

Chapter 1

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{I + v^TA^{-1}u}$$

$$\Rightarrow I = (A + uv^T)(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{I + v^TA^{-1}u})$$

$$= I + uv^TA^{-1} - \frac{uv^TA^{-1}}{I + v^TA^{-1}u} - \frac{uv^TA^{-1}uv^TA^{-1}}{I + v^TA^{-1}u}$$

$$\Rightarrow (I + v^TA^{-1}u)uv^TA^{-1} - uv^TA^{-1} - uv^TA^{-1}uv^TA^{-1}$$

$$= v^TA^{-1}u \ uv^TA^{-1} - (uv^TA^{-1})^2$$

$$(uv^TA^{-1}) uv^TA^{-1} = v^TA^{-1}u \ uv^TA^{-1}$$

αu in both cases

$$uv^TA^{-1}uv^TA^{-1} = v^TA^{-1}u \ uv^TA^{-1} \quad \checkmark$$

Schur Complement

$$M_{NN} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -M_{22}^{-1}M_{21} & I \end{pmatrix} = \begin{pmatrix} S & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

$$S = M_{11} - M_{12} M_{22}^{-1} M_{21} = [M^{-1}]_{11}$$

$$\begin{pmatrix} I & -M_{12}M_{22}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S & M_{12} \\ 0 & M_{22} \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & M_{22} \end{pmatrix}$$

$$\Rightarrow M = \begin{pmatrix} I & M_{12}M_{22}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ M_{22}^{-1}M_{21} & I \end{pmatrix}$$

$$\Rightarrow M^{-1} = \begin{pmatrix} I & 0 \\ -M_{22}^{-1}M_{21} & I \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & M_{22}^{-1} \end{pmatrix} \begin{pmatrix} I & -M_{12}M_{22}^{-1} \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} S^{-1} & -S^{-1}M_{12}M_{22}^{-1} \\ -M_{22}^{-1}M_{21}S^{-1} & M_{22}^{-1} + M_{22}^{-1}M_{21}S^{-1}M_{12}M_{22}^{-1} \end{pmatrix}$$

When M_1 is 1×1

$$u = \begin{pmatrix} M_{11} \\ M_{21}/M_{11} \end{pmatrix}$$

$$V = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\frac{M_{11}}{\det M_{11}}$$

$$(M_{22} - \frac{M_{21} M_{12}}{M_{11}})^{-1} = M_{22}^{-1} + \frac{M_{22}^{-1} M_{21} M_{12} M_{22}^{-1}}{M_{11} - M_{12} M_{22}^{-1} M_{21}}$$

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & M_{22} - \frac{M_{21} M_{12}}{M_{11}} \end{pmatrix} \xrightarrow{\text{unr}}$$

$$B = A^{-1}$$

$$G := \frac{\partial B_{kl}}{\partial A_{ij}} = \frac{1}{2} [A^{-1}]_{ik} [A^{-1}]_{jl} + \frac{1}{2} [A^{-1}]_{il} [A^{-1}]_{jk}$$

$$\Rightarrow \sum_{kl} G_{ij;kl} \underbrace{[v_{\alpha k} v_{\beta l} + v_{\alpha l} v_{\beta k}]}_{(\lambda_\alpha \lambda_\beta)^{-1}} \quad \text{at eig of } A$$

$$\Rightarrow \det_{\text{sym}} G = \prod_{\alpha, \beta} \frac{1}{\lambda_\alpha \lambda_\beta} \Rightarrow \log \det_{\text{sym}} G = -\frac{1}{2} \sum_{\alpha, \beta} \log \lambda_\alpha \lambda_\beta - \frac{1}{2} \sum_{\alpha} \log \lambda_\alpha^2$$

$$= -(N+1) \log \det A$$

$$\Rightarrow \det G = (\det A)^{-N-1}$$

$\mathbb{R}^{N(N+1)/2}$ space
of symmetric matrices

Chapter 2

2.1 Normalized Trace

$$\tau(A) := \frac{1}{N} E[\text{Tr } A]$$

Then we look at $N \rightarrow \infty$ limit

$$\frac{1}{N} \text{Tr } F(A) = \frac{1}{N} \sum_i F(\lambda_i) =$$

$$\langle F(\lambda) \rangle = \frac{1}{N} \sum_i F(\lambda_i)$$

$$\text{Concentration: } \tau(F(A)) = \langle F(\lambda) \rangle_A$$

When $\lambda_i \rightarrow g(\lambda)$

$$\Rightarrow \tau(F(A)) = \int g(\lambda) F(\lambda) d\lambda$$

$$m_k := \tau(A^k) \quad \|A\|_F^2 = m_2$$

2.2 Wigner

$$X_{i \neq j} \sim N(0, \sigma_{ad}^2)$$

$$X_{i=j} \sim N(0, \sigma_d^2)$$

$$\tau(X) = 0$$

$$\tau(X^2) = \sigma_d^2 + (N-1) \sigma_{ad}^2$$

$$\sigma_{ad} = \sigma^2 / N \quad \Rightarrow \quad \tau(X^2) = \frac{\sigma^2}{2N}$$

$$\sigma_d^2 = 2\sigma^2 / N$$

$$\Rightarrow X = H + H^T \quad H_{ij} \sim N(0, \frac{\sigma^2}{2N})$$

$$\begin{aligned} \Rightarrow \tau(X^2) &= 2 \tau(H^2) \\ &= \frac{2}{N} \cdot N^2 \cdot \frac{\sigma^2}{2N} = \sigma^2 \end{aligned}$$

$$\tau(X^3) = 0$$

$$\tau(X^4) = 2\sigma^2 \text{ as we will show} \Rightarrow \text{non-Gaussian}$$

$$\text{In fact } g(\lambda) = \frac{\sqrt{1-\lambda^2}}{2\pi\sigma^2}$$

For $v \sim N(0, \sigma^2)$

$w = Ov$ is still $\sim N(0, \sigma^2)$

$$E[w_i w_j] = \sum O_{ik} O_{jl} E[v_k v_l] = \sigma^2 O^T O = \sigma^2 I$$

$$\text{Now } X = Hv$$

$$p(OH) = p(H) \quad \forall H$$

$$\Rightarrow p(OXO^T) = p(X)$$

2.3 Resolvent

$$G_A(z) = \frac{1}{zI - A}$$

Stiljetes:

$$g_N(z) = T(G_A(z)) = \frac{1}{N} \sum_{k=1}^N \frac{1}{z - \lambda_k}$$

will write as g_N

Empirical spectral density

$$p_N(\lambda) = \frac{1}{N} \sum_k \delta(\lambda - \lambda_k)$$

$$g_N = \int_{-\infty}^{\infty} d\lambda \frac{p_N(\lambda)}{z - \lambda} \quad g_N \text{ as } N \rightarrow \infty$$

$$\text{near } z \in \infty \text{ as } N \rightarrow \infty \quad g(z) = \sum_{k=0}^{\infty} \frac{1}{z - \lambda_k} m_k(A)$$

$$T(F(A)) = \int_{-\lambda}^{\lambda} p(\lambda) F(\lambda) d\lambda$$

$$E_A[g_N(A)]$$

$$T(A^k) = \lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr } A^k$$

Concretely we will use the fact that we only know $g(z)$ for large z

$$\exists \min \lambda_c > c > 0$$

$g(z)$ has a $z=0$ expansion

$$g(z) = - \sum_{k=0}^{\infty} z^k \tau(A^{-k-1})$$

$$g(0) = -\tau(A^{-1})$$

Ex 2.3.1

$$g_{\alpha A}(z) = \frac{1}{N} \sum_k \frac{1}{z-\alpha \lambda} = \frac{1}{\alpha} g\left(\frac{z}{\alpha}\right)$$

$$g_{A+\beta I\!\!I}(z) = \frac{1}{N} \sum_k \frac{1}{z-\beta-\lambda} = g(z-\beta)$$

23.2 Stigies for Wigner

Cavity method: Relate g_N to g_∞

and then equate as $N \rightarrow \infty$

$$M = z - X$$

$$G_X = \frac{1}{z-X}$$

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}_{\overset{(N-1) \times (N-1)}{\square}}$$

$$\left[\frac{1}{G_X} \right]_{11} = M_{11} - \sum_{k,l} M_{1k} \left(M_{22} \right)_{kl}^{-1} M_{l1}$$

\Leftarrow Schur complement

concentrates at large N

$$S^{-1} = [M^{-1}]_{11}$$

$$\underset{x}{\mathbb{E}}[M_{11}] = z$$

$$\underset{x}{\mathbb{E}} \left[M_{1k} \left[M_{22} \right]_{kl}^{-1} M_{l1} \right] = \frac{\sigma^2}{N} \left(M_{22} \right)_{kl}^{-1} \delta_{k,l}$$

$$\Rightarrow \underset{x_0}{\mathbb{E}} \left[\sum_{k,l} M_{1k} \left(M_{22}^{-1} \right)_{kl} M_{l1} \right] = \sigma^2 \tau(M_{22}^{-1})$$

$\underbrace{\phantom{\sum_{k,l} M_{1k} \left(M_{22}^{-1} \right)_{kl} M_{l1}}} \sigma^2 g$

$$\Rightarrow \frac{1}{g(z)} = z - \sigma^2 g(z) \Rightarrow g(z) = \frac{z}{z^2 - 4\sigma^2} \pm \frac{\sqrt{z^2 - 4\sigma^2}}{2\sigma^2}$$

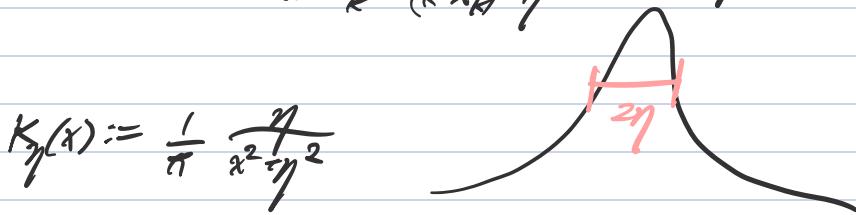
Need this to be analytic for large $z \Rightarrow$ onto not into ω
and $\rightarrow 0$

$$\Rightarrow g(z) = \frac{z - z \sqrt{1 - \frac{4\eta^2}{z^2}}}{2\eta^2}$$

$$g(z) = \int d\lambda \frac{g(\lambda)}{z-\lambda}$$

$$g_N(x-i\eta) = \frac{1}{N} \sum_k \frac{\frac{x-\lambda_k + i\eta}{(x-\lambda_k)^2 + \eta^2}}{(x-\lambda_k)^2 + \eta^2}$$

$$\operatorname{Im}(g_N) = \frac{1}{N} \sum_k \frac{\eta}{(x-\lambda_k)^2 + \eta^2} = (\pi K_\eta) * p$$



$\eta \ll N^{-1} \Rightarrow 0$ to 1 cog in the interval $\Rightarrow g_N$ wildly fluctuates

$$N^1 \ll \eta \ll 1 \quad (\text{e.g. } \eta \sim N^{1/2})$$

Then for $\Delta x \gg \eta$ p is locally constant with

$$\pi \sim N p(x) \Delta x \Rightarrow N \eta \gg 1$$

$$\Rightarrow \frac{1}{N} \sum_{k: x_k \in [x \pm \Delta x]} \frac{i\eta}{(x-\lambda_k)^2 + \eta^2} \rightarrow i \int_{x-\Delta x}^{x+\Delta x} d\lambda \frac{g(\lambda) \eta}{(x-\lambda)^2 + \eta^2} \rightarrow i \pi p(x)$$

↑
need $\Delta x \gg \eta$ for this

23.4

$$\lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \operatorname{Im} g(x-i\eta) = p(x)$$

At finite size $N^{-1} \ll \eta \ll 1$, $\eta \sim N^{1/2}$ works best

Want η as small as possible to avoid blur

$$\text{Error} \sim \eta p(x)$$

Want eta as large as possible for low statistical error

$$\text{Error} \sim \frac{1}{N\eta}$$

$$E_{\text{tot}} = \frac{1}{N\eta} + \eta p(\lambda) \Rightarrow \eta \sim \frac{1}{\sqrt{N}} \cdot p(x)$$

Density of Eigenvalues of a Wigner Matrix

$$p(x) = \frac{1}{\pi} \lim_{\eta \rightarrow 0^+} \text{Im } g(x-i\eta) = \frac{\sqrt{40^2 - x^2}}{2\pi\eta^2} \quad \leftarrow \text{semicircle law}$$

\Rightarrow Asymptotically no $\lambda > 20$
 $\lambda < -20$

+ square root singularities

Will later show 3 sing in region of size $N^{2/3}$ beyond thresh

2.3.3 Assume A has $\tau(A^k \sim \gamma_k)$

a) $\Rightarrow g(z) \sim \sum_k \frac{1}{k} z^{-k-1}$

b) $-\frac{1}{z} \log(1 - \frac{1}{z})$

c) sing at $0 < z < 1$

d) $-\frac{1}{x-i\eta} \log\left(1 - \frac{1}{x-i\eta}\right)$

$\frac{1}{\pi} \text{Im} \left(\log\left(1 - \frac{1}{x-i\eta}\right) \right) \Rightarrow \frac{1}{x} \Rightarrow g(x) = \frac{1}{x}$

as before

e) $\int_0^1 \frac{1}{x} \frac{1}{z-\lambda} d\lambda = -\frac{1}{z} [\log(z-ez) - \log(z-e)] = -\frac{\log e}{z} - \frac{1}{z} \log\left(\frac{z-1}{z}\right)$

div

f) $\sum_{z=1}^{\infty} \frac{1}{z^{2k+1}} = \frac{z}{z^2-1} \Rightarrow \text{singular at } z=1$

$k=0$

$$1-i\eta \Rightarrow \frac{1-i\eta}{1-2i\eta - 1} \Rightarrow \frac{1}{2\eta} \Rightarrow \pm \frac{1}{2} \text{ mass}$$

$$-1-i\eta \Rightarrow \frac{-1-i\eta}{1-2i\eta - 1} \Rightarrow \frac{1}{2\eta} \quad \begin{pmatrix} \text{consistent} \\ \text{with odd=0} \\ \text{even=1} \end{pmatrix}$$

$$\int dx \frac{\pm S(x \pm 1)}{z-x} = \frac{1}{2} \left[\frac{1}{z-1} + \frac{1}{z+1} \right] = \frac{z}{z^2-1} \quad \checkmark$$

2.3.6 Stiltjes transform on \mathbb{R}

$g_N \rightarrow \infty$ as $z \rightarrow x_i$

as $N \rightarrow \infty$ z is close to resonant pole

$$d_i := |z - x_i|$$

$$P[d_i < \epsilon/N] = 2\epsilon p(z)$$

$$g_N(z) \approx \pm \frac{1}{N} d_i \quad \text{as } N \rightarrow \infty$$

nearest one

$$P[g_N > \epsilon^{-1}] = P[d_i < \epsilon/N] = 2\epsilon p(z)$$

$\Rightarrow g_N$ decays as $\frac{p(z)}{g^2}$

$$P[g_N > \epsilon] \sim \frac{p(z)}{\epsilon}$$

$$P(g_N) \sim \frac{p(z)}{g_N^2} \text{ for } g_N \text{ big}$$

Nontrivial Claim $[G_{11}]_{11}$ is distributed like $\frac{1}{N} \text{Tr } G$

$$\Rightarrow P^{(n)}(g) = \int_{-\infty}^{\infty} dg' P^{(n-1)}(g') \delta(g - \frac{1}{z - \sigma^2 g'})$$

$(z - A_K)^{-1}$

$$[G_X]_{11} \sim \frac{1}{M_{11} - \sum_{k,l} M_{1k} [M_{22}]_{kk}^{-1} M_{21}}$$

$$g = \frac{1}{z - \sigma^2 g'} \quad \Rightarrow \sigma^2 g'$$

This functional iteration gives rise to the fixed point:

$$P^\infty(g) = \frac{p(z)}{\left(g - \frac{z}{\sigma^2}\right)^2 + \pi^2 p(z)}$$

$$g \sim P^{(n-1)}$$

as $g \rightarrow \infty$ get $\frac{p(z)}{g^2}$

If we just had that $x_i \sim \frac{1}{x^2}$ iid we'd expect Cauchy

The fact that we get it even w/ correlated x_i s is super-universality

Chapter 3

GUE GME GSE

Because only $\mathbb{R}, \mathbb{C}, \mathbb{H}$ have division ring property

\Rightarrow only 3 possible Gaussian Ensembles

Same $N \rightarrow \infty$ behavior

Correlations & large- N deviations differ only through their dependence on the Dyson index β

$$A = A^\dagger \Rightarrow A = U \Lambda U^\dagger$$

$$z = x_r + i x_i \quad x_r, x_i \sim N(0, \sigma^2/2)$$

$$\Rightarrow \mathbb{E}[|z|^2] = \sigma^2$$

\hat{x} a vector of complex centered Gaussians

$y = Ux$ is Gaussian w/ same var

Take H a matrix of complex centered Gaussians

$$X = H^\dagger H$$

$$UXU^\dagger \sim X$$

$$\text{Want } \mathbb{E}(X^2) = \sigma^2 \cdot \mathcal{O}(N)$$

$$\Rightarrow \mathbb{E}[|H_{ij}|^2] = \frac{\sigma^2}{2N}$$

$$X_{ii} \text{ real w/ variance } \mathbb{E}[X_{ii}^2] = \gamma_N$$

$$X_{ij} \text{ complex w/ total var } \mathbb{E}[|X_{ij}|^2] = \gamma_N$$

i.e. real & im parts have var $\frac{\gamma_N}{2N}$

$$\mathbb{P}\{X_{ij}\} = \exp\left[-\frac{N}{2\sigma^2} \text{Tr } X^2\right]$$

For $\mathbb{R}, \mathbb{C}, \mathbb{H}$:

$$P\{\{X_{ij}\}\} = \exp\left[-\frac{N\beta}{\sigma^2} \operatorname{Tr} X^2\right]$$

$\beta = 1, 2, 4$ resp.

Results of prior chapter apply
 $\Rightarrow p(\lambda) = \frac{\sqrt{4\sigma^2 - \lambda^2}}{2\pi\sigma^2}$

3.1.2 Quaternionic Matrices

$$i^2 = j^2 = k^2 = ijk = -1 \quad ij = -ji = k \quad \text{etc}$$

$$h = x_r + ix_i + jx_j + kx_k \quad h^* = -h \quad i^* = -i \quad \text{etc}$$

$$\|h\|^2 = hh^* = x_r^2 + x_i^2 + x_j^2 + x_k^2$$

$A = A^+ \Rightarrow$ Hermitian quaternionic

$SS^+ = I \Rightarrow$ Symplectic

Using Pauli matrices $i = i\sigma_z$, $j = i\sigma_y$, $k = i\sigma_x$
can think of A as $\underbrace{2N \times 2N}_{:= Q(A)}$ *Imaginary unit*

$$Q(I-j) := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$\Rightarrow +$ acts just like hermitian conjugation

Need 2 properties for Hermitian Quaternionic:

$$1) Q^+ = Q$$

$$2) Q^R := ZQ^TZ^{-1} = Q^+ = Q$$

In this $2N \times 2N$ rep'n symplectic matrices have

$$SS^+ = SS^R = I$$

Off-diag $E[|X_{ij}|^2] = 1/N \Rightarrow$ each part of X_{ij} has var $\frac{1}{4N}$

diag are real $E[X_{ii}^2] = \frac{1}{N} \leftarrow \frac{1}{2N}$?

$$\Rightarrow P[\sum X_{ij}^2] \propto \exp\left[-\frac{N}{\sigma^2} \operatorname{Tr} X^2\right]$$

$$V[(X_{ij})^2] = \frac{2}{\beta} \quad \text{by LLN}$$

$\Rightarrow \beta \rightarrow \infty$ would imply $|X_{ij}|^2 = 1$

$$\text{Ex 3.1.1} \quad 1) \quad Z = j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow Z^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = Z^T$$

$$\Rightarrow ZZ^T Z^{-1} = Z^T$$

$$2) \text{ For } \mathbf{I}, \mathbf{0}_x, \mathbf{0}_y, \mathbf{0}_z \quad Z Z^T Z^{-1} = \mathbf{0}^+$$

3) By linearity so does the \mathbb{R} -span

4) $\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$ does not satisfy this

3.1.3 Ginibre Ensemble

H_{ij} indep from $H_{ji} \Rightarrow$ eigs uniform on $D(0, \sigma^2) \subset \mathbb{C}$

$E[H_{ij} H_{ji}] = p\sigma^2 \Rightarrow$ ellipse $(1+p)\sigma$ along real $(1-p)\sigma$ along imag

HTH Wishart $\rightarrow \sqrt{\lambda_j}$ has quarter circle (next chapter)

Ex 3.1.2

3.2 Moments

$$\alpha(X^Y) = \frac{1}{N} \sum_{ijkl} E[X_{ij} X_{jk} X_{kl} X_{li}]$$

For nonzero:

- 1) All Y are equal
- 2) They are equal pairwise

$$1) \Rightarrow \frac{3\sigma^4}{N^2} \cdot N^2 \cdot \frac{1}{N} \rightarrow 0$$

2) 3 ways

$$\text{i)} \quad X_{ij} = X_{jk} \quad X_{ke} = X_{ei} \quad j \neq k$$

$$\Rightarrow \frac{1}{N} \sum_{i,j \neq k} E[X_{ij}^2 X_{ik}^2] = \frac{1}{N} \cdot N \cdot N(N-1) \left(\frac{\sigma^2}{N} \right)^2 \\ \Rightarrow \sigma^4$$

$$\text{ii)} \quad X_{ij} = X_{el} \quad X_{jk} = X_{el} \Rightarrow \sigma^4$$

$$\text{iii)} \quad X_{ij} = X_{el} \quad X_{jk} = X_{li} \Rightarrow i \neq k$$

$$\Rightarrow i = l \quad j = k \quad \Rightarrow i = j \quad k = l \quad \Rightarrow i = k \Leftarrow$$

$$\Rightarrow \tau(X^4) = 2\sigma^4$$

$$\text{Can show } \tau(X^{2n}) = C_k \sigma^{2k} \\ \tau(X^{2n+1}) = 0$$

$$C_k = \sum_{j=0}^{k-1} C_j C_{k-j-1} \quad C_0 = C_1 = 1$$

$$= \binom{2k}{k} \frac{1}{k+1}$$

3.2.2 Catalan

$$\tau(X^{2k}):$$

$$\begin{array}{ccccccc} \alpha_1 \alpha_2 & \alpha_2 \alpha_3 & \alpha_3 \alpha_4 & \dots & & \alpha_{2k} \alpha_1 \\ | & | & | & & & | \\ 1 & 2 & 3 & & & 2k \end{array}$$

\Rightarrow Only need to look @ 2-point pairings $\alpha_i = \alpha_j$

since $\alpha_i = \alpha_j = \alpha_k = \alpha_l$ will be sub-leading in n

$\Rightarrow (2k-1)!!$ pairings

Crossing vs non-crossing:

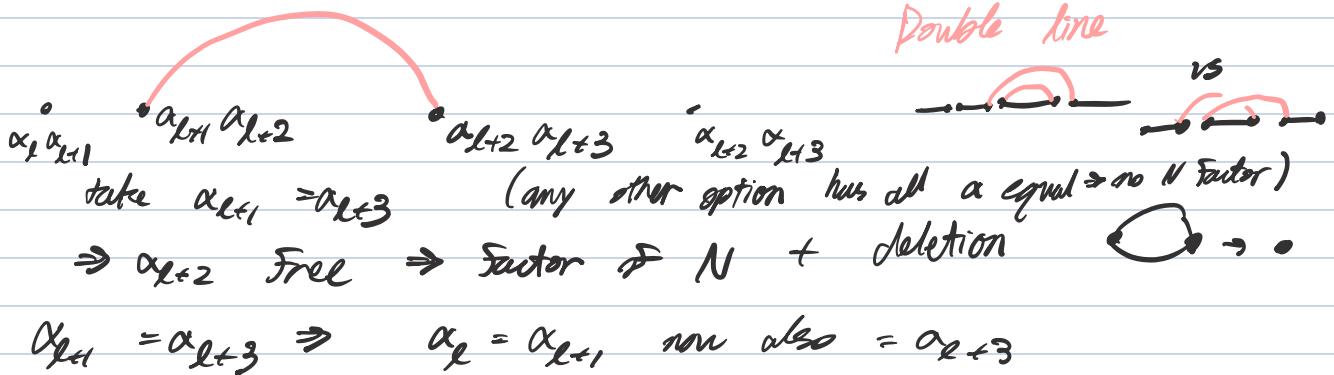
$\overline{-} = E$

$- =$ matrix multiplication



Prove by induction non-crossing contribute
 $k=1, 2$ taken care of

Always \exists a pair of consecutive points that are joined



\Rightarrow left w/ α^2 . a partition of $2k-2$
 $\approx \alpha^{2k-2}$ by induction

The other index match $\alpha_{k+1} = \alpha_{k+2} = \alpha_{k+3}$ is a subset of the first

$t(x^{2k})$ involves $2k$ matrices, $4k$ indices w/ Tr and material forcing $2k$ to be equal

$$\frac{1}{N} \text{Tr} + k \text{ var terms} \Rightarrow \frac{\sigma^{2k}}{N^{k+1}}$$

\Rightarrow Need $k+1$ free indices

$$Eg \quad k=1 \quad \text{Diagram} \Rightarrow \frac{1}{N^2} \sum \text{Tr}[X_{ii_1} X_{i_2 ii}]$$

$$k=2 \quad \text{+ } \dots$$

For crossing, we match indices that are not equal
a priori \Rightarrow

$$C_k = \sum_i$$

$$= \sum_j C_{j-1} C_{k-j}$$

$$g(z) = \sum_{k=0}^{\infty} \frac{C_k \sigma^{2k}}{z^{2k+1}}$$

$$\Rightarrow g(z) - \frac{1}{z} = \sum_{k=1}^{\infty} \frac{\sigma^{2k}}{z^{2k+1}} \sum_{j=0}^{k-1} C_j C_{k-j-1}$$

$$= \frac{\sigma^2}{z} \sum_{j=0}^{\infty} \frac{C_j \sigma^{2j}}{z^{2j+1}} \sum_{k=j+1}^{\infty} \frac{C_{k-j-1} \sigma^{2k-2j-2}}{z^{2(k-j-1)+1}}$$

$$= \frac{\sigma^2}{z} \sum_{j=0}^{\infty} \frac{C_j \sigma^{2j}}{z^{2j+1}} \sum_{k=0}^{2j} \frac{C_k}{z^{2k+1}} \sigma^{2k}$$

$$= \frac{\sigma^2}{z} g(z)^2$$

$$\Rightarrow g(z) = \frac{z - z\sqrt{1 - 4\sigma^2/z^2}}{2\sigma^2}$$

$$\text{Using } \sqrt{1-x} = 1 - \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} \left(\frac{x}{4}\right)^{k+1}$$

$$\text{we get } \frac{z - z\sqrt{1 - 4\sigma^2/z^2}}{2\sigma^2} = \frac{z}{\sigma^2} \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} \left(\frac{\sigma^2}{z^2}\right)^{k+1}$$

$$= \sum_{k=0}^{\infty} \frac{C_k \sigma^{2k}}{z^{2k+1}}$$

Chapter 4: Wishart

$N \times T$
 ↗ ↘
 # dims # samples

$$E_{ij} := \frac{1}{T} \sum_t x_i^t x_j^t \in \mathbb{R}^{N \times N}$$

↑ sample cov

$$E = \frac{1}{T} H H^T \quad H \text{ is } N \times T \text{ data matrix}$$

$H_{it} = x_i^t$

E sym psd

$$\Rightarrow \lambda_k^E \geq 0$$

$$F = \frac{1}{N} H^T H \in \mathbb{R}^{T \times T}$$

↑ observation covariance

$$q := N/T$$

$$q \leq 1 \Rightarrow F \text{ has } \begin{cases} N & \text{nonzero } \lambda \\ T-N & \lambda=0 \end{cases}$$

$$\lambda^F = q^{-1} \lambda^E$$

$$\begin{aligned} g_F^F(z) &= \frac{1}{T} \sum_k \frac{1}{z - \lambda_k^F} = \frac{T^N}{T} \frac{1}{z} + \frac{1}{T} \sum_k \frac{1}{z - q^{-1} \lambda_k^E} \\ &= \frac{1-q}{z} + q^2 g_E(qz) \end{aligned}$$

3.1.2

$$\mathbb{E}[H_{it} H_{js}] = C_{ij} \delta_{ts}$$

$$\Rightarrow \mathbb{E}[E_{ij}] = \frac{1}{T} \sum_t \mathbb{E}[H_{it} H_{jt}] = C_{ij} \Rightarrow \mathcal{U}(E) = \mathcal{U}(C)$$

$$\mathcal{U}(E^2) = \frac{1}{N} \frac{1}{T^2} \mathbb{E} \operatorname{Tr} [H H^T H H^T]$$

$$= \frac{1}{NT^2} \sum_{i,j,t,s} \mathbb{E}[H_{it} H_{jt} H_{js} H_{is}]$$

$$\boxed{\Gamma \Gamma} = \tau(C^2)$$

$$\boxed{\Gamma \Gamma} = \frac{1}{M^2} \sum_{i,j} C_{ii} C_{jj} = \frac{N}{T} \tau(C)^2 = q \tau(C)^2$$

$$\boxed{\Gamma \Gamma \Gamma} = \frac{1}{M^2} \sum_{i,j} (C_{ij})^2 = \frac{1}{T} \tau(C^2) \rightarrow 0$$

$$\Rightarrow \tau(E^2) = \tau(C^2) + q \tau(C)^2$$

$$C = a \mathbb{I} \Rightarrow \tau(C^2) - \tau(C)^2 = 0$$

$$\tau(E^2) - \tau(E)^2 = q a^2$$

4.1.3 Law of Wishart Matrices

$$P(SHSE) \propto \exp\left[-\frac{1}{2} \text{Tr}(H^T C^{-1} H)\right]$$

$$\propto \exp\left[-\frac{T}{2} \text{Tr}(EC^{-1})\right]$$

$$H \rightarrow E = \frac{HH^T}{T}$$

$$\text{Jac}'(E) = \int dH \delta(E - \frac{HH^T}{T})$$

$$= \int dH dA \exp\left\{i \text{Tr}\left(\hat{H}E - A \frac{HH^T}{T}\right)\right\}$$

integrate out H now

\curvearrowleft *Integrating out E diagonal*

(can always
rotate
from $H \rightarrow OH$)

$$= \int d\hat{H} \exp\left\{i \text{Tr} \hat{H}E - \frac{T}{2} \log \det \hat{H}\right\}$$

$$\hat{H} \rightarrow E^{-1/2} \hat{H} E^{-1/2} \quad (\det E)^{-\frac{N+1}{2}}$$

$$d\hat{H} = \prod_i d\hat{H}_{ii} \prod_{j>i} d\hat{H}_{ij} \rightarrow \prod_i E_{ii}^{-1} \prod_{j>i} (E_{ii} E_{jj})^{-1/2} d\hat{H} = (\det E)^{-1} \prod_{j>i} (E_{ii} E_{jj})^{-1/2} d\hat{H} = (\det E)^{-1} (\det E)^{-\frac{N+1}{2}} d\hat{H}$$

$$= \int d\hat{H} \exp\left\{i \text{Tr} \hat{H} - \frac{T}{2} \log \det \hat{H} + \frac{T}{2} \log \det E - \frac{N+1}{2} \log \det E\right\}$$

$$\propto (\det E)^{-\frac{N+1}{2}} (\det E)^{\frac{T}{2}}$$

$$\Rightarrow P(E) = \frac{(\det E)^{\frac{T-N-1}{2}}}{(\det C)^{T/2}} \exp \left[-\frac{1}{2} \text{Tr}[EC^{-1}] \right] \times \frac{(T/2)^{NT/2}}{\Gamma_N(T/2)}$$

Wishart multivariate Γ fn
generalizes Γ -distribution

$$P(E) \propto \frac{1}{(\det C)^{T/2}} \exp \left[-\frac{T}{2} \text{Tr}[EC^{-1}] + \frac{T-N-1}{2} \text{Tr}[\log E] \right]$$

$C=I$ $E \rightarrow W$ as $NT \rightarrow \infty$

$$P(W) \propto \exp \left[-\frac{N}{2} \text{Tr}[V(W)] \right] - \frac{N}{2}(1-q^{-1})$$

$$V(W) = q^{-1}W + (1-q^{-1})\log W$$

rotationally
inv. when $C=I$

4.2 Cavity derivation of Marcenko-Pastur

4.2.1

$$g_W(z) = \tau(G_W(z)) \quad G_W(z) = \frac{1}{z-W} = M^{-1} \quad M = zI - \frac{1}{T} HHT$$

$$\begin{aligned} \frac{1}{G_{11}} &= M_{11} - M_{12} M_{22}^{-1} M_{21} \\ &= z - W_{11} - \frac{1}{T^2} \sum_{j,s}^T \sum_{j,k=2}^N H_{it} H_{jt} [M_{22}^{-1}]_{jk} H_{ks} H_{ts} \end{aligned}$$

Schur
 off-diag W $N(0,1)$
 $N(0,1)$ $\Omega_{ts} = \frac{1}{T} H^T M_{22}^{-1} H$

$$= z - W_{11} - \frac{1}{T^2} \sum_{j,k} \text{Tr} \left[M_{00} [M_{22}^{-1}]_{jk} H_{k0} \right] + O(T^{-1/2})$$

$$= z - W_{11} - \frac{1}{T} \sum_{j,k \geq 2} W_{kj} [M_{22}^{-1}]_{jk} \quad \gamma^2 = \frac{1}{T} \text{Tr}[\Omega^2]$$

$$= z - W_{11} - \frac{1}{T} \text{Tr}[W_{22} G_{22}]$$

$O(T^{-1/2})$

$$\text{Tr} \left[W_2 (z - \lambda_2)^{-1} \right] = \sum_i \frac{\lambda_i}{z - \lambda_i} = \sum_i \left(\frac{z}{z - \lambda_i} - 1 \right)$$

$$= -\text{Tr} I + z \text{Tr} G_{22}$$

$$\Rightarrow \frac{1}{G_{11}(z)} = z - 1 + q - qzg + O(N^{-1/2})$$

↖ ↗ concentrates

$$\frac{1}{G_{11}} = \frac{1}{EG_{11}} + O(N^{-1/2})$$

$$\mathbb{E} G_{11} = \frac{1}{N} \mathbb{E} \text{Tr } G = g(z) \quad \text{by rotational invariance}$$

$$\Rightarrow \frac{1}{g(z)} = z - 1 + q - qzg(z)$$

4.2.2

$$\Rightarrow g(z) = \frac{z+q-1 - \sqrt{(z+q-1)^2 - 4qz}}{2qz}$$

For asymptotic $\frac{1}{z}$

$$= \frac{z - (1-q) - \sqrt{z - \lambda_+} \sqrt{z - \lambda_-}}{2qz} \quad \lambda_{\pm} = (1 \pm \sqrt{q})^2$$

finding correct branch is subtle!

$$\Rightarrow p(x) = \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2\pi q x} \quad \lambda_- < x < \lambda_+$$

If $q > 1 \Rightarrow$ pole @ $z=0 \Rightarrow \underbrace{\frac{f'(-1)}{q}}_{\text{N-T trivial zero eigs}} \delta(x)$

$$p_{1/q}(x) = q^2 p(qx)$$

N-T trivial zero eigs

$$\Rightarrow p(x) = \underbrace{\frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2\pi q x}}_{\text{N-T trivial zero eigs}} + \frac{q-1}{q} \delta(x) \Theta(q-1)$$

Ex 9.2.1

$$a) (z+q-1)^2 - 4qz = z^2 + 2(q-1)z + (q-1)^2 - 4qz \\ = z^2 - 2(q+1)z + (q-1)^2 \\ \Rightarrow z = q+1 \pm \sqrt{(q+1)^2 - (q-1)^2} = q+1 \pm 2\sqrt{q} = (1 \pm \sqrt{q})^2$$

$$b) \frac{z+q-1 - z\sqrt{1-\frac{1}{z}}\sqrt{1-\frac{1}{z}}}{2qz} \Rightarrow \frac{q-1}{2q} \frac{1}{z} + \frac{\lambda+\lambda_r}{4qz} = \frac{1}{z} = \tau(x^0) \checkmark \therefore O(z^{-1})$$

$$\alpha(z^{-2}) \frac{1}{2q} \left(-\lambda_r \lambda_l - \frac{\lambda_r^2 + \lambda_l^2}{8} \right) = 1 \Rightarrow \tau(x') = 1$$

c) Regular @ $z=0$ if $q < 1$

$$\frac{z+q-1 - \sqrt{z-\lambda_r}\sqrt{z-\lambda_l}}{2qz} = \frac{z+q-1 +}{2qz}$$

$\xrightarrow{z \approx 0}$

$$= \frac{z+q-1 + \sqrt{\lambda_r \lambda_l} - \sqrt{\left(\frac{-\lambda_r}{q}\right)\left(\frac{-\lambda_l}{q}\right)}}{2qz}$$

$$q < 1 \quad \text{at } \alpha(z^0) \quad \lambda_r \lambda_l = 1-q \quad = \frac{1}{2q} + \frac{\lambda_r^{1/2} \lambda_l^{1/2}}{2q} = \frac{1}{q} = \tau(X')$$

$$\frac{q-1 + \sqrt{\lambda_r \lambda_l}}{2qz} = \frac{q-1 + \sqrt{(1+\sqrt{q})(\sqrt{q}-1)}}{2qz} = \frac{q-1}{q} \cdot \frac{1}{z} \Rightarrow S(x) \cdot \frac{q-1}{q}$$

$$d) \int_{\lambda_-}^{\lambda_+} \frac{\sqrt{(x-\lambda_-)(\lambda_+-x)}}{2\pi q x} dx = \frac{(\sqrt{\lambda_+}-\sqrt{\lambda_-})^2}{4q} = \begin{cases} \frac{(1+\sqrt{q}-(-\sqrt{q}))^2}{4q} & q < 1 \\ \frac{(1+\sqrt{q}-\sqrt{q}-1)^2}{4q} & q = 0 \\ \frac{1}{q} & q > 1 \end{cases} \Rightarrow \begin{cases} 1 & q < 1 \\ \frac{1}{q} & q = 0 \\ \frac{q-1}{q} & q > 1 \end{cases}$$

$$\frac{1}{\pi q} \int_{\lambda_-}^{\lambda_+} \frac{ds}{s} \sqrt{s^2 - \lambda_-^2} \Rightarrow \Rightarrow \text{need } S(1) - \frac{q-1}{q}$$

$$s = \sqrt{x} \Rightarrow ds = \frac{1}{2\sqrt{x}} dx \Rightarrow dx = 2s ds$$

$$e) q=1 \Rightarrow \lambda_- = 0 \quad \lambda_+ = 4$$

$$\frac{1}{2\pi q} \int_0^4 dx \frac{\sqrt{x(4-x)}}{2\pi x} = \frac{1}{\pi q} \int_0^2 ds \sqrt{4-s^2}$$

$s = \sqrt{x}$ semicircle

$$\frac{ds}{s} = \frac{1}{2} \frac{dx}{x}$$

f) ...

Chapter 5: Joint Distributions

$$P(M) \sim Z_N^{-1} \exp \left[-\frac{N}{2} \text{Tr } V(M) \right]$$

↑
matrix potential

$$V(x) = \frac{x^2}{2\sigma^2} \quad \text{for Wigner}$$

$$V(x) = \frac{x + (q-1) \log x}{q} \quad \text{for Wishart}$$

$$V(x) = \frac{x^2}{2} + g \frac{x^q}{q} \quad \leftarrow \text{Physics}$$

5.1.2 Matrix Jacobian

M (symm) has $\frac{N(N+1)}{2}$ vars

$M = O \Lambda O^T$ has $O \sim \frac{N(N-1)}{2}$

$M \rightarrow (\Lambda, O)$ introduces $\det(\Lambda)$

$$\Delta := \left[\frac{\partial M}{\partial \Lambda}, \frac{\partial M}{\partial O} \right] \in \mathbb{R}^{\frac{N(N+1)}{2}, \frac{N(N+1)}{2}}$$

$$[DM] = d^{N(N+1)/2} \quad [D\Lambda] = d^N \quad [DO] = d^0$$

$$[\det \Delta] \sim d^{N(N-1)/2}$$

WLOG take M diagonal

$M \rightarrow OMOT$ has $\text{jac} = 1$

$$O = I + \epsilon S O$$

$$SO = -SOT \Rightarrow \epsilon SO = \sum_{k < l} \theta_{kl} A^{(kl)}$$

$\uparrow [A^{(kl)}]_{kl} = 1$
 $[A_{kl}]_{lk} = -l$
 else 0

$$M + SM = (I + \theta_{kl} A^{(kl)}) (I + \epsilon A) (I - \theta_{kl} A^{(kl)})$$

$$\Rightarrow SM = \epsilon A + \theta_{kl} [A^{(kl)} I - I A^{(kl)}]$$

$$\Rightarrow \frac{\partial M_{ij}}{\partial \lambda_m} = \delta_{im} \delta_{jn}$$

$$\Rightarrow I = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \lambda_2 - \lambda_1 & \\ & & & \ddots \\ & & & & \lambda_N - \lambda_{N-1} \end{pmatrix}$$

First N

$$\frac{\partial M_{ij}}{\partial \theta_{kl}} = \delta_{ik} \delta_{jl} (\lambda_l - \lambda_k)$$

k < l always!

$$\Rightarrow \det I = \prod_{k < l} (\lambda_l - \lambda_k)$$

↑ Vandermonde det

$$\Rightarrow P(\{x_i\}) \propto \prod_{k < l} (\lambda_k - \lambda_l) \exp \left[-\frac{N}{2} \sum_i V(x_i) \right]$$

↑
not independent!
repulsion!

