

Hong Liu lectures on:

[Holographic Duality
 &
Applications]

Lecture Notes

by

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Table of Contents

Chapter 1: Black Holes & the Holographic Principle · · · 1

1.1 General Remarks on Gravity · · · · ·	9
1.2 Classical Black Hole Gravity · · · · ·	17
1.3 Black Hole Temperature · · · · ·	17
1.3.1 Hawking & Unruh Temperatures from Black Hole Analytic Continuation	
1.3.2 Unruh Temperature from Entanglement · · · · ·	22
1.3.3 Free Field Theories Derivation · · · · ·	35
1.4 Black Hole Thermodynamics · · · · ·	41
1.5 Quantum Nature of Black Holes and the Holographic Principle · · · · ·	45

Chapter 2: Matrices & Strings · · · · · 57

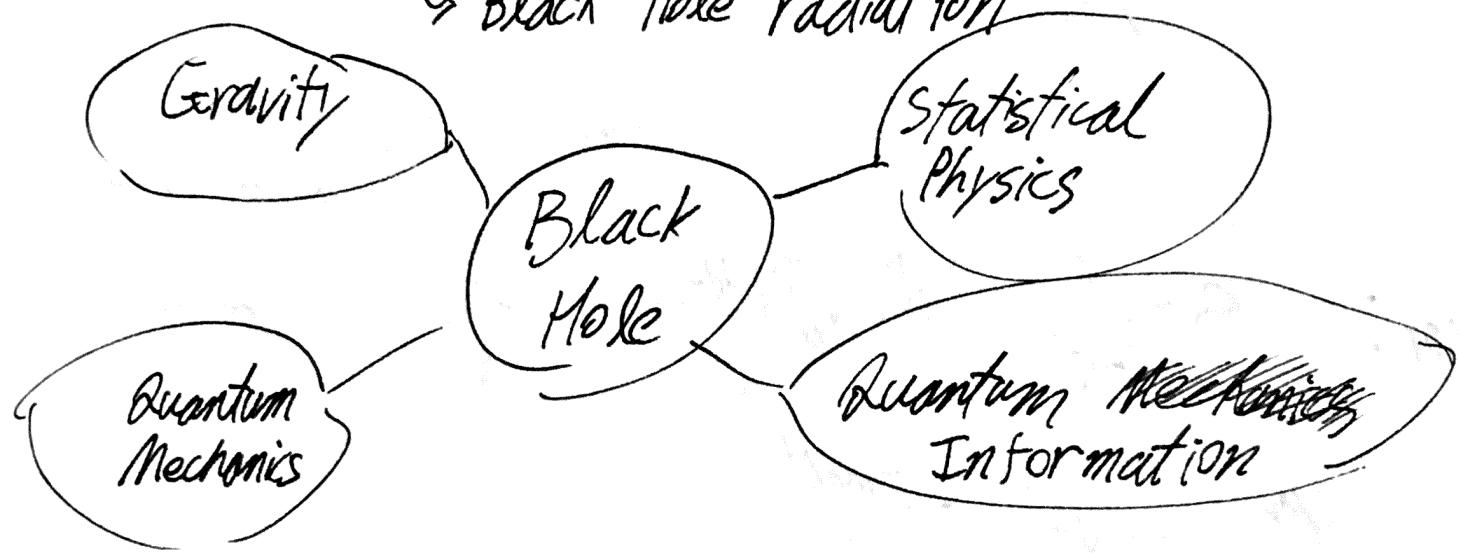
2.1 Path Integrals of Strings · · · · ·	57
2.2 Matrix Integrals in the Large-N Limit · · · · ·	65
2.3 Strings & Matrices · · · · ·	75
2.4 String Description of a Gauge Theory · · · · ·	85
2.5 Anti De-Sitter Spacetime · · · · ·	105

Chapter 3: Holographic Duality · · · · · 117

3.1 General Aspects · · · · ·	117
3.1.1 IR/UV Connection · · · · ·	121
3.1.2 Matching of the Spectrum · · · · ·	137
3.1.3 Euclidean Correlation Functions · · · · ·	147
3.1.4 Wilson Loops · · · · ·	153
3.2 Finite Temperature · · · · ·	161
3.3 Holographic Entanglement Entropy · · · · ·	

Chapter 1: Black Holes & the Holographic Principle

- BHs:
- 1) Key object - astrophysically ubiquitous
 - 2) Quantum Matter around BH
 - Hawking's 1975 paper
 - Black Hole radiation



→ Black holes bring Quantum Gravity to a Macroscopic level.

1.1 General Remarks on Gravity

all other interactions: probed to 10^{-33} cm (Large Hadron Collider)
gravity: only 10^{-2} cm

General Relativity: "gravity = spacetime"

Quantum Gravity: " = quantum spacetime"

Question: What is the relationship between quantum gravitational effects and the nature of spacetime?

Answer: A) Einstein Gravity & Gravitons

line element: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Einstein's Equations: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) = 8\pi G_N T_{\mu\nu} \quad (*)$

↑
cosmological const.
↑
matter
(i.e. stress-energy tensor)

Action: $S = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} (R - 2\Lambda) + S_{\text{matter}} \quad (**)$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}$$

take $\Lambda = 0, T_{\mu\nu} = 0$

simplest solution to $(*) \rightarrow g_{\mu\nu} = \eta_{\mu\nu} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

"Weak gravity" $g_{\mu\nu} = \eta_{\mu\nu} + \chi h_{\mu\nu}^{(1)}$ with $\chi^2 = 8\pi G_N$

putting (1) into (**), we get

$$S = \int d^d x \left[L_2 + \chi L_3 + \chi^2 L_4 + \dots \right] + \frac{\chi}{2} h_{\mu\nu} T^{\mu\nu} + O(\chi^3)$$

canonically normalized

starts at $O(\chi^2) \approx O(h^2)$

since $g_{\mu\nu}$ solves E.O.M.

cancels $16\pi G_N$

EDM from \mathcal{L}_2 give plane wave solutions

These are what we call "gravitational waves"

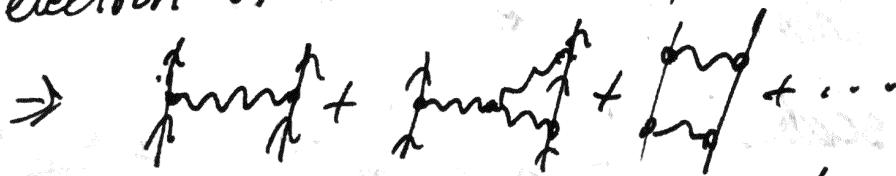
\mathcal{L}_2 is quadratic in $h \Rightarrow$ free field theory for h

\hookrightarrow quantize $\mathcal{L}_2 \Rightarrow$ spin-2 massless particle
"graviton"

$\rightsquigarrow \mathcal{L}_3, \mathcal{L}_4, \dots \rightarrow$ self-interaction of the graviton
gravitational interaction of matter

is by exchange of an $h_{\mu\nu}$

e.g. electron as matter has $T_{\mu\nu} \sim \bar{\psi}\psi$

\Rightarrow 

Treated as QFT, S is non-renormalizable.

\rightsquigarrow so we can treat $S[h]$ as an effective field theory
not fundamental!

B) Important Scales of Gravity

► Planck Scales: $\hbar, G_N, c = 1 \Rightarrow M_p = \sqrt{\frac{\hbar}{G_N}} \approx 1.2 \cdot 10^{19} \text{ GeV}$

$$l_p = \frac{\hbar}{M_p} = \sqrt{\hbar G_N} = 1.6 \cdot 10^{-33} \text{ cm}$$

For reference: top quark mass: $\approx 10^{17} M_p$ $t_p = l_p \approx 5.4 \cdot 10^{-44} \text{ sec}$
electron mass: $\approx 10^{-23} M_p$

Largeness of $M_p \leftrightarrow$ weakness of gravity at microscopic scales

\Rightarrow consider two particles of mass m brought to nearest possible distance

$$\rightsquigarrow \lambda_C := \frac{V_g(r_c)}{m} = \frac{1}{m} \frac{G_N m^2}{r_c} = \frac{G_N m^2 - m_e^2}{\hbar M_p^2} \quad r_c = \hbar/m \quad \text{"Compton wavelength"}$$

We see $\lambda_G = \frac{m^2}{M_p^2} = \frac{k_p^2}{r_c^2}$. For $m \ll M_p$, $\lambda_G \ll 1$ e.g. $c \rightarrow \lambda_G \sim 10^{-10}$

For $m \sim M_p$, $\lambda_G \sim O(1) \Rightarrow$ Quantum Gravity
Relativistic calculation: $\lambda_G \sim \frac{E^2}{M_p^2}$, E c.o.m. energy

If we take $m \gg M_p$, does $\lambda_G \gg 1$? No!

Different question: Take point particle of mass m .

At what distance r_s from it does classical gravity become strong?

$$\text{probe } m' \rightarrow \frac{\frac{Gmm'}{r_s}}{m'} \sim 1 \Rightarrow r_s = G_N m$$

Remarks: 1) In Newtonian gravity, at r_s escape velocity \sim

2) In GR, $r_s \sim$ Schwarzschild radius of Black Hole

$\Rightarrow r_s \sim$ minimal distance we can probe an obj. in classical gravity.

Two important scales: $r_c \sim k/m \Rightarrow \frac{r_s}{r_c} = \frac{G_N m^2}{k} = \left(\frac{m}{M_p}\right)^2$

1) $m \ll M_p \Rightarrow r_s \ll r_c$ so Compton wavelength outside r_s
 \Rightarrow gravitation is weak, negligible [r_s not important
quantum effects dominate]

2) $m \sim M_p$, $r_s \sim r_c$, $\lambda_G \sim 1$ Quantum Gravity becomes important

3) $m \gg M_p$, $r_s \gg r_c \Rightarrow r_c$ not relevant \Rightarrow classical gravity dominates

The relationship between Black Holes and quantum gravity, however, affects much more than Planck scale physics.

Last time.

$$\lambda_G \sim \frac{G_N E^2}{\hbar} \sim \frac{E^2}{M_p^2}$$

$$\propto h_{\mu\nu} T^{\mu\nu}, \kappa^2 = 8\pi G_N$$

REEMBEDDING

$$\frac{r_s}{r_c} \sim \frac{m^2}{M_p^2}$$

Corollary: l_p is minimal localization length

$$\text{non-grav: } \underline{s}_L \sim \underline{E}$$

$$\text{with gravity: } \underline{E} \sim M_p$$

$$r_s \sim G_N E \sim r_c \sim l_p$$

$$E \gg M_p$$

$$r_s \gg r_c$$

$$S_p \sim \frac{\hbar}{S_L} \Rightarrow S_L > G_S p \sim \frac{G \hbar}{S_X}$$

$$\Rightarrow S_X > \sqrt{G \hbar} \sim l_p$$

$E \ll M_p$, ignore: (1) grav. interaction

(2) fluctuations of $g_{\mu\nu}$

(3)

\Rightarrow QFT in rigid spacetime
(can be curved)
i.e. on earth.

~~Wick~~ Mathematical Treatment:

E fixed, t fixed, $G_N \rightarrow 0$
($\ell_p \approx 0$, $M_p \rightarrow \infty$)

C low energy expansion in G_N

$$\mathcal{Z} = \int Dg D\chi e^{iS[g, \chi]}$$

$$S = \frac{1}{16\pi G_N} S_{\text{grav}}[g] + \frac{1}{\lambda} S_m[g, \chi]$$

λ : matter coupling

$$\lambda \gg G_N$$

$G_N \rightarrow 0 \Rightarrow$ saddle point: $\delta S_{\text{grav}}[g] = 0 \Rightarrow g_{\text{classical}}$

$$\text{Expand } g = g_{\text{classical}} + \chi h$$

$$\Rightarrow S = \frac{1}{16\pi G_N} S_{\text{grav}}[g_c] + \underbrace{\frac{1}{\lambda} S_m[\chi, g_c]}_{\text{RFT in curved spacetime}} + S[h] + \dots + \chi h_{\mu\nu}$$

Small G_N expansion breaks down at $\frac{E^2}{M_p^2} \sim O(1)$.

For a sphere of radius L , p is quantized as $\frac{1}{L}$

$$\Rightarrow E^2 \sim p^2 \sim \frac{1}{L^2} \sim R$$

~~REVIEW~~

67

$$\frac{G_N E^2}{\pi}$$

D. gravity in general dimensions

$$S_{\text{grav}} = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g} (R - 2\Lambda)$$

$$[G_d] = \frac{L^{d-1}}{MT^2} \Rightarrow M_{pd}^{d-2} = \frac{\hbar^{d-3}}{G_d}, \quad l_{pd}^{d-2} = \hbar G_d$$

$$\lambda_G \sim \frac{G_d \hbar^{d-2}}{\hbar^{d-3}} \sim \frac{E^{d-2}}{M_{pd}^{d-2}}$$

$$r_s \sim (G_N m)^{\frac{1}{d-3}}$$

consider $M_d = M_D \times Y$

M_D non-compact, D-dimensional

Y compact, d-D

suppose Y is too small to be detected.

Q.R x S' The effective Newton constant G_d for an observer is not the same as the fundamental

$$\frac{1}{G_d} = \frac{1}{G_D} V_Y \leftarrow \text{volume of } Y$$

$$l_{PD}^{D-2} = \frac{l_{pd}^{d-2}}{L^{d-D}}, \quad \text{expect } L > l_{pd}$$

$\Rightarrow l_{PD} < l_{pd}$

Einstein gravity as E.P.T.

gravity tested to 10^{-2} cm

we're going down to 10^{-33} cm

1) Extra dimensions

will see d-dim gravity
before reaching l_p

2) string theory

l_s string length



3) Suppose new physics appears at some scale $\ell \sim \frac{1}{M}$

$$S = \frac{1}{16\pi G_N} \int d^D x \sqrt{g} [R - 2\Lambda + \frac{\alpha_1}{\ell^2} R^2 + \frac{\alpha_2}{\ell^2} R_{\mu\nu} R^{\mu\nu} + \dots]$$

1.2 Classical BH Geometry I

consider a spherically symmetric, electrically neutral object of mass M

The Schwarzschild solution (4D) can be analytically found to be:

$$ds^2 = -F dt^2 + \frac{1}{F} dr^2 + r^2 d\Omega^2$$

$$= g_{\mu\nu} dx^\mu dx^\nu$$

with $F = 1 - \frac{2G_N M}{r} = 1 - \frac{r_s}{r}$

$$r_s := 2G_N M$$

most important features:

- 1) $r \rightarrow \infty, F \rightarrow 1, g_{\mu\nu} \rightarrow \eta_{\mu\nu}$
- 2) $r = r_s, F \rightarrow 0, g_{tt} = 0$
 $g_{rr} = \infty$

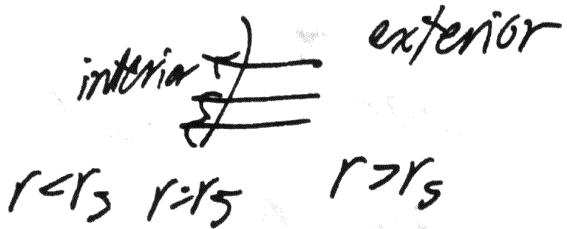
will see + (Schwarzschild time) becomes singular at $r = r_s$

$r = \text{const} > r_s \Rightarrow$ time-like hypersurface

$r = \text{const} < r_s \Rightarrow$ space-like hypersurface

$r = r_s \Rightarrow$ null hypersurface

3) $r=r_s$: event horizon



4) $r=r_s$: hypersurface of infinite redshift

consider an observer ∂_h at $r=r_h \approx r_s$

proper time for ∂_∞ at $r=\infty$
proper time for ∂_h

$$dt_h = f^{1/2} dt = \left(1 - \frac{r_s}{r_h}\right)^{1/2} dt$$

consider a physical process at $r=r_h$
with local proper energy ϵ

$$\partial_\infty \text{ sees energy } E_\infty = \epsilon \left(1 - \frac{r_s}{r_h}\right)^{1/2}$$

as $r_h \rightarrow r_s$, $E_\infty \rightarrow 0$ "infinite redshift"

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2$$

$$f = 1 - \frac{r_s}{r}, \quad r_s = 2GM$$

causal structure & Rindler spacetime

$$\text{consider } r \geq r_s \quad \frac{r-r_s}{r_s} \ll 1$$

proper distance ρ from $r=r_s$

$$\text{s.t. } d\rho^2 = \frac{dr^2}{f} \Rightarrow d\rho = \frac{dr}{\sqrt{f}}$$

$$f(r) = f(r_s) + f'(r_s) \frac{(r-r_s)}{\frac{dr}{f}} + \dots$$

$$\Rightarrow \rho = \frac{2}{\sqrt{f'(r_s)}} \sqrt{r-r_s}$$

$$\Rightarrow f(r) = \left[\frac{1}{2} \left(\frac{2}{\sqrt{f'(r_s)}} \right)^2 \rho^2 \right] = k\rho^2$$

$$\Rightarrow ds^2 = -k^2 \rho^2 dt^2 + d\rho^2 + r_s^2 d\Omega^2$$

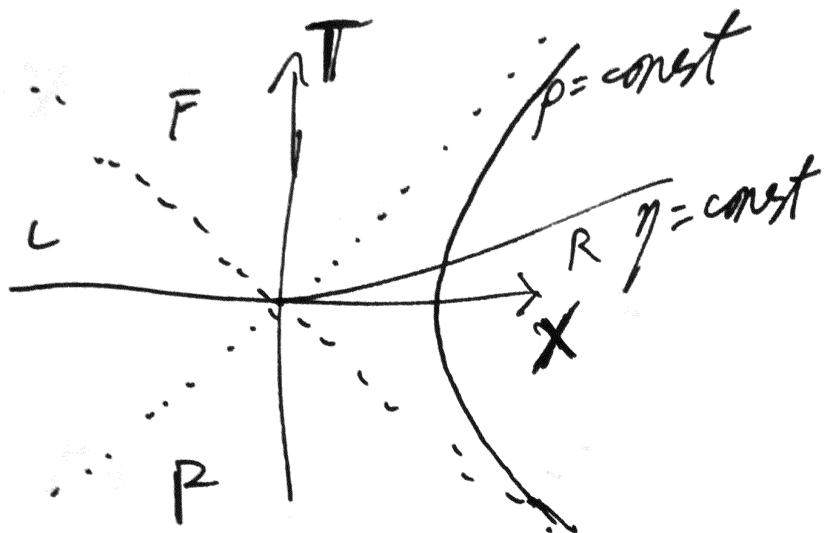
$$= -\underbrace{\rho^2 d\eta^2}_{\text{Mink}_2} + d\rho^2 + r_s^2 d\Omega^2$$

$$\eta = kt$$

(Rindler spacetime)

$$ds^2 = -dt^2 + d\Omega^2 = -\rho^2 d\eta^2 + d\rho^2$$

$$X = \rho \cosh \eta \quad T = \rho \sinh \eta$$



$$P^2 = X^2 - T^2$$

$$\tanh \eta = \frac{T}{X}$$

$$X = T = 0 \Rightarrow \begin{cases} P \rightarrow 0 \\ \eta \text{ finite} \end{cases}$$

At

$$X = T \Rightarrow \begin{cases} P \rightarrow 0 \\ \eta \rightarrow +\infty \text{ s.t. } Pe^\eta \text{ finite} \end{cases}$$

$$X = -T \Rightarrow \begin{cases} P \rightarrow 0 \\ \eta \rightarrow -\infty \text{ s.t. } Pe^{-\eta} \text{ finite} \end{cases}$$

Rindler observers: $p = \text{const}$ ($\Rightarrow r = \text{const}$)

$$\Rightarrow d\eta/dt = c_{\text{prop}} = 1/p$$

Note: No signal can propagate from F to R

(can only go in)

$X = T$: future horizon

(can only come out)

$X = -T$: past horizon

(can only come out)

$$1) r = r_s \leftrightarrow P = 0 \leftrightarrow X = \pm T$$

null hypersurface

(P, η) singular at $P = 0 \Leftrightarrow (T, r)$ singular at $r = r_s$

2) $r = \text{const}$ observer
 $\Leftrightarrow p = \text{const}$ Rindler observer
 their accelerations agree

3) free-fall observer in BH \Leftrightarrow inertial observer in Rindler Minkowski

4) Using (T, X) , we can extend the black hole geometry from $r \gg r_s$ to four regions with the near-horizon metric

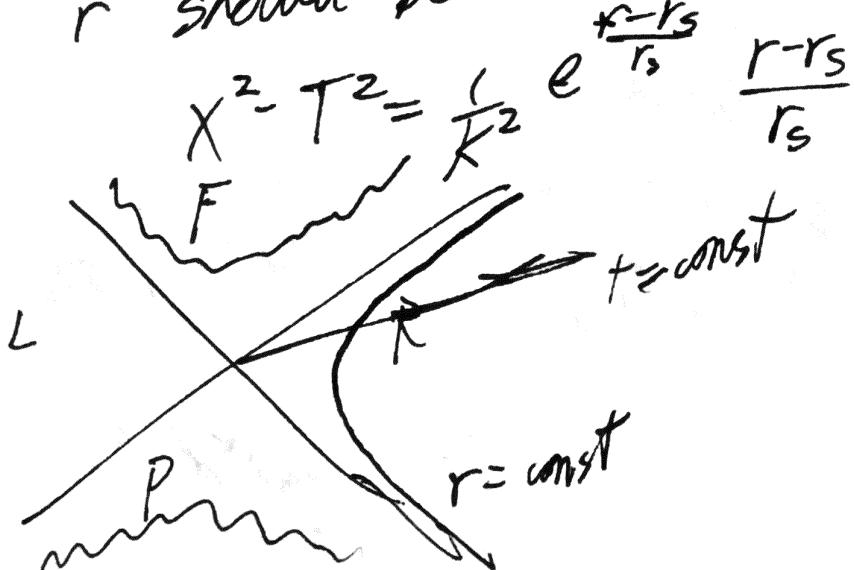
$$ds^2 = -dT^2 + dX^2 + r_s^2 d\Omega^2$$

T, X as coordinate transform of (r, t) and then extend them to full spacetime

(derive) $ds^2 = g(r)(-dT^2 + dX^2) + r_s^2 d\Omega^2$

$$g(r) = \frac{r_s}{r} e^{-\frac{r-r_s}{r_s}}$$

r should be considered as a function of (x, T)



a) $g(r_s) = 1$
 $X^2 - T^2 = 0$
 $(r = r_s)$

b) singularity at $r = 0$
 $\Leftrightarrow T^2 - X^2 = \frac{1}{r^2} > 0 / \sqrt{13}$

