a \bar{\gamma}_5$ in Dolan & Osborne
- Understand Lorentz vs Euclidean ← what is there to understand
- Main goal ATM is to recover Volovich
- Work through coeff $\times$ $\Psi \propto \mu$