Ex 5.1.1

$$O = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad I = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

$$\Rightarrow O I O^T = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \lambda_1 \cos \theta & -\lambda_1 \sin \theta \\ \lambda_2 \sin \theta & \lambda_2 \cos \theta \end{pmatrix}$$

$$= \lambda_1 \cos^2 \theta + \lambda_2 \sin^2 \theta \quad (\lambda_2 - \lambda_1) \cos \theta \sin \theta \\ (\lambda_2 - \lambda_1) \cos \theta \sin \theta \quad \lambda_1 \sin^2 \theta + \lambda_2 \cos^2 \theta$$

$$\frac{\partial M}{\partial \{ \lambda_1, \lambda_2, \theta \}} = \begin{matrix} \cos^2 \theta & \sin^2 \theta & 2(\lambda_2 - \lambda_1) \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & 2(\lambda_1 - \lambda_2) \cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos \theta \sin \theta & (\lambda_2 - \lambda_1) \cos^2 \theta - \sin^2 \theta \end{matrix}$$

$$|\det(\dots)| = |\lambda_2 - \lambda_1| \quad \therefore$$

5.1.2

$$a) |\lambda_2 - \lambda_1| \exp \left[- \frac{\lambda_1^2}{25^2} - \frac{\lambda_2^2}{20^2} \right]$$

$$b) |\lambda_-| \exp \left[- \frac{\lambda_-^2}{q_0^2} \right]$$

$$\approx |\lambda_-| \exp \left[- \frac{\lambda_-^2}{q_0^2} \right] \quad \text{take } x = |\lambda_-| > 0$$

$$c) \Rightarrow P(x) = \frac{\pi}{2} x \exp \left(- \frac{\pi x^2}{4} \right) \quad \text{for } x \geq 0 \quad \mathbb{E}[x] = 1$$

d) $\beta = 2$ affects Vandermonde det $\rightarrow x^\beta$

$$\# x^2 \exp \left[- \frac{x^2}{\sigma^2} \right] \rightarrow \frac{32}{\pi^2} x^2 \exp \left[- \frac{4}{\pi} x^2 \right]$$

5.2 Coulomb Gas

For β ensemble

$$P(\{\lambda_i\}) = Z_{N,\beta}^{-1} \prod_{k \neq l} (\lambda_k - \lambda_l)^\beta \exp \left[- \frac{\beta}{2} N \sum_i V(\lambda_i) \right]$$

$$= Z_{N,\beta}^{-1} \exp \left[- \frac{\beta}{2} \left[\sum_i N V(\lambda_i) - \sum_{i \neq j} \log |\lambda_i - \lambda_j| \right] \right]$$

$T = \frac{2}{\beta}$ with N particles in potential $NV(x)$

interacting w/ $V(x,y) = -\log |x-y|$

$\beta \rightarrow \infty$ has evals "freeze"

Also for $N \rightarrow \infty$, β fixed

log likelihood

$$P(\{\lambda_i\}) \propto \exp \left[\frac{1}{2} \beta N Z \right]$$

$$Z = - \sum_i V(\lambda_i) + \frac{1}{N} \sum_{i \neq j} \log |\lambda_i - \lambda_j|$$

$$\frac{\partial L}{\partial \lambda_i} = 0 \Rightarrow V'(\lambda_i) = \frac{2}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$$

Multiply both sides by $\frac{1}{z - \lambda_i}$ and sum

$$\begin{aligned} \Rightarrow \frac{1}{N} \sum_i \frac{V(\lambda_i)}{z - \lambda_i} &= \frac{2}{N^2} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} \frac{1}{z - \lambda_i} \\ &= \frac{1}{N^2} \sum_{i \neq j} \frac{1}{(z - \lambda_i)} \frac{1}{(z - \lambda_j)} = g_N^2(z) - \frac{1}{N^2} \sum_i \left(\frac{1}{z - \lambda_i} \right)^2 \\ &= V'(z) g_N(z) - \frac{1}{N} \sum_i \frac{V(z) - V(\lambda_i)}{z - \lambda_i} = g_N^2(z) - \frac{1}{N} g_N'(z) \\ &= V'(z) g_N(z) - \overbrace{\Pi_N(z)}^{\text{red}} \end{aligned}$$

IF V is poly of degree k
 Π_N is also poly of degree $k-1$

Ex 5.2.1

a) $\lambda_1, \lambda_2, \lambda_3$ - expect $\lambda_3 = -\lambda_1$ $\lambda_2 = 0$

b) $V'(\lambda_2) = \frac{2}{3} \left(\frac{1}{\lambda_2 - \lambda_1} + \frac{1}{\lambda_2 + \lambda_1} \right)$ $\lambda_2 = 0 \Rightarrow V'(0) = 0$ ✓

$$\Rightarrow V'(\lambda_1) = \frac{2}{3} \left(\frac{1}{\lambda_1} + \frac{1}{2\lambda_1} \right) = \frac{1}{\lambda_1} \Rightarrow \lambda_1 = \pm 1$$

c) $g_3(z) = \frac{z^2 - 1/3}{z^3 - z}$ ✓

d) $\Rightarrow \Pi_3(z) = 1$

e) $z g_3(z) - 1 = \frac{2}{3} \frac{1}{z^2 - 1} = g(z)^2 - \frac{1}{N} g'(z)$

5.2.3 As $N \rightarrow \infty$ g is self-averaging

so $\langle g_N \rangle = \langle g_{MLE} \rangle_{\text{mode}}$

$$V(z)g(z) - \bar{P}(z) = g^2(z)$$

$$\Rightarrow g(z) = \frac{V'(z)}{2} \pm \sqrt{\frac{V'(z)^2 - 4\bar{P}(z)}{2}} \quad \leftarrow p(x) > 0 \Rightarrow \text{discriminant} < 0$$

for $x \in W$ $p(x) \neq 0$

$$\operatorname{Re} g(x) = \int \frac{p(\lambda) d\lambda}{z-\lambda} = \frac{V'(z)}{z} \quad \text{"Hilbert Transform"}$$

Inverse question: given P does there exist a generalized orthogonal
one (or β one) with that β
if Hilbert transform exists, then yes

Ex 5.2.2

$$a) \quad g(z) = -\log \frac{z-1}{z} \quad z \in (0,1)$$

$$= -i\pi - \log \frac{1-z}{z}$$

$$\sqrt{V(z)/2}$$

$$b) \quad \int_0^x \frac{dl}{z-l} = -\log(z-1) + \log x$$

$$= \log \frac{x}{1-x}$$

$$c) \Rightarrow V(x) = 2 \left[x \log x + (1-x) \log(1-x) \right] + C$$

$$\underbrace{4 D_{KL}(x || \beta)}$$

5.3 Applications

$$V(z) = \frac{z^2}{2\sigma^2} \Rightarrow V'(z) = \frac{z}{\sigma^2} \Rightarrow \bar{P}(z) = \frac{1}{\sigma^2}$$

$$\Rightarrow g(z) = \frac{z - \sqrt{z^2 - 4\sigma^2}}{2\sigma^2}$$

$$V(z) = \frac{1}{q} \left(1 + \frac{q-1}{z}\right)$$

$\Rightarrow zV(z)$ is poly of degree 1 $\Rightarrow z\bar{\Pi}(z)$ of degree 0

$$\Rightarrow \bar{\Pi}(z) = \frac{c}{z}$$

$$\Rightarrow \frac{z+q-1 - \sqrt{(2q-1)^2 - 4cq^2z}}{2qz}$$

$$\text{as } z \rightarrow \infty \quad \frac{cq}{z} \in O(z^{-2})$$

$$g(z) \sim z^{-1} \Rightarrow c = q^{-1}$$

5.3.2

Assume limiting $\rho(\lambda)$ has no gaps

(ie $\rho(z)$ has only one cut)

Expect this if V is convex

$$\Rightarrow g(z) = \int_{\lambda_-}^{\lambda_+} \frac{\rho(\lambda)}{z-\lambda} d\lambda$$

$\Rightarrow g$ is singular near λ_{\pm}
 $g(x)$ has imaginary part only when $x \in (\lambda_-, \lambda_+)$
 analytic elsewhere

$$D := V'(z)^2 - 4\bar{\Pi} < 0 \quad \text{there}$$

D poly of degree $2k$

only even zeros away from λ_{\pm}

$$\Rightarrow D = Q(z)^2 (z - \lambda_+) (z - \lambda_-)$$

Q poly of deg $k-1$

$$\Rightarrow g(z) = \frac{V'(z) \pm Q(z) \sqrt{(z - \lambda_+) (z - \lambda_-)}}{2}$$

By enforcing $g(z) \sim \frac{1}{z}$ as $z \rightarrow \infty$

get $k+2$ constraints

→ get all k-th coeffs of Q and λ_{\pm}

$$\Rightarrow g(\lambda) = \frac{Q(\lambda) \sqrt{\lambda_+ - \lambda} (\lambda - \lambda_-)}{2\pi}$$

Near edge $g(\lambda_{\pm} \mp \delta) \sim \sqrt{\delta}$ unless Q has root of order n then

→ gives rise to $N^{-\frac{2}{3}}$
effects (ch 14) $\xrightarrow{N^{-\frac{2}{3}(6+n)}}$

$$5.3.3 \quad M^2 \leftarrow M^4$$

$$V(z) = \frac{x^2}{2} + \gamma x^4$$

$$\text{Symmetry} \Rightarrow \lambda_+ = -\lambda_- =: 2a$$

$$V = z + \gamma z^3 \quad Q(z) = a_0 + a_1 z + \gamma z^2$$

$$\Rightarrow g(z) = \frac{z + \gamma z^3 - (a_0 + a_1 z + \gamma z^2) z \sqrt{1 - \frac{4a^2}{z^2}}}{2}$$

$$z^3 \text{ coeff} \Rightarrow a_2 = \gamma$$

$$z^2 \text{ coeff} \Rightarrow a_1 = 0$$

$$z \text{ coeff} \Rightarrow 1 - a_0 + 2\gamma a^2 = 0$$

$$z^{-1} \text{ coeff} \Rightarrow 2a^4 \gamma + 2a^2 a_0 = 2$$

$$\Rightarrow g(z) = \frac{z + \gamma z^3 - ((1+2\gamma a^2 + \gamma z^2) z \sqrt{1 - 4a^2/z^2})}{2}$$

$$3\gamma a^4 + a^2 - 1 = 0 \Rightarrow a^2 = \frac{\sqrt{1+12\gamma} - 1}{6\gamma}$$

$$\Rightarrow g(\lambda) = \frac{(1+2\gamma a^2 + \gamma)^2 \sqrt{4a^2 - \lambda^2}}{2\pi} \quad \text{for } \gamma > -\frac{1}{12}$$

$$\text{At } \gamma = -\frac{1}{12} \Rightarrow a = \sqrt{2} \quad g(\lambda) = \frac{(8-\lambda^2)^{3/2}}{2\sqrt{\pi}}$$

$$g(z) = \frac{z^3}{2^4} \left(\left(1 - \frac{8}{z^2} \right)^{3/2} - 1 + \frac{12}{z^2} \right)$$

5.4 Fluctuations at large N

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = \frac{2}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} - V'(\lambda_i) \underset{\sim O(N)}{\sim} O(1)$$

Coulomb repulsion \Rightarrow force of order $\delta^2 \sim N$ on each eig
 $\text{d}\lambda \sim \sqrt{N}$

V' is of order $O(1) \Rightarrow$ repulsion dominates on small scales

\Rightarrow equidistributed on small scales

$$[\lambda - \frac{L}{2N}, \lambda + \frac{L}{2N}]$$

$$L \ll 1 \quad g'(\lambda) \frac{L}{N} \ll p(\lambda)$$

No fluctuations $\Rightarrow n(L) = g(\lambda)L + O(1)$

$$\text{For Poisson process } n(L) = g(\lambda)L + \xi \sqrt{g(\lambda)L} \underset{\sim N(0, 1)}{\sim}$$

$$\text{We get } n(L) = g(\lambda)L + \sqrt{D} \underset{\text{super small for large } \lambda}{\approx} \underset{\lambda := \frac{2}{L^2} [\log \tilde{n} + C] + O(\tilde{n}^{-1})}{\rightarrow \text{quasi-crystalline arrangement}}$$

Let's prove this:

$$H_{ij} = -\frac{\partial \mathcal{L}}{\partial \lambda_i \partial \lambda_j} = \begin{cases} V''(\lambda_i) + \frac{1}{N} \sum_{k \neq i} \frac{2}{(\lambda_i - \lambda_k)^2} & i=j \\ -\frac{2}{N} \frac{1}{(\lambda_i - \lambda_j)^2} & i \neq j \end{cases}$$

ϵ_i/N is the deviation of λ_i from equilibrium

$$\Rightarrow P(\{\epsilon_i\}) \propto \exp \left[-\frac{\beta}{q_N} \epsilon_i H_{ii} \epsilon_i \right]$$

recall $P(\{\lambda_i\}) \propto e^{-\frac{1}{2} \beta N \mathcal{L}}$

$$\Rightarrow e^{-\frac{1}{4} \beta N \delta^2 H_{ii}} \quad \delta = \frac{\epsilon}{N}$$

$$\Rightarrow \mathbb{E} \epsilon_i \epsilon_j = \frac{2N}{\beta} H_{ij}^{'}$$

$$\text{Locally } \lambda_i - \lambda_j \approx \frac{i-j}{Np}$$

$$\text{Take } V(x) = \frac{x^2}{2} \Rightarrow V'' = 1$$

$$\Rightarrow H = \begin{pmatrix} 1+4Np^2 + \frac{4Np^2}{q} & -2Np^2 & -\frac{2Np^2}{q} & \dots \\ -2Np^2 & 1+4Np^2 + \frac{4Np^2}{q} & \dots & \dots \\ -\frac{2Np^2}{q} & \dots & \dots & \dots \end{pmatrix}$$

$\hat{x}_q = \frac{1}{\sqrt{N}} \sum_{k=1}^N e^{-2\pi i \frac{kq}{N}} x_k$
 translation acts as $e^{-2\pi i q/N}$
 Up to bdy terms

$$\Rightarrow \mu_q = 1 + 2Np^2$$

$$\lambda \hat{x}_q = 1 + 2Np^2(2 - q - q + \frac{2}{q^2} - \frac{q^2}{q^2} - \frac{q^2}{q^2} - \dots)$$

$$du = \frac{2\pi q}{N} \quad u = \frac{2\pi lq}{N} \Rightarrow \frac{1}{l^2} = \frac{(2\pi q)^2}{(Nu)^2}$$

$$\Rightarrow \frac{du}{du} \frac{1}{l^2} = \frac{N}{2\pi q} \cdot \frac{(2\pi q)^2}{(Nu)^2} du$$

$$\Rightarrow \mu_q = 1 + 4Np^2 \times \frac{2\pi}{N} |q| \int_0^\infty du \frac{1 - \cos u}{u^2} = 1 + 4\pi^2 p^2 / q$$

$$\Rightarrow \text{evals of } H' \sim \frac{1}{\mu_q}$$

$$n = j - i \quad \frac{2\pi q}{N} = u \quad \frac{du}{du} = du \frac{N}{2\pi}$$

$$\mathbb{E}[\epsilon_i \epsilon_j] = \frac{2N}{\beta} \frac{1}{N} \sum_q \frac{e^{2\pi i q \eta/N}}{1 + 4\pi^2 p^2 / q} = \frac{2N}{\beta} \int_{-\pi}^{\pi} \frac{du}{2\pi} \frac{e^{iun}}{1 + 2\pi p^2 / u N} = \frac{2}{\beta} \int_{-\pi}^{\pi} \frac{du}{2\pi} \frac{e^{iun}}{N^2 + 2\pi p^2 / u}$$

$$\Rightarrow \mathbb{E}[(\epsilon_i - \epsilon_j)^2] = 2\mathbb{E}[\epsilon_i^2] - 2\mathbb{E}[\epsilon_i \epsilon_j] = \frac{4}{\beta} \int_{-\pi}^{\pi} \frac{du}{2\pi} \frac{1 - \cos un}{N^2 + 2\pi p^2 / u}$$

$$d_{ij} = \frac{n}{Np} + \frac{\epsilon_i - \epsilon_j}{N}$$

$$= \frac{8}{\beta \cdot (2\pi p)^2} \int_0^\pi du \frac{1 - \cos un}{u} \approx \frac{2}{\beta \pi^2 p^2} \log n$$

$$\text{Var} \sim \frac{2}{\beta \pi^2 p^2 N^2} \log n$$

$$\Rightarrow \frac{n}{Np} \approx \frac{2}{Np} \sqrt{N}$$

$$\Delta \sim \frac{2}{\pi^2 \beta} \log n$$

$$\frac{1}{N} \frac{n}{d_{ij}} = \frac{1}{N} \frac{1}{\frac{n}{Np} + \frac{\epsilon_i - \epsilon_j}{N}} = \frac{1}{\frac{1}{Np} + \frac{\epsilon_i - \epsilon_j}{n}} \approx p + p^2 \frac{\epsilon_i - \epsilon_j}{n}$$

$$\Rightarrow \text{Var } p = \frac{2p^2}{\pi^2 \beta} \frac{\log n}{n^2}$$

$$\Rightarrow \text{Var } n = \text{Var } pL = \frac{2}{\pi^2 p} \log \bar{n} \quad \bar{n} = pL$$

5.4.2 Top eig deviations

$$P(\lambda_{\max}; \{\lambda_i\}) = P_{N-1}(\{\lambda_i\}) \exp \left[-\frac{NB}{2} \left[V(\lambda_{\max}) - \frac{2}{N} \sum_{i=1}^{N-1} \log \lambda_{\max} - \lambda_i^* \right] \right]$$

The most likely positions of the $N-1$ λ_i are changed by $O(\frac{1}{N^2})$ from their eq values under P_{N-1}

$$\frac{P(\lambda_{\max}; \{\lambda_i\})}{P(\lambda_+; \{\lambda_i\})} \approx \exp \left[-\frac{NB}{2} \Phi(\lambda_{\max}) \right] \quad \text{← Large deviation principle}$$

$$\Phi(x) = V(x) - V(\lambda_+) - \frac{2}{N} \sum_{i=1}^{N-1} \log \left(\frac{x - \lambda_i^*}{x - \lambda_i^*} \right)$$

$$\Phi'(x) = V'(x) - \frac{2}{N} \sum_{i=1}^{N-1} \frac{1}{x - \lambda_i^*} \rightarrow V'(x) - 2g(x)$$

$$\Rightarrow \Phi(x) = \int_{\lambda_+}^x (V(s) - 2g(s)) ds$$

$$= \int_{\lambda_+}^x \sqrt{V(s)^2 - 4\pi(s)} ds$$

$\pi(s)$ of order $k-1 \Rightarrow$ for large s $\Phi(x) \approx V(x)$

$(V)^2$ of order $2k$

(repulsion plays no role)

For $x - \lambda_+$ small $\sqrt{V^2 - 4\pi} \sim (s - \lambda_+)^{\theta} \cdot c$
but $\gg \frac{1}{N}$

$$\Phi(\lambda_{\max}) \approx \frac{c}{\theta+1} (\lambda_{\max} - \lambda_+)^{\theta+1}$$

$$\text{recall } \pi_p(\lambda) \approx c(\lambda_+ - \lambda)^{\theta}$$

$$\Rightarrow \mathbb{E}(\lambda_{\max}) \sim (\lambda_{\max} - \lambda_+)^{\frac{\theta}{2}} \text{ generically}$$

$$\Rightarrow P(\lambda_{\max}) \sim \exp \left(-\frac{2}{3} \beta c u^{\frac{1}{2}} \right) \quad u = N^{\frac{2}{3}} (\lambda_{\max} - \lambda_+)$$

Ex 5.4.1

Wigner

$$V(x) = \frac{x^2}{2} \Rightarrow V' = x \quad \pi = 1 \quad \lambda_+ = 2$$

$$\Rightarrow \int_2^x \sqrt{s^2 - 4} \, ds = \frac{1}{2} x \sqrt{x^2 - 4} - 2 \log \left[\frac{\sqrt{x^2 - 4} + x}{2} \right]$$

$$\approx \frac{1}{2} x^2 \text{ as } x \rightarrow \infty$$

$$\sim 2\sqrt{x^2} \text{ as } x \rightarrow 2$$

$$V'(x) = 1 \quad \pi(x) = \sqrt{x}$$

$$\Rightarrow \int_2^x \sqrt{1 - \frac{4}{s}} \, ds = \sqrt{x(x-4)} + 2 \log \left[\frac{x - \sqrt{(x-4)x}}{2} - 2 \right]$$

$$\approx x \text{ as } x \rightarrow \infty$$

$$\sim 2\sqrt{x-4} \text{ as } x \rightarrow 4$$

5.5 Eigen Density Saddle Point

$$w(x) = \frac{1}{N} \sum_i \delta(x_i - x) \quad \text{"density field"}$$

$$P(\{w_i\}) = z^{-1} \exp \left\{ -\frac{BN^2}{2} \left[\int dx w \cdot V - \int w(x) w(y) \log |x-y| \right] - N \int dx w(x) \log x \right\} \quad \left. \begin{array}{l} \text{comes from change of vars} \\ \text{often neglected} \\ \text{since } N \rightarrow \infty \end{array} \right\} \quad \text{??}$$

$$\frac{\delta}{\delta w} = 0 \Rightarrow V(x) = 2 \int dy w(y) \log |x-y| + G \quad \left. \begin{array}{l} \text{comes from change of vars} \\ \text{often neglected} \\ \text{since } N \rightarrow \infty \end{array} \right\}$$

$$\Rightarrow V(x) = 2 \int dy \frac{w(y)}{x-y} \quad \left. \begin{array}{l} \text{Tricomi type} \end{array} \right\}$$

Can solve using Tricomi-type equation ansatz:

$$f(x) = \int dx' \frac{g(x')}{x-x'}$$

$$\Rightarrow P(x) = -\frac{1}{\pi^2} \frac{1}{\sqrt{(x-a)(b-x)}} \int_a^b dx' \sqrt{(x-a)(b-x')} \frac{f(x')}{x-x'} + C$$

$$f(x) = \frac{\sqrt{x}}{2} = \frac{x}{2} \Rightarrow -\frac{1}{\pi^2} \frac{1}{\sqrt{a^2-x^2}} \int_{-a}^a dx' \sqrt{a^2-(x')^2} \frac{x'}{x-x'} = \frac{1}{2\pi^2} \frac{1}{\sqrt{a^2-x^2}} \left[\frac{1}{2} e^{a^2\pi} + x \int_{-a}^a dx' \frac{\sqrt{a^2-(x')^2}}{x'-x} \right]$$

$\underbrace{-\pi x^2}_{-\pi x^2}$

$$= \frac{1}{2\pi} \sqrt{a^2-x^2} \Rightarrow a=2 \text{ for normalization}$$

$$\text{For } V = \int_0^\infty dy \frac{w^*(y)}{x-y} \Rightarrow \frac{1}{\pi^2} \frac{1}{\sqrt{x(a-x)}} \int_0^a dx' \sqrt{x'(a-x')} \frac{x'}{x'-x}$$

$$= \frac{1}{2\pi^2} \frac{1}{\sqrt{x(a-x)}} \left[\underbrace{\int_0^a dx' \sqrt{x'(a-x')}}_{\frac{1}{8} a^2 \pi} + x \underbrace{\int_0^a dx' \frac{\sqrt{x'(a-x')}}{x'-x}}_{\frac{1}{2} x \pi(a-2x)} \right]$$

$$\rightarrow \frac{\pi}{2} (a+2x)(a-x)$$

$$= \frac{1}{q\pi} \sqrt{\frac{a-x}{x}} (a+2x)$$

$$a=\lambda_+ \Rightarrow w(x) = \frac{1}{q\pi} \sqrt{\frac{\lambda_+-x}{x}} (\lambda_++2x) \quad \lambda_+ = \frac{q}{\sqrt{3}} \text{ for norm}$$

$$2 \rightarrow \frac{q}{\sqrt{3}}$$

$$P(\text{matrix all } > 0) \sim \exp[-\beta CN^2] \quad C \sim \log \frac{3}{4}$$

For $V=0$ between two walls

$$w^* \sim \frac{1}{\pi} \frac{1}{\sqrt{(x-l_+)(x-l_-)}}$$

Chapter 8: Addition of Random Variables & Brownian Motion

$X = X_1 + \dots + X_N$ has

$$P(x) = \int \prod_i dx_i P(x_i) \delta(x - \sum x_i)$$

$$\Rightarrow P(k) = \mathcal{D}_1(k) \cdots \mathcal{D}_N(k)$$

$$\Rightarrow \mu(k) = \log \mathcal{D}(k) = \sum_i \mu_i(k)$$

"cumulants simply add"

In what follows we consider dB gaussian

$$\mathbb{E}[dB] = 0 \quad \mathbb{E}[(dB)^2] = dt$$

8.2 Stochastic Calculus

Brownian Motion = Weiner process

X_t is Gaussian of mean μt
and variance $\sigma^2 t$

Because of the infinite divisibility of Gaussians, can write:

$$X_{t_k} = \sum_{\ell=0}^{k-1} \mu \delta t + \sum_{\ell=0}^{k-1} \sigma \delta B_\ell \quad t_k = \frac{k t}{N} \quad \delta t = \frac{T}{N}$$

$$\delta B_\ell \sim N(0, \delta t) \text{ for each } \ell$$

$$\text{As } N \rightarrow \infty \quad \delta t \rightarrow dt \quad \delta B_k \rightarrow dB$$

$$X_{t_N} = X_t$$

$X_{t_k} \rightarrow$ continuous time Weiner

$$dX_t = \mu dt + \sigma dB_t \quad X_0 = 0$$

X_t depends on $X_{t'}$ but $X_{t'}$ & $X_t - X_{t'}$ are indep when $t' < t$

By convention X_{t_k} does not depend on dB_k
 $\Rightarrow X_t \perp dB_t$

$$\Rightarrow F(X(t+\delta t)) = F(X_t) + \delta X F'(X_t) + \frac{(\delta X)^2}{2} F''(X_t) + o(\delta t)$$

$$\begin{aligned}\delta X &= \mu \delta t + \sigma \delta B \\ \Rightarrow (\delta X)^2 &= \sigma^2 \delta t + \underbrace{\sigma^2 (\delta B)^2}_{\text{mean } 0} - \delta t + o(\delta t)\end{aligned}$$

$$\Rightarrow dF = \frac{\partial F}{\partial X} dX + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial X^2} dt$$

More generally when $\mu(X_t, t)$ $\sigma(X_t, t)$ "general Itô process"

$$dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dB_t$$

$$\Rightarrow dF_t = \frac{\partial F}{\partial X} dX + \left[\frac{\partial F}{\partial t} + \frac{\sigma^2(X_t, t)}{2} \frac{\partial^2 F}{\partial X^2} \right] dt$$

$$\text{For } N \text{ indep } \{X_{i,t}\} = \vec{X}_t$$

$$dX_{i,t} = \mu_i(\vec{X}_t, t) dt + dW_{i,t}$$

$$\mathbb{E}[dW_{i,t} dW_{j,t}] = C_{ij}(\vec{X}_t, t) dt$$

$$\Rightarrow dF_t = \frac{\partial F}{\partial X_i} dX_{i,t} + \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial X_i} \frac{\partial}{\partial X_i} C_{ij}(\vec{X}_t, t) \right] dt$$

$$\text{When noises are indep } C_{ij} \propto \mathbb{I} \Rightarrow \text{only } \left(\frac{\partial}{\partial X_i} \right)^2 F$$

8.2.3 Var as a function of t

$$\text{Take } \mu=0 \quad F(X) = X^2$$

$$\Rightarrow dF_t = 2X_t dX_t + \sigma^2 dt$$

$$\Rightarrow F_t = 2\sigma \int_0^t X_s dB_s + \sigma^2 t$$

$$\Rightarrow \mathbb{E} F_t = \sigma^2 t$$

8.2.4 Gaussian Addition

Use Itô's Lemma for $Z = X + Y$

Gaussian

$Z \rightarrow Z_t$ brownian

$Z_t = Y$

$$dZ_t = \mu dt + \sigma dB_t$$

$$F(Z_t) = \exp[ikZ_t]$$

$$\begin{aligned} dF_t &= ik e^{ikZ_t} dZ_t - \frac{k^2 \sigma^2}{2} F dt \\ &= \left(ik\mu - \frac{k^2 \sigma^2}{2}\right) F dt + ik F \sigma dB_t \end{aligned}$$

$$\Rightarrow dE[F_t] = \left(ik\mu - \frac{k^2 \sigma^2}{2}\right) F dt$$

$$\Rightarrow \log E[F_t] = \log E F_0 + \left(ik\mu - \frac{k^2 \sigma^2}{2}\right) t$$

take $t \rightarrow 1$

$\Rightarrow Z$ remains Gaussian

8.2.5 Langevin Equation

Goal: Construct a process s.t. steady state $X_t \sim P(X)$ for a given P

$$\text{For } dX_t = dB_t$$

Var X_t grows unboundedly

Instead, rescale at each step:

$$X_{t+1} = \frac{X_t + dB_t}{\sqrt{1 + dt}} \Rightarrow dX_t = dB_t - \frac{1}{2} X_t dt$$

"Ornstein-Uhlenbeck"

Converges to $N(0, 1)$

$$\text{Replace } -\frac{1}{2} X_t dt \text{ by } -\frac{V'(X_t)}{2} dt$$

For μdt we had $V = -\mu x$

$$\Rightarrow dF = \frac{\partial F}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} \overset{?}{\cancel{dt}}$$

$$= f'(X_t) \left[dB_t - \frac{1}{2} V'(X_t) dt \right] + \frac{1}{2} f''(X_t) dt$$

Demanding $\frac{d}{dt} E[F] = 0$

$$\Rightarrow 0 = -E[F'(X_t)V(X_t)] + E[F''(X_t)]$$

$$\Rightarrow E[F'(X_t)V(X_t)] = E[F''(X_t)]$$

$$\Rightarrow \nabla F \int dx p(x) f(x) V(x) = \int dx p(x) f''(x)$$

$$= \left[p(x) f'(x) \right]_{-\infty}^{\infty} - \int dx p(x) f'(x)$$

$$\Rightarrow \nabla F \int dx p(x) f(x) V(x) = - \int dx p(x) f'(x)$$

$$- p(x) V'(x) = p''(x)$$

$$\Rightarrow p(x) \propto \exp[-V(x)]$$

\Rightarrow Given $p(x)$, define $V(x) = -\log p(x)$

$$dX_t = dB_t - \frac{1}{2} V''(X_t) dt$$

\uparrow_{+1} \uparrow_0 \uparrow'

$$+ \rightarrow \sigma^2 t \Rightarrow dX_t = \sigma dB_t - \frac{\sigma^2}{2} V''(X_t) dt$$

$V'(X_t)$ also acts like $\frac{\partial}{\partial X_t} \Rightarrow V$ conjugate to X

For N vars: $dX_{jt} = \sigma dB_j + \frac{\sigma^2}{2} D_j \log P(\tilde{x}) dt$

Ex 8.2.1 Student's t

a) $P_\mu = \tilde{z}_\mu^{-1} \left(1 + \frac{x^2}{\mu}\right)^{-\frac{\mu+1}{2}}$

$$\Rightarrow V(x) = \frac{\mu+1}{2} \log \left(1 + \frac{x^2}{\mu}\right) + C$$

$$V'(x) = \frac{\mu+1}{2} \frac{2x/\mu}{1 + x^2/\mu} = (\mu+1) \frac{x}{\mu+x^2}$$

b) $E\left[\frac{x^2}{x^2+\mu}\right] = E\left[\frac{xV'}{1+x^2}\right] = \frac{E[1]}{1+\mu} = \frac{1}{1+\mu}$

c) $dX_t = dB_t - \frac{\mu+1}{\mu x^2} x dt$ $\mu \rightarrow \infty \Rightarrow dB - x dt$
Gaussian

d) Simulate

e) $V'(x) = \begin{cases} x & \\ \frac{\mu+1}{\mu x^2} x & \sim \frac{1}{x} \text{ as } x \rightarrow \infty \text{ (less restoring)} \\ \sim (\frac{1-\mu}{\mu})x \text{ as } x \rightarrow 0 \text{ (stronger force)} \end{cases}$

8.2.6 Fokker-Planck

$P(x, t)$

$$E[F(X_t) dB_t] = 0$$

$$dE[F(X_t)] = E[F(X_t) F(X_t)] dt + \frac{\sigma^2}{2} E[F''(X_t)] dt$$

\uparrow
 $-\frac{V'}{2}$

$$\Rightarrow \int f(x) \frac{\partial P(x, t)}{\partial t} dx = \int f'(x) F(x) P(x, t) dx - \frac{\sigma^2}{2} \int f''(x) P(x, t) dx$$

$$\Rightarrow \frac{\partial P}{\partial t} = - \frac{\partial}{\partial x} (F(x) P(x, t)) + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial x^2}$$

Fokker-Planck

$$\frac{\partial P}{\partial t} = 0 \Rightarrow P(x, t) \propto \exp\left[\frac{-F}{\sigma^2}\right] = \exp\left[-\frac{V'}{\sigma^2}\right]$$

Chapter 9 Dyson Brownian Motion

9.1 Perturbation Theory

$$H = H_0 + \epsilon H_1$$

$$\lambda_i = \lambda_{i,0} + \sum_k \epsilon^k \lambda_{i,k}$$

$$\vec{v}_i = \vec{v}_{i,0} + \sum_k \epsilon^k \vec{v}_{i,k} \quad \|v_i\| = \|v_{i,k}\| = 1$$

\Rightarrow 1st order variation is 0
 $\Rightarrow \vec{v}_{i,1} \perp \vec{v}_{i,0}$

$$\lambda_i = \lambda_{i,0} + \epsilon (H_1)_{ii} + \epsilon^2 \sum_{j \neq i} \frac{|(H_1)_{ij}|^2}{\lambda_{i,0} - \lambda_{j,0}} + O(\epsilon^3)$$

$$\vec{v}_i = \vec{v}_{i,0} + \epsilon \sum_{j \neq i} \frac{(H_1)_{ij}}{\lambda_{i,0} - \lambda_{j,0}} \vec{v}_{j,0} + O(\epsilon^2)$$

Using this, take M_0 initial X , Wigner

$$M = M_0 + \sqrt{dt} X,$$

Using rotational inv of Wigner, wlog X diag

$$(X_i)_{jj} \sim N(0, \frac{2}{N})$$

$$(X_i)_{ji} \sim N(0, \gamma_{ij})$$

Structures negligible over $dt \rightarrow 0 \Rightarrow$ deterministic

$$\Rightarrow |X_{ij}|^2 = \frac{1}{N} + O\left(\frac{1}{N}\right)$$

$$d\lambda_i = \sqrt{\frac{2}{\rho N}} dB_i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_i - \lambda_j} \quad \text{b.c. } |(X_i)_{jj}|^2 \text{ can be treated deterministically}$$

$$d\vec{v}_i = \frac{1}{N} \sum_{j \neq i} \frac{dB_{ij}}{\lambda_i - \lambda_j} v_j - \frac{1}{2N} \sum_{j \neq i} \frac{dt}{(\lambda_i - \lambda_j)^2} v_i$$

Perturbation theory becomes exact.
 Works at any N

$$dB_{ij} = dB_{ji} \text{ otherwise independent}$$

Ex 9.1.1

$$d\lambda_i = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_i - \lambda_j}$$

$$F(\lambda_i) = \frac{1}{N} \sum_i \lambda_i^2$$

a) $dF = \frac{\partial F}{\partial \lambda_i} d\lambda_i + \frac{2}{N} \frac{1}{2} \frac{\partial^2 F}{\partial \lambda_i^2} dt \cdot N$

$$= \frac{1}{N} \sum_i 2\lambda_i \left[\sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_i - \lambda_j} \right] + \frac{2}{N} dt$$

$$= \frac{1}{N} \sum_i 2\lambda_i \sqrt{\frac{2}{N}} dB_i + \frac{1}{N^2} \sum_{i,j \neq i} \frac{2\lambda_i}{\lambda_i - \lambda_j} dt + \frac{2}{N} dt$$

$$= \frac{1}{N} \sum_i 2\lambda_i \sqrt{\frac{2}{N}} dB_i + \left(\frac{MN-1}{N^2} + \frac{2}{N} \right) dt$$

b) $E[dF] = \frac{N+1}{N} dt$

$$\Rightarrow F(t) = F(0) + \frac{N+1}{N} t$$

Ito Calculus

$$dX_{kk} = \sqrt{\frac{2}{N}} dB_{kk} \quad dX_{ke} = \sqrt{\frac{2}{N}} dB_{ke}$$

Taking $X_0 = \text{diag}(\lambda_1(0), \dots, \lambda_N(0))$

$$d\lambda_i = \sum_k \frac{\partial \lambda_i}{\partial X_{kk}} \sqrt{\frac{2}{N}} dB_{kk} + \sum_{k \neq e} \frac{\partial \lambda_i}{\partial X_{ke}} \sqrt{\frac{1}{N}} dB_{ke} + \sum_k \frac{\partial^2 \lambda_i}{\partial X_{kk}^2} \frac{dt}{N} + \sum_{k \neq e} \frac{\partial \lambda_i}{\partial X_{ke}^2} \frac{dt}{2N}$$

$$X_0 + \delta X \Rightarrow \begin{pmatrix} \lambda_k & \delta X_{ke} \\ \delta X_{ek} & \lambda_e \end{pmatrix} \Rightarrow \lambda_k + \lambda_e \pm \frac{\lambda_k - \lambda_e}{2} \sqrt{1 + \frac{4(\delta X_{ke})^2}{(\lambda_k - \lambda_e)^2}}$$

$$\lambda_k \rightarrow \lambda_k + \frac{(\delta X_{ke})^2}{\lambda_k - \lambda_e} \Rightarrow \frac{\partial \lambda}{\partial X_{ke}} = 0 \quad \frac{\partial^2 \lambda_i}{\partial X_{ke}^2} = \frac{2\delta_{ik} - 2\delta_{ie}}{\lambda_k - \lambda_e}$$

$$\Rightarrow d\lambda_i = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_i - \lambda_j} \quad \text{as before!}$$

*

9.3 Dyson BM for the Resolvent

$$M_+ = M_b + X_+$$

$$g_N(z, \delta \lambda; \zeta) = \frac{1}{N} \sum_i \frac{1}{z - \lambda_i}$$

$$\frac{\partial g_N}{\partial \lambda_i} = \frac{1}{N} \frac{1}{(z - \lambda_i)^2}, \quad \frac{\partial^2 g_N}{\partial \lambda_i^2} = \frac{2}{N} \frac{1}{(z - \lambda_i)^3}$$

$$\begin{aligned} dg_N &= \frac{\partial g_N}{\partial \lambda_i} d\lambda_i + \frac{1}{2} \frac{\partial^2 g_N}{\partial \lambda_i^2} \frac{2}{N} dt \\ &= \frac{1}{N} \sqrt{\frac{2}{N}} \sum_i \frac{d\lambda_i}{(z - \lambda_i)^2} + \underbrace{\frac{1}{N^2} \sum_{i \neq j} \frac{dt}{(z - \lambda_i)^2 (\lambda_i - \lambda_j)}}_{= \frac{1}{2N^2} \sum_{i \neq j} \left[\frac{1}{(z - \lambda_i)^2 (\lambda_i - \lambda_j)} + \frac{1}{(z - \lambda_j)^2 (\lambda_j - \lambda_i)} \right]} + \frac{2}{N^2} \sum_i \frac{dt}{(z - \lambda_i)^3} \\ &= \frac{1}{2N^2} \sum_{i \neq j} \frac{2z - \lambda_i - \lambda_j}{(z - \lambda_i)^2 (z - \lambda_j)^2} = \frac{1}{N^2} \sum_{j \neq i} \frac{1}{(z - \lambda_i)(z - \lambda_j)^2} \\ &= \frac{1}{N^2} \sum_{i,j} \frac{1}{(z - \lambda_i)(z - \lambda_j)^2} - \frac{1}{N^2} \sum_j \frac{1}{(z - \lambda_j)^3} \\ &= -g_N \frac{\partial g_N}{\partial z} + \frac{1}{2N} \frac{\partial^2 g_N}{\partial z^2} \end{aligned}$$

$$\Rightarrow dg_N = \frac{1}{N} \sqrt{\frac{2}{N}} \sum_i \frac{d\lambda_i}{(z - \lambda_i)^2} - g_N \frac{\partial g_N}{\partial z} dt + \frac{1}{2N} \frac{\partial^2 g_N}{\partial z^2} dt$$

$$\Rightarrow \mathbb{E} dg_N = - \mathbb{E} \left[g_N \frac{\partial g_N}{\partial z} \right] dt + \underbrace{\frac{1}{2N} \frac{\partial^2}{\partial z^2} \mathbb{E} g_N dt}_{\text{viscosity}} \quad \} \text{exact } \forall N$$

$$\Rightarrow \frac{\partial}{\partial t} g = -gg' \quad \text{as } N \rightarrow \infty$$

inviscid Burgers'

- can develop singularities
- needs a viscosity term to regularize it (exactly the last term, dropped)

9.3.2 Evolution of Resolvent

$$G_+ = (z \mathbb{I} - M_+)^{-1}$$

$$M_+ = M_b + X_+$$

$$dG_{ij} = \sum_{kl} \frac{dG_{ij}}{dM_{kl}} dX_{kl} + \frac{1}{2} \sum_{klmn} \frac{\partial^2 G_{ij}}{\partial M_{kl} \partial M_{mn}} \text{cov}(X_{kl}, X_{mn}) dt$$

by M symmetric: $\frac{1}{2}[G_{ik}G_{jl} + G_{jk}G_{il}]$ $\frac{1}{N}(b_{km}\delta_{jn} + \delta_{kn}\delta_{jm})$

$$\Rightarrow \frac{\partial^2 G_{ij}}{\partial M_{kl} \partial M_{mn}} = \frac{1}{N} [(G_{im}G_{kn} + G_{in}G_{km})G_{jl} + \dots]$$

$$\Rightarrow dG_{ij} = \sum_{kl} G_{ik} G_{jl} dX_{kl} + \frac{1}{N} \sum_{kl} (G_{ik} G_{kl} G_{jl} + G_{il} G_{jl} G_{ik}) dt$$

$$\Rightarrow \mathbb{E}\left[\frac{dG}{dt}\right] = \underbrace{\frac{1}{N} \text{Tr } G}_{g} \underbrace{\mathbb{E}G^2}_{-\partial_z \mathbb{E}G} + \underbrace{\frac{1}{N} \mathbb{E}G^3}_{\frac{1}{2N} \partial_z^2 \mathbb{E}G}$$

$$\Rightarrow \mathbb{E} \frac{dG}{dt} = -g \partial_z \mathbb{E}G + \underbrace{\frac{1}{2N} \partial_z^2 \mathbb{E}G}_{\rightarrow 0}$$

linear in G if g
is known $\quad \text{as } N \rightarrow \infty$

9.4 DBM with V

$$P(\{\lambda_i\}) = z' \exp\left[-\frac{\beta}{2} \left(\sum_i NV(\lambda_i) - \sum_{j \neq i} \log |\lambda_j - \lambda_j| \right)\right]$$

be wary of the factor of 2

Langevin $\Rightarrow d\lambda_k = \sqrt{\frac{2}{N}} dB_k + \frac{1}{N} \left(-\frac{\beta}{2} NV'(\lambda_k) + \sum_{j \neq k} \frac{\beta}{\lambda_k - \lambda_j} \right) dt$

$$\sigma^2 = \frac{2}{N} = \sqrt{\frac{2}{N}} dB_k + \left[\frac{1}{N} \sum_{j \neq k} \frac{1}{\lambda_k - \lambda_j} - \frac{\beta}{2} V'(\lambda_k) \right] dt$$

Modified Burgers':

$$V(\lambda) = \frac{\lambda^2}{2} \Rightarrow \frac{\partial g}{\partial t} = -g \partial_z g + \frac{1}{2} \partial_z^2 (zg)$$

Ex 9.4.1

$$F_k = \frac{1}{N} \sum_i \lambda_i^k$$

a) $dF_k = \frac{1}{N} \sum_i k \lambda_i^{k-1} d\lambda_i + \frac{1}{2} \sum_i k(k-1) \lambda_i^{k-2} dt$

$$= \frac{1}{N} \sum_i k \lambda_i^{k-1} \left[\frac{\sqrt{2}}{N} dB_i + \left(\frac{1}{N} \sum_{j \neq i} \frac{\beta}{\lambda_i - \lambda_j} - \frac{\beta}{2} V'(\lambda_i) \right) dt \right] + \frac{2}{N} \frac{1}{2N} \sum_i k(k-1) \lambda_i^{k-2} dt$$

b) $E \frac{dF_2}{dt} = \frac{1}{N^2} \sum_{i,j} \underbrace{\frac{2\lambda_i}{\lambda_i - \lambda_j}}_{N(N-1)} - \frac{1}{N} \sum_i \lambda_i V'(\lambda_i) + \frac{2}{N} \sum_i \frac{1}{N} dt$
 $\beta=1$
 $= 1 + E \frac{1}{N} \sum_i \lambda_i V'(\lambda_i) + \frac{1}{N}$

c) For Wigner $V(x) = x$

$$\Rightarrow O = 1 + \frac{1}{N} - E F_2$$

$$\Rightarrow E F_2 = 1 + \frac{1}{N}$$

d) For general V we have as $N \rightarrow \infty$

$$1 = E \left[\frac{1}{N} \sum_i \lambda_i V'(\lambda_i) \right] = \mathbb{E}[XV'(X)]$$

e) For Wishart:

$$V(x) = q^{-1}x + (1-q^{-1}) \log x$$

$$\Rightarrow V'(x) = q^{-1} + \frac{1-q^{-1}}{x}$$

$$\Rightarrow X V'(X) = q^{-1}X + (1-q^{-1})$$

$$1 = \mathbb{E}(q^{-1}X) + 1 - q^{-1}$$

$$\Rightarrow q^{-1} \mathbb{E}(X) = q^{-1} \Rightarrow \mathbb{E}(X) = 1$$

f) $O = E \frac{dF_{k+1}}{dt} = \frac{1}{N} \sum_i k \lambda_i^k \left(\frac{1}{N} \sum_{j \neq i} \frac{\beta}{\lambda_i - \lambda_j} - \frac{\beta}{2} V'(\lambda_i) \right) + \frac{2}{N} \frac{1}{2N} \sum_i \overbrace{k(k+1) \lambda_i^{k-1}}^0 dt$

$$\Rightarrow \mathbb{E}(V(X)X^k) = \frac{1}{N} \sum_i V'(\lambda_i) \lambda_i^k = \frac{2}{N^2} \sum_i \frac{\lambda_i^k}{\lambda_i - \lambda_j}$$

$$= \frac{1}{N^2} \sum_{\ell=0}^{k-1} \sum_{i \neq j} \lambda_i^\ell \lambda_j^{k-\ell-1}$$

$$k=2 \Rightarrow \tau(V(X)X^2) = \frac{2}{N^2} \sum_{i \neq j} \lambda_i = \frac{2(N-1)}{N^2} \sum \lambda_i \\ \Rightarrow 2 \tau(X)$$

$$k=3 \Rightarrow \tau(V(X)X^3) = \frac{2}{N^2} \sum_{i \neq j} \lambda_i^2 + \frac{1}{N^2} \sum_{i \neq j} \lambda_i \lambda_j \\ = 2 \tau[X^2] + \tau[X]^2$$

g) For $V' = x \Rightarrow \tau(X^{k+1}) = \sum_{\ell=0}^{k-1} \tau(X^\ell) \tau(X^{k-\ell-1})$ 3 catalan

9.4.2 Fokker Planck for DBM:

$$\partial_t P = -\nabla \cdot (\vec{F}P) + \frac{\sigma^2}{2} \nabla^2 P$$

$$\sigma^2 = \frac{2}{N} \Rightarrow \dot{P} = \frac{1}{N} \sum_i \frac{\partial}{\partial \lambda_i} \left[\frac{\partial P}{\partial \lambda_i} - F_i P \right]$$

joint force $F = -\nabla \tilde{V}/2$

now
(no longer cumulative)

$$\tilde{V} = \frac{\beta}{N \sigma^2} \left[- \sum_{i \neq j} \log |\lambda_i - \lambda_j| + N \sum_i V(\lambda_i) \right]$$

$$\Rightarrow F_i = \beta \left[\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} - \frac{N \lambda_i}{2} \right]$$

$$\begin{aligned} & \frac{\partial}{\partial \lambda_k} \sum_i \log |\lambda_i - \lambda_j| \\ &= \sum_{ij} \frac{\delta_{ik} - \delta_{jk}}{\lambda_i - \lambda_j} \frac{1}{\lambda_i - \lambda_j} \\ &= \sum_j \frac{1}{\lambda_k - \lambda_j} - \sum_i \frac{1}{\lambda_i - \lambda_k} \\ &= 2 \sum_j \frac{1}{\lambda_k - \lambda_j} \end{aligned}$$

$$P(\{\lambda_i\}_{i=1}^N) = \exp \left[\frac{\beta}{2} \left[\sum_{i,j} \log |\lambda_i - \lambda_j| - N \sum_i \frac{\lambda_i^2}{2} \right] \right] W(\{\lambda_i\}_{i=1}^N)$$

$U := \int F/2 \quad u := F/2$

$$e^u W = \frac{1}{N} \sum_i \frac{\partial}{\partial \lambda_i} \left[P(e^u W) - F_i e^u W \right]$$

$$= \frac{1}{N} \sum_i e^u W_i'' + 2 \cancel{U' e^u W'} + U_i'' e^u W - \cancel{F_i e^u W'} - \cancel{F_i' e^u W} - \cancel{F_i U' e^u W}$$

$$\Rightarrow W = \frac{1}{N} \sum_i W'' - (-\underbrace{U_i'' + F_i' + F_i U_i'}_{V_i}) W$$

$$V_i = \frac{1}{2} F_i' + \frac{F^2}{2}$$

$$V_i = -\frac{\beta}{2} \left[\sum_{j \neq i} (\lambda_i - \lambda_j)^2 + \frac{N}{2} \right] + \frac{\beta^2}{2} \left[\frac{N \lambda_i}{2} - \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right]^2$$

$$1) \frac{1}{N} \times 2 \times \left(-\frac{\beta^2 N}{4} \right) \times \sum_{\substack{i,j \\ j \neq i}} \frac{\lambda_j}{\lambda_i - \lambda_j} = -\frac{\beta^2}{4} \frac{1}{2} \frac{N(N-1)}{2} \times 2 = -\frac{\beta^2 N(N-1)}{8}$$

cancel, sum

$$2) \sum_{\substack{j \neq i \\ k \neq i}} \frac{1}{\lambda_i - \lambda_j} \frac{1}{\lambda_i - \lambda_k} = \sum_{\substack{j \neq k \neq i \\ j \neq i}} \frac{1}{\lambda_i - \lambda_j} \frac{1}{\lambda_i - \lambda_k} + \sum_{\substack{i,j \\ j \neq i}} \frac{1}{(\lambda_i - \lambda_j)^2} \Rightarrow \frac{\beta^2}{4} \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2}$$

$$3) \frac{\beta^2}{4} \frac{N^2 \lambda_i^2}{4} = \frac{\beta^2 N^2 \lambda_i^2}{16}$$

$$\Rightarrow V_i = \frac{\beta^2 N^2 \lambda_i^2}{16} - \frac{\beta(2-\beta)}{4} \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2} - \frac{N\beta}{4} \left(1 + \frac{\beta(N-1)}{q} \right)$$

