

# The ADHM Construction, Hilbert Schemes, and the Heisenberg Algebra

## Lecture 2

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### Abstract

Continuing from the previous lecture, we consider the moduli space  $\mathcal{M}^{reg}(n, k)$  of framed “genuine” instantons and extend it to its compactification  $\hat{\mathcal{M}}(n, k)$  of so-called *ideal* instantons. This space has singularities, and we define its resolution  $\tilde{\mathcal{M}}(n, k)$  in terms of the original ADHM data. From here, we study the special case  $\mathcal{M}(1, k)$  as a Hilbert scheme of points on  $\mathbb{C}^2$ . This gives us a way to categorify Heisenberg algebras directly through the homology groups of these spaces.

## 1 Compactification and Resolution of $\mathcal{M}^{reg}(n, k)$

From before we have the ADHM construction:

**Definition 1.1** (ADHM datum). Two hermitian vector space  $V, W$  with  $\dim V = k, \dim W = n$  and maps  $B_1, B_2 \in \text{End}(V), I \in \text{Hom}(W, V), J \in \text{Hom}(V, W)$ . We have that the maps satisfy

$$[B_1, B_2] + IJ = 0 \tag{1}$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \tag{2}$$

Moreover  $U(V)$  acts on this space by

$$(gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \quad (3)$$

**Definition 1.2** ( $\mathcal{M}^{reg}(n, k)$ ). The moduli space of framed genuine instantons is given by taking the set of ADHM data  $[B_1, B_2, I, J]$  with trivial stabilizer under the action of  $U(V)$  and quotienting out by  $U(V)$ .

Equivalently, however, we can “forget” about the Hermitian structure on  $V$  (losing the notion of adjoints) and deal with a general vector space:

**Theorem 1.3.** *Let  $V, W$  be vector spaces of dimension  $\dim V = k, \dim W = n$ , and maps  $B_1, B_2 \in \text{End}(V), I \in \text{Hom}(W, V), J \in \text{Hom}(V, W)$ , satisfying only*

$$[B_1, B_2] + IJ = 0 \quad (4)$$

*And  $\text{GL}(n, \mathbb{C})$  acts on this space in the same way as before, by:*

$$(gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \quad (5)$$

*Then the moduli space  $\mathcal{M}(n, k)$  of all such data with trivial  $\text{GL}_n$  stabilizer, quotiented by  $\text{GL}(n, \mathbb{C})$  is equivalent to  $\mathcal{M}^{fr}(n, k)$ .*

This process of forgetting complex structure and then quotienting out by a larger group is similar to the identification:

$$\text{U}(n)/T_n \cong \text{GL}(n)/B_n \quad (6)$$

Where  $T_n$  is the toroidal subgroup of diagonal matrices and  $B_n$  is the Borel subgroup of upper-triangular matrices.

**Proposition 1.4.** *If ADHM datum  $[B_1, B_2, I, J]$  satisfies either*

1. *(Stability) There is no proper subspace  $S \subset V$  s.t.  $B_i(S) = S$  and  $I(W) \subset S$*
2. *(Co-stability) There is no proper subspace  $S \subset V$  s.t.  $B_i(S) = S$  and  $S \subset \ker J$*

*Then it has both nontrivial stabilizer and a closed orbit under  $\text{GL}(V)$ .*

*Proof.* Assume the stabilizer is nontrivial, so there is a  $g$  s.t.  $gI = I$ . Then  $\text{im } I \in \ker(g - 1_V) =: S$  and moreover since  $g^{-1}B_i g = B_i$ , we have that  $S$  is invariant under the  $B_i$  and contains  $\text{im } I$ . Similarly,  $Jg^{-1} = J \Rightarrow J(g - 1_V) = 0$  so  $\text{im}(g - 1_V) \in \ker J$ , violating co-stability as well.  $\square$

An ADHM datum that is both stable and co-stable is called **regular**.

