# Instantons and the ADHM Construction
# Introduction to Gauge Theory

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**Abstract**

We explore gauge groups using the language of bundles, and obtain the Yang-Mills equations by introducing the idea of action. We study solutions to these equations posessing nonzero action, known as instantons, and demonstrate how we can construct all instantons in this moduli space of solutions. This is the ADHM construction of Atiyah et al.

## 1 Background

We assume an undergraduate-level familiarity with Lie Theory, as well as a good understanding of fiber bundles and their properties.

### 1.1 Review: Principal Bundles

For a fiber bundle $\pi : E \to M$ with fiber $F$, we have local triviality, so that we can make an open cover $M = \bigcup\{U_\alpha\}$ with **local trivializations** $\psi_\alpha$ so that the following diagram commutes:

$$\begin{tikzcd} U_\alpha \times F \arrow[rr, "\psi_\alpha"] \arrow[dr, "p_1"'] & & \pi^{-1}(U_\alpha) \arrow[dl, "\pi"] \\ & U_\alpha & \end{tikzcd}$$

For a fiber bundle, we have a Lie group $G \circlearrowright F$ called the **structure group** acting freely on each fiber so that if $\forall f \in F \; gf = f$, $g$ is trivial.

**Example 1.1.** The Mobius strip is the classical example of a fiber bundle.

**Example 1.2.** Pick a Riemannian manifold $M$ of dimension $n$. At each point $p$, coordinate frame transformations in $\mathrm{SO}_n(\mathbb{R})$ form a structure group for the vector bundle $\mathrm{Vect}(M)$ of orthogonal reference frames at $p$ known as the **orthogonal frame bundle**. This should be our first grounding in the intuition behind gauge theory.

The structure group is a vast generalization of the notion of "reference frame". With this structure group, it is extremely worthwhile to study the **principal bundles** $P(M, G)$, of such reference frames over $M$, with fiber $G$.

For a given bundle $\pi : E \to M$, we define a **global section** as an injective map $s : M \to E$ such that $\pi \circ s = id|_M$. The set of all smooth sections of $\pi : E \to M$ is denoted by $\Gamma^\infty(E)$. Sections of fiber bundles, in particular vector bundles, generalize the idea of vector fields on $M$ (which can themselves be viewed as $\Gamma^\infty(TM)$).

The reason for this formalism is that while before we could write a vector field in $V$ over $M$ at each point $p$ as:

$$X : M \to V \Rightarrow X(p) \in V. \tag{1}$$

Although the $V_p$ are isomorphic as vector spaces, they are not one and the same. So therefore we want separate vector spaces fibered over every point $p$. Note the existence of global sections is not guaranteed:

**Example 1.3.** Let $\pi : E \to S^2$ be a circle bundle over $S^2$ of orthogonal frames. This does not admit a global section (hairy ball theorem).

**Lemma 1.4.** *Let $\pi : E \to M$ be a fiber bundle so that the fibers $F$ have group structure. Then $G \circlearrowright E$ and $E \circlearrowleft G$.*

*Proof.* Take $h \in G$. We have a canonical action fiberwise by looking at $\pi^{-1}(p)$ in a trivialization as elements $[p, g]$. Then $h[p, g] = [p, hg]$ and $[p, g]h = [p, gh]$ ∎

Analogously if $F$ has vector structure, then $E$ obtains vector structure fiberwise and in particular so does $\Gamma^\infty(E)$.

## 1.2 Review: Lie Groups and their Tangent Spaces

For a Lie group $G$, we define $\mathfrak{g}$ to be the identity's tangent space $T_eG$. We can identify this tangent space with that at any other point $h \in G$ by noting that the group acts by pushforward of the left action on its own tangent bundle, inducing isomorphism between any two tangent spaces:

$$(L_h)_* : TG \to TG \tag{2}$$

Because of this, each element $\xi \in \mathfrak{g}$ gives rise to a $G$-invariant vector field $X_\xi$ such that $L_* X_\xi = X_\xi$. More explicitly, we get this vector field by considering

$$\frac{d}{dt}\left[e^{tx}g\right]_{t=0} \tag{3}$$

This also means we can take any vector on $T_gG$ *back* to the Lie algebra, caninically, by mapping:

$$\theta : T_gG \to T_eG = \mathfrak{g} \;\; s.t. \;\; \theta(g) = [L_(g^{-1})]_* \tag{4}$$

This is the Maurer-Cartan $\mathfrak{g}$-valued 1-form on $G$: $\theta \in \Omega^1(G, \mathfrak{g}) = C^\infty(G, T^*G \otimes \mathfrak{g})$.

