Mixed Correlators in 3D Ising-like Conformal Field Theories

Alexander B. Atanasov
Department of Physics, Yale University, New Haven, CT, USA

A background is given on the idea of the conformal bootstrap, and the implications of conformal symmetry to the correlators of the quantum fields involved. From here, we use the conformal bootstrap together with semidefinite programming to draw further results on the space of Ising-like conformal field theories investigated previously. In particular, we make use of symmetry in the three-point coefficients to exclude a significant portion of previously-possible points from being valid CFTs, and further successfully implement a scan over a variable θ related to the ratios of the three-point coefficients to further exclude and draw information about points away from the 3D Ising conformal field theory.

I. INTRODUCTION

Quantum theory theory (QFT) studies fields that transform under the Poincare group of spacetime symmetries. By contrast, conformal field theory (CFT) studies quantum fields transforming under the action of the larger conformal group in D-dimensions. Such conformal symmetry arises is hypothesized to arise in a large variety of statistical models at critical phase, as well as being associated to the fixed-points of Wilsonian renormalization group flows [1]. In this sense, we can view UV-complete QFTs as flows between two fixed point CFTs in the UV and IR [2]. Thus, understanding the space of CFTs may allow us to gain further insight into quantum field theory. CFT has been especially promising in $D = 2$ dimensions for solving a wide variety of statistical models at their critical phase [3].

On the other hand, in statistical physics, the phenomenon of critical universality implies that large classes of distinct statistical systems appear to converge to the same model at their critical points [4]. An example of this is that the scaling dimensions describing water at its critical point approach those describing a 3D ferromagnet with spins along one dimension. The latter system is known more broadly as the 3D Ising model, and the statistical model that both systems approach at critical phase is called the 3D Ising CFT. This theory has two relevant operators: σ, ϵ, corresponding to the spin and energy densities, as well as a $\mathbb{Z}_2$ symmetry (corresponding to flipping all the spins) under which σ is odd and ϵ is even.

In this paper, we study the space of 3D "Ising-like" CFTs. That is, CFTs with two relevant operators (i.e. operators with scaling dimensions less than 3), one being $\mathbb{Z}_2$-odd and the other being $\mathbb{Z}_2$-even as before. We denote these operators again as σ, ϵ, respectively. However, we allow for the *scaling dimensions* $(\Delta_\sigma, \Delta_\epsilon)$ to vary. The conformal bootstrap, described in the next section, allows us to use semidefinite programming methods to exclude points in $(\Delta_\sigma, \Delta_\epsilon)$ space from being scaling dimensions of valid CFTs.

II. OVERVIEW OF THE CONFORMAL BOOTSTRAP

Alongside the usual generators of the Poincare group, the conformal group's Lie algebra contains the generators of inversions, K^μ as well as the dilation operator $D = x^\mu \partial_\mu$. A primary quantum field $\mathcal{O}$ (c.f. [2]), i.e. one annihilated by K^μ, has a dilation eigenvalue Δ so that

$$[D, \mathcal{O}(0)] = \Delta \mathcal{O}. \tag{1}$$

Here, Δ is called the scaling dimension of $\mathcal{O}$.

A. Correlation Functions

The observables of a conformal field theory are correlation functions of the fields:

$$\langle 0 | \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) | 0 \rangle \tag{2}$$

Using conformal symmetry, all two-point correlation functions of the theory can be written in the form [2]:

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \frac{C_{12}}{x_{12}^{\Delta_1 + \Delta_2}}. \tag{3}$$

Where C_{12} is a constant depending on only the two fields, Δ_i denotes the scaling dimension of the ith field, and x_{ij} will be shorthand throughout the paper of $|x_i - x_j|$. Further, the three-point correlation functions can also be written exactly:

$$\begin{aligned} &\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle \\ &\quad = \frac{\lambda_{123}}{x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{23}^{\Delta_2 + \Delta_3 - \Delta_1} x_{31}^{\Delta_3 + \Delta_1 - \Delta_2}} \end{aligned} \tag{4}$$

The case of four-point correlators is not so trivial. For a scalar operator, ϕ

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \frac{g(u, v)}{x_{12}^{\Delta_\phi} x_{34}^{\Delta_\phi}} \tag{5}$$

where $g(u, v)$ is a known analytic function that depends, on (u, v) which are themselves functions of the four coordinates x_i.

B. The Operator-Product Expansion

In quantum field theory, the operator-product expansion (OPE) allows us to locally expand a product of quantum fields action on the vacuum as a sum [2]

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\left|0\right\rangle = \sum_k C_{ijk}(x_{12}, P_2)\mathcal{O}_k(x_2)\left|0\right\rangle \qquad (6)$$

although this is only approximate for short distances in QFT, in CFT this is exact, and forms a powerful tool for understanding the correlators. Here, k ranges over the primary operators.

