The ADHM Construction, Hilbert Schemes, and the Heisenberg Algebra

Lecture 2

Alex Atanasov

December 6, 2016

Abstract

Continuing from the previous lecture, we consider the moduli space $\mathcal{M}^{reg}(n,k)$ of framed "genuine" instantons and extend it to its compactification $\hat{\mathcal{M}}(n,k)$ of so-called *ideal* instantons. This space has singularities, and we define its resolution $\tilde{\mathcal{M}}(n,k)$ in terms of the original ADHM data. From here, we study the special case $\tilde{\mathcal{M}}(1,k)$ as a Hilbert scheme of points on $\mathbb{C}^2$. This gives us a way to categorify Heisenberg algebras directly through the homology groups of these spaces.

1 Compactification and Resolution of $\mathcal{M}^{reg}(n,k)$

From before we have the ADHM construction:

Definition 1.1 (ADHM datum)**.** Two hermitian vector space V, W with $\dim V = k, \dim W = n$ and maps $B_1, B_2 \in \mathrm{End}(V), I \in \mathrm{Hom}(W,V), J \in \mathrm{Hom}(V,W)$. We have that the maps satisfy

$$[B_1, B_2] + IJ = 0 \tag{1}$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \tag{2}$$

Moreover $U(V)$ acts on this space by

$$(gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \tag{3}$$

Definition 1.2 ($\mathcal{M}^{reg}(n,k)$)**.** The moduli space of framed genuine instantons is given by taking the set of ADHM data $[B_1, B_2, I, J]$ with trivial stabilizer under the action of $U(V)$ and quotenting out by $U(V)$.

Equivalently, however, we can "forget" about the Hermitian structure on V (losing the notion of adjoints) and deal with a general vector space:

Theorem 1.3. *Let V, W be vector spaces of dimension $\dim V = k, \dim W = n$, and maps $B_1, B_2 \in \mathrm{End}(V), I \in \mathrm{Hom}(W,V), J \in \mathrm{Hom}(V,W)$, satisfying only*

$$[B_1, B_2] + IJ = 0 \tag{4}$$

And $\mathrm{GL}(n,\mathbb{C})$ acts on this space in the same way as before, by:

$$(gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}) \tag{5}$$

Then the moduli space $\mathcal{M}(n,k)$ of all such data with trivial GL_n stabilizer, quotiented by $\mathrm{GL}(n,\mathbb{C})$ is equivalent to $\mathcal{M}^{fr}(n,k)$.

This process of forgetting complex structure and then quotienting out by a larger group is similar to the identification:

$$\mathrm{U}(n)/T_n \cong \mathrm{GL}(n)/B_n \tag{6}$$

Where T_n is the toroidal subgroup of diagonal matrices and B_n is the Borel subgroup of upper-triangular matrices.

Proposition 1.4. *If ADHM datum $[B_1, B_2, I, J]$ satisfies either*

1. *(Stability) There is no proper subspace $S \subset V$ s.t. $B_i(S) = S$ and $I(W) \subset S$*
2. *(Co-stability) There is no proper subspace $S \subset V$ s.t. $B_i(S) = S$ and $S \subset \ker J$*

Then it has both nontrivial stabilizer and a closed orbit under $\mathrm{GL}(V)$.

Proof. Assume the stabilizer is nontrivial, so there is a g s.t. $gI = I$. Then im $I \in \ker(g - 1_V) =: S$ and moreover since $g^{-1}B_ig = B_i$, we have that S is invariant under the B_i and contains im$I$. Similarly, $Jg^{-1} = J \Rightarrow J(g - 1_V) = 0$ so $\mathrm{im}(g - 1_V) \in \ker J$, violating co-stabillity as well.

□

An ADHM datum that is both stable and co-stable is called **regular**.

For the hermitian case, it clear that (B_1, B_2, I, J) is stable iff $(B_1^\dagger, B_2^\dagger, I^\dagger, J^\dagger)$ is co-stable. Equation (2) demands compatibility between these descriptions, so in fact it is enough to just require stability on the system, and co-stability follows:

$$\mathcal{M}^{reg}(n,k) = \{\text{Solutions to (1) + (2) + regularity}\}/\mathrm{U}(V)$$

It is not too hard to see that all solutions of the ADHM equations giving rise to instantons are both stable and co-stable. As a result $\mathcal{M}^{reg}(n,k)$ can be written as

$$\mathcal{M}^{reg}(n,k) = \{\text{Solutions to (1) + regularity}\}/\mathrm{GL}(V)$$

The non-regular case is taken by compactifying this to allow for *all* ADHM data in the quotient:

Definition 1.5. $\hat{\mathcal{M}}(n,k)$, the compactification of $\mathcal{M}^{reg}(n,k)$ is defined as

$$\hat{\mathcal{M}} := \{\text{Solutions to (1)}\}/\mathrm{GL}(V)$$

This is a singular space in the sense that any Riemannian metric defined on the entire space will have a singularity at some point. For this reason, we define the **resolution** of this space.