Potential

$$\Rightarrow \frac{\partial W}{\partial f} = -\Gamma W_f$$

$$\frac{2}{N} \sum_{i=1}^N \left(-\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + \frac{1}{2} V_i \right) W_f = \Gamma W_f \quad \leftarrow \text{eval eq'n}$$

"Real Schrödinger Equation"

Calayers Model

$$\Gamma(n_1 \dots n_N) = \frac{\beta}{2} \left(\sum_i n_i - \frac{N(N-1)}{2} \right) \quad 0 \leq n_1 < \dots < n_N$$

$W_f = 0$ when $\lambda_i = \lambda_j \Rightarrow W$ is fermionic

$$n_1=0 \dots n_N=N-1 \Rightarrow \Gamma = 0$$

next excitation is

$$n_1=0 \dots n_{N-1}=N-2 \quad n_N=N \Rightarrow \Gamma = \frac{\beta}{2}$$

$$\tau_{eq} = \frac{2}{\beta} \sim O(1)$$

\Rightarrow converges in $O(1)$ time!

$$\beta=2 \Rightarrow \rightarrow$$

9.5 Karlin McGregor

Brownian motion:

$$p(y, t|x) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x-y)^2}{2t}\right]$$

$$\frac{\partial p}{\partial t} = -\frac{1}{2} \frac{1}{\sqrt{2\pi t}} \left[\frac{1}{t} - \frac{(x-y)^2}{t^2} \right] \exp\left[-\frac{(x-y)^2}{2t}\right]$$

$$\frac{\partial^2 p}{\partial y^2} = \frac{\partial}{\partial y} \left[\frac{1}{\sqrt{2\pi t}} \left(\frac{x-y}{t} \right) \exp\left[-\frac{(x-y)^2}{2t}\right] \right] = \frac{1}{\sqrt{2\pi t}} \left[\left(\frac{x-y}{t} \right)^2 - \frac{1}{t} \right]$$

$$\Rightarrow \frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial y^2} \quad \checkmark$$

N independent Brownian motions starting at $\vec{x} = (x_1, \dots, x_N)$

$$x_1, \dots, x_N$$

$$P_{km}(\vec{y}, t | \vec{x}) = \left| \det p(y_i, t | x_i) \right|$$

sum of \sum_i terms, each is a product \prod involving only one y_i for each i

$$\Rightarrow \frac{\partial}{\partial t} \Pi = \frac{1}{2} \sum_i \frac{\partial^2 \Pi}{\partial y_i^2}$$

$$\Rightarrow \frac{\partial}{\partial t} P_{km} = \frac{1}{2} \sum_i \frac{\partial P_{km}}{\partial y_i^2}$$

$$\text{Survival probability } p(t|\vec{x}) := \int dy \cdot p(y, t | \vec{x})$$

decreases w/ time

$$\text{can show } p(t|\vec{x}) \propto t^{-N(N-1)/2}$$

$$P(\vec{y}, t | \vec{x}) =: \text{IP(end at } \vec{y} \text{ @ } t \mid \text{start at } \vec{x} \text{ & never intersect)}$$

Not proven in book

$$\Rightarrow = \frac{\Delta(x)}{\Delta(y)} P_{kn}(y, t | \vec{x})$$

$$\Delta(x) := \prod_{i < j} (x_i - x_j)$$

Without confining potential @ $\beta=2$ this is

$$\exp\left[\frac{1}{2} \sum_{i,j} \log|x_i - x_j|\right] W = \Delta(\vec{x}) W(\vec{x}, t)$$

W obeys the diffusion equation here
& vanishes linearly when two eigs meet

\Rightarrow same as P_{kn} w/ $\vec{y} = \vec{x}$

Chapter 10: Addition of large random matrices

$A + B \Rightarrow$ what is $P_{A+B}(A)$ in terms of P_A, P_B

When A is Wigner, can use DBM from before

10.1 Wigner:

$$\text{Burgers': } \partial_t g = -g \partial_z g$$

$$g_0 := g_{M_0}(z)$$

Method of characteristics:

$$g_+(z) = g_0(z - t g_+(z))$$

$$\partial_t g_+ = g'_0(z - t g_+(z)) [-g_+(z) - + \partial_t g_+(z)]$$

$$\Rightarrow \partial_t g_+ = -\frac{g_+ g'_0(z - t g_+(z))}{1 + g'_0(z - t g_+(z))}$$

comes from
Method of
Characteristics

$$\partial_z g_+ = g'_0(z - t g_+(z)) (1 - g'_0(z - t g_+(z)))$$

$$\Rightarrow \partial_z g_+ = \frac{g'_0(z - t g_+(z))}{1 + g'_0(z - t g_+(z))} = -g_+ \partial_t g_+$$

$$\text{Eq } M_0 = 0 \Rightarrow g_0 = z^{-1}$$

$$\Rightarrow g_+ = [z - t g_+]^{-1}$$

$$\text{def } \beta_+(g_+(z)) = z \Rightarrow \beta_+ \text{ inv of } g_+$$

$$\Rightarrow g = g_+(z) = g_0(z - t g_+)$$

$$z = \beta_+(g)$$

$$\Rightarrow \mathcal{Z}_0(g) = z - tq = \mathcal{Z}_+(g) - tq$$

$$\Rightarrow \mathcal{Z}_+(g) = \mathcal{Z}_0(g) + tq \quad \Leftarrow \text{additive shift}$$

Eg 2 Mo Wigner wr var σ^2

$$\frac{1}{g(z)} = z - \sigma^2 g$$

$$\mathcal{Z}_0(g) = \sigma^2 g + \frac{1}{g}$$

$$\Rightarrow \mathcal{Z}_+(g) = \mathcal{Z}_0(g) + tq = (\sigma^2 + t)g + \frac{1}{g}$$

Generally

$$B = M_+$$

$$A = M_0$$

$$\begin{aligned} \mathcal{Z}_B(g) &= \mathcal{Z}_A(g) + tq \\ &= \mathcal{Z}_A(g) + \mathcal{Z}_{X_+}(g) - \frac{1}{g} \end{aligned}$$

$$\text{def } R(g) := \mathcal{Z}(g) - \frac{1}{g}$$

$$\Rightarrow R_B(g) = R_A(g) + R_{X_+}(g)$$

Eg White Wishart

$$qzg^2 - (z-1+q)g + 1 = 0$$

$$\Rightarrow z(qg-1)g = (q-1)g - 1$$

$$\Rightarrow z = \frac{q-1}{qg-1} - \frac{1}{(qg-1)g}$$

$$= \frac{(q-1)g-1}{(qg-1)g} = \frac{1}{g} + \frac{1}{1-qg}$$

$$\Rightarrow \mathcal{Z}(g) = \frac{1}{g} + \frac{1}{1-qg}$$

$$\Rightarrow R_W = \frac{1}{1-qg}$$

Ex 10.1.1

$$g(z) = \mathcal{C}(z\mathbb{I} - M)^{-1} = \int_{\text{supp } P} \frac{\rho(\lambda) d\lambda}{z - \lambda}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}} \quad m_0 = 1$$

a) $g = \frac{1}{z} + \frac{m_2}{z^2} + \frac{m_3}{z^3} + \frac{m_4}{z^4} + \dots$

$$\Rightarrow z^4 g = z^3 + z^2 m_2 + z m_3 + m_4$$

$$\epsilon = \frac{1}{z} \Rightarrow g = \frac{1}{z} + \frac{m_2}{z^2} + \dots$$

$$\Rightarrow \frac{1}{g} = \frac{z}{1 + m_2 z}$$

$$\Rightarrow z = \frac{1}{g} + \frac{1}{g} \frac{m_2}{z}$$

$$z \rightarrow \infty \Rightarrow g \rightarrow 0 \text{ near } g \approx 0 \quad \Im(g) \sim \frac{1}{g}$$
$$\Im(g) - \frac{1}{g} \sim \frac{m_2}{zg} \sim \frac{m_2}{1 + m_2 z} \rightarrow 1 \Rightarrow R(0) = m_2$$

b) Line by line $x_1 = 0$
in pert theory $x_2 = m_2$

$$\Rightarrow z^{(2)} = \frac{1}{g} + x_2 g$$

$$g^{(2)} = \frac{1}{z} + \frac{m_2}{z^2}$$

$$\Rightarrow z = \frac{1}{\frac{1}{z} + \frac{m_2}{z^2}} + \frac{x_2}{z} + \frac{x_2}{z^2} = z \left(1 - \frac{m_2}{z^2} \right) + \frac{x_2}{z} + \frac{x_2}{z^2}$$
$$\Rightarrow m_2 = x_2$$

determined by: $\frac{1}{z}$ $\frac{1}{z^2}$ $\frac{1}{z^3}$

$$z = \frac{1}{g} + m_2 g + x_3 g^2 + x_4 g^3$$

$$g = \frac{1}{z} + \frac{m_2}{z^2} + \frac{m_3}{z^3}$$

$$\Rightarrow z = \frac{z}{1 + \frac{m_2}{z^2} + \frac{m_3}{z^3} + \frac{m_4}{z^4}} + m_2 \left(\frac{1}{z} + \frac{m_2}{z^2} + \frac{m_3}{z^3} \right) + x_3 \left(\frac{1}{z} + \frac{m_2}{z^2} + \frac{m_3}{z^3} \right)^2 + x_4 \left(\frac{1}{z} \right)^3$$

$$= z - \underbrace{\frac{m_2}{z} - \frac{m_3}{z^2} - \frac{m_4}{z^3} + \frac{m_2^2}{z^3}}_{\text{red}} + \underbrace{\frac{m_2}{z} + \frac{m_3^2}{z^3} + \frac{m_2 m_3}{z^4} + \frac{K_3}{z^2}}_{\text{red}} + \dots + \underbrace{\frac{x_4}{z^3}}_{\text{blue}}$$

$$k_3 = m_3$$

$$x_y = m_y - 2m_2^2 \leftarrow \text{Free Cumulant!}$$

Ex 10.1.2

$$g_{\alpha A}^{(z)} = \tau((2I - \alpha A)^t) = \alpha^{-1} g_A(\alpha^{-1} z)$$

$$g_{A+bI}^{(z)} = g_A(z-b)$$

$$\mathcal{Z}_A(g_A) = z$$

$$\mathcal{Z}_A(g_A(z-b)) = z-b$$

$$\mathcal{Z}_A(g_{A+bI}(z)) = z-b$$

$$\Rightarrow \mathcal{Z}_{A+bI}(g_{A+bI}(z)) = \mathcal{Z}_A(g_{...}) + b$$

$$\Rightarrow \mathcal{Z}_{A+bI}(g) = \mathcal{Z}_A(g) + b$$

$$\mathcal{Z}_A(g_A(\alpha^{-1} z)) = \alpha^{-1} z$$

$$\alpha \mathcal{Z}_A(\alpha g_{\alpha A}(z)) = z$$

$$\mathcal{Z}_{\alpha A}(g_{\alpha A}) = z$$

$$\mathcal{Z}_{\alpha A}(g) = \alpha \mathcal{Z}_A(\alpha g)$$

$$\Rightarrow R_{A+bI} = R_A + b$$

$$\Rightarrow R_{\alpha A} = \mathcal{Z}_{\alpha A} - \frac{1}{\alpha}$$

$$= \alpha \mathcal{Z}_A(\alpha g) - \frac{\alpha}{\alpha g} = \alpha R_A(\alpha g)$$

Ex 10.1.3

M orthogonal symmetric
 X Wigner

$$E = M + X$$

a) evals of M are ± 1 each with $P = \frac{1}{2}$

$$\Rightarrow P(\lambda) = \frac{1}{2} \delta(\lambda \pm 1)$$

$$\Rightarrow \frac{1}{2} \left(\frac{1}{z-1} + \frac{1}{z+1} \right) = \frac{z}{z^2-1} = g_M(z)$$

$$\begin{aligned} b) \quad g_t(z) &= g_0(z - tg_t(z)) \\ &= \frac{z - tg_t(z)}{(z - tg_t(z))^2 - 1} \end{aligned}$$

$$\Rightarrow g_+(z)(z - tg_+(z))^2 - g_+(z) = z - tg_+(z)$$

$$t^2g^3 - 2tg^2 + (z^2 - 1)g = z - tg$$

c) $g_0 = \frac{z}{z^2 - 1} \Rightarrow z^2g_0 - z - g_0 = 0$

$$\Rightarrow z = \frac{1 \pm \sqrt{1 + 4g_0^2}}{2g_0}$$

$$\Rightarrow \exists_0(g) = \frac{1 + \sqrt{1 + 4g^2}}{2g} \quad \leftarrow \text{+ bc } \frac{1}{2} \text{ as } g \rightarrow \infty$$

d) $\exists_+(g) = \exists_0(g) + tg$

$$\Rightarrow z = \frac{1}{2g} + \frac{\sqrt{1 - 4g^2}}{2g} + tg$$

$$(2zg - 1 - 2tg^2)^2 = 1 - 4g^2$$

$$\cancel{4z^2g^2} - \cancel{1} + \cancel{4t^2g^4} - \cancel{4zg} - \cancel{4tg^2} - \cancel{8 + 2g^3} = \cancel{1 - 4g^2}$$

$$\Rightarrow \cancel{z^2g} + \cancel{t^2g^3} - \cancel{z} + \cancel{tg} - \cancel{2 + 2g^2} = \cancel{-g}$$

$$+ 2g^3 - 2tg^2 + (z^2 - 1)g = z - tg$$

\Leftarrow almost identical

e) Need nonreal g for real z

$$z=0 \Rightarrow t^2g^2 - 1 \neq$$

$$g = \pm \sqrt{\frac{1-t}{t}}$$

$\Rightarrow t > 1 \Rightarrow$ nonreal \Rightarrow eq density @ 0

$$\Rightarrow t^2 > 1$$

f) $\sigma^2 = 1 \Rightarrow g^3 - 2g^2z + g^2z^2 - z$
 $(t=1)$

$$\Rightarrow \Delta = -2z^2z^2 + 4z^4 = z^2(4z^2 - 2z)$$

$$(z=0) \times 2 \quad z = \pm \frac{\sqrt{2z}}{2}$$

$$\Delta < 0 \Rightarrow |z| < \frac{\sqrt{2z}}{2} = \frac{3\sqrt{3}}{2}$$

```
In[515]:= -g + g t + g^3 t^2 - z - 2 g^2 t z + g z^2 /. t → 1
```

```
Out[515]= g^3 - z - 2 g^2 z + g z^2
```

```
In[520]:= a = 1; b = -2 z; c = z^2; d = -z;
```

```
18 a b c d - 4 b^3 d + b^2 c^2 - 4 a c^3 - 27 a^2 d^2
```

```
Out[521]= -27 z^2 + 4 z^4
```

$$g) \quad \sigma^2 = 1 \quad \text{thus } g(\theta) \sim 0$$

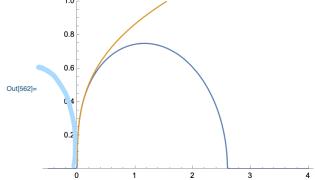
$$\Rightarrow g = z^{1/3} + \alpha(z)$$

$$\Rightarrow g(x) = \frac{\sqrt[3]{x}}{2} \quad \text{?} \quad \text{Im } \sqrt[3]{\epsilon} = \frac{\sqrt[3]{4\pi}}{2} \sqrt[3]{|\epsilon|}$$

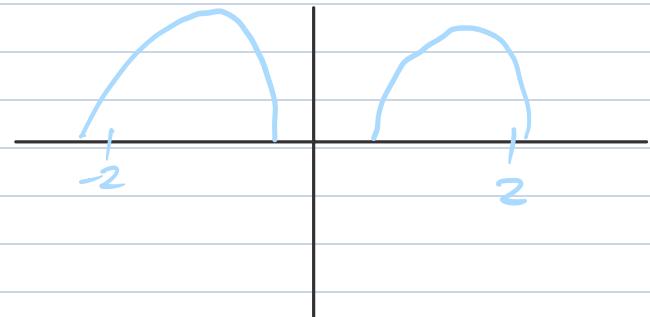
$$h) \quad t = 1:$$

```
In[601]:= soln = Solve[{-g + g t + g^3 t^2 - z - 2 g^2 t z + g z^2 /. t → 1} == 0, g];
```

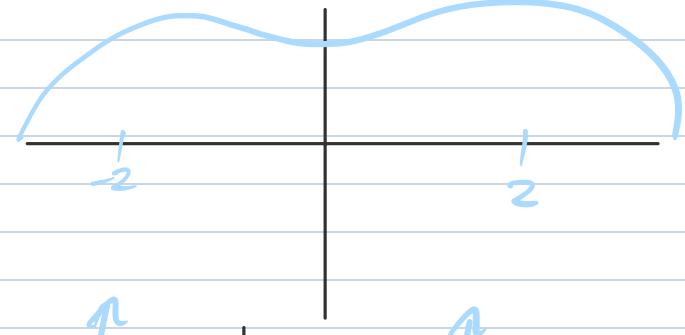
```
Plot[{Im[g /. soln[[2]] /. z → x + I 0.001], Sqrt[3] x^{1/3}}, {x, -0.5, 4}, PlotRange → {0, 1}]
```



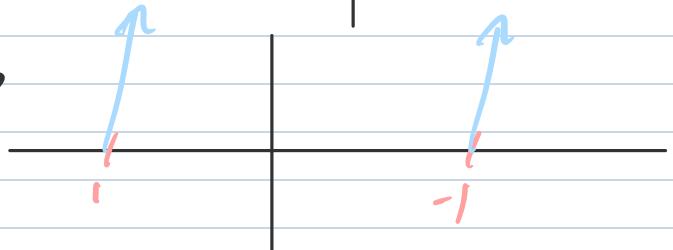
$$t = \frac{1}{2} \Rightarrow$$



$$t = 2 \Rightarrow$$



$$t = 0 \Rightarrow$$



10.2 Generalization to Non-Wigner:

IF A, B have common eigs

$A+B$ has $v_i = \lambda_i + \mu_i$

Assume evecs are Haar distributed \Rightarrow unlikely overlap

\Rightarrow define "free addition"

$$C = B + O A O^T, \quad O \sim \text{Haar}(O(N))$$

Candidate for CGF:

$$I(X, T) := \left\langle \exp\left(\frac{N}{2} \text{Tr } T O X O^T\right) \right\rangle_O \quad \text{depends only on evals of } X$$

$$\begin{aligned} I(B + O_A O_A^T, T) &= \left\langle \exp\left(\frac{N}{2} \text{Tr } T O B O^T + \frac{N}{2} \text{Tr } T O O_A O_A^T O^T\right) \right\rangle_O \\ &= I(B, T) I(A, T) \end{aligned}$$

$\Rightarrow \log I$ is additive!

Start with $T = \gamma \gamma^T$ rank 1

$$I(T, B) = \exp\left[\frac{N}{2} H_B(T)\right]$$

$$H_B := \lim_{N \rightarrow \infty} \frac{2}{N} \log \left\langle \exp\left[\frac{N}{2} \text{Tr } \gamma \gamma^T O B O^T\right] \right\rangle_O$$

For A, B randomly rotated: $H_C = H_A + H_B$

Calculate H_D : WLOG B diag $O D O^T = \gamma \gamma^T$ random projector

$$Z_T(B) = \int \frac{d^N \gamma}{(2\pi)^{N/2}} \delta(\|\gamma\|^2 - Nt) \exp\left(\frac{1}{2} \gamma^T B \gamma\right)$$

$Z_T(t) \neq 1$ atm

$$\delta(x) = \int_{-\infty}^{\infty} e^{izx} \frac{dz}{2\pi} = \int_{-i\infty}^{i\infty} e^{-zx/2} \frac{dz}{4\pi i}$$

$$\Rightarrow Z_T(B) = \int_{1-i\infty}^{1+i\infty} \frac{dz}{4\pi i} \int \frac{d^N \gamma}{(2\pi)^{N/2}} \exp\left[\frac{1}{2} \gamma^T (B - z\mathbb{1}) \gamma + \frac{zNt}{2}\right]$$

$$= \int_{\Gamma - i\infty}^{\Gamma + i\infty} \frac{dz}{2\pi i} \exp \left[\frac{N}{2} \left(zt - \frac{1}{N} \sum_i \log(z - \lambda_i) \right) \right]$$

$\underbrace{F_t(z, B)}$

$$\partial_z F \Rightarrow t - \frac{1}{N} \sum_i \frac{1}{z - \lambda_i} \Rightarrow t - g_N^B(z) = 0$$

$$\Rightarrow z = g^{-1}(t)$$

For $x > \lambda_{\max}$ g is monotonically decreasing $\Rightarrow t < \lambda_{\max}$ works

likewise for $x < \lambda_{\min}$ but

$F_t(z, B)$ is analytic in z only for $\operatorname{Re}(z) > \lambda_{\max}$

↑

$\sim \bullet * z_{\text{crit}} = g^{-1}(t)$

↓

$\sqrt{\frac{2\pi}{N \cdot H}}$

$$\Rightarrow Z_t(B) = \frac{1}{\sqrt{N} \partial_z^+ F_t(z(t), B)} \exp \left[\frac{N}{2} [G(t) t - \frac{1}{N} \sum_i \log G(t) - \lambda_i(B)] \right]$$

$$= \frac{1}{z \sqrt{\pi N |g'(z(t))|}}$$

$$B = 0 \Rightarrow g = z^{-1} \Rightarrow z(t) = t^{-1}$$

$$\Rightarrow Z_t(0) = \frac{1}{2 + \sqrt{N}} \exp \left[\frac{N}{2} [1 + \log t] \right] \Rightarrow H_B(t) = \frac{d}{dt} \ln(Z_t(t))$$

$$\frac{d}{dt} \ln(Z_t(t), t) = zt - 1 - \log t - \frac{1}{N} \sum_i \log(z - \lambda_i)$$

not super easy to work with

Note $\partial_z \frac{d}{dt} \ln(Z_t(t), t) = 0 \Rightarrow \frac{d}{dt} H_B(t) = \frac{d}{dt} \frac{d}{dt} \ln(Z_t(t)) = \frac{\partial}{\partial t} \frac{d}{dt} \ln(Z_t(t), t)$

$$\Rightarrow H_B(t) = \int_0^t dt' R_B(t')$$

$\leftarrow \text{ensures } H(0) = 0$

$$= z - \frac{1}{t} = R_B(t)$$

H is additive $\Rightarrow R$ is too!

$$R_C(t) = R_A(t) + R_B(t) \quad \text{for } A, B \text{ relatively free}$$

Whenever rank $T \ll N$, can show

$$\Rightarrow I(T, B) \approx \exp \left[\frac{N}{2} \sum_i H_B(t_i) \right] = \exp \left[\frac{N}{2} \operatorname{Tr} H_B(T) \right]$$

Needed $+$ small $\Rightarrow z_{\text{crit}} = \beta(T)$ large

10.4 Invertibility of Steltjes:

$$g_N^A(z) = \sum_j \frac{1}{z - \lambda_j}$$

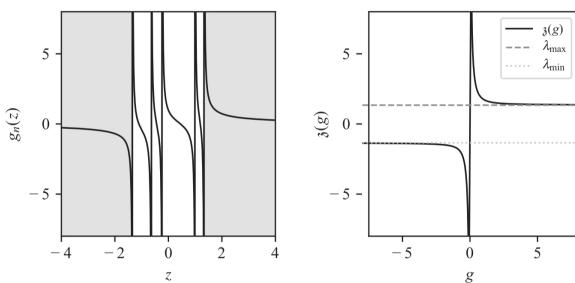
For $z < \lambda_{\min}$ or $z > \lambda_{\max}$ g_N is monotonically decreasing

$$g_N'(z) = \frac{1}{z} + O(\frac{1}{z^2})$$

$\Rightarrow g_N(z)$ is invertible for large z , behaves as $\frac{1}{z} + \text{regular}$

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Addition of Large Random Matrices



$z > \lambda_{\max} \Rightarrow g$ takes all values > 0 exactly once
 $z < \lambda_{\min} \Rightarrow$ " " < 0 "

$\Rightarrow \exists(g)$ exists by $g \neq 0$

$$\lim_{g \rightarrow -\infty} \exists(g) = \lambda_{\min}$$

$$\lim_{g \rightarrow \infty} \exists(g) = \lambda_{\max}$$

As $N \rightarrow \infty$ $g(z) = \frac{1}{z} + O(\frac{1}{z^2})$ still
 and $\exists(g) = \frac{1}{g} + \text{reg.}$

But now at eg λ , $g(z) = \int \frac{P(x) dx}{z-x}$

$$g(z) \sim (z - \lambda)^{\theta} \quad \theta > 0 \quad \Rightarrow \quad \int \frac{(z - \lambda)^{\theta}}{z - x} dx \Rightarrow \int (z - x)^{\theta-1} dx$$

Branch cut below λ_+

at $z = \lambda_+$
finite!

λ_+ is essential singularity for g

$g(\lambda_+)$ is well-defined

(what do they really mean by essential here...)

$\exists z(g)$ exists for $g_- \leq g \leq g_+$

$$\exists z(g_{\pm}) = \lambda_{\pm}$$

Wigner: $\lambda_{\pm} = \pm 2 \quad g_{\pm} = \pm 1 \Rightarrow z$ exists on $[-1, 1]$

10.4.3 Extending domain of $z(g)$

For e.g. MCIZ integral want to know z beyond g_{\pm}

$$\text{Wigner: } g + \frac{1}{g} - z = 0$$

$$\Rightarrow z(g) = g + \frac{1}{g} \leftarrow \text{not the inverse of } g \text{ for } |g| > 1$$

Wrong!

Realize we use g as an approximator to g_N for large N

For $z > \lambda_+$ this converges $g_N \rightarrow g$
but not on supp g

For finite N there is $\lambda_{\max} = \lambda_+$ at leading order in $1/N$

$$\Rightarrow g_N(z) \approx g(z) + \frac{1}{N} \frac{1}{z - \lambda_{\max}} \approx g(z) + \frac{1}{N} \frac{1}{z - \lambda_+}$$

At any finite distance above λ_+ , $g_N \rightarrow g$ as $N \rightarrow \infty$

$$\lim_{z \rightarrow \lambda_+} g_N(z) \rightarrow \infty \Rightarrow z(g) = \lambda_+ \text{ by } g \geq g_+$$

$$\Rightarrow z(g) = \begin{cases} \lambda_- & g < g_{\min} \\ z_{\text{bulk}}(g) & g \in [g_{\min}, g_{\max}] \\ \lambda_+ & g \geq g_{\max} \end{cases}$$

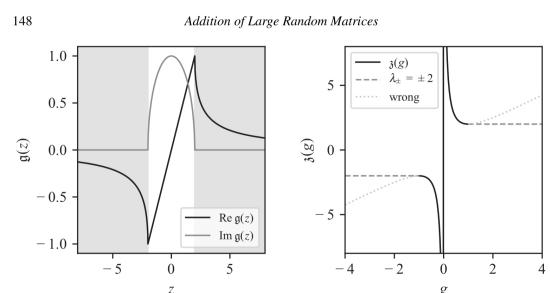


Figure 10.3 (left) The limiting function $g(z)$ for a Wigner matrix, a typical density that vanishes at its edges. The function is plotted against a real argument. In the white part of the graph, the function is ill defined and it is shown here for a small negative imaginary part of its argument. In the gray part ($z < \lambda_-$ and $z > \lambda_+$) the function is well defined, real and monotonic. It is therefore invertible. (right) The inverse function $z(g)$ only exists for $g_- \leq g \leq g_+$ and has a $1/g$ singularity at zero. The dashed lines show the extension of $z(g)$ to all values of g that are natural when we think of $g(z)$ as the limit of $g_N(z)$ with maximal and minimal eigenvalues λ_{\pm} . The dotted lines indicate the wrong branch of the solution of $z(g) = g + 1/g$.

10.4.4 Large t behavior of $I_t(B)$

$$I_t(B) := \left\langle \exp\left(\frac{N}{2} + \mathbf{V}^T \mathbf{B} \mathbf{V}\right) \right\rangle_{\mathcal{V}}$$

$$\exp\left[\frac{N}{2} + \lambda_{\min}\right] < I_t(B) < \exp\left[\frac{N}{2} + \lambda_{\max}\right]$$

$$\Rightarrow H_B \leq +\lambda_{\max}$$

For Wigner $R_W(t) = t \Rightarrow H_W = t^2/2$

but $\lambda_{\max} = 2 \Rightarrow$ fails since $H_W > 2t$ for $t > 4$

But our calculation is valid only for $t < g_t = g(2) = 1$

For $t > g_t, \zeta = \lambda_{\max}$

$$\Rightarrow \frac{dH}{dt} = \begin{cases} 3(H - \zeta t) = R_B(t) & t \leq g_{\max} \\ \lambda_{\max} - \zeta t & t > g_{\max} \end{cases}$$

Will later derive this w/ Replicas

10.5 Full-Rank HCIZ

$$I_\beta(A, B) := \int_{G(N)} dU \exp\left[\frac{\beta N}{2} \text{Tr } A U B U^\dagger\right]$$

$$G(N) = O(N), U(N), Sp(N)$$

$$\beta = 1, 2, 4$$

For $\beta=2$ the famous HCIZ result:

$$I_2(A, B) = \frac{c_N}{N^{N(N-1)/2}} \frac{\det(e^{Nv_i x_i})}{\Delta(A) \Delta(B)}$$

*Vandermonde
dets*

$$c_N = \prod_{i=1}^{N!} i!$$

Can be obtained from Karlin-McGregor

10.5.1 Derivation

Interpret $e^{N \text{Tr } A U B U^\dagger}$ for $\beta=2$ as a diffusion propagator in the space of unitary matrices

$$P(B|A) \propto N^{N/2} e^{-N/2 \text{Tr}(B-A)^2}$$

↑
 det factor ↑
 multi dim gaussian

eigs follow

$$dx_i = \sqrt{\frac{1}{N}} dB_i + \frac{1}{N} \sum_j \frac{dt}{x_i - x_j}$$

$$\sigma^2 = 1/N \quad \beta = 2$$

$$P(\{x_i\} / \{v_i\}) = \frac{\Delta(B)}{\Delta(A)} P(\lambda, t=1 | v)$$

det(p(x_i, t=1 | v_i))

$$\sqrt{\frac{N}{2\pi}} P^N \begin{vmatrix} e^{-(x_1 - v_1)^2/2} & e^{-(x_1 - v_2)^2/2} & \dots \\ e^{-(x_2 - v_1)^2/2} & \dots & \dots \\ \dots & \dots & e^{-(x_N - v_N)^2/2} \end{vmatrix} = \left(\frac{N}{2\pi}\right)^{N/2} e^{\frac{N}{2}(\sum x_i^2 + \sum v_i^2)} \det(e^{N\lambda_i v_i})$$

$$\widehat{P(B|A)} := \frac{\int dU P(U B U^\top | A)}{\Omega_N} = N^{N/2} e^{-\frac{N}{2}(\text{Tr} A^2 + \text{Tr} B^2)} \frac{I_2(A, B)}{\Omega_N}$$

$$\Omega_N = \int_{\mathcal{U}_N} dU$$

$$\frac{B}{dB} \rightarrow \lambda_i \quad \frac{d\lambda_i}{d\lambda_j} \quad \Delta^2(B) \rightarrow P(\{x_i\} / \{v_i\}) \propto N^{N/2} e^{-\frac{N}{2}(\text{Tr} A^2 + \text{Tr} B^2)} I_2(A, B) \Delta^2(B)$$

$$\Rightarrow I_2(A, B) \propto N^{N(N-1)/2} \frac{\det(e^{N\lambda_i v_i})}{\Delta(A) \Delta(B)}$$

can recover c_N from $A \rightarrow 1$

For $v_i = t$ $v_{i \neq i} = 0$ get rank 1 HClZ:

$$I_2(t, B) = \frac{(N-1)!}{(Nt)^{N-1}} \sum_j \frac{e^{Nt\lambda_j}}{\prod_{k \neq j} (\lambda_j - \lambda_k)}$$

10.5.2 HClZ at large N

DBM for $\widehat{P(B|A)}$

Start at v_i for $t=0$ end at λ_i

$$dx_i = \sqrt{\frac{I}{N}} dB_i - \partial_{x_i} V dt \quad V(x_i, z) = -\frac{1}{N} \sum_{j \neq i} \log |x_i - x_j|$$

$$\Rightarrow P(x_i, z) = \frac{1}{Z} \exp \left[-\frac{N}{2} \int_0^t dt \sum_i (x_i - \partial_{x_i} V)^2 \right] =: Z^{-1} e^{-N^2 S}$$

Neglecting Jac

Return this ^{to}

Chapter 11: Free Probabilities

III. 3.3 Additivity of the R-transform

$$\text{Def: } g_A(z) = \sum_k \frac{\tau(A^k)}{z^{k+1}}$$

$$\Rightarrow R_A(g) := \beta_A(g) - \frac{1}{g} \quad \Leftarrow \beta_A(g) \text{ formal power series satisfying } \beta_A(\beta_A(z)) = z \text{ to all orders}$$

scalar g :

$$\tau(g\mathbb{1}) = g = g_A(\beta_A(g)) = \tau[(\beta_A(g) - A)^{-1}]$$

$$\text{Def } gX_A = (\bar{z}_A - A)^{-1} - g\mathbb{1}$$

$$\Rightarrow A - \bar{z}_A = -\frac{1}{g} (1 + X_A)^{-1}$$

Take B free from A $\bar{z}_B := \beta_B(g)$

$$B - \bar{z}_B = -\frac{1}{g} (1 + X_B)^{-1}$$

$$\begin{aligned} X_A, X_B \text{ also free } \Rightarrow A + B - \bar{z}_A - \bar{z}_B &= -\frac{1}{g} (1 + X_A)^{-1} - \frac{1}{g} (1 + X_B)^{-1} \\ &= -\frac{1}{g} (1 + X_A)^{-1} (2 + X_A + X_B) (1 + X_B)^{-1} \end{aligned}$$

$$2 + X_A + X_B = (1 + X_A)(1 + X_B) + 1 - X_A X_B$$

$$\Rightarrow A + B - \bar{z}_A - \bar{z}_B + \frac{1}{g} = -\frac{1}{g} (1 + X_A)^{-1} (1 - X_A X_B) (1 + X_B)^{-1}$$

$$\begin{aligned} \Rightarrow [A + B - (\bar{z}_A + \bar{z}_B - \frac{1}{g})]^{-1} &= -g (1 + X_B) (1 - X_A X_B)^{-1} (1 + X_A) \\ &= -g (1 + X_B) \sum_k (X_A X_B)^k (1 + X_A) \end{aligned}$$

$$= \tau[A + B - (\bar{z}_A + \bar{z}_B - \frac{1}{g})]^{-1} = -g \tau(1) + \tau(X_A X_B \dots) \quad \text{---}$$

$$g_{A+B}(\bar{z}_A + \bar{z}_B - g^{-1}) = g$$

$$\Rightarrow \beta_{A+B}(g) = \beta_A(g) + \beta_B(g) - \frac{1}{g}$$

$$\Rightarrow R_{A+B}(g) = R_A(g) + R_B(g)$$

11.3.4 R-transform & Cumulants

$$R_A(g) = \sum_k x_k g^{k-1}$$

$$R_A(g) = g(g_A) - \log_A \Rightarrow z g_A(z) - 1 = g_A^{(2)} R_A(g_A(z))$$

$$\underbrace{\sum_{k=1}^{\infty} \frac{m_k}{z^k}}_{m} = \sum_{k=1}^{\infty} x_k \left[\underbrace{\frac{1}{z} + \sum_{\ell=1}^{\infty} \frac{m_\ell}{z^{\ell+1}}}_{R_A(g_A(z))} \right]^k$$

$$\begin{aligned} \Rightarrow z^{-1}: m_1 &= x_1 & \Rightarrow m_1 = x_1 \\ \bar{z}^{-2}: m_2 &= x_2 + x_1 m_1 & \Rightarrow m_2 = x_2 + x_1^2 \\ \bar{z}^{-3}: m_3 &= x_3 + 2x_2 m_1 + x_1 m_2 & \Rightarrow m_3 = x_3 + 3x_2 x_1 + x_1^3 \end{aligned}$$

$m_k = x_k + \text{lower } x_\ell \cdot m_\ell \text{ terms}$

$\Rightarrow x_k(A) = C(A^k) + \text{homogenous products of lower order}$

+ additive \Rightarrow uniquely defines the cumulant

11.3.5 Cumulants from non-crossing

$$m_n = \sum_{\pi \in NC(n)} x_{\pi_1} \cdots x_{\pi_{l(\pi)}} \quad \pi \text{ is a non-crossing partition of } n \text{ elements}$$

$$\sum_{k=1}^{l(\pi)} \pi_k = n$$

Eg: $m_4 = \overbrace{\text{||||}}^{x_1^4} + \overbrace{\text{|||}}^{x_2 x_1^2} + \overbrace{\text{|||}}^{x_3 x_1} + \overbrace{\text{|||}}^{x_2^2} + \text{|||} + \text{|||} + \text{|||} + \text{|||} + \text{|||} + \text{|||} + \text{|||}$

$$= x_1^4 + 6x_2 x_1^2 + 2x_2^2 + 4x_3 x_1 + x_4$$

$$\Rightarrow m_n = \sum_{l=1}^n x_l \prod_{\substack{k_1 \dots k_l=0 \\ k_1 + \dots + k_l = n-l}} m_{k_1} \dots m_{k_l}$$

size of l's partition

$$\Rightarrow \sum_{n=1}^{\infty} \frac{m_n}{z^n} = \sum_{n=1}^{\infty} \frac{1}{z^n} \sum_{k=1}^n x_k \prod_{l=1}^{k-1} m_{x_l} \cdots m_{x_1}$$

$$= \sum_{k=1}^{\infty} x_k \left[\sum_{n=k}^{\infty} \frac{m_k}{z^{k+1}} \right]^k = g(z) R(g(z))$$

For z^{-n} need $\sum k_i \cdot l_i = n$ w/ const $x_k m_{x_1} \cdots m_{x_k}$

N.B. at

In commutative case

$$\sum_{n=0}^{\infty} \frac{m_n z^n}{n!} = \exp \left[\sum_{r=1}^{\infty} x_r \frac{(zt)^r}{r!} \right]$$

this stage
there is nothing
to do w/ freeness
as we focus only
on 1 variable A

II.3.6 Freeness as vanishing of mixed cumulants

$$\tau(A_1 \cdots A_n) = \sum_{\pi \in MC(n)} \chi_{\pi}(A_1, \dots, A_n)$$

mixed cumulant

eg $\tau(A_1 A_2 A_3) = \boxed{\boxed{\boxed{}} + \boxed{\boxed{\boxed{}} + \boxed{\boxed{\boxed{}} + \boxed{\boxed{\boxed{}} + \boxed{\boxed{\boxed{}}$

$$\begin{aligned} & \chi_1(A_1) \chi_1(A_2) \chi_1(A_3) + \chi_2(A_1 A_2) \chi_1(A_3) + \chi_2(A_1 A_3) \chi_1(A_2) \\ & + \chi_3(A_1) \chi_2(A_2 A_3) + \chi_3(A_1 A_2) \chi_1(A_3) \end{aligned}$$

A set of variables is free iff all of their mixed cumulants vanish

Free $\rightarrow \chi$'s additive

$$\chi_k(A+B, \dots, A+B) = \chi_k(A) + \chi_k(B)$$

II.3.7 CLT for Free Vars

Sum of K free identically distributed (FID) variables $\sqrt{K} \Rightarrow$ constant w/ same mean

Define Wigner var as $x_2 \geq 0$, all other $x_i = 0$

$$\Rightarrow R(g) = x_2 g$$

$$\Rightarrow \xi = x_2 g - \frac{1}{2}g$$

$$\Rightarrow g = \frac{z - z\sqrt{1 - \frac{4}{x_2}}}{2x_2} \quad x_2 = \sigma^2 \Rightarrow \text{Wigner w/ } \sigma^2 \text{ exists}$$

CLT says $\sqrt{K} \sum_{i=1}^K x_i / K$ are $x_i = 0 \quad x_2 > 0$ FID
 \rightarrow Wigner with x_2

Again, pairwise-freeness is not enough

11.3.8

$$J_A(g) + R_B(g) = J_{A+B}(g)$$

$\underbrace{}_{:=z}$

$$\Rightarrow J_A(g) = J_{A+B}(g) - R_B(g)$$

$$\text{set } g = J_{A+B}(z) \Rightarrow J_A(J_{A+B}(z)) = z - R_B(J_{A+B}(z))$$

$$\Rightarrow J_{A+B}(z) = J_A(z - R_B(J_{A+B}(z)))$$

"Subordination relation"

To get from $J_A(z)$ to $J_{A+B}(z)$, shift z by $-R_B(J_{A+B}(z))$

11.4 Free Product

IF A, B are free & $\tau(A) = \tau(B) = 0$

$$\tau((AB)^k) = \tau(ABAB\dots) = 0 \Rightarrow \text{trivial!}$$

11.4.1 Low moments

$C = AB$ A, B free $\tau(A) \neq 0 \quad \tau(B) \neq 0$ wlog $\tau(A) = \tau(B) = 1$

$$\tau(C) = \tau[(A - \tau(A))(B - \tau(B))] + \tau(A)\tau(B) = 1$$

$\cancel{\text{mean}}$ $\cancel{0 \text{ vars}}$

$\Rightarrow 0$

$$\tau(C) = x_2(A, B) + x_1(A)x_1(B) = 1$$

mixed cumulant

$$\begin{aligned}\tau(C^2) &= \tau(ABAB) = x_1(A)^2 x_1(B)^2 + x_2(A)x_2(B)^2 + x_2(B)x_1(A)^2 \\ &= 1 + x_1(A) + x_2(B)\end{aligned}$$

$$\Rightarrow x_2(C) = \tau(C^2) - \tau(C)^2 = x_2(A) + x_2(B)$$

$$\tau(C^3) = \tau(ABABAB)$$

$$\begin{aligned}&= \boxed{1111} + \boxed{1111} + \boxed{1111} + \boxed{1111} + \boxed{1111} + \boxed{1111} + \boxed{1111} \\ &\quad + \boxed{1111} + \boxed{1111} + \boxed{1111} + \boxed{1111}\end{aligned}$$

$$= 1 + 3x_2(A) + 3x_2(B) + 3x_2(A)x_2(B) + x_3(A) + x_3(B)$$

$$\Rightarrow x_3(C) = \tau(C^3) - 3\tau(C^2)\tau(C) + 2\tau(C^2)$$

$$= x_3(A) + x_3(B) + 3x_2(A)x_2(B)$$

11.4.2 S-transform defn

First T-transform:

$$\begin{aligned}t_A(s) &= \tau[(1-s^{-1}A)^{-1}] - 1 \\ &= s g_A(s) - 1 \\ &= \sum_{k=1}^{\infty} \frac{m_k}{s^k}\end{aligned}$$

} same singularities as $g_A(z)$
except maybe at 0

$$\lim_{\gamma \rightarrow 0^+} \operatorname{Im} t(x-i\gamma) = \pi x \rho(x)$$

$$g \text{ regular } @ 0 \Rightarrow t_A(0) = -1$$

shifts from this indicate dirac mass there

$$t_A(s) = \tau \left[\frac{1}{1-s^{-1}A} - 1 \right] = \tau \left[\frac{s^{-1}A}{1-s^{-1}A} \right] = \tau[A(s-A)^{-1}]$$

For $m_1 \neq 0$ t_A is invertible for large ζ

$$\Rightarrow \xi_A(t) \text{ s.t. } t_A(\xi_A(t)) = t$$

Define $S_A(t) := \frac{t+1}{t+\xi_A(t)}$

$$S_A(t) \text{ has } t_A(\zeta) = \frac{1}{\zeta-1} \Rightarrow \xi_A(t) = t^{-1} + 1 \Rightarrow S_A(t) = 1$$

Identity is free wrt any var

$$t_{\alpha A}(\zeta) = \tau \left[(I - (\alpha' \zeta)^{-1} A)^{-1} \right] - 1 = t_A(\zeta/\alpha)$$

$$\Rightarrow \xi_{\alpha A}(t) = \alpha \xi_A(t)$$

$$\Rightarrow S_{\alpha A}(t) = \alpha^{-1} S_A(t) \quad \leftarrow \text{surprising}$$

recall though $S_A(0) = \mathcal{E}(A)$

Can show:

$$S_A(t) = \frac{1}{R_A(t) S_A(t)}$$

$$R_A(g) = \frac{1}{S_A(g) R_A(g)}$$

11.4.3 Multiplicativity of S

Fix t , ξ_A , ξ_B are inv T-transforms of t_A , t_B

$$E_A := (I - A/\xi_A)^{-1} - 1 - t$$

$$E_B := (I - B/\xi_B)^{-1} - 1 - t$$

$$A/\xi_A = I - (E_A + 1 + t)^{-1}$$

$$\begin{aligned} \frac{AB}{\xi_A \xi_B} &= [I - (E_A + 1 + t)^{-1}] [I - (E_B + 1 + t)^{-1}] \\ &= (1 + t + E_A)^{-1} \underbrace{[(t + E_A)(t + E_B)]}_{t^2 + t(E_A + E_B) + E_A E_B} (1 + t + E_B)^{-1} \end{aligned}$$

$$t(E_A + E_B) = \frac{+}{1+t} \left[(1+t+E_A)(1+t+E_B) - (1+t)^2 - E_A E_B \right]$$

$$\Rightarrow \frac{AB}{S_A S_B} = \frac{+}{1+t} + (1+t+E_A)^{-1} \left[-t(1+t) - E_A E_B \frac{+}{1+t} + t^2 + E_A E_B \right] (1+t+E_B)^{-1}$$

$$= \frac{+}{1+t} + (1+t+E_A)^{-1} \left[-t + \frac{E_A E_B}{1+t} \right] (1+t+E_B)^{-1}$$

$$1 - \frac{1+t}{+} \frac{AB}{S_A S_B} = (1+t) (1+t+E_A)^{-1} \left[1 - \frac{E_A E_B}{1+t} \right] (1+t+E_B)^{-1}$$

$$\Rightarrow \left(1 - \frac{1+t}{+} \frac{AB}{S_A S_B} \right)^{-1} = \frac{1}{1+t} (1+t+E_A) \left[1 - \frac{E_A E_B}{1+t} \right]^{-1} (1+t+E_B)$$

$$\sum_{k=0}^{\infty} \left(\frac{E_A E_B}{1+t} \right)^k$$

$$\Rightarrow \tau \left[\left(1 - \frac{AB}{S_A S_B} \frac{1+t}{+} \right)^{-1} \right] = 1+t$$

$$\Rightarrow t_{AB} \left(\frac{+}{1+t} S_A S_B \right) = + \Rightarrow S_{AB}(+) = \frac{+}{1+t} S_A S_B$$

$$\Rightarrow S_{AB}(+) = \left(\frac{+}{1+t} \right)^2 S_A S_B = S_A(+) S_B(+) \quad \text{X}$$

11.4.4 Subordination

$$S_{AB}(+) = \frac{S_A(+)}{S_B(+)}$$

$$t = t_{AB}(S_{AB}(+)) = t_A(S_{AB}(+) S_B(+))$$

$$\Rightarrow t_{AB}(z) = t_A(z S_B(t_{AB}(z))) \quad \text{X}$$

Exercise 11.4.1

$$1) \quad S_A(+) = \frac{+1}{+ S_A(+)} \quad t = j R_A(j) = zj - 1$$

$$S(gR(g)) = \frac{zg}{gR_A(g)} S_A(zg - 1)$$

$$\Rightarrow R_A(g) = \frac{1}{S(gR(g))}$$

2)

Chapter 12 : Free Random Matrices

Now we concretely look at large symmetric matrices

$$A, B \text{ free} \Leftrightarrow \tau(p_1(A)q_1(B)p_2(A)\cdots q_n(B)) = 0$$

$$\tau = \frac{1}{N} \operatorname{Tr}[\cdot]$$

$$A = U \Lambda U^T \quad B = V \Lambda' V^T \Rightarrow \tau(\Lambda, 0, \Lambda', 0^T, \Lambda_2, 0, \dots, \Lambda_n, 0^T) = 0 \quad *$$

$$O = U^T V$$

We will show * holds whenever we average over O so long as Λ_i, Λ'_i are traceless

12.1.2 Integration over $O(N)$

Want

$$\langle \tau(\Lambda, 0, \Lambda', 0^T, \Lambda_2, 0, \dots, \Lambda_n, 0^T) \rangle_O$$

$$\text{First: } I(i, j, n) := \langle O_{i_1 j_1} O_{i_2 j_2} \cdots O_{i_n j_n} \rangle_O$$

Worked out very recently in the general case (Weingarten functions)

2003 - 2008

When $N \rightarrow \infty$

$$I(i, j, n) = N^n \sum_{\substack{\text{pairings} \\ \pi}} S_{\pi(1)-\pi(2)} S_{\pi(3)-\pi(4)} \cdots S_{\pi(2n-1)-\pi(2n)} + O(N^{-n})$$

$$S_{\alpha-\beta} := S_{i_1 j_1 i_2 j_2 \cdots i_n j_n}$$

$$n=1 \Rightarrow N \langle O_{i_1 j_1}, O_{i_2 j_2} \rangle = \delta_{i_1 j_2} \quad \text{exact}$$

$$n=2 \Rightarrow N^2 \langle O_{i_1 j_1} O_{i_2 j_2} \rangle = \delta_{i_1 j_2} \delta_{i_2 j_1} + \delta_{i_1 j_1} \delta_{i_2 j_2} + \delta_{i_1 j_1} \delta_{i_2 j_2}$$

$$\Rightarrow \langle \tau(\Lambda, 0, \Lambda', 0^T) \rangle_O = \frac{1}{N} \sum_{i,j} \underbrace{\Lambda_i}_{\frac{1}{N} \delta_{ii} \delta_{jj}} \langle O_{i_1 j_1} \Lambda'_{i_2 j_2} \rangle_O$$

$$= \tau(\Lambda) \tau(\Lambda')$$

$$\Rightarrow \lim_{N \rightarrow \infty} \langle \bar{\epsilon}(A, O A^T, O^T A_1^T O A_2^T O A_2^T O^T) \rangle_O *$$

$$= \frac{1}{N} \sum_{ijkl} A_1^i A_j^l A_k^m A_l^n \langle \delta_{ij}^0 \delta_{kl}^0 \delta_{km}^0 \delta_{nl}^0 \rangle$$

$$= \frac{1}{N^3} \sum_{ijkl} A_1^i A_j^l A_k^m A_l^n (\delta_{ik}^2 + \delta_{jl}^2 + \delta_{jk}^2 + \delta_{il}^2) \quad \text{subleading}$$

$$= \bar{\epsilon}(A^i A^j) \bar{\epsilon}(A^k A^l) \bar{\epsilon}(A^m A^n) + \bar{\epsilon}(A^i A^j) \bar{\epsilon}(A^k A^l) \bar{\epsilon}(A^m A^n)$$

$$A_1 = A_2 = \mathbb{I} \Rightarrow \langle \bar{\epsilon}(\dots) \rangle = 1 \text{ but this gives } 2 !!!$$

Subleading terms matter!