From the state-operator correspondence, we can write directly:

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ijk}(x_{12}, \partial_2)\mathcal{O}_k(x_2) \qquad (7)$$

C. The Conformal Bootstrap

Together with the operator product expansion, we can write

$$\begin{aligned}&\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle\\ &= \sum_{\mathcal{O},\mathcal{O}'} \lambda_{\phi\phi\mathcal{O}}\lambda_{\phi\phi\mathcal{O}'}C_a(x_{12},\partial_2)C_b(x_{34},\partial_4)\left\langle\mathcal{O}^a(x_2)\mathcal{O}'^b(x_4)\right\rangle\end{aligned} \qquad (8)$$

where we sum over primaries. We can choose an orthonormal basis of primaries so that

$$\langle\mathcal{O}^a(x_2)\mathcal{O}'^b(x_4)\rangle = \delta_{OO'}\frac{I^{ab}}{x_{24}^{2\Delta_{\mathcal{O}}}} \qquad (9)$$

and I^{ab} is a tensor that takes into account spins $\ell_{\mathcal{O}}$ in general. Then Equation (8) becomes

$$\frac{1}{x_{12}^{\Delta_\phi}x_{34}^{\Delta_\phi}}\sum_{\mathcal{O}}\lambda^2_{\phi\phi\mathcal{O}}g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u,v) \qquad (10)$$

here $g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}$ is called a conformal block, and is an analytic function defined as

$$g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}} := x_{12}^{\Delta_\phi}x_{34}^{\Delta_\phi}C_a(x_{12},\partial_2)C_b(x_{34},\partial_4)\frac{I^{ab}_\ell}{x_{24}^{2\Delta}} \qquad (11)$$

This, together with Equation (5) gives the conformal block decomposition of $g(u,v)$

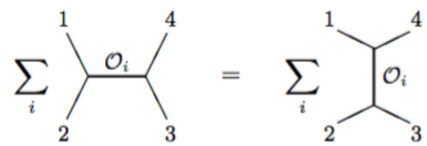
$$g(u,v) = \sum_{\mathcal{O}}\lambda^2_{\phi\phi\mathcal{O}}g_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u,v) \qquad (12)$$

This is the conformal block decomposition.

The critical observation is that the OPE must be associative:

$$(\mathcal{O}_1\mathcal{O}_2)\mathcal{O}_3 = \mathcal{O}_1(\mathcal{O}_2\mathcal{O}_3) \qquad (13)$$

Contracting both sides with a fourth $\mathcal{O}_4$ gives rise to the following diagrammatic result from section 10 of [2]:



The four-point correlators should be independent of order of contraction.

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \qquad (14)$$

At the level of the conformal blocks, this becomes a simple idea:

$$\sum_{\mathcal{O}}\lambda^2_{\phi\phi\mathcal{O}}(v^{\Delta_\phi}g_{\Delta,\ell}(u,v) - u^{\Delta_\phi}_{\Delta,\ell}g(v,u)) = 0. \qquad (15)$$

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D. Reduction to Semidefinite Programming

Equation (15) can be interpretted in vector-space language as

$$\sum_{\Delta,\ell}p_{\Delta\ell}\vec{F}^{\Delta_\phi}_{\Delta,\ell} \qquad (16)$$

where now $\vec{F}^{\Delta_\phi}_{\Delta,\ell} = v^{\Delta_\phi}g_{\Delta,\ell}(u,v) - u^{\Delta_\phi}_{\Delta,\ell}g(v,u)$ is interpreted as a vector on the space of functions of u, v, and Δ, ℓ run over the scaling dimensions of the primary operators. Note further that $p_{\Delta,\ell} = \lambda^2_{\phi\phi\mathcal{O}}$ is a set of positive real numbers.

Therefore, if we can find a separating hyperplane in this vector space, so that all $\vec{F}$ lie on one side, we cannot have a nontrivial combination of $\vec{F}$ be zero. This is the task of finding a linear functional α that is non-negative on all $\vec{F}_{\ell,\Delta}$ and strictly positive on at least one. If such an α exists, then the hypothesis is wrong and no CFT exists. If an α cannot be found, then a CFT could possibly exist, for this scaling dimension.

Recent work on this semidefinite programming approach [5] has led to the development of the SDPB program, into which coefficient tables associated to the conformal blocks for a given CFT can be fed. For a given order in the expansion, SDPB will determine whether this set of conformal blocks gives rise to a contradiction.

III. RESULTS USING THREE-POINT SYMMETRY

The conformal field theory describing the 3D Ising model has two operators of dimension < 3, namely σ and ϵ. Recall that operators of dimensions less than 3 are particularly important, because they correspond to observable macroscopic fields, and so are called relevant.