Definition 1.6 (Resolution)**.** The resolution of a singular space X is a birational map π, together with a smooth variety, $\tilde{X}$ such that $\pi : \tilde{X} \to X$.

Definition 1.7. We define $\tilde{\mathcal{M}}(n,k)$ to be the set of data: satisfying

$$\hat{\mathcal{M}} := \{\text{Solutions to (1)} + \textit{stability}\}/\mathrm{GL}(V)$$

From the proof before, we see stability alone is enough to ensure a trivial stabilizer, so that this quotient is well-defined topologically.

Because stability on $[B_1, B_2] + IJ$ is the same as co-stability on a corresponding dual system, it would have worked to also only have co-stability. Either definition works, so long as we have one stability condition but not the other.

Theorem 1.8. *$\tilde{\mathcal{M}}(n,k)$ is the minimal resolution of $\hat{\mathcal{M}}(n,k)$ for all n,k.*

Observation 1.9. *Although $\mathcal{M}^{reg}(1,k)$ is empty (for, as we know, there are no $U(1)$ instantons), the definition of $\tilde{\mathcal{M}}(n,k)$ gives rise to a nonempty set of solutions, as we shall see in the next section.*

2 Hilbert Schemes of Points on $\mathbb{C}^2$

Definition 2.1 (Hilbert Scheme)**.** A Hilbert scheme $\mathrm{Hilb}_n X$ of n points on an algebraic variety X is given as the space of all ideals of codimension n in the $\mathbb{C}[X]$. That is, the set of ideals I so that $\mathbb{C}[X]/I \cong V$ a vector space of dimension n.

This can be thought of as the moduli space of arrangements of n points on X, with subtleties when the points coincide. Grothendieck showed, through a much more general result, that this space is in fact a scheme.

Example 2.2. When $\dim X = 1$ we have $\mathrm{Hilb}_n(X) \cong S^n X$ as the set of arrangements of n points modulo the symmetric group acting on these points by interchange.

Example 2.3. When $X = \mathbb{C}$, since $\mathbb{C}[z]$ is a PID, we are interested in ideals I that are generated by a polynomial f of degree n. We are then looking all possible spaces $V = \mathbb{C}[z]/I \cong \mathbb{C}^n$.

Every such ideal gives rise to a map $\varphi : \mathbb{C}[z]/I \to \mathbb{C}$ mapping

$$z \mapsto B \in \mathrm{End}(V),\ 1 \mapsto v_0 \in V \tag{7}$$

That is, we can represent multiplication by z as an operator $B \circlearrowleft V$ satisfying $f(B) = 0$.

The eigenvalues of B are exactly the points corresponding geometrically to this ideal (through nullstellensatz). We care about the coordinate-independent data:

$$(B, v_0)/GL(V) \tag{8}$$

Note that because of the existence of a cyclic vector, this system has the stability property: any space containing v_0 and closed under the action of B is all of V.This constrains the Jordan form of B. We can go further and show that there is a 1-to-1 correspondence between such solutions and arrangements of n points.

In general the form of B will be

$$\begin{pmatrix} \lambda_1 & 1 & 0 & \dots & \dots \\ 0 & \lambda_1 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & \dots & \lambda_k & 1 \\ 0 & \dots & \dots & 0 & \lambda_k \end{pmatrix} \tag{9}$$

in jordan blocks, where $\lambda_i \neq \lambda_j$ unless $i = j$. Then we can form a map from this data into $S^n\mathbb{C}$ by $B \to \sum_{\lambda \in \mathbb{C}} [\lambda]\lambda$, where $[\lambda]$ stands for the multiplicity. This is called the **Hilbert-Chow morphism**. For the case of $\mathbb{C}$, it is an isomorphism, as B can be recovered from the n points on $\mathbb{C}$

We will be more interested in the case $X = \mathbb{C}^2$, the 2-dimensional complex space.