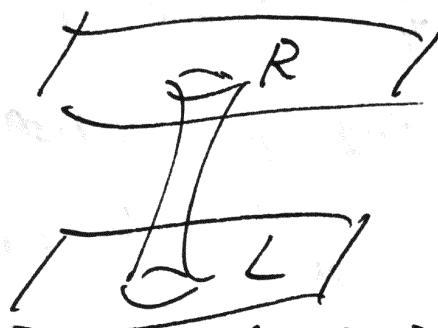
c) symmetries

(i) $T \leftrightarrow -T \quad X \leftrightarrow -X$

(ii) boost in $(T, X) \Leftrightarrow T \rightarrow T + \text{const}$

d) L is a mirror of R w/ another asymptotic flat region

e) $T=0$ slice



f) F: interior of BH
(future horizon)

g) P: white hole
(past horizon)

h) LP not present in
collapse of a star

wormhole (E-R)
non-traversable

A digression: Penrose diagrams

Procedure: i. choose a metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

x^μ covers full spacetime

2. find $x^\mu = x^\mu(y^\alpha)$ s.t. y^α has finite range

3. construct a new metric

$$ds'^2 = \Omega^2(r) ds^2 = \tilde{g}_{\alpha\beta} dy^\alpha dy^\beta$$

TU

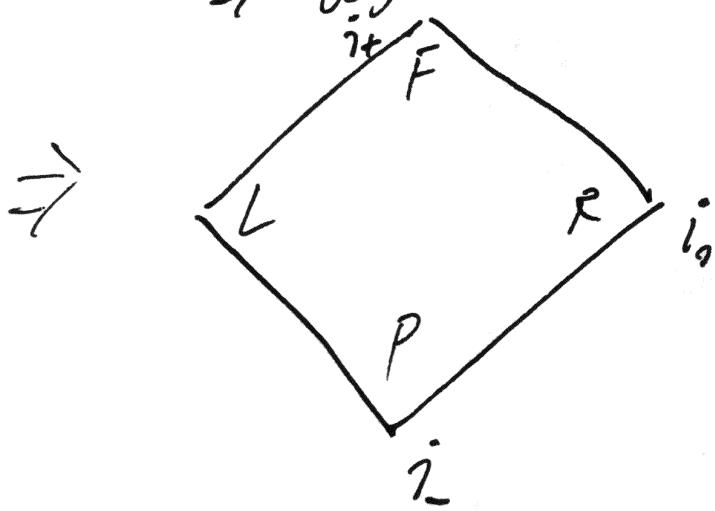
so that the causal structure of \tilde{g} is known

$$\text{Mink}_2: ds^2 = -dT^2 + dX^2 \\ = -dUdV \quad u = T-X \\ v = T+X$$

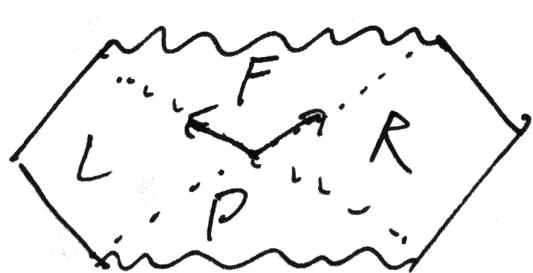
$$x_1: U = \tan u \Rightarrow u, v \in [-\pi/2, \pi/2]$$

$$V = \tan v \quad ds^2 = -\frac{1}{\cos^2 u \cos^2 v} du dv$$

$$\Rightarrow \tilde{ds}^2 = -du dv$$

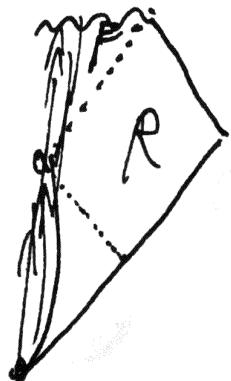


Black hole:



Kruskal coordinates
(X, T)

Stellar collapse:



Formulas we will use going forward

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2$$
$$= -k^2 \rho^2 dt^2 + dp^2 + r_s^2 d\Omega^2, \dots \leftarrow \text{near horizon}$$

$$k = \frac{1}{2} f'(r_s) = \frac{1}{2r_s} = \frac{1}{4GM}$$

1.3 Black Hole temperature

1975: Hawking

1976: Unruh

Bisognano-Wichmann

Both phenomena at level of leading order
in low-energy approx

QFT in a rigid curved spacetime

This effect is universal insofar as it would apply
to any QFT regardless of interactions
of matter content.

$$\text{e.g. (*) } S = - \int d^4x \sqrt{g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right]$$

1.3.1 Hawking & Unruh temperatures from Euclidean
analytic continuation

$$S_\beta = \frac{1}{Z_\beta} e^{-\beta H}$$

$$\text{with } Z_\beta = \text{Tr } e^{-\beta H} = \text{Tr} (e^{-iHt/\hbar})$$

$$t = -i\beta \hbar$$

$$ds^2 = -dt^2 + dx^2$$

$$t \rightarrow -it \quad \tau = \tau + \beta \hbar$$

$$ds_E^2 = dt^2 + dx^2 \quad (1)$$

Thermal equilibrium at $T = \frac{1}{\beta}$ described by path integrals in (1) with periodicity $\beta\tau$

(A) Hawking temp: take BFL metric
 $\rightarrow -\tau\tau$

$$\begin{aligned} ds_E^2 &= f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2 \\ &= k^2 \rho^2 dt^2 + d\rho^2 + r_s^2 d\Omega^2 \\ &= \underbrace{\rho^2 d\theta^2}_{\text{locally } R^2} + d\rho^2 + r_s^2 d\Omega^2 \end{aligned}$$

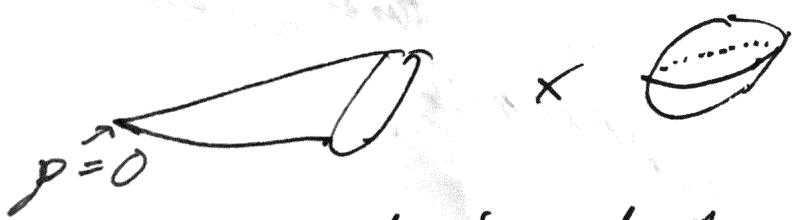
$\theta = kT$ is like
an angular wind

Global structure: depends on periodicity of θ

$$\theta \sim \theta + 2\pi \Rightarrow \text{globally } R^2$$



other periodicity:



\approx ALE?

gravitational
instantons

$\rho = 0 \Rightarrow$ conical singularity

smoothness of Euclidean geometry

$$\Rightarrow \tau \rightarrow \tau = \frac{2\pi}{k} \quad (\text{uniquely determined})$$

Different from on Mink.

$$R \times R^3 \rightarrow S^1 \times R^3$$

↑
any period
allowed

→ in a black hole geometry, quantum matter can be in equilibrium only at a single temperature $T_H = \frac{1}{\beta_H}$

$$\hbar \beta_H = \frac{2\pi}{\chi} \Rightarrow T_H = \frac{\hbar \chi}{2\pi} = \frac{\hbar}{8\pi G M}$$

Remarks:

1) T_H should be considered as temperature measured in ~~proper~~ units of proper time at $r = \infty$.

2) \Rightarrow BH must have temperature T_H

3) field theory on a cone

\Rightarrow observables can be singular at the singularity.

4) suppose $\tau \sim e^{\beta H}$ $\beta \neq \beta_H$

must be singular

screen difference



$$T \neq T_H$$

$P=0 \Leftrightarrow$ horizon

"Force" → the equilibrium

5) You can put any matter at any T outside the black hole (including nothing, $T=0$)

\rightsquigarrow non-eq. state

but euclidean A.C. you can only desc. the equilibrium state.

$$6) ds^2 = g(r) (-dT^2 + dX^2) + r^2 d\Omega_2^2$$

$$r = r(T^2 - X^2) \rightsquigarrow T_E^2 - X^2 = \\ T \rightarrow -iT_E$$

7) For a stationary observer at r

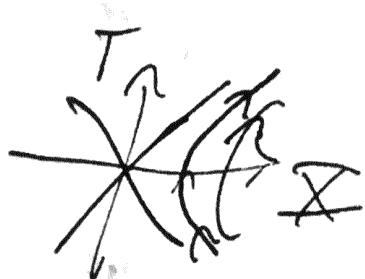
$$t_{loc} = \sqrt{f(r)} dt \\ \Rightarrow T_{loc} = \frac{\hbar k}{2\pi} f^{-1}(r) \quad (2)$$

$$r \rightarrow r_s \quad T_{loc} \rightarrow \infty$$

B. Unruh temp

$$ds^2 = -dT^2 + dX^2$$

$$\rightsquigarrow \text{Rindler space} \quad ds^2 = -r^2 d\eta^2 + d\rho^2$$



$$\eta \rightarrow -i\theta \\ f(r_E^2) = \rho^2 dT^2 + d\rho^2$$

Smoothness of Euclidean space

$$\Rightarrow \theta = \theta + 2\pi$$

local time: $d\tau_{loc} = \beta d\eta$

$$d\tau_{loc} = \beta d\theta$$

$$\tau_{loc} \sim \tau_{loc} + 2\pi p - k \beta_{loc}$$

$$\Rightarrow T_u(p) = \frac{k}{2\pi p} = \frac{ka}{2\pi}, \quad a = \frac{1}{p}$$

\Rightarrow a uniformly accelerated obs. in Mink can be in thermal equilibrium only at $T_u(p)$, otherwise one finds singular behavior at $T = \pm \infty$ ($p=0$)

Remarks:

1) ② and ③ agree when $r=r_s$, as expected

BH: $r \rightarrow \infty \quad T \rightarrow T_H \quad (a_{prop} \neq 0)$
 Rindler: $p \rightarrow 0 \quad T \rightarrow 0 \quad (a_{prop} \neq 0)$

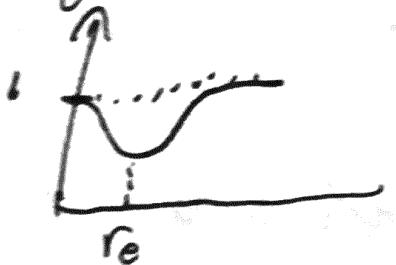
2) Does this happen to all accelerated observers?

$$ds^2 = -g(r) dt^2 + \frac{1}{g(r)} dr^2 + r^2 d\Omega^2$$

$$g(r) = (-\frac{2G_N m(r)}{r})$$

$$m(r) = \begin{cases} M_{\text{earth}} & r > r_e \\ \propto r_3 & r < r_e \end{cases}$$

$$\Rightarrow g(r)$$



$$t \rightarrow -it$$

t can have any periodicity

Planck, Unruh require $g_H \rightarrow 0$

3) Rindler $T \neq T_u$

singular behavior at $T = \pm \infty$

1.3.2 Unruh temp. from entanglement

1) Clarifies physical origin of temp.

2) Gives deeper understanding of the quantum state of matter

A1 digression — an alternative (Lorentzian) way
to describe thermal states

$$\mathcal{H}, H, E_n, |n\rangle, \rho = \frac{1}{Z_\beta} e^{-\beta H}$$

→ double the system

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$\mathcal{H}_1 \cong \mathcal{H}_2 \cong \mathcal{H}$$

typical state:

$$\sum_{m,n} a_{mn} |m\rangle_1 \otimes |n\rangle_2$$

non-factorizable

$$\notin \mathcal{H}_1 \otimes \mathcal{H}_2$$

⇒ entangled

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\beta E_n} |n\rangle_1 \otimes |\tilde{n}\rangle_2 \quad \leftarrow \text{normalized}$$

$$\langle \Psi_\beta | \Psi_\beta \rangle = 1$$

$|\tilde{n}\rangle$ is T-reversal of $|n\rangle$

$$Z_\beta = \text{Tr}(e^{-\beta H}) \quad \leftarrow \begin{matrix} \text{either} \\ \text{system} \end{matrix}$$

Consider \hat{X}_i which acts only on \mathcal{H}_i

$$\Rightarrow \langle \psi_B | \hat{X}_i | \psi_B \rangle = \frac{1}{Z_B} \sum_n e^{-\beta E_n} \langle n | \hat{X}_i | n \rangle \\ = \text{Tr}(\rho_B \hat{X}_i)$$

$$\text{Tr}_2 (\langle \psi_B | \langle \psi_B |) = \rho_B$$

$|\psi_B\rangle$: thermal field double

Unmezawa (1968?)

Remarks: Finite-T effects come from:

- 1) Special entangled structure of $|\psi_B\rangle$
- 2) Ignorance of the other system
- 3) Purification of ρ_B
- 4) $(H_1 - H_2)|\psi_B\rangle = 0 \Rightarrow e^{-i(H_1 - H_2)t} |\psi_B\rangle = |\psi_B\rangle$
- 5) $H = \hbar\omega (a^\dagger a + \frac{1}{2})$ for harmonic oscillator
 $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$

$$\Rightarrow |\psi_B\rangle = \frac{e^{-q\beta\hbar\omega}}{\sqrt{Z_B}} e^{-\frac{i}{2}\beta\hbar\omega a_1^\dagger a_2^\dagger} |0\rangle_1 \otimes |0\rangle_2$$

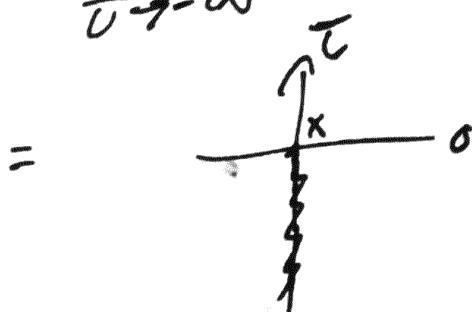
Recall: Path integral for vacuum state

$$\underline{\psi(x) = \langle x | 0 \rangle} \quad + \rightarrow -i\tau$$

$$= c \int^{\underline{x(\tau=0)=x}} D X(\tau) e^{-S_E[x(\tau)]/\hbar}$$

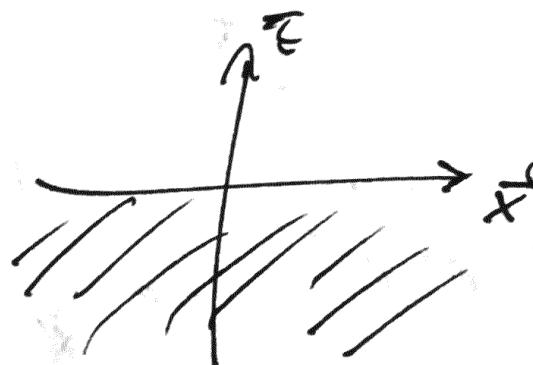
$$\begin{aligned} x(\tau_0) &= 0 \\ \tau_0 &\rightarrow -\infty \end{aligned}$$

$$= \lim_{\tau \rightarrow -\infty} \langle x | e^{\tau H} | 0 \rangle$$



$$ds^2 = -dt^2 + dx^2$$

$$ds_E^2 = dt^2 + d\vec{x}^2$$



$$\begin{aligned} \psi[\phi(x)] &= \langle \phi(x) | 0 \rangle \\ \phi(\tau=0, \vec{x}) &= \phi(x) \end{aligned}$$

$$= c \int^{\phi(\tau \rightarrow -\infty) \rightarrow 0} D\phi e^{-S_E[\phi]/\hbar}$$

$$\begin{aligned} \langle x_2 | \hat{n}_2^\dagger &= \frac{\langle n | x_2 \rangle}{2} \\ &= \langle n | x_2 \rangle \end{aligned}$$

$$\begin{aligned} \psi_\beta(x_1, x_2) &= \langle x_1, x_2 | \psi_\beta \rangle \\ &= \frac{1}{\sqrt{Z_\theta}} \sum_n e^{-\frac{i}{\hbar} E_n} \langle x_1 | n \rangle \langle x_2 | \hat{n}_2 \rangle \end{aligned}$$

$$= \sum_n e^{-\frac{1}{2}\beta E_n} \langle x_1 | n \rangle \langle n | x_2 \rangle$$

$$= \langle x_1 | e^{-\frac{1}{2}\beta \delta H} | x_2 \rangle$$

$$= \langle x_1 | e^{-i\frac{\pi}{\hbar} \Delta t} | x_2 \rangle \Big|_{\Delta t = -\frac{i\pi\beta}{2}}$$

$$\rightsquigarrow = \frac{1}{Z_\beta} \int \overset{x(0)=x_1}{DX(\tau)} e^{-\frac{i}{\hbar} S_E[x(\tau)]} \\ \overset{x(-\frac{1}{2}\beta\hbar)=x_2}{}$$

$$\begin{matrix} x_1 \\ x_2 \end{matrix} \begin{pmatrix} 0 \\ -\frac{1}{2}\beta\hbar \end{pmatrix}$$

$$= \begin{matrix} x_2 \\ x_1 \end{matrix}$$

$$\begin{matrix} x_1 \\ x_2 \end{matrix} \begin{pmatrix} x(0)=x_1 \\ \tilde{x}(\tau)=x_2 \end{pmatrix}$$

$$\Rightarrow \langle \psi_\rho | \psi_\rho \rangle = \frac{1}{Z_\beta} \times \int \overset{x(0)=x_1}{DX(\tau)} \overset{x(-\frac{1}{2}\beta\hbar)=x_2}{\tilde{x}(\frac{1}{2}\beta\hbar)=x_2} D\tilde{x}(\tau) e^{-S_E[x]}$$

assume S_E invariant under
 $\tau \rightarrow -\tau$

$$= \frac{1}{Z_\beta} \int DX(\tau) e^{-S_E[x]} = 1$$

$x(-\frac{1}{2}\beta\hbar)=x(\frac{1}{2}\beta\hbar)$

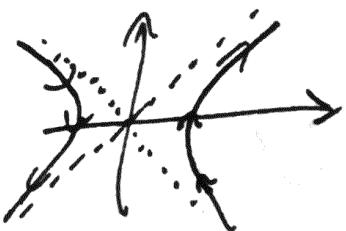
$$\begin{matrix} x_2 & x_1 \\ x_2 & x_1 \end{matrix} = \text{circle}$$

Field theory:

$$\langle \phi_1(\vec{x}) \phi_2(\vec{x}) | \Psi_{\text{in}} \rangle$$
$$= \overline{\mathbb{E}_{\text{in}}} \int D\phi(\vec{c}, \vec{x}) e^{-S_E[\phi]} \quad (*)$$
$$\phi(\vec{c}, \vec{x}) = \phi(\vec{x})$$

B. Unruh temperature from entanglement

$$ds^2 = -dT^2 + dX^2$$
$$= -\rho^2 d\eta^2 + d\rho^2$$
$$X = \rho \cosh \eta \quad \leftarrow R \text{ patch}$$
$$T = \rho \sinh \eta$$



$$\eta \rightarrow \eta + \text{const} \Rightarrow \text{boost in } (T, X)$$

similarly:

$$X = -\rho \cosh \eta \quad \leftarrow L \text{ patch}$$
$$T = -\rho \sinh \eta$$

R, L are causally disconnected

Three sets of observers:

Mink: "see" the entire Mink space

Rind_R: R patch

Rind_L: L patch

Mink: Cauchy slice: $T=0$

$\mathcal{H}_{\text{Mink}}$: $\text{span}\{\varphi(\vec{x})\}$

$$\varphi(\vec{x}) = \varphi(T=0, \vec{x})$$

H_m : using T as time

$$10\rangle_m$$

Rind_R: Cauchy slice $\eta=0$ ($\vec{x} > 0$ axis)

$\mathcal{H}_{\text{Rind}_R}$: $\text{span}\{\varphi_R(p)\}$

$$\varphi_R(p) = \varphi(T=0, \vec{x}=p>0)$$

H_R : obtained from S restricted to R
with η as time

$$10\rangle_R$$

Rind_L: Cauchy slice: $\eta=0$ ($\vec{x} < 0$ axis)

$\mathcal{H}_{\text{Rind}_L}$: $\text{span}\{\varphi_L(p)\}$

$$\varphi_L(p) = \varphi(T=0, \vec{x}=-p<0)$$

Since $\mathcal{H}(X) = (\mathcal{H}_L(p), \mathcal{H}_R(p))$

$$|\Psi(\Phi)\rangle = |\mathcal{H}_L(p)\rangle \otimes |\mathcal{H}_R(p)\rangle$$

$$\Rightarrow \mathcal{H}_{\text{Mink}} = \mathcal{H}_{\text{Rind}} \overset{\sim}{\otimes} \mathcal{H}_{\text{Rind}}^{\text{PT}}$$

Question:

is $|\Psi_m\rangle$ equivalent to $|\Psi_L\rangle \otimes |\Psi_R\rangle$?

Answer:

It turns out, no.

Note: any field theory is CPT-invariant.

$$\Rightarrow \mathcal{H}_R \overset{\sim}{\leftrightarrow} \mathcal{H}_L$$

(R,L) form a double

Claim: $|\Psi\rangle_m$ is a TFD for $\mathcal{H}_{\text{Rind}} \otimes \mathcal{H}_{\text{Rind}}^{\text{PT}}$
strategy for proof: coordinate space wavefunction

Note: (T_E, \mathbb{I}) -LHP in fact coincides with Euclidean analytic continuation of Rindler

$$\eta \rightarrow -i\theta$$

$$\theta \in (-\pi, 0)$$

$$\Rightarrow \psi_0[\phi(x)] = \int_{\phi(\theta=0, p) = \phi_R(p)}^{\phi(\theta=-\pi, p) = \phi_L(p)} D\phi(\theta, p) e^{-S_E[\phi]} \quad (\star\star)$$

compare (*) w/ (**)

$$\Rightarrow \psi_0[\phi(x)] = \langle \phi_R(p) \phi_L(p) | \psi_B \rangle$$

$$\text{with } \frac{1}{2}\beta\pi = \pi$$

$$\Rightarrow \beta = \frac{2\pi}{\hbar}$$

$$\Rightarrow Z_0^{(\text{Mink})} = Z_{\beta=\frac{2\pi}{\hbar}}^{(\text{Rind})}$$

We conclude:

$$|\psi\rangle_M = \left| \psi_{\beta=\frac{2\pi}{\hbar}} \right\rangle$$

$$Z_0 = \text{Tr} \left(e^{-\frac{2\pi}{\hbar} H_{\text{Rind}}} \right)$$

since β is associated with η

$$\begin{aligned} dt_{\text{loc}} &= p d\eta \\ \Rightarrow \boxed{\beta_{\text{loc}} &= \frac{2\pi p}{\hbar}} \end{aligned}$$

just as we derived last time
but this time from real-time wavefunction

Sept 24 (Missed, got notes from Sam)

Remarks

- 1) Euclidean method: Regularity of analytic continuation
⇒ only have equilibrium at T_E
Now: When system is at $|0\rangle_m$
⇒ R/L observers both thermal at T_E

$$\frac{Z_0}{\text{Mink}} = \frac{Z_{\text{Rind}}}{e^{-\beta = 2\pi/k}}$$

- 2) Thermal nature comes from
(a) Special entangled structure of $|0\rangle_m$
(b) Tracing out the other half
- 3) Both derivations used a simple geometric feature.