## 1.3 Gauge Theory

Take a $G$-principal bundle $\pi : P \to M$. As with Lie groups (i.e. principal bundles over a point), $G$ acts by right $R_g$ and left $L_g$ action, by the previous lemma.

Just like $\xi \in \mathfrak{g}$ gives rise to a vector field $X_\xi$ on $G$, it also canonically gives rise to a vector field $\sigma(\xi)$ on $P(M, G)$.

**Definition 1.5** (**Fundamental Vector Field** of $\xi$)**.** Let $\xi \in \mathfrak{g}$ and consider $\exp(t\xi) \in G$ so that for $p \in P(M, G)$ we get $c_p(t) = R_{\exp(t\xi)}p$ which depends smoothly on $p$. Note $c_p'(0) \in T_pP(M, G)$ at each point.

$$\sigma : \mathfrak{g} \to \mathrm{Vect}(P(G, M)), \;\; [\sigma(\xi)](p) \mapsto \left[\frac{d}{dt} p e^{t\xi}\right]_{t=0} \tag{5}$$

The **vertical subspace** $V_pP$ at a point $p$ of a fiber bundle is the tangent space at $p$ of the fiber over $x$, i.e. $T_p(\pi^{-1}(x))$. We can also write it as $\ker \pi_*$ because. Note

$$\pi_* \circ \sigma(x) = \frac{d}{dt}(\pi \circ c_p(t))|_{t=0} = \frac{d}{dt}(p) = 0 \tag{6}$$

so $\sigma(x) \in V_p P$. In fact:

**Proposition 1.6** ($V_p \leftrightarrow \mathfrak{g}$)**.** *At each point $P$ we have $\mathfrak{g} \cong \ker \pi_*|_p = V_p P$.*

*Proof.* Since $E$ is a manifold of dimension $\dim M + \dim G$, $\pi_* : T_p E \to T_{\pi(p)} M$ has a kernel of dimension $\dim G = \dim \mathfrak{g}$. $\square$

**Corollary 1.7.** *$\sigma_p$ is the Lie algebra isomorphism between $\mathfrak{g}$ and $V_p P$*

Note since $G$ acts freely, that $\sigma$ is injective, so in fact gives this isomorphism.

**Lemma 1.8** (Properties of $\sigma$)**.** *We get that $\sigma$ satisfies:*

1. $[R_g]_* \sigma(x) = \sigma(ad_{g^{-1}} x)$
2. $[g_i]_* \sigma(x) = g_i(p) x$

*Proof.* 1. We have

$$\begin{aligned} [R_g]_* [\sigma(x)](p) &= \frac{d}{dt}(R_g p e^{tx}) \\ &= \frac{d}{dt} pg \mathrm{Ad}_{g^{-1}} e^{tx} \\ &= \frac{d}{dt} pg \exp[t(\mathrm{ad}_{g^{-1}} x)] \\ &= [\sigma(\mathrm{ad}_{g^{-1}} x)](pg) \end{aligned} \tag{7}$$

2. And

$$\begin{aligned} [g_i]_* [\sigma(x)](p) &= \frac{d}{dt} g_i p e^{tx} \\ &= g_i(p) x \end{aligned} \tag{8}$$

$\square$

Now $\sigma$ respects the Lie algebra structure and forms a homomorphism from $\mathfrak{g}$ to $\mathrm{Vect}(P(M, G))$ so that in fact

**Corollary 1.9.** $(R_g)_* V_p = V_{pg}$*: pushforward acts equivariantly on vertical subspaces.*

*Proof.* Let $X(p) \in V_p$ pick $A \in \mathfrak{g}$ s.t. the corresponding fundamental vector field $\sigma(A)(p) = X(p)$. Then we just look at

$$(R_g)_* \sigma(A)(p) = \sigma(\mathrm{ad}_{g^{-1}} A)(pg) \tag{9}$$

which is vertical. It's easy to go back from $pg$ to $g$ as well by picking $A \in \mathfrak{g}$ so that $X(pg) = \mathrm{ad}_{g^{-1}} A$. $\square$

Now note:

$$0 \longrightarrow V_p P \longrightarrow T_p P \overbrace{\longrightarrow}^{\pi_*} T_{\pi(p)} M \longrightarrow 0 \tag{10}$$

Since this splits, there is an injection of $T_{\pi(p)} P$ into $P$, called a **horizontal subspace** $H_p P$.