Really: $\mathcal{I}(\vec{i}, \vec{j}, n) = \sum_{\pi, \sigma} W_n(\pi, \sigma) \tilde{\delta}_{i_1, j_1} \cdots \tilde{\delta}_{i_n, j_n}$

$$\tilde{\delta}_{\alpha\beta} = \delta_{i_1 \pi(\alpha), j_1 \pi(\beta)} \delta_{i_2 \sigma(\alpha), j_2 \sigma(\beta)}$$

$$\begin{array}{c} \pi \\ \sigma \end{array} \prod \boxed{\prod} \quad \# \text{ loops} = 3$$

W_n is matrix on space of perms π, σ

$\sigma = \pi$

$$\begin{array}{c} \pi \\ \sigma \end{array} \prod \boxed{\prod} \quad \# \text{ loops} = 1 < 2$$

pseudoinverse of $M_n = N^{l(n, \sigma)}$

$$\begin{aligned} G = \pi \Rightarrow l(\pi, \sigma) = n &\Rightarrow W \sim N^{-n} \\ \sigma \neq \pi \Rightarrow l(\pi, \sigma) < n \end{aligned}$$

Can show:

$$W_n(\pi, \sigma) = N^{l(\pi, \sigma) - 2n} \sum_{g=0}^{\infty} Q_g(\pi, \sigma) N^{-g} \quad \leftarrow \text{genus?}$$

→ Subleading term in * is $\pi \neq \sigma$ $l(\pi, \sigma) = 1 \Rightarrow N^{-3}$

Ex 12.1.1 For $n=2$ there are 3 perms

$$1. \quad l(\pi_1, \pi_1) = 2 \quad \checkmark$$

$$l(\pi_1, \pi_2) = 1$$

$$2. \quad M_2 = \begin{pmatrix} N^2 & N & N \\ N & N^2 & N \\ N & N & N^2 \end{pmatrix} \Rightarrow (N^2 - N)\delta + N\mathbb{I}^T$$

$$\Rightarrow N^2 \left(\mathbb{I} + \frac{1}{N} \mathbb{I}^T \right)$$

$$\Rightarrow M^{-1} \sim N^2 \mathbb{I} - N^{-3} \mathbb{I}^T$$

$$\Rightarrow W_2 = \begin{pmatrix} N^{-2} & -N^{-3} & -N^{-3} \\ -N^{-3} & N^{-2} & -N^{-3} \\ -N^{-3} & N^{-3} & N^{-2} \end{pmatrix}$$

3.

$$\begin{aligned}
 4. \langle \tau(OO^T OO^T) \rangle &= \frac{1}{N} \sum_{ijkl} \langle O_{ij} O_{kj} O_{kl} O_{il} \rangle \\
 &= \frac{1}{N^3} \sum (\delta_{ik}^2 + \delta_{jl}^2 + \delta_{ik} \delta_{jl}) \text{ AAAA} \\
 &\quad - \frac{1}{N^4} \sum (2\delta_{ik} \delta_{jl} + \delta_{ik} \delta_{jk} \delta_{jl} + 1 + \delta_{jl}) \underbrace{\text{AAA} \atop \substack{\text{1st row} \\ \text{2nd row}}} \\
 &\quad \underbrace{+ \delta_{ik} \delta_{jl} + 1}_{3\delta_{ik} \delta_{jl} + \delta_{ik} \delta_{jl} + 1} \\
 &= \tau(A_1 A_2) \tau(A_1') \tau(A_2') + \tau(A_1' A_2') \tau(A_1) \tau(A_2) \\
 &\quad - \tau(A_1) \tau(A_2) \tau(A_3) \tau(A_4) - \cancel{3 \tau(A_1 A_2) \tau(A_1' A_2')} \underbrace{\text{---}}_{n^2}
 \end{aligned}$$

\Rightarrow 1 to leading order
(can show to subleading as well)

12.1.4 Freeness of large matrices

$$\langle \tau(A_1 O A_1' O^T \dots A_n O A_n' O^T) \rangle_O$$

$$= \frac{1}{N} \sum_{\substack{i, j \\ i \neq j}} I(i, j, n) [A_1]_{i_{2n}, i} [A_2]_{i_2, i_3} \dots [A_n]_{j_{2n-1}, j_{2n}}$$

i & j never mix contractions

like closed strings

\Rightarrow Focus on i index

By assumption $\tau(A_i) = 0 \Rightarrow$ have $= L^{\frac{n}{2}}$ traces
 $\rightarrow L^{\frac{n}{2}} 1$ is max power of N

Combining $i, j \Rightarrow$ max power is $< N^n$

$$\begin{aligned}
 \frac{1}{N} \sum_{\sigma, \sigma'} \sum_{\substack{i, j \\ i \neq j}} W_n \tilde{\delta}_{i-2} \dots \tilde{\delta}_{2n-1-2n} [A_1]_{i_{2n}, i} [A_2]_{i_2, i_3} \dots [A_n]_{j_{2n-1}, j_{2n}}
 \end{aligned}$$

$O(\frac{1}{n!^2})$ $\uparrow \leq N^n$ $\underbrace{\sum L^{\frac{n}{2}} \text{ free vars over } i, \text{ over } j}_{\leq N^{2L^{\frac{n}{2}}}}$

N -indep

$$\leq O(N^{-1-n+2L\frac{1}{N^2}}) \rightarrow 0 \text{ as } N \rightarrow \infty$$

Can easily be adapted to UN $Sp(N)$

12.2 R-Transforms from Perturbation Theory

$$A + B^R \quad B^R = O B O^T$$

$$g(z) = \langle \tau_R[(zI - A - B^R)^{-1}] \rangle_0 =: \tau_R[(zI - A - B^R)^{-1}]$$

expand in B^R

$$g(z) = \tau_R(G_A) + \tau_R(G_A B^R G_A) + \dots$$

$\uparrow \uparrow$
free \Rightarrow mixed cumulants vanish

Three types of mixed moments:

$$\left. \begin{array}{l} m_n^{(1)} := \tau_R[BG \dots BG] \\ m_n^{(2)} := \tau_R[BG \dots GB] \\ m_n^{(3)} := \tau_R[BGS \dots BG] = \tau_R[GB \dots GS] \end{array} \right\} \begin{array}{l} B \text{ appears} \\ n \text{ times} \end{array}$$

$$m_0^{(1)} = \tau_R[G_A]$$

$$\tilde{M}^{(a)}(u) = \sum_{n=0}^{\infty} m_n^{(a)} u^n$$

$$m_0^{(2)} = m_0^{(3)} = 0$$

$$\text{care about } g(z) = \tilde{M}'(u=1)$$

Sum over size l of first group G_A belongs to

$$m_n^{(1)} = \sum_{k_1 \dots k_l} \chi_{G_A k_l} \prod_{\sum k_i + l = n} m_{k_1}^{(2)} \dots m_{k_{l-1}}^{(2)} m_{k_l}^{(3)}$$

$$m_n^{(2)} = \sum_{k_1 \dots k_l} \chi_{B k_l} \prod_{\sum k_i + l = n} m_{k_1}^{(1)} \dots m_{k_{l-1}}^{(1)} m_{k_l}^{(3)}$$

$$m_n^{(3)} = \sum_{k_1 \dots k_l} \chi_{B k_l} \prod_{\sum k_i + l = n} m_{k_1}^{(1)} \dots m_{k_l}^{(1)}$$

$$\Rightarrow M^{(1)}(u) = g_A(z) + \sum_{\ell=1} \chi_{G_A, \ell} u^\ell \tilde{M}^{(2)(\ell-1)} \tilde{M}^{(3)} = g_A + u \tilde{M}^{(3)} R_{G_A}(u \tilde{M}^{(2)})$$

$$\tilde{M}^{(2)}(u) = \sum_{\ell=1} \chi_{B, \ell} u^\ell \tilde{M}^{(1)(\ell-1)} M^{(3)} = u \tilde{M}^{(3)} R_B(u \tilde{M}^{(1)})$$

$$\tilde{M}^{(3)}(u) = \sum_{\ell=1} \chi_{B, \ell} u^\ell \tilde{M}^{(1)\ell} = u \tilde{M}^{(1)} R_B(u \tilde{M}^{(1)})$$

$$\Rightarrow g(z) = g_A(z) + g(z) R_B(g(z)) R_{G_A}(g(z) R_B(g(z))^2)$$

$$\text{Take } B = b \mathbb{I} \Rightarrow R_B = b$$

$$g = g_A + b g(z) R_{G_A}(g(z) b^2)$$

$\overbrace{g_A(z-b)}$

$$\Rightarrow R_{G_A}(b^2 g(z-b)) = \frac{g_A(z-b) - g_A(z)}{b g(z-b)}$$

$$\Rightarrow R_{G_A}[b^2 g(z)] = \frac{g(z) - g(z+b)}{b g(z)}$$

$$\text{setting } b = R_B(g)$$

$$\Rightarrow R_{G_A}[R_B^2(g) g] \cdot R_B(g) g = g - \underbrace{g(z+R_B(g))}_{}$$

$$\text{need } g_A(z) = g(z+R_B(g(z)))$$

$$\rightarrow g(z) = g_A(z - R_B(g(z))) \Leftarrow \text{Subordination}$$

12.3 CLT

$$M_K = \frac{1}{\sqrt{K}} \sum O_i A_i O_i^T \quad \text{Tr } A_i = 0$$

$$\mathbb{E}(A_i^k) = m_k$$

$$\Rightarrow R_{A_i} = \sum_{l=2}^{\infty} x_l z^{l-1}$$

$$\Rightarrow R_{M_K} = \sum_{l=2}^{\infty} K^{1-\frac{l}{K}} x_l z^{l-1}$$

At Finite K $R_{M_K} \approx \sigma^2 z + \frac{x_3}{\sqrt{K}} z^2 + \dots$

$\sigma = 1$ wlog $\underbrace{R_X}_{\sim} \underbrace{m}_{R_3}$

assume small x_3 expansion exists: $\Rightarrow g_{M_K}(z) = g_X(z) + \frac{x_3}{\sqrt{K}} g_3(z) + \dots$

$$g_X(z) = \frac{1}{2} (z - \sqrt{z^2 - 4}) \quad \Delta = z^2 - 4$$

$$R_{M_K}(g_{M_K}(z)) = R_X \left[g_X(z) + \frac{x_3}{\sqrt{K}} g_3(z) \right] + R_3(g_X(z))$$

in

$$3g - \frac{1}{g} \Rightarrow z - \frac{1}{g_X + \frac{x_3}{\sqrt{K}} g_3} = g_X + \frac{x_3}{\sqrt{K}} \frac{g_3}{g_X^2} = g_X + \frac{x_3}{\sqrt{K}} g_3(z) + R_3(g_X(z))$$

$$\Rightarrow \frac{g_3(1-g_X^2)}{g_X^2} = R_3(g_X)$$

$$\Rightarrow g_3 = -\frac{1}{2} \left[1 - \frac{z}{\sqrt{z^2 - 4}} \right] [g_X(z)]^2$$

$$= -g_X'(z) g_X(z)^2$$

$$= -\frac{1}{2} \left(1 - \frac{z\sqrt{\Delta}}{\Delta} \right) (z^2 - 2 - z\sqrt{\Delta})$$

$$\begin{aligned} z &= \lambda e^{i\theta} \\ \Rightarrow z^2 &= \lambda^2 e^{2i\theta} \end{aligned}$$

$$\Rightarrow z^2 - 4 = \lambda^2 - 4 + 2i\lambda\theta$$

$$\frac{\operatorname{Im} P}{\pi} = \frac{x_3}{2\pi\sqrt{x}} \frac{\lambda(x^2-3)}{\sqrt{4-x^2}}$$

$\underbrace{\phantom{\frac{x_3}{2\pi\sqrt{x}} \frac{\lambda(x^2-3)}{\sqrt{4-x^2}}}_{\delta p}$

$$\int_{-2}^2 \lambda^3 \delta p(\lambda) = \frac{x_3}{5K} \checkmark$$

For $x_3=0$ $x_4 \neq 0$

$$R_{M_k}(z) = \underbrace{\sigma^2}_1 z + \frac{x_4}{K} z^3 \quad g_{M_k} = g_x + \frac{x_4}{K} g_y + \dots$$

$$z - \frac{1}{g_x + \frac{x_4}{K} g_y} \approx g_x + \frac{x_4}{K} \frac{g_y}{g_x^2} = g_x + \frac{x_4}{K} g_y + \frac{x_4}{x} (g_x)^3$$

$$\Rightarrow g_y \left(\frac{1-g_x^2}{g_x^2} \right) = g_x^3$$

$$\text{In[642]:= } \frac{1}{2} \left(-1 + \frac{z}{\sqrt{-4+z^2}} \right) g x^3 // \text{Simplify}$$

$$\text{Out[642]:= } \frac{(z - \sqrt{-4+z^2})^4}{16 \sqrt{-4+z^2}}$$

$$\text{In[643]:= } \frac{(z - \sqrt{-4+z^2})}{16 \sqrt{-4+z^2}} // \text{Simplify}$$

$$\text{In[645]:= } \frac{1}{16} \left(-1 + \frac{z}{\sqrt{-4+z^2}} \right) (z - \sqrt{-4+z^2})^3 // \text{Expand}$$

$$\text{Out[645]:= } z - \frac{z^3}{2} + \frac{z^4}{16 \sqrt{-4+z^2}} - \frac{1}{4} \sqrt{-4+z^2} + \frac{7}{16} z^2 \sqrt{-4+z^2}$$

$$\delta p = \frac{x_4}{2\pi K} \frac{\lambda^4 - 4\lambda^2 + 2}{\sqrt{4-\lambda^2}}$$

$$\delta p_n(\lambda) = \frac{x_n}{\pi K^{n+1}} \frac{T_n(\lambda \epsilon)}{\sqrt{4-\lambda^2}}$$

$$\text{In[646]:= Assuming } [-2 < \lambda < 2 \& \epsilon > 0,$$

$$\text{Simplify}[\text{Im}\left[\frac{z^4}{16 \sqrt{-4+z^2}} - \frac{1}{4} \sqrt{-4+z^2} + \frac{7}{16} z^2 \sqrt{-4+z^2} /., z \rightarrow \lambda + I \epsilon\right]]]$$

$$\text{Out[646]:= } \frac{1}{2} \text{Im}\left[\frac{2 + \epsilon^4 - 4 i \epsilon^3 \lambda - 4 \lambda^2 + \lambda^4 + \epsilon^2 (4 - 6 \lambda^2) + 4 i \epsilon \lambda (-2 + \lambda^2)}{\sqrt{-4 + (\epsilon + \lambda)^2}} \right]$$

12.4 Finite Free Convolutions

$$p(z) = \prod_i (z - \lambda_i) = \det(z \mathbb{1} - A)$$

$$= \sum (-1)^k a_k z^{N-k}$$

$$a_k = \sum_{i_1, i_2, \dots, i_k} \lambda_{i_1} \dots \lambda_{i_k}$$

$\overset{\text{ordered k-tuple}}{i_1 < i_2 < \dots < i_N}$

$$\mu(\lambda_i) = \frac{a_1}{N} \quad \sigma^2(\lambda_i) = \frac{1}{N-1} \sum_j (\lambda_i - \mu)^2 = \frac{1}{N-1} \sum_i \lambda_i^2 - \frac{N \mu^2}{N-1}$$

$$= \frac{1}{N-1} (a_1)^2 - \frac{2}{N-1} a_2 - \frac{N}{N-1} \left(\frac{a_1}{N} \right)^2$$

$$= \left[\frac{N-1}{N(N-1)} - \right] a_1^2 - \frac{2}{N-1} a_2$$

$$= \frac{a_1^2}{N} - \frac{2a_2}{N-1}$$

$p(z)$ will often be $E[\det(zI - A)]$

\Rightarrow think of λ_i as deterministic

$$\lambda_i \text{ indep} \Rightarrow E[p(z)] = (z - E\mu_i)^N$$

$$\Rightarrow \lambda_i = E(\mu_i) + \epsilon_i$$

$$P_{M+\alpha I}(z) = P_M(z-\alpha)$$

$$P_{M+\alpha I}(z) = \alpha^N P_M(\frac{z}{\alpha} - z) \Rightarrow \tilde{a}_k = \alpha^k a_k$$

$$P_{M^{-1}} = \frac{(-z)^N}{a_N} P_M(\frac{1}{z}) \Rightarrow \tilde{a}_k = \frac{a_{N-k}}{a_N}$$

$$= \frac{z^N}{\lambda_1 \cdots \lambda_N} (\frac{1}{z} - \lambda_1) \cdots (\frac{1}{z} - \lambda_N)$$

$$= (\frac{1}{\lambda_1} - z) \cdots (\frac{1}{\lambda_N} - z)$$

$$p(z) = \hat{p}(\alpha_z) z^N$$

$$\hat{a}_k = (-1)^N \frac{k!}{N!} a_{N-k}$$

comes from $(-1)^{kk} \alpha_z^{kk} N! \text{ yielding } (-1)^k z^k$

12.4.2 Finite free addition:

p_1, p_2 monic

$$p_1 \boxplus p_2(z) = \langle \det[zI - A_1 - \alpha_1 \alpha_2^T] \rangle_0 \quad \text{"free additive convolution"}$$

$O \in O(N)$ or $U(N)$ or even S_N
result is the same!

$$\text{Will show soon: } p_1 \boxplus p_2 = \hat{p}_1(\alpha_z) p_2(z) = \hat{p}_2(\alpha_z) p_1(z) = \hat{p}_1(\alpha_z) \hat{p}_2(\alpha_z) z^N$$

Turns out $p_1 \boxplus p_2$ still has real roots!

$\Rightarrow \boxplus$ is bilinear in the coeffs

\Rightarrow If p_1, p_2 are polys of indep random matrices

$$E[p_1 \boxplus p_2] = E[p_1] \boxplus E[p_2]$$

$$a_k^s = \sum_{i+j=k} \frac{(N-i)!(N-j)!}{N! (N-k)!} a_i^{(1)} a_j^{(2)}$$

$$a_0^S = 1$$

$$a_1^S = a_1^{(1)} + a_1^{(2)}$$

$$a_2^S = a_2^{(1)} + a_2^{(2)} + \frac{N-1}{N} a_1^{(1)} a_1^{(2)}$$

\Rightarrow sample mean & variance adds under \boxplus

$$\text{let } p_0(z) = z^N \leftarrow a_0 = 1 \quad a_i = 0 \quad b_i > 0 \quad \Rightarrow \quad p \boxplus p_0 = p(z)$$

$$\text{let } p_\mu(z) = (z - \mu)^N \Rightarrow p \boxplus p_\mu = p(z - \mu)$$

Can show Hermite polynomials are stable under \boxplus

$$H_N \boxplus H_N = 2^{N/2} H_N(2^{-N/2} z)$$

$$\text{because } \widehat{H}_N(\partial_z) = e^{-\partial_z^2/2} \quad \text{by the differential operator rep'n of } H_N$$

$$\text{Wishart rank 1: } M = X X^T \quad X \sim N(0, \mathbb{I} \cdot N)$$

$$\text{IF char poly } \Rightarrow p(z) = z^{N-1}(z - N) = (1 - \partial_z) z^N$$

$$\Rightarrow \widehat{p}(\partial_z) = 1 - \partial_z$$

For Wishart of param t $\sum_T X_T X_T^T$ is the free sum of T such rank 1 projectors

$$\Rightarrow \widehat{p}_T(\partial_z) = (1 - \partial_z)^T$$

$$\Rightarrow p_T(z) = (1 - \partial_z)^T z^N$$

12.4.3 Finite R-Transform

$\log \widehat{p}(\partial_z)$ is additive under \boxplus

Define all $\widehat{p}(\partial_z)$ mod $\partial_z^{N+1} \Rightarrow$ finite dim ring

$$\widehat{p}(\partial_z) = 1 + O(\partial_z)$$

\Rightarrow define $\log \widehat{p}(u)$ as a formal series truncated beyond u^N

$$L(u) := -\log \widehat{p}(u) \text{ mod } u^{N+1}$$

$$\text{Eq 1} \quad p_2 = (z - 1)^N = \sum_k (-1)^k \binom{N}{k} z^{N-k} = \sum_k \frac{(-\partial_z)^k}{k!} z^N = \exp(-\partial_z) z^N$$

$$\Rightarrow \widehat{p}_2(u) = \exp(-u) \text{ mod } u^{N+1} \Rightarrow L_2(u) = u$$

Eg 2 For Wigner E char poly is

$$p_X(z) = N^{-N/2} H_N(\sqrt{N}z) = \exp\left[-\frac{1}{2N} z^2\right] z^N$$

$$\Rightarrow L_X(u) = \frac{u^N}{2N}$$

Eg 3 For Wishart E char poly is normalized Laguerre:

$$p_W(z) = \left(1 - \frac{1}{T} \frac{\partial}{\partial z}\right)^T z^N \quad T = N/q$$

$$\Rightarrow L_W(u) = -\frac{N}{q} \log\left(1 - \frac{qu}{N}\right) \bmod u^{N+1}$$

In these three cases:

$$L'(u) = [R(y_N)] \bmod u^{N+1}$$

Generally:

$$\lim_{N \rightarrow \infty} L'(Nu) = R(u)$$

12. 9. 4 Finite free product

$$p_1 \boxtimes p_2 := \langle \det[zI - A_1 O A_2 O^T] \rangle_O$$

O over $O(N), U(N), S_N$ - doesn't matter

Will show:

$$a_k^{(m)} = \binom{N}{k}^{-1} a_k^{(1)} a_k^{(2)}$$

$$p_{AII} = (z - \alpha)^N \Rightarrow a_k = \binom{N}{k} \alpha^k$$

$\Rightarrow p \boxtimes p_{AII}$ has $a_k \rightarrow \alpha^k a_k$
 ie $\lambda_i \rightarrow \alpha \lambda_i$

$$a_1^{(m)} = \frac{1}{N} a_1^{(1)} a_1^{(2)} \Rightarrow \mu^m = \mu^{(1)} \cdot \mu^{(2)}$$

$$\text{Assume } \mu^{(1)}, \mu^{(2)} = 1 \Rightarrow \mu^m = 1, \quad \sigma_i^2 = N - \frac{2\alpha_i}{N-1} \Rightarrow \frac{\sigma_{(1)}^2 \sigma_{(2)}^2}{N} = N - \frac{4 a_2^{(1)} a_2^{(2)}}{(N-1)^2 N} - \frac{2(a_2^{(1)} + a_2^{(2)})}{(N-1)}$$

$$\sigma_{(m)}^2 = N - 2 \frac{a_2^{(m)}}{N-1} = N - \frac{4 a_2^{(1)} a_2^{(2)}}{N(N-1)^2} = \sigma_{(1)}^2 + \sigma_{(2)}^2 - \frac{\sigma_{(1)}^2 \sigma_{(2)}^2}{N}$$

$$\text{Ex 12.4.1} \quad p(z) = z^m (z-1)^m$$

$$a) \quad p(z) = z^m \sum_{k=1}^m (-1)^k \binom{m}{k} z^{m-k} \Rightarrow a_k = \binom{m}{k} (-1)^m$$

$$b) \quad p_m = p \otimes p \quad \text{has} \quad a_k^{(m)} = \begin{cases} \binom{m}{k}^2 \binom{2m}{k}^{-1} & k \leq m \\ 0 & \text{else} \end{cases}$$

c) p_m still has zeros of mult $M \Rightarrow q(z) = z^{-m} p_m(z)$ poly w/ M roots

$$d) \quad \text{Average root is } a_1 = \frac{M^2}{N} = \frac{m}{2}$$

$$e) \quad z^2(z-1)^2 \Rightarrow p_m = z^2 \cdot \left(z^2 + \frac{M}{2}z + \frac{(M(M-1))^2}{2^2} \frac{2}{2M(2M-1)} \right) \\ = z^2 \left(z^2 + z + \frac{2}{12} \right)$$

$$\Rightarrow r_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1-2/3}{2}} = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$$

$$f) \quad z^4(z-1)^4 \Rightarrow p_m = z^4 \left(z^4 + \frac{M}{2} z^3 + \frac{(M(M-1))^2}{4!} \frac{2}{2M(2M-1)} z^2 + \frac{M(M-1)(M-2)}{6!} \dots \right)$$

pointless

12.4-5 Derivation of Results

$$p_S(z) = p_1 \oplus p_2 = \frac{1}{N!} \sum_{\text{perms } \sigma} \prod_{i=1}^N (z - \lambda_i^{(1)} - \lambda_{\sigma(i)}^{(2)})$$

det(zI - A^1 - \sigma A^2) $\sigma \in S_N$

$$a_1^{(S)} = \frac{1}{N!} \sum_{\sigma} \sum_i (\lambda_i^{(1)} + \lambda_{\sigma(i)}^{(2)}) = \sum_i (\lambda_i^{(1)} + \lambda_i^{(2)}) = a_1^{(1)} + a_1^{(2)}$$

For other $a_k^{(S)}$:

$$(z - \lambda_1^{(1)} - \lambda_{\sigma(1)}^{(2)}) \dots (z - \lambda_N^{(1)} - \lambda_{\sigma(N)}^{(2)})$$

choose z $N-k$ times, $\lambda^{(1)}$ i times, $\lambda^{(2)}$ $k-i$ times

Once averaged over $\sigma \in S_N$ the product of the $\lambda^{(1)} \lambda^{(2)}$ must be completely symmetric in $\lambda^{(1)}, \lambda^{(2)}$ and therefore a $a_i^{(1)} a_{k-i}^{(2)}$

$$a_k^{(5)} = \sum_{j=0}^k C(i, k, N) a_j^{(1)} a_{k-j}^{(2)}$$

\Rightarrow TBD

$$A_2 = \mathbb{I} \Rightarrow P_2(z) = (z-1) = \sum_k (-1)^k \binom{n}{k} z^{n-k}$$

$\underbrace{\hspace{10em}}$
 $\underbrace{\hspace{10em}}_{a_k}$

$$p_s = p \text{ 由 } p_{\bar{x}} = p(z-1) = \sum_k (-1)^k a_k (z-1)^{N-k}$$

$$\Rightarrow a_k^{(5)} = \sum_i \binom{N-i}{N-k} a_i$$

$$\Rightarrow a_k^{(s)} = \sum_k (-1)^k z^{(k-k)} \sum_{j=0}^k \binom{k-j}{k-k} a_j$$

$$= \sum_k (-1)^k z^{N-k} \sum_{j=0}^k C(i, k, N) a_i^{(1)} a_{t-j}^{(2)} \Rightarrow C(i, k, N) = \binom{N-i}{N-k} \binom{N}{k-i}^{-1}$$

C(i, k, N) =
 $\frac{(N-i)! (N-k+i)! (k-i)!}{N! (N-k)! (k-i)!}$

Now see next exercise

Non for $O(N)$, WN

Expand det in powers of λ , $\lambda^{(1)}$

After averaging $\lambda_i^{(1)}$, appearances must be perm-invariant

$$\Rightarrow a_k^{(s)} = \sum_{j=1}^k C(i, k, N, \{a_j^{(2)}\}) a_j^{(1)}$$

By dimensional analysis + En-gym on $\lambda^{(2)}$ we get

$$= \sum_{j=1}^k \tilde{C}(i, k, N) a_i^{(1)} a_{k-j}^{(2)}$$

Because $P_s = P \oplus P_{\bar{x}}$ must be the same by
regardless of the group average, we get $\tilde{C} = C$

Now for \otimes :

$$P_m(z) = \frac{1}{N!} \sum_{\sigma} \prod_{i=1}^N (z - \lambda_i^{(1)} \lambda_{\sigma(i)}^{(2)}) = \frac{1}{N!} \sum_{\sigma} p_{\sigma}(z)$$

$$a_k^{\sigma} = \underbrace{\sum_{i_1 \in T_k} \lambda_{i_1}^{(1)} \dots \lambda_{i_k}^{(1)}}_{\text{ordered } k\text{-tuple}} \lambda_{\sigma(i_1)}^{(2)} \dots \lambda_{\sigma(i_k)}^{(2)}$$

ordered k -tuple

After σ -arg we get $a_k^{(s)} \propto a_k^{(1)} a_k^{(2)}$

proportionality const must be $\left[\sum_{i \in T_k} \right]^{-1} = \binom{N}{k}^{-1}$

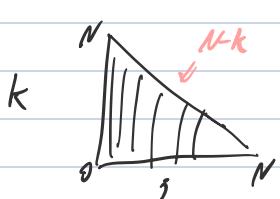
or by requiring $P \otimes P_2 = P$

Ex 12.4.2

a) $\hat{p}(\partial_z) z^N = \sum_k \frac{(N-k)!}{N!} a_k (-1)^k \underbrace{\partial_z^k}_{\frac{N!}{(N-k)!} z^{N-k}} z^N = p(z)$

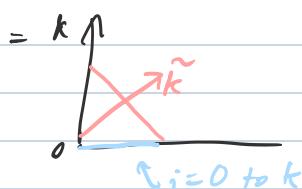
b) $p_1 \boxplus p_2 = \hat{p}_1(\partial_z) p_2$

$$\begin{aligned} \hat{p}_1(\partial_z) p_2(z) &= \sum_{j=0}^N (-1)^k \frac{(N-j)!}{N!} a_j^{(1)} \partial_z^j \sum_{k=0}^{N-j} (-1)^i a_k^{(2)} z^{N-k} \\ &= \sum_{i+k=N} (-1)^{i+k} \frac{(N-i)!}{N!} \frac{(N-k)!}{(N-i-k)!} a_i^{(1)} a_k^{(2)} z^{N-i-k} \end{aligned}$$



$$= \sum_{k=0}^N \sum_{i=0}^k (-1)^k \frac{(N-i)!}{N!} \frac{(N-k+i)!}{(N-k)!} a_i^{(1)} a_{k-i}^{(2)} z^{N-k}$$

12.90 defining \boxplus



12.5 Freeness for 2×2 Matrices

$N=1 \Rightarrow 1 \times 1$ matrices commute \Rightarrow only constants are free

$N=2 \Rightarrow$ Def $\sigma = \frac{1}{2} \mathbb{E} \text{Tr } A$

Take A with evals deterministic
evecs random

$$A = a\mathbb{1} + \sigma O \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} O^T$$

$$O = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \Rightarrow A = a\mathbb{1} + \sigma \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

12.5.1

\Rightarrow traceless polynomials take the form:

$$p_k = a_k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad q_k = b_k \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$\mathbb{E} \left[\prod_k p_k(A) q_k(B) \right] = \frac{1}{2} \sum_k a_k b_k \text{Tr } \mathbb{E} \left(\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}^n \right)$$

0 by symmetry \Rightarrow Free!

Ex 12.5.1 a) A_1 has $\lambda = \pm \sigma$

$$\Rightarrow \frac{1}{2} \text{Tr} \frac{1}{z-A} = \frac{1}{2} \left[\frac{1}{z-\sigma} + \frac{1}{z+\sigma} \right] \Rightarrow \frac{2}{z^2 - \sigma^2}$$

$$\Rightarrow g = \frac{1 + \sqrt{1+4g^2\sigma^2}}{2g} \Rightarrow R = \frac{-1 + \sqrt{1+4g^2\sigma^2}}{2g}$$

b)

```
In[792]:=  $\left[ z \text{ /. Solve}\left[g = \frac{z}{z^2 - \sigma^2}, z\right] \right] \text{ // Simplify}$ 
Out[792]=  $\frac{-1 + \sqrt{1 + 4 g^2 \sigma^2}}{2 g}$ 

In[800]:=  $\text{Solve}\left[z = \frac{-1 + \sqrt{1 + 4 g^2 \sigma^2} \text{ a1}^2}{2 g} + \frac{-1 + \sqrt{1 + 4 g^2 \sigma^2} \text{ a2}^2}{2 g} + \frac{1}{g}, g\right] \text{ // Simplify}$ 
... Solve: There may be values of the parameters for which some or all solutions are not valid.
Out[800]=  $\left\{ g \rightarrow \frac{\text{a1} z}{\sqrt{z^4 + (\sigma 1^2 - \sigma 2^2)^2 - 2 z^2 (\sigma 1^2 + \sigma 2^2)}} \right\}$ 
```

c)

```
In[807]:= -Det[ $\begin{bmatrix} \sigma 1 + \sigma 2 \text{ Cos}[2 \theta] & \sigma 2 \text{ Sin}[2 \theta] \\ \sigma 2 \text{ Sin}[2 \theta] & -\sigma 1 - \sigma 2 \text{ Cos}[2 \theta] \end{bmatrix}$ ] // FullSimplify
Out[807]=  $\sigma 1^2 + \sigma 2^2 + 2 \sigma 1 \sigma 2 \text{ Cos}[2 \theta]$ 
```

d)

$$\text{In[813]:= } \frac{1}{2} \left(\frac{1}{z - \sqrt{a_1^2 + a_2^2 + 2 a_1 a_2 \cos(2\theta)}} + \frac{1}{z + \sqrt{a_1^2 + a_2^2 + 2 a_1 a_2 \cos(2\theta)}} \right) // \text{FullSimplify}$$

$$\text{Out[813]:= } -\frac{z}{z^2 + a_1^2 + a_2^2 + 2 a_1 a_2 \cos(2\theta)}$$

$$\text{In[814]:= } \frac{1}{2\pi} \text{Integrate}\left[\frac{z}{z^2 - (a_1^2 + a_2^2 + 2 a_1 a_2 \cos(2\theta))}, (\theta, 0, 2\pi) \right]$$

$$\text{Out[814]:= } -\frac{1}{2\pi z} \cdot \left(\left(2\pi \left(-z^2 + a_1^2 + a_2^2 + \sqrt{-4 a_1^2 a_2^2 + (-z^2 + a_1^2 + a_2^2)^2} \right) \right) / \right.$$

$$\left. \left(-4 a_1^2 a_2^2 + (-z^2 + a_1^2 + a_2^2)^2 - z^2 \sqrt{(z^4 - 2 z^2 a_1^2 + a_1^4 - 2 z^2 a_2^2 - 2 a_1^2 a_2^2 + a_2^4) + \right. \right.$$

$$\left. \left. a_1^2 \sqrt{(z^4 - 2 z^2 a_1^2 + a_1^4 - 2 z^2 a_2^2 - 2 a_1^2 a_2^2 + a_2^4)} + a_2^2 \sqrt{(z^4 - 2 z^2 a_1^2 + a_1^4 - 2 z^2 a_2^2 - 2 a_1^2 a_2^2 + a_2^4)}) \right) \right) // \text{FullSimplify}$$

$$\text{Out[820]:= } \frac{z}{\sqrt{(z - a_1 - a_2)(z + a_1 - a_2)(z - a_1 + a_2)(z + a_1 + a_2)}}$$

↑ as before

Ex 12.5.2

$$A_1 = O \begin{pmatrix} 0 & 0 \\ 0 & a_1 \end{pmatrix} O^T$$

$$a) \quad t_A = \tau \left[\frac{1}{1 - S^2 A} \right] - 1 = \frac{1}{2} \left(\frac{1}{1} + \frac{1}{1 - S^2 a_1} \right) - 1 = \frac{1}{2} \frac{a_1}{S^2 - a_1}$$

$$\Rightarrow G_A(t) = a_1 \frac{(1+2t)}{2t} \Rightarrow S_A(t) = \frac{2}{a_1} \frac{1+t}{1+2t}$$

$$b) \quad S_{AA_2} = \frac{2}{a_1 a_2} \frac{(1+t)^2}{(1+2t)^2} \Rightarrow G_{AA_2} = \frac{a_1 a_2}{2} \frac{(1+2t)^2}{(1+t)t}$$

$$\Rightarrow t = -\frac{1}{2} - \frac{1}{2} \sqrt{\frac{3}{S^2 - a_1 a_2}}$$

$$\Rightarrow \frac{1}{2} (t+1) = \frac{1}{2z} - \frac{1}{2} \frac{1}{\sqrt{z^2 - a_1 a_2} z}$$

$$\Rightarrow p(\lambda) = \frac{1}{2} \delta(\lambda) + \frac{1}{2\pi} \frac{1}{\sqrt{\lambda(a_1 a_2 - \lambda)}} \quad 0 \leq \lambda \leq a_1 a_2$$

↑ shifted arcsine

$$c) \quad A_1 = O \begin{pmatrix} 0 & 0 \\ 0 & a_1 \end{pmatrix}$$

$$A_2 = O \begin{pmatrix} 0 & 0 \\ 0 & a_2 \end{pmatrix} O^T = a_2 \begin{pmatrix} \sin^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \cos^2 \theta \end{pmatrix}$$

$$\sqrt{A_1} A_2 \sqrt{A_1} = a_1 a_2 \begin{pmatrix} 0 & 0 \\ 0 & \cos^2 \theta \end{pmatrix}$$

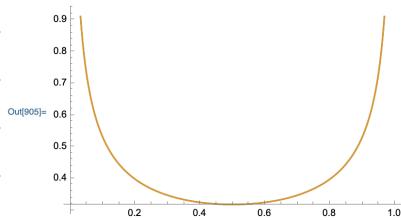
$$\Rightarrow \theta = \arccos \sqrt{\frac{2}{a_1 a_2}} \quad \theta \sim \text{Unif}(0, \pi/2) \quad p(\theta) = (2\pi)^{-1}$$

$$p(\lambda) = \frac{d\theta}{d\lambda} = \frac{1}{\pi \cdot \cos \theta \sin \theta a_1 a_2}$$

↑ no double cover

$$\frac{d\lambda}{d\theta} = 2 \cos \theta \sin \theta \quad \alpha_1, \alpha_2$$

In[905]:= Plot[{(1/\pi) Csc[2 ArcSec[(1/(Sqrt[\lambda]))]], (1/(2\pi Sqrt[\lambda(1-\lambda)]))}, {\lambda, 0, 1}]



12.5.3 Pairwise Freeness \neq Joint Freeness

Take A, B, C traceless $\sigma^2 = 1$, deterministic λ_i :

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \quad B = 0, A\Omega^T \quad C = \Omega_2 B \Omega_2^T$$

$$ABC = \begin{pmatrix} \cos 2(\theta-\phi) & \sin 2(\theta-\phi) \\ \sin 2(\theta-\phi) & -\cos 2(\theta-\phi) \end{pmatrix} \Rightarrow \tau(ABC) = 0$$

$$\begin{aligned} x_1 &= 0 \\ x_3 &= 0 \\ x_6 &= 1 \end{aligned}$$

$$(ABC)^2 = \mathbb{1} \Leftarrow \tau \neq 0 \Rightarrow \text{not free as a collection}$$

$$\text{Can show } \chi_{\epsilon}(A+B, A+B, C, A+B, A+B, C) = q \neq 0$$

$\Rightarrow A+B$ not free of C

\Rightarrow Do not satisfy free CLT

\Rightarrow Sums of $A_{2 \times 2}$ w/ λ_i deterministic do not yield Wigner f(x)

Ex 12.5.3

$$\begin{aligned} a) \quad P(\xi \lambda; \beta) &= \prod_{k=1}^n |\lambda_k - \lambda| \exp\left[-\frac{n}{2} \sum_i V(\lambda_i)\right] \\ &\Rightarrow P(\lambda_1, \lambda_2) = |\lambda_1 - \lambda_2| e^{-\frac{\lambda_1^2}{2} - \frac{\lambda_2^2}{2}} \end{aligned}$$

b)

$$\begin{aligned} \text{In[936]:= } & \text{Integrate}[\text{Exp}[-\lambda 1^2/2 - \lambda 2^2/2], \{\lambda 1, \lambda 2, \text{Infinity}\}] + \\ & \text{Integrate}[\text{Exp}[-\lambda 1^2/2 - \lambda 2^2/2], \{\lambda 2, \lambda 1, \{-\text{Infinity}, \lambda 2\}\}] // \text{FullSimplify} \\ \text{Out[936]\!:= } & e^{-\lambda 2^2} \left(2 + e^{-\frac{\lambda 2^2}{2}} \sqrt{2\pi} \lambda 2 \text{ Erf}\left[\frac{\lambda 2}{\sqrt{2}}\right]\right) \\ \text{In[937]\!:= } & \text{Integrate}[\text{Exp}[-\lambda 1^2/2 - \lambda 2^2/2] \text{ Abs}[\lambda 1 - \lambda 2], \{\lambda 1, -\text{Infinity}, \text{Infinity}\}, \\ & \{\lambda 2, -\text{Infinity}, \text{Infinity}\}] \\ \text{Out[937]\!:= } & 4 \sqrt{\pi} \end{aligned}$$

$$\Rightarrow \frac{e^{-\lambda^2}}{\sqrt{4\pi}} \left(2 + e^{\frac{\lambda^2}{2}} \sqrt{2\pi} \lambda \text{ Erf}\left(\frac{\lambda}{\sqrt{2}}\right)\right)$$

$$c) \int d\lambda_1 (\lambda_1 - \lambda_2)^2 e^{-\lambda_1^2 - \lambda_2^2}$$

$$d) = \frac{e^{-\lambda^2}}{\sqrt{\pi}} (\lambda^2 + \frac{1}{2})$$

not semicircle

Chapter 13: The Replica Method

Let's now recover all major results so far using replicas!

$$\langle \log Z \rangle = \lim_{n \rightarrow 0} \frac{\langle Z^n \rangle - 1}{n}$$

We can usually only compute $\lim_{n \rightarrow 0} \lim_{N \rightarrow \infty} Z_N^n$

But we want $\lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} Z_N^n$

13.1 Stieltjes Transform

13.1.1 General setup:

$$\mathbb{E} g_A(z) = \frac{1}{N} \mathbb{E} \left[\sum_k \frac{1}{z - \lambda_k} \right] = \frac{1}{N} \mathbb{E} \left[\frac{d}{dz} \log \det(z\mathbb{I} - A) \right]$$

$$= \frac{1}{N} \frac{d}{dz} \mathbb{E}_A \frac{1}{z} \log Z$$

$$= \frac{d}{dz} \mathbb{E}_A \frac{Z^{n-1}}{2Nn}$$

$$\mathbb{E} Z^n = \int \frac{d^N \gamma_\alpha}{(2\pi)^{Nn/2}} \mathbb{E}_A \exp \left[-\frac{1}{2} \gamma_\alpha^\top (z\mathbb{I} - A) \gamma_\alpha \right]$$

$$g := \lim_{N \rightarrow \infty} g_A = \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} [\dots]$$

We can use replicas for $\lim_{n \rightarrow 0} \lim_{N \rightarrow \infty} [\dots]$

13.1.2 Wigner Case

$A = X$ symmetric Gaussian

$$\int \frac{d^N \gamma_\alpha^i}{(2\pi)^{Nn/2}} e^{-\frac{z}{2} \gamma_\alpha^T \gamma_\alpha^i} \mathbb{E}_{\substack{X_{ij} \\ X_{ii} \\ \sim N(0, \sigma^2/N)}} \exp \left[\gamma_\alpha^i X_{i \neq j} \gamma_\alpha^j \right] \mathbb{E}_{\substack{X_{ii} \\ \sim N(0, \frac{2\sigma^2}{N})}} \exp \left[\frac{1}{2} \gamma_\alpha^i X_{ii} \gamma_\alpha^i \right]$$

$$= \int \frac{d^N \gamma_\alpha^i}{(2\pi)^{Nn/2}} e^{-\frac{z}{2} \gamma_\alpha^T \gamma_\alpha^i} \exp \left[\frac{\sigma^2}{2N} \sum_{ij} \left(\sum_\alpha \gamma_\alpha^i \gamma_\alpha^j \right)^2 + \frac{\sigma^2}{4N} \sum_i \left(\sum_\alpha \gamma_\alpha^i \gamma_\alpha^i \right)^2 \right]$$

$$= \int [E] e^{-\frac{1}{2} \bar{\Psi}^T \bar{\Psi}} \exp \left[\frac{\sigma^2}{N} \sum_{i,j} \left(\sum_{\alpha} \psi_{\alpha}^i \psi_{\alpha}^j \right)^2 \right] \quad \text{← Factorizes over } i, j$$

$\psi_{\alpha}^i \psi_{\beta}^i \psi_{\alpha}^j \psi_{\beta}^j$

$$\frac{\sigma^2}{N} \sum_{\alpha, \beta} \left(\sum_i \psi_{\alpha}^i \psi_{\beta}^i \right)^2 \quad \text{← Factorizes in } \alpha, \beta \quad \text{(allows us to define } q_{\alpha \beta})$$

$$= \frac{\sigma^2}{2N} \sum_{\alpha < \beta} \left(\sum_i \psi_{\alpha}^i \psi_{\beta}^i \right)^2 + \frac{\sigma^2}{N} \sum_{\alpha} \left(\sum_i \frac{\psi_{\alpha}^i \psi_{\alpha}^i}{2} \right)^2$$

$q_{\alpha \beta}$ $q_{\alpha \alpha}$

$$= \int dq_{\alpha \beta} d\bar{\Psi}_{\alpha} \exp \left[-\frac{N \operatorname{Tr} q^2}{q_0^2} - \frac{1}{2} \sum_{\alpha, \beta, i} (\epsilon_{\alpha \beta} - q_{\alpha} q_{\beta}) \psi_{\alpha}^i \psi_{\beta}^i \right] \quad \text{← Factors over } i$$

$$= \int dq_{\alpha \beta} \exp \left[-\frac{N \operatorname{Tr} q^2}{q_0^2} - \frac{N}{2} \operatorname{Tr} \log(z \mathbb{I} - q) \right]$$

- $\frac{NF(q)}{2}$

$$= \int dq_{\alpha} \exp \left\{ -N \left[\sum_{\alpha} \left(\frac{q_{\alpha}^2}{2\sigma^2} + \log(z - q_{\alpha}) \right) - \frac{1}{N} \sum_{\alpha < \beta} \log(q_{\alpha} - q_{\beta}) \right] \right\}$$

Vandermonde

\nearrow
eigs of q

$$F(q_{\alpha}) = \sum_{\alpha} \left[\frac{q_{\alpha}^2}{2\sigma^2} + \log(z - q_{\alpha}) \right]$$

$$\Rightarrow \frac{q_{\alpha}}{\sigma^2} - \frac{1}{z - q_{\alpha}} - \frac{1}{N} \sum_{\beta \neq \alpha} \frac{2}{(q_{\alpha} - q_{\beta})} = 0 \quad \text{as } N \rightarrow \infty$$

$$\Rightarrow q_{\alpha} = \frac{\sigma^2}{z - q_{\alpha}} \quad \text{S.C.Eq} \quad \text{same } \forall \alpha \rightarrow \text{Factor of } n \text{ out front}$$

$$\Rightarrow E[z^n] \approx \exp \left[-\frac{Nn}{2} F_1(z, q^*(z)) \right]$$

$$\Rightarrow \lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{E[z^n] - 1}{Nn} = -\frac{1}{2} F_1(z, q^*(z))$$

$$\Rightarrow g_X(z) = \frac{d}{dz} F_1(z, q^*(z)) = \frac{\partial}{\partial z} F_1 = \frac{1}{z - q^*} = \frac{q^*}{\sigma^2}$$

B.2 Resolvent Matrix

B.2.1 General Case

$$\begin{aligned} M^{-1} &= \frac{1}{Z} \int d^N \psi \psi \psi^\top \exp\left[-\frac{1}{2} \psi^\top M \psi\right] \\ &= \lim_{m \rightarrow 1} Z^m \int d^N \psi \psi \psi^\top \exp\left[-\frac{1}{2} \psi^\top M \psi\right] \\ &= \lim_{n \rightarrow 0} \int d^{Nn} \vec{\psi}_\alpha \vec{\psi}_i \vec{\psi}_j \exp\left[-\frac{1}{2} \vec{\psi}_\alpha^\top M \vec{\psi}_\alpha\right] =: \langle \vec{\psi}_i \vec{\psi}_j^\top \rangle_{n=0} \end{aligned}$$

For Gaussian Wigners can use S.P. config

$$\mathbb{E}[G_X(z)]_{ij} = \langle \vec{\psi}_i \vec{\psi}_j \rangle = -2 \lim_{n \rightarrow 0} \frac{\partial}{\partial z \delta_{ij}} \frac{z^n - 1}{n} = \delta_{ij} g_X(z)$$

B.2.2 Free Addition

$$C = A + OBO^\top$$

Want: $\mathbb{E}_O[(zI - A - OBO^\top)^{-1}]$

wlog A, B diagonal

$$\mathbb{E}[G_C(z)] = \lim_{n \rightarrow 0} \int \frac{d^N \vec{\psi}_\alpha}{(2\pi)^{N/2}} \vec{\psi}_i \vec{\psi}_j^\top \exp\left[-\frac{\vec{\psi}_\alpha^\top (zI - A) \vec{\psi}_\alpha}{2}\right] \mathbb{E}_O \exp\left[\frac{1}{2} \vec{\psi}_\alpha^\top OBO^\top \vec{\psi}_\alpha\right]$$

$$\mathbb{E}_O \exp\left[\frac{N}{2} \sum_\alpha \text{Tr} Q_{\alpha\alpha} OBO^\top\right]$$

$$\delta\left(\frac{\vec{\psi}\vec{\psi}^\top}{N} - Q\right) = \int_{-\infty}^{\infty} \frac{N^{\frac{n(n+1)}{2}}}{2^n (2\pi)^{n/2}} d\vec{Q} \exp\left[-\frac{N}{2} \text{Tr} Q\vec{Q} + \frac{1}{2} \text{Tr} \vec{Q} \vec{\psi} \vec{\psi}^\top\right]$$

low rank \Rightarrow HCIZ formula applies

$$\exp\left[\frac{N}{2} \text{Tr}_n H_B(Q)\right] \quad \int_0^2 R_B(Q)$$

Can now do $\vec{\psi}$ integral

$$J_{ij} = \int \frac{d^N \vec{\psi}_\alpha}{(2\pi)^{N/2}} \vec{\psi}_i \vec{\psi}_j \exp\left[-\frac{1}{2} \vec{\psi}_{ik} (z\delta_{\alpha\beta} - a_k \delta_{\alpha\beta} - \hat{Q}_{\alpha\beta}) \vec{\psi}_{\beta k}\right]$$

$$= -\frac{1}{2} \prod_{k=1}^N \det[(z - a_k) \delta_{\alpha\beta} - \hat{Q}_{\alpha\beta}] + [(z - a_i - \hat{Q})^{-1}]_{ii} \delta_{ij}$$

why? \Leftrightarrow will imply G_C is diagonal

$$\Rightarrow \boxed{\mathbb{E} G_C}_{ij} = \lim_{n \rightarrow \infty} \int d[\hat{Q}, \hat{q}] \delta_{ij} [(z - a_i - \hat{q})^{-1}] \exp \left[\frac{N}{2} [\text{Tr } U_B(Q) - \text{Tr } Q \hat{Q} - \frac{1}{N} \sum_{k=1}^N \text{Tr } \log(z - a_k - \hat{q})] \right]$$

counts here
 don't matter
 for this calc'n

no power
 of n
 does
 not determine
 saddle

saddle
 $\frac{N}{2}$ soft

$$\frac{\partial S_{\text{soft}}}{\partial Q_{\alpha\beta}} = - Q_{\alpha\beta} + R_B(Q)_{\alpha\beta} \Rightarrow \boxed{\hat{Q} = R_B(Q)}$$

$$\frac{\partial S_{\text{soft}}}{\partial \hat{Q}_{\alpha\beta}} = - Q_{\alpha\beta} + \frac{1}{N} \sum_{k=1}^N [(z - a_k) I_n - \hat{Q}]_{\alpha\beta}^{-1}$$

Work in basis that diagonalizes Q

$$Q_{\alpha\alpha} = \frac{1}{N} \sum_{k=1}^N \frac{1}{z - a_k - \hat{Q}_{\alpha\alpha}} =: g_A(z - \hat{Q}_{\alpha\alpha})$$

at large z the solution is unique $\forall \alpha \Rightarrow Q = q^* \mathbf{1} \quad \hat{Q} = \hat{q}^* \mathbf{1}$

$$\hat{q} = R_B(q) \quad q = g_A(z - \hat{q})$$

For large z q, \hat{q} are small \Rightarrow justifies deforming \hat{Q} contour How?

$$\Rightarrow \mathbb{E} G_C(z) = \lim_{n \rightarrow \infty} \frac{\delta_{ij}}{z - a_i - \hat{q}^*} \exp \left[\frac{\pi N}{2} [-q^* \hat{q}^* + U_B(q^*) - \frac{1}{N} \sum_{k=1}^N \log(z - a_k - \hat{q}^*)] \right]$$

$$\Rightarrow E G_C(z) = G_A(z - R_B(q^*)) \quad q^* = g_A(z - R_B(q^*)) \quad \boxed{*}$$

13.2.3

After taking τ ~~of~~ * we get

$$q^* = g_C(z) = g_A(z - R_B(q^*)) \quad q^* = g(z - R_B(q^*))$$

~~Subordination~~

$\Rightarrow *$ is a more powerful version!

$$\begin{aligned} \mathbb{E} G_C(z) &= G_A(z - R_B(g_C(z))) \\ &= G_B(z - R_A(g_C(z))) \end{aligned}$$

Can use replica theory to show for $C = A^{\frac{1}{2}} B A^{\frac{1}{2}}$

$$\begin{aligned} \mathbb{E} T_C(\xi) &= T_A [S_B(t_C(\xi))\xi] \\ \Rightarrow \mathbb{E} G_C(z) &= S^* G_A(z S^*) \quad S^* = S_B(z g_C(z) - 1) \end{aligned}$$

13.2.4

Because of strong repulsion λ_i barely fluctuate

\Rightarrow Annealed approximation $\langle \log Z \rangle \approx \log \langle Z \rangle$ is exact
for getting

$$g_X(z)$$

$$\mathbb{E} G = \langle \text{rep} \rangle$$

$$\mathbb{E} Z^n = \exp\left(-\frac{nN}{2} F_i\right)$$

$$\Rightarrow \frac{1}{N} \log \mathbb{E} Z = -\frac{1}{2} F_i \quad \text{as required}$$

This will not be the case for the following rank 1 HCIZ integral beyond some range

13.3 Rank 1 HCIZ

$$T = +\hat{e}_i$$

$$H_B(t) := \lim_{N \rightarrow \infty} \frac{2}{N} \log \left\langle \exp \left[\frac{N}{2} \text{Tr } T O B O^T \right] \right\rangle_0$$

Will then average H_B over B

Annealed approx:

$$\tilde{A}(t) := \lim_{N \rightarrow \infty} \frac{2}{N} \log \left\langle \exp \left[\frac{N}{2} \text{Tr } T O B O^T \right] \right\rangle_{0,B}$$

For small t $A(t) = H_B(t) = \langle H_B(t) \rangle_B$

Beyond t_c there is a phase transition

13.3.1 Annealed Average

$$\left\langle \exp\left[\frac{N}{2} \text{Tr } TX\right] \right\rangle_X = \left\langle \exp\left[\frac{Nt}{2} e^r X e_r\right] \right\rangle_X$$

$$= \int \frac{dX_{11}}{\sqrt{4\pi\sigma^2/N}} \exp\left[\frac{Nt}{2} X_{11} - \frac{N}{4\sigma^2} X_{11}^2\right]$$

$$= \exp\left[\frac{Nt^2\sigma^2}{4}\right]$$

$$\Rightarrow \hat{H}_{\text{wig}}(t) = \frac{t^2\sigma^2}{2} \Leftarrow \text{same as } \int R_X(t) dt$$

13.3.2 Quenched Average

Set $\sigma^2 = 1$ for simplicity

$$Z_t(X) = \int \frac{d^N \gamma}{(2\pi)^{N/2}} \delta(Nt - \gamma^T \gamma) \exp\left(\frac{1}{2} \gamma^T X \gamma\right)$$

$$E H_X(t) = \lim_{N \rightarrow \infty} \frac{2}{N} \lim_{n \rightarrow 0} \frac{Z_t^n(X) - 1}{n} - \frac{2}{N} \log Z_t(0)$$

$$\text{Vol } S_{N-1}(2\sigma)^{N/2} = e^{N/2} + N^{1/2}$$

$$\approx \frac{2}{N} \log \dots = 1 + \log t$$

$$Z_t^n(X) = \int \frac{d^{Nn} \gamma}{(2\pi)^{Nn/2}} d\gamma_\alpha \exp\left[\frac{1}{2} \gamma_\alpha^T X \gamma_\alpha + \frac{1}{2} (Nz_\alpha t - z_\alpha \gamma_\alpha^T \gamma)\right]$$

$$E_x \exp\left(\frac{1}{2} \gamma^T X \gamma\right) = E_i \exp\left(\frac{1}{2} X_{ii} \gamma_\alpha^i \gamma_\alpha^i\right) E_{i,j} \exp(X_{ij} \gamma_\alpha^i \gamma_\alpha^j)$$

$$= \exp\left[\frac{1}{4N} \sum_i \left(\sum_\alpha \gamma_\alpha^i \gamma_\alpha^i\right)^2\right] \exp\left[\frac{1}{2N} \sum_{i,j} \left(\sum_\alpha \gamma_\alpha^i \gamma_\alpha^j\right)^2\right]$$

$$= \exp\left[\frac{1}{4N} \sum_{i,j} \left(\sum_\alpha \gamma_\alpha^i \gamma_\alpha^j\right)^2\right]$$

$$= \exp\left[\frac{1}{4N} \sum_{\alpha,\beta} \left(\sum_i \gamma_\alpha^i \gamma_\beta^i\right)^2\right] \leftarrow \text{now Hubbard}$$

$$= \int dq \exp \left[-\frac{N}{4} \text{Tr } q^2 + \sum_{\alpha\beta} \frac{q_{\alpha\beta} \gamma_\alpha^\top \gamma_\beta}{2} \right]$$

(cn)
numerical
eff $\leftarrow N\text{-indep} \Rightarrow$ no contrib

Do γ integral

$$\int \frac{d^{Mn} \gamma}{(2\pi)^{Mn}} \exp \left[-\frac{(z_\alpha \delta_{\alpha\beta} - q_{\alpha\beta}) \gamma_\alpha^\top \gamma_\beta}{2} \right] \leftarrow \text{decouples over } i$$

$$= \exp \left[-\frac{N}{2} \text{Tr} \log z - q \right]$$

↑
di ~~stage~~

$$\Rightarrow E[z^n] = \int dq dz \exp \left[\frac{N}{2} \left[t \text{Tr } z - \frac{\text{Tr } q^2}{2} - \text{Tr } \log z - q \right] \right]$$

$F_n(q, z; t)$

$$n=1 \Rightarrow t z - \frac{q^2}{2} - \log z - q$$

$$\partial_q = 0 \Rightarrow q = \frac{1}{z - q} \Rightarrow q = t \Rightarrow F_1(t) = t^2 + 1 - \frac{t^2}{2} \log t$$

$$\partial_z = 0 \Rightarrow t = \frac{1}{z - q} \Rightarrow z = t + t^{-1} \Rightarrow H_{\text{rig}} = t^2/2 \quad \checkmark$$

General n

$$\begin{aligned} \partial_q = 0 \Rightarrow q &= (z - q)^{-1} \Rightarrow q = \begin{pmatrix} t & b & \cdots & b \\ b & t & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & \cdots & \cdots & t \end{pmatrix} \\ \partial_z = 0 \Rightarrow t &= [(z - q)^{-1}]_{\alpha\alpha} \end{aligned}$$

$$q = (t - b) \mathbb{1} + nb \frac{\mathbb{1} \mathbb{1}^\top}{n} \Rightarrow \lambda_q = \begin{cases} t - b + nb & \times 1 \\ t - b & \times (n-1) \end{cases}$$

$$\Rightarrow (z - q)^{-1} = \frac{\mathbb{1}}{z - t + b} + \frac{1}{(z - t + b)^2} \frac{b \mathbb{1} \mathbb{1}^\top}{1 - nb(z - t + b)^{-1}}$$

$$\Rightarrow t - b = \frac{1}{z - t + b} \Rightarrow z = (t - b) + (t - b)^{-1}$$

$$\Rightarrow b = \frac{(t - b)^2 b}{1 - nb(t - b)}$$

$$1) b=0 \Rightarrow z = t + t^{-1}$$

$$2) b \neq 0 \Rightarrow (b+t)^2 - nb(b+t) - 1 = 0$$

$$b^2(1-n) + (n-2)t + b + t^2 - 1 = 0$$

$$\Rightarrow b = \frac{(n-2)t \pm \sqrt{(n-2)(n+1)t^2 - 4(n-1)(t-n)}}{2(n-1)}$$

$$= \frac{(n-2)t \pm \sqrt{n^2t^2 - 4(n-1)}}{2(n-1)}$$

$$z = (t+b) + (t+b)^{-1}$$

$$= \frac{n^2t \pm (n-2)\sqrt{n^2t^2 - 4(n-1)}}{2(n-1)} =: z_{\pm}$$

$$n=1 \Rightarrow \frac{nt(n \pm (n-2))}{2(n-1)} \quad \begin{cases} z_+ = t \\ z_- \text{ under} \end{cases} \Rightarrow \text{cancelled}$$

$n \geq 2 \Rightarrow t \leq 2\sqrt{\frac{n-1}{n}}$ has no solutions $\Rightarrow z_0$ is solution there

$t > t_s := \frac{\sqrt{n-1}}{n}$ has $z_+ > z_-$ always \Rightarrow compare z_0 & z_+

$$z_0 = t + t^{-1} \quad \text{vs} \quad \frac{n^2t + (n-2)\sqrt{n^2t^2 - 4(n-1)}}{2(n-1)}$$

$$n=2 \Rightarrow t_s = 1 \quad z_+ = 2t \Rightarrow t + t^{-1} < 2t \quad \text{when } t^{-1} < t \Rightarrow t > 1 \Rightarrow t_c = 1$$

For $n > 2 \quad t_s < t_c < 1$

$$\frac{d}{dt} F_n = \frac{\partial F_n}{\partial t} = \text{Tr } z^* = nz^*$$

$$\Rightarrow \text{For } n \geq 1 \quad \log E z_+'' \sim \frac{N_n}{2} \bar{f}_n$$

$$\partial_t \bar{F}_n = \begin{cases} t + t^{-1} & t < t_c \\ \frac{n^2t + (n-2)\sqrt{n^2t^2 - 4(n-1)}}{2(n-1)} & 2 < t_c \end{cases} \Rightarrow \partial_t \bar{F}_n = \begin{cases} t + t^{-1} & t < t_c \\ 2 & t > t_c \end{cases}$$

\uparrow
we must have $t_c = 1$ for \bar{F}_n to be continuous

Note $n \rightarrow 0$ slips maximizing \mathcal{F} w/ minimizing it!

$$\Rightarrow \frac{d}{dt} \mathbb{E}_X H_X(t) = \begin{cases} + & t \leq 1 \\ -t^{-1} & t > 1 \end{cases}$$

$$\Rightarrow H_X(t) = \begin{cases} \frac{t^2}{2} & t \leq 1 \\ 2t - \log t & t > 1 \end{cases}$$

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The Replica Method

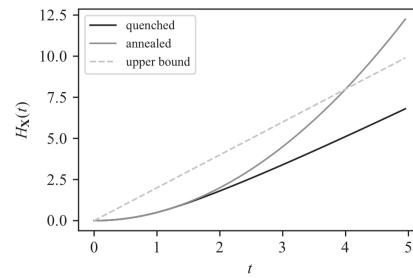


Figure 13.2 The function $H_X(t)$ for a unit Wigner computed with a "quenched" average (HCIZ integral) and an "annealed" average. We also show the upper bound given by Eq. (10.59). The annealed and quenched averages are identical up to $t = t_c = 1$ and differ for larger t . The annealed average violates the bound, which is expected as in this case λ_{\max} fluctuates and exceptionally large values of λ_{\max} dominate the average exponential HCIZ integral.

13.4 Spin Glasses & Low-Rank HCIZ

$$H(\{S_i\}) = \frac{1}{2} \sum_{i,j=1}^N J_{ij} S_i S_j \quad J = O_1 O^T$$

Take $J=X$ Wigner \Rightarrow SK model

$$F = -T \mathbb{E}_J \log Z \quad Z = \mathop{\text{Tr}}_S e^{H(\{S_i\})/T}$$

$$\mathbb{E}_J Z^n = \mathbb{E}_J \sum_{S^1 \dots S^n} \exp \left[\frac{1}{2T} \sum_{\alpha, i, j} J_{ij} S_i^\alpha S_j^\alpha \right]$$

$$K_{ij}^{(\alpha)} := \sum_{\alpha} \frac{S_i^\alpha S_j^\alpha}{N} \quad \text{is a rank } n \ll N \text{ matrix}$$

\Rightarrow HCIZ yields

$$\begin{aligned} \mathbb{E}_J \exp \left[\frac{N}{2T} \text{Tr}_{\alpha} \underbrace{K^{(\alpha)} O_1 O^T}_{J_{ij}} \right] &\approx \exp \left[\frac{N}{2} \text{Tr}_n H_J(K^n/T) \right] \\ &= \exp \left[\frac{N}{2} \text{Tr}_n H_J(\alpha/T) \right] \end{aligned}$$

$$\Rightarrow \mathbb{E}_J Z^n = \sum_{S^1 \dots S^n} \exp \left[\frac{N}{2} \text{Tr}_n H_J(\alpha/T) \right]$$

$$= \sum_{S^1 \dots S^n} * \int dQ d\bar{Q} \exp \left[\frac{N}{2} \text{Tr}_n H_J(Q) - N \text{Tr}_n Q \bar{Q} + \sum_{\alpha \beta} \hat{Q}_{\alpha}^{\dagger} (S^{\alpha})^T S^{\beta} \right] \quad \text{decouples over } i$$

$$= \frac{1}{N!} \int d\hat{\alpha} d\hat{\beta} \exp \left[\frac{N}{2} \text{Tr} H_f(\frac{\hat{\alpha}}{T}) - N \text{Tr} \hat{\alpha} \hat{\beta} + NG(\hat{\beta}) \right]$$

$$G(\hat{\beta}) = \log \sum_{S^1 \dots S^n} e^{S^T \hat{\beta} S}$$

↑
scalar

Saddle:

$$\partial_{\hat{\alpha}} \Rightarrow \hat{\alpha} = \frac{1}{2T} R_J(\frac{\hat{\beta}_T}{T})$$

$$SCEq: \quad \partial_{\hat{\alpha}} \Rightarrow \hat{\alpha}_{\alpha\beta} = \langle S^\alpha S^\beta \rangle_T := \frac{1}{Z} \sum_S S^\alpha S^\beta \exp \left[\frac{1}{2T} S^T R_J(\frac{\hat{\beta}}{T}) S \right]$$

Replica Symmetry:

$$\hat{\alpha}_{\alpha\beta} = (1-q) \mathbf{1} \mathbf{1}^T + q \mathbf{1} \mathbf{1}^T \Rightarrow \hat{\alpha} = \begin{cases} 1 + (n-1)q & x_1 \\ 1-q & x_{(n-1)} \end{cases}$$

$$\Rightarrow R_J(\hat{\beta}) = \begin{cases} R_J\left(\frac{1+(n-1)q}{T}\right) \\ R_J\left(\frac{1-q}{T}\right) \end{cases} \Rightarrow r = n' \left(R_B\left[\frac{1+(n-1)q}{T}\right] - R_B\left[\frac{1-q}{T}\right] \right)$$

$$r_d = R_J\left[\frac{1-q}{T}\right] + r$$

$$\tilde{Z} = \sum_S \exp \left[\frac{1}{2T} \left(nr_d + r \sum_{\alpha \neq \beta} S^\alpha S^\beta \right) \right] = \sum_S e^{\frac{n}{2T} (r_d - r)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + x \sqrt{\frac{r}{T}} \sum_\alpha S_\alpha}$$

$$\underbrace{\left(\sum_\alpha S_\alpha \right)^2 - n}_{= e^{\frac{n}{2T} (r_d - r)}} = e^{\frac{n}{2T} (r_d - r)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \left[2 \cosh x \sqrt{\frac{r}{T}} \right]^n$$

$$\Rightarrow \log Z = \frac{n}{2T} (r_d - r) + n \log 2 + \log \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \cosh^n(x \sqrt{\frac{r}{T}})$$

$$2T \frac{\partial \log Z}{\partial r} = \sum_{\alpha \neq \beta} \langle S_\alpha S_\beta \rangle = n(n-1)q$$

$$= -n + \frac{n}{Z} \frac{2T}{2\sqrt{T}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \times \frac{\sinh(\sqrt{\frac{r}{T}} x)}{\cosh(\sqrt{\frac{r}{T}} x)^{n-1}}$$

$$= -n + \frac{n}{Z} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{\cosh(\sqrt{\frac{r}{T}} x)^2} = -n \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \tanh^2(\sqrt{\frac{r}{T}} x)$$

$$\Rightarrow q = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \tanh(\sqrt{\frac{r}{T}} x) \quad \left. \right\} \langle \tanh^2(\sqrt{\frac{r}{T}} x) \rangle_{x \sim N(0,1)}$$

$$\text{From before: } r = \bar{n}' \left(R_B \left[\frac{1+(n-1)q}{T} \right] - R_B \left[\frac{1-q}{T} \right] \right)$$

$$\lim_{n \rightarrow \infty} r = \frac{q}{T} R'_B \left[\frac{1-q}{T} \right]$$

$$q=0 \Rightarrow r=0 \Rightarrow q=0 \in SC$$

uncorrelated phase (high +)

$q \neq 0$ but small \leftarrow assumes that phase transition is continuous

$$r \approx \frac{q}{T} R' \left(\frac{1}{T} \right) - \frac{q^2}{T^2} R'' \left(\frac{1}{T} \right)$$

$$\Rightarrow \langle \tanh^2 \sqrt{\frac{r}{T}} x \rangle \approx \cancel{\langle x^2 \rangle \frac{r}{T} - \frac{2}{3} \cancel{\langle x^4 \rangle} \frac{r^2}{T^2}}$$

$$\approx \frac{q}{T^2} R'_S \left(\frac{1}{T} \right) - \underbrace{\frac{q^2}{T^3} R''_S \left(\frac{1}{T} \right) - \frac{2q^2}{T^4} R' \left(\frac{1}{T} \right)^2}_{\text{assume } < 0}$$

(true for $R_S(x)=x$ in Wigner case)

$$q \approx \frac{q}{T^2} R'_S \left(\frac{1}{T} \right) + (>0)$$

$$T_c^2 = R'_S \left(\frac{1}{T_c} \right) = \frac{1}{T_c} \quad \text{for Wigner}$$

$$\Rightarrow T_c = 1$$

For random orthogonal model where $\lambda_j = \pm 1$ equally
transition is discontinuous

Below T_c we have stages of RSB

Chapter 14: Edge Eigenvalues & Outliers

14.1 Tracy-Widom

λ_{\max} maximum eig of Wigner or Wishart

λ_+ denotes upper edge

$$* P[\lambda_{\max} \leq \lambda_+ + \gamma N^{-\frac{2}{3}}] = F(u)$$

\uparrow
 $\beta=1$ Tracy-Widom

Wigner: $\lambda_+ = 2$ $\gamma = 1$

Wishart: $\lambda_+ = (1+\sqrt{q})^2$ $\gamma = \sqrt{q} \lambda_+^{2/3}$

* Holds for symmetric IID matrices w/ finite 4^{th} moment

$$F_i := F'_i \rightarrow \begin{cases} \log F_i \propto -u^{3/2} & u \rightarrow \infty \\ \propto -|u|^3 & u \rightarrow -\infty \end{cases} \leftarrow \text{much thinner}$$

(harder to push λ_{\max} in & compress the spectrum)

From 5.4.2: $\sqrt{V(s) - qT(s)} \approx C(s - \lambda_+)^{\theta}$

$$\Rightarrow \underline{\Phi}(x) = \int_{\lambda_+}^x \sqrt{V(s) - qT(s)} \approx C \int_{\lambda_+}^x (s - \lambda_+)^{\theta+1}$$

$$\Rightarrow P(\lambda_{\max}) = \exp \left[-\frac{N\beta}{2} \underline{\Phi}(\lambda_{\max}) \right] = \exp \left[-\frac{2\beta C}{3} u^{3/2} \right] \quad u = N^{2/3} (\lambda_{\max} - \lambda_+)$$

IF $\beta(\lambda) \rightarrow 0$ as $(\lambda_+ - \lambda)^{\theta}$ I think they mean $\gamma/3$

The probability $P(\lambda > \lambda_+) \propto (\lambda_+ - \lambda)^{\theta+1}$

When this is $\sim 1/N \Rightarrow |\lambda_+ - \lambda| \sim N^{-\frac{1}{\theta+1}}$

For Gaussian ensembles

$$P_N(\lambda \approx \lambda_+) = N^{-\frac{2}{3}} \underline{\Phi}(N^{2/3}(\lambda - \lambda_+))$$

as $N \rightarrow \infty$ $\underline{\Phi} \sim N^{-1/2}$

14.2 Additive Low-Rank

$$M \rightarrow M + auu^T \quad \|u\|=1$$

$$G_a = (z - M - auu^T)^{-1} = G(z) + a \frac{Guu^TG}{1 - au^TGu}$$

G_a has a pole $\Rightarrow 1 - au^T G(\lambda_1) u = 0$ G, uu^T free

$$u^T G(z) u = \text{Tr}[G(z) uu^T] = \frac{\text{Tr}[G]}{N} = g_N(z) \rightarrow g(z)$$

pole when $g(\lambda_1) = 1/a$

$$\Rightarrow \lambda_1 = 3(1/a)$$

↑ monotonically increasing in a

$$\lambda_1 = \lambda_+ \text{ when } a = g(\lambda_+)^{-1}$$

↑ critical value of a

Generally $\frac{d\lambda(g_+)}{dg} = 0$ ie for Wigner $\lambda(g) = \sigma^2 g + g^{-1}$

$$\Rightarrow \frac{\partial}{\partial g} \lambda(g) = \sigma^2 - g^{-2} \Rightarrow g = \sigma^{-1}$$

$$\lambda(\sigma^{-1}) = 2\sigma = \lambda_+$$

$$\lambda_1 = a + R(1/a)$$

$$\Rightarrow \lambda_1 = a + \tau(M) + \frac{x_e(N)}{a} + \dots$$

$$\text{For Wigner} \quad \lambda_1 = a + \sigma^2/a \quad a > a^* = \sigma$$

$$\left. \frac{d\lambda_1}{da} \right|_{a^*} = \left. \frac{d\lambda}{dg} \right|_{g_1=1/a^*} = 0$$

$$\Rightarrow \lambda_1 = \lambda_+ + (a - a^*)^2 + \dots \quad \text{For square root singularities}$$

Fluctuations of the outlier around λ_1 can be shown to go as $N^{-1/2}$

While fluctuations of $\lambda_{\max} \sim N^{-2/3}$

Transition is called BBP transition

1)

2) For rank k , can show

$$\lambda_k = \alpha_k + R\left[\frac{1}{\alpha_k}\right] \quad \alpha_k > \frac{1}{g_+}$$

14.2.2 Outlier Eigenvectors

For $\alpha \gg 1$ $v \approx u$

For $\alpha \ll 1$ u will mix w/ the rest of M 's evecs

$$G_a = \sum \frac{v_i v_i^T}{z - \lambda_i} \Rightarrow \lim_{z \rightarrow \lambda_i} u^T G_a u (z - \lambda_i) = |v_i^T u|^2$$

$$= \lim_{z \rightarrow \lambda_i} \left[g(z) + a \frac{g(z)^2}{1 - ag(z)} \right] (z - \lambda_i)$$

$$= \lim_{z \rightarrow \lambda_i} \frac{g(z)(z - \lambda_i)}{1 - ag(z)} = \lim_{z \rightarrow \lambda_i} \left[-\frac{g(z)}{a g'(z)} \right] \quad \frac{1}{a} = g(z)$$

$$\Rightarrow |v_i^T u|^2 = -\frac{g(z)^2}{g'(z)} \quad *$$

$$z = R(g(z)) + g^{-1}(z)$$

$$\Rightarrow 1 = R'(g(z))g'(z) - g^{-2}(z)g'(z) \Rightarrow \frac{1}{g'(z)} = R'(g) - \frac{1}{g^2}$$

$$\Rightarrow |v_i^T u|^2 = 1 - g^2 R'(g)$$

$$= 1 - \frac{R'\left[\frac{1}{a}\right]}{a^2}$$

$$\alpha \rightarrow \infty \Rightarrow R'[0] \sim x_2(M)$$

$$\Rightarrow |v_i^T u|^2 = 1 - \frac{x_2(M)}{a^2}$$

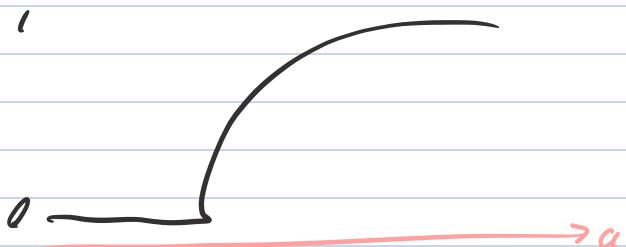
$$\text{As } \lambda_i \rightarrow \lambda_+ \quad g(z) = \int_z^{\lambda_+} \frac{p(x)dx}{z-x} \Rightarrow g'(z) = - \int_{\lambda_-}^{\lambda_+} \frac{p(x)dx}{(z-x)^2}$$

For $p(\lambda) \sim (\lambda - \lambda_+)^{\theta}$ $\theta < 1$ $g(\lambda_+)$ Finite
but g' diverges as $(z - \lambda_+)^{\theta-1}$

$$* \Rightarrow |v_i^T u|^2 \propto (\lambda_+ - \lambda_i)^{1-\theta} \quad \text{as } \alpha_+ \rightarrow a$$

For Wigner $R(x) = \sigma^2 x \Rightarrow R' = \sigma^2$

$$\Rightarrow |V^\top u|^2 = 1 - q^2 \sigma^2 = 2 - \frac{z^2 - \sqrt{1 - 4\sigma^2/z^2}}{2\sigma^2} = 1 - \left(\frac{\alpha^*}{\alpha}\right)^2 \quad \alpha > \alpha^* = \sigma$$



The fact that $|V^\top u|^2 \sim \alpha(1)$ for $\lambda_i > \lambda_4$
is not general (see Ch 19)

Ex 14.21

a) $\frac{II^T}{N}$

b) $\lambda_i = a + R \left[\frac{1}{a} \right] = a + \frac{1}{1 - qa^{-1}} = \frac{1+a-q}{1-qa^{-1}}$

c) $|V^\top u|^2 = 1 - a^{-2} R'(a^{-1})$

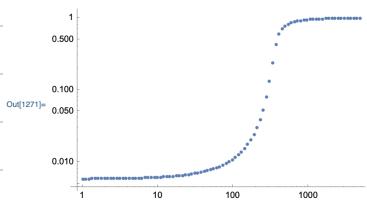
$$R' = (1 - qa)^{-2} q$$

$$= 1 - \frac{q}{(a-q)^2}$$

d)

```
In[126]:= n = 100;
W = RandomVariate[WishartMatrixDistribution[4 n, IdentityMatrix[n]]];

One = ConstantArray[1, {n, n}] / n;
uv2 = {};
as = {};
For[a = 1, a < 5000, a = a + 1.1,
  M1 = W + a One;
  as = as ~Join~ {a};
  uv2 = uv2 ~Join~ {(ConstantArray[1/Sqrt[n], {n}].Eigensystem[M1][[2]][[1]])^2}];
ListPlot[Transpose[{as, uv2}], ScalingFunctions -> {"Log", "Log"}]
```



14.3 Fat Tails

Let X_{ij} come from wide-tailed dist

$$X_{ij} = \frac{x_{ij}}{\sqrt{N}} \quad x_{ij} \sim P(x) \text{ of Mean=0, Var=1} \quad \text{decays as } \mu|x|^{-1-\mu}$$

$$P(|X_{ij}| > 1) \approx 2 \int_{\sqrt{N}}^{\infty} dx \frac{\mu}{x^{1+\mu}} = \frac{2}{N^{1/\mu}}$$

$$\Rightarrow \# \text{ entries} > 1 \approx \frac{N(N+1)}{N^{1/\mu}} \sim N^{2-1/\mu}$$

$$\begin{aligned} \mu > 4 &\Rightarrow 0 \\ &\Rightarrow \text{No outliers} \end{aligned}$$

Finiteness of x_y is enough for Tracy-Widom!