Euclidean analytic cont. of Mink₂

Euclidean analytic cont. of Rind
+ special periodicity

~ This is very general, applies to any QFT

4) Entanglement method: no need to deal w/ critical singularity

$$5) (H_L - H_R)|\psi\rangle_B = 0 \Rightarrow e^{-i\eta(H_L - H_R)}|\psi\rangle_B = |\psi\rangle_B$$

$$\text{Boost inv. of vacuum} \rightarrow e^{-i\eta(H_L - H_R)/2}|0\rangle_B = |0\rangle_B$$

$$H_R = \int_0^\infty d\tau \Im T_{00} \quad (\text{on } T=0 \text{ Cauchy slice})$$

$$H_L = \int_{-\infty}^0 d\tau (-X) T_{00} \quad (\text{on } '')$$

C: Hawking temperature from entanglement

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \\ &= g(r) (-dt^2 + dX^2) + r^2 d\Omega^2 \quad \cancel{\text{metric}} \\ X^2 - t^2 &= \frac{1}{2^a} e^{\frac{r-r_s}{r_s}} \left(\frac{r-r_s}{r_s} \right)^2 \quad \cancel{R} \end{aligned}$$

Similarity: Kruskal observers $\rightarrow t$
Schwarzschild observers $\rightarrow t$

$$\mathcal{H}_K = \mathcal{H}_L \otimes \mathcal{H}_R$$

Important difference from Rindler
 \rightarrow Metric is not T -independent $\Rightarrow H_K$ is not either \Rightarrow energy not conserved

32| no notion of vacuum state

Nevertheless we can define the counterpart of $\langle D \rangle_M$ using path integrals to get $\langle D \rangle_{HH}$ "Hartle-Hawking vacuum"

Key Kruskal metric allows a sensible $T \rightarrow -iT_E$

(1) Euclidean manifold is again the same as taking $t \rightarrow it$ with $\tau \sim t + \frac{2\pi}{\kappa x}$ that is obtained by

$\Rightarrow (\text{def}) \langle D \rangle_{HH} := \text{path integral over } T_E < 0$

$\langle D \rangle_{HH} = \langle \chi_{\beta_H} \rangle \leftarrow \text{Thermal field double with } \beta_H = \frac{2\pi}{\kappa x}$

D: Geometry & Entanglement

Previously: From perspective of Rindler or Schwarzschild observers, there is a singular behaviour at the horizon unless they are at T_H

\rightarrow explain this using entanglement

Rindler: $\eta \rightarrow i\theta$, zero temp \rightarrow not compact $\rightarrow \langle D \rangle_R \otimes \langle D \rangle_L$ (*)
in this state L and R are unentangled

For any smooth wavefunction of any QFT in $Mink_2$

$\frac{1}{L+R} \rightarrow$ always entangled for any finite energy state

For L, R not entangled, we'd need a barrier at $I=0$
 \Rightarrow singular behavior at $I=0$ that causally propagates

Note (*) is not a state in Mink_2

$|L\rangle \otimes |R\rangle \nrightarrow$ No F/P regions

\Rightarrow Presence of F/P regions in Mink_2

Entanglement of L, R.

Generalize:

- 1) Any finite energy state in Mink_2 requires ~~specific~~
entanglement between L and R <sup>with all
of regions</sup>
- 2) Generic state doesn't have that entanglement structure
 \Rightarrow Cannot be interpreted as sensible in Mink_2
- 3) Similarly, generic state in $\mathcal{H}_L \otimes \mathcal{H}_R$ for Schwarzschild
will have "fire wall" at the horizon
- 4) discussion is at the level of states, independent
of details about $\mathcal{H}_N, \mathcal{H}_L, \mathcal{H}_R$, etc.

1.3.3 Free Field Theories Derivation

$$|\psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{\epsilon}{2}\beta E_n / \hbar\omega} |\alpha^n\rangle_R = \frac{e^{-\frac{\epsilon}{2}\beta \hbar\omega}}{\sqrt{Z_\beta}} e^{\frac{i}{2}\beta \hbar\omega a_1^\dagger a_2^\dagger} |0\rangle \otimes |0\rangle_2$$

Note $a e^{\alpha a^\dagger} |0\rangle = a e^{\alpha a^\dagger} |0\rangle$ $a_1 |0\rangle = 0$
 $a_2 |0\rangle_2 = 0$

$$\Rightarrow (a_1 - e^{-\frac{\epsilon}{2}\beta \hbar\omega} a_2^\dagger) |\psi_\beta\rangle = 0$$

$$(a_2 - e^{-\frac{\epsilon}{2}\beta \hbar\omega} a_1^\dagger) |\psi_\beta\rangle = 0$$

def: $b_1 = \cosh \theta a_1 + \sinh \theta a_2^\dagger$

$$b_2 = \cosh \theta a_2 - \sinh \theta a_1^\dagger$$

$$\Rightarrow \text{state } \cosh \theta = \frac{1}{\sqrt{1 - e^{-\frac{\epsilon}{2}\beta \hbar\omega}}}, \sinh \theta = \frac{e^{-\frac{\epsilon}{2}\beta \hbar\omega}}{\sqrt{1 - e^{-\frac{\epsilon}{2}\beta \hbar\omega}}}$$

then $[b_1, b_1^\dagger] = [b_2, b_2^\dagger] = 1$

else $[\cdot, \cdot] = 0$

and $b_1 |\psi_\beta\rangle = b_2 |\psi_\beta\rangle = 0$

$\Rightarrow |\psi_\beta\rangle$ is vacuum for b_1, b_2

Free Field Theories \rightarrow a bunch of harmonic oscillators

$$|0\rangle_1, |0\rangle_2 \rightarrow |0\rangle_L |0\rangle_R, |\psi_\beta\rangle \rightarrow |0\rangle_m$$

Free Massless scalar:

$$S = -\frac{1}{2} \int d^2x \partial^\mu \phi \partial_\mu \phi$$

Minkowski observers

$$(-\partial_T^2 + \partial_X^2) \phi = 0$$

$$\Rightarrow u_p = \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p T + ipX}$$

$$\omega_p = |p|$$

$$u = T - X \Rightarrow u_p = \begin{cases} \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p T - ipX} \\ \frac{1}{\sqrt{2\omega_p}} e^{-i\omega_p T + ipX}, p > 0 \end{cases}$$

$$V = T + X$$

$$(P_2, P_1) := i \int \left(P_2^* \partial_1 \phi - (\partial_T P_2) \phi_1 \right)$$

$$(u_p, u_{p'}) = 2\pi \delta(p-p')$$

$$(u_p^*, u_{p'}^*) = -2\pi \delta(p-p')$$

$$(u_p, u_{p'}^+) = 0$$

CCR with $\phi = \sum_p (a_p u_p + a_p^\dagger u_p^*)$

$$[a_p, a_p^\dagger] = 2\pi \delta(p-p')$$

$$a_p \langle 0 \rangle_R = 0$$

Rindler "R"-observers

$$ds^2 = -dT^2 + d\xi^2 = -\rho^2 d\eta^2 + d\rho^2$$

$$\rho = e^{\xi} \text{ Mink} \quad \Rightarrow \quad ds^2 = e^{2\xi} (-d\eta^2 + d\xi^2) \text{ Rind}$$

$$(-\partial_\eta^2 + \partial_\xi^2) \phi = 0$$

$$u = \eta - \xi \Rightarrow u_K = \begin{cases} \frac{1}{\sqrt{2\omega_K}} e^{-i\omega_K u} \\ \frac{1}{\sqrt{2\omega_K}} e^{-i\omega_K u} \end{cases}$$

$$v = \eta + \xi \Rightarrow v_K = \begin{cases} \frac{1}{\sqrt{2\omega_K}} e^{-i\omega_K v} \\ \frac{1}{\sqrt{2\omega_K}} e^{-i\omega_K v} \end{cases}$$

$$P_R = \sum_K (b_K^{(R)} v_K + b_K^{(R)\dagger} v_K^*)$$

$$[b_K, b_K^\dagger] = \frac{2\pi}{\text{metric}} \delta(K-K')$$

$$b_K^{(R)} \langle 0 \rangle_R = 0$$

36

Sept 26 (Missed, got notes from Sam)

Recall $U = -e^{-u}$, $V = e^v$ (so $U < 0$, $V > 0$)

→ Rindler "R" modes become

$$V_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k u} & k > 0 \\ \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k v} & k < 0 \end{cases} = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k \log(-U)} & k > 0 \\ \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k \log(V)} & k < 0 \end{cases}$$

$$\phi_R = \sum_k b_k^{(R)} V_k + b_k^{(R)*} V_k^* \Rightarrow [b_k^{(R)}, b_k^{(R)*}] = 2\pi S(k)$$

Define right vacuum $\boxed{b_k^{(R)} |D\rangle_R = 0}$ Note $(\phi_R^{(2)}, \phi_R^{(1)}) = i \int_{-\infty}^{\infty} dE (\partial_R^{(2)*} \partial_R^{(1)} - \partial_R^{(2)} \partial_R^{(1)*})$

V_k is singular at $U=0$ and $V=0$ since the modes are only supported in the R region
→ potential singular behavior in physical quantities

Rindler "L":

$$T = -e^{\eta} \sinh \eta, X = -e^{\eta} \cosh \eta, U = e^{-\eta}, V = -e^{\eta}$$

$$V_k = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k u} & k > 0 \\ \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k v} & k < 0 \end{cases} = \begin{cases} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k \log(U)} & k > 0 \\ \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k \log(V)} & k < 0 \end{cases}$$

PT reversal ensures

$k > 0$ has U
 $k < 0$ has V

Note: $(\phi_L^{(2)}, \phi_L^{(1)}) = -i \int_{-\infty}^{\infty} dE (\partial_L^{(2)*} \partial_L^{(1)} - (\partial_L^{(2)} \partial_L^{(1)*}))$

we have chosen positive-frequency modes to be the PT reversed of V_k
(recall $|V_k\rangle = \sum_n e^{\frac{2\pi i n}{\lambda} k \omega_n} |n\rangle$)

$$\text{So } \rho_L = \sum_k (b_k^{(L)} w_k + b_k^{(L)\dagger} w_k^*) \Rightarrow [b_k^{(L)}, b_k^{(L)\dagger}] = 2\pi \delta_{k,k}$$

Left vacuum $\overline{[b_k^{(L)} | 0\rangle_L]} = 0$

On Mink₂, $\Phi(T, X) = (\rho_L, \rho_R)$

$$\Rightarrow \rho = \sum_p (a_p u_p + a_p^\dagger u_p^*) = \sum_k \left(b_k^{(L)} w_k + b_k^{(L)\dagger} w_k^* + b_k^{(R)} v_k + b_k^{(R)\dagger} v_k^* \right)$$

To find relation between $\{a_p, u_p^*\}$ and

$\{b_k^{(L)}, b_k^{(L)\dagger}, b_k^{(R)}, b_k^{(R)\dagger}\}$, need to find relations between $\{u_p, u_p^*\}$ and $\{w_k, w_k^*, v_k, v_k^*\}$.

Two possibilities:

$$(1) v_k = \sum_p c_{kp} u_p, w_k = \sum_p \tilde{c}_{kp} u_p \quad \text{without } u^* \text{ involved}$$

\Rightarrow positive frequency modes of both L and R observers are related to only positive freq. Minkowski modes.

$$\Rightarrow b_k^{(R)} = \sum_p d_{pk} a_p, b_k^{(L)} = \sum_p \tilde{d}_{pk} a_p$$

$\Rightarrow |0\rangle_M$ coincides with $|0\rangle_L \otimes |0\rangle_R$

$$(2) \text{ suppose } u_p = \sum_k (d_{pk} v_k + \tilde{d}_{pk} w_k + e_{pk} v_k^* + \tilde{e}_{pk} w_k^*)$$

$$\text{then } b_k^{(R)} = \sum_p (d_{pk} a_p + \tilde{e}_{pk} a_p^*), b_k^{(L)} = \sum_p (\underbrace{d_{pk} a_p + \tilde{e}_{pk} a_p^*}_{\text{Bogoliubov Transformation}})$$

33] $\Rightarrow |0\rangle_M \neq |0\rangle_L \otimes |0\rangle_R$

Requiring $b_k^{(R)} |0\rangle_L \otimes |0\rangle_R = b_k^{(L)} |0\rangle_L \otimes |0\rangle_R = 0$
 we get: $|0\rangle_L \otimes |0\rangle_R \sim e^{\# \text{at} \#}$

$$|0\rangle_R \sim e^{\# \text{at} \#} |0\rangle_R$$

Focus on right-moving modes:

$$v_p = \frac{1}{\sqrt{2w_p}} e^{i w_p u}, \quad v_p = \begin{cases} \frac{1}{\sqrt{2w_p}} e^{i w_p \log(-u)} & \text{for } u < 0 \\ 0 & \text{for } u > 0 \end{cases} \quad \text{for } u < 0$$

$$w_k = \begin{cases} \frac{i}{\sqrt{2w_k}} e^{-i w_k \log u} & \text{for } u > 0 \\ 0 & \text{for } u < 0 \end{cases} \quad \text{for } u > 0$$

argument for possibility (2):

v_p is analytic in lower complex u plane

\rightarrow So is any linear superposition

Neither v_k or w_k is analytic there

$\Rightarrow v_k, w_k$ must involve both v_p, v_p^* (which is doable)

Instead of finding $d_{pk}, \tilde{d}_{pk}, e_{pk}, \tilde{e}_{pk}$ explicitly,
 consider another basis equivalent to (v_p, v_p^*)
 (i.e. has the same vacuum) but related to

v_k, v_k^*, w_k, w_k^* in a simple way:

construct χ_k from analytic cont.

of v_k to lower half plane

$$\chi_k = \frac{1}{\sqrt{2 \sinh \pi w_k}} [e^{\frac{i\pi}{2} w_k} v_k + e^{-\frac{i\pi}{2} w_k} v_k^*]$$

$$\chi_k = \frac{1}{\sqrt{2 \sinh \pi w_k}} [e^{\frac{i\pi}{2} w_k} w_k + e^{-\frac{i\pi}{2} w_k} w_k^*]$$

\checkmark analytic continuation
 in LH plane \Rightarrow

$\{\lambda_k, \chi_k\}$ share the same vacuum, $|0\rangle_m$ as $\sum p$

$$\Rightarrow \mathcal{J} = \sum_k [c_k \lambda_k + d_k \chi_k + h.c.], c_k |0\rangle_m = d_k |0\rangle_m = 0$$

$$c_k = \cosh \theta_k b_k^{(R)} - \sinh \theta_k b_k^{(L)\dagger}$$

$$d_k = \cosh \theta_k b_k^{(L)} - \sinh \theta_k b_k^{(R)\dagger}$$

where $\cosh \theta_k = \frac{e^{\frac{i\pi}{2}\omega_k}}{\sqrt{2 \sinh \pi \omega_k}}$

$$\sinh \theta_k = \frac{e^{\frac{i\pi}{2}\omega_k}}{\sqrt{2 \sinh \pi \omega_k}}$$

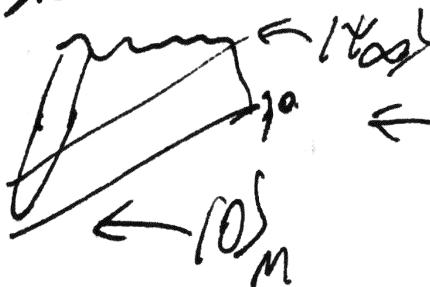
$$\Rightarrow |0\rangle_m = \prod_k \left(\frac{e^{-\frac{i\pi}{2}\omega_k}}{\sqrt{z_k}} \right) \exp \left[\sum_k e^{-\pi \omega_k} b_k^{(R)\dagger} b_k^{(L)\dagger} \right] |0\rangle_L |0\rangle_R$$

with $z_k = \frac{1}{2 \sinh \pi \omega_k}$ } this is exactly the result
for each k
for a 2D at $B = \frac{2\pi}{\lambda}$

(Free) Massive scalar in Schwarzschild background

$|0\rangle_{HH}$ & squeezed state for Schwarzschild obs.

Realistic BH



get temp.
from free
Field theory

1.4 BH Thermodynamics

BH has a temperature

$$T_H = \frac{kK}{2\pi} = \frac{k}{8\pi G_N M} \quad (1) \Rightarrow \text{natural to interpret it as a thermodynamic system}$$

suppose it has an entropy S .

Expect it to satisfy 1st law:

$$dE = T dS \quad (2)$$

identify $E = M$, integrate (2) to find S

$$dS = \frac{dM}{T} \Rightarrow \boxed{S = \frac{4\pi G_N M^2}{k}}$$

$$\text{but } r_S = \frac{2GM}{c^2}$$

$$\Rightarrow \boxed{S = \frac{4\pi r_S^2}{G_N k} = \frac{A}{4\pi G_N}} \quad (3)$$

so using (1) and (3), we can rewrite (2) as

$$dM = \frac{kK}{2\pi} \frac{A}{4\pi G_N} = \frac{k}{8\pi G_N} dA \quad (4)$$

(4) is a pure geometric relation

Eq. (4) is part of a set of four laws on general BHs called "Four laws of BH mechanics"

- No hair theorem:

A stationary asymptotically flat BH is solely characterized by:

- 1) mass M
- 2) angular momentum J
- 3) electric or magnetic charges \mathcal{Q}

- Four laws: (1972)

0th law: surface gravity K is constant over the horizon

1st law: $dM = \frac{K}{8\pi G_N} dA + \Omega dJ + \Phi dQ$

Ω : angular frequency at the horizon

Φ : electric potential (s.t. $\Phi(\infty)=0$)

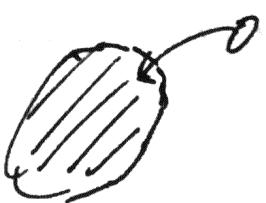
2nd law: Horizon area never decreases classically.

3rd law: surface gravity of a BH cannot be reduced to 0 in a finite sequence of operations

With (1) + (3), the four laws of BH mechanics become the standard laws of thermodynamics

Beckenstein (1972-1974):

BH should have an entropy $\geq A$



otherwise the second law of thermodynamics would be violated in the presence of a BH

Define

$$S_{\text{tot}} := S_{\text{matter}} + S_{\text{BH}}$$

\leadsto Generalized second law

$$\Delta S_{\text{tot}} \geq 0$$

with (1975) Hawking radiation, GSL becomes standard 2nd law.

Remarks:

1) in classical limit $\hbar \rightarrow 0$ (env fixed)

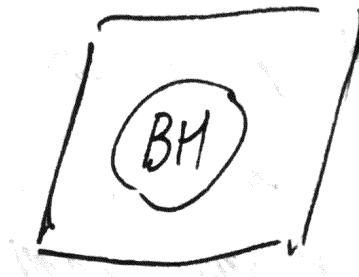
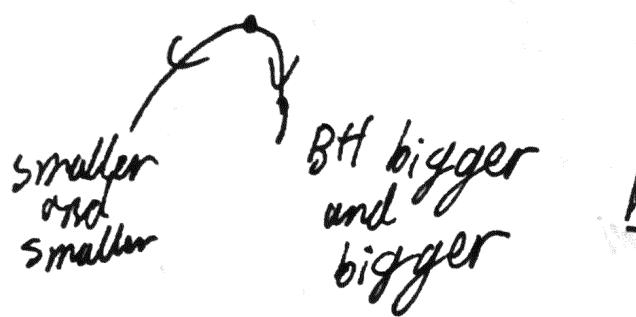
$$T_H \rightarrow 0, S_{\text{BH}} \rightarrow \infty$$

2) $T_H \propto \frac{1}{M}, M \uparrow \Rightarrow T_H \downarrow$

$$\Rightarrow C = \frac{\partial M}{\partial T} = -\frac{k}{8\pi G_N} \frac{1}{T^2} < 0 \Rightarrow \begin{array}{l} \text{negative} \\ \text{specific heat} \\ \text{specific heat} \end{array}$$

\Rightarrow T_H not a stable equilibrium. Why? Consider small fluctuations in T_{BH} , say $T_{BH} \uparrow$, radiate a bit more to env $\Rightarrow M \downarrow \Rightarrow T_{BH} \uparrow \Rightarrow$ radiate even more and similarly unstable in the other direction. \square_B

BH + infinite bath



Stable equilibrium
is possible in
a finite box

3) $T_H = \frac{\hbar x}{2\pi}$ and $S = \frac{A}{4\hbar G_N}$

Universal

Apply to any matter coupled to Einstein gravity. (AdS, dS, Mink, all spacetimes)

4) With higher derivative corrections to Einstein gravity

(i.e. $R^2 + \lambda^2 R_{\mu\nu}^2 + \mu R^2 + \dots$)

These ~~equations no longer apply~~ but S, T_H can still be expressed in terms of horizon quantities

1.5 Quantum Nature of Black Holes and the Holographic Principle

$$\text{BH thermodynamics} + T_H \propto k \\ S_{\text{BH}} \propto k$$

\Rightarrow Natural to treat BH as a macroscopic quantum statistical system.

Questions:

- (1) What is the statistical interpretation of the entropy of a black hole?

From standard stat Mech.

$$\# \text{ microstates} = \Omega$$

(consistent w/ a given macroscopic equilibrium)

$$\Rightarrow S = k_B \log \Omega$$

$$\text{For BH, expect } \Omega = e^{\frac{A}{kT_{\text{BH}}}} = e^{\frac{A}{kT_p}}$$

Heuristically:



entropy ~ put 1 d.o.f. in each Planckian cell.
(e.g. spin)

(a) This is a huge entropy
 For $M_{BH} = M_\odot$, $r_s = 3\text{ km} \Rightarrow \frac{A}{4\pi G_N} \approx 1.1 \times 10^{77}$
 $\therefore Q \approx e^{10^{77}}$
 The sun itself has entropy $\frac{S}{k} \approx 10^{57}$

(b) When a star collapses to form a BH,
 there is a huge increase in the # available
 microstates

no hair theorem: all these states must be
 quantum mechanical in nature.

huge increase \Leftrightarrow gravity is (Planck scale)
 weak (is very small)

(c) In string theory, there are black holes
 whose microstates can be precisely counted.
 giving (after complicated combinatorics), exactly
 an entropy of $S = \frac{A}{4\pi G_N}$

(d) In holographic duality, for black holes in AdS ,
 the statistical origin is again known

(2) Hawking's information loss paradox

Hawking:

- 1) To an excellent approximation
BH radiates thermally for $M \gg M_{\text{Pl}}$
"white noise"
- 2) BH loses mass
- 3) should disappear

But when $M \sim M_{\text{Pl}}$, not enough d.o.f.
to encode all the information put into it

Another way to say this:

Suppose a star is in a pure state

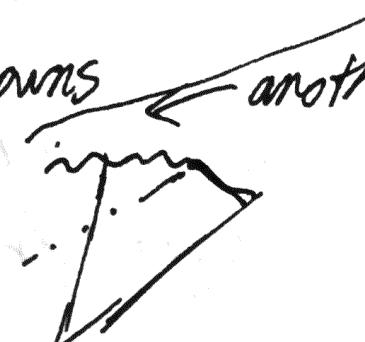
\Rightarrow BH

\Rightarrow Radiates

\Rightarrow Radiation (mixed (thermal) density matrix)

\Rightarrow 3 logical possibilities.

- 1) Information is lost \Rightarrow QM must be modified
- 2) Hawking radiation stops at $M \sim O(M_{\text{Pl}})$ $\underbrace{\text{R}}_{\Rightarrow \text{Planckian mass remnant is left, which encodes all information}}$ \otimes Radiation
- 3) No remnants, unitary evolution \Rightarrow information comes out from radiation

- 1) Is the most radical. It is also fiendishly difficult to modify QM
- 2) Blames unknowns ~~ignores~~ another universe
 A variant:
- 

- 3) Is the most conservative
- ~ significant challenges still to explain how the information comes out from radiation
 - ⇒ imply quantum gravity puts highly nontrivial constraints/implications on low-energy physics
- simpler question: Burning of coal

Preparation: A typical highly excited pure state in a non-integrable many-body system looks thermal if one only probes a small part of it.