**Definition 1.10** (Horizontal Subspace)**.** A horizontal subspace is a subspace $H_p P$ of $T_p P$ s.t.

$$T_p P = V_p P \oplus H_p P \tag{11}$$

We'll abbreviate this by $H_p$ and the vertical subspace by $V_p$ when our principal bundle is unambiguous.

Now $H_p$ is not canonical, reflecting a more broad fact there is no god-given way to compare local gauges between different points. For a gauge $g$ at $x$, a vector on $T_x M$ should lift to a vector on $T_{[x,g]} P$ given by lifting to a horizontal subspace.

**Definition 1.11.** An **Ehresmann connection** is a choice of horizontal subspace at each point $p \in P(M, G)$ so that

1. Any smooth vector field $X$ splits as a sum of two smooth vector fields: a **vertical field** $X_V$ and a **horizontal field** $X_H$ so that at each point $p \in P(M, G)$ we have $X_V \in V_p$, $X_H \in H_p$. That is, the choice of $H_p$ varies smoothly.

2. $G$ acts equivariantly on $H_{pg}$:

$$H_{pg} = (R_g)_* H_p \tag{12}$$

We will denote the collection of our choice of $H_p P$ by $HP$ and similarly define $VP$ to be the (always canonical) collection of vertical subspaces. We say any vector field can be split into a vector field $X^H \in HP$ and $X^V \in VP$.

Naturally, for any choice of $HP$, we have a corresponding projection operator $\pi_H$ on vector fields $\pi_H : \mathrm{Vect}(P(M, G)) \to HP$ and similarly $\pi_V = id - \pi_H$, both with corresponding equivariance conditions.

**Proposition 1.12.** *We have the following correspondence:*

$$\begin{array}{c} \text{Ehresman} \\ \text{Connections } HP \end{array} \longleftrightarrow \begin{array}{c} \text{Horizontal/Vertical} \\ \text{Projection Operators } H/V \end{array} \longleftrightarrow \begin{array}{c} \mathfrak{g}\text{-valued} \\ \text{1-forms } \omega \end{array}$$

*Each of the above are smooth on $E$, and have appropriate Equivariance conditions:*

- $R_g H_p = H_{pg}$*: Horizontal subspaces are $G$-equivariant*
- $[R_g]_* H = H[R_g]$*: Projection commutes with $G$ action: changing gauge*
- $\omega(pg) = R_g^* \omega = g^{-1}\omega(p)g$*: 1-form is $G$-covariant*

We can lift $k$-forms on $M$ to $k$-forms on $P$ by $\pi^*$. Such a $k$-form depends only on the horizontal subspace:

$$H^\vee \alpha = \alpha(HX_1, \dots, XH_k) = \alpha(X_1, \dots X_k) = \alpha$$

and is called **horizontal**. To define differentiation consistent with the gauge group, we need to keep the exterior derivative of a horizontal form horizontal. For a given connection $\omega$, we do this by restricting:

$$d_\omega := d|_H : \Omega^k(P, \mathfrak{g}) \to \Omega^{k+1}(P, \mathfrak{g}) \tag{13}$$

Where $d|_H$ is the exterior derivative on $P$ restricted to the horizontal subspace by:

$$d|_H \alpha = d\alpha(HX_1, \dots, HX_k) \tag{14}$$

we also write this using the pushforward of the projection map on forms by:

$$\begin{aligned} H^*\alpha(X_1 \dots X_k) &= \alpha(HX_1, \dots, HX_k) \\ H^\vee d\alpha :=& d(H^*\alpha) = d|_H \alpha \end{aligned} \tag{15}$$

We can then define the **curvature** 2-form as:

$$\Omega := d_\omega \omega \tag{16}$$

**Proposition 1.13.** *The curvature form can be written as*

$$\Omega = d\omega + \omega \wedge \omega$$

*Proof.* Consider $\Omega(X,Y)$. When $X, Y$ are both horizontal we have $\Omega = d\omega$. When $X, Y$ are both vertical we have that $d_\omega \omega = 0$ so that $\Omega = 0$. But note also that if $\sigma(A) = X, \sigma(B) = Y$:

$$\begin{aligned} d\omega(X,Y) &= X\omega(Y) - Y\omega(X) - \omega([X,Y]) \\ &= XB - YA - \omega([X,Y]) \\ &= 0 - \omega[X,Y] \\ &= -\omega \wedge \omega(X,Y) \end{aligned} \tag{17}$$

an analogous argument follows when one is vertical and the other is horizontal. □

**Proposition 1.14.** *$\Omega$ is $G$-contravariant just like $\omega$:*

$$R_g^* \Omega = g \Omega g^{-1} \tag{18}$$

*Proof.* Since horizontal projection commutes with $G$-pullback, and the exterior derivative commutes with pullback, $R_g^* \Omega = R_g^* H^\vee d\omega$ commutes across and gives us the desired result. □

**Proposition 1.15** (Bianchi Identity)**.** $d_\omega \Omega = 0$

*Proof.*

$$\begin{aligned} H^\vee d\Omega &= H^\vee d(d\omega + \omega \wedge \omega) \\ &= H^\vee (d^2\omega + d(\omega \wedge \omega)) \\ &= H^\vee (d\omega \wedge \omega - \omega \wedge d\omega) \\ &= H^* d\omega \wedge H^* \omega - H^* \omega \wedge H^* d\omega \\ &= 0 \end{aligned} \tag{19}$$

because $\omega = 0$ on the horizontal space. □

## 1.4 The Vector Potential

For a given local section $s_i$ on an open set $U_i \in M$, we can pull back the connection and curvature forms to give the **vector potential** and **field strength** forms on $M$:

$$\mathcal{A}_i := s_i^* \omega, \ \mathcal{F}_i := s_i^* \Omega \tag{20}$$

**Observation 1.16.** *Because of the $G$-equivariance properties of $\omega, \Omega$, these two forms on $M$ contain the exact same information as their lifts in $P$.*

**Proposition 1.17.** *$\mathcal{F}_i$ satisfies the following*

- $\mathcal{F}_i = d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i$
- $\mathrm{d}\mathcal{F}_i + \frac{1}{2}[\mathcal{A}_i, \mathcal{F}_i]$

*Proof.* These all follow from the previous identities for $\omega$ and $\Omega$ by noting that exterior derivatives commute with pullback. □

**Observation 1.18.** *In coordinate language, we can write:*

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [A_\mu, A_\nu] \tag{21}$$

Now that we have a connection on the principal bundle, we can define one on an associated bundle $E_\rho$ to a representation $\rho$ of $G$:

$$\begin{aligned} \rho &: G \to \mathrm{GL}(F) \\ \mathrm{d}\rho &: \mathfrak{g} \to \mathrm{End}(F) \end{aligned} \tag{22}$$

by having $\omega$ act as $\rho(\omega)$.

Now for forms $\alpha \in \Omega^k(M, F)$, we define the covariant derivative as:

$$\mathrm{d}_{\mathcal{A}}\alpha = \mathrm{d}\alpha + \rho(\mathcal{A}) \wedge \alpha \tag{23}$$

This can be obtained in a straightforward manner from pulling back the covariant derivative on a corresponding form in $\Omega^k(P, F)$ in the associated bundles. Given any associated bundle $E$, we have an associated connection $\omega_E$ and curvature form $\mathcal{F}_E$ on $E$.

**DRAW A DIAGRAM HERE TO ELABORATE THIS**.

## 1.5 The Gauge Group

A diffeomorphism $\Phi : P \to P$ is a **gauge transformation** if it satisfies

1. $\pi \circ \Phi = \pi$, so $\Phi$ acts fiberwise
2. $R_g \circ \Phi = \Phi \circ R_g$, so $\Phi$ is $G$-equivariant.

the group of all such diffeomorphisms is called the **gauge group** of $P$ and denoted by $\mathcal{G}$.

# 2 Yang-Mills Theory and Instantons

## 2.1 Introduction to Yang-Mills

We take $M$ Riemannian so that we are given the Hodge-star operator $\star$. We define the action, from which we will obtain all information about the dynamics, by:

$$S_E[\mathcal{A}] = -\int_M \mathrm{Tr}(\mathcal{F}_i \wedge \star\mathcal{F}_i) \tag{24}$$

**Proposition 2.1.** $\mathrm{Tr}(\mathcal{F}_i \wedge \star\mathcal{F}_i)$ *is globally-defined and gauge invariant*

*Proof.* This follows directly from the cyclic properties of the trace, and the transformation laws on $F$ making it transform under ad. We will from now on drop the subscripts $i$. $\square$