Even for $\mu > 4$ large elements still dominate tail at finite N for such power law distributions

When $2 < \mu < 4$ x_2 is finite

$\Rightarrow p(\lambda) \rightarrow \text{semicircle by CLT}$

$O(N^{2-1/\mu})$ outliers with density $X_{ij} = X_{ji} = a \Rightarrow 1$ contributes $\lambda = \pm(a + a^{-1})$

$$p_{\text{out}}(\lambda > 2) = N^{1-1/\mu} \int_1^{\infty} dx \frac{\mu}{x^{1+\mu}} \delta(\lambda - x - \frac{1}{x})$$

$$\begin{aligned} p &\rightarrow 0 \text{ as } N \rightarrow \infty \\ \lambda_{\max} &\rightarrow \infty \end{aligned}$$

$\mu < 2 \Rightarrow \text{"Lévy"}$

$$X_{ij} = \frac{x_{ij}}{N^{1/\mu}} \quad P(x) \sim \frac{\Gamma(1+\mu)}{(x)^{1+\mu}} \sin \frac{\pi \mu}{2} \quad \text{as } x \rightarrow \infty$$

$p(\lambda)$ becomes unbounded at $N \rightarrow \infty$

Tail of $p(\lambda)$ turns out to be the same as the tail of $P(x)$

14.4 Multiplicative Perturbation

True \rightarrow Sample cov is free prod w/ white Wishart

$$E = B^{1/2} C_0^{1/2} (I + \alpha u u^\top) C_0^{1/2} B^{1/2}$$

↑
 wish
 ↑
 true
 ↑
 free
 ↙
 of C_0

$\tau(B) = 1$ B is a white Wishart \Rightarrow Noisy obs of pert. cov
w/ one mode amplified by $\alpha \alpha$

$$E_0 = B^{1/2} C_0 B^{1/2}$$

$$\det(\lambda, \mathbb{1} - E_0 - \alpha B^{1/2} C_0 u u^\top B^{1/2}) = 0$$

λ outside Spec $E_0 \Rightarrow$ invert $\lambda, \mathbb{1} - E_0$

$$\Rightarrow \det(\cdots) = \det(\lambda, \mathbb{1} - E_0) \left(1 - \alpha u^\top C_0^{1/2} B^{1/2} G_B(\lambda) B^{1/2} C_0^{1/2} u \right) = 0$$

$$\Rightarrow \alpha \lambda_0 u^\top B^{1/2} G_B B^{1/2} u = 1$$

$$\Rightarrow \alpha \lambda_0 N^i \text{Tr}[G_B B] = 1$$

Using $\mathbb{E} G_C = S^* G_A (zS^*)$ $S^* = S_B(zg_0 - 1)$ for $C = B^{1/2} A B^{1/2}$

$$\text{For } E_0 = B^{1/2} C_0 B^{1/2} \Rightarrow \mathbb{E} G_0 = S^* G_B (zS^*) \quad S^* = S_{C_0}(zg_0 - 1)$$

$$\Rightarrow \mathbb{E}[G_B B] = S^* \mathbb{E}[G_B (zS^*) B]$$

$$= S^* t_B(\lambda, S^*)$$

$$\Rightarrow \alpha \lambda_0 S^* t_B(\lambda, S^*) = 1$$

$$\text{Assume } C_0 = \mathbb{1} \Rightarrow g_0 = \frac{1}{z-1} \Rightarrow zg_0 - 1 = \frac{1}{z-1} = g_0 \quad S^*[g_0] = 1 \quad \forall z$$

$$\Rightarrow \lambda_0 = 1$$

$$\Rightarrow \alpha t_B(\lambda_0) = 1 \quad \Rightarrow \text{need } t_B \text{ invertible}$$

$$f_B(\zeta) = \int_{-\infty}^{\zeta} \frac{B(x)x}{\zeta - x} dx \quad \leftarrow \text{decreasing for } \zeta > \lambda_+$$

$$\lambda_+ = \zeta(a^{-1}) = \frac{a+1}{S(a^{-1})} \quad a > \frac{1}{f_B(\lambda_+)}$$

Let $B=W$ a Wishart

$$\zeta_W = (I + qX)^{-1} \quad \lambda_{\pm} = (I \pm \sqrt{q})^2$$

$$\Rightarrow \lambda_+ = (a+1)(1 + q/a) \\ \approx 1 + a + q$$

$\Rightarrow a+1$ in the Cov will appear shifted by q

Ex 14.4.1

a) Known

$$b) (I + (\sqrt{a}-1)uu^\top)^2 = I + [(\sqrt{a}-1)^2 + 2\sqrt{a}-2] uu^\top \\ = I + (a-1)uu^\top$$

$$c) \text{ Apply prev result: } \text{Spec } B^{\frac{1}{2}}(I + (a-1)uu^\top)B^{\frac{1}{2}} = \text{Spec } (I + (\sqrt{a}-1)uu^\top)B(I + (\sqrt{a}-1)uu^\top) \\ \Rightarrow \lambda_+ = 1 + q + a - 1 = q + a$$

$$c = \sqrt{a} - 1 \quad q = 1 \Rightarrow = 1 + a \\ (c+1)^2 = a$$

Ex 14.4.2 Inverse Wishart Mult Pert

$$M_p := (1-q)W_q^{-1} \quad p = \frac{q}{1-q} \quad \text{var of } W^{-1}$$

$$\Rightarrow S_{M_p}(1) = (1-pt)$$

$$a) D = I + (d-1)\hat{e}_1\hat{e}_1^\top$$

$$b) S_{M_p}(a^{-1}) = (1-p/a) \Rightarrow \lambda_+ = \frac{a+1}{1-p/a} \approx 1 + a + p$$

$$\text{use } d-1 \Rightarrow M_p^{\frac{1}{2}}(I + (d-1)uu^\top)M_p^{\frac{1}{2}}$$

$$\Rightarrow d = d^2 - 1$$

$$\lambda_1 \approx d^2 + p$$

c)
d)

Finish

14.5 Phase Retrieval & Outliers

$$\hat{x} = \underset{x}{\operatorname{argmin}} \|(\alpha_k^T x)^2 - y\|, \quad x \in \mathbb{R}^n \quad k=1 \dots T$$

Non convex in x w/ many local minima

Need x_0 st. $x^* \cdot x_0 > 0$

In high dim $\log P[x^* \cdot x_0 > \epsilon] \propto -N\epsilon$

Want nonzero overlap w/ x^*

$$\text{Take } M = \frac{1}{T} \sum_k f(y_k) \alpha_k \alpha_k^T$$

Assume $\alpha_k \sim N(0, I)$

\Rightarrow WLOG $x^* = \hat{e}_1$

$\Rightarrow f(y_k)$ related to $|\alpha_k \cdot \hat{e}_1|^2$

$$M = \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix}$$

$$N g_M(z) = \operatorname{Tr} G_{22} + \frac{\operatorname{Tr} G_{22} M_{21} M_{12} G_{22}}{z - M_{11} - M_{12} G_{22} M_{21}}$$

outlier

$$\lambda_1 = M_{11} + M_{12} G_{22} (\lambda_1) M_{21}$$

self-averaging

$$|V^T x|^2 = \lim_{z \rightarrow \lambda_1} \frac{z - \lambda_1}{z - M_{11} - M_{12} G_{22} M_{21}}$$

$$M_{11} = \frac{1}{T} \sum_{k=1}^T f(y_k) (a_k)_i^2 \xrightarrow{T \rightarrow \infty} E[f(y) (a_i)^2]$$

$$M_{12} G_{22} M_{21} = \frac{1}{T} \sum_{k,l} f(y_k) f(y_l) [a_k]_i [a_l]_j \sum_{i,j} [a_k]_i [a_l]_j [G_{22}]_{ij}$$

$$\Rightarrow \frac{1}{T} \sum_{k,l} f_k f_l [a_k]_i [a_l]_j \underbrace{\left(\frac{M G_{22} H^T}{T} \right)_{kl}}_{S_{KL} (\tilde{a}_i)^2}$$

$$\Rightarrow q \underbrace{E[f^2(a_i)]^2}_{\alpha} \tau \underbrace{(H H^T G_{22})}_{h(z)} \quad q = N/T$$

$$\frac{z - \lambda}{z - c_1 - q c_2 h(z)}$$

$$\lambda_1 = c_1 + q c_2 h(0)$$

$$|x^* \cdot v|^2 = \frac{1}{1 - q c_2 h(\lambda_1)}$$

$$q \rightarrow 0 \Rightarrow T \rightarrow \infty \text{ really fast}$$

$$\Rightarrow M \propto E f(y) \mathbb{I} = m_1 \quad \Rightarrow g_m(z) = \frac{1}{z - m_1} \quad H H^T \rightarrow T \cdot \mathbb{I}$$

$$\Rightarrow h(z) = \frac{1}{z - m_1}$$

Linear correction in q :

$$\lambda_1 = c_1 + \frac{q c_2}{c_1 - m_1} \quad \text{why?}$$

FINISH

Chapter 15 ← and ×: Recipes & Examples

$$g_A(z) = \tau[(z-A)^{-1}]$$

$$t_A(z) = \tau[A(S-A)^{-1}] = \tau[(1-S^{-1}A)^{-1}] - 1 = \zeta g_A(\zeta) - 1$$

$$R_A(g) = \beta_A(g) - \frac{1}{2}g \quad S_A(t) = \frac{t+1}{t-g_A(t)} \leftarrow \tau(A) \neq 0$$

$$R_{\alpha A}(x) = \alpha R_A(\alpha x)$$

$$S_{\alpha A}(t) = \alpha^{-1} S_A(t)$$

$$R_{A+\alpha I}(x) = \alpha + R_A(x)$$

$$S_{A^{-1}}(x) = \frac{1}{S_A(-x-1)}$$

$$S_A(t) = \frac{1}{R_A(t+S_A(t))}$$

$$R_A(x) = \frac{1}{S_A(x R_A(x))}$$

$$g_1 = \frac{1}{z-1} \Rightarrow R_1(x) = 1$$

$$t_1 = \frac{1}{\zeta-1}$$

$$R_A(x) = \sum_{i=1}^{\infty} x_i x^{i-1}$$

$$S_A(x) = \frac{1}{x_1} - \frac{x_2}{x_1^3} x + \frac{2x_2^2 - x_1 x_3}{x_1^5} x^2 + \dots$$

$$\tau(A^{-1}) = [S_{A^{-1}}(0)]^{-1} = S_A(-1)$$

$$\begin{aligned} \tau(A^{-2}) &= -S_{A^{-1}}'(0)[S_{A^{-1}}(0)^3]^{-1} = -S_A(-1)^3 \frac{d}{dt} \Big|_{t=0} [S_A(-t-1)]^{-1} + S_A(-1)^2 \\ &\quad + \tau(A^{-1})^2 \\ &= S_A(-1)(S_A(1) - S_A'(-1)) \end{aligned}$$

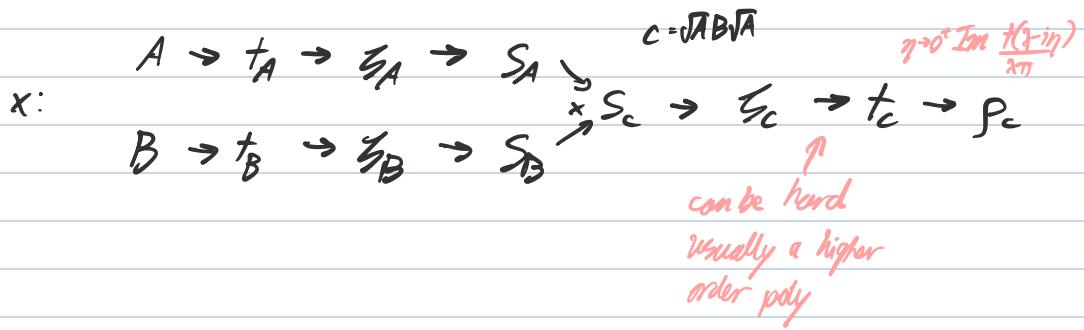
15.1.2 Computing ρ

$t:$

$$A \rightarrow g_A \rightarrow \beta_A \rightarrow R_A \xrightarrow{C=A+B} R_C \rightarrow \beta_C \rightarrow g_C \rightarrow \rho_C$$

$\gamma \rightarrow \text{Im } \frac{g(A-i\gamma)}{T}$

$$B \rightarrow g_B \rightarrow \beta_B \rightarrow R_B$$



Trick: $\text{TP} = \max_{\text{sols}} \text{Im} \dots$ for cubic/quadratic
since solns come in pairs

15.2 R, S Transforms of Useful Ensembles

15.2.1 Wigner:

$$R(x) = \sigma^2 x$$

Free sum has $\sigma_c^2 = \sigma_1^2 + \sigma_2^2$

$\tau(X) = 0 \Rightarrow$ No S transform

But $X \neq m\mathbb{1}$ has $R_{X \neq m\mathbb{1}} = m + \sigma^2 x$

$$\Rightarrow S(t) = \frac{1}{m + \sigma^2 t S} \Rightarrow S^2 t \sigma^2 + m S - 1 = 0$$

$$\Rightarrow S = \frac{\sqrt{m^2 + 4t\sigma^2} - m}{2\sigma^2 t} = \frac{m}{2\sigma^2 t} \left(\sqrt{1 + \frac{4t\sigma^2}{m^2}} - 1 \right)$$

$$\tau(X^{2k}) = \frac{(2k)!}{k!(k+1)} \sigma^{2k} \Rightarrow (\sigma\sqrt{t})^{-1} \text{ as } m \rightarrow 0$$

$$\tau(X^{-k}) = \infty$$

15.2.2 Wishart

$$q = N/T \quad R_{W_q} = \frac{1}{1-qx} \Rightarrow \tau(W_q) = 1 \quad \tau(W_q^2) = 1+q$$

g satisfies

$$g^{-1} = z - 1 + q - qzg \Rightarrow 1 = \left(\frac{z+1}{z}\right)(z-1) - q + \left(\frac{z+1}{z}\right)$$

$$\underbrace{\left(\frac{z+1}{z}\right)^{-1}}_{-q+} \quad \underbrace{-q+}_{=} \Rightarrow z = (z+1)(z-1) - q + (z+1)$$

$$(t+1)(\zeta - 1 - qt) - \zeta = 0$$

$$\Rightarrow \zeta t - (t+1)(1 - qt) = 0$$

$$\frac{1}{1-q} + \frac{1}{(1-q)^2}$$

$$\Rightarrow S = \frac{1}{1 - qt} \Rightarrow \tau(W_q^{-1}) = \frac{1}{1 - q} \quad \tau(W_q^{-2}) = \frac{1}{(1 - q)^3}$$

15.2.3 Inverse Wishart

$$S(-t-1)^{-1} \Rightarrow 1 - q - qt$$

$$\tau(W_q^{-1}) = \frac{1}{1 - q} \Rightarrow M_p = (1 - q) W_q^{-1}$$

$$S_{M_p}(t) = (1 - q)^{-1} (1 - q - qt) = 1 - pt \quad p = \frac{q}{1 - q}$$

$$\Rightarrow x_1(M_p) = S(O) = 1$$

$$x_2(M_p^2) = -S'(O) S(O)^{-3} = p$$

$$S''(O) = 0 \Rightarrow x_3 = 2x_2^2 = 2p^2$$

$$R_{M_p}(x) = [S_{M_p}(x R_{M_p})]^{-1}$$

$$\Rightarrow R(1 - pxR) = 1 \Rightarrow R = \frac{1 - \sqrt{1 - 4px}}{2px}$$

$$\begin{aligned} \zeta &= \frac{t+1}{t} (1 - pt) \Rightarrow t+1 = \zeta(1 - pt)^2 \\ &\Rightarrow p\zeta t^2 + (1 - \zeta)t + 1 = 0 \end{aligned}$$

$$\Rightarrow t = \frac{\zeta - 1 \pm \sqrt{(\zeta - 1)^2 - 4p\zeta}}{2p\zeta}$$

$$g = \frac{t+1}{z} = \frac{(1+2p)z - 1 \pm \sqrt{(z-1)^2 - 4pz}}{2pz^2}$$

$\zeta \rightarrow 0 \Rightarrow \sqrt{1 - 2(1-p)z} = 1 - (1-p)z$

$$\Rightarrow p = \frac{\sqrt{(1-p)(1-2p)}}{2pz^2} \quad z_{\pm} = 2p + 1 \pm 2\sqrt{p(1-p)}$$

$$\tau(M_p^{-1}) = \lim_{z \rightarrow \infty} g_{M_p}(z) = 1 + p$$

From $z=\infty$ expansion

$$g(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1+p}{z^3} + \dots$$

$$\tau(M_p) = 1 \quad \tau(M_p^{-2}) = 1+p$$

From $z=0$

$$g(z) = -(1+p) - (1+p)(1+2p)z + \dots$$

$$\tau(M_p^{-2}) = (1+p)(1+2p)$$

Using $\operatorname{Re} g(x) = \int \frac{g(\lambda)}{z-\lambda} d\lambda = \frac{V'(x)}{z}$

$$\Rightarrow V' = \frac{(1+2p)z - 1}{2pz^2} \Rightarrow V = \frac{1}{M_p} \frac{1}{px} + \frac{1+2p}{p} \log x$$

Recall for Wishart:

$$P(E) = \frac{(\frac{T}{2})^{NT/2}}{\Gamma_N(T/2)} \frac{(\det E)^{(T-N)/2}}{(\det C)^{T/2}} \exp \left[-\frac{1}{2} \operatorname{Tr}[E C^{-1}] \right]$$

cov

$N \times N$ Wishart
w/ T datapoints

$$E[E^{-1}] = \frac{T}{T-N-1} C^{-1} =: \Sigma$$

$\approx \frac{1}{1-q}$

$$E[M] = \sum$$

$$M = E^{-1} \Rightarrow \det M = \det E^{-N-1} = \det M^{N+1}$$

$$P(M) = \frac{((T-N-1)/2)^{NT/2}}{\Gamma_N(T/2)} \frac{(\det \Sigma)^{T/2}}{(\det M)^{(T+N+1)/2}} \exp \left[\frac{T-N-1}{2} \operatorname{Tr}[M^{-1} \Sigma] \right]$$

Jac

$$N=1 \quad \Rightarrow P(m) = \frac{b^a}{\Gamma(a)} m^{-a-1} e^{-b/m}$$

$$b := \frac{T-2}{2} \sum$$

$$a = T/2$$

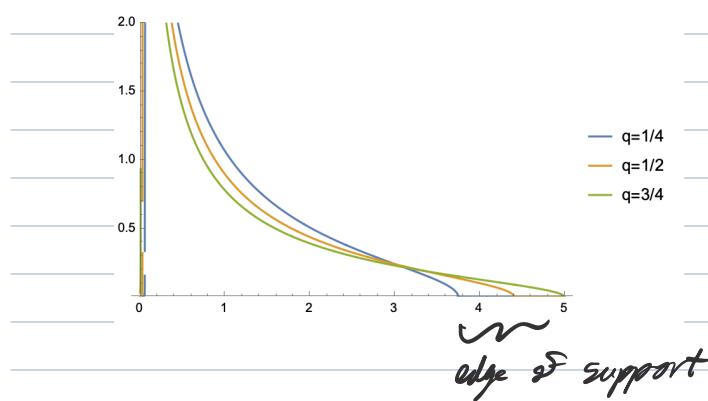
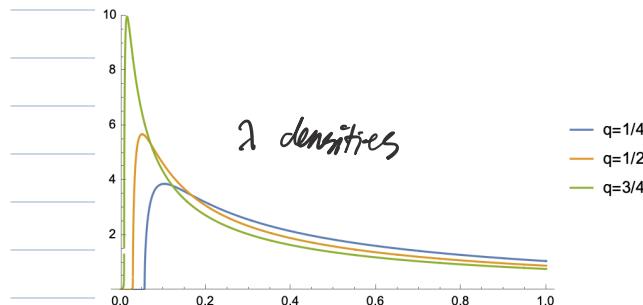
Ex 15.2.1

$$E = W_{q_0}^{1/2} W_q W_{q_0}^{1/2} \quad W_{q_0} \text{ is true covar}$$

$$1) S_E = S_{W_0} S_W = \frac{1}{(1-q_0 t)(1-q_1 t)}$$

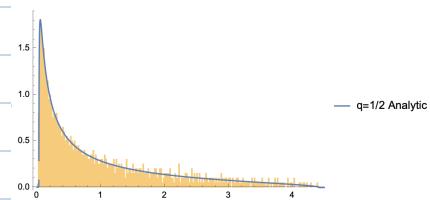
$$2) \Rightarrow S_E = \frac{t+1}{t} (1-q_0 t)(1-q_1 t) \Leftarrow \text{cubic to solve for } S$$

3) $t \rightarrow 0 \Rightarrow MP$



Y

```
W0 = RandomVariate[WishartMatrixDistribution[2000, IdentityMatrix[1000] / 2000]];
W1 = RandomVariate[WishartMatrixDistribution[2000, IdentityMatrix[1000] / 2000]];
evals0 = Eigenvalues[W1.W0];
Show[Histogram[evals0, 200, "PDF"],
Plot[Max[Im@{soln[[1]], soln[[2]], soln[[3]]}] /. q \[Rule] 1/2] /.\. z \[Rule] x - I 0.0001 // Evaluate,
{x, 0, 10}, PlotRange \[Rule] {0, 10}, PlotPoints \[Rule] 1000, PlotLegends \[Rule] {"q=1/2 Analytic"}]
```



15.3 Worked-out Examples: Addition

15.3.1 Arcsine Law

M_1, M_2 Symm orth $\Rightarrow \lambda_i = \pm 1$

char poly of $M_1 = (z-1)^{N_2}(z+1)^{N_2}$

$$\frac{1}{N} \frac{\partial}{\partial z} \log p_{M_1} = \frac{1}{2} \frac{1}{z-1} - \frac{1}{2} \frac{1}{z+1} \Rightarrow \frac{z}{z^2-1} = g(z)$$

$$\Rightarrow z = \frac{1 + \sqrt{1+4g^2}}{2g} \Rightarrow R = \frac{-1 + \sqrt{1+4g^2}}{2g}$$

$$\Rightarrow M = \frac{1}{2}(M_1 + M_2) \text{ has } R_M = \frac{R}{\frac{1}{2}(M_1 + M_2)}$$

$$= 2 R_{\frac{1}{2}M} = \frac{\sqrt{1+g^2} - 1}{g}$$

$$\Rightarrow z = \sqrt{1+g^2} \Rightarrow \frac{1}{\sqrt{z^2-1}} = g$$

\Downarrow

$$\frac{1}{z\sqrt{1-\frac{1}{z^2}}}$$

$$\Rightarrow p(\lambda) = \frac{1}{\pi} \frac{1}{\sqrt{1-\lambda^2}} \quad \lambda \in (-1, 1)$$

Jacobi arcsine law

15.3.2 Sums of Uniform

$$M = U + DUU^T$$

U is diag w $\lambda \sim \text{Unif}(-1, 1)$

$$P_U = \frac{1}{2} \Rightarrow g(z) = \frac{1}{2} \int_{-1}^1 \frac{d\lambda}{z-\lambda} = \frac{1}{2} \log\left(\frac{z+1}{z-1}\right)$$

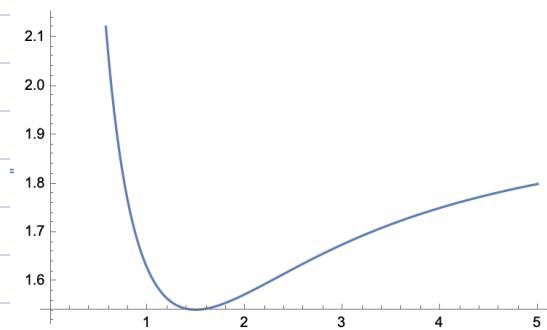
$$e^{2g} = \frac{z+1}{z-1} \Rightarrow z = \coth g$$

$$\Rightarrow R_U = \coth g - \frac{1}{g}$$

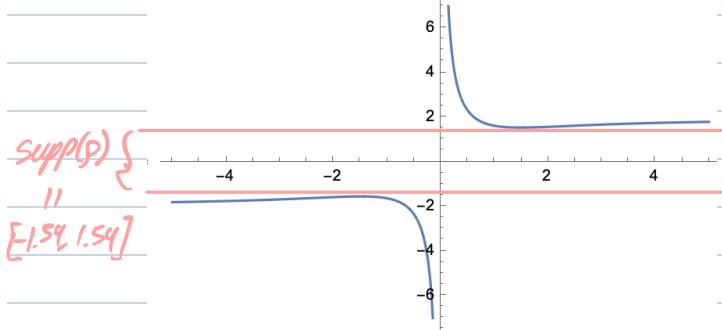
$$R_M = 2 \coth g - \frac{2}{g}$$

$$\Rightarrow z_m = 2 \coth g - \frac{1}{g} \quad \leftarrow \text{transcendental!}$$

= Plot[2 Coth[g] - 1/g, {g, 0, 5}]



: Plot[2 Coth[g] - 1/g, {g, -5, 5}]



15.4 Example: Multiplication

$$E = \sqrt{M_p} W_q \sqrt{M_p}$$

↑
indep

$$S_{M_p} = 1-pt \quad S_{W_q} = \frac{1}{1+qt} \Rightarrow S_E = \frac{1-pt}{1+qt}$$

$$\Rightarrow S_E = \frac{1+t}{t} \frac{1-pt}{1-pt} \Rightarrow \text{quadratic in } t$$

$$t_E = \frac{2-q-1 - \sqrt{(q+1-\lambda)^2 - 4(q+2p)}}{2(q+2p)} \Rightarrow p_E = \frac{\sqrt{4(q+2p) - (1+q-\lambda)^2}}{2\pi\lambda(q+p)}$$

$$\text{edges at } \lambda_{\pm} = 1+2p - q \pm 2\sqrt{(fp)(q+p)}$$

$$p \rightarrow 0 \Rightarrow 1-q \approx 2\sqrt{q} = (1 \pm \sqrt{q})^2 \text{ as expected}$$

Ex 15.4.1 $p=1/4 \Rightarrow q=1/5$

```
In[45]:= n = 1000; t = 5000;
M = RandomVariate[InverseWishartMatrixDistribution[t, (t - n - 1) IdentityMatrix[n]]];
n = 1000; t = 4000;
W = RandomVariate[WishartMatrixDistribution[t, 1/t IdentityMatrix[n]]];
E0 = MatrixPower[M, 1/2].W.MatrixPower[M, 1/2];
```

a)

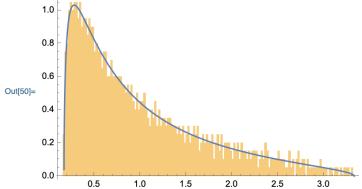
```
In[46]:= Tr[M] / n
Tr[M.M] / n
Tr[E0] / n
Tr[E0.E0] / n
evals0 = Eigenvalues[E0];
Show[Histogram[evals0, 200, "PDF"],
Plot[Sqrt[4 (p \lambda + q) - (1 + q - \lambda)^2] / . p \rightarrow 1/4 /. q \rightarrow 1/4, {\lambda, 0, 4}]]
```

Out[45]= 0.999972

Out[46]= 1.25124

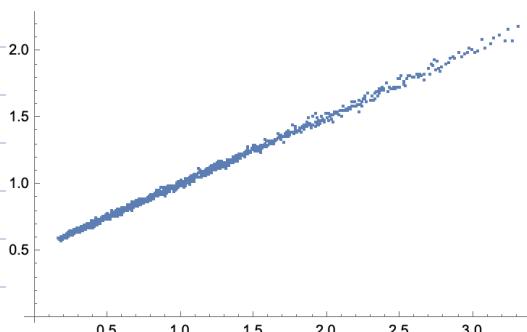
Out[47]= 0.999669

Out[48]= 1.50124



b)

```
overlaps = U.M.Transpose[U];
overlaps2 = Table[overlaps[[i, i]], {i, 1, 1000}];
ListPlot[Transpose[{evals0, overlaps2}]]
```



15.4.2 Free Product of Projectors

$$P_1 = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 & \ddots & 0 \end{pmatrix} \quad \text{Projector} \quad N \rightarrow N_1$$

$$P_2: N \rightarrow N_2$$

$P_1 P_2$ free product

$$g_{P_a} = \frac{N_a/N}{z-1} + \frac{N-N_a}{z} = \frac{q_a z + (1-q_a)(z-1)}{z(z-1)} = \frac{q_a + z - 1}{z(z-1)}$$

$$\Rightarrow t_{P_a} = \frac{q_a}{z-1} \Rightarrow g_{P_a} = \frac{q_a}{t} + 1 \Rightarrow s_{P_a} = \frac{t+1}{t+q_a}$$

$$\Rightarrow s_p = \frac{(t+1)^2}{(t+q_a)(t+q_b)}$$

$$\xi_p = \frac{(t+q_a)(t+q_b)}{t(t+1)} \Rightarrow \text{quadratic}$$

$$\Rightarrow t = \frac{q_1 + q_2 - \xi + \sqrt{\xi^2 - 2\xi(q_1 + q_2 - 2q_1 q_2) + (q_1 - q_2)^2}}{2(\xi - 1)}$$

$$\lambda_{\pm} = q_1 + q_2 - 2q_1 q_2 \pm 2\sqrt{q_1 q_2 (1-q_1)(1-q_2)}$$

$$\lambda_- \geq 0 \quad \lambda = 0 \text{ iff } q_1 = q_2 \quad \Rightarrow g = \frac{1}{z} + \frac{t}{z}$$

Jacobi ensemble with $c_i = \frac{q_{\max}}{q_{\min}}$ $c_t = \frac{1}{q_{\min}}$

$$\Rightarrow p(\lambda) = \frac{\sqrt{(t-\lambda)(\lambda-\lambda_-)}}{2\pi i(1-\lambda)} + A_0 \delta(z) + A_1 (\delta(z-1))$$

$$A_0 = 1 - \frac{|q_1 + q_2 - |q_1 - q_2||}{2} = 1 - \min(q_1, q_2)$$

$$A_1 = \frac{q_1 + q_2 - |q_1 + q_2 - 1|}{2} = \max(q_1 + q_2 - 1, 0)$$

15.4.3 Jacobi Revisited

$$S_{W_{12}^{-1}} = \frac{T_{12}^{-1}}{1 + \frac{N}{C_{12} + t}} = \frac{N^{-1}}{C_{12} + t} \quad c_i = T_i / N_i$$

unnormalized

$$S_{W_1^{-1}} = N (c_{12} - 1)$$

$$S_E = S_{W_1^{-1}} S_{W_2} = \frac{c_1 - 1}{c_2 + t}$$

Hardest part is the shift

$$(c_2 + t) S_E = c_1 - 1$$

$$(c_2 + xR) = (c_1 - xR - 1) R$$

$$R(x) = \frac{1}{S_E(xR(x))}$$

$$\begin{aligned} t &= xR(x) \\ S(t) &= 1/R(x) \end{aligned}$$

$$\Rightarrow c_2 + R\left(1 + x - c_1 + xR\right) = 0$$

$$c_2 + \left(\frac{R-1}{E}\right)\left(1 - c_1 + xR\right) = 0 \quad S(t) = \frac{1}{R(tS)} \Rightarrow R \rightarrow \frac{1}{S} \quad x \rightarrow +t$$

$$\Rightarrow c_2 + \left(S_{E+1}^{-1} - 1\right)\left(1 - c_1 + t\right) = 0$$

$$\Rightarrow (c_2 - c_1 - t - 1) S_{E+1} + 1 - c_1 + t = 0$$

$$\Rightarrow S_{E+1} = \frac{1 + t - c_1}{t + 1 - c_1 - c_2}$$

$$\Rightarrow S_S = (S_E(-1))^{-1} = \frac{t + c_1 + c_2}{t + c_1} \quad \text{quad } \underbrace{\quad}_{\quad}$$

$$\Rightarrow t(\zeta) = \frac{1 + c_1 - c_2 \zeta + \sqrt{c_1^2 \zeta^2 - 2(c_1 c_2 - 2c_1) \zeta + (c_1 - 1)^2}}{2(\zeta - 1)}$$

$$c_{\pm} = c_1 \pm c_2$$

$$x_1 = \frac{c_1}{c_1 + c_2} \quad x_2 = \frac{c_1 c_2}{(c_1 + c_2)^3} \dots$$

From 5:

$$R_J = \frac{x - c_1 - \sqrt{(c_1 - x)^2 + 4c_1 x}}{2x}$$

$$c_1 = c_2 = 1 \Rightarrow S = \frac{t+2}{t+1}$$

$$R = \frac{x - 2 - \sqrt{x^2 + 4}}{2x}$$

$$\text{For centered: } R_c(t) = 2R_c(2t) - 1$$

Chapter 16: Products of Many Random Matrices

16.1

$$M_K := A_K A_{K-1} \cdots A_1 A_1^T \cdots A_K^T$$

$$S_{M_K} = \prod_{j=1}^K S_j(z) = S_1(z)^K$$

↑
S-transform of $A_i A_i^T$

Assume $M_K \propto \mu^K$ w/ μ a random var

$$t_K = \int \frac{\mu^K}{z - \mu^K} P_{\mu}(d\mu) = - \int \frac{1}{1 - z\mu^K} P_{\mu}(d\mu)$$

$$\text{Take } z = u^K \Rightarrow - \int \frac{1}{1 - (\frac{u}{\mu})^K} P_{\mu}(d\mu)$$

$$\begin{aligned} u > \mu &\Rightarrow 0 \\ u < \mu &\Rightarrow \end{aligned} \quad t_K(u) = - \int_u^\infty P_{\mu}(d\mu) = - P_u = - P(z^{u/K})$$

$$S_K(t) = [P^{-1}(-t)]^K$$

$$\Rightarrow S = \frac{t+1}{t + [P^{-1}(-t)]^K} = S_1^K$$

$$K \rightarrow \infty \text{ has } S_1 = \frac{1}{P^{-1}(-t)} \Rightarrow P(\mu) = -S_1^{-1}(\frac{1}{\mu})$$

$$\Rightarrow P_{\mu} = -P(\mu) = \frac{\partial}{\partial \mu} S_1^{-1}(\frac{1}{\mu})$$

$$\text{For Wishart: } S_1 = \frac{1}{1+q\mu} \Rightarrow S_1^{-1}(\frac{1}{\mu}) = \frac{\mu^{-1}-1}{q}$$

$$\Rightarrow -S_1^{-1}(\frac{1}{\mu}) = \frac{1-\mu}{q}$$

$$\Rightarrow P_{\mu} = \frac{1}{q} \text{ for } \mu \in (1-q, 1)$$

Log Normal: $S_{LN}^0 = e^{-\alpha(z+\frac{1}{2})} \Rightarrow S'(\mu) = -\frac{1}{\alpha} \log \mu - \frac{1}{2}$

 $\Rightarrow P(\mu) = -S'(\frac{1}{\mu}) = -\frac{1}{\alpha} \log \mu + \frac{1}{2} \Rightarrow p = -P'(\mu) = \frac{1}{\alpha \mu} \quad \mu \in (e^{-\alpha}, e^{\alpha})$

Lyapunov Exponent:

$$\Lambda := \lim_{K \rightarrow \infty} \frac{1}{K} \log M_K$$

e.g. for log-normal

Λ is distributed uniformly on $(-\frac{\alpha}{2}, \frac{\alpha}{2})$

So far the spectrum of the K matrices is K -indep
 \Rightarrow leads to dist that depends on $S(z)$ explicitly
 \Rightarrow Non-universal

Instead assume $AAT = \left(1 + \frac{a}{2K}\right)I + \frac{B}{\sqrt{K}}$ $\mathbb{E}(B) = 0$
 $\mathbb{E}(B^2) = b$

$$\Rightarrow S(z) = 1 - \frac{a}{2K} - \frac{b}{K}z + o(K^{-1})$$

$$\Rightarrow S_{M_K}(z) = \left(1 - \frac{a}{2K} - \frac{b}{K}z\right)^K \rightarrow e^{-\frac{a}{2} - bz} \quad \text{Multiplicative CLT}$$

$b = a \Rightarrow$ free log-normal

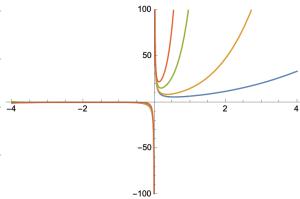
Ex 16.1.1 Edges of Spectrum

$$a) S_i = \frac{1}{1+q^x} \Rightarrow S_m = (1+q^t)^{-K}$$

$$\Rightarrow S_m = \frac{t+1}{t} (1+q^t)^K$$

b)

```
Plot[(t+1)/t (1+q t)^k /. q -> 1/2 /. k -> {3, 5, 10, 15} // Evaluate, {t, -4, 4}, PlotRange -> 100]
```



$$c) S'_m = \frac{(1+q^t)^{K-1}}{t^2} (-1+q t(-1+k+kt))$$

$$S' = 0 \Rightarrow t = \frac{q(1-K) \pm \sqrt{q^2 K + q - 2Kq + K^2 q}}{2Kq} = \frac{-qK \pm Kq}{2qK} + \frac{q \pm (2-q)q}{2Kq}$$

$\frac{-q^2 + 3q}{2Kq} \quad \frac{q^2 - q}{2Kq}$

$$S = \Rightarrow 0$$

$$d) \lim_{K \rightarrow \infty} \frac{1}{K} \log S_m = * + \sum \frac{1}{1-q}$$

16.2 Free Log-Normal

$$S_N = \exp(-a_2 - b_2)$$

Product of lognormals $AB=C$ is lognormal

$$a_A + a_B = a_C$$

$$b_A + b_B = b_C$$

$$S_N \approx e^{-a_2} \left[1 - b_2 + \frac{b_2^2}{2} z^2 \right]$$

$$X_1 = e^{a_2}$$

$$X_2 = e^{a_2} b$$

$$X_3 = e^{2a_2} \frac{3b_2^2}{2}$$

$$\text{When } b=a \quad S[-z-1]^{-1} = e^{\frac{az}{2}} e^{-a(z+1)} = e^{-\frac{a}{2}-az}$$

$$g(t) = \frac{t+1}{t} e^{\frac{a}{2}+at}$$

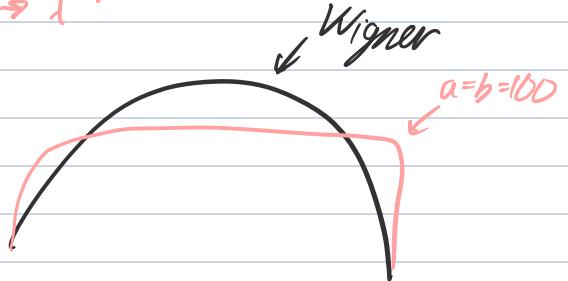
$$g' = 0 \Rightarrow t = \frac{-1 \pm \sqrt{1+4a}}{2}$$

$$\Rightarrow \lambda_+ = e^{\frac{1+a^2}{4}} \left(\sqrt{\frac{a}{2}} + \sqrt{1+\frac{a}{2}} \right)^2 = \lambda_-$$

$$a=0 \Rightarrow \lambda_+ = \lambda_- = 0$$

symmetric in $\lambda \Rightarrow \lambda^{-1}$

$\lambda = \log \lambda$ has even density



$a \neq b \Rightarrow \lambda$ is shifted by $\frac{a-b}{2}$

16.3 Multiplicative Dyson Brownian Motion

$$M_{n+1} = \sqrt{M_n} \left[\left(I + \frac{a\varepsilon}{2} \right) \mathbb{I} + \sqrt{\varepsilon} B_n \right] \sqrt{M_n}$$

Noise
 $D(B_n) = 0$

$$\lambda_{i,n+1} = \lambda_{i,n} \left(1 + \frac{a\varepsilon}{2} + \sqrt{\varepsilon} V_{i,n}^T B_n V_{i,n} \right) + \varepsilon \sum_{j \neq i} \frac{\lambda_{i,n} \lambda_{j,n} (V_{i,n}^T B_n V_{j,n})^2}{\lambda_{i,n} - \lambda_{j,n}}$$

Since M, B free, $E(B) = 0 \Rightarrow E[V_i^T B V_j] = 0$

$$E[(V_i^T B V_j)^2] = \frac{b}{N} \quad b = E(B_n^2)$$

Take $\epsilon = dt \Rightarrow \frac{d\lambda_i}{dt} = \frac{a}{2} \lambda_i + \frac{b}{N} \sum_{j \neq i} \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} + \sqrt{\frac{b}{N}} \lambda_i \xi_i$

$$g(z,t) = \frac{1}{N} \sum_i \frac{1}{z - \lambda_i(t)} \Rightarrow \frac{dg}{dt} = \frac{1}{N} \sum_i \left(\frac{1}{(z - \lambda_i)^2} \right) \frac{d\lambda_i}{dt} = -\frac{1}{N} \frac{\partial}{\partial z} \sum_i \frac{1}{z - \lambda_i} \frac{d\lambda_i}{dt}$$

$$-\frac{a}{2N} \sum_i \frac{\lambda_i}{z - \lambda_i} - \frac{b}{N^2} \sum_{i,j} \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)(z - \lambda_i)} - \sqrt{\frac{b}{N}} \frac{1}{N} \sum_i \frac{\lambda_i}{z - \lambda_i} g_i$$

$\cancel{E=0}$

$$-\frac{a}{2} z g(z)$$

$$\frac{1}{N^2} \frac{1}{2} \sum_{i,j} \left[\frac{\lambda_i \lambda_j}{(z - \lambda_i)(\lambda_i - \lambda_j)} + \frac{\lambda_i \lambda_j}{(z - \lambda_j)(\lambda_j - \lambda_i)} \right] = \frac{1}{2N} \sum_{i \neq j} \frac{\lambda_i \lambda_j}{(z - \lambda_i)(z - \lambda_j)} = \frac{1}{2} \left(\frac{1}{N} \sum_i \frac{\lambda_i}{z - \lambda_i} \right)^2$$

$$-\sum_i \frac{\lambda_i}{z - \lambda_i} = \sum_i 1 - \sum_i \frac{z}{z - \lambda_i}$$

$$\Rightarrow \frac{1}{2} (1 - z g(z))^2 = \cancel{\frac{1}{2}} - z g + \frac{z^2 g^2}{2}$$

$$bzg - \frac{b}{2} z^2 g^2$$

$$\Rightarrow \frac{\partial g}{\partial t} = \frac{1}{2} \frac{\partial}{\partial z} \left[(2b-a) zg - bz^2 g^2 \right]$$

$$h = e^t g(e^t, t) + \frac{a}{2b} - 1$$

$$e^t g(e^t, t) = h - \frac{a}{2b} + 1$$

$$\Rightarrow \frac{\partial h}{\partial t} = e^t \frac{\partial g}{\partial t}(e^t)$$

$$= \frac{e^t}{2} \frac{\partial}{\partial e^t} \left[(2b-a)e^t g - b e^{2t} g^2 \right]$$

$$= \frac{1}{2} \frac{\partial}{\partial t} \left[(2b-a) \left(h - \frac{a}{2b} + 1 \right) - b \left(h - \frac{a}{2b} + 1 \right)^2 \right]$$

$\cancel{5h^2 + (2b-a)h + const}$

$$= -\frac{1}{2} \frac{\partial}{\partial t} (bh^2)$$

$$= -bh \frac{\partial h}{\partial t} \quad \leftarrow \text{after } t' = bt, \text{ Burgers'}$$

\Rightarrow Method of characteristics

$$g(z, 0) = (z-1)^{-1}$$

$$h(l, t) = h_0(l - bt + h(l, t))$$

$$h_0(l) = h(l, 0) = \frac{1}{1-e^{-l}} + \frac{a}{2b} - 1$$

$$\Rightarrow g(z, t) = \frac{1}{z - \exp[t(bzg + \frac{a}{2} - b)]}$$

$$g = \frac{t+1}{t} \exp(g_2 + bt)$$

$$t = zg - 1 \Rightarrow z = \frac{zg}{zg - 1} e^{g_2 - b + bzg} \Rightarrow zg - 1 = ge^{g_2 - b + bzg} \Rightarrow g = (z - e^{g_2 - b + bzg})^{-1}$$

some as $g(z, t=1)$

16.4 The Matrix Kesten Problem

$$Z_{n+1} = z_n(1 + Z_n)$$

$$z_n = 1 + \varepsilon m + \sqrt{\varepsilon} \sigma \eta_n$$

$$Z_n = U_n/\varepsilon$$

$$U_{n+1} = \varepsilon (1 + \varepsilon m + \sqrt{\varepsilon} \sigma \eta_n) (1 + U_n/\varepsilon)$$

$$= U_n + \varepsilon m U_n + \sqrt{\varepsilon} \sigma \eta_n U_n + \varepsilon$$

$$\Rightarrow \frac{dU}{dt} = 1 + mU + \sigma \eta U$$

$\underbrace{m}_{\frac{1}{2}}$ $\underbrace{\sigma \eta}_{\text{part of the noise (multiplicative)}}$

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial U} [(1+mU)P] + \frac{\sigma^2}{2} \frac{\partial^2}{\partial U^2} [U^2 P]$$

$$\frac{\partial P}{\partial t} = 0 \Rightarrow (1+mU)P = \frac{\sigma^2}{2} \frac{\partial}{\partial U} (U^2 P)$$

$\underbrace{m}_{-\hat{m}}$

$$\Rightarrow P_{eq} \propto \frac{e^{-\frac{\sigma^2}{2}U}}{U^{1+\mu}} \quad \mu = 1 + \frac{2\hat{m}}{\sigma^2}$$

Power law tail $U^{-1-\mu}$

w/ non-universal exponent

Now for matrices:

$$U_{n+1} = \varepsilon \sqrt{1 + \frac{U_n}{\varepsilon}} ((1+m\varepsilon)\mathbb{1} + \sqrt{\varepsilon} \sigma B) \sqrt{1 + \frac{U_n}{\varepsilon}}$$

$$\Rightarrow U_{n+1} - U_n = \varepsilon (1 + mU_n) + \sigma \sqrt{\varepsilon} B \sqrt{U_n}$$

$\underbrace{\varepsilon \dots}_{E=0} \leftarrow \sum_{j \neq i} \frac{\lambda_{i,n} \lambda_{j,n}}{\lambda_{i,n} - \lambda_{j,n}} (V_i^T B V_{j,n})^2$

$$\Rightarrow \frac{d\lambda_i}{dt} = 1 - \hat{m} \lambda_i + \frac{\sigma^2}{N} \sum_{j \neq i} \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} \leftarrow \text{as in Mult Dyson motion}$$

$$\Rightarrow \frac{\partial g}{\partial t} = \frac{\partial}{\partial z} \left[-g + (\sigma^2 + \hat{m}) zg - \frac{1}{2} \sigma^2 z^2 g^2 \right] \quad \checkmark$$

as before
↓

$$\Rightarrow (1 - z(\sigma^2 + \hat{m}))g + \frac{1}{2} \sigma^2 z^2 g^2 + C = 0$$

$$zg \rightarrow 1 \quad \text{or} \quad z \rightarrow \infty \Rightarrow C = \frac{1}{2} \sigma^2 + \hat{m}$$

$$\Rightarrow g = \frac{1}{\sigma^2 z^2} \left[(\sigma^2 + \hat{m})z - 1 - \sqrt{\hat{m}^2 z^2 - 2(\sigma^2 + \hat{m})z + 1} \right]$$

$$\lambda_{\pm} = \underbrace{\sigma^2 + \hat{m}}_{\hat{m}^2} \pm \sqrt{\sigma^2(\sigma^2 + 2\hat{m})}$$

$\hat{m} > 0 \Rightarrow$ unbounded spec

else truncated at $2\sigma^2/\hat{m}^2$

Universal exponent $\mu = 1/2$ at $m=0$ Unlike scalar case!

$$X = \frac{2}{(\sigma^2 + \hat{m})\lambda} \Rightarrow \text{Marčenko-Pastur} \quad q = \frac{\sigma^2}{\sigma^2 + 2\hat{m}} = \frac{1}{\mu} < 1$$

Chapter 17: Sample Covariance Matrices

Sample Cov: $E = \frac{1}{T} H H^T$ $H \in \mathbb{R}^{N \times T}$

$T \gg N \Rightarrow E \rightarrow C$ ← true Cov

C_{ij} are spatial correlations

The T samples yield temporal correlations

Care about λ_k of E and singular values s_k of H $s_k = \sqrt{T \lambda_k}$

17.1 Spatial Correlations

$$H \sim N(0, C)$$

$\Rightarrow E$ is Wishart with $\text{cov} = C$

$$E = \sqrt{C} W_q \sqrt{C}$$

$$\Rightarrow S_E = \frac{S_C(t)}{1-q+t}$$

Recall: $t_{AB} = t_A(z) S(t_{AB}(z))$

$$\Rightarrow t_E = t_c(z) \quad Z(z) = \frac{z}{1-q+t_E(z)}$$

$$zg_E(z) = Zg_c(z) \quad Z = \frac{z}{1-q+qzg_E(z)}$$

$$\Rightarrow g_E = \frac{\int g_c(\mu) d\mu}{z - \mu(1-q+qzg_E(z))}$$

True $\mathcal{W}H$ with finite 2nd moment \Rightarrow Universality

$$q \rightarrow 0 \Rightarrow g_E = g_c \quad f_E = f_c$$

Wishart

$$C = \mathbb{I} \Rightarrow g_c = \frac{1}{z-1} \Rightarrow zg_E = \frac{z}{z-1+q-qzg_E} \Rightarrow \frac{1}{f_E} = z + q - qzg_E$$

$$t_E \rightarrow \sum_{k=1}^{\infty} \tau(E^k) z^{-k} \Rightarrow \mathbb{Z}(z) = \frac{z}{1+q \sum_{k=1}^{\infty} \tau(E^k) z^{-k}}$$

$$\Rightarrow t_C(\mathbb{Z}(z)) \rightarrow \sum_{k=1}^{\infty} \frac{\tau(C^k)}{z^k} \left(1 + q \sum_{l=1}^{\infty} \frac{\tau(E^l)}{z^l} \right)^k = \sum_{k=1}^{\infty} \frac{\tau(E^k)}{z^k}$$

$$\tau(E) = \tau(C)$$

$$\tau(E^2) = \tau(C^2) + q$$

$$\tau(E^3) = \tau(C^3) + 3q\tau(C^2) + q^2$$

$$q < 0 \Rightarrow E \text{ invertible}$$

$$g_E(z) = \sum_{k=1}^{\infty} z^{k-1} \tau(E^{-k}) \quad z = \frac{z}{1-q+q \sum_k \tau(E^{-k}) z^k}$$

$$\Rightarrow \sum_{k=1}^{\infty} z^k \tau(E^{-k}) = \sum_k \tau(C^{-k}) \left(\frac{z}{1-q} \right)^k \left[1 - \frac{q}{1-q} \sum_{\ell=1}^{\infty} \tau(E^{-\ell}) z^\ell \right]^{-k}$$

$$\Rightarrow \tau(E^{-1}) = \frac{\tau(C^{-1})}{1-q} \quad \checkmark$$

related to general

$$\tau(E^{-2}) = \frac{\tau(C^{-2})}{(1-q)^2} + \frac{q \tau(C^{-1})^2}{(1-q)^3} \quad \checkmark$$