Say I separate the system in two:

$$P_A = \text{Tr}_B (\langle \psi | \psi \rangle)$$

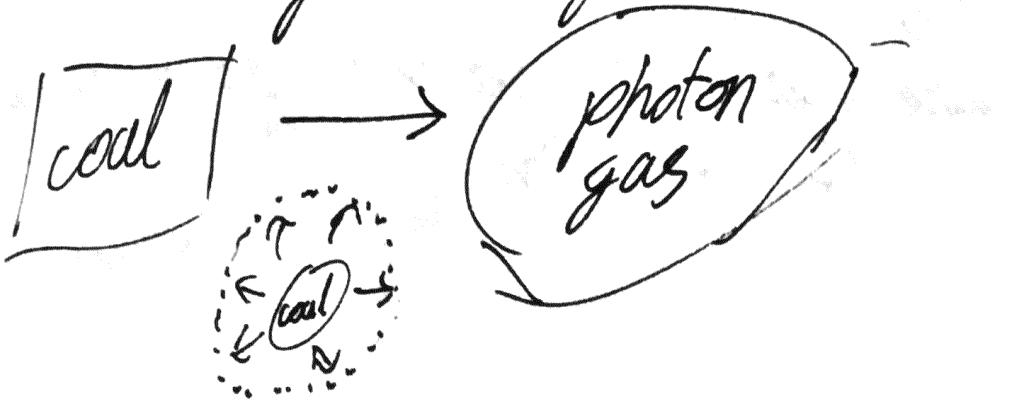
$A + B$

is very close to thermal being thermal

trace distance is exponentially suppressed in $\propto \frac{1}{\sqrt{N}}$

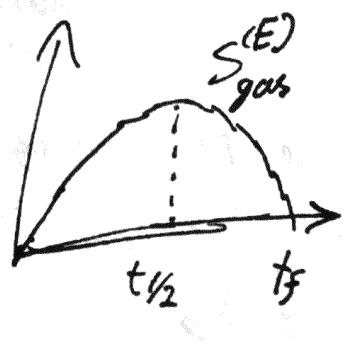
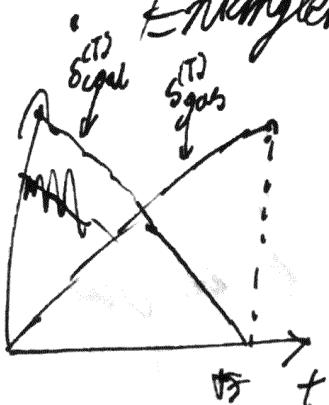
provided $|A| \ll |B|$

~~3 cont.~~ \rightarrow One reveals a given state is pure only by having full global information



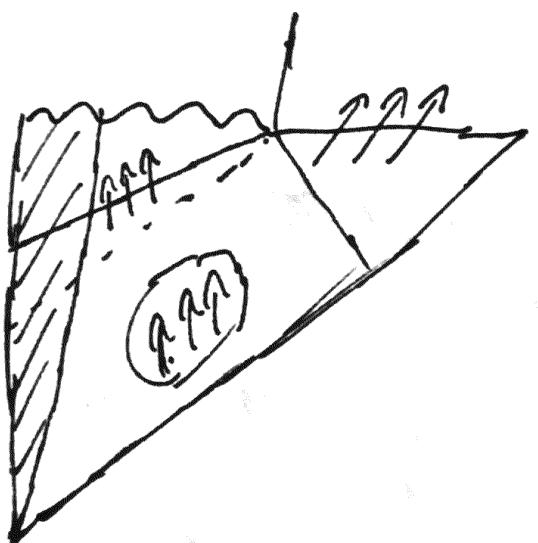
some remarks:

- (a) At a given time, emitted photons look almost perfectly thermal.
- (b) Nevertheless, they do contain information, but in a very subtle way.
The information is encoded in the entanglement with the rest of the system. $\rightarrow e^{-N} e^{K_{\text{ent}}}$ & respectively
- (c) Consider the following quantities:
 - . Thermal entropy of photon gas: $S_{\text{gas}}^{(T)}$ } "coarse-grained"
 - . Thermal entropy of the coal: $S_{\text{coal}}^{(T)}$
 - . Entanglement entropy of photon gas: $S_{\text{gas}}^{(E)}$ } "fine-grained"
 - . Entanglement entropy of the coal: $S_{\text{coal}}^{(E)}$



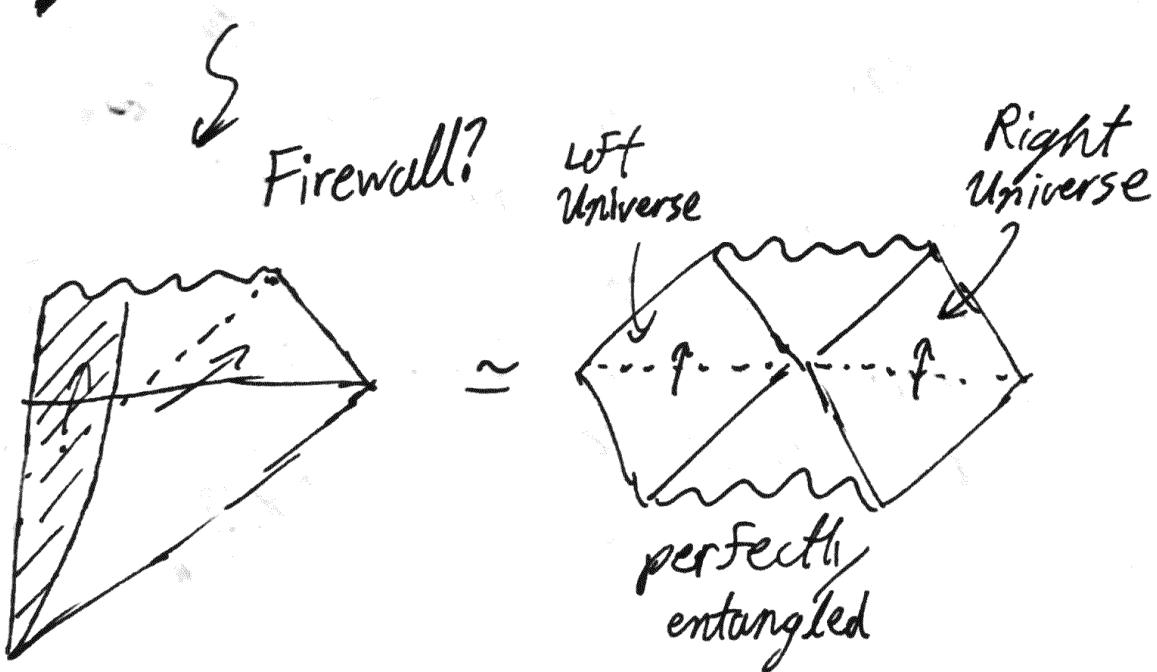
$$S(p_A) = -\text{Tr}(p_A \log p_A)$$

This is a good paradigm for BH evaporation
but BH is not coal.
Coal is causally connected with emitted radiation.
Black hole's infalling matter is causally disconnected
with the emitted radiation.



Would either violate

- No-cloning theorem (QM)
- Locality (QFT)



Holographic duality tells us that information
evaporation for a Black Hole should be just like
the burning of coal

• Entropy bounds and holographic principle
starting point:

BH is a quantum statistical system
+ a couple of "facts"
⇒ entropy bounds and holography

Facts:

(1) A sufficiently massive object in a compact volume always collapses to form a black hole
Rule of thumb: if $2GM > L \Rightarrow$ BH

(2) Entropy reflects # of degrees of freedom

$$\rho \rightarrow S = -\text{Tr}[\rho \log \rho]$$

for a system with N -dimensional Hilbert space

$$S_{\max} = \log N$$

(a) For a system of spins $N = 2^n \Rightarrow S_{\max} \sim n$

(b) If for h.o. is infinite-dimensional, but for finite energy, \mathcal{H} is f.d.

* Spherical entropy bound

→ Take an isolated system of energy E , entropy S_0 in asymptotically flat spacetime

Let A be the area of smallest sphere enclosing the system, M_A be the mass of a black hole with that area. $\boxed{S_0}$

Then $E \leq M_A$

\Rightarrow Maximal energy one could add (keeping T_{sys})
is $M_A - E$

$$S_{\text{final}} = S_{\text{BH}} + S_{\text{init}} \\ = S_0 + S'$$

$$S_0 \leq S_{\text{BH}} = \frac{A}{4\pi G_N} \Rightarrow S_{\text{max}} = \frac{A}{4\ell p^2}$$

Remarks:

- 1) S is ~~classically~~ extensive $\propto V$
 \Rightarrow QG behaves very differently from non-gravitational systems
- 2) A cubic lattice of spins (size L , spacing a) has $S_{\text{max}} = \frac{L^3}{a^3} \log 2$ (*)

At $G_N = 0$

Now slowly increase $G_N(\ell p)$ ^{i.e.}

then (*) violates the bound when

$$\frac{L^3}{a^3} \log 2 \geq \frac{A \sim L^2}{4\ell p^2}$$

$$\frac{\ell p^2}{L^2} > \frac{a}{L} \cdot * = \lambda$$

Suppose each site has mass m

$$M = \frac{L^3}{a^3} m$$

Total system is a black hole when

$$\pi \frac{\ell p}{\hbar} \approx M > L \Rightarrow \ell p^2 \frac{L^3}{a^3} \frac{m}{\hbar c} > L$$
$$\Rightarrow \frac{\ell p^2}{a^2} > \frac{a}{L} \frac{\hbar c}{L} = \lambda_2$$

$$\lambda_2 < \lambda_1$$

$$\text{so } \partial S \propto A$$

$$\text{and } \partial S \propto V$$

But both conditions are important.

a) In a closed universe S^3

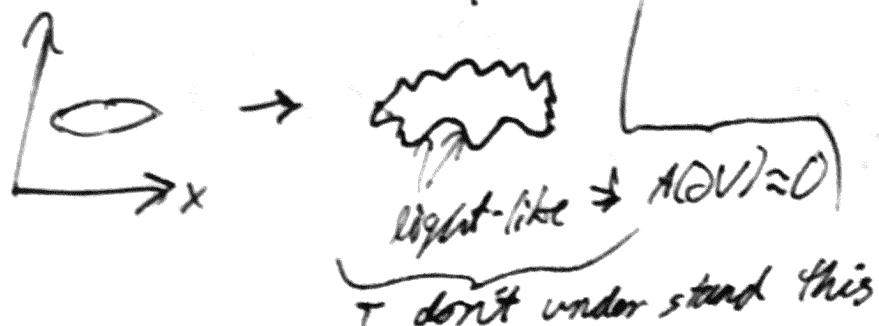
$$A=0, S \neq 0$$

trivially violated

b) consider non-spherical region in asymptotically flat universe
take V space-like, with boundary ∂V

$$S_{\text{max}} \stackrel{?}{=} \frac{A(\partial V)}{4G\hbar}$$

(a)+(b): For a general region, V , with bdry ∂V ,
in general ∂V has
not much to do with
physics inside V



light-like $\Rightarrow A(\partial V) \approx 0$

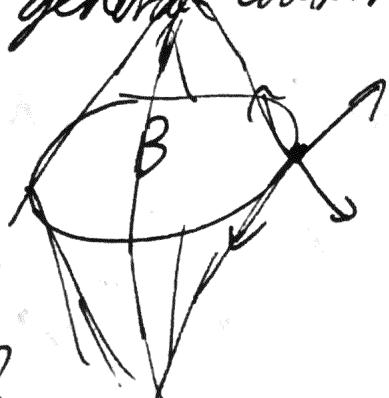
I don't understand this

A generalization for asymptotic flat spacetime universes

- Consider a general codimension 1 spacelike closed surface

\Rightarrow 4 light rays

causal diamond of B



future in
out

past in
out

c.f. hep-th/0203101

Take D to denote the causal diamond

- any point in D is fully determined by the information enclosed in D

conjecture:

$$\text{Entropy on any cauchy slice } \leq \frac{A_B}{4\pi G_N}$$

This doesn't work in cosmological settings
or inside a black hole.

Most general formulation:

For any B , construct light-sheet from B : null hypersurface formed by non-expanding light rays from B .

$$\Rightarrow S[L(B)] \leq \frac{A_B}{4\pi G_N}$$



54 entropy of "matter"
d.o.f. passing through
light sheet.

Entropy bound + entropy associated with #d.o.f.

⇒ statement on # d.o.f.

⇒ Holographic principle

"When something is so new, that you don't have the language to describe it, you describe it in whatever language you can. It may not be precise, it may not even be true, but it's better than nothing."

~ Xiao-gang Wen

⇒ A spherical region of boundary area A can be fully described by no more than

$$\frac{A}{4\pi l_p^2} = \frac{A}{4l_p^2} \text{ d.o.f.}$$

i.e. \sim one degree of freedom per Planck-area

Chapter 2:

Matrices & Strings

2.1 Path integrals of strings

QFT: a theory of "particles"
 If particle interactions are weak, we can consider
 the first quantized approach



$$x^\mu(\tau) \quad \mu = 0, 1, \dots, d-1$$

$$S_{\text{particle}} = -m \int_m^l \frac{dl}{\text{proper length}}$$

$$= -m \int dt \frac{d\ell}{dt} = -m \int dt \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt}}$$

- 1) Lorentz invariant
 - 2) Correct equation of motion
 - 3) Correct non-relativistic limit
 - 4) Reparameterization invariant in τ
- $\tau \rightarrow \tau'(\tau)$
- $$x^\mu \rightarrow x'^\mu \text{ s.t. } x^\mu(\tau) = x'^\mu(\tau')$$

For a quantum particle:

$$G(x, x') = \sum_{\substack{\text{path} \\ \text{from } x \text{ to } x'}} e^{iS_{\text{particle}}} \quad \text{but "}\int\text{" is awkward to deal with}$$

rewrite S_{particle} as

$$S = \frac{1}{2} \int d\tau (e^i(\tau) \eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} - e(\tau) m^2)$$

Upon eliminating e , we go back to (x)

$$\Rightarrow G(x, x') = \int_{\substack{x_F''=x \\ x_i''=x'}} D X''(x) D e^{iS_{\text{particle}}}$$

$\Rightarrow =$ Feynman propagator for a scalar field of mass m

Note:

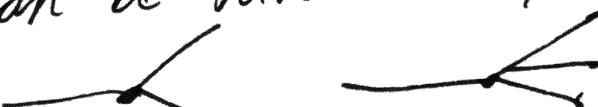
1) $e(\tau)$ is an intrinsic "vielbein" along the worldline

$$h_{\mu\nu}^{TE} = -e^2(\tau)$$

$\underline{h}_{\mu\nu}^{TE}$ intrinsic metric on the worldline

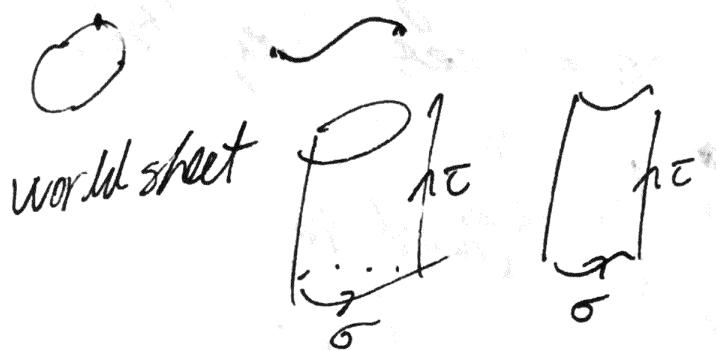
2) For curved spacetime, simply do
 $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(X(\tau))$

3) Interactions can be introduced by including vertices



4) No general principle to restrict allowed types
of vertices \leftrightarrow interactions in a QFT have to
be specified by hand

- Strings:
1-d objects



$$\Sigma: X^{\mu}(\xi^a) \quad a = 0, 1 \\ \xi^a = (0, \tau)$$

(*) can be immediately generalized

$$S_{NG} = -T \int_{\Sigma} d\xi^a dA$$

$$= -T \int_{\Sigma} d^2 \xi \sqrt{-\det \left(\eta_{\mu\nu} \frac{dx^\mu}{d\xi^a} \frac{dx^\nu}{d\xi^b} \right)}$$

has induced metric

$$= -T \int_{\Sigma} d^2 \xi \sqrt{-h}$$

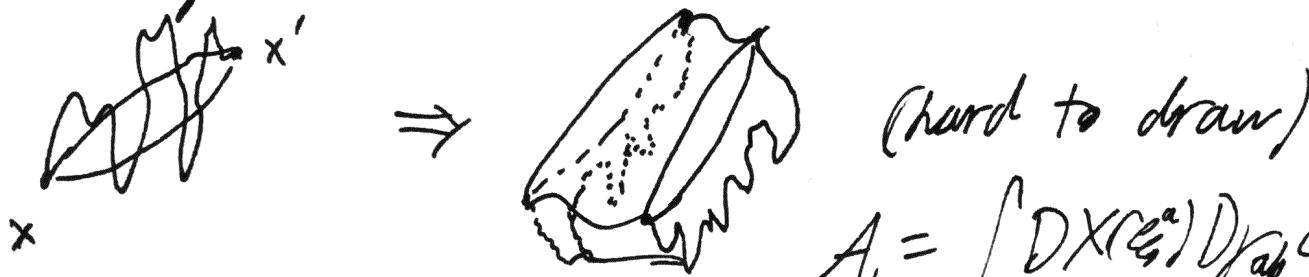
Alternative form:

Introduce new auxiliary metric γ^{ab} "intrinsic metric on Σ "

$$S_p = -\frac{T}{2} \int_{\Sigma} d^2\xi \sqrt{\gamma} \gamma^{ab} h_{ab}$$

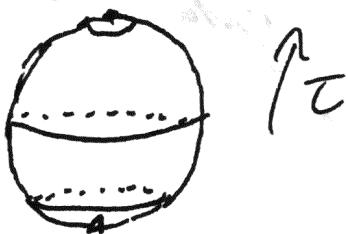
Quantum dynamics of string:

Integrate over all possible string trajectories
 \Leftrightarrow Integrate over all 2-d surfaces (weighted by e^{iS_p})

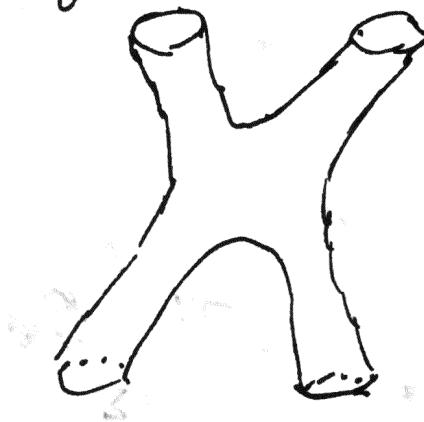


$$A = \int D X(\xi^a) D_{ab} e^{iS_p}$$

Some examples:



P_C

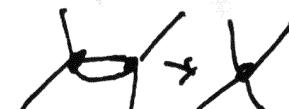
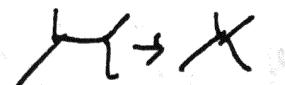


Remarks:

- (a) Each such string diagram should be understood as representing integration of all continuous deformations of the corresp. surface, i.e. view each diagram as a rubber sheet (integrating over γ_{ab} , $\tilde{\Sigma}$ corresponds to arbitrary stretching)

b) Stretching a 2-d surface is much richer than stretching a line, leading to many important new features of string theory

(1) No sharp interacting vertices

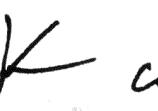
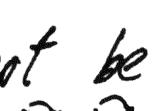
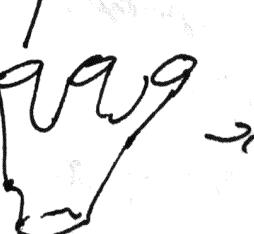
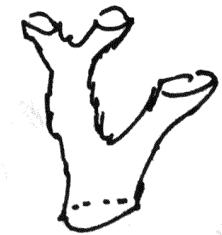
✓, UV divergences come from  

vs

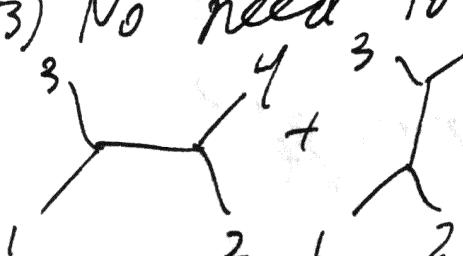
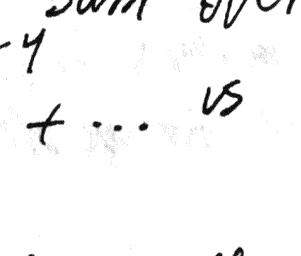
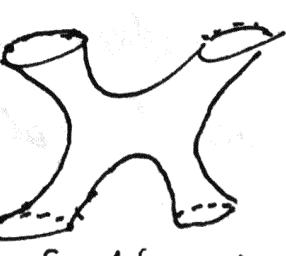
 \Rightarrow string theory has excellent UV behavior
(in fact UV finite)

(2) Interactions of strings are essentially unique:
All 2-D surfaces can be built up from:

  $\underline{g_s} \leftarrow$ key fundamental constant
in string theory

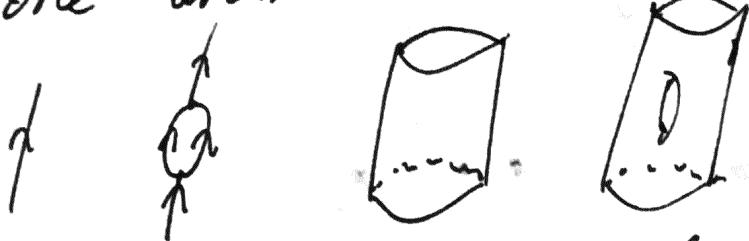
For particles,  cannot be reduced to 
but for strings  

(3) No need to sum over different channels

 +  + ... vs 

\rightarrow Tree level scattering of strings has a single diagram

c) Diagrams of different loops are given by surfaces of different topologies, which cannot be continuously deformed to one another



d) Basic theorem of topology:

Orientable 2-d surfaces are classified by a closed single integer h called genus, which is the number of holes (handles)

genus 0:



1:



2:



genus = # of loops

sum over loops = sum over topologies

\Rightarrow at each loop order we have a single diagram

$\cdot g_s$ -dependence \Rightarrow at each loop multiply by $g_s^2 \Rightarrow$ for h -loops g_s^{2h}

tree-level amplitude for n strings: g_s^{n-2}

$$A_n = g_s^{n-2} (A_n^{(0)} + g_s^2 A_n^{(1)} + \dots + g_s^{2h} A_n^{(h)} + \dots)$$

$$= \sum_{n=0}^{\infty} g_s^{n-2+2h} A_n^{(h)}$$

Path integrals over surfaces

of genus h with n external strings

also applies to $n=0, 1, 2$ from unitarity

$$n=0 \quad \text{---} \sim g_s^{-2}$$

$$n=1 \quad \text{---} \sim g_s^{-1} \quad \text{---} \quad \text{(i)}$$

$$A_n^{(0)} = \text{---} \Rightarrow \text{Each external leg can be considered associated with } g_s$$

Note: $2-n-2h = \chi_{n,h}$: Euler character for a surface of h handles and n boundaries

$$\Rightarrow A_n = \sum_{h=0}^{\infty} g_s^{-\chi_{n,h}} A_n^{(h)}$$

Open strings are the same story:



$|0\rangle \Rightarrow$ adding a loop \Leftrightarrow adding a boundary
 $\propto g_0^2$

$$f_n^{(\text{open})} = \sum_{L=0}^{\infty} g_0^{n-2+2L} A_n^{(L)}$$

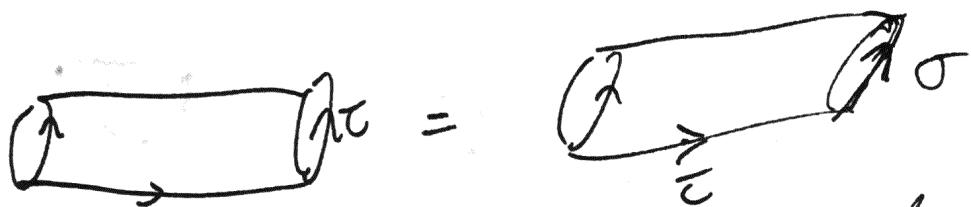
$$\begin{matrix} n = 0 \\ L = 0 \end{matrix}$$



$$g_0^{-2} \quad (\text{ii})$$

Now a profound statement:

Now a ~~proper~~
A theory of open strings must contain
closed strings



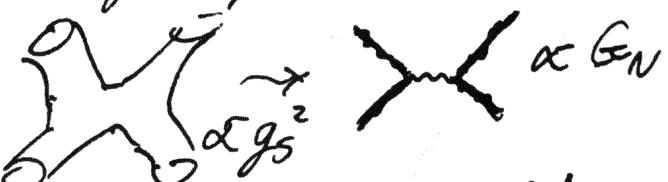
\Rightarrow 1-loop open string = tree level propagation
of closed string

For a theory with both open and closed strings, comparing diagrams (i) and (ii) gives $g_s \propto g_0^2$.