Example 2.4. As before, an element in the Hilbert scheme would correspond to an ideal $I \in \mathbb{C}[x, y]$ s.t. $V := \dim \mathbb{C}[x, y]/I = n$. Consider the map

$$\varphi : \mathbb{C}[x, y] \to V \ s.t.\ x \mapsto B_1,\ y \mapsto B_2,\ 1 \mapsto v_0 = I(1). \tag{10}$$

Then clearly we have $[B_1, B_2] = 0$, so we can write the B_k simultaneously in upper-trianglar form as:

$$B_1 = \begin{pmatrix} \lambda_1 & \dots & \dots & \dots \\ 0 & \lambda_2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{pmatrix},\ B_2 = \begin{pmatrix} \mu_1 & \dots & \dots & \dots \\ 0 & \mu_2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \mu_n \end{pmatrix} \tag{11}$$

Moreover we identify the vector v_0 with the embedding operator $I : \mathbb{C} \to V$ from a 1-dimensional space into V, so $v_0 = I(1)$.

We further have the Hilbert-Chow morphism to $S^n\mathbb{C}^2$:

$$\pi : (B_1, B_2, I) \to \{(\lambda_i, \mu_i)\}_{i=1}^n, \tag{12}$$

but now we no-longer preserve all information about the matrices, as the upper-triangular structure is unknown given the projection point.

Note that the orbit of v_0 under B_1, B_2 is all of V, so that any (B_1, B_2)-stable subspace of V containing $v_0 = \operatorname{im} I$ is all of V. This is exactly the stability condition from before. In fact, aside from $[B_1, B_2] = 0$ instead of $[B_1, B_2] + IJ = 0$, this is exactly $\tilde{\mathcal{M}}(1, n)$. In fact, it is exactly this space:

Theorem 2.5. *We have an isomorphism of smooth spaces:*

$$\tilde{\mathcal{M}}(1,n) \cong \mathrm{Hilb}_n(\mathbb{C}^2) \tag{13}$$

To prove this theorem, it is enough to just show $J = 0$, and we'll be done.

Proposition 2.6. *Consider the resolved space $\tilde{\mathcal{M}}(1,k)$ from before (i.e. only stability), then any solution of ADHM has $J = 0$.*

Proof. Because stability implies that $\mathbb{C}[B_1, B_2]I = V$, it's enough to show $Jp(B_1, B_2)I = 0$ for any monomial. We do this by induction on degree. For degree 0 we have $JI = \mathrm{Tr}(JI) = \mathrm{Tr}(IJ) = -\mathrm{Tr}([B_1, B_2]) = 0$.

For higher degree, we can use $[B_1, B_2] = -JI$ to commute B_1, B_2 across one another to get

$$\begin{aligned} JB_{\alpha_1} \dots B_2B_1 \dots B_{\alpha_m}I &= JB_{\alpha_1} \dots IJ \dots B_{\alpha_m}I + JB_{\alpha_1} \dots B_1B_2 \dots B_{\alpha_m}I \\ &= JB_{\alpha_1} \dots B_1B_2 \dots B_{\alpha_m}I \end{aligned} \tag{14}$$

so we can reduce this to $JB_1^aB_2^bI$ and then use trace properties to get zero. □

This proves the theorem.

3 Homology Theory of $\mathrm{Hilb}_n(\mathbb{C}^2)$

For a closed, oriented manifold X of dimension n, we have Poincare duality

$$H_i(X) \cong H^{n-i}(X) \tag{15}$$

When the manifold is not compact, we must pair the cohomology of *compactly-supported forms* with the homology.

$$H_i(X) \cong H_c^{n-i}(X) \tag{16}$$

and similarly, we define the **Borel-Moore homology** of locally finite chains by $H_i^{lf} \cong H_c^{n-i}(X)$

Definition 3.1. The Borel-Moore (locally finite) homology H^{lf} is equivalent to the relative homology:

1. $H_i^{lf}(X) := H_i(X \cup \{\infty\}, \{\infty\})$, the one-point compactification of X

This definition immediately yields

Proposition 3.2. *$H_{2m}^{lf}(\mathbb{C}^m) = \mathbb{Z}$ and otherwise is equal to zero.*

by recognizing $\mathbb{C}^m \cup \{\infty\}$ as S^{2m}. Further:

Proposition 3.3. *For a space X that is a disjoint union of a finite number of open subspaces X_α: $H_i^{lf}(X) = \bigoplus_\alpha H_i^{lf}(X_\alpha)$*