Ex 12.11 EMA-SCM

$$E(t) = \gamma_c \sum_{t'=-\infty}^t (1-\gamma_c)^{t-t'} x_t x_t^T$$

$$\Rightarrow E(t) = (1-\gamma_c)E(t-1) + \gamma_c x_t x_t^T \quad \leftarrow T=1 \text{ Wishart} \Rightarrow q=N$$

$$E \stackrel{\text{def}}{=} (1-\gamma_c)E + \gamma_c x x^T \quad \underbrace{\gamma_c}_{\sim (1-N\gamma_c x)^{-1}}$$

a) By R trans: $R_E[x] = (1-\gamma)R_E[(1-\gamma)x] + \gamma R_{w,p=N}[x]$

b) Let $\gamma_c = \lambda \tau_c$ $q := N\tau_c$ Fixed

$$\Rightarrow R_E[x] = \left(1 - \frac{q}{N}\right) R_E\left[\left(1 - \frac{q}{N}\right)x\right] + \frac{q}{N} (1-qx)^{-1}$$

$$R_E[x] = \cancel{R_E[x]} - \frac{q}{N} R_E[x] - \frac{qx}{N} R'_E(x) + \frac{q}{N} \frac{1}{1-qx}$$

$$\Rightarrow R_E[x] = x R'_E[x] + \frac{1}{1-qx}$$

c) $\tau(x_{xx^T}) = 1 \Rightarrow \tau(E) = 1 \Rightarrow R[0] = 1$

$$R_E[x] = - \frac{\log(1-qx)}{qx}$$

$$\Rightarrow = 1 + \frac{qx}{2} + \frac{(qx)^2}{3} + \dots$$

$$\Rightarrow x_2 = q^2/3$$

d) $\beta(g) = R[g] + \frac{1}{g} + \dots$

FINISH

17.2 Temporal Correlations

T samples are correlated $\Rightarrow T_{\text{eff}} < T$

Assume $C = \mathbb{I}$ (White spatial correlates)

$$\Rightarrow E[H_{it} H_{js}] = \delta_{ij} K_{ts}$$

$\underbrace{\quad}_{T \times T} \text{ cov matrix}$

assume $\tau(K) = 1$

$$\Rightarrow H = H_0 K^{1/2} \quad H_0 \text{ white}$$

$$\Rightarrow E := \frac{1}{T} H H^T = \frac{1}{T} H_0 K H_0^T \leftarrow \text{not quite a free prod of } K \text{ & a Wishart}$$

$$F := \frac{1}{N} H^T H = \frac{1}{N} K^{1/2} H_0^T H_0 K^{1/2} = K^{1/2} W_{q,1} K^{1/2} \leftarrow \text{free prod}$$

$$\Rightarrow S_F = \frac{S_K}{1 + t/q}$$

$$\text{Recall (Eq 1.5)} \quad g_F(z) = q^2 g_E(qz) \cdot \frac{1-q}{z}$$

$$\begin{aligned} \Rightarrow f_F(z) &= z q^2 g_E(qz) - q \\ &= q f_E(qz) \end{aligned}$$

$$\Rightarrow S_E = q S_F(qt) \quad \Rightarrow \quad \frac{t+1}{1 + qt} = \frac{t+1}{q + S_F(qt)}$$

$$\Rightarrow S_E = q^{-1} S_F(qt) \quad \Rightarrow \quad S_E = \frac{t+1}{1 - qt} S_F(qt)$$

$$\Rightarrow S_E(t) = \frac{S_K(qt)}{1 - qt}$$

$$\Rightarrow g_E = \frac{t+1}{t} \frac{1 - qt}{S_K(qt)} = q(t+1) S_K(qt)$$

$$t_K \left[\frac{z}{q(t_E+1)} \right] = q f_E(z) \quad f_E(z) = 0^0 - 1 - z \varepsilon(E^{-1}) + O(z^2) \quad \text{by defn}$$

$$\Rightarrow t_K \left[-\frac{1}{q \varepsilon(E^{-1})} \right] = -q \quad \Rightarrow \quad \varepsilon(E^{-1}) = -\frac{1}{q S_K(-q)}$$

17.2.2 Exponential Correlations

$$K_{ts} = a^{|t-s|}$$

$$\tau = \frac{1}{|\log a|} \approx \frac{1}{1-a}$$

[↑]
Toeplitz

$$K = \begin{pmatrix} 1 & a & a^2 & \dots & a^{T-1} \\ a & 1 & a & & \\ \vdots & & 1 & & \vdots \\ a^{T-1} & \dots & & 1 & \end{pmatrix}$$

Take $T \rightarrow \infty$

$$\sum_{t=-\infty}^{\infty} K_{ts} e^{2\pi i k s} = e^{2\pi i k t} \sum_{l=-\infty}^{\infty} a^{|kl|} e^{2\pi i k l}$$

$$\tau_c := \frac{1}{1-a}$$

First modify $K \rightarrow \tilde{K}$ so $|s-t|$ lies on a circle

$$\tilde{K}_{ts} = a^{-\min\{|t-s|, |t-s+T|, |t-s-T|\}}$$

[↑]
Only changes bottom right & top left change drastically
Circular

$$\Rightarrow [V_k]_l = e^{2\pi i k l / T} \quad 0 \leq k \leq T/2$$

Each v_k is really 2 evecs: real & im of v_k

$$\text{Except } v_0 = \vec{1}, \quad v_{T/2} = (+ - + - \cdots)$$

$$k=0 \Rightarrow \lambda_+ = \lambda_- = 1 + 2 \sum_{k=1}^{T/2-1} a^k + a^{T/2} \approx \frac{1-a}{1-a}$$

$$\lambda_{T/2} = \lambda_- = 1 + 2 \sum_{k=1}^{T/2-1} (-a)^k + (-a)^{T/2} \approx \frac{1-a}{1-a} = \frac{1}{\lambda_+}$$

$$\lambda_+ = 2\tau_c - 1$$

let $x_k = \frac{2k}{T} \in [0, 1]$ index the eigs

$$\lambda(x) = \frac{1-a^2}{1+a^2+2a \cos \pi x} \quad \leftarrow \text{SHOW}$$

For general $K_{ts} = K(|t-s|)$ $\lambda = 1 + 2 \sum_{l=1}^{\infty} K(l) \cos \pi xl$

$$\text{returning: } f_K(z) = \int_0^1 \frac{1-a^2}{z(1+a^2+2a \cos \pi x) - (1-a^2)} dx = \frac{1}{z - \lambda}$$

$$\text{Can show } t_k = \frac{1}{\sqrt{2-\lambda} \sqrt{2-\lambda_+}} \Rightarrow p(\lambda) = \frac{1}{\pi \lambda} \cdot \frac{1}{\sqrt{(\lambda_+-\lambda)(\lambda-\lambda_-)}}$$

$$\int_0^1 \frac{dx}{c-d \cos \pi x} = \frac{1}{\sqrt{c-d} \sqrt{c+d}}$$

$$\lambda_- < \lambda < \lambda_+$$

$$t^2 \zeta_k^2 - 2bt^2 \zeta_k + t^2 - 1 = 0 \quad b = \frac{1+\alpha^2}{1-\alpha^2}$$

$$\Rightarrow \zeta_k = \frac{bt^2 + \sqrt{(b^2-1)t^2 + t^2-1}}{t^2}$$

$$\Rightarrow S_k = \frac{t+1}{bt + \sqrt{(b^2-1)t^2 + 1}} \approx 1 - (b-1)t + O(t^2)$$

$$\Rightarrow \tau(K) = 1$$

$$\chi_2(K) = b-1 = \frac{2\alpha^2}{1-\alpha^2}$$

Using $S_E = \frac{S_k(qt)}{1+qt} = (bqt + \sqrt{(b^2-1)(qt)^2+1})^{-1}$

$$\Rightarrow \zeta_E = (bqt + \sqrt{(b^2-1)(qt)^2+1}) \frac{t+1}{t}$$

$\text{4th order in } t \text{ to invert } \therefore$

$$\tau(E^{-1}) = \frac{-(-q)}{qb(-q) + q\sqrt{1+(b^2-1)q^2}}$$

$$= \frac{1}{\sqrt{1+(b^2-1)q^2} - qb} =: \frac{1}{1-q^*}$$

$$q^* := \frac{N}{T^*}$$

\uparrow
effective length
of time series

$$a \rightarrow 0 \Rightarrow b=1 \Rightarrow q=q^*$$

$$q^* = q(1+2\alpha^2 + \dots)$$

$$a \rightarrow 1 \Rightarrow b=\infty \Rightarrow \frac{1}{b} \xrightarrow{b \rightarrow \infty} \infty \Rightarrow q=1$$

for a small

$b \gg 1 \Rightarrow a \rightarrow 1$, long-range correlations

$$S_E \approx (2bqt)^{-1}$$

S_E depends only on bq jointly

Define $qb = \sigma^2$

Take $b \rightarrow \infty$, $q \rightarrow 0$, $qb = \sigma^2$ fixed

$$\Rightarrow S(t) = \frac{1}{\sqrt{1+\sigma^2 t^2 + qt}}$$

$$\Rightarrow R(z) = \frac{1}{\sqrt{1-\sigma^2 z}} = 1 - \sigma^2 z + \frac{3}{2} \sigma^4 z^2 + O(z^3)$$

$$\Rightarrow x_2 = \sigma^2$$

$$x_3 = \frac{3}{2} \sigma^4 \leftarrow \text{for Wishart } x_3 = \sigma^4$$

Unlike MP, no support or Dirac mass @ 0 even for $\sigma^2 > 1$

λ_{\pm} is y th order in z here

Intuitive understanding:

N Ornstein-Uhlenbeck process $w \tau_c$

recorded over $T \gg \tau_c$

* observations = T/Δ Δ = window size

$\Rightarrow N \times N$ sample cov

$\Delta \gg \tau_c \Rightarrow$ samples uncorrelated

\Rightarrow MP with $q = N\Delta/T$

$\Delta \ll \tau_c \Rightarrow$ samples correlated

E depends not on Δ but on τ_c

$$\Rightarrow \sigma^2 = qb = N\tau_c/T$$

} Not MP

17.2.3 Spatial & Temporal correlations

$$E = \frac{1}{T} HH^T = \frac{1}{T} C^{1/2} H K H^T C^{1/2}$$

$$\text{Get: } S_E(t) = \frac{S_C(t) S_K(qt)}{1+qt}$$

$$\Rightarrow S_E = \frac{1+q}{1+S_C(t)} \frac{1+q}{S_K(qt)}$$

$$= q S_C(t) S_K(qt)$$

$$\Rightarrow q t_E(z) = t_K \left(\frac{z}{q t_E(z) S_C(t_E(z))} \right) *$$

$$C=1 \Rightarrow S_C = \frac{1+t}{t} \Rightarrow q t_E = t_K \left(\frac{z^2}{q(1+q t_E(z))} \right) \text{ as before}$$

Now specialize to $K_t = q^{1s+1}$

$$t_k(z) = \frac{1}{\sqrt{z^2 - 2zb + 1}}$$

$$\Rightarrow \frac{1}{q^2 t_E^2} = \frac{z^2}{q^2 t_E^2 \xi_c(t_E)^2} - \frac{2zb}{q t_E \xi_c(t_E)} + 1$$

$$\Rightarrow \xi_c(t_E)^2 = z^2 - 2zbq + \frac{q^2 t_E^2 \xi_c(t_E)^2}{z^2} \xrightarrow{q \rightarrow 0} 0$$

$$C = \mathbb{I} \Rightarrow \left(\frac{1-t}{t}\right)^2 = z^2 - 2zb^2 \left(\frac{1-t}{t}\right) \leftarrow \text{Same Eq for } t \text{ as in previous subsection}$$

$$C = W^{-1} \Rightarrow \xi_c(t) = \frac{t+1}{t(1-pt)} \Rightarrow \text{y'th order eq for } t$$

$$z \rightarrow 0 \Rightarrow \text{by * } -q = t_k \left(\frac{z}{q(-1)\xi_c(-1+q_E(0))} \right) \Rightarrow \xi_k(-q) = \frac{\bar{c}(C^{-1})}{-q \tau(E^{-1})}$$

$$t_E(0) = -1$$

$$1 + t_E(z) = zg_E(0) = z\tau(E^{-1}) \quad \left(1 + \frac{1}{t}\right) \frac{1}{z}$$

$$\xi(-1 + zg_E(0))$$

$$\Rightarrow \tau(E^{-1}) = -\frac{\bar{c}(C^{-1})}{q\xi_k(-q)}$$

$$t_c(z) = -1 + zg_c(z) \approx -1 + z\tau(C^{-1})$$

$$\Rightarrow \xi(-1+\epsilon) = \frac{\epsilon}{\tau(C^{-1})}$$

For matrix w/ purely spatial correlations we had

$$\tau(E^{-1}) = \frac{\bar{c}(C^{-1})}{1-q}$$

$$\Rightarrow q^* = 1 + q\xi_k(-q)$$

Ex 17.2.1 Futility of oversampling

T indep obs, true corr = C

Repeat them m times \rightarrow mt columns

a) Temporal correlation = $\begin{pmatrix} \cdots & | & 0 \\ \cdots & | & \ddots \\ 0 & | & \ddots \end{pmatrix} \Bigg) T \text{ blocks} \Rightarrow T \text{ eigs} = m$
 m columns
 $(T - Dm \text{ eigs} = 0)$

$$b) t_K = \frac{1}{Tm} \left(\frac{Tm}{z-m} + \frac{(T-1)0}{z} \right) = \frac{1}{z-m}$$

$$c) \bar{s} = \frac{1-mt}{t} \Rightarrow s_k = \frac{1+t}{1-mt}$$

$$d) q_m = N/m \quad \text{naively but}$$

$$S_E = S_C(t) \frac{S_K(q_m t)}{1+q_m t} = \frac{S_C(t)}{1+q_m m t} = \frac{S_C(t)}{1+q t} \quad m \text{ cancels!}$$

17.3 Time-dependent variance

N time series are heteroscedastic

$$x_i^t = \sigma_t H_{it} \quad E[H_{it} H_{js}] = S_{ts} C_{ij}$$

↑
time-dep

$$\Rightarrow E = \sum_{t=1}^T P_t, \quad P_t := \frac{1}{T} \sigma_t^2 H_t H_t^T$$

$$\Rightarrow R_E(g) = \sum_t R_t(g) \quad \text{← need } N \text{ corrections since } T \text{ terms!}$$

$$g_t = \frac{1}{N} \left[\frac{N-1}{z} + \frac{1}{z - q\sigma_t^2} \right] = \frac{1}{z} + \frac{1}{N} \frac{q\sigma_t^2}{z(z - q\sigma_t^2)}$$

$$\Rightarrow \beta_t(g) = \frac{1}{g} + \frac{1}{N} \frac{q\sigma_t^2}{1 - q\sigma_t^2} \quad \frac{g}{N} = \frac{1}{T}$$

$$\Rightarrow R_E = \frac{1}{T} \sum_t \frac{\sigma_t^2}{1 - q\sigma_t^2 g} \quad \text{encode } s = \sigma^2$$

$$\rightarrow \int_0^\infty \frac{s P(s) ds}{1 - qs g}$$

$$P(s) = \delta(s-1) \Rightarrow \frac{1}{1 - qg} = R_{\text{without}}$$

$$\text{Generally } R_E = t_s \left(\frac{1}{qg} \right)$$

$$\text{When } C \neq \mathbb{I} \quad \text{can write } S_E = S_C(t) S_K(t)$$

View σ_t as diagonal temporal cov w/ entries drawn from $P(s)$

$$\Rightarrow S_E = \frac{S_s(qt)}{1+qt} \Rightarrow \tilde{S}_E = \frac{S_e(t) S_s(qt)}{1+qt}$$

When $P(s)$ is inv-gamma $E[x^t]$ is Gaussian $\Rightarrow \sigma_t x^t$ is a multivariate Student t

12.4 Empirical Cross-Covariance

$$x^t \in \mathbb{R}^{N_1} \quad y^t \in \mathbb{R}^{N_2} \quad E_{xy} = \frac{1}{T} \sum_{t=1}^T x^t y^t \in \mathbb{R}^{N_1 \times N_2}$$

Assume "true" $E[x y^T] = 0$

What is the singular value spec of E_{xy}

$q_1 = N_1/T$ $q_2 = N_2/T$, take asymptotic limit

Consider $E_{xy} E_{xy}^T \in N_1 \times N_1$

$$\text{Def } \hat{E}_x = x^T x \in \mathbb{R}^{T \times T}$$

$$\hat{E}_y = y^T y \in \mathbb{R}^{T \times T}$$

E_{xy} has same nonzero λ as $\hat{E}_x \hat{E}_y$

Consider first "sample-normalized" PCs

$$x \rightarrow \hat{x} \Rightarrow \hat{E}_{\hat{x}} \text{ has } N_1 \text{ eigs} = 1, T - N_1 = 0$$

$$y \rightarrow \hat{y} \Rightarrow \hat{E}_{\hat{y}} \text{ has } N_2 \text{ eigs} = 1, T - N_2 = 0$$

$$\Rightarrow \text{Singular spectrum is } p(s) = \max(q_1, q_2 - 1, 0) \delta(s-1) + \text{Re} \frac{\sqrt{(s^2 - \gamma_+)(\gamma_+ - s^2)}}{\pi s(1-s^2)}$$

$$\gamma_{\pm} = q_1 \pm q_2 - 2q_1 q_2 \pm 2\sqrt{q_1 q_2 (1-q_1)(1-q_2)}$$

$$0 \leq \gamma_{\pm} \leq 1$$

$\Rightarrow s$'s are linear combinations of x s correlated into linear combinations of y s

$$T \rightarrow \infty \text{ at fixed } N_1, N_2 \Rightarrow \text{all } s \in [\sqrt{q_1} - \sqrt{q_2}, \sqrt{q_1} + \sqrt{q_2}] \Rightarrow s \sim T^{-1/2} \Rightarrow 0$$

Gives rise to statistical tests for cross-correlations

Chapter 18: Bayesian Estimation

18.1.2 A Simple Estimation Problem

$$y = x + \epsilon$$

$$P(y|x) = P_{\epsilon}(y-x)$$

Assume ϵ Gaussian $\Rightarrow P(y|x) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left[-\frac{(y-x)^2}{2\sigma_n^2}\right]$

$$\Rightarrow P(x|y) \propto P_0(x) \exp\left[-\frac{x^2}{2\sigma_n^2} + \frac{2xy}{2\sigma_n^2}\right]$$

P_0 is Gaussian $N(x_0, \sigma_s^2) \Rightarrow P(x|y) = N(\hat{x}, \sigma^2)$

$$\hat{x} = x_0 + r(y - x_0) = (1-r)x_0 + ry$$

$$\sigma^2 = (\sigma_n^{-2} + \sigma_s^{-2})^{-1} = r\sigma_n^{-2} \quad r = \frac{\sigma_s^2}{\sigma_n^2 + \sigma_s^2} \quad \text{"SNR" (not really)}$$

Linear Shrinkage Estimator

$$\begin{aligned} \text{Var } \hat{x} &= r^2 \text{Var } y = r^2 [\text{Var } y|x + \text{Var } x] \\ &= r^2 \left(\frac{\sigma_s^4}{\sigma_n^2 + \sigma_s^2} \right) = \frac{\sigma_s^4}{\sigma_n^2 + \sigma_s^2} \leq \sigma_s^2 \leq \text{Var } x \end{aligned}$$

\Rightarrow estimator's variance is smaller than the truth!

Ex 18.1.1 Optimal Affine Estimator

$$\hat{x} = ay + b$$

min over a, b

$$\begin{aligned} a) \text{ If } E[x] = E[y] = 0 \Rightarrow E[(x-\hat{x})^2] &= E[x^2] + E[\hat{x}^2] - 2E[x\hat{x}] \\ &= \sigma_x^2 + a^2\sigma_y^2 + b^2 - 2ab\sigma_{xy}^2 \end{aligned}$$

$$b) \partial_a = 0 \Rightarrow a = \sigma_{xy}^2 / \sigma_y^2 \quad \partial_b = 0 \Rightarrow b = 0$$

$$c) \text{ For } (x - E[x]) = a(y - E[y]) + b, \quad b = 0$$

$$\Rightarrow b = E[x] - aE[y]$$

$$d) \quad y = x + \varepsilon \quad \varepsilon \perp x$$

$$\sigma_y^2 = E[\varepsilon^2] + \sigma_x^2$$

$$\sigma_{xy}^2 = E[yx] = E[x(x+\varepsilon)] = \sigma_x^2$$

$$e) \quad a = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2} =: r \Rightarrow \hat{x} = ry \quad \text{when } E[\varepsilon] = 0$$

$$\text{For } b \neq 0 \quad \& \quad E[\varepsilon] = 0 \quad \Rightarrow \quad \hat{x} = x_0 + r(y - x_0) \quad \leftarrow \quad 18.10$$

$$\Rightarrow E[x] = E[y] =: x_0$$

Bernoulli

$$\text{If } P_0(x) = \frac{1}{2} [\delta(x-1) + \delta(x+1)]$$

$$P(y|x) \propto \exp\left[-\frac{(y-x)^2}{2\sigma_n^2}\right] \Rightarrow P(x|y) \propto \exp\left[\frac{(x-y)^2}{2\sigma_n^2}\right] \delta_{x,1} + \exp\left[\frac{(x+y)^2}{2\sigma_n^2}\right] \delta_{x,-1}$$

$$= \underbrace{e^{\frac{y-x}{\sigma_n^2}} \delta_{x,1} + e^{-\frac{y+x}{\sigma_n^2}} \delta_{x,-1}}_{e^{\frac{y-x}{\sigma_n^2}} + e^{-\frac{y+x}{\sigma_n^2}}}$$

$$= \sigma(\frac{y-x}{\sigma_n^2}) \delta_{x,1} + \sigma(-\frac{y+x}{\sigma_n^2}) \delta_{x,-1}$$

$$\Rightarrow \hat{x}_{MAP} = \text{sgn}(y)$$

$$\hat{x}_{MMSE} = \tanh\left(\frac{y}{\sigma_n^2}\right) \quad \leftarrow \text{Eq. 15}$$

$$\text{Var}(\hat{x}_{MMSE}) <$$

$$\text{Var}(x) = 1$$

Laplace

$$P_0(x) = \frac{b}{2} e^{-b|x|} \quad \text{Var} = \frac{2}{b^2}$$

$$P(x|y) = P_0(x) P(y|x)$$

$$\propto \exp\left[\frac{2xy - x^2}{2\sigma_n^2} - b|x|\right]$$

MMSE & MAVE estimators have [↑] posterior median *closed-form*

MAP is just $\hat{x} = \begin{cases} 0 & |y| < b\sigma_n^2 \\ y - b\sigma_n^2 \text{sgn}(y) & \text{else} \end{cases}$

LASSO

Non-Gaussian Noise

If x is a centered Gaussian & ϵ is Cauchy noise

$$P(x|y) \propto \frac{e^{-x^2/2}}{(y-x)^2 + 1}$$

Prior ϵ, y don't have first moment!

$$E[x|y] = y + \frac{\operatorname{Im} \Phi}{\operatorname{Re} \Phi} \quad \Phi = e^{iy} \operatorname{erf} \frac{1+iy}{\sqrt{2}}$$

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Bayesian Estimation

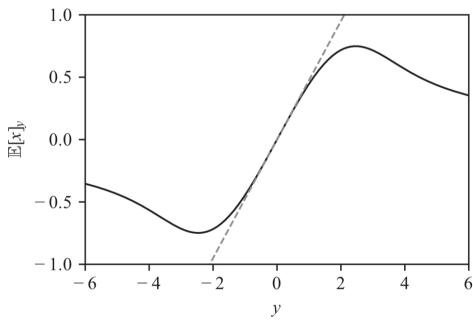


Figure 18.1 A non-monotonic optimal estimator. The MMSE estimator of a Gaussian variable corrupted by Cauchy noise (see Eq. (18.21)). For small absolute observations y , the estimator is almost linear with slope $2 - \sqrt{2/\pi} \operatorname{erfc}(1/\sqrt{2}) \approx 0.475$ (dashed line).

For small y , assume moderate $x, \sigma \Rightarrow$ locally linear $r \approx 1/2$

For huge y , must be all noise \Rightarrow regress to 0

18.1.3 Conjugate Priors

$$P(c|y) \propto P_0(c) c^{-T/2} \exp\left(-\frac{y^T y}{2c}\right)$$

$$\Rightarrow P_0(c) \propto c^{-a-1} \exp(-b/c)$$

$$\Rightarrow a_p = a + T/2$$

$$b_p = b + \frac{y^T y}{2}$$

$$\Rightarrow \text{MMSE} = E(c|y) = \frac{b_p}{a_p - 1} = \frac{2b + y^T y}{2(a-1) + T}$$

$$= (1-r)c_0 + r \frac{y^T y}{T} \quad r = \frac{T}{2(a-1) + T}$$

$$T \rightarrow \infty \Rightarrow r \rightarrow 1$$

Ex 18.1.2

$$y_i \sim \text{Lap}(0, b)$$

$$a) P(y|b) = \left(\frac{b}{2}\right)^T \exp(-b \sum |y|)$$

$$b) P(b) = \text{Gamma}(a_0, b_0) \sim b^{a_0-1} e^{-b_0 b}$$

$$P(b|y) \propto b^T b^{a_0-1} e^{-b(b_0 + \sum |y|)}$$

$$a_p = a_0 + T$$

$$b_p = b_0 + \sum |y|$$

$$c) \text{MMSE} = \frac{a_0 + T}{b_0 + \sum |y|}$$

$$d) T \rightarrow 0 \Rightarrow a_p/b_p \quad T \rightarrow \infty \Rightarrow m^{-1}, \quad m = \frac{1}{T} \sum |y|$$

$$e) \text{MMSE} = \frac{a_0 + T}{b_0 + Tm} = \frac{a_0}{b_0} (1-r) + m^{-1} r$$

18.2 Ridge & LASSO

$$y = \sum_i a_i x_i + \epsilon$$

$$\Rightarrow y = H^T a + \epsilon \quad a \in \mathbb{R}^N$$

$$H \in \mathbb{R}^{N \times T}$$

$$E[\epsilon \epsilon^T] = \sigma_n^2 \mathbb{I}$$

$$\frac{1}{T} E[H H^T] = C$$

can be arbitrary
will assume $= \mathbb{I}$
(in practice how could you
be sure?)

Centring won't work if $T \ll N$

$$a_{\text{reg}} = (H H^T)^{-1} H y$$

When $\eta = N/T < 1$, $H H^T$ is invertible

$$P(a|y) \propto P_a(a) \exp \left[-\frac{1}{2\sigma_n^2} \|y - H^T a\|^2 \right]$$

Ridge: $P_a \sim N(0, \sigma_s^2)$

$$\Rightarrow P(a|y) \propto \exp \left[-\frac{1}{2\sigma_n^2} \left[a^T \left(H H^T + \frac{\sigma_n^2}{\sigma_s^2} \mathbf{I} \right) a - 2a^T H y \right] \right]$$

$$E[a] = \left(\frac{H H^T}{T} + \frac{\sigma_n^2}{\sigma_s^2} \mathbf{I} \right)^{-1} \frac{H y}{T}$$

$$\xi = \frac{\sigma_n^2}{T \sigma_s^2}$$

$$a_{\text{ridge}} = \underset{a}{\operatorname{argmin}} \|y - H^T a\|^2 + T \xi \|a\|^2$$

Set ξ by C.V.

LASSO

$$P(a|y) \propto \exp \left[-b \sum |a_i| - \frac{1}{2\sigma_n^2} \|y - H^T a\|^2 \right]$$

$$a_{\text{LASSO}} = \underset{a}{\operatorname{argmin}} \underset{\uparrow}{2b\sigma_n^2 |a_i|} + \|y - H^T a\|^2 \Rightarrow a_{\text{LASSO}} \text{ is sparse}$$

pick by CV

18.2.3 In-Sample ε Out-of-sample error

$$a_{\text{reg}} = E^{-1} b \quad E = \frac{1}{T} H_i H_i^T \quad b = \frac{1}{T} H_i^T y_i$$

best in-sample estimator

$$\text{Generally: } \hat{a} = \tilde{E}^{-1} b \quad \tilde{E} = E \quad \text{or} \quad \tilde{E} = E + \xi \mathbf{I}$$

R_{in}^2 :

$$\begin{aligned} R_{in}^2 &= \frac{1}{T} E \frac{1}{\varepsilon} \|y - H_i^T \tilde{E}^{-1} H_i y\|^2 \\ &= \frac{1}{T} E \frac{1}{\varepsilon} \|H_i^T a + \varepsilon - H_i^T \tilde{E}^{-1} H_i (H_i^T a + \varepsilon)\|^2 \end{aligned}$$

$$= \frac{1}{T} \left[\sigma_n^2 (T - 2 \operatorname{Tr} \Sigma^{-1} E + \operatorname{Tr} \Sigma^{-1} E \Sigma^{-1} E) \right. \\ \left. + a^T (E - 2 E \Sigma^{-1} E + E \Sigma^{-1} E \Sigma^{-1} E) a \right]$$

When $\Sigma = E \in \mathbb{R}^{N \times N}$

$$= \frac{1}{T} \left[\sigma_n^2 (T - N) + 0 \right] = \sigma_n^2 (1-q) \neq \text{No RMT needed!}$$

R^2_{out} :

$$R^2_{\text{out}} = \frac{1}{T_2} \mathbb{E}_{H_2, \varepsilon_2} \|H_2^T a + \varepsilon_2 - H_2^T \bar{a}\|^2$$

$$= \frac{1}{T_2} \mathbb{E}_{H_2, \varepsilon_2} \|H_2^T a + \varepsilon_2 - H_2^T \Sigma^{-1} E a - H_2^T \Sigma^{-1} \frac{H_2 \varepsilon_1}{T}\|^2$$

$$E = T^{-1} H_1 H_1^T$$

$$\mathbb{E}_{H_2} T_2^{-1} H_2 H_2^T = C$$

$$\text{For } \Sigma_1 = E_1$$

$$\Rightarrow R^2_{\text{out}} = \frac{1}{T_2} \mathbb{E}_{H_2, \varepsilon_2} \left\| \varepsilon_2 - H_2^T \Sigma^{-1} \frac{H_1 \varepsilon_1}{T} \right\|^2 = \sigma_n^2 + \frac{1}{T_2} \mathbb{E}_{\substack{\varepsilon_1 \\ H_2}} \operatorname{Tr} \left[H_2 H_2^T E^{-1} \frac{H_1 \varepsilon_1 \varepsilon_1^T H_1^T E^{-1}}{T^2} \right] \\ = \sigma_n^2 + \frac{\sigma_n^2}{T} \operatorname{Tr}[CE^{-1}]$$

$$\operatorname{Tr}[CE^{-1}] = \operatorname{Tr}[C C^{-1/2} W_q^{-1} C^{-1/2}] = \operatorname{Tr}[W^{-1}]$$

in Wishart

$$S_{W_q} = \frac{1}{T-q+1} \Rightarrow S_{W_q^{-1}} = 1 - q - q + 1 \Rightarrow \operatorname{Tr}[W^{-1}] = \frac{1}{S_{W_q^{-1}}(0)} = \frac{1}{1-q}$$

$$\Rightarrow R^2_{\text{out}} = \sigma_n^2 \left(1 + \frac{N}{T} \frac{1}{1-q} \right) = \frac{\sigma_n^2}{1-q} = \frac{R^2_m}{(1-q)^3}$$

$$R^2_{\text{in}} < \sigma_n^2 < R^2_{\text{out}}$$

For $\Sigma = E_1 + \xi I$

$$\begin{aligned} R_{\text{out}}^2 &= \frac{1}{T_2} \mathbb{E}_{H_2, \varepsilon_2} \left\| H_2^T a + \varepsilon_2 - H_2^T \Sigma^{-1} E_1 a - H_2^T \Sigma^{-1} \frac{H_2 \varepsilon_1}{T} \right\|^2 \\ &= \sigma_n^2 + E \left\| H_2^T \Sigma^{-1} (\Sigma - E_1) a \right\|^2 + \frac{\sigma_n^2}{T} \text{Tr}(\Sigma^{-1} C \Sigma^{-1} E) \\ &= \sigma_n^2 + \xi^2 \text{Tr} C \Sigma^{-1} a a^T \Sigma^{-1} + \frac{\sigma_n^2}{T} \text{Tr}(\Sigma^{-1} C \Sigma^{-1} E) \end{aligned}$$

At $O(\xi)$

$$\Sigma^{-1} = E^{-1} - \xi E^{-1}$$

$$R_{\text{out}}^2(a_{\text{ridge}}) = R_{\text{out}}^2(a_{\text{reg}}) - \frac{2\sigma_n^2}{T} \text{Tr}(CE^{-2}) \xi$$

$\frac{N}{T} \cdot \frac{1}{N} \text{Tr} C \cdot \frac{1}{N} \text{Tr} E^{-2}$

$$\text{Tr } W_q^{-2} = (1-q)^{-3}$$

$$\Rightarrow R_{\text{ridge}}^2 - R_{\text{reg}}^2 = - \frac{2\sigma_n^2 q}{(1-q)^3} \mathcal{O}(C^{-1}) \xi + O(\xi^2)$$

For ξ large this reverses

Note for $C = I$

$$\Sigma = \xi I + E$$

$$\begin{aligned} R_{\text{ridge}}^2 &= \sigma_n^2 + \frac{\sigma_n^2}{T} \text{Tr}(\Sigma^{-1} C \Sigma^{-1} E) + \xi^2 \text{Tr} C \Sigma^{-1} a a^T \Sigma^{-1} \\ &= \sigma_n^2 + \frac{\sigma_n^2}{T} \text{Tr}(\Sigma^{-1}) - \xi \frac{\sigma_n^2}{T} \text{Tr}(\Sigma^{-2}) + \xi^2 \frac{\|a\|^2}{N} \text{Tr}(\Sigma^{-2}) \\ &= \sigma_n^2 (1 - q g_{W_q}(-\xi)) + \xi (q \sigma_n^2 - \xi \|a\|^2) g'_{W_q}(-\xi) \end{aligned}$$

$$\xi = 0 \Rightarrow g_{\vartheta}(W_q) = -\frac{1}{1-q}$$

$\Rightarrow \xi_{\text{opt}}$ depends only on $\|a\|^2$

$$\begin{aligned} \partial_{\xi} R_{\text{ridge}}^2 &= q \sigma_n^2 g'_{W_q} + q \sigma_n^2 g'_{W_q} - 2\xi \|a\|^2 g'(\xi) + \xi (q \sigma_n^2 - \xi \|a\|^2) g'' \\ &\Rightarrow 0 = 2(q \sigma_n^2 - \|a\|^2 \xi) [g'(\xi) + \frac{\xi}{2} g''(\xi)] \\ &\Rightarrow \xi = \frac{q \sigma_n^2}{\|a\|^2} \quad \|a\|^2 = \sigma_s^2 \end{aligned}$$

$$\Rightarrow R_{\text{ridge}}^2 = \sigma_n^{-2} \left[1 - q g_{\text{reg}} \left(-\frac{q \sigma_n^{-2}}{\text{Tr} C^2} \right) \right]$$

- $g(-\xi)$ is monotonically decreasing \Rightarrow optimal ridge is better than standard regression

18.3 Bayesian Estimation of C

Wishart Law (Ch 4) $P(E|C) \propto (\det C)^{-T/2} \exp \left[-\frac{I}{2} \text{Tr } C^{-1} E \right]$

Conj Prior: $P_0(C) = (\det C)^a \exp[-b \text{Tr } C^{-1} X]$

Inverse Wishart $a = \frac{T^* + N + 1}{2}$ $b = \frac{T^* - N - 1}{2}$

$\Rightarrow E_C = X$

$$\Rightarrow P(C|E) = (\det C)^{-\frac{T+T^*+N+1}{2}} \exp \left[-\frac{I}{2} \text{Tr}[C^{-1} E^*] \right]$$

$$E^* = E + \frac{T^* - N - 1}{T} X$$

$$\Rightarrow E[C|E] = \frac{TE^*}{T+T^*-N-1} = rE + (1-r)X$$

$$r = \frac{T}{T+T^*-N-1}$$

Chapter 19 Rotationally Invariant Estimators

19.1 Eigenvector Overlaps

$$(v_i^T u_j)^2 \quad \text{Note} \quad \sum_i (v_i^T u_j)^2 = u_j \cdot \mathbb{1} \cdot u_j = 1$$

same with \sum_j \Rightarrow "bi-stochastic"

v even \mathcal{E}

$$\Rightarrow G_E = \sum_i \frac{v_i v_i^T}{z - \lambda_i} \Rightarrow u^T G_E u = \sum_i \frac{(u \cdot v_i)^2}{z - \lambda_i}$$

v_i continue to fluctuate as $N \rightarrow \infty$ unlike $p(\lambda)$

$$v_i \cdot u_j \sim O\left(\frac{\sigma_{ij}}{\sqrt{N}}\right) \Rightarrow N(v_i \cdot u_j)^2 \sim O(\sigma_{ij})$$

$$\Phi(\lambda_i, \mu_j) := N \mathbb{E}(v_i \cdot u_j)^2$$

\mathbb{E} is either over realizations of E

or otherwise avg over interval of width $d\lambda = \eta$
 \uparrow
 \uparrow $N^1 \ll \eta \ll 1$ eq $N^{-1/2}$

These are the same by self-averaging property

$$\text{Im } u_j^T G(\lambda_i - i\eta) u_j = \pi p_E(\lambda_i) \Phi(\lambda_i, \mu_j)$$

For E, E' (with same population cov C)

$$\Psi(\lambda_i, \lambda'_j) := N \mathbb{E}[v_i \cdot v'_j]^2$$

$$\text{Consider } \Psi := \frac{1}{N} \text{Tr } G_E(z) G_{E'}(z') = \frac{1}{N^2} \sum_{i,j} \frac{N(v_i^T v'_j)^2}{(z - \lambda_i)(z' - \lambda'_j)}$$

$$\Rightarrow \text{Re}[\Psi(\lambda_i - i\eta, \lambda'_j + i\eta) - \Psi(\lambda_i - i\eta, \lambda'_j - i\eta)] = 2\pi^2 p_E(\lambda_i) p_{E'}(\lambda'_j) \Psi(\lambda_i, \lambda'_j) *$$

$$[i\pi p(\lambda_i) \cdot (-i\pi p(\lambda'_j)) - i\pi p(\lambda_i) i\pi p(\lambda'_j)] \Psi(\lambda_i, \lambda'_j)$$

19.1.2 Overlaps in the Additive Case

From replica calcn:

$$\mathbb{E} G_E(z) = G_C(z - R_X(g_E(z)))$$

$$u_j^T G_E(\lambda_i - i\eta) u_j = \frac{1}{\lambda_i - i\eta - R_X(g_E(\lambda_i - i\eta)) - \mu_j}$$

↑
evec of C

as $\eta \rightarrow 0$ Im gives $\Phi(\lambda_i, \mu_j)$

For $R_X(z) = \sigma^2 z$:

$$u_j^T G_E(\lambda_i - i\eta) u_j = \frac{1}{\lambda_i - i\eta - \sigma^2 g_E(\lambda_i - i\eta) - \mu_j} = \frac{1}{\lambda_i - \mu_j - \sigma^2 h_E - i\pi\sigma^2 p_E - i\eta}$$

$g_E := h_E + i\pi p_E$

$$\Rightarrow \Phi(\mu, \lambda) = \frac{\sigma^2}{(\mu - \lambda + \sigma^2 h_E(\lambda))^2 + \pi^2 \sigma^4 p_E^2}$$

\Rightarrow Overlap peaks for $\mu = \lambda - \sigma^2 h_E(\lambda)$

$\sigma^2 \rightarrow 0 \Rightarrow \Phi(\mu, \lambda) = \delta(\mu - \lambda)$ if evecs are equal

\Rightarrow Overlaps of $u_i \cdot v_j$ are $\sim N^{-1/2}$ for $\sigma > 0$

Let $E = C + X \quad E' = C + X' \quad X \perp X'$

$$* \Psi(z, z') = \frac{1}{N} \text{Tr } G_E(z) G_{E'}(z') \quad EG_X = \mathbf{1} g_X$$

$$G_E(z) = (z - E)^{-1} = (z - \sigma^2 g_E - C)^{-1}$$

$$\Rightarrow \Psi = \text{Tr} [(z - E)^{-1} (z' - E')^{-1}]$$

1603.04364

$$= \text{Tr} [(\xi - C)^{-1} (\bar{\xi} - C)^{-1}]$$

Key step

$$= \frac{1}{\xi' - \xi} \text{Tr} [(\xi - C)^{-1} - (\bar{\xi} - C)^{-1}]$$

$$= - \frac{g_E(z) - g_E(z')}{\xi(z) - \xi(z')} \quad \text{for } \Psi(\xi, \bar{\xi})$$

$$\Psi(\lambda - i\eta, \lambda + i\eta) - \Psi(\lambda - i\eta, \lambda - i\eta)$$

$$= \frac{g(\xi' - \bar{\xi}') + \xi(\bar{g}' - g') + g\bar{\xi}' - \bar{g}\xi'}{(\xi - \xi')(\bar{\xi} - \bar{\xi}')} \rightarrow \frac{2\text{Re}[\sigma^2(\lambda - \lambda')(\xi - \bar{\xi}')]}{(\xi - \bar{\xi}')^2} = \frac{2\sigma^2}{\partial_\lambda \xi}$$

$$\Rightarrow \Psi(\lambda, \lambda) = \frac{1}{2\pi^2 p_E(\lambda)^2} \operatorname{Re} \left[\frac{1}{\lambda(\lambda - \sigma^2 g_E)} \right]$$

19.1.3 Overlaps in Multiplicative case

$$E = C^{1/2} W_q C$$

$$G_E = \frac{Z(z)}{z} G_C(Z(z)) \quad Z = \frac{z}{1 - q + q z g_E(z)}$$

$$u_j^T G_E(\lambda_i - i\eta) u_j = \frac{Z(\lambda_i - i\eta)}{\lambda_i - i\eta} \frac{1}{Z(\lambda_i - i\eta) - \mu_j}$$

$$= \frac{1}{\mu(q-1) - g_E q \mu + \lambda - i\eta} = \frac{q \mu \lambda \pi p_E}{(\mu(1-q) - \lambda + q \mu \lambda h_E)^2 + q^2 \mu^2 \lambda^2 \pi^2 p_E^2}$$

$q \rightarrow 0 \Rightarrow$ sharp peak near $\lambda = \mu$

$$\text{Now } E = C^{1/2} W_q C^{1/2} \quad E' = C^{1/2} W_{q'} C^{1/2}$$

→ Annoying computation 1603.04364

For $C \rightarrow \mathbb{I}$, $\tau(C^2) = 1 + \epsilon$

$$\Rightarrow \Psi(\lambda, \lambda') = 1 + \epsilon [2h_E(\lambda) - 1][2h_E(\lambda') - 1] + O(\epsilon^2)$$

Overlaps again depend only on $g_E \Rightarrow$ hypothesis that E, E' come from same C only needs λ_E, λ'_E

19.2 Rotationally Invariant Estimators

Our priors are rotationally invariant

$$P_o(C) = P_o(O C O^T)$$

$$P(C|E) \propto \det C^{-1/2} \exp \left[-\frac{1}{2} \operatorname{Tr} C^T E \right] P_o(C)$$

$$\mathbb{E}(C|OEO^T) = O \mathbb{E}(C|E) O^T$$

More generally $\Sigma(C)$ is a RIE if

$$\Sigma(OEO^T) = O \Sigma(E) O^T$$

$\Sigma(E)$ can be diagonalized in the same basis as E
up to a fixed rotation Ω

There is no natural guess for Ω except $\Omega = \mathbb{I}$

$$\Rightarrow \Sigma(E) = \sum_{i=1}^N \xi_i v_i v_i^\top \quad \begin{matrix} \text{evecs of } E \\ \text{function} \\ \text{of empirical } \lambda_i \end{matrix}$$

Goal is to choose ξ_i optimally
so that $\Sigma(E)$ is as close to C as possible

If $\vec{v}_E = \vec{v}_C$ then $\xi_i = \mu_i$

Generally want to minimize:

$$\begin{aligned} \text{Tr}[(\Sigma(E) - C)^2] &= \sum_i v_i^\top (\Sigma(E) - C)^2 v_i \\ &= \sum_i \xi_i^2 - 2\xi_i v_i^\top C v_i - v_i^\top C^2 v_i \end{aligned}$$

$$\frac{\partial}{\partial \xi_i} = 0 \Rightarrow \xi_i = v_i^\top C v_i \quad \text{"Oracle estimator"}$$

Can't compute ξ_i without knowing C it seems!