For the hermitian case, it clear that  $(B_1, B_2, I, J)$  is stable iff  $(B_1^\dagger, B_2^\dagger, I^\dagger, J^\dagger)$  is co-stable. Equation (2) demands compatibility between these descriptions, so in fact it is enough to just require stability on the system, and co-stability follows:

$$\mathcal{M}^{reg}(n, k) = \{\text{Solutions to (1) + (2) + regularity}\}/\text{U}(V)$$

It is not too hard to see that all solutions of the ADHM equations giving rise to instantons are both stable and co-stable. As a result  $\mathcal{M}^{reg}(n, k)$  can be written as

$$\mathcal{M}^{reg}(n, k) = \{\text{Solutions to (1) + regularity}\}/\text{GL}(V)$$

The non-regular case is taken by compactifying this to allow for *all* ADHM data in the quotient:

**Definition 1.5.**  $\hat{\mathcal{M}}(n, k)$ , the compactification of  $\mathcal{M}^{reg}(n, k)$  is defined as

$$\hat{\mathcal{M}} := \{\text{Solutions to (1)}\}/\text{GL}(V)$$

This is a singular space in the sense that any Riemannian metric defined on the entire space will have a singularity at some point. For this reason, we define the **resolution** of this space.

**Definition 1.6** (Resolution). The resolution of a singular space  $X$  is a birational map  $\pi$ , together with a smooth variety,  $\tilde{X}$  such that  $\pi : \tilde{X} \rightarrow X$ .

**Definition 1.7.** We define  $\tilde{\mathcal{M}}(n, k)$  to be the set of data: satisfying

$$\hat{\mathcal{M}} := \{\text{Solutions to (1) + stability}\}/\text{GL}(V)$$

From the proof before, we see stability alone is enough to ensure a trivial stabilizer, so that this quotient is well-defined topologically.

Because stability on  $[B_1, B_2] + IJ$  is the same as co-stability on a corresponding dual system, it would have worked to also only have co-stability. Either definition works, so long as we have one stability condition but not the other.

**Theorem 1.8.**  $\tilde{\mathcal{M}}(n, k)$  is the minimal resolution of  $\hat{\mathcal{M}}(n, k)$  for all  $n, k$ .

**Observation 1.9.** Although  $\mathcal{M}^{reg}(1, k)$  is empty (for, as we know, there are no  $U(1)$  instantons), the definition of  $\tilde{\mathcal{M}}(n, k)$  gives rise to a nonempty set of solutions, as we shall see in the next section.

## 2 Hilbert Schemes of Points on $\mathbb{C}^2$

**Definition 2.1** (Hilbert Scheme). A Hilbert scheme  $\text{Hilb}_n X$  of  $n$  points on an algebraic variety  $X$  is given as the space of all ideals of codimension  $n$  in the  $\mathbb{C}[X]$ . That is, the set of ideals  $I$  so that  $\mathbb{C}[X]/I \cong V$  a vector space of dimension  $n$ .

This can be thought of as the moduli space of arrangements of  $n$  points on  $X$ , with subtleties when the points coincide. Grothendieck showed, through a much more general result, that this space is in fact a scheme.

**Example 2.2.** When  $\dim X = 1$  we have  $\text{Hilb}_n(X) \cong S^n X$  as the set of arrangements of  $n$  points modulo the symmetric group acting on these points by interchange.

**Example 2.3.** When  $X = \mathbb{C}$ , since  $\mathbb{C}[z]$  is a PID, we are interested in ideals  $I$  that are generated by a polynomial  $f$  of degree  $n$ . We are then looking all possible spaces  $V = \mathbb{C}[z]/I \cong \mathbb{C}^n$ .

Every such ideal gives rise to a map  $\varphi : \mathbb{C}[z]/I \rightarrow \mathbb{C}$  mapping

$$z \mapsto B \in \text{End}(V), \quad 1 \mapsto v_0 \in V \quad (7)$$

That is, we can represent multiplication by  $z$  as an operator  $B \in \text{End}(V)$  satisfying  $f(B) = 0$ .