For a theory with both open and closed strings, comparing diagrams (i) and (ii) gives $g_s \propto g_0^2$.
 Altogether: An α' contributions go as $g_s^{n_c + \frac{n_2}{2} - 2 + 2h}$ the
 $\begin{cases} \# \text{closed strings} & \# \text{open strings} \\ \alpha' \text{-handles} & \# \text{boundaries} \end{cases}$

Closed string excitations: graviton, ...

$$\Rightarrow \text{gravity} \propto G_N \sim g_s^2$$



5/5

Open string excitations: An gauge Field

$\frac{g^2}{g_{YM}^2} \propto g^2$

2.2 Matrix integrals in the large- N limit
 A non-abelian gauge theory with $SU(N)$ gauge group

$$\mathcal{L} = -\frac{1}{4} \frac{1}{g_{YM}^2} \text{Tr } F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - i [A_\mu, A_\nu]$$

$$A_\mu = (A_\mu)^\alpha_b \quad a, b = 1, \dots, N$$

$N \times N$ traceless hermitian matrices

$N=3$: gluon sector of QCD

t'Hooft 1974: $N \rightarrow \infty$
 expand in $1/N$

Consider first a matrix scalar theory:

$$\mathcal{L} = -\frac{1}{g^2} \text{Tr} \left[\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{m^2}{2} \Phi^2 + \frac{1}{4!} \Phi^4 \right]$$

$\Phi = \Phi^\alpha_b$ $N \times N$ hermitian

$$(\Phi^\alpha_b)^* = \Phi^b_a$$

$$\text{i.e. } \mathcal{L} = -\frac{1}{g^2} \text{Tr} \left[\frac{1}{2} \partial_\mu \Phi^\alpha_b \partial^\mu \Phi^b_a + \frac{1}{2} m^2 \Phi^\alpha_b \Phi^b_a + \frac{1}{4!} \Phi^\alpha_b \Phi^b_c \Phi^c_d \Phi^d_a \right]$$

For any spacetime dimension d

$d=0$: Matrix integral

$d=1$: QM (Matrix)

\mathcal{L} is invariant under a $U(N)$ global symmetry

$$\Phi(x) \rightarrow U\Phi(x)U^\dagger$$

$$U \in U(N) \text{ const.}$$

Feynman rules:

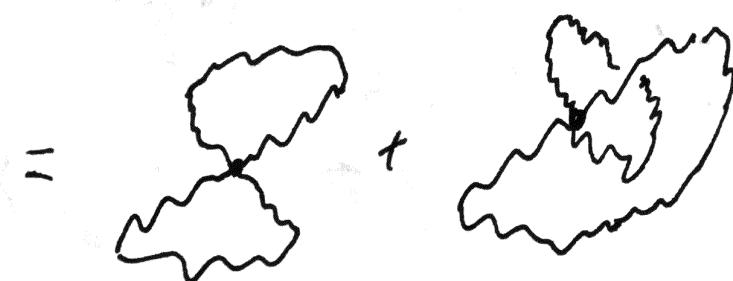
$$\begin{aligned} & \left\langle \Phi_b^a(x) \Phi_d^c(y) \right\rangle \\ &= \begin{array}{c} a \\ b \\ \sim \sim \sim \\ d \end{array} \\ &= g^2 \delta_d^a \delta_b^c G(x-y) \quad \begin{array}{l} \text{scalar} \\ \text{propagator} \end{array} \end{aligned}$$

$$= \frac{1}{g^2} S_h^a S_b^c S_d^e S_g^f$$

vacuum process:

$$Z = \int D\Phi(x) e^{i \int d^4x \mathcal{L}}$$

$\log Z = \text{sum of all connected diagrams with no external legs}$



66 | (a)

(b)

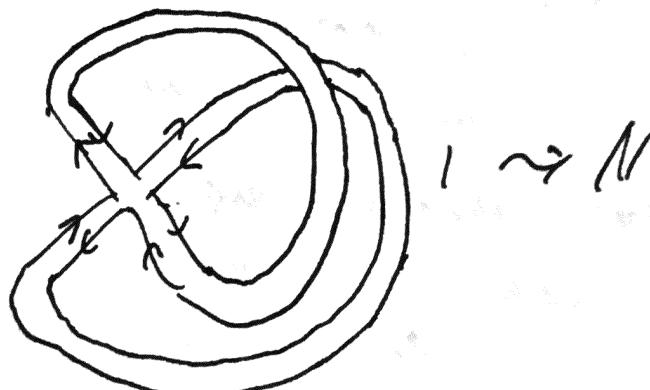
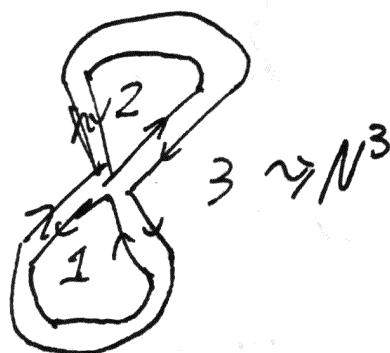
- (a) $\propto g^2 N^3$ (planar)
- (b) $\propto g^2 N$ (non-planar)

Trick (introduced by 't Hooft): double line notation

$$a \overbrace{~~~~~}^d_c = \begin{array}{c} a \\[-1ex] b \\[-1ex] c \end{array} \quad \begin{array}{c} d \\[-1ex] c \end{array}$$

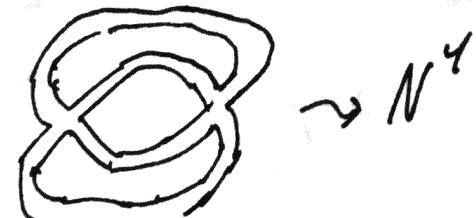
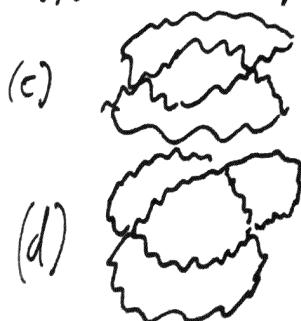
$$\begin{array}{c} a \\[-1ex] b \\[-1ex] c \end{array} \quad \begin{array}{c} h \\[-1ex] g \\[-1ex] f \\[-1ex] e \\[-1ex] d \end{array} = \begin{array}{c} a \\[-1ex] b \\[-1ex] c \end{array} \quad \begin{array}{c} h \\[-1ex] g \\[-1ex] f \\[-1ex] e \\[-1ex] d \end{array}$$

- (i) a single line: index line
indices connected by a single line are contracted
- (ii) direction: from upper to lower indices



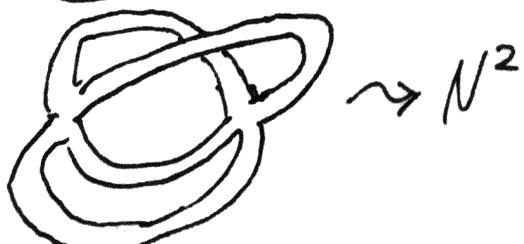
Each index loop gives $\sum_a s_a^a = N$

another example



Empirical evidence:
non-planar diagrams
have smaller
 N -dependence

(d)



Hints:

(1') These non-planar diagrams can be drawn on the torus without crossing lines

e.g. (b) can be drawn as:



(d) can be drawn as:



(2') The power of N for each diagram is equal to the number of faces after we straighten it.

Face: A connected subregion bounded by propagators in a diagram

For planar diagrams, the outside region also counts as a face.

double line: each face is bounded by an index loop

recall: an orientable closed 2-d surface is classified by its number of handles (genus) h

a genus- h surface \Leftrightarrow polygon of $4h$ sides
with opposite pairs identified

Generalize 1' and 2' to:

1. For any non-planar diagram, $\exists h$ s.t. the diagram can be drawn on a genus- h surface but not lower
 \Rightarrow diagrams can be classified by genus h

2. N -dependence is given by # of faces on genus h surface

For a general diagram:

$$\lambda \propto (g^2)^E (g^{-2})^V (N)^F \quad (*)$$

of
propagators # of
interaction
vertices # of faces

note this is unbounded
~~as N → ∞~~
 \Rightarrow (naively) there is no sensible large N limit

However, note that the Feynman diagrams triangulate the surface. For any triangulation, the Euler characteristic is an invariant of the surface:

$$\chi = F + V - E = 2 - 2h$$

$$\Rightarrow \lambda \propto (g^2)^{E-V} N^{F+V-E} E^{E-V}$$

$$= (g^2 N)^{E-V} N^{2-2h}$$

Now let $g^2 \rightarrow 0, N \rightarrow \infty$
 but $\lambda = g^2 N$ finite

$$\underbrace{E-V}_{\substack{\text{number of indep. momenta} \\ \# of loops}} = L-1 \Rightarrow \lambda \propto \lambda^{L-1} N^{2-2h}$$

sum over genus-h diagrams

$$\Rightarrow (+\text{most limit}) \log Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda)$$

$$f_h(\lambda) = \sum_{L=1}^{\infty} f_{hL} \lambda^{L-1}$$

In the $N \rightarrow \infty$ limit, Planar diagrams dominate

N.B. Though for general diagrams, the number of contributing diagrams increase factorially in N , for planar diagrams, this instead grows polynomially.

We see $\log Z \propto N^2$, and from the Lagrangian we can see this too:

$$Z = \frac{1}{g^2} \text{Tr}\left(\frac{1}{2}(D\phi)^2 + \dots\right) = -\frac{N}{\lambda} \text{Tr}(-\dots) \sim O(N^2)$$

2) This discussion only depends on structure, not on detailed ~~derivation~~ ^{form} of L or its fields (spinor, vector, etc..)

$$Z = N \text{Tr}(-\dots)$$

Last time:

$$\mathcal{Z} = N \text{Tr}(\dots) \quad U(N) \text{ symmetry}$$
$$\Rightarrow \boxed{\log \mathcal{Z} = \sum_{n=0}^{\infty} N^{2-2h} f_n(s_2, s_3)}$$

Correlation Functions

Will restrict our discussion to singlet operators
i.e. operators invariant under $U(N)$ symmetry

Such an operator must involve traces

$$\text{Tr } \overset{\alpha}{\Phi^2}, \text{Tr } \overset{\alpha}{\Phi^4}, \text{Tr}(\partial_\mu \overset{\alpha}{\Phi} \partial^\mu \overset{\alpha}{\Phi})$$

single-trace $\text{Tr } \overset{\alpha}{\Phi^2} \text{Tr } \overset{\alpha}{\Phi^4} \dots$

multi-trace

Suppose $\{\overset{\alpha}{\mathcal{O}_k}\}_{k=1,2,\dots}$ denote the set of all single-trace operators, then general singlet operators can be generated from them

Enough to restrict to correlation functions of single-trace operators.

$$\langle \overset{\alpha}{\mathcal{O}_1}(x_1) \dots \overset{\alpha}{\mathcal{O}_N}(x_N) \rangle_{c \sim \text{connected}}$$

What is the leading order N expansion?

There is a simple trick:

$$Z[\{J_i\}] = \int D\Phi \exp[iS_0 + i\int J_i \partial_i(\chi)]$$

$$= \int D\Phi e^{iS_{\text{eff}}}, \quad S_{\text{eff}} = S_0 + \frac{1}{N} \int J_i(x) \partial_i(x)$$

$$G_n = \frac{1}{i^n N^n} \frac{\delta^n \log Z}{\delta J_1(x_1) \dots \delta J_n(x_n)} \Big|_{J_i=0}$$

$$\underline{S_{\text{eff}} = N \text{Tr}(\dots)}$$

$$\Rightarrow \log Z[\{J_i\}] = \sum_{n=0}^{\infty} N^{2-2h} \cdot F_h(\{J_i, \lambda_a\})$$

$$\Rightarrow G_n = \sum_{h=0}^{\infty} N^{2-2h-n} G_n^{(h)} \xrightarrow{\text{without } N \text{ prefactor}}$$

contribution from genus h diagrams

As $N \rightarrow \infty$, at leading order $(1) \sim O(N^2)$

$$(0) \sim O(N)$$

$$(0, 0_2) \sim O(1)$$

$$(0, 0_2 0_3)_c \sim O(N^{-1})$$

$$(0, \dots, 0_n)_c \sim O(N^{2-n})$$

Remarks:

$$1) \mathcal{L} = -\frac{1}{g^2} \text{Tr} \left[\frac{i}{2} (\partial_i \Phi)^2 + \frac{1}{4} \Phi^4 \right]$$

contains other observables which are not singlets under $U(N)$

e.g. $\Phi_b^\alpha \Phi_d^\beta(x)$.

In general, such operators do

$\boxed{F_2}$ not have nice scaling with $N \rightarrow \infty$

- 2) For YM theory, $O(N)$ symmetry is local.
only singlets are allowed. $\Rightarrow \text{--}$
- 3) Almost all theories of interest to us are gauge theories $\Rightarrow \text{--}$
- 4) For gauge theories, there are also nonlocal singlet operators such as Wilson loops:

$$W(C) = \text{Tr} \left(P \exp(i \oint A) \right)$$

↑ path-ordering

They have the same large- N scaling of single-trace local operators

- The physical nature of the $N \rightarrow \infty$ limit
- (a) $\langle \phi \rangle \sim O(N) \neq 0$
 Variance of ϕ : $\sigma_\phi^2 = \langle (\phi - \bar{\phi})^2 \rangle = \langle \phi^2 \rangle - \langle \phi \rangle^2$
- we include connected
and disconnected
- ✓
✓
↳ $\alpha!$
- disconnected
cancel with this
- $\Rightarrow \frac{\sigma_\phi^2}{\langle \phi \rangle} \sim \frac{1}{N} \rightarrow 0 \quad \text{as } N \rightarrow \infty$
- i.e. no fluctuations.

similarly, n -point functions factorize

$$\langle \phi_1 \dots \phi_n \rangle = \langle \phi_1 \rangle \dots \langle \phi_n \rangle + \dots$$

is dominated by product of 1-pt Functions
"classical"

$$(b) \text{ ReDefine } \theta \rightarrow \theta - \bar{\theta}$$

$$\Rightarrow \langle \theta \rangle = 0$$

Then to leading order in large N

$$\langle g, \theta_2 \rangle = O(1)$$

$$\langle \theta, \dots \theta_n \rangle = \langle \theta, \theta_2 \rangle \langle \theta_3 \theta_4 \rangle \dots + \text{all "contractions"} \sim O(1)$$

"Gaussian Theory"

is a "Generalized Free Field Theory"

$\theta_i(t, \vec{x})$, but no com connecting $\theta_i(t, \vec{x})$ with $\theta_i(t_2, \vec{x})$

(c) Consider any connected part of the correlation functions.

$$\langle \theta_1 \dots \theta_n \rangle_c \sim O(N^{2-h})$$

This is like a tree-level theory of interacting "particles" with coupling Y_N

I imagine $\theta(x)|0\rangle$ "create a single particle"

$\theta(x_1)\theta(x_2)|0\rangle$ "two-particle"

$\theta(x_1) \dots \theta(x_n)|0\rangle$ "n-particle"

$$k Y^j \sim \frac{1}{N} \sim g, \quad X \sim g^2 \sim \frac{1}{N^2}$$

$$\langle \theta_i \theta_j \rangle \sim O(1)$$

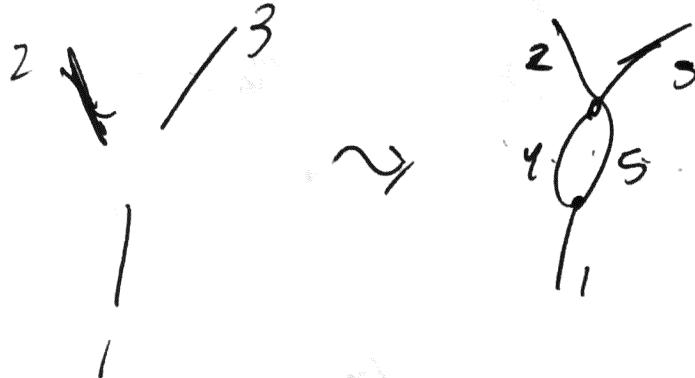
$$\langle \theta_1 \dots \theta_n \rangle \sim N^{2-h} \sim g^{h-2}$$

"tree-level"
n-particle scattering
amplitude

(d) Adding loop of O's:

Add intermediate states with more than one O's

$$\langle \theta_1 \theta_2 \theta_3 \rangle = \langle \theta_1 \theta_4 \theta_5 \rangle \langle \theta_4 \theta_5 \theta_2 \theta_3 \rangle - N^1 N^{-2} \sim N^{-3}$$



Adding loop \Rightarrow adding $1/N^2$

subleading terms in $1/N^2$, "loop" corrections

2.3 Strings and Matrices

$$A_n = \sum_{h=0}^{\infty} g_s^{n-2+2h} A_n^{(h)}$$

$$G_n = \sum_{h=0}^{\infty} N^{2-n-2h} G_n^{(h)}$$

scattering of strings \longleftrightarrow of single-trace operators

Topology of w.s. \longleftrightarrow Topology of Feynman diagrams

$$g_s \longleftrightarrow 1/N$$

$A_0^{(h)}$: integrate over
ws of genus h $\leftrightarrow G_n^{(h)}$ sum over genus h
Feynman diagrams

external string \leftrightarrow single-trace op

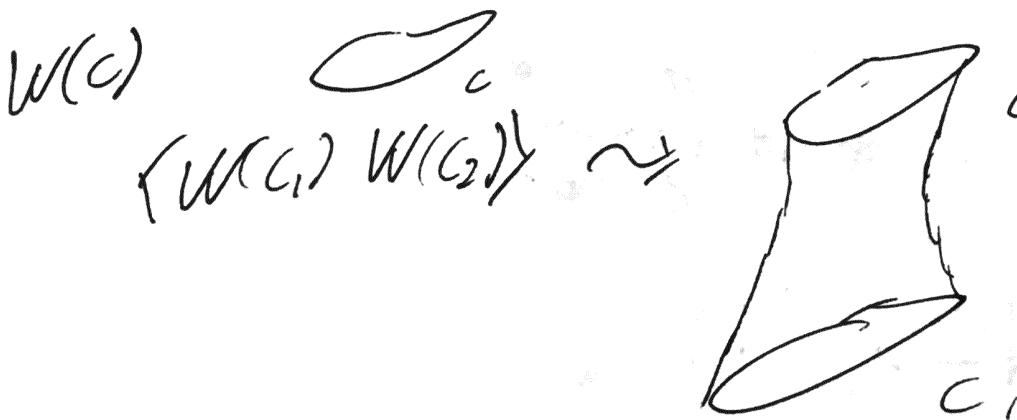
loops of string \leftrightarrow "loops" of single-trace op

Rough argument:

$$A_0^{(h)} = \sum_{\substack{\text{sum} \\ \text{over genus} \\ h \text{ surfaces}}} e^{iS_{\text{string}}} = \sum_{\substack{\text{triangulations} \\ \text{of genus } h \\ \text{surfaces}}} e^{iS_{\text{string}}}$$

$$\Downarrow$$

$$G_0^{(h)} = \sum_{\substack{\text{genus-}h \\ \text{Feynman} \\ \text{diagram}}} \tilde{G} = \sum_{\substack{\text{triangulations} \\ \text{of genus } h \\ \text{surfaces}}} \tilde{G}$$



$$\begin{aligned} A_{ab}^d &= a \sim b \\ W(C) = e^{iS_C} &\rightarrow \tilde{G} \\ \varphi^d &= a \sim b \\ b &= a \sim b \\ \text{Tr } \Phi^2 &\rightarrow \tilde{G}_b \end{aligned}$$

$\Rightarrow W(C)|0\rangle$ can be considered as macroscopic string states
 $i\int d^3x \sim \partial(x)|0\rangle$; microscopic string state

Oct 24, 2018 (Notes from Sam Zentner)

$\int_{\Gamma}^{(h)}$: sum over random triangulations
of a genus-h surface (and its embeddings)
weighted by $e^{iS_{\text{string}}}$

\int_{gen} : sum over triangulations
of genus h surfaces
weighted by E

Conjecture:

a duality of:

a large- N matrix-like \longleftrightarrow a string theory
theory

$1/N \longleftrightarrow g_s$

single trace operator \longleftrightarrow single string

To establish this duality:

S_{String} [$\gamma_{ab}, X^{\mu}, \dots$], $X^{\mu}(0, t)$: Worldsheet $\rightarrow M$ "spacetime"
manifold

\rightsquigarrow Continuum string picture should emerge in regime
where "infinitely complicated" Feynman diagrams dominate

Remarks:

(1) So far, only matrix-valued fields ^{have been} considered.
This also includes fields transforming in the fundamental
representation of $U(N)$ $q = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$ "quarks"

$$\langle q^a q_b \rangle = \overbrace{\qquad}^a \overbrace{\qquad}^b$$

Precise mapping to a string theory with both closed and open strings (i.e. quarks add open strings)

(2) In addition to $U(N)$, can also consider

$$SO(N), Sp(N)$$

$$\langle \Phi_{ab} \Phi_{cd} \rangle = \overbrace{\qquad}^a \overbrace{\qquad}^d \quad (\text{no arrows})$$

\Rightarrow include non-orientable surfaces

\sim maps to non-orientable string theory

Explicit example: (0-dimensional)

$$e^{-Z} = \int dM \exp\left[-\frac{N}{g} \text{Tr}[V(M)]\right]$$

M = hermitian matrix

$$V(M) = \frac{1}{2}M^2 + \sum_{k \geq 3} a_k M^k$$

$$\sim Z = Z_0 + Z_1 + \dots$$

$$Z_0 \sim O(N^2)$$

$$dM = \prod_{a,b} dM_a^a \delta_{ab} \quad (\text{for } a=b \text{ this is the usual } dM_a^a \in \mathbb{R})$$

for off-diagonal $dM_a^b = dM_a^a \frac{\partial}{\partial M_b^a}$

Since $\text{Tr}(V(M))$ depends only on eigenvalues,
 write $M = U^\dagger \Lambda U$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$
 \rightsquigarrow Measure might depend on λ_i :
 $\Rightarrow \text{Tr}(V(M)) = \sum_{i=1}^N V(\lambda_i)$, $dM = \prod_{i=1}^N d\lambda_i \Delta^2(\Lambda) "DU"$

It turns out $\Delta(\Lambda) = \prod_{i < j} (\lambda_i - \lambda_j)$ (Vandermonde determinant)
shown in problem set

$$\Rightarrow e^{-Z} = \int \prod_{i=1}^N d\lambda_i \Delta^2(\Lambda) e^{-\frac{N}{2} \sum_i V(\lambda_i)} \leftarrow \underbrace{N\text{-sum}}_{\text{in } O(N^2)}$$

\rightsquigarrow Naive saddle point: $V'(\lambda) = 0$ (incorrect)

$$\Delta^2(\Lambda) = \exp \left[\sum_{i < j} \log(\lambda_i - \lambda_j)^2 \right] \leftarrow \begin{array}{l} \text{cant have } \lambda_i = \lambda_j \\ \text{as } \log +\infty \end{array}$$

double sum
 $\Rightarrow O(N^2)$ too

"level repulsion"