19.2.3 The Large Dimension Miracle

$$\begin{aligned} t_{\lambda_k} &= \sum_j v_k^\top u_j \mu_j u_j^\top v_k \\ &= \sum_j \mu_j (v_k^\top u_j)^2 \rightarrow \int d\mu p_c(\mu) \Phi(\lambda_k, \mu) \\ &= \frac{1}{\pi p_E(\lambda_k)} \lim_{\eta \rightarrow 0^+} \operatorname{Im} \sum_j \mu_j u_j^\top G_E(\lambda_k - i\eta) u_j \\ &= \frac{1}{\pi p_E(\lambda_k)} \lim_{\eta \rightarrow 0^+} \operatorname{Im} \text{Tr}[C G_E(\lambda_k - i\eta)] \end{aligned}$$

In both the additive ξ_i multiplicative case:

$$G_E(z) = Y(z) G_C(z) \quad \left. \right\} \text{Subordination}$$

$$\begin{aligned} Y(z) &= 1 \quad \text{for } + \Rightarrow \tau[C G_E] = Y(z) \tau[C G_C(z)] \\ Y(z) &= \frac{z(z)}{z} \quad \text{for } \times \quad = Y(z) z(z) g_C(z) - Y(z) \\ &= z(z) g_E(z) - Y(z) \end{aligned}$$

But $Z(z)$ depends only on g_E

$\Rightarrow \mathbb{E}[C(g_E)]$ doesn't depend on g_C !

$$\xi_k = \frac{1}{\pi P_E(z)} \lim_{\gamma \rightarrow 0^+} \operatorname{Im} Z(z_k) g_E(z_k) - Y(z_k) \quad \text{estimable from data alone!}$$

19.2.4 Additive Case:

$$Z(z) = z - R_X(g_E(z)) \quad Y = 1$$

$$\xi(\lambda) = \lambda - \frac{\lim_{\gamma \rightarrow 0} \operatorname{Im} R_X(g_E(z)) g_E(z)}{\lim_{\gamma \rightarrow 0} \operatorname{Im} g_E}$$

$$X=0 \Rightarrow R=0 \Rightarrow \xi = \lambda \quad \text{as expected}$$

$$X \text{ small} \Rightarrow R = \varepsilon X + \dots$$

$$\mathbb{E}[X^2] \quad \mathbb{E}[X] = 0$$

$$= \lambda - \frac{2 \mathbb{E}[h_E(\lambda) \pi P_E(\lambda)]}{\pi P_E} = \lambda - 2 \mathbb{E}[h_E(\lambda)]$$

$$\text{Exact for Wigner noise} \quad R_X(\lambda) = \sigma_n^2 \delta$$

IF C is also Wigner with $\sigma_s^2 \Rightarrow E$ is Wigner with $\sigma_s^2 + \sigma_n^2$

$$h_E(z) = \frac{z}{2\sigma^2} \quad \text{for Wigner} \quad \sigma^2 = \sigma_s^2 + \sigma_n^2$$

$$\Rightarrow \xi(\lambda) = \lambda \left(1 - \frac{\sigma_n^2}{\sigma^2}\right) = \lambda \left(\frac{\sigma_s^2}{\sigma^2}\right)$$

linear shrinkage from last chapter!

$$(x_0 = 0)$$

$$\Rightarrow \Sigma(E)_{ij} = r E_{ij}$$

Ex 19.2.1

$$E = C + X \quad C, X \text{ from same dist mutually free}$$

$$a) R_X(g) + R_C(g) = R_E(g)$$

$$\Rightarrow 2R_X(g) = R_E(g)$$

b) $g_E R_E(g) = zg_E - 1$

$$\Rightarrow g_E R_X(g) = \frac{1}{2}(zg_E - 1)$$

c) $\Rightarrow \xi(\lambda) = \lambda - \frac{1}{2} \frac{\lim_{z \rightarrow 0^+} \operatorname{Im} zg_E}{\lim_{z \rightarrow 0^+} \operatorname{Im} g_E} = \frac{1}{2}$

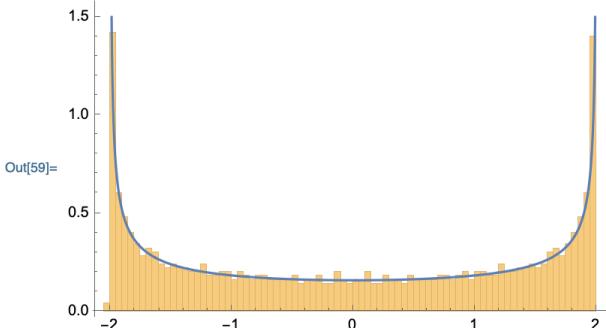
d) $\Xi = E[C|E] \Rightarrow \Xi = E/\lambda$

e)

```
In[48]:= n = 1000;
X1 = RandomVariate[GaussianOrthogonalMatrixDistribution[1, n]] / Sqrt[n/2];
X2 = RandomVariate[GaussianOrthogonalMatrixDistribution[1, n]] / Sqrt[n/2];
{vals, vecs} = Eigensystem[X1];
sX1 = Transpose[vecs].DiagonalMatrix[Sign[vals]].vecs;
{vals, vecs} = Eigensystem[X2];
sX2 = Transpose[vecs].DiagonalMatrix[Sign[vals]].vecs;
{vals, vecs} = Eigensystem[sX1 + sX2];
```

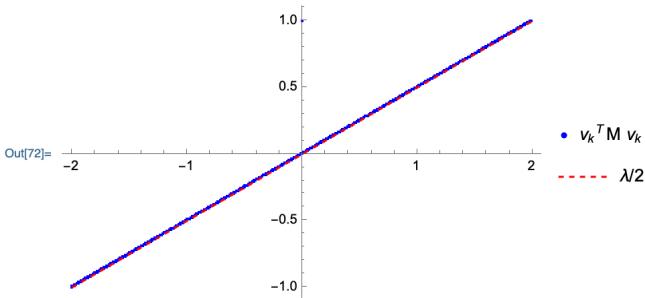
f)

```
In[59]:= Show[Histogram[vals, 100, "PDF"], Plot[1/(π Sqrt[4 - λ^2]), {λ, -2, 2}, PlotRange → 1.5]]
```



g)

```
In[72]:= Show[ListPlot[{Transpose[{vals, Diagonal[vecs.sX1.Transpose[vecs]}]}], PlotStyle → Blue, PlotLegends → {"v_k^T M v_k"}], Plot[λ/2, {λ, -2, 2}, PlotStyle → {Red, Dashed}, PlotLegends → {"λ/2"}]]
```



19.2.5 Multiplicative Case

$$\xi(\lambda) = \frac{1}{\pi P_E(\lambda_k)} \lim_{\eta \rightarrow 0^+} \operatorname{Im} \tau [C G_E(\lambda_k - i\eta)] = \lim_{\eta \rightarrow 0^+} \operatorname{Im} \frac{1}{\lambda_k - i\eta} \frac{\tau[C T_E(\lambda_k - i\eta) + C]}{\pi P_E(\lambda_k)}$$

$$= \frac{\lim_{\eta \rightarrow 0^+} \operatorname{Im} \tau [C T_E(z)]}{\lim_{\eta \rightarrow 0^+} \operatorname{Im} T_E(z)}$$

$$T_E(z) = T_C[z S_W(T_E(z))]$$

$$\Rightarrow \xi(\lambda) = \frac{\lim_{\eta \rightarrow 0^+} \operatorname{Im} \tau [C T_C(z S_W(T_E(z)))]}{\lim_{\eta \rightarrow 0^+} \operatorname{Im} T_E(z)}$$

$$= \tau[C^2(z S_W(T_E(z)) - C)^{-1}]$$

$$= -\tau[C] + z S_W(T_E(z)) T_C(z S_W(T_E(z)))$$

$$= -\tau[C] + z S_W(T_E(z)) T_E(z)$$

real

$$\Rightarrow \xi(\lambda) = \lambda \lim_{\eta \rightarrow 0^+} \frac{\operatorname{Im} S_W(T_E(z)) T_E(z)}{\operatorname{Im} T_E(z)} \quad z = \lambda - i\eta \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{general}$$

$$E = C^{1/2} W_q C^{1/2} \Rightarrow S_W = (I - qE)^{-1} \quad \operatorname{Im} T_E = 0$$

$$\xi(\lambda) = \lambda \lim_{\eta \rightarrow 0^+} \frac{\operatorname{Im} \frac{T_E}{I - qT_E}}{\operatorname{Im} T_E} = \frac{\lambda}{|I - qT_E|^2} \frac{\operatorname{Im} T_E(I - qT_E)}{\operatorname{Im} T_E}$$

nonlinear
"shrinkage"

$$= \frac{\lambda}{|I - qT_E(\lambda - i\eta)|^2} \Big|_{\eta \rightarrow 0^+}$$

$$T_E = \lambda g_E(\lambda) - I$$

$$\lambda > 0 \text{ for cov} \Rightarrow f(\lambda_-) < 0 \\ f(\lambda_+) > 0$$

$$= \frac{\lambda}{|I - q + qg_E(\lambda - i\eta)|^2} \Big|_{\eta \rightarrow 0^+}$$

For $C = M_p$ inverse Wishart

$$\Rightarrow t_E = \frac{z-q-1 \pm \sqrt{(z-q-1)^2 - 4(q+2p)}}{2(q+2p)} \quad \text{from 15.4}$$

$$\Rightarrow \xi(\lambda) = \frac{q + \lambda p}{p+q} = r\lambda + (1-r) \quad r := \frac{p}{p+q}$$

↑ Bayesian
Shrinkage as in 18.3

for $\lambda \in [\lambda_-, \lambda_+]$

for λ outside this, ξ is nonlin in λ

Ex 19.2.2 RIE for C Wishart

$$C = W_q$$

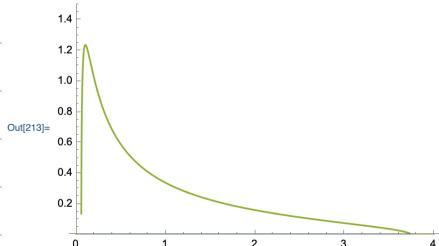
$$E = C^{\frac{1}{2}} W_q C^{\frac{1}{2}}$$

$$a) S_E = \frac{1}{(1+q_0t)(1+qt)}$$

$$\Rightarrow t_E = (1+q_0t)(1+qt)(1-t) \quad \text{3 Cubic}$$

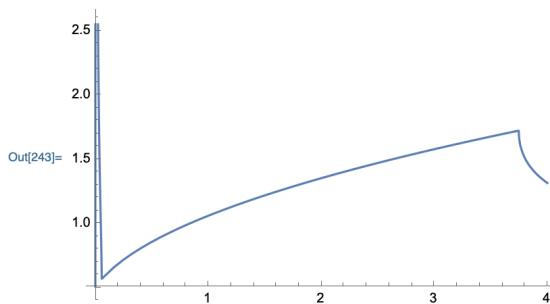
b)

```
In[212]:= solns = Solve[t g == (1 + q0 t) (1 + q t) (1 + t), t] /. q0 → 1/4 /. q → 1/2;
Plot[1/π Im[t + 1/g] /. solns // Evaluate, {g, 0, 4}, PlotPoints → 400, PlotRange → {0, 1.5}]
```



c)

```
In[243]:= Plot[Abs[1 + q t]^2 /. q → 1/2 /. solns[[2]] /. g → λ - I 0.0001, {λ, 0, 4}]
```



d)

```
In[172]:= n = 1000;
q = 1/4;
q = 1/2;
t0 = n/q0;
t1 = n/q;
C0 = RandomVariate[WishartMatrixDistribution[t0, IdentityMatrix[n] / t0]];
W1 = RandomVariate[WishartMatrixDistribution[t1, IdentityMatrix[n] / t1]];
W2 = RandomVariate[WishartMatrixDistribution[t1, IdentityMatrix[n] / t1]];
SqrtC = MatrixPower[C0, 1/2];
E1 = SqrtC.W1.SqrtC;
E2 = SqrtC.W2.SqrtC;

In[183]:= τ[M_] := Tr[M] / Length[M]
{τ[C0], τ[W1], τ[W2], τ[E1], τ[E2]}
{τ[C0.C0], τ[W1.W1], τ[W2.W2], τ[E1.E1], τ[E2.E2]}

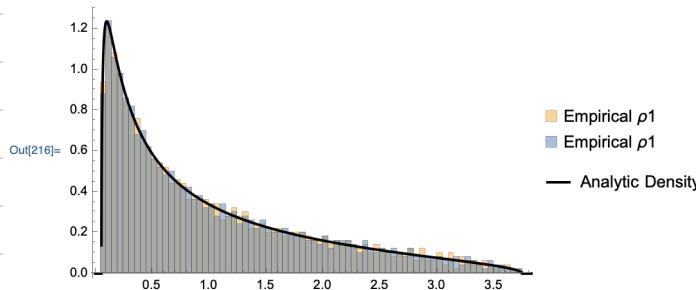
Out[184]= {0.999531, 0.998954, 1.00131, 0.998242, 1.00131}

Out[185]= {1.24894, 1.49946, 1.50357, 1.74519, 1.75685}
```

e)

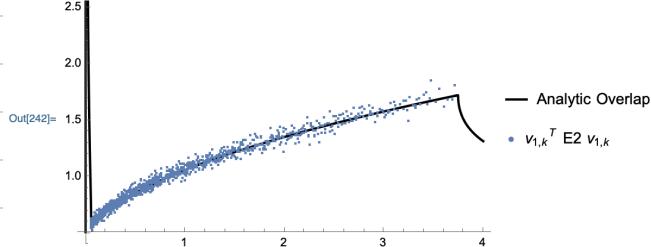
```
In[216]:= Show[Histogram[{Eigenvalues[E1], Eigenvalues[E2]}, 100, "PDF",
ChartLegends -> {"Empirical ρ1", "Empirical ρ1"},

Plot[ $\frac{1}{\pi} \operatorname{Im}\left[\frac{t-1}{s}\right] / . \text{solns}[3] // \text{Evaluate}, \{\xi, 0, 4\}, \text{PlotPoints} \rightarrow 400,$ ,
PlotStyle -> {Black, Thick}, PlotLegends -> {"Analytic Density"}]]
```



f)

```
In[240]:= {evals1, evecs1} = Eigensystem[E1];
overlaps = Diagonal[evecs1.E2.Transpose[evecs1]];
Show[Plot[ $\frac{\lambda}{\operatorname{Abs}[1+q t]^2 / . q \rightarrow 1/2 / . \text{solns}[2] / . \xi \rightarrow \lambda - I 0.0001}$ , {\lambda, 0, 4},
PlotStyle -> {Thick, Black}, PlotLegends -> {"Analytic Overlap"}],
ListPlot[Transpose[{evals1, overlaps}], PlotLegends -> {" $v_{1,k}^T E2 v_{1,k}$ "}]]
```



Ex 19.2.3

a) For W, C drawn from same ensemble

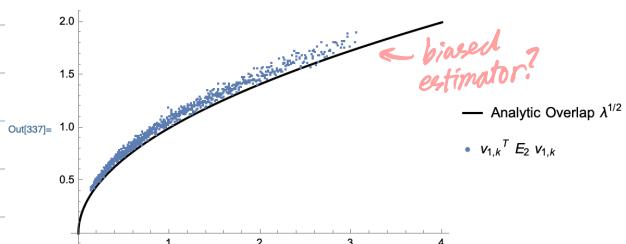
$$S_E = S_C^2 = S_W^2 \Rightarrow S_W = \sqrt{S_E}$$

$$\Rightarrow t_E \sqrt{S_E(t_E(z))} = \sqrt{\frac{t_E + 1}{t_E z}} \cdot t_E \Rightarrow \sqrt{\lambda} \cdot \frac{\operatorname{Im} \sqrt{t_E^2 + t_E}}{\operatorname{Im} t_E}$$

$$= \sqrt{\lambda} \cdot \frac{\operatorname{Im} t_E \sqrt{1 + t_E^{-1}}}{\operatorname{Im} t_E} \quad \text{real!}$$

b)

```
In[336]:= {evals1, evecs1} = Eigensystem[E1];
overlaps = Diagonal[evecs1.E2.Transpose[evecs1]];
Show[Plot[Sqrt[\lambda], {\lambda, 0, 4}, PlotStyle -> {Thick, Black},
PlotLegends -> {"Analytic Overlap \lambda^{1/2}"}],
ListPlot[Transpose[{evals1, overlaps}], PlotLegends -> {"v_{1,k}^T E_2 v_{1,k}"}]]
```



19.2.6 RIE for outliers

Assume C has outliers that appear as outliers of E

For z outside the bulk g_E, t_E analytic

$$\Rightarrow \operatorname{Im} g_E(\lambda - i\eta) = -\eta g'_E(\lambda)$$

$$\operatorname{Im} t_E(\lambda - i\eta) = -\eta t'_E(\lambda)$$

Additive Case:

$$\begin{aligned} \Rightarrow \xi(\lambda) &= \lambda - \frac{(R(g)g)'}{g'} = \lambda - R(g) - R'(g)g' \\ &= \lambda - \frac{d}{dg}[gR_x(g)] \end{aligned}$$

Multiplicative Case:

$$\Rightarrow \xi(\lambda) = \lambda \frac{d}{dt} [\cdot S_w(t)]$$

19.3 Properties of optimal RIE for Covariance Matrices

What is the effect of $\lambda_i \rightarrow \xi(\lambda_i)$

$$\operatorname{Tr} E = \sum_j \mu_j u_j^T (\sum_i v_i v_i^T) u_j = \operatorname{Tr} C$$

\Rightarrow Cleaning preserves trace

$$\text{Tr } \Sigma = \sum_{j,k} \mu_j \mu_k \sum_i (u_j \cdot v_i)^2 (u_k \cdot v_i)^2$$

\sum_i (u_j \cdot v_i)^2 (u_k \cdot v_i)^2 \approx A_{jk}

$$\begin{aligned} \sum_j A_{jk} &= \sum_{j,i} v_i^T u_j u_j^T v_i v_i^T u_k u_k^T v_i \\ &= \sum_i v_i^T u_k u_k^T v_i = \text{Tr } u_k u_k^T = 1 \end{aligned}$$

Similarly for $\sum_k A_{jk} = 1$

$$\Rightarrow \sum_{j,k} A_{jk} \mu_j \mu_k \leq \sum_j \mu_j^2$$

$$\Rightarrow \text{Tr } \Sigma^2 \leq \text{Tr } C^2 \leq \text{Tr } E^2$$

Analogue of optimal ridge?

Be more cautious than just "bringing back" the sample λ_i

Always shrink top λ down
bottom λ up as in $\frac{1}{\sqrt{\lambda}}$ for additive case
 $\sqrt{\lambda}$ for mult. case

Asymptotic behavior:

Assume outlier to the left of lower bound ρ_E
 $\epsilon q < 1 \Rightarrow E$ has no 0 modes

$\lambda g_E(\lambda)$ is $O(N)$ for $\lambda \geq 0$

$$\Rightarrow \text{Tr } q T_E(\lambda) = 1 - q + O(\lambda)$$

$$\Rightarrow g(\lambda) = \frac{\lambda}{(1-q)^2} + O(\lambda^2)$$

small λ grow ✓

Now assume $\lambda \rightarrow \infty$

$$\lim_{\lambda \rightarrow \infty} \lambda T_E \sim \frac{\tau(E)}{\lambda} \Rightarrow g = \frac{\lambda}{\left| 1 + q \frac{\tau(E)}{\lambda} + O(\lambda^{-2}) \right|^2} \approx \lambda - 2q \tau(E) + O(\lambda^{-2})$$

$$\tau(E) = 1 \Rightarrow g = \lambda - 2q + O(\lambda^{-1})$$

From 19.54 an outlier eig will be shifted by q

$$\lambda = \underbrace{a+1+q}_{\mu}$$

$$\Rightarrow \xi \approx \mu - q$$

$$\Rightarrow \xi < \mu < \lambda$$

19.4 Conditional Average in Free Probability

Alternative derivation of

$$\xi_k = \sum_{\pi_{PE}(\lambda_k)} \lim_{\eta \rightarrow 0^+} \operatorname{Im} \tau [C G_E(\lambda_k - i\eta)]$$

E obtained by free operations on C

Best MSE estimator is $\hat{E}(E) = E[C|E]$

\hat{E} is a function of E only (since only E is known)

$$\Rightarrow [\hat{E}, E] = 0$$

$$\Rightarrow \text{compute } m_k = \tau [\hat{E} E^k]$$

Generating function: $F(z) = \tau [\xi(E) (z - E)^{-1}]$
at $z \rightarrow \infty$

$$= \tau [E[C|E] (z - E)^{-1}]$$

But τ contains E

$$\Rightarrow \tau(E[\cdot]) = \tau(\cdot) = \tau [C (z - E)^{-1}] = \int_{\rho_E(\lambda)} \frac{\xi(\lambda)}{z - \lambda} d\lambda$$

$$\Rightarrow \lim_{\eta \rightarrow 0^+} \operatorname{Im} F(\lambda - i\eta) = \pi_{PE} \xi(\lambda)$$

$$\Rightarrow \xi(\lambda) = \lim_{\eta \rightarrow 0^+} \frac{\operatorname{Im} F(\lambda - i\eta)}{\operatorname{Im} g_E(\lambda - i\eta)}$$

19.5 Real Data

19.5.1 Parametric Approach

Postulate a form for p_c

Simplest is $p_c \sim \text{Inverse Wishart}$

$$\Rightarrow E = \sqrt{M_p} W_q \sqrt{M_p} \Rightarrow p_E = \frac{\sqrt{4(p\lambda+q)-(1+q-\lambda)^2}}{2\pi\lambda qp}$$

$$\tau(E^2) = 1+p+q$$

$$\tau(E^{-1}) = \frac{1-p}{1-q} \quad \{ \tau(W_q^{-1} M_p^{-1}) = \tau(W_q^{-1}) \tau(M_p^{-1}) \}$$

When the empirical p seems to be bounded above and below
a more general ansatz:

$$S_c = \frac{(1-p_1t)(1-p_2t)}{(1+q_1t)} \Leftrightarrow S_E(t) = \frac{(1-p_1t)(1-p_2t)}{(1+q_1t)(1+q_2t)}$$

\rightarrow Cubic for t (and hence p)
 p_1, p_2, q_1, \dots fitted from moments of E, E^{-1} (equiv. from density)

Doesn't work super well...

Alternatively, postulate a form eg

$$p_E = z^{-1} \frac{(1-a_1\lambda + a_2\lambda^2)\sqrt{(\lambda - \lambda_-)(\lambda_+ - \lambda)}}{1+b_1\lambda + b_2\lambda^2}$$

Through eg MSE on CDF

\rightarrow Then reconstruct $g_E(x-i0^\pm)$ numerically
from fitted p & its Hilbert transform $\int \frac{P(x)}{z-\lambda} d\lambda$

But even when p fits the sample density,
it cannot be obtained as a free prob of Wishart w/ some population density

\rightarrow Approximate estimator is non-monotonic in true estimator
Believed that this should never be the case

For unbounded support (eg sharp left edge but unbounded right tail)

$$p_c = C(\lambda - \lambda_-) \frac{2}{\alpha \sqrt{\pi} \mu} \frac{\Gamma(\frac{1+\mu}{2})}{\Gamma(\frac{\mu}{2})} \left(1 + \frac{(\lambda - \lambda_+)^2}{\alpha^2 \mu} \right)^{-\frac{1+\mu}{2}}$$

$$\Rightarrow p(\lambda) \sim \lambda^{-\mu-1} \text{ as } \lambda \rightarrow \infty$$

19.5.2 Kernel Methods

$$p_s(\lambda) = \frac{1}{N} \sum_{k=1}^N K_{\eta_k}(\lambda - \lambda_k)$$

possibly k-dep width η_k

$$\int du K_\eta(u) = 1 \Rightarrow \int dx p_s(x) = 1$$

$$\Rightarrow g_s(z) := \frac{1}{N} \sum_{k=1}^N g_{K_{\eta_k}}(z - \lambda_k)$$

$$g_{K_{\eta_k}} = \int_{-\infty}^{\infty} du \frac{K_{\eta_k}(u)}{z-u} \quad \text{Im}(z) \neq 0$$

$$\text{Im } g_{K_{\eta}}(x-i0^+) = i\pi K_\eta(x)$$

$$\Rightarrow \text{Im } g_s(x-i0^+) = i\pi g_s(x) \quad \forall K_\eta$$

$$\Rightarrow h_s(x) := \text{Re } g_s(x) = \int d\lambda \frac{p_s(\lambda)}{z-\lambda}$$

Particularly relevant kernels are:

$$\text{Cauchy: } K_\eta^C(x) = \frac{1}{\pi} \frac{\eta}{u^2 + \eta^2}$$

$$\Rightarrow g_{K_\eta^C}(z) = \frac{1}{z \pm i\eta} \Rightarrow g_{s,C}^c(z) = \frac{1}{N} \sum_k \frac{1}{z - \lambda_k - i\eta_k} \quad \text{Im}(z) < 0$$

Another choice: Semicircle "Wigner" Kernel

$$K_\eta^W(u) = \frac{\sqrt{4\eta^2 - u^2}}{2\pi\eta^2}$$

$$g_s^W(z) = \frac{1}{N} \sum_{k=1}^N \frac{z - \lambda_k}{2\eta_k^2} \left[1 - \sqrt{1 - \frac{4\eta_k^2}{(z - \lambda_k)^2}} \right]$$

1. Can rectify nonmonotonicity of $\xi(\lambda)$ by hand

2. Usually want $\xi(\lambda)$ exactly at $\lambda = \lambda_k$
 empirically, excluding λ_k from kernel estimator gives consistently better results

19.6 Validation & RIE

$$\xi_x(\lambda_i) := v_i E' v_i$$

v_i computed from train set E

E' , ξ_x "out of sample"

IF C is the same $E = \sqrt{C} W \sqrt{C}$ $E' = \sqrt{C} W' \sqrt{C}$ $V' \perp W$

$$\Rightarrow \xi_x = \sum_{k=1}^N (v_i^\top v_k')^2 \lambda_k' \rightarrow \int d\lambda' p_E(\lambda') \lambda' \Psi(\lambda, \lambda')$$

$$\Psi(\lambda, \lambda') = \frac{1}{N} \sum_j \underbrace{\Phi(\lambda, \mu_j)}_{E, C} \underbrace{\Phi(\lambda', \mu_j)}_{E', C} \quad \left. \right\} \text{exact}$$

$$\rightarrow \int p_C(\mu) \Phi(\lambda, \mu) \Phi(\lambda', \mu) d\mu$$

Proof:

$$v_i^E = \frac{1}{\sqrt{N}} \sum_j \epsilon_{ij} \sqrt{\Phi(\lambda_i, \mu_j)} u_j \quad v_i^{E'} = \frac{1}{\sqrt{N}} \sum_j \epsilon_{ij}' \sqrt{\Phi(\lambda_i', \mu_j)} u_j$$

$$\underbrace{E[\epsilon \epsilon']}_{=} = 0 \quad E[\epsilon] = E[\epsilon'] = 0 \quad E[\epsilon_{ij} \epsilon_{kl}] = E[\epsilon_{ij}' \epsilon_{kl}'] = \delta_{ik} \delta_{jl}$$

"Ergodic assumption" (Justifiable by DBM of evecs)

$$\Rightarrow E[(v_i^\top v_k')^2] = \frac{1}{N} \sum_j \Phi(\lambda_i, \mu_j) \Phi(\lambda_k, \mu_j)$$

$$\Rightarrow \xi_x(\lambda) = \frac{1}{N^2} \sum_{k,j} \Phi(\lambda, \mu_j) \Phi(\lambda_k, \mu_j) \lambda_k'$$

$$= \frac{1}{N^2} \sum_j \Phi(\lambda, \mu_j) \sum_k \Phi(\lambda_k, \mu_j) \lambda_k'$$

$$= N u_j^T E' u_j$$

$$= N u_j^T \sqrt{C} W \sqrt{C} u_j$$

$$= N u_j^T u_j^T W' u_j \quad \leftarrow \text{since } u \text{ is eig of } W$$

Since $C \perp W'$ $\int_{\text{interval}} du p(u) [\dots]$

involves W' wedged between randomly oriented axes

$$\mathbb{E}[u^T W u] = \tau(W) \tau(u u^T) = 1$$

$$\Rightarrow \xi_x(\lambda) = \frac{1}{N} \sum_j \Phi(\lambda, \mu_j) \mu_j \rightarrow \int p_C(\mu) \Phi(\lambda, \mu) d\mu = \xi(\lambda)$$

\Rightarrow Can approximate ξ by considering even when E, E' have different q $v_i^T E' v_i$

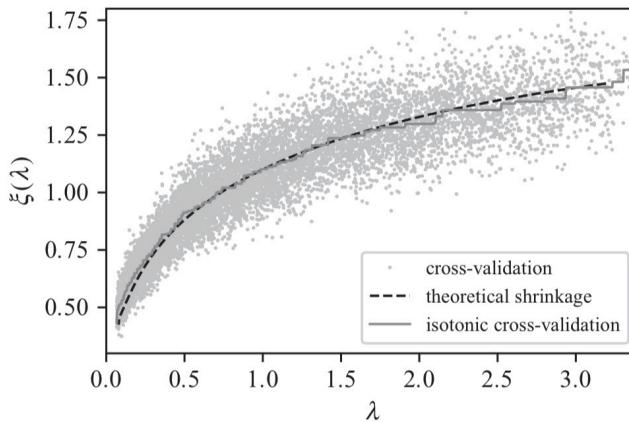


Figure 19.5 Shrinkage function $\xi(\lambda)$ computed for the same problem as in Figure 19.3, now using cross-validation. The dataset is divided into $K = 10$ blocks of equal length. For each block, we compute the $N = 1000$ eigenvalues λ_i^b and eigenvectors v_i^b of the sample covariance matrix using the rest of the data (of new length $9T/10$), and compute $\xi_X(\lambda_i^b) := v_i^{b T} E' v_i^b$, with E' the sample covariance matrix of the considered block. The dots correspond to the 10×1000 pairs $(\lambda_i^b, \xi_X(\lambda_i^b))$. The full line is an isotonic regression through the dots. The procedure has a slight bias as we in fact compute the optimal shrinkage for a value of q equal to $q_X = 10N/9T$, but otherwise the agreement with the optimal curve is quite good.

Chapter 20: Applications to Finance

N assets at time t with prices $p_i t$

$$\text{Returns: } r_{it} := \frac{p_{it} - p_{i,t-1}}{p_{i,t-1}}$$

Total capital: C

$$\text{Return at } t: R_t = \sum_i \pi_i r_{it} + (C - \sum_i \pi_i) r_0$$

risk free rate

$$\Rightarrow \text{Excess return} = R_t - Cr_0$$

$$= \sum_i \pi_i (r_{it} - r_0)$$

From now on, denote $r_{it} - r_0$ by r_{it}

assume $\vec{r}_t \sim \vec{g}$ "gains"

$$C_{ij} = \text{Cov}[r_i, r_j]$$

assume gains are known

Not wise to swap E for C
 \rightarrow evals can differ meaningfully

Limited data $\Rightarrow T$ is not huge

20.1.2 Portfolio Risk

$$\mathcal{R}^2 := \text{Var } R = \sum_{ij} \pi_i \pi_j \text{Cov}[r_i, r_j] = \pi^T C \pi$$

Expected shortfall at p th quantile

$$S_p = -\frac{1}{p} \int_{-\infty}^{R_p} dR R P(R)$$

$$R_p \text{ s.t. } \int_{-\infty}^{R_p} dR P(R) = p$$

20.1.3 Markowitz Portfolio Theory

$$\begin{array}{ll} \min & \pi^T C \pi \\ \text{s.t.} & \pi^T g \geq G \end{array} \quad \text{expected return target}$$

$$\Rightarrow \min_{\pi} \pi^T C \pi - \gamma \pi^T g$$

$$\Rightarrow \pi_c = G \frac{C^{-1}g}{g^T C^{-1}g} \quad \text{Needs knowledge of } C, g$$

$$R_{\text{true}}^2 = \frac{G^2}{g^T C^{-1}g}$$

20.1.4 Predicted & Realized Risk

Naive: Sub E for C

$$\Rightarrow \pi_E = G \frac{E^{-1}g}{g^T E^{-1}g}$$

$$R_{\text{in}}^2 = \frac{G^2}{g^T E^{-1}g} \quad \text{"in-sample risk"}$$

$g^T E^{-1}g$ is convex w.r.t. E

$$\Rightarrow E[g^T E^{-1}g] \geq g^T E[E]^{-1}g = g^T C^{-1}g$$

$$\Rightarrow R_{\text{in}}^2 \leq R_{\text{true}}^2$$

For E' out of sample

$$R_{\text{out}}^2 := \pi_E^T E' \pi_E^T = G^2 \frac{g^T E'^{-1} E' E^{-1} g}{(g^T E'^{-1} g)^2}$$

$$\pi_E \perp E' \Rightarrow \text{as } N \rightarrow \infty \quad \pi_E^T E' \pi_E = \pi_E^T C \pi_E$$

$$\text{From optimality of } \pi_c : \pi_c^T C \pi_c \leq \pi_E^T C \pi_E = \pi_E^T E' \pi_E$$

$$\Rightarrow R_{\text{in}}^2 \leq R_{\text{true}}^2 \leq R_{\text{out}}^2$$

20.2 The High-Dimensional Limit

20.2.1 R_{in}^2 vs R_{out}^2 : Exact Results

Let C, gg^T free (not a natural assumption unless predictors are market neutral & idiosyncratic characteristics)

$$M \text{ pos definite} \Rightarrow \frac{g^T M g}{N} = \frac{1}{N} \text{Tr}[M g g^T] = \frac{g^2}{N} \tau(M)$$

Can normalize $g^2/N = 1$

$$1) R_{in}^2 = \frac{\xi^2}{N \tau(E^{-1})}$$

$$2) R_{true}^2 = \frac{\xi^2}{N \tau(C)}$$

$$3) R_{out}^2 = \frac{\xi^2 \tau(E^{-1} C E^{-1})}{N \tau(E^{-1})^2}$$

$$1) \text{ & } 2): q < 1$$

$$\tau(C^{-1}) = (1-q) \tau(E^{-1}) \Rightarrow R_{in}^2 = (1-q) R_{true}^2 \quad \leftarrow 0 \text{ as } q \rightarrow 1$$

$$3): E = C^{1/2} W_q C^{1/2}$$

$$\Rightarrow R_{out}^2 = \frac{\xi^2 \tau(C^{-1} W_q^{-2})}{N \tau(E^{-1})^2} \quad \begin{matrix} \text{~} \\ \text{precise} \\ \text{quantification} \\ \text{of over-optimism} \end{matrix}$$

$$\begin{matrix} \text{W.C} \\ \text{asymptotically} \\ \text{free} \end{matrix} = \frac{\xi^2 (1-q)^2}{N} \frac{\tau(C^{-1}) \tau(W_q^{-2})}{\tau(E^{-1})^2} \quad \begin{matrix} \text{~} \\ \text{~} \\ \text{~} \end{matrix}$$

$$= \frac{R_{true}^2}{1-q} \quad \leftarrow \text{diverges as } q \rightarrow 1$$

$$\frac{R_{in}^2}{1-q} < R_{true}^2 < (1-q) R_{out}^2$$

$q \rightarrow 1$ is most dangerous

20.2.2 Out-of sample Risk Minimization

Naively since Markowitz uses C^{-1} , may think to estimate that
but $E R_{\text{out}}^2$ depends linearly on $C \Rightarrow$ estimate that

Proof:

$$\Sigma = \sum_i \xi(\lambda_i) v_i v_i^T$$

\uparrow
errors of Σ

$$R_{\text{out}}^2 = G^2 \frac{\text{Tr}(\Sigma^{-1} C \Sigma^{-1})}{(\text{Tr } \Sigma^{-1})^2} = G^2 \sum_i \frac{v_i^T C v_i}{\xi^2(\lambda_i)} \left(\sum_i \frac{1}{\xi(\lambda_i)} \right)^{-2}$$

$$\frac{\partial R_{\text{out}}^2}{\partial \xi_j} = 0 \Rightarrow -2 \frac{v_j^T C v_j}{\xi(\lambda_j)^3} \left(\sum_i \frac{1}{\xi(\lambda_i)} \right)^{-2} + \frac{2}{\xi(\lambda_j)^2} \sum_i \frac{v_i^T C v_i}{\xi^2(\lambda_i)} \left(\sum_i \frac{1}{\xi(\lambda_i)} \right)^{-3}$$

$$\Rightarrow \xi(\lambda_j) = A v_j^T C v_j$$

$\downarrow \text{const}$

$\text{Tr}[\Sigma] = \text{Tr}[C] \Rightarrow A=1 \Rightarrow \xi$ is oracle estimator

\Rightarrow RIE minimizes R_{out}^2 !

$$\text{Tr}[\Sigma^n C] = \sum_i \xi(\lambda_i)^n \text{Tr}[v_i v_i^T C] = \sum_i \xi^n(\lambda_i) v_i^T C v_i = \text{Tr}[\Sigma^{n+1}]$$

$\forall n \in \mathbb{Z}$

$$\Rightarrow R_{\text{out}}^2(\Sigma) = \frac{G^2}{\text{Tr}(\Sigma^{-1})}$$

20.2.3 Inverse Wishart Model

$$C = M_p \quad p > 0$$

$$\mathcal{I}(C^{-1}) = -g_C(0) = 1+p$$

$$\Rightarrow R_{\text{true}}^2 = \frac{G^2}{N} \frac{1}{1+p} \quad \text{as } N \rightarrow \infty$$

$$\text{Can show } \mathcal{I}(\Sigma^{-1}) = -(1+\frac{q}{p}) g_E(-\frac{q}{p}) = 1 + \frac{p^2}{p+q+pq}$$

$$\Rightarrow R_{\text{out}}^2(\Sigma) = \frac{G^2}{N} \frac{p+q+pq}{(p+q)(1+p)}$$

$$\Rightarrow \frac{R_{\text{out}}^2(\Xi)}{R_{\text{true}}^2(\Xi)} = 1 + q \frac{pq}{(p+q)(1+p)} \geq 1$$

$$R_{\text{in}}^2(\Xi) = \frac{G^2}{N} \frac{\tau(\Xi^{-1} E \Xi^{-1})}{\tau(\Xi^{-1})^2} \quad \left. \right\} \text{Bun et al 2017}$$

$$= \frac{G^2}{N} \frac{p+q}{(1+p)(p+q(1+p))}$$

$$\Rightarrow \frac{R_{\text{in}}^2(\Xi)}{R_{\text{true}}^2(\Xi)} = 1 - \frac{pq}{p+q(1+p)} \leq 1$$

Can see $R_{\text{in}}^2(\Xi) - R_{\text{in}}^2(E) \geq 0$

$$R_{\text{out}}^2(\Xi) - R_{\text{out}}^2(E) \leq 0$$

20.2 The High-Dimensional Limit

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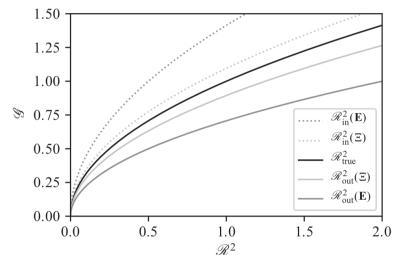


Figure 20.1 Efficient frontier associated with the mean-variance optimal portfolio (20.10) for $\mathbf{g} = \mathbf{1}$ and \mathbf{C} an inverse-Wishart matrix with $p = 0.5$, for $q = 0.5$. The black line depicts the expected gain as a function of the true optimal risk (20.11). The gray lines correspond to the realized (out-of-sample) risk using either the SCM \mathbf{E} or its RIE version Ξ . Both estimates are above the true risk, but less so for RIE. Finally, the dashed lines represent the predicted (in-sample) risk, again using either the SCM \mathbf{E} or its RIE version Ξ . \mathcal{R} and \mathcal{G} in arbitrary units, such that $R_{\text{true}} = 1$ for $\mathcal{G} = 1$.

20.3 Statistics of Price Changes

Bachelier's thesis: price variations $\propto \sqrt{\text{time}}$

$$V(\tau) := E \left[\left(\log \frac{p_{t+\tau}}{p_t} \right)^2 \right]$$

$$\Rightarrow V(\tau) = \sigma^2 \tau$$

Now let $\log p_t = \log p_0 + \sum_{t'=1}^t r_{t'}$

$$\text{Cov}(r_t, r_{t''}) = \sigma^2 C_r (1 + t' - t'')$$

$C_r(u) = \delta(u)$ for uncorrelated walk

Trending $\Rightarrow C_r(u) > 0$

Mean-reverting $\Rightarrow C_r(u) < 0$

Implications for Bachelier's first law:

$$\sigma^2(\tau) = \frac{V(\tau)}{\tau} = \sigma^2(1) \left[1 + \sum_{n=1}^{\infty} (1 - \frac{n}{\tau}) C_n(u) \right]$$

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Applications to Finance

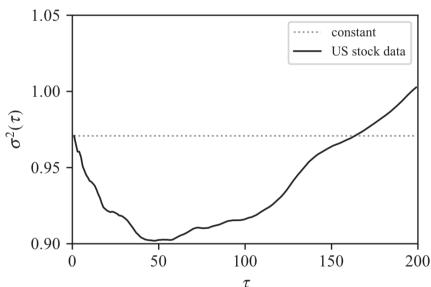


Figure 20.2 Average signature plot for the normalized returns of US stocks, where the x-axis is in days. The data consists of the returns of 1725 US companies over the period 2012–2019 (2000 business days), returns are normalized by a one-year exponential estimate of their past volatility. To a first approximation $\sigma^2(\tau)$ is independent of τ . The signature plot allows us to see deviations from this pure random walk behavior. One can see that stocks tend to mean-revert slightly at short times ($\tau < 50$ days) and trend at longer times. The effect is stronger on the many low liquidity stocks included in this dataset.

Returns have fat power law tails:

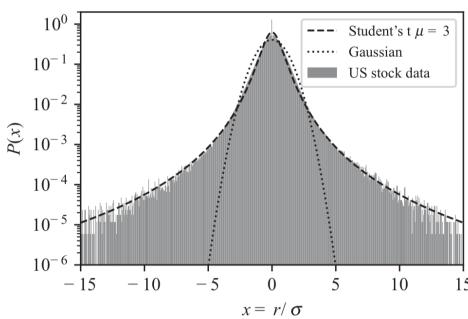


Figure 20.3 Empirical distribution of normalized daily stock returns compared with a Gaussian and a Student's t-distribution with $\mu = 3$ and the same variance. Same data as in Figure 20.2.

$\mu=3$ consistently across markets

Cool!

However, returns are far from IID draws from Student's t

Uncorrelated but dependent

Because time-aggregated returns do not revert to a Gaussian

Volatility is itself time-varying \Rightarrow heteroskedastic returns

$$\eta_t = \sigma_\eta \varepsilon_t$$

↑↑
not indep \Rightarrow Leverage Effect

ε_t are IID non-gaussian
if $\text{Var } \varepsilon_t = 1$

$$\text{For } E[\sigma_t^2 \sigma_{t+c}^2]$$

$$E[\varepsilon_t \sigma_{t+c}] < 0$$

$$E[\varepsilon_t \sigma_{t-c}] \approx 0$$

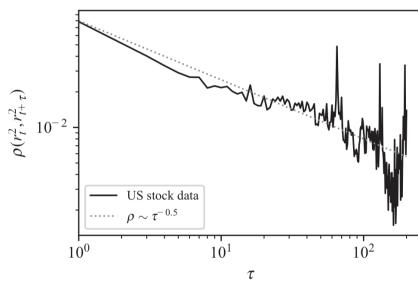


Figure 20.4 Average autocorrelation function of squared daily returns for the US stock data described in Figure 20.3. The autocorrelation decays very slowly with the time difference τ . A power law $\tau^{-\gamma}$ with $\gamma = 0.5$ is plotted to guide the eye. Note the three peaks at $\tau = 65, 130$ and 195 business days correspond to the periodicity of highly volatile earning announcements.

$$\mathbb{E}[\sigma_t^2 \sigma_{t+c}^2]$$

20.4 Empirical Covariance

Choose $N = 500$ stocks over $T = 2000$ days

$$\Rightarrow q = \frac{1}{q}$$

$$\lambda_i = \lambda_{\max} \sim 100 \times \langle \lambda \rangle$$

outlier "market mode"

$$v_i \approx 1/\sqrt{N} \quad (v_i \cdot \frac{1}{\sqrt{N}}) \approx 0.95$$

One-factor model:

$$r_{i,t} = \beta_i f_t + \epsilon_{i,t} \quad f_t, \epsilon_{i,t} \text{ uncorrelated mean 0}$$

$$\Rightarrow C_{ij} = \beta_i \beta_j \sigma_f^2 + \epsilon_{ij} \sigma_\epsilon^2 \quad \beta \approx \frac{1}{\sqrt{N}}$$

We know the spectrum is

$$1) \text{ MP "sea" between } (1 \pm \sqrt{q})^2 \sigma_\epsilon^2$$

$$2) \text{ Outlier at } \sigma_\epsilon^2 (1 - a)(1 + \frac{q}{a}) \quad a = \frac{\sigma_f^2 / |\beta|^2}{\sigma_\epsilon^2}$$

$$|\beta|^2 \sim O(N) \text{ for large portfolios} \Rightarrow a \sigma_\epsilon^2 q = \sigma_f^2 / |\beta|^2$$

$$\text{For } q = \frac{1}{q} \text{ get } \lambda_- = 0.2 \quad \lambda_+ = 1.8$$

We get ≈ 20 outliers \Rightarrow need more factors \nrightarrow market sectors

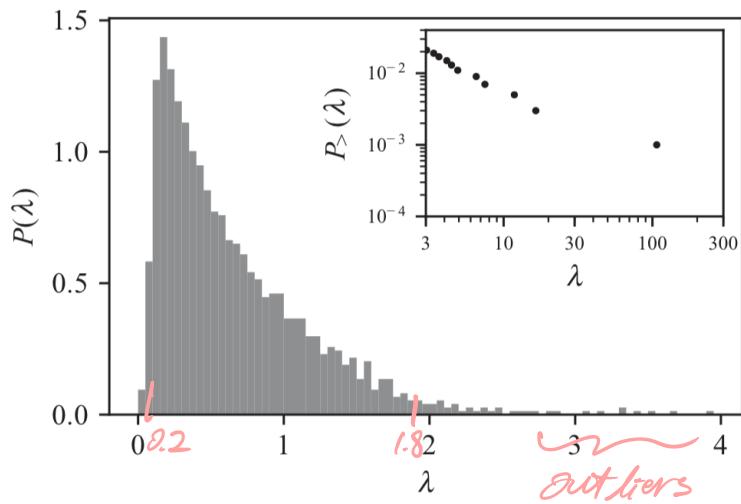


Figure 20.5 Eigenvalue distribution of the SCM, averaged over three random sets of 500 US stocks, each measured on 2000 business days. Returns are normalized as in Figure 20.3, corresponding to $\bar{\lambda} = 0.97$. The inset shows the complementary cumulative distribution for the largest eigenvalues indicating a power-law behavior for large λ , as $P_>(\lambda) \approx \lambda^{-4/3}$. Note the largest eigenvalue $\lambda_1 \approx 0.2N$, which corresponds to the “market mode” i.e. the risk factor where all stocks move in the same direction.

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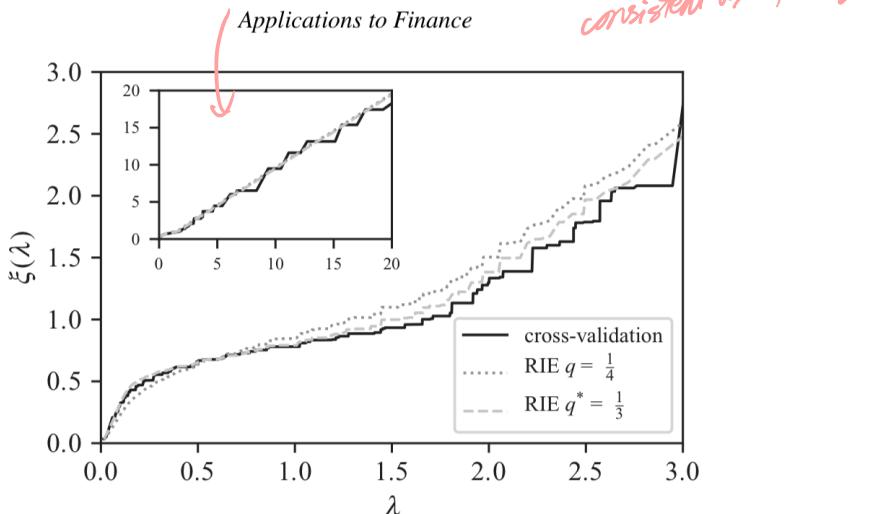


Figure 20.6 Non-linear shrinkage function $\xi(\lambda)$ computed using cross-validation and RIE averaged over three datasets. Each dataset consists of 500 US stocks measured over 2000 business days. Cross-validation is computed by removing a block of 100 days (20 times) to compute the out-of-sample variance of each eigenvector (see Eq. (19.88)). RIE is computed using the sample Stieltjes transform evaluated with an imaginary part $\eta = N^{-1/2}$. Results are shown for $q = N/T = 1/4$ and also for $q^* = 1/3$, chosen to mimic the effects of temporal correlations and fluctuating variance that lead to an effective reduction of the size of the sample (cf. Section 17.2.3). All three curves have been regularized through an isotonic fit, i.e. a fit that respects the monotonicity of the function.