The eigenvalues of  $B$  are exactly the points corresponding geometrically to this ideal (through nullstellensatz). We care about the coordinate-independent data:

$$(B, v_0)/GL(V) \quad (8)$$

Note that because of the existence of a cyclic vector, this system has the stability property: any space containing  $v_0$  and closed under the action of  $B$  is all of  $V$ . This constrains the Jordan form of  $B$ . We can go further and show that there is a 1-to-1 correspondence between such solutions and arrangements of  $n$  points.

In general the form of  $B$  will be

$$\begin{pmatrix} \lambda_1 & 1 & 0 & \dots & \dots \\ 0 & \lambda_1 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & \dots & \lambda_k & 1 \\ 0 & \dots & \dots & 0 & \lambda_k \end{pmatrix} \quad (9)$$

in jordan blocks, where  $\lambda_i \neq \lambda_j$  unless  $i = j$ . Then we can form a map from this data into  $S^n\mathbb{C}$  by  $B \rightarrow \sum_{\lambda \in \mathbb{C}} [\lambda] \lambda$ , where  $[\lambda]$  stands for the multiplicity. This is called the **Hilbert-Chow morphism**. For the case of  $\mathbb{C}$ , it is an isomorphism, as  $B$  can be recovered from the  $n$  points on  $\mathbb{C}$

We will be more interested in the case  $X = \mathbb{C}^2$ , the 2-dimensional complex space.

**Example 2.4.** As before, an element in the Hilbert scheme would correspond to an ideal  $I \in \mathbb{C}[x, y]$  s.t.  $V := \dim \mathbb{C}[x, y]/I = n$ . Consider the map

$$\varphi : \mathbb{C}[x, y] \rightarrow V \text{ s.t. } x \mapsto B_1, y \mapsto B_2, 1 \mapsto v_0 = I(1). \quad (10)$$

Then clearly we have  $[B_1, B_2] = 0$ , so we can write the  $B_k$  simultaneously in upper-triangular form as:

$$B_1 = \begin{pmatrix} \lambda_1 & \dots & \dots & \dots \\ 0 & \lambda_2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}, \quad B_2 = \begin{pmatrix} \mu_1 & \dots & \dots & \dots \\ 0 & \mu_2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \mu_n \end{pmatrix} \quad (11)$$

Moreover we identify the vector  $v_0$  with the embedding operator  $I : \mathbb{C} \rightarrow V$  from a 1-dimensional space into  $V$ , so  $v_0 = I(1)$ .

We further have the Hilbert-Chow morphism to  $S^n\mathbb{C}^2$ :

$$\pi : (B_1, B_2, I) \rightarrow \{(\lambda_i, \mu_i)\}_{i=1}^n, \quad (12)$$

but now we no-longer preserve all information about the matrices, as the upper-triangular structure is unknown given the projection point.

Note that the orbit of  $v_0$  under  $B_1, B_2$  is all of  $V$ , so that any  $(B_1, B_2)$ -stable subspace of  $V$  containing  $v_0 = \text{im } I$  is all of  $V$ . This is exactly the stability condition from before. In fact, aside from  $[B_1, B_2] = 0$  instead of  $[B_1, B_2] + IJ = 0$ , this is exactly  $\tilde{\mathcal{M}}(1, n)$ . In fact, it is exactly this space:

**Theorem 2.5.** *We have an isomorphism of smooth spaces:*

$$\tilde{\mathcal{M}}(1, n) \cong \text{Hilb}_n(\mathbb{C}^2) \quad (13)$$

To prove this theorem, it is enough to just show  $J = 0$ , and we'll be done.

**Proposition 2.6.** *Consider the resolved space  $\tilde{\mathcal{M}}(1, k)$  from before (i.e. only stability), then any solution of ADHM has  $J = 0$ .*

*Proof.* Because stability implies that  $\mathbb{C}[B_1, B_2]I = V$ , it's enough to show  $Jp(B_1, B_2)I = 0$  for any monomial. We do this by induction on degree. For degree 0 we have  $J I = \text{Tr}(J I) = \text{Tr}(I J) = -\text{Tr}([B_1, B_2]) = 0$ .

For higher degree, we can use  $[B_1, B_2] = -J I$  to commute  $B_1, B_2$  across one another to get

$$\begin{aligned} J B_{\alpha_1} \dots B_2 B_1 \dots B_{\alpha_m} I &= J B_{\alpha_1} \dots I J \dots B_{\alpha_m} I + J B_{\alpha_1} \dots B_1 B_2 \dots B_{\alpha_m} I \\ &= J B_{\alpha_1} \dots B_1 B_2 \dots B_{\alpha_m} I \end{aligned} \quad (14)$$

so we can reduce this to  $J B_1^a B_2^b I$  and then use trace properties to get zero.  $\square$

This proves the theorem.

### 3 Homology Theory of $\text{Hilb}_n(\mathbb{C}^2)$

For a closed, oriented manifold  $X$  of dimension  $n$ , we have Poincare duality

$$H_i(X) \cong H^{n-i}(X) \quad (15)$$

When the manifold is not compact, we must pair the cohomology of *compactly-supported forms* with the homology.

$$H_i(X) \cong H_c^{n-i}(X) \quad (16)$$

and similarly, we define the **Borel-Moore homology** of locally finite chains by  $H_i^{lf} \cong H_c^{n-i}(X)$

**Definition 3.1.** The Borel-Moore (locally finite) homology  $H^{lf}$  is equivalent to the relative homology:

1.  $H_i^{lf}(X) := H_i(X \cup \{\infty\}, \{\infty\})$ , the one-point compactification of  $X$

This definition immediately yields

**Proposition 3.2.**  $H_{2m}^{lf}(\mathbb{C}^m) = \mathbb{Z}$  and otherwise is equal to zero.

by recognizing  $\mathbb{C}^m \cup \{\infty\}$  as  $S^{2m}$ . Further:

**Proposition 3.3.** *For a space  $X$  that is a disjoint union of a finite number of open subspaces  $X_\alpha$ :  $H_i^{lf}(X) = \bigoplus_\alpha H_i^{lf}(X_\alpha)$*

**Proposition 3.4.** *The Hilbert scheme  $\text{Hilb}_n(\mathbb{C}^2)$  is a disjoint union of open spaces  $C_\mu$  indexed by the partitions of  $n$ . Moreover,  $C_\mu \cong \mathbb{C}^{n+\ell(\mu)}$ , where  $\ell$  is the number of parts in the partition  $\mu$ .*

*Sketch.* Consider the action of the torus  $T^2 = \langle (t, q) \rangle \curvearrowright \text{Hilb}_n(\mathbb{C}^2)$  by its action on an ideal element  $((t, q)f)(x, y) = f(t^{-1}x, q^{-1}y)$ . The fixed points of this are the ideals

$$I_\mu = (x^a y^b : (a, b) \notin \mu) \quad (17)$$

Where  $\mu$  is viewed as its corresponding Young tableau on the plane  $\mathbb{N}^2$ . We then have corresponding complement ideals:

$$B_\mu = (x^a y^b : (a, b) \in \mu) \quad (18)$$

This gives rise to open sets  $U_\mu$  that cover  $\text{Hilb}_n(\mathbb{C}^2)$  defined by

$$U_\mu := \{I \in \text{Hilb}_n(\mathbb{C}^2) : B_\mu \text{ spans } \mathbb{C}[x, y]/I\} \quad (19)$$

and the closed sub-cells of these open sets are defined by:

$$C_\mu := \{I \in \text{Hilb}_n(\mathbb{C}^2) : \lim_{t \rightarrow 0} \lim_{q \rightarrow 0} (t, q)I = I_\mu\} \quad (20)$$

This limiting process picks out exactly the greatest monomials with nonzero coefficients from all the polynomials of the ideal.  $\square$

**Remark.** This type of idea, of decomposing a space into its different orbits, is universally used in studying not just Hilbert schemes but also projective spaces, Grassmannians, flag varieties, and other such spaces.

**Corollary 3.5.**

$$H_*^{lf}(\text{Hilb}_n(\mathbb{C}^2)) = \bigoplus_\mu [\mathbb{C}_\mu] = \bigoplus_\mu \mathbb{C} \quad (21)$$

where  $[\mathbb{C}_\mu]$  denotes the fundamental Borel-Moore class (the top homology ring), which in this case is exactly the only nonzero one.

**Observation 3.6.** *The dimension of this space  $H_*^{lf}(\text{Hilb}_n(\mathbb{C}^2))$  is  $p(n)$ , the number of partitions of  $n$ .*

Note however, that unlike  $S^n(\mathbb{C}^2)$ , which can also be covered by cells indexed by the partition type,  $\mu$ ,  $\text{Hilb}_n(\mathbb{C}^2)$  has the property that it has **constant rank** for its tangent space, equal to  $2n$ .

**Proposition 3.7.** *We have a graded dimension for the algebra of homology rings for all Hilbert schemes of points over  $\mathbb{C}^2$  of:*

$$\bigoplus_n H_*^{lf}(\text{Hilb}_n(\mathbb{C}^2)) = \sum_n p(n)q^n = \prod_m \frac{1}{1 - q^m} \quad (22)$$

This is exactly the graded dimension of the space of symmetric polynomials  $S(p_1, p_2, \dots)$  where  $\deg p_i = i$ . This is exactly a representation of the Heisenberg algebra, suggesting a connection between these two objects beyond just an isomorphism as graded vector spaces.

**Optional:**

**Theorem 3.8** (From Fulton). *The Poincare polynomial*

$$P_t^{lf}(X) := \sum_{n \geq 0} t^n \dim H_n^{lf}(X) \quad (23)$$

*is equal to*

$$\sum_{\mu} t^{2n+2\ell(\mu)} \quad (24)$$

*for  $X = \text{Hilb}_n(\mathbb{C}^2)$*

**Corollary 3.9.** *We have an identity for the polynomial*

$$\sum_{n=1}^{\infty} q^n P_t^{lf}(H_n) = \prod_{m=1}^{\infty} \frac{1}{1 - t^{2m+2}q^m} \quad (25)$$

**Theorem 3.10.** *The homology group  $H_*[\text{Hilb}_n(\mathbb{C}^2)]$  vanishes on odd degrees and otherwise is torsion free, with betti number;*

$$b_{2i}(\text{Hilb}_n(\mathbb{C}^2)) = p(n, n - i) \quad (26)$$

*where  $p(n, a)$  is the number of partitions of  $n$  into  $a$  parts.*

**Corollary 3.11.** *The Hilbert polynomial*

$$P_t(X) := \sum_{n \geq 0} t^n b_n(X) \quad (27)$$

*is equal to*

$$\sum_{\mu} t^{2n-2\ell(\mu)} \quad (28)$$



**Corollary 3.12.** *We have an identity for the polynomial*

$$\sum_{n=1}^{\infty} q^n P_t(\text{Hib}_n(\mathbb{C}^2)) = \prod_{m=1}^{\infty} \frac{1}{1 - t^{2m-2} q^m} \quad (29)$$

We have constructed  $H_*^{lf}$  and  $H_*$ , and in fact all the nontrivial topological information is contained in the zero fiber  $Z_n$  of  $\pi : \text{Hilb}_n(\mathbb{C}^2) \rightarrow S^n(\mathbb{C}^2)$ .

## 4 Hilbert Schemes and the Heisenberg Algebra

The Heisenberg Lie Algebra  $\mathfrak{h}$  is defined by generators  $p_i, q_i$  so that:

$$[q_j, p_i] = c_j \delta_{ij}, \quad c_j \in \mathbb{C}^\times \quad (30)$$

so that on  $S(p_1, p_2, \dots)$  the  $p_j$  act by multiplication and the  $q_j$  by derivation  $c_j \frac{\partial}{\partial p_j}$ . We can pair  $\bigoplus_j p_j$  and  $\bigoplus_i q_i$  as dual vector spaces by:

$$\langle q_j, p_i \rangle = c_j \delta_{ij} \quad (31)$$

and with this can define the dual space

$$S^* = S(q_1, q_2, \dots) \quad (32)$$

with  $\deg q_j = j$  as before, and satisfying:

$$\langle q_1^{n_1} q_2^{n_2} \dots, p_1^{m_1} p_2^{m_2} \rangle = n_1! c_1^{n_1} \delta_{n_1, m_1} n_2! c_2^{n_2} \delta_{n_2, m_2} \dots \quad (33)$$

Of course we have multiplication on  $S$  and comultiplication on  $S^*$ . In fact, since the commutation relations give a bilinear pairing:

$$S^* \otimes S \rightarrow S \quad (34)$$

by interpreting  $c_i$  as an element in  $S$ , we get

$$1 \otimes p_j = p_j, \quad q_i \otimes p_j = c_i \delta_{ij}, \quad (35)$$

and this gives **comultiplication** of the Heisenberg algebra:

$$\Delta : p_i \mapsto 1 \otimes p_i + p_i \otimes 1 \quad (36)$$

together with our previous multiplication. This makes this into a **Hopf algebra**

## 5 Geometric Realization of the Heisenberg Algebra

Going back to topology: we note that on any Homology ring  $H_*X$  we have that a continuous map  $\varphi : X \rightarrow Y$  between manifolds  $X$  and  $Y$  descends to a map between chains on  $X$  to chains on  $Y$ , and thus gives a map  $\varphi_* : H_*X \rightarrow H_*Y$  called the **pushforward** on homology rings, just by considering the homology of the mapped chains.

Similarly, any map  $\varphi : X \rightarrow Y$  induces a contravariant map  $\varphi^*\omega = \omega \circ \varphi$  on forms over  $Y$  called the **pullback**.

Now assume that  $f : X \rightarrow Y$  is a **proper map** (i.e.) the inverse image of any compact subset is compact. Then the pushforward descends to the locally-finite homology rings:

$$f_* : H_*^{lf}(X) \rightarrow H_*^{lf}(Y) \quad (37)$$

By defining  $\hat{f} : \hat{X} \rightarrow \hat{Y}$  on the 1-point compactifications of  $X, Y$  s.t.  $\hat{f}(\infty) = \infty$ . Then by properness,  $f$  is continuous on this space, so we can define the pushforward  $\hat{f}_*$  and pass this to the relative homology to get  $f_*$ .

From this, we can associate Hopf algebra structure to

$$\mathbb{H}^{lf} = \bigoplus_n \text{Hilb}_n(\mathbb{C}^2) \quad (38)$$

by considering a pair of three nested Hilbert schemes<sup>1</sup>:

$$\begin{aligned} H_{m+n \geq n} &:= \{V \in H_{m+n}, V'' \in H_n, V' = V/V'' \in H_m\} \\ &\subset H_m \times H_n \times H_{m+n} \end{aligned} \quad (39)$$

where for shorthand  $H_n$  denotes  $\text{Hilb}_n(\mathbb{C}^2)$ . We then have projections onto the components as the following diagram illustrates:

$$\begin{array}{ccc} & H_{m+n \geq n} & \\ p' \times p'' \swarrow & & \searrow p \\ H_n \times H_m & & H_{n+m} \end{array}$$

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<sup>1</sup>There is an issue when points collide that  $V'$  is not guaranteed to have a well-defined vector  $v_0$ . This can be resolved by looking at the open sets of distinct points first, and then taking the closure of this constructed space.

The following lemma can be proven by checking the preimages of points in  $S_\mu \mathbb{C}^2$  and noting that the fibers of the Hilbert-Chow morphism  $\pi^{-1}(\mu_i[x_i])$  are compact:

**Lemma 5.1.** *These projections  $p', p'', p$  are proper.*

So the projections therefore give rise to pushforwards on the Borel-Moore homology. Moreover, by this fact together with Poincare duality, we can identify elements in the homology rings with elements in the cohomology rings that we can *pull back*. Thus, we have maps:

$$\mu : H_*^{lf}(H_n) \otimes H_*^{lf}(H_m) \rightarrow H_*^{lf}(H_{n+m}) \quad (40)$$

given exactly by mapping homology elements

$$c_1 \otimes c_2 \mapsto p_*((p' \times p'')^*(c_1 \otimes c_2)) \quad (41)$$

This mirrors how multiplication in the Heisenberg algebra respects the degree grading. Similarly we have

$$\Delta : H_*^{lf}(H_{m+n}) \rightarrow H_*^{lf}(H_n) \otimes H_*^{lf}(H_m) \quad (42)$$

by

$$c \mapsto (p' \times p'')_*(p^*(c)) \quad (43)$$

**Theorem 5.2.** *The operations  $\mu, \Delta$  defined as above give  $\mathbb{H}^{lf}$  the structure of a graded Hopf algebra.*

*Proof.* We have already seen that these operations respect the grading. The associativity and coassociativity conditions follow from the functoriality of pushforward and pullback in the following diagrams:  $\square$

and

$$\begin{array}{ccccc}
 & & H_{m+n+r \geq n+r \geq r} & & \\
 & \swarrow p' \times p'' & & \searrow p & \\
 H_{m+n \geq n} \times H_r & & & & H_{n+m+r \geq r} \\
 \swarrow p' \times p'' \times id & \searrow p \times id & & \swarrow p' \times p'' & \searrow p \\
 H_m \times H_n \times H_r & & H_{m+n} \times H_r & & H_{m+n+r}
 \end{array}$$

$$\begin{array}{ccccc}
 & & H_{m+n+r \geq n+r \geq r} & & \\
 & \swarrow p' \times p'' & & \searrow p & \\
 H_m \times H_{n+r \geq r} & & & & H_{n+m+r \geq n+r} \\
 \swarrow id \times p' \times p'' & \searrow id \times p & & \swarrow p' \times p'' & \searrow p \\
 H_m \times H_n \times H_r & & H_m \times H_{n+r} & & H_{m+n+r}
 \end{array}$$

We can in fact go further and define the fiber  $F_\mu = \pi^{-1}S_\mu(\mathbb{C}^2)$  of arrangements of  $n$  points on  $\mathbb{C}$  of partition type  $\mu$ . We have that  $[F_\mu]$  is in fact well-defined and that

$$[F_\mu] \in H_{2(n+\ell(\mu))}^{lf}(\text{Hilb}_n(\mathbb{C}^2)) \quad (44)$$

**Theorem 5.3.** *The  $[F_\mu]$  form a basis for  $H_*^{lf}(\text{Hilb}_n(\mathbb{C}^2))$ . Picking  $[F_n] \in H_*^{lf}(\text{Hilb}_n(\mathbb{C}^2))$  corresponding to the fiber class with all  $n$  points coincident gives a multiplication operator  $P_n : H_m \rightarrow H_{n+m}$  corresponding to  $p_m$ .*

By Poincare duality, there is the intersection pairing

$$\cap : H_* \times H_*^{lf} \rightarrow \mathbb{C} \quad (45)$$

so that  $H_*$  can in fact be made to correspond to the dual space  $S^*$  of derivations. We can obtain fundamental classes in the regular homology  $[E_\mu]$  in a similar way, and have them form a basis for  $H_*\text{Hilb}_n(\mathbb{C}^2)$ . The remaining relations for the Heisenberg algebra can be obtained through careful calculation.

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