Note however that it is not necessarily independent of connection. This is what we want: we want to find $\mathcal{A}$ so that $S_E[\mathcal{A}]$ is a minimum. To do this, we use standard calculus of variations. Consider a local perturbation $A + t\alpha$

$$\begin{aligned} \mathcal{F}[\mathcal{A} + t\alpha] &= \mathrm{d}(\mathcal{A} + t\alpha) + \mathcal{A} \wedge \mathcal{A} + t[\mathcal{A}, \alpha] + O(t^2) \\ &= \mathcal{F}[\mathcal{A}] + t(d\alpha + [\mathcal{A}, \alpha]) \\ &= \mathcal{F}[\mathcal{A}] + \mathrm{d}_{\mathcal{A}}\alpha \end{aligned} \tag{25}$$

so that to order $t$:

$$\begin{aligned} ||\mathcal{F}[\mathcal{A} + t\alpha]||^2 = ||\mathcal{F}[\mathcal{A} + t\alpha]||^2 + 2t(\mathcal{F}[\mathcal{A}], \mathrm{d}_{\mathcal{A}}\alpha) \\ \Rightarrow (\mathcal{F}[\mathcal{A}], \mathrm{d}_{\mathcal{A}}\alpha) = 0 \ \forall\alpha \end{aligned} \tag{26}$$

By taking adjoints, this gives:

$$\begin{aligned} &\Rightarrow \delta\mathcal{F}[\mathcal{A}] = 0 \\ &\Rightarrow \mathrm{d}_{\mathcal{A}} \star \mathcal{F} = 0 \end{aligned} \tag{27}$$

This, together with the tautological Bianchi identity: $\mathrm{d}_{\mathcal{A}}\mathcal{F} = 0$ form the Yang-Mills equations. These equations are very difficult to solve in all but abelian gauges, where they become linear.

## 2.2 Instantons and Topological Charge

First notice that:

**Proposition 2.2.** *Let* $\dim M = 4$*. Then* $\int_M \mathrm{Tr}(\mathcal{F}\wedge\mathcal{F})$ *is independent of* $\delta A$*.*

*Proof.* Following the same variational procedure will give us $\mathrm{d}_{\mathcal{A}}\mathcal{F}$, which is zero always, independent of any condition on $\mathcal{A}$. □

We define the **topological charge** n of the theory by

$$\mathrm{n} = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(\mathcal{F}^2) \tag{28}$$

**Proposition 2.3.** *When* $M = S^4$*, we have that* n *is an integer.*

*Proof.* The proof lies in simple ideas from Chern Cohomology classes and classifying bundles over $S^4$.
**PERHAPS THIS PROOF IS WORTH ADDING?** □

Now note that on $\mathbb{R}^4$, we have $\star\star = 0$ (not no $\mathbb{R}^{3,1}$). This means that $\star$ has eigenvalues $\pm 1$ and so $\Omega^2(U,\mathfrak{g})$ splits as a direct sum of two orthogonal spaces:

$$\Omega^2(U,\mathfrak{g}) = \Omega^2_+ \oplus \Omega^2_- \tag{29}$$

called **self-dual** and **anti-self-dual** spaces respectively. We can "symmetrize" any form to become a sum of a self-dual and an anti-self dual one. In particular, if we write:

$$\mathcal{F} = \mathcal{F}_+ + \mathcal{F}_- \tag{30}$$

then we have

$$\begin{aligned}
8\pi^2\mathrm{n} &= \int_M \mathrm{Tr}[(\mathcal{F}_+ + \mathcal{F}_-)\wedge(\mathcal{F}_+ + \mathcal{F}_-)]\mathrm{dVol} \\
&= \int_M \mathrm{Tr}[(\mathcal{F}_+)\wedge(\mathcal{F}_+)]\mathrm{dVol} + \int_M \mathrm{Tr}[(\mathcal{F}_-)\wedge(\mathcal{F}_-)]\mathrm{dVol} \\
&= \int_M ||\mathcal{F}_+||^2\,\mathrm{dVol} - \int_M ||\mathcal{F}_-||^2\,\mathrm{dVol}
\end{aligned} \tag{31}$$

Note that the absolute value of this gives:

$$8\pi^2|\mathrm{n}| \leq \int_M ||\mathcal{F}||^2 = |S_{\mathcal{A}}[\mathcal{F}]| \tag{32}$$

**Proposition 2.4.** *The action is bounded below by this topological charge and is in fact* equal *to it exactly when one of* $\mathcal{F}_+ = 0$ *or* $\mathcal{F}_- = 0$.

We call a solution an **instanton** of the theory. Its action is equal to the topological charge, and in fact we call this the **instanton number** k when appropriate.