Proposition 3.4. *The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ is a disjoint union of open spaces C_μ indexed by the partitions of n. Moreover, $C_\mu \cong \mathbb{C}^{n+\ell(\mu)}$, where ℓ is the number of parts in the partition μ.*

Sketch. Consider the action of the torus $T^2 = \langle (t,q) \rangle \circlearrowleft \mathrm{Hilb}_n(\mathbb{C}^2)$ by its action on an ideal element $((t,q)f)(x,y) = f(t^{-1}x, q^{-1}y)$. The fixed points of this are the ideals

$$I_\mu = (x^a y^b : (a,b) \notin \mu) \tag{17}$$

Where μ is viewed as its corresponding Young tableau on the plane $\mathbb{N}^2$. We then have corresponding complement ideals:

$$B_\mu = (x^a y^b : (a,b) \in \mu) \tag{18}$$

This gives rise to open sets U_μ that cover $\mathrm{Hilb}_n(\mathbb{C}^2)$ defined by

$$U_\mu := \{I \in \mathrm{Hilb}_n(\mathbb{C}^2) : B_\mu spans \mathbb{C}[x,y]/I\} \tag{19}$$

and the closed sub-cells of these open sets are defined by:

$$C_\mu := \{I \in \mathrm{Hilb}_n(\mathbb{C}^2) : \lim_{t\to 0}\lim_{q\to 0}(t,q)I = I_\mu\} \tag{20}$$

This limiting process picks out exactly the greatest monomials with nonzero coefficients from all the polynomials of the ideal. □

Remark. This type of idea, of decomposing a space into its different orbits, is universally used in studying not just Hilbert schemes but also projective spaces, Grassmannians, flag varieties, and other such spaces.

Corollary 3.5.

$$H_*^{lf}(\mathrm{Hilb}_n(\mathbb{C}^2)) = \bigoplus_\mu [\mathbb{C}_\mu] = \bigoplus_\mu \mathbb{C} \tag{21}$$

where $[C_\mu]$ denotes the fundamental Borel-Moore class (the top homology ring), which in this case is exactly the only nonzero one.

Observation 3.6. *The dimension of this space $H_*^{lf}(\mathrm{Hilb}_n(\mathbb{C}^2))$ is $p(n)$, the number of partitions of n.*

Note however, that unlike $S^n(\mathbb{C}^2)$, which can also be covered by cells indexed by the partition type, μ, $\mathrm{Hilb}_n(\mathbb{C}^2)$ has the property that it has **constant rank** for its tangent space, equal to $2n$.

Proposition 3.7. *We have a graded dimension for the algebra of homology rings for all Hilbert schemes of points over $\mathbb{C}^2$ of:*

$$\bigoplus_n H_*^{lf}(\mathrm{Hilb}_n(\mathbb{C}^2)) = \sum_n p(n)q^n = \prod_m \frac{1}{1-q^m} \tag{22}$$

This is exactly the graded dimension of the space of symmetric polynomials $S(p_1, p_2, \dots)$ where $\deg p_i = i$. This is exactly a representation of the Heisenberg algebra, suggesting a connection between these two objects beyond just an isomorphism as graded vector spaces.

Optional:

Theorem 3.8 (From Fulton). *The Poincare polynomial*

$$P_t^{lf}(X) := \sum_{n\geq 0} t^n \dim H_n^{lf}(X) \tag{23}$$

is equal to

$$\sum_{\mu} t^{2n+2\ell(\mu)} \tag{24}$$

for $X = \mathrm{Hilb}_n(\mathbb{C}^2)$

Corollary 3.9. *We have an identity for the polynomial*

$$\sum_{n=1}^{\infty} q^n P_t^{lf}(H_n) = \prod_{m=1}^{\infty} \frac{1}{1-t^{2m+2}q^m} \tag{25}$$

Theorem 3.10. *The homology group* $H_*[\mathrm{Hilb_n}(\mathbb{C}^2)]$ *vanishes on odd degrees and otherwise is torsion free, with betti number;*

$$b_{2i}(\mathrm{Hilb}_n(\mathbb{C}^2)) = p(n, n-i) \tag{26}$$

where $p(n,a)$ *is the number of partitions of* n *into* a *parts.*

Corollary 3.11. *The Hilbert polynomial*

$$P_t(X) := \sum_{n\geq 0} t^n b_n(X) \tag{27}$$

is equal to

$$\sum_{\mu} t^{2n-2\ell(\mu)} \tag{28}$$

Corollary 3.12. *We have an identity for the polynomial*

$$\sum_{n=1}^{\infty} q^n P_t(\mathrm{Hib}_n(\mathbb{C}^2)) = \prod_{m=1}^{\infty} \frac{1}{1-t^{2m-2}q^m} \tag{29}$$

We have constructed H_*^{lf} and H_*, and in fact all the nontrivial topological information is contained in the zero fiber Z_n of $\pi : \mathrm{Hilb}_n(\mathbb{C}^2) \to S^n(\mathbb{C}^2)$.

4 Hilbert Schemes and the Heisenberg Algebra

The Heisenberg Lie Algebra $\mathfrak{s}$ is defined by generators p_i, q_i so that:

$$[q_j, p_i] = c_j \delta_{ij}, \;\; c_j \in \mathbb{C}^\times \tag{30}$$

so that on $S(p_1, p_2, \dots)$ the p_j act by multiplication and the q_j by derivation $c_j \frac{\partial}{\partial p_j}$. We can pair $\bigoplus_j p_j$ and $\bigoplus_i q_i$ as dual vector spaces by:

$$\langle q_j, p_i \rangle = c_j \delta_{ij} \tag{31}$$

and with this can define the dual space

$$S^* = S(q_1, q_2 \dots) \tag{32}$$

with $\deg q_j = j$ as before, and satisfying:

$$\langle q_1^{n_1} q_2^{n_2} \dots, p_1^{m_1} p_2^{m_2} \rangle = n_1! c_1^n \delta_{n_1,m_1} \; n_2! c_2^n \delta_{n_2,m_2} \dots . \tag{33}$$

Of course we have multiplication on S and comultiplication on S^*. In fact, since the commutation relations give a bilinear pairing:

$$S^* \otimes S \to S \tag{34}$$

by interpreting c_i as an element in S, we get

$$1 \otimes p_j = p_j, \; q_i \otimes p_j = c_i \delta_{ij}, \tag{35}$$

and this gives **comultiplication** of the Heisenberg algebra:

$$\Delta : p_i \mapsto 1 \otimes p_i + p_i \otimes 1 \tag{36}$$

together with our previous multiplication. This makes this into a **Hopf algebra**

5 Geometric Realization of the Heisenberg Algebra

Going back to topology: we note that on any Homology ring H_*X we have that a continuous map $\varphi : X \to Y$ between manifolds X and Y descends to a map between chains on X to chains on Y, and thus gives a map $\varphi_* : H_*X \to H_*Y$ called the **pushforward** on homology rings, just by considering the homology of the mapped chains.

Similarly, any map $\varphi : X \to Y$ induces a contravariant map $\varphi^* \omega = \omega \circ \varphi$ on forms over Y called the **pullback**.

Now assume that $f : X \to Y$ is a **proper map** (i.e.) the inverse image of any compact subset is compact. Then the pushforward descends to the locally-finite homology rings:

$$f_* : H_*^{lf}(X) \to H_*^{lf}(Y) \tag{37}$$

By defining $\hat{f} : \hat{X} \to \hat{Y}$ on the 1-point compactifications of X, Y s.t. $\hat{f}(\infty) = \infty$. Then by properness, f is continuous on this space, so we can define the pushforward $\hat{f}_*$ and pass this to the relative homology to get f_*.

From this, we can associate Hopf algebra structure to

$$\mathbb{H}^{lf} = \bigoplus_n \mathrm{Hilb}_n(\mathbb{C}^2) \tag{38}$$

by considering a pair of three nested Hilbert schemes[1]:

$$\begin{aligned} H_{m+n \geq n} :=& \{V \in H_{m+n}, V'' \in H_n, V' = V/V'' \in H_m\} \\ &\subset H_m \times H_n \times H_{m+n} \end{aligned} \tag{39}$$

[1] There is an issue when points collide that V' is not guaranteed to have a well-defined vector v_0. This can be resolved by looking at the open sets of distinct points first, and then taking the closure of this constructed space.

where for shorthand H_n denotes $\mathrm{Hilb}_n(\mathbb{C}^2)$. We then have projections onto the components as the following diagram illustrates:

$$\begin{tikzcd}
 & H_{m+n\geq n} \arrow[dl, "p'\times p''"'] \arrow[dr, "p"] & \\
H_n \times H_m & & H_{n+m}
\end{tikzcd}$$

The following lemma can be proven by checking the preimages of points in $S_\mu\mathbb{C}^2$ and noting that the fibers of the Hilbert-Chow morphism $\pi^{-1}(\mu_i[x_i])$ are compact:

Lemma 5.1. *These projections p', p'', p are proper.*

So the projections therefore give rise to pushforwards on the Borel-Moore homology. Moreover, by this fact together with Poincare duality, we can identify elements in the homology rings with elements in the cohomology rings that we can *pull back*. Thus, we have maps:

$$\mu : H_*^{lf}(H_n) \otimes H_*^{lf}(H_m) \to H_*^{lf}(H_{n+m}) \tag{40}$$

given exactly by mapping homology elements

$$c_1 \otimes c_2 \mapsto p_*((p' \times p'')^*(c_1 \otimes c_2)) \tag{41}$$

This mirrors how multiplication in the Heisenberg algebra respects the degree grading. Similarly we have

$$\Delta : H_*^{lf}(H_{m+n}) \to H_*^{lf}(H_n) \otimes H_*^{lf}(H_m \tag{42}$$

by

$$c \mapsto (p' \times p'')_*(p^*(c)) \tag{43}$$

Theorem 5.2. *The operations μ, Δ defined as above give $\mathbb{H}^{lf}$ the structure of a graded Hopf algebra.*

Proof. We have already seen that these operations respect the grading. The associativity and coassicativity conditions follow from the functoriality of pushforward and pullback in the following diagrams: $\square$

$$\begin{tikzcd}
 & & H_{m+n+r\geq n+r\geq r} \arrow[dl, "p'\times p''"'] \arrow[dr, "p"] & & \\
 & H_{m+n\geq n} \times H_r \arrow[dl, "p'\times p''\times id"'] \arrow[dr, "p\times id"] & & H_{n+m+r\geq r} \arrow[dl, "p'\times p''"'] \arrow[dr, "p"] & \\
H_m \times H_n \times H_r & & H_{m+n} \times H_r & & H_{m+n+r}
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & H_{m+n+r\geq n+r\geq r} \arrow[dl, "p'\times p''"'] \arrow[dr, "p"] & & \\
 & H_m \times H_{n+r\geq r} \arrow[dl, "id\times p'\times p''"'] \arrow[dr, "id\times p"] & & H_{n+m+r\geq n+r} \arrow[dl, "p'\times p''"'] \arrow[dr, "p"] & \\
H_m \times H_n \times H_r & & H_m \times H_{n+r} & & H_{m+n+r}
\end{tikzcd}$$

We can in fact go further and define the fiber $F_\mu = \pi^{-1}S_\mu(\mathbb{C}^2)$ of arrangements of n points on $\mathbb{C}$ of partition type μ. We have that $[F_\mu]$ is in fact well-defined and that

$$[F_\mu] \in H^{lf}_{2(n+\ell(\mu))}(\mathrm{Hilb}_n(\mathbb{C}^2)) \tag{44}$$

Theorem 5.3. *The $[F_\mu]$ form a basis for $H^{lf}_*(\mathrm{Hilb}_n(\mathbb{C}^2))$. Picking $[F_n] \in H^{lf}_*(\mathrm{Hilb}_n(\mathbb{C}^2))$ corresponding to the fiber class with all n points coincident gives a multiplication operator $P_n : H_m \to H_{n+m}$ corresponding to p_m.*

By Poincare duality, there is the intersection pairing

$$\cap : H_* \times H^{lf}_* \to \mathbb{C} \tag{45}$$

so that H_* can in fact be made to correspond to the dual space S^* of derivations. We can obtain fundamental classes in the regular homology $[E_\mu]$ in a similar way, and have them form a basis for $H_*\mathrm{Hilb}_n(\mathbb{C}^2)$. The remaining relations for the Heisenberg algebra can be obtained through careful calculation.

References

[1] Igor B. Frenkel *Compiled Lecture Notes #15, on Hilbert Schemes and the Heisenberg Algebra* 2016.

[2] Igor B. Frenkel and Marcos Jardim *Complex ADHM equations, sheaves on* $\mathbb{P}^3$ 2008.

[3] Alexander Kirillov Jr. *Quiver Varieties and Quiver Representations*: 2016: Graduate Studies in Mathematics, AMS.

[4] Hiraku Nakajima *Hilbert Schemes of Points on Surfaces* 1991: University Lecture Series, AMS.

[5] Hiraku Nakajima *Heisenberg Algebra and Hilbert Schemes of Points on Projective Surfaces* 1995.