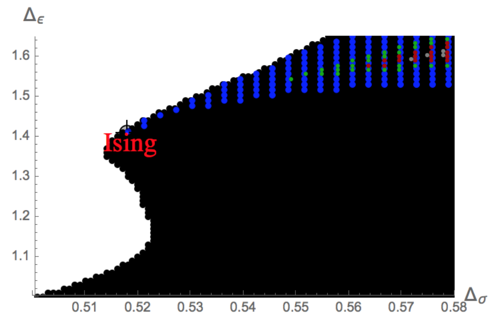

FIG. 1: A plot of the points marking non-excluded CFTs in the $(\Delta_\sigma, \Delta_\epsilon)$ space. The black region are those allowed after using mixed correlators at up to expansion order $\Lambda = 9$ in ∂_u, ∂_v. The colored regions impose three point symmetry at various orders. Note that the regions from $\Lambda = 11$ onwards seise to be connected the Ising island.

While the prior results for the space of Ising-like CFTs used only the constraint from the $\langle\sigma\sigma\sigma\sigma\rangle$ correlator [6], more recent results [7] have made use of $\langle\epsilon\epsilon\epsilon\epsilon\rangle$ and the mixed correlator $\langle\sigma\sigma\epsilon\epsilon\rangle$ as well. For a more complete discussion on the treatment of correlators of mixed scaling dimensions, see [7].

Here, we further make use of this mixed-correlator constraint by making use of the three-point function symmetry

$$\lambda_{\sigma\sigma\epsilon} = \lambda_{\sigma\epsilon\sigma} \tag{17}$$

to further constrain the possible coefficients $p_{\Delta,\ell}$ in the linear combination of Equation (16). This was implemented by modifying pre-existing code in [8].

This has been quite a powerful constraint, as can be seen in Figure 1. Further, a global scan over all values of the scaling dimensions was taken, and reveals that although the three point constraint is effective in excluding a wide variety of CFTs away from the Ising model, there are many CFTs left to be excluded in the region $(\Delta_\sigma, \Delta_\epsilon) \geq (1, 1)$.

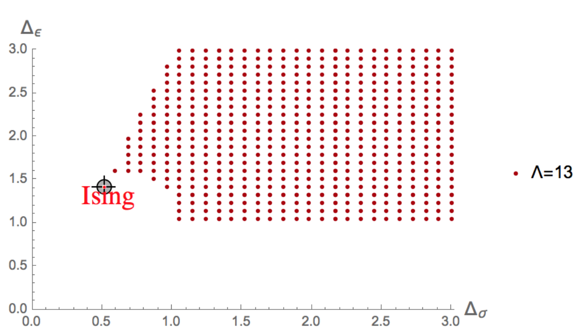

FIG. 2: A global plot of all Ising-like CFTs still not excluded by the three-point symmetry at $\Lambda = 13$.

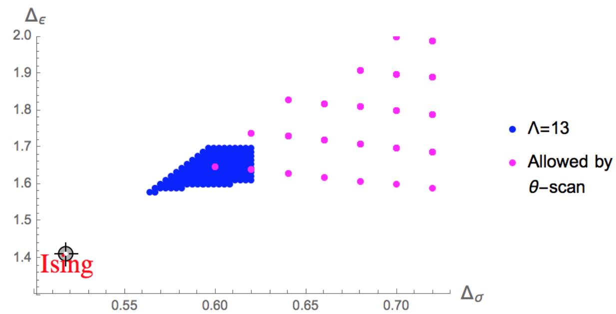

FIG. 3: A plot of the points allowed by θ scan near the 3D Ising model, as well as the non-theta scan three point symmetry constrain from Figure 1. Both are at $\Lambda = 13$. Note the the blue region cuts off at the top and right not because all further points are excluded, but because of computational limits.

IV. RESULTS USING THETA-SCAN

Further, because by $\mathbb{Z}_2$-symmetry, the only nonvanishing three-point functions in σ, ϵ are $\langle\epsilon\epsilon\epsilon\rangle$ and $\langle\sigma\sigma\epsilon\rangle$ = perms., we can define a parameter:

$$\tan\theta := \lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}. \tag{18}$$

In the semidefinite programming, one of the prior constraints required that a certain matrix be positive definite. In fact, it is enough to make sure that this matrix acts positively on the vector $(\lambda_{\sigma\sigma\epsilon}, \lambda_{\epsilon\epsilon\epsilon})$. By scanning over all direction θ and making seeing if this matrix acts positively always, we can exclude the CFT. Note this is a weaker constraint, because the matrix is allowed to vary as theta varies, so it is easier to allow SDPB to find a matrix on a case-by-case basis rather than globally over all theta.

This theta scan has been very fruitful for pinpointing the exact scaling dimensions of the ising model [7]. Here, we use it more globally to push back the allowed CFTs up to $\Delta_\sigma = 0.8$.

V. CONCLUSION AND NEXT STEPS

We have implemented both the three-point symmetry constraint and theta-scan in the cboot package of Ohtsuki, to form conformal block coefficients to be fed into SDPB.

We will now attempt to use the theta-scan at significantly higher Λ, near 30 or 40 which will likely be a significant challenge in high-performance computing. Further, we will trace the size of the region as Λ increases and attempt to draw a conclusion of what will be excluded in the $\Lambda \to \infty$ limit.

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