**Proposition 2.5.** *For a principal bundle* $P$ *the instanton number is an invariant.*

*Proof.* This follows since for instantons $S_{\mathcal{A}} = 8\pi \mathrm{k}$ is independent of the connection. □

**Corollary 2.6.** *There are no instantons when* $G$ *is abelian.*

*Proof.* $\mathcal{F} = dA \Rightarrow ||\mathcal{F}|| = (\star dA, dA) = (\delta \star A, dA) = (\star A, d^2 A) = 0$ □

# 3 The ADHM Construction

## 3.1 Motivating Example

**Perhaps motivate it, but its unlikely you will have time**

## 3.2 The Data

Let $x_1, x_2, x_3, x_4$ parameterize a $\mathbb{R}^4$, and write this as $\mathbb{C}^2$ using $z_1 = x_2 + ix_1, z_2 = x_4 + ix_3$. We can then write all the $(\mathrm{d}_{\mathcal{A}})_\mu$ (from now on just $\mathcal{D}_\mu$). Moreover in terms of the complex coordinates, we get

$$\begin{aligned} D_1 &= \frac{1}{2}(\mathcal{D}_2 - i\mathcal{D}_1) \\ D_2 &= \frac{1}{2}(\mathcal{D}_4 - i\mathcal{D}_3) \end{aligned} \tag{33}$$

We can express anti-self duality of $\mathcal{F}_{\mu\nu}$ in terms of these $D_\mu$ through two equations:

$$\begin{aligned} [D_1, D_2] &= 0 \\ [D_1, D_1^\dagger] + [D_2, D_2^\dagger] &= 0 \end{aligned} \tag{34}$$

The idea behind ADHM is to take "Fourier transforms" of these $D_i$ to matrices $B_i$.

**Definition 3.1** (ADHM Data)**.** Let $U$ be a 4-dimensional space with complex structure. An **ADHM System** on $U$ is a set of linear data:

1. Vector spaces $V, W$ over $\mathbb{C}$ of dimensions $k, n$ respectively.
2. Complex $k \times k$ matrices $B_1, B_2$, a $k \times n$ matrix $I$, and an $n \times k$ matrix $J$.

**Definition 3.2** (ADHM System)**.** A set of ADHM Data is an ADHM system if it satisfies the following contraints:

1. The ADHM equations:

$$\begin{aligned} +IJ &= 0 \\ [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J &= 0 \end{aligned} \tag{35}$$

2. For $(x, y) \in U^2$ with $x = (z_1, z_2), y = (w_1, w_2)$, the map:

$$\alpha_{x,y} = \begin{pmatrix} w_2 J - w_1 I^\dagger \\ -w_2 B_1 - w_1 B_2^\dagger - z_1 \\ w_2 B_2 - w_1 B_1^\dagger + z_2 \end{pmatrix} \tag{36}$$

is injective from $V$ to $W \oplus (V \otimes U)$ while

$$\beta_{x,y} = \begin{pmatrix} w_2 I + w_1 J^\dagger & w_2 B_2 - w_1 B_1^\dagger + z_2 & w_2 B_1 + w_1 B_2^\dagger + z_1 \end{pmatrix} \tag{37}$$

is surjective from $W \oplus (V \otimes U)$ to $V$.

**Lemma 3.3.** *If $B_1, B_2, I, J$ satisfy the above conditions, then for $g \in \mathrm{U(k)}, h \in$ $\mathrm{SU}(n)$*

*Proof.* Check directly through algebra that equations are preserved. $\square$

**Proposition 3.4.** *The ADHM equations are satisfied iff*

$$V \overset{\alpha_{x,y}}{\hookrightarrow} W \oplus (V \otimes U) \overset{\beta_{x,y}}{\twoheadrightarrow} V \tag{38}$$

*is a complex*

*Proof.* We need both $\beta\alpha = 0$ as well as surjectivity of $\beta$ and injectivity of $\alpha$. The actual equation $\beta\alpha = 0$ does reduce exactly to a quadratic polynomial in the $w_1, w_2$ with the two ASD equations emerging as coefficients. $\square$

**Theorem 3.5.** *There is a one-to-one correspondence between equivalence classes of solutions to this system and gauge equivalence classes of anti-self-dual* $\mathrm{SU}(n)$*-connections* $\mathcal{A}$ *with instanton number* $k$*.*

We'll prove one direction: getting instantons from a solution to this system in the next lecture.