$\Delta^2(\Lambda)$ i.: 1) $O(N^2)$

2) Repulsion between λ_i 's

$\Rightarrow \lambda_i$ cannot all sit at minimum

BDM for λ_i :

$$2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \frac{N}{g} V'(\lambda_i) \quad (*) \quad (\text{get this by directly differentiating})$$

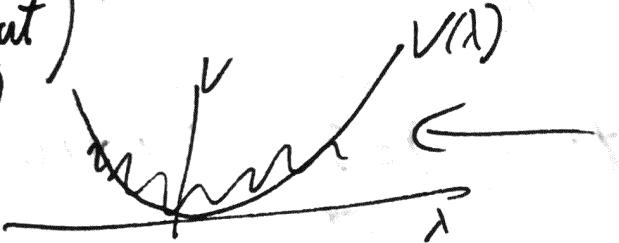
$N \rightarrow \infty$ limit (expect $p(\lambda)$ to form a continuous function)

$$\int_{-\infty}^{\infty} p(\lambda) d\lambda = 1, \quad p(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i)$$

$$\Rightarrow \left[2P \int d\lambda' \frac{p(\lambda')}{\lambda - \lambda'} \right] = \frac{1}{g} V'(\lambda) \quad (**)$$

principal value
(i.e. throw out)
 $i=j$

Note:



λ 's pushed out, but not to ∞ as this would cause infinite energy so as $N \rightarrow \infty$, we get continuous distribution of λ : (not infinite range)

Now assume $p(\lambda)$ only supported on finite interval in \mathbb{R} , denoted by I

Introduce:

$$F(\xi) = \int_I d\lambda' \frac{p(\lambda')}{\xi - \lambda'} \quad \text{for general complex } \xi$$

Then $(**)$ is equivalent to: $P F(\xi) = \frac{1}{2g} V'(\lambda)$

$$\text{at } \xi = \lambda - i\varepsilon$$

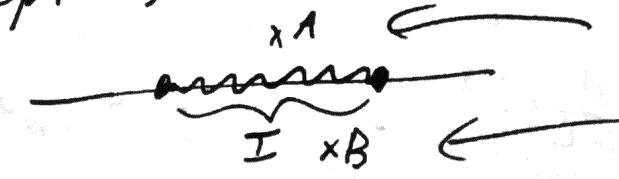
for $\lambda \in I$

The preceding equation comes from the identity:

$$\frac{1}{x-i\epsilon - \lambda} = P \frac{1}{x-\lambda} + i\pi S(x-\lambda) \quad (\star\star\star)$$

$F(\xi)$ satisfies the following analyticity properties:

- (1) Analytic function on complex plane except for branch cut at I



compare A and B values using $(\star\star\star)$ to see branch cut

- (2) On real axis, $F(\xi)$ is real for $x \notin I$

(3) $F(\xi) = \frac{1}{\xi} + \dots$ as $\xi \rightarrow \infty$

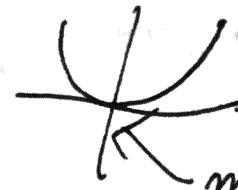
since for $\xi \notin I$, $\frac{1}{\xi-x} \rightarrow \frac{1}{\xi} \frac{1}{1-\frac{x}{\xi}}$, $\int_0^1 = 1$

(4) ~~Re~~ $\operatorname{Re}(F(x-i\epsilon)) = \frac{1}{2\pi} V(x)$

$$\operatorname{Im}(F(x-i\epsilon)) = \pi P(x)$$

These conditions determine $F(\xi)_n$ (and therefore $P(\lambda)$) completely

Example: $V(\lambda) = \frac{\lambda^2 + 1}{2}$



min at 0

take $I = [-a, a]$, a to be determined

White ansatz for $F(\xi)$:

$$F(\xi) = \frac{1}{2g} V'(\xi) + f(\xi) \sqrt{\xi^2 - a^2} \quad f \text{ is TBD}$$

\leftarrow pure imaginary for $\xi \in I$

$$(3) + \text{analyticity gives } F(\xi) = -\frac{1}{2g} (1 + g x^2 + 2a^2)$$

$$a^2 = \frac{1}{6} (\sqrt{1+48g} - 1)$$

$$\Rightarrow \rho(\lambda) = \frac{1}{2\pi g} (4\lambda^2 + 1 + 2a^2) \sqrt{a^2 - \lambda^2} \quad \leftarrow \lambda \in [-a, a]$$

$$\Rightarrow Z_0 = N^2 \left[-\frac{1}{g} \int_{-a}^a d\lambda \rho(\lambda) V(\lambda) - P \right] \int_{-a}^a d\lambda \int_a^\infty du \rho(\lambda) \rho(u) \log(\lambda u)$$

from potential from Vandermonde

Maps to string theory in 2 dimensions!

Oct 29 (Notes from Sam Leethusser)

Remarks

- (0) $g \rightarrow 0$, find $\rho(\lambda) = \frac{1}{2\pi} \sqrt{4-\lambda^2} \leftarrow$ Wigner distribution for N gaussian
- (1) $Z_0[g]$ is analytic in g for small g . When expanded as a power series, there is a finite radius of convergence.
 ~ If g flips sign, one would expect $\leftarrow \rightarrow$
- essential sing. at $g=0$ (theory)
 (comes from total & Feynman diagrams $\sim n!$)
 But planar diagrams are polynomial and contribute most at large N is unstable)
- (2) a^2 has branch point in g at $\leftarrow g=g_c = -\sqrt{g_8}$
 Perturbation theory breaks down there \leftarrow ratio of conv. = $\sqrt{g_8}$
 near g_c , $Z_0[g] \sim$ analytic in $g + \epsilon (g_c - g)^{1/2} \dots$
- (3) From perspective of summing planar diagrams, to see this non analytic behavior one must sum full series (all powers of g)
 $(g_c - g)^{1/2} \sim \sum_n \begin{matrix} \text{planar diagram} \\ \text{singularity} \end{matrix} \begin{matrix} (g) \\ (g_c) \end{matrix}^n \leftarrow$ at large n
 $\begin{matrix} \text{non-analytic} \\ \text{behavior} \end{matrix}$
- (4) Only in $g \rightarrow g_c$ limit can we expect a continuum description of string theory to emerge (need arbitrarily complicated triangulations for continuum)
- (5) Consistency check: all Z_h need non-analytic behavior at $g=g_c$ for continuum limit to be $Z_n[g]$ non-analytic at $g=g_c = -\sqrt{g_8}$ \leftarrow string theory $\boxed{183}$

near g_c , $\beta_n[g] \sim |g-g_c|^{\frac{\chi}{2}(2-\Gamma)}$ $\chi=2-2g$ (Euler char)
 $\Gamma=-\frac{1}{2}$ in this case

(6) String theory dual

$X: (\sigma, \tau) \rightarrow M$, g_{ab} , \varPhi : internal worldsheet d.o.f.

0-d string theory: M is a point, \mathcal{I} is trivial

$$Z_{\text{string}} = \int Dg_{ab} D\varPhi e^{-S_{\text{string}}[g, \varPhi]} \quad \text{spacetime is a point so work in Euclidean picture}$$

$$S_{\text{string}} = \mu \underbrace{\int d^2\sigma \delta g}_{\text{Area, } A} + \lambda \underbrace{\frac{1}{4\pi} \int d^2\sigma \sqrt{R}}_{\text{Euler number, } \chi=2-2h} + S_{\text{matter}}[\varPhi]$$

$$\Rightarrow S_{\text{string}} = \mu A + \lambda \chi + S_m[\varPhi]$$

Identify $\frac{1}{N} \sim e^\gamma$ (controls expansion in genus)

$\mu \propto (g-g_c)^n$ \leftarrow power ends up being 1 since μ is chemical potential for area: $\int_\mu Z = (A)$
 only param $\Rightarrow g \rightarrow g_c \Rightarrow n$ in expansion of $(g-g_c)^n$

$$\Rightarrow Z_{\text{string}} = \sum_h \int D\gamma D\varPhi e^{-\mu h - \lambda \chi - S[\varPhi]} \quad \text{is } \simeq \text{area}$$

After gauge fixing, reparam. invariance of γ allows us to take \varPhi to be the remaining d.o.f. of γ

$$\frac{1}{84} \Rightarrow Z_{\text{string}} = \sum_h \int D\varPhi D\varPhi e^{\frac{1}{84} \int d^2\sigma [-S_L(\varPhi) - S_{\text{matter}}(\varPhi)]}$$

Here $S_L = \int d^2\sigma \sqrt{f_0} [(\partial\varphi)^2 + \varphi \cdot \nabla R + \mu e^{2\varphi}]$

\uparrow
in
gauge
fixed parameter

$\Rightarrow \varphi$ behaves like an inhomogeneous "emergent" spatial dimension. Even though X^a is trivial, we grow a dimension.

$$Z_h^{(\text{string})} \propto \mu^{\frac{X}{2}} Z_{\text{string}}$$

For Smaller μ trivial, we find $\Gamma_{\text{string}} = -\frac{1}{2}$.
(matches matrix theory)

2.3 String Description of a Gauge theory

Non-Abelian $SU(N)$ gauge theory \Leftrightarrow String theory?
 $N \times \infty$ in d -dim

A simplest guess: Maybe a string theory in Mink_d
 \rightarrow This does not work!

(1) Such a string theory appears inconsistent
A string theory in Mink_d is consistent only for $d=26$
 $\begin{cases} d=26 & (\text{bosonic}) \\ d=10 & (\text{superstring}) \end{cases}$

(2) How about Mink_d $\times N$ compact
so string theory has $SO(3,1)$ symmetry
But all string theories on Mink_d $\times N$ have 4d graviton \rightarrow violates
W-N applies because $SU(N)$ theory has gauge-in conserved
stress tensor: 185

→ We'd want a string theory "without gravity"?
 → impossible!

Hints:

- (1) \varPhi with Minky combines to Form 5d curved spacetime
 → 5d non-compact spacetime
 → 5d gravity
 → Does not contradict Weinberg-Witten

(2) Holographic Principle

4d gauge theory could in principle be related to 5-d gravity

→ One could try a string theory in $15 \times N$

⇒ Y_5 should have all the Minky symmetries
 $\sim ds^2 = g(z)(dt^2 - dx^2 + dz^2)$

$$f(z)$$

$$= Q(z)(-dt^2 + dx^2 + dz^2) \quad (\text{after redefinition})$$

Suppose the 4-d theory is scale-invariant

$$(\star\star) \quad (t, \vec{x}) \rightarrow \lambda(t, \vec{x})$$

(*) must be invariant under (**)

\Rightarrow must have $z \rightarrow \lambda z$, ~~$\Omega(z) \neq \Omega(\lambda z)$~~

$$\Omega(\lambda z) = \frac{1}{\lambda} \Omega(z)$$

This fixes $\Omega(z) = c/z$

so, for scale-invariant lower dimensional theory, we would have

$$ds^2 = \frac{R^2}{z^2} [-dt^2 + d\vec{x}^2 + dz^2]$$

R is some constant

This is AdS.

AdS₅ has isometry group SO(2, 4)

188

Last time:

$$Y_5 \times \mathcal{N}$$

non-compact

does not contradict
Weinberg - Witten

$$ds^2 = Q(z)(-dt^2 + dx^2 + dz^2) \quad \leftarrow \begin{array}{l} \text{1997 Polyakov wrote this down to} \\ \text{study QCD.} \end{array}$$

$$\text{scale-invariant: } (t, \vec{x}) \rightarrow (\lambda t, \vec{x}) \\ z \rightarrow \lambda z \Rightarrow Q(\lambda z) = \frac{1}{\lambda} Q(z)$$

$$\Rightarrow Q(z) = \frac{R}{z} \quad (R: \text{const})$$

$$\Rightarrow ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2)$$

determined uniquely
(up to R)

AdS_5 space

scale invariance \rightsquigarrow conformal invariance $SO(2, 4)$
isometry

Outline of string theory
& derivation of AdS/CFT

(a) closed strings

quantization of a closed string in a fixed spacetime:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu \quad (*)$$

$$X^\mu(0, \tau) = X^\mu(0+2\pi, \tau)$$

Tricky to quantize, but the procedure is well-known

\Rightarrow String theory excitation spectrum

(1) Not all spacetime allow a consistent string propagation quantum mechanically

(2) For (1), we have bosonic string theory:

Mink.: $d=26$

Taking (X^μ, ψ^α) : superstring
super coordinates

Mink.: $d=10$

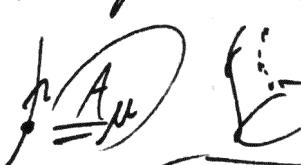
(2) Spectrum:

Oscillation excitation of a string \leftrightarrow spacetime particle

Massless: $h_{\mu\nu}, B_{\mu\nu}, \varPhi$,
universal to all string theories

Massive: $m^2 = \frac{\mu}{\alpha'}$
(infinite towers of massive modes)
 $\alpha' = l_s^2$ l_s : string length

$h_{\mu\nu}$: massless spin-2 (graviton)

$B_{\mu\nu} = -B_{\nu\mu}$:  "gauge field" for a string (WII)

antisymmetric tensor

90]

\varPhi : free parameter

$g_s = \langle e^\varPhi \rangle$ e^\varPhi can vary if \varPhi does coupling of the string

$$G_N \propto g_s^2 \Rightarrow G_N = \kappa g_s^2 \alpha'$$

At low energies: $E^2 \ll \gamma$
 effective theory: Einstein gravity + matter (massless)
 + higher derivative corrections

Due to presence of graviton;
 spacetime metric becomes dynamical
 $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

\Rightarrow closed strings must themselves be excitations
of spacetime (*this logic is still not clear to me)

(3) Bosonic string:
 there exists excitation with negative N
 i.e. $\frac{m^2}{l^2} < 0$
 can be

Superstrings ~~are~~ stable open question

Why 5 different perturbative superstrings
 We now know they are related nonperturbatively
 Their spectra contain spacetime fermions

At low energies: Supergravity

interesting note:
 there are "other" superstring theories
 with only bosons but emergent fermions.

Massless: ($d=10$)

IIA: $h, B^{(2)}, \mathcal{L}, C_\mu, C_{\mu\nu}^{(3)}$, + fermions

IIB: $h, B^{(2)}, \mathcal{L}, \chi, C_{\mu\nu}^{(2)}, C_{\mu\nu\rho}^{(4)}$ + fermions

$C^{(2)}, C^{(3)}, C^{(4)}$ are fully anti-symmetric

$C^{(4)}$: self-dual

$$F^{(5)} := dC^{(4)}, \quad F^{(5)} = *F^{(5)}$$

(b) Open string

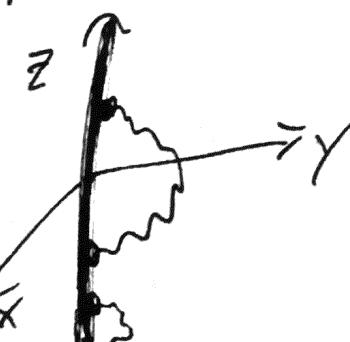
need to impose boundary conditions
at ~~all~~ end points

They can only end on some "special places"
which can be considered as some kind of
"defects" in spacetime

"D-branes"

a D_p -brane has p spatial dimensions

$D1$ -brane



$D3$ -brane

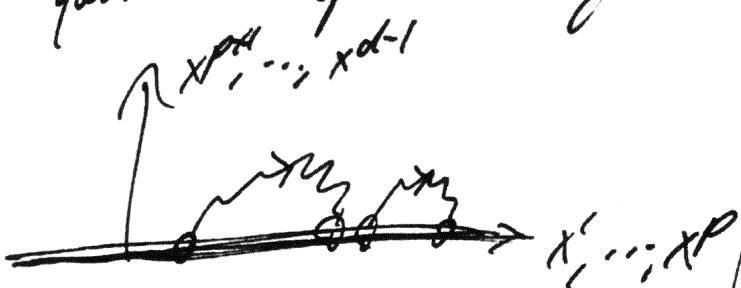
"spacetime-filling"
~ can end anywhere

Classically, D-brane can be considered as a proxy of specifying boundary conditions on an open string

D0-brane



Given a D-brane configuration, you can quantize open string ending on it.



quantize open strings:

massless: $A_\mu(x^\nu)$

$$\mu\nu = 0, \dots, P$$

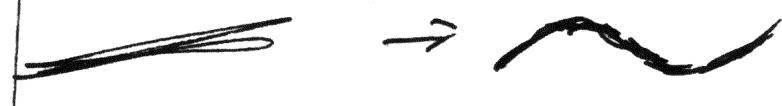
$$\varphi^i(x^\alpha) \quad i = p+1, \dots, d-1$$

massive: $m^2 \equiv N/\alpha'$

endpoint of a string:
charged particle

A_μ : gauge field for the charge

φ^i : transverse motion
of the brane itself



- ⇒ D-branes are dynamical objects
- ⇒ open strings are excitations of D-branes

Some D-branes have $N < 0 \Rightarrow$ unstable excitations
⇒ decay into closed string modes

There are stable-situations

I A 0, 2, 4, 6, (8) (even-dimensional)

I B (-1), 1, 3, 5, 7, (9) (odd-dimensional)

An object is stable if it carries some sort of "gauge" charge.

A p-dimensional object can carry charge of $(p+1)$ -form



Nov 5, 2018

$$x^\mu(\sigma, \tau) \sim x^\mu(0, \tau) \Rightarrow \begin{cases} \text{Dirichlet} \\ \partial_\sigma x^\mu = 0 \text{ Newman} \end{cases}$$

$\sigma \in [0, \pi]$

$\sigma=0$

$$x^\mu(0, \tau) \quad \mu = 0, 1, \dots, p \quad \text{Newman}$$

$$x^i(0, \tau) = a^i \quad i = p+1, \dots, d-1 \quad \text{Dirichlet}$$

"no momentum exits from the string"

Massless excitations:

$$A_\mu(x^\nu) \quad \mu, \nu = 0, 1, \dots, p$$

$$\Phi^i(x^\mu) \quad i = p+1, \dots, d-1$$

Massive: $m^2 = N \alpha'$ ← again can be negative \Rightarrow unstable D-branes

- ⇒ D-branes are dynamical objects
- ⇒ Open strings are excitations of D-branes

Stable D-branes:

I A: even dimensions
0, 2, 4, 6, (8)

I B: odd-dimensional ones

(-1), 1, 3, 5, 7, (9)

$$h_{\mu\nu}, B_{\mu\nu}, \Phi + \begin{cases} C_{\mu}^{(1)}, C_{\mu\nu}^{(3)} \\ X, C_{\mu\nu}^{(2)}, C_{\mu\nu\rho}^{(4)} \end{cases} \quad \begin{array}{l} IIA \\ IIB \end{array} \quad \begin{array}{l} \text{Massless,} \\ \text{closed string} \\ \text{spectrum} \end{array}$$

Anti-symmetric potentials are generalization of U(1) Maxwell field to higher forms

$$A_\mu \quad A = A_\mu dx^\mu$$

$$F = dA \quad \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$C^{(n)} = C_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

$$F^{(n+1)} = dC^{(n)} \quad \mathcal{L} = -\frac{1}{2} \frac{1}{n!} (F \wedge \star F)$$

$$C^{(n)} \rightarrow C^{(n)} + dA dx^{(n+1)}$$

\rightsquigarrow point charged particle with path x^μ couples to A as $\int_C A_\mu \frac{dx^\mu}{d\tau} d\tau = \int_C A$

p -dimensional object has $(p+1)$ -world volume Σ

$$\int_{\Sigma} C^{(p+1)} = \int_{\Sigma}^{(p+1)} C_{\mu_1 \dots \mu_{p+1}}^{(p+1)} \frac{\partial X^{\mu_1}}{\partial \xi^1} \wedge \dots \wedge \frac{\partial X^{\mu_{p+1}}}{\partial \xi^{p+1}}$$

with $X^\mu(\xi)$ denoting embedding of Σ in spacetime

$$\text{For any } C^{(n)} \rightarrow F^{(n+1)} = dC^{(n)} \rightarrow \tilde{F}^{(d-n-1)} = \star F^{n+1}$$

$$\tilde{F}^{d-n-1} = dC^{(d-n-2)}$$

Given $C^{(n)}$ there is a $p=n-1$ dimensional object charged under "electric" and a $d-n-3$ dimensional object charged under "magnetic"

B_{MN} : String electric
 NS5-brane magnetic

IIA:	$C^{(1)}$	DO-brane	(E)
		D6-brane	(M)
$C^{(3)}$		D2-brane	(E)
		D4-brane	(M)
IIB:	$\oplus X$	D(-)-brane	(E)
	$\oplus Y$	D7+brane	(M)
	$C^{(2)}$	D-string	(E)
		D5-brane	(M)
$C^{(4)}$		DY brane	(EM)

Properties of D-branes:

- At low energies

A single D-brane $\Rightarrow U(1)$ Maxwell + Φ^i (free scalar)
 $\text{on } d-p-1$

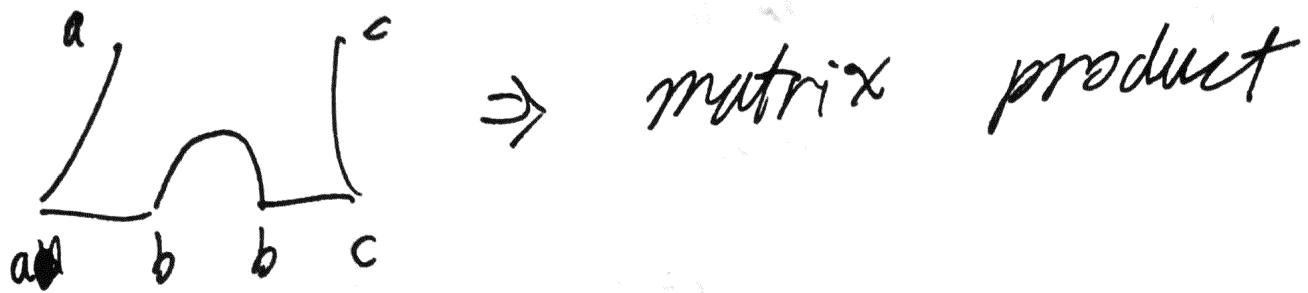
N coincidental D-branes:

N D-branes right on top of one another

\rightsquigarrow Each string has two endpoints \nexists two labels $1, \dots, N$

\Rightarrow massless fields $(A_M)_b^a$ $a, b = 1, \dots, N$

$(\Phi^i)_b^a$ you can show these are hermitian
 $\Rightarrow U(N)$ /97



low-energy limit:

$$S = -\frac{1}{g_{YM}^2} \int d^4x \text{Tr} \left(\frac{1}{4} F^2 + D_\mu \Phi^i D^\mu \Phi_i + [\Phi^i, \Phi^j] \right) + \dots$$

(fermionic terms)

$$F_{\mu\nu} = \partial_\mu A_\nu - i [A_\mu, A_\nu]$$

$$D_\mu \Phi^i = \partial_\mu \Phi^i - i [A_\mu, \Phi^i]$$

$$g_{YM}^2 \sim g_S$$

Maximally supersymmetric YM

D3-brane : $\Rightarrow p+1=4$ $N=4$ SYM

When we separate the branes:

$$\langle \Phi^i \rangle_a^a \neq 0$$

$$U(N) \rightarrow U(n_1) \times U(n_2) \cdots \times U(n_R)$$



D-branes gravitate

a charged point particle in $D=4$

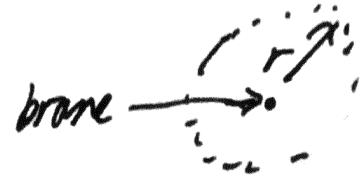
$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G_N} R - \frac{1}{4} F^2 \right] - m \int_C ds + q \int A$$

$$\partial_\mu F^{\mu\nu} = j^\nu \quad (\text{particle})$$

$$G_{\mu\nu} = 8\pi G_N (T_{\mu\nu}(\text{particle}) + T_{\mu\nu}(\text{EM}))$$

E.g. Consider a D3-brane

1,2,3 4,5,6,7,8,9



N D3-branes

$$ds^2 = f(r)(-dt^2 + d\vec{x}^2) + g(r)(dr^2 + r^2 d\Omega_5^2)$$

$$f^{-1} = g = \left(1 + \frac{R^4}{r^4}\right)^{1/2}, \quad R^4 = 4\pi g_s (\alpha')^2 N \alpha'^{-1} T_3$$

$r=0 \rightarrow$ location of D3-brane

$r \rightarrow \infty \rightarrow$ "Minkowski space"

tension
of D3-brane

R characterizes when gravitational effect becomes strong.

note that

Schwarzschild metric has a \equiv sign

$$\text{At } r \rightarrow 0, f(r) = \frac{r^2}{R^2}, \quad g(r) = \frac{R^2}{r^2}$$

$$\Rightarrow ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}(dr^2 + r^2 d\Omega_5^2)$$

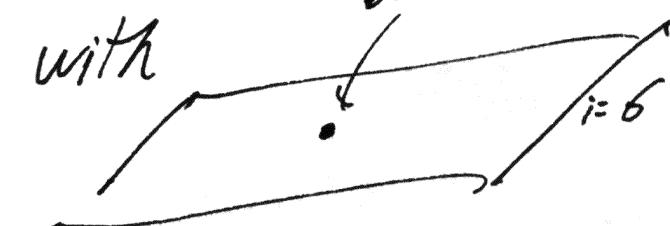
$$= \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \underbrace{\frac{R^2}{r^2} dr^2}_{r} + R^2 d\Omega_5^2$$

$AdS_5 \times S^5$

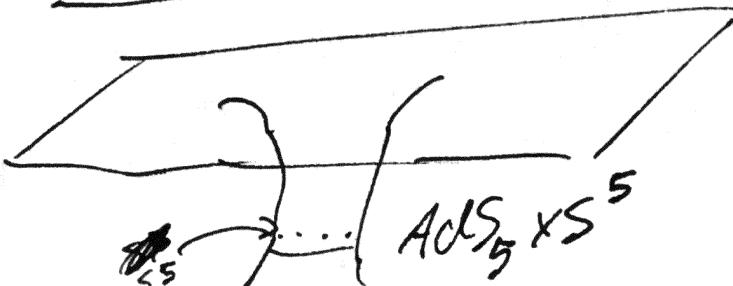
AdS_5

S^5 never shrinks to 0 size
(const radius) $= R$

Started with



Became:



$$\frac{dr^2}{r^2} = dl^2$$

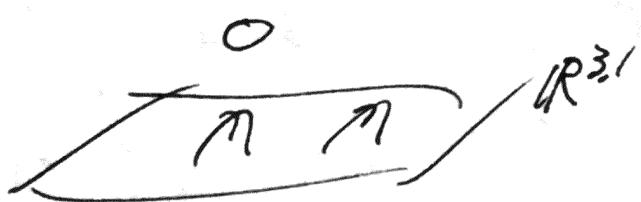
$$l = \log r$$

($r=0$ still an infinite proper distance $\propto 1/l$)

1001

We thus have two descriptions of D3-branes

- (A) D-branes in flat $Mink_{10}$ where open strings can live

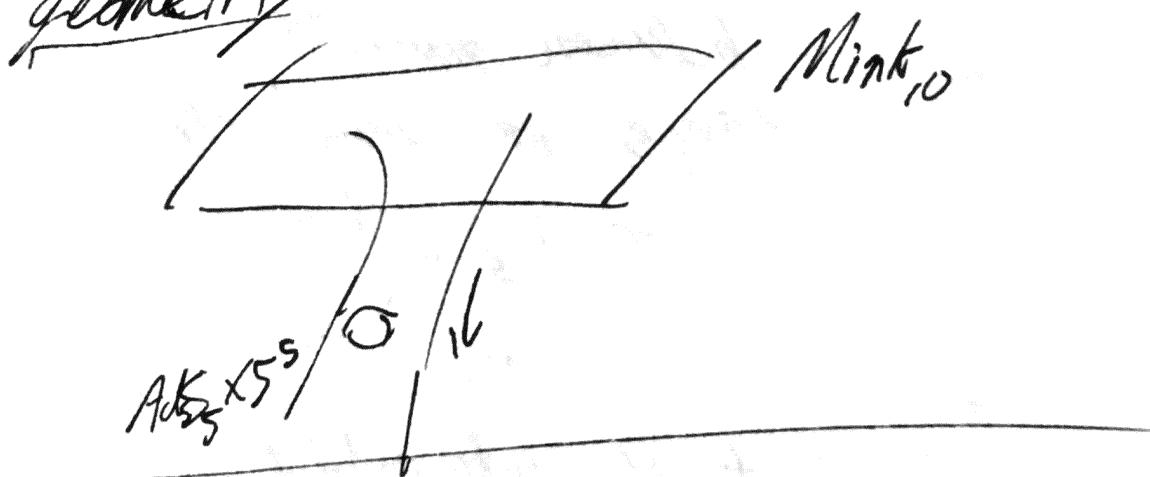


- (B) Using spacetime metric:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + R^2 dr^2 + R^2 d\vec{\theta}_S^2$$

(+ F_5 flux on S^5)

\leadsto only closed strings that see a curved geometry



$$A = B$$

Both descriptions can in principle be valid
For all α' and g_S

Maldacena (1997)

low energy limit \Rightarrow AdS/CFT

What is the low energy limit?

Fix E , take $\alpha' \rightarrow 0$
(Fix α' , $E \rightarrow 0$) $\Rightarrow \alpha' E^2 \rightarrow 0$

(A): Open string \Rightarrow $N=4$ Super Yang-Mills
sector with gauge group $U(N)$

$$g_{YM}^2 = 4\pi g_s$$

Closed string \Rightarrow graviton, dilaton
sector

couplings between massless closed and
open strings, or closed strings themselves

$$G_N \propto g_s^2 \alpha'^4$$

$$E \rightarrow 0 \Rightarrow G_N E^8 \rightarrow 0$$

So we get the interacting $N=4$ theory
+ free massless modes

as $E \rightarrow 0$

As before:

$$(B): ds^2 = f(r)(-dt^2 + d\vec{x}^2) + g(r)(dr^2 + r^2 d\Omega_S^2)$$

$$f^{-1} = g = \left(1 + \frac{R^4}{r^4}\right)^{1/2}$$

Curved spacetime: must be careful to specify what "energy" to use

E in (A): defined w.r.t. t (i.e. time at $r=\infty$)

At r : local proper time

$$dt = \sqrt{g} dt$$

$$\Rightarrow E_T = \sqrt{g} E$$

For $r \gg R$: ~~$\sqrt{g} \sim 1$~~

$E_{\alpha'}^2 \rightarrow 0 \Rightarrow$ all massive closed strings decouple

For $r \ll R$ $g \sim \frac{R^2}{r^2}$

$$E_{\alpha'}^2 \rightarrow 0 \Rightarrow E_T^2 \frac{r^2}{R^2} \alpha' \rightarrow 0$$

$$\Rightarrow E_T^2 \frac{r^2}{\sqrt{4\pi g_S N}} \rightarrow 0$$

\Rightarrow For any E_T , low energy limit has $r \rightarrow 0$ Minkowski
 So low energy limit gives free gravitons at $r = \infty$
 & full string theory in $AdS_5 \times S^5$

(with flux) 103

A
↓
 $N=4$ SYM theory
+ free graviton

= B
↓ (at energy limit)
IIB String in $AdS_5 \times S^5$
+ free graviton

\Rightarrow $N=4$ SYM
theory with gauge
group $U(N)$ = IIB String in
 $AdS_5 \times S^5$

Wednesday Nov. 14, 2018

$N=4$ SYM theory with gauge group $U(N)$ in $M_4 = (1,3)$
15

IIB string theory in AdS

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx^2) + \frac{R^2 dr^2}{r^2} + R^2 d\Omega_5^2$$

plus F_5^+

$$g_{ym}^2 = 4\pi g_s \rightarrow \lambda = g_{ym}^2 N = \frac{R^4}{(\alpha')^2}$$

$$\underbrace{16\pi G_N}_{\text{fundamental}} = (2\pi)^7 g_s^2 (\alpha')^4 \Rightarrow \frac{G_N}{R^8} = \frac{\pi^4}{2N^2}$$

2.5 Anti-de Sitter spacetime AdS_{d+1}

Homogeneous spacetime of constant negative curvature. Consider hyperboloid in $(2, d)$

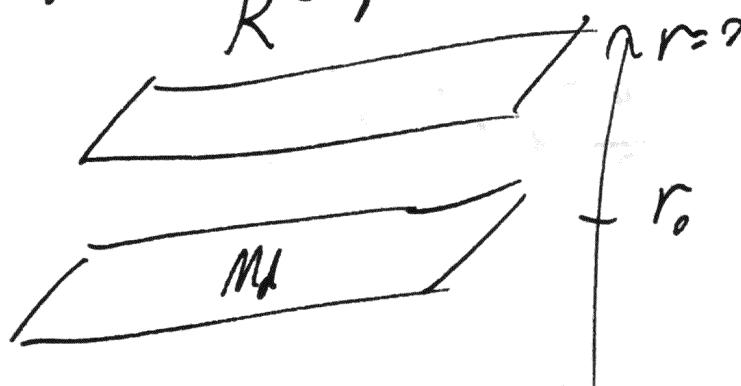
$$x_{-1}^2 + x_0^2 - \sum_{i=1}^d x_i^2 = R^2$$

with metric $ds^2 = -dx_{-1}^2 - dx_0^2 + \sum_{i=1}^d dx_i^2$
 $\approx SO(2, d)$ isometry

(i) Poincare coordinates

$$r = X_{-1} + X_d, \quad X^{\mu} = R \frac{X^{\mu}}{r} \quad \mu = 0, \dots, d$$

$$\Rightarrow ds^2 = \frac{r^2}{R^2} \eta_{\mu\nu} dx^{\mu} dx^{\nu} + \frac{R^2}{r^2} dr^2$$



(ii) Global coordinates take $X_{-1} = R\sqrt{1+\rho^2} \cos\chi \quad X_0 = R\sqrt{1+\rho^2} \sin\chi$

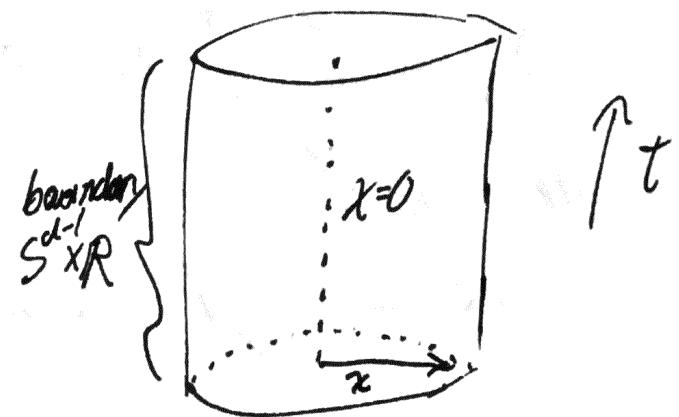
$$\sum_{i=1}^d X_i^2 = R^2 \rho^2 \Rightarrow X_{-1}^2 + X_0^2 = R^2(1+\rho^2)$$

$$\Rightarrow ds^2 = R^2 \left[-\left(1+\rho^2\right) dt^2 + \frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega_{d-1}^2 \right]$$

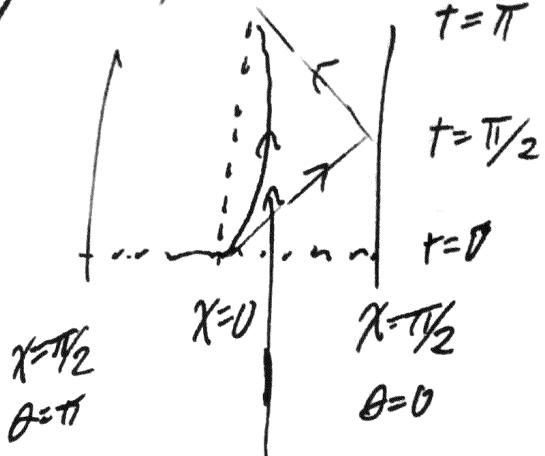
$$\rho = \tan \chi \quad (\chi \in [0, \pi/2])$$

$$ds^2 = \frac{R^2}{\cos^2 \chi} \left[-dt^2 + d\chi^2 + \sin^2 \chi d\Omega_{d-1}^2 \right]$$

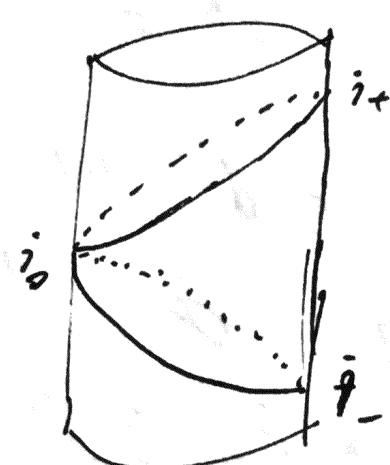
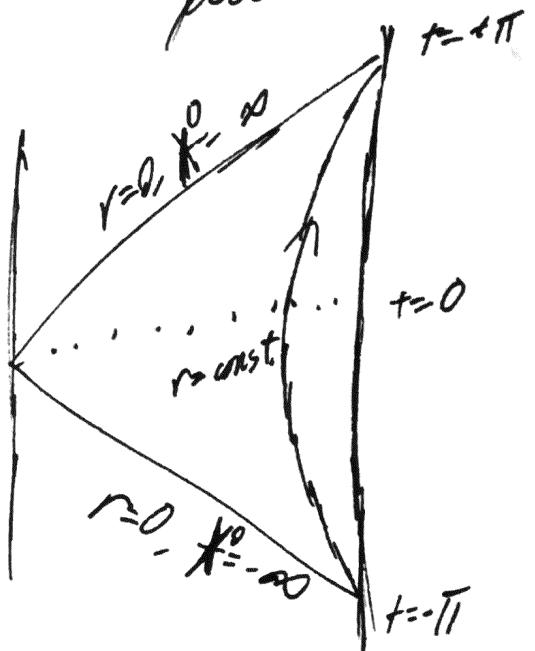
The causal structure
of AdS_{d+1} is that
of a solid
cylinder:



Light ray can reach the boundary in finite time



a massive particle cannot. It will be pulled back by gravitational pull.



global AdS
contains infinite
of copies
of Poincare
patch.

Symmetries of AdS_{d+1}

isometry: $SO(2, d)$, $\frac{1}{2}(d+2)(d+1)$ generators

$$ds^2 = \frac{r^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^2} dr^2 = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) \quad (*)$$

P^μ : Translation along x^μ d

$M^{\mu\nu}$: Lorentz trans. for x^μ $\frac{1}{2}d(d-1)$

scaling: $z \rightarrow \lambda z$, $x^\mu \rightarrow \lambda x^\mu$

(*) is also invariant under

$$I = \begin{aligned} z &\rightarrow \frac{z}{z^2+x^2} \\ x^\mu &\rightarrow \frac{x^\mu}{z^2+x^2} \end{aligned} \quad \begin{array}{l} \text{not connected to} \\ \text{identity} \end{array}$$

~ special conformal $I \circ P^\mu(b_\mu) \circ I$ d

$$\hookrightarrow z' = \frac{z}{1+2b \cdot x + b^2(z^2+x^2)}$$

$$x'^\mu = \frac{x^\mu + b^\mu(x^2+z^2)}{1+2b \cdot x + b^2(z^2+x^2)}$$

$$\frac{1}{2}(d+2)(d+1) = d + \frac{1}{2}d(d-1) + 1 + d \quad \checkmark$$

. Symmetries of S^n : $SO(n+1)$

~ $AdS_5 \times S^5$: $SO(2, 4) \times SO(6)$

String Theory in $AdS_5 \times S^5$

$$g_s, \alpha', R$$

~ Two dimensionless parameters

$$g_s, \frac{\alpha'}{R^2} \text{ i.e. } \left(\frac{g_N}{R^3}, \frac{\alpha'}{R^2} \right)$$

. classical gravity as $g_s \rightarrow 0$, $\frac{\alpha'}{R^2} \rightarrow 0$

↑
string weakly
interacting

?
massive modes
de couple

\Rightarrow IIB supergravity

= Einstein gravity + finite # of matter fields

"classical" string limit

$$\frac{\alpha'}{R^2} = \text{finite}, g_S \rightarrow 0$$

$$\rho(x^\mu, z, \Omega_S) = \sum_l \phi_l(x^\mu, z) Y_l(\Omega_S)$$

fields in AdS₅ harmonics on S⁵

\Rightarrow 5-dimensional gravity

$$S_{\text{gravity}} = \frac{1}{16\pi G_S} \int d^5x \sqrt{g} R_S + \text{matter}$$

$$G_S = \frac{G_N^{(D)}}{V_S} = \frac{G_N}{\pi^3 R^5}$$

Volume of S⁵

N=4 SYM (3+1)

Field content: A_μ ϕ^i x_α^A $A=1\dots 4$

all in adjoint rep'n of U(N)
≈ all N×N hermitian matrices

altogether: $(8b + 8f) \times N^2$
on-shell d.o.f.

Interacting part: $SU(N)$

$U(1)$ part: Free

$$\mathcal{L} = -\frac{1}{g_{YM}^2} \text{Tr} \left(\frac{1}{4} F^2 + \frac{1}{2} (\partial_\mu \phi^i) (\partial^\mu \phi^i) + [\phi^i, \phi^j]^2 \right) + \text{fermionic part}$$

Properties:

(1) Has $N=4$ supersymmetries

Susy: boson \leftrightarrow fermion

\Rightarrow conserved fermionic charges

trans parameter: spinor (Weyl)

4 such indep. spinor parameters

The "simplest strongly-interacting 4-D theory"

(2) g_{YM} is a dimensionless quantity

and
 β -function is 0

(3) Conformally-invariant.

continuing: (Nov 19, 2018)

IIB String theory = $N=4$ SYM
in $AdS_5 \times S^5$ with $U(N)$ in Minky

5-dimensional
quantum gravity
↓
classical limit

($A_\mu, \phi^i, i=1,\dots,6, \chi_\alpha^A, A=1,\dots,4$)
• Maximally supersymmetric theory in $d=4$
(4 susys)

- β -function g_{YM} is zero
 g_{YM} : true dimensionless param.
- conformally invariant

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$x^\mu \rightarrow x'^\mu(x) \text{ s.t. } g_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x)$$

For $g_{\mu\nu} = \eta_{\mu\nu}$, such transforms overall scaling

$$\text{are: } x'^\mu = x^\mu + a^\mu$$

$$x'^\mu = \Omega^\mu_\nu x^\nu$$

$$x'^\mu = \lambda x^\mu$$

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b^\mu x^\mu + b^\mu x^2}$$

"special conformal"
 $S = I \cdot T(b) \cdot I$

inversion
translation

$SO(2, d)$

conformal group:
Poincaré: $P^\mu, M^{\mu\nu}$

scaling: D

special conformal: K^μ

(super) Conformal Field Theory

The full bosonic symmetries are

$$SO(2,4) \times SO(6)$$

rotate ϕ^i (and x_α^A)

so with SUSY included, the full (super) group of symmetries is

$$PSU(2,2|4)$$

• Remarks on CFT's

basic objects: local operators with definite scaling dimensions

$$O(x) \rightarrow O'(x') = \lambda^\Delta O(\lambda x)$$

(Δ : dimension)

Hilbert space: fall into reps of $SO(2, d|4)$

typical observables: correlation functions of local operators

conformal symmetry determines the 2 and 3 point correlation functions up to constants:

$$\langle O_1(x), O_2(y) \rangle = \frac{C}{|x-y|^{2\Delta_1}} \delta_{\Delta_1, \Delta_2}$$

$$\langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1} |x_{31}|^{\Delta_3 + \Delta_1 - \Delta_2}}$$

Remarks:

- (1) Isometries of $AdS_5 \times S^5$ form a subgroup of general coordinate transformations, which are local symmetries or gravity side
- (2) Isometry: This is the subgroup of coordinate transformations that leave the asymptotic form of the metric invariant "large gauge transformations"

(by analogy: $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x)$)
 $\Lambda(x) \rightarrow 0$ as $|x| \rightarrow \infty$ usually
but if $\Lambda(x) \rightarrow \text{const.}$ as $|x| \rightarrow \infty$
it is "large"

Match parameters:

gravity:

$$4\pi g_s$$

=

$$N = 41$$

$$R^4 / (\alpha')^2$$

=

$$g_{YM}^2$$

$$(E_S / R)^3$$

=

$$\lambda = g_{YM}^2 N$$

$$\frac{\pi}{2N^2}$$

classical gravity:

$$E_S / R^3 \rightarrow 0$$

$\Rightarrow N \rightarrow \infty$

$$(\alpha')^2 / R^4 \rightarrow 0$$

$\Rightarrow \lambda \rightarrow \infty$

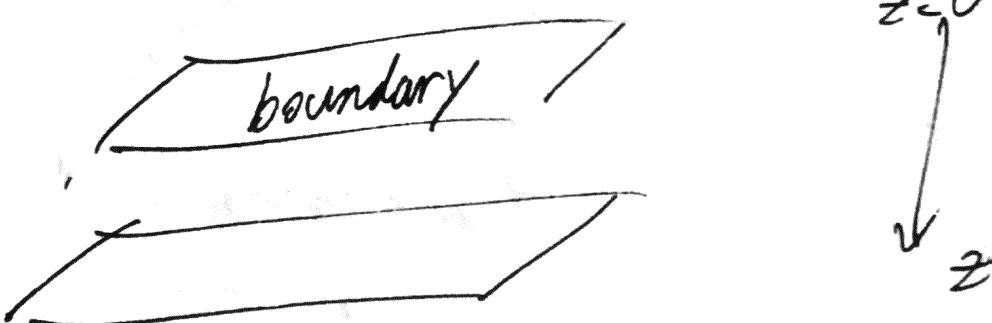
strong coupling
and large N limit

$1/N^2 \leftrightarrow$ RG corrections
 $1/R^2 \leftrightarrow$ string corrections

$1/\alpha' \leftrightarrow$ string corrections

- An example of an equivalence between matrices and strings
- Can also be considered as an example of the holographic principle

In Poincaré coordinates:



$$ds^2 = \frac{R^2}{z^2} (-dt^2 + dx^2 + dz^2) \quad z \in (0, \infty)$$

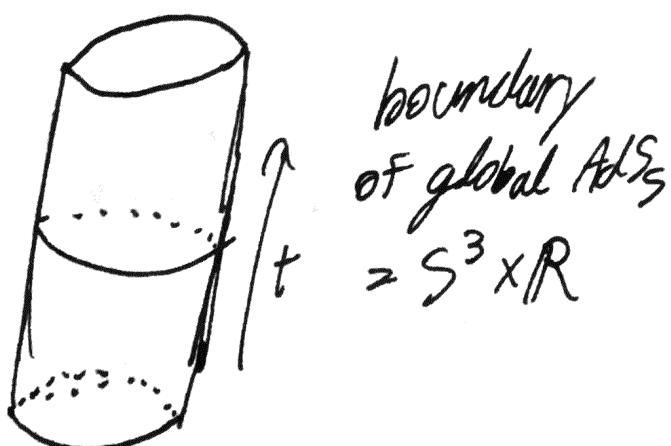
boundary of AdS_5 = Mink₄

Holographic perspective \Rightarrow prediction:

quantum gravity in global

AdS_5

$N=4$ SYM on $S^3 \times R$



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• 176

Chapter 3: Holographic Duality

$$\text{quantum gravity in } \text{AdS}_{d+1} = \text{CFT}_d$$

Equivalence between two quantum systems

→ guess the dictionary

⇒ verify

→ make more guesses

• parameters, symmetries should match

e.g. $U(1)$ gauge $\leftrightarrow U(1)$ global

3.1 General aspects

3.1.1 IR/UV connection

$$ds^2 = \frac{R^2}{z^2} (-dt^2 + d\vec{x}^2 + dz^2) \quad (*)$$

(*) is invariant under $(t, \vec{x}) \rightarrow \lambda(t, \vec{x})$
 $z \rightarrow \lambda z$

⇒ extra dimension \leftrightarrow scale

Note: (t, ξ) defined in boundary units

$$d\tau = \frac{R}{z} dt \quad dl = \frac{R}{z} dx$$

→ For some process of local energy E_{loc} and local length d_{loc} at z

$$d_{YM} = \frac{z}{R} d_{loc}, \quad E_{YM} = \cancel{\frac{R}{z}} \frac{R}{z} E_{loc}$$

→ For the same process at different z :

(boundary) $z \rightarrow 0$: $E_{YM} \rightarrow \infty$, $d_{YM} \rightarrow 0$ (UV process)

$z \rightarrow \infty$: $E_{YM} \rightarrow 0$, $d_{YM} \rightarrow \infty$ (IR process)

⇒ typical bulk process, $E_{loc} \sim 1/R$

$$\Rightarrow E_{YM} \sim 1/z$$

IR - UV connection

Wednesday Nov 29, notes from Maoya Guo

From before:

$$ds^2 = \frac{R^2}{z^2} (dt^2 + d\vec{x}^2 + dz^2)$$

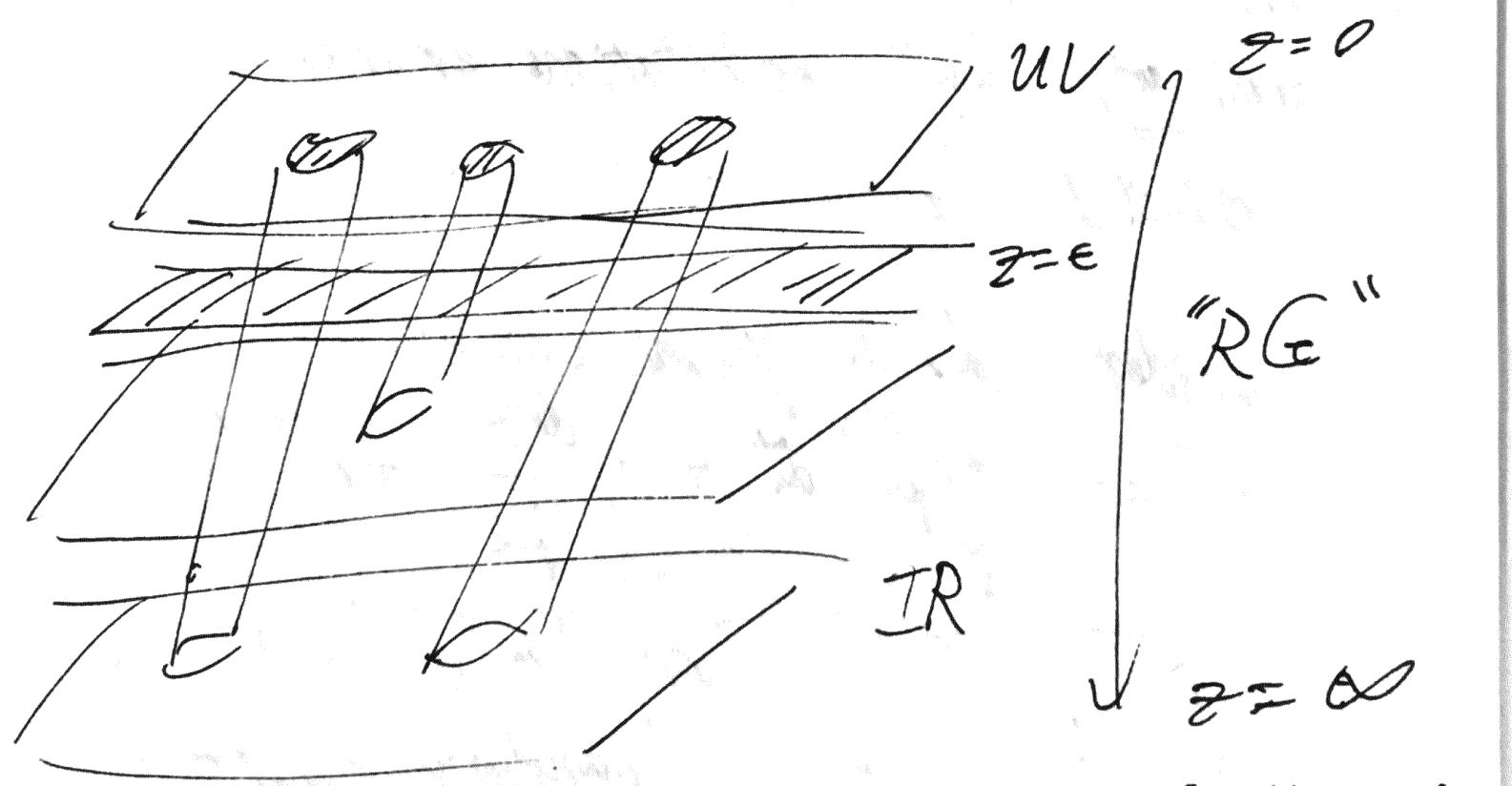
$$E_{YM} = \frac{R}{z} E_{loc}$$

$$d_{YM} = \frac{z}{R} d_{loc}$$

$$E_{YM} \propto \frac{1}{z}$$

$$d_{YM} \propto z$$

~~$$E_{YM} \propto \frac{1}{z} d_{YM} dt/dz$$~~
$$E_{loc} \sim 1/R, d_{loc} \sim R$$



radial direction: geometrization of "scale"

IR - UV connection

Remarks:

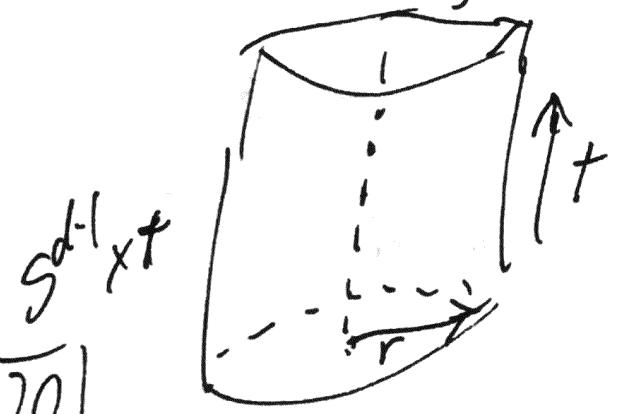
1) $z \rightarrow 0$ $F_{\text{dyn}} \rightarrow \infty$
 $d_{\text{dyn}} \rightarrow 0$

put on IR -cutoff on gravity side at $z = \epsilon$
 \Rightarrow UV cutoff $\propto 1/\epsilon$ (energy)
distance $\propto \epsilon$

check: basic idea of holographic principle
(pset)

2) In a CFT on $R^{3,1}$ there exist
arbitrarily low excitations energies.
reflected in $z \rightarrow \infty$ on the gravity side

3) Consider AdS in global coordinates

$$ds^2 = - \underbrace{\left(1 + \frac{r^2}{R^2}\right)}_f dt^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2 d\Omega^2$$


on field theory side
boundary sphere has
radius \sqrt{R}

$$E_{\text{ym}} \sim \frac{1}{R} \quad (\text{energy gap})$$

$$\begin{aligned} E_{\text{ym}} &= f^{\frac{1}{2}} E_{\text{loc}} \\ &= \left(1 + \frac{r^2}{R^2}\right)^{\frac{1}{2}} E_{\text{loc}} \\ &= \begin{cases} \infty & \text{as } r \rightarrow \infty \\ E_{\text{loc}} \sim \frac{1}{R} & \text{as } r \rightarrow 0 \end{cases} \end{aligned}$$

4) This works in more general asymptotic AdS metric

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2 d\Omega_{d-1}^2$$

away from boundary f decreases
 $\Rightarrow E_{\text{ym}}$ decreases

3.1.2 Matching of the spectrum

$$\text{QG in AdS}_{d+1} = \text{CFT}_d$$

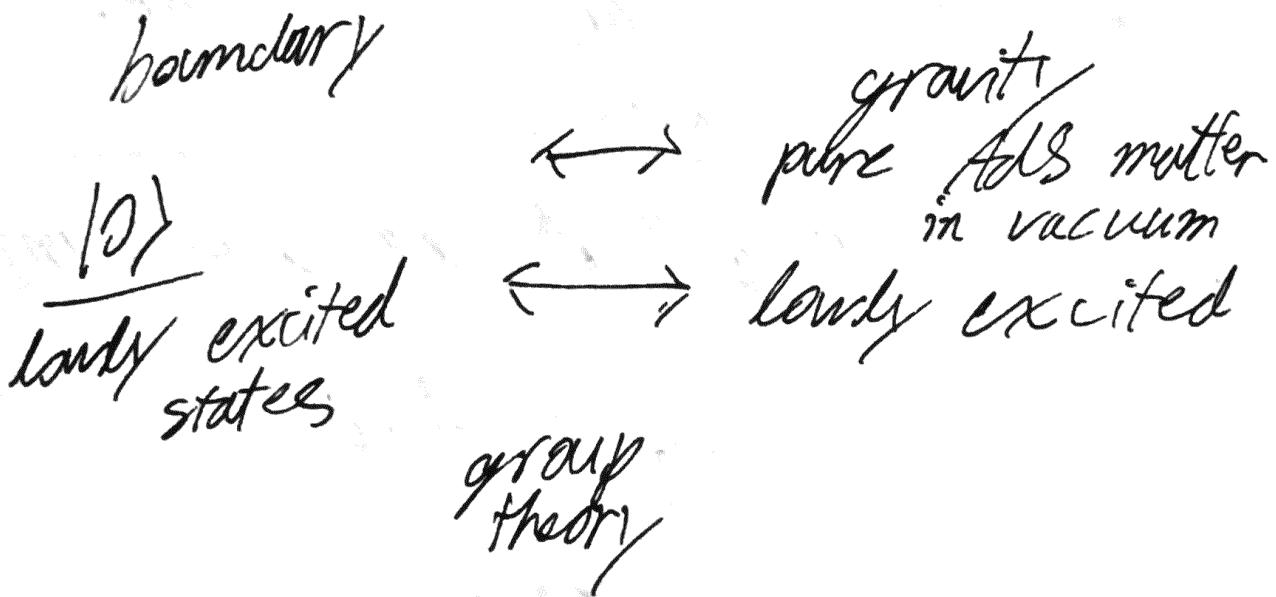
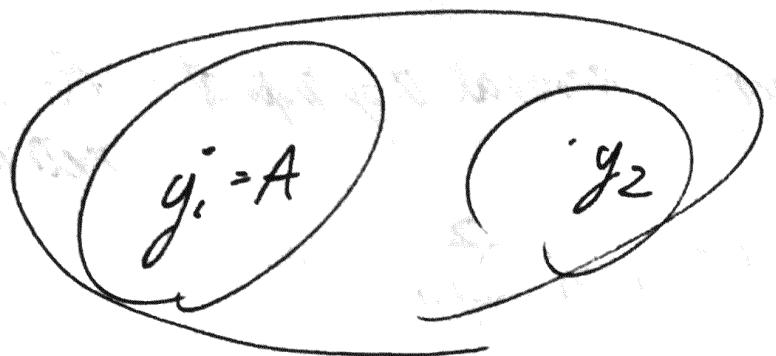
same Hilbert space

physical states \longleftrightarrow physical states

Classical gravity \rightarrow classical solutions
(states)

geometry \Rightarrow state

given a geometry : quantize matter
fields \Rightarrow a subset of quantum states
on gravity side



boundary

gravity

$\phi(0)$

$\longleftrightarrow \phi(0)$

conformal
operators

\longleftrightarrow bulk fields

scalar
operators

\longleftrightarrow scalar fields

$J_\mu \longleftrightarrow A_\mu$

all quantum numbers of any symmetry should match

lets use a simple scalar as an example
to see how this mapping works

Recall: in a matrix-type field theory, key objects:

key objects: single-trace operators

$$\langle \phi \phi \rangle \sim O(1) \quad \langle \phi \phi \phi \phi \rangle \sim \frac{1}{N}$$

$$\langle \phi \phi \phi \phi \phi \phi \rangle \sim \frac{1}{N^2}$$

leading order in large N :

Gaussian Theory

on gravity side:

$$S = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g} [R - 2\Lambda + L_{\text{matt}}]$$

$$L_{\text{matt}} = -\frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} m^2 \Phi^2 + O(\Phi^3) + \dots$$

$$g_{\mu\nu} = \underbrace{g_{\mu\nu}^0}_{\substack{\text{pure} \\ \text{AdS}}} + \chi h_{\mu\nu}$$
$$16\pi G_N = 2\kappa^2$$

$$\Phi = 0 + \chi \Phi$$

$$\Rightarrow S = \int d^{d+1}x \sqrt{-g} \left[-\frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} m^2 \Phi^2 + \frac{1}{2} \Phi^3 + \frac{1}{2} \chi^2 \Phi^2 - (\partial h)^2 - \frac{1}{2} \partial_\mu \partial^\mu h^2 - \frac{1}{2} k h^3 + \chi^2 h \Phi^2 \right] + \dots$$

$$G_N \sim \frac{1}{N^2}, \quad \kappa \sim \frac{1}{N}$$

leading order in $1/N \rightarrow$ quadratic theory
(standard)
 \rightsquigarrow quantization of Φ

com for ϕ

$$\int \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) - m^2 \phi = 0$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{R^2}{z^2} (-dt^2 + d\vec{x}^2 + d\vec{z}^2)$$

$$\phi(z, x^\mu) = \int \frac{dk}{(2\pi)^d} e^{ik \cdot x} \phi(z, k)$$

$$z^{d+1} \partial_z (z^{1-d} \partial_z \phi) - k^2 k^2 \phi - m^2 R^2 \phi = 0$$

$$k^2 = -w^2 + \vec{k}^2$$

$$k^\mu = (w, \vec{k})$$

consider $z \rightarrow 0$

$$\sim z^2 \partial_z^2 \phi + (1-d) z \partial_z \phi - m^2 R^2 \phi = 0$$

let $\phi \sim z^\alpha$

$$\alpha(\alpha-1) + (1-d)\alpha - m^2 R^2 = 0$$

$$\Rightarrow \alpha = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2}$$

$$\Delta \equiv \frac{d}{2} + V$$

$$\alpha_+ = \Delta$$
$$\alpha_- = d - \Delta$$

$$\Rightarrow \phi(k, z) = \underbrace{A(k)}_{\alpha_+} z^{\Delta} + \underbrace{B(k)}_{\alpha_-} z^d$$

as $z \rightarrow 0$ 125

Remarks:

(1) Exponents are real
provided $m^2 R^2 \geq -\frac{d^2}{y} (\chi)$

one can show: (a) a theory is well-defined
if $(*)$ is satisfied
(b) if $(*)$ is violated,
there exist exponentially
growing terms in time
 \Rightarrow instabilities

B.F. bound

contrast: in Mink

$$\begin{aligned} \partial^2 \Phi - m^2 \Phi &= 0 \\ \Rightarrow \omega^2 = k^2 + m^2 & \\ \underline{\underline{m^2 < 0}} \quad \omega^2 < 0 \text{ for } k = 0 & \\ \Rightarrow \omega \text{ pure imaginary} & \end{aligned}$$

~~$|A|^2 = R^2 g_{\mu\nu} A_\mu A_\nu$~~

In AdS, due to spacetime curvature,
 the constant modes are not allowed
 \Rightarrow A field is forced to have some kinetic
 energy, compensating for some negative m^2

- (2) AdS has a boundary, and light rays
 reach this boundary in finite time.
 \Rightarrow Energy can be exchanged at the boundary.
 \Rightarrow need to impose appropriate boundary
conditions.

~~universe~~
 Canonical quantization: expand Ψ in a
 complete set of normalizable
 modes, satisfying appropriate
 boundary conditions

Inner Product:

(Klein-Gordon) $(\Psi_1, \Psi_2) = -i \int_{\Sigma_+} dz dx \sqrt{g} g^{ij} (\Psi_1^* \partial_j \Psi_2 - \Psi_2 \partial_j \Psi_1^*)$

const. $\rightarrow \Sigma_+$
time slice

Can check: (\varPhi_1, \varPhi_2) independent of t .
(this was done in problem set 2, problem 1)

$z_j^\Delta \rightarrow 0$ as $z \rightarrow 0$ is always normalizable

$$\Delta = \frac{d-v}{2} > 0$$

z^{Δ} is non-normalizable for $v \geq 2$
is normalizable for $0 \leq v < 1$

Boundary conditions.

$$v \geq 1: A = 0$$

$$0 \leq v < 1: \begin{aligned} A &= 0 && \text{"standard quantization"} \\ \text{or } B &= 0 && \text{"alternative quantization"} \\ &\text{(or mixed)} && \end{aligned}$$

"normalizable" behavior specified by quantization.

(3) Normalizable modes: Used to build up Hilbert space in the bulk



States of the boundary theory

4) Non-normalizable modes are not part of the Hilbert space. If present, they should be considered/viewed as defining the background.

In standard quantization: $A \neq 0$

→ A is boundary "value" of the field

If $A(x) = \phi(x) \Rightarrow S_{\text{boundary}}$ should contain a term: $\int d^d x \phi(x) \partial(x)$

⇒ non-normalizable modes determine the boundary theory itself

i.e. two solutions with the same non-normalizable modes describe different states of the same theory.

but two solutions with different ^{non-}normalizable modes describe different theories

$$\int d^d x \phi(x) \partial(x) \leftrightarrow \phi(x) = \lim_{z \rightarrow 0} z^{1-d} \bar{\phi}(2/x) (x)$$

(5) Relation (*) implies that Δ is the scaling dimension of θ

$$x \rightarrow x'^\mu = \lambda x^\mu$$

$$\theta(x) \rightarrow \theta(x') = \lambda^{-\Delta} \theta(x)$$

Δ : scaling dimension of θ

boundary scaling: \leftrightarrow bulk isometry:

$$x^\mu \rightarrow \theta x'^\mu = \lambda x^\mu$$

$$z + z' = \lambda z$$

$$\begin{array}{ccc} \varphi(z, x) & \leftrightarrow & \theta(x) \\ \sim & \downarrow & \downarrow \\ & \varphi'(z', x') & \leftrightarrow \theta'(x') \end{array}$$

$$\int d^d x' \varphi'(x') \theta(x) = \int d^d x \varphi(x) \theta(x)$$

φ : scalar

$$\varphi'(z', x') = \varphi(z, x)$$

$$\begin{aligned} \varphi'(x') &= \lim_{z' \rightarrow 0} (z')^{\Delta-d} \varphi'(z', x') \\ &= \lambda^{\Delta-d} \varphi(x) \end{aligned}$$

130

Further, $d^d x' = \lambda^d d^d x$

$$\Rightarrow \delta(x') = \lambda^{-d} \delta(x)$$

For a scalar (standard quantization)

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{q} + m^2 R^2}$$

- (i) $m=0 \Leftrightarrow \Delta=d$ marginal operator
- (ii) $m^2 < 0 \Leftrightarrow \Delta < d$ relevant operator
- (iii) $m^2 > 0 \Leftrightarrow \Delta > d$ irrelevant operator

$$\int d^d x \delta(x) \delta(x) \leftrightarrow z^{d-\Delta} f(z) + \dots$$

$$UV \leftrightarrow z \rightarrow 0$$

- (i) does not change const.
- (ii) less and less important $\rightarrow 0$
- (iii) more and more important $\rightarrow \infty$

Before:

$$\vartheta(x) \longleftrightarrow \varPhi(z, x)$$

scalar field at m^2

$$z \rightarrow 0, \quad \vartheta(z, x) = A(x) z^{d-1} + B(x) z^d$$

$$\Delta = \frac{d}{2} + v, \quad v = \sqrt{\frac{d^2}{4} + m^2 R^2}$$

normalizable modes \longleftrightarrow states

non-normalizable modes \longleftrightarrow action (theory)

standard: (A term non-normalizable)

$$A(x) \longleftrightarrow \int \vartheta(x) \partial(x) \quad (\vartheta = A)$$

$$B(x) \longleftrightarrow \langle \vartheta \rangle$$

$$\delta = \Delta$$

alternative quant: $(0 \leq v < 1)$

choose $B(x)$ term to be non-normalizable

$$B(x) \longleftrightarrow \int \vartheta(x) \partial \quad \vartheta = B$$

$$A(x) \longleftrightarrow \langle \vartheta \rangle, \quad \delta = d - \frac{1}{\int \vartheta}$$

different CFTs 503



different gravities with $\{\phi\}$

conserved currents:

- 1) $\underline{J^\mu}$ (global $U(1)$ internal symmetry) $\leftrightarrow A_\mu$ gauge field
 $Q = \int d^d x J^0 \Rightarrow [J] = d-1$
- 2) $\underline{T^{\mu\nu}}$ $\rightarrow E = \int \sum_{\Sigma} d^{d-1} T^{00} \rightsquigarrow [E] = 1 \Rightarrow [T] = d$
 $\downarrow h_{\mu\nu}$ (metric perturbations)

(1) Suppose we deform the CFT

by $\int a_\mu(x) J^\mu(x) dx (\star)$

$$a_\mu \equiv A_\mu \Big|_{\partial(\text{AdS})}$$

since J^μ is conserved, (\star) is invariant

under $A_\mu \rightarrow a_\mu(x) + \partial_\mu \Lambda(x)$

$\rightsquigarrow A_\mu$ should have some gauge transformation

139

\Rightarrow Maxwell field

Conversely : Start on gravity side with:

$$-\frac{1}{4} \int d^d x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}$$

$$A_\mu = (A_2, A_\mu)$$

$$\Rightarrow \delta = d-1$$

$$z \rightarrow 0 \quad A_\mu = a_\mu + b_\mu z^{d-2}$$

(2) Add $\int h_{\mu\nu} T^{\mu\nu} d^d x$ to the boundary action

\Leftrightarrow deforming boundary metric $\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + h_{\mu\nu}$

$$ds^2 = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2)$$

$$\underset{z=0}{\rightarrow} \frac{R^2}{z^2} \underbrace{\eta_{\mu\nu} dx^\mu dx^\nu}_{\text{boundary metric}} \cancel{ds^2 = R^2}$$

so now:

$$ds^2 = \frac{R^2}{z^2} ((g_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu + dz^2)$$

$$z \rightarrow 0$$

$\Rightarrow T_{\mu\nu}$ should correspond to bulk metric perturbations

Conversely:

Using linearized Einstein equations
and finding boundary behaviour:

$$g_{MN} = g_{MN}^{\text{AdS}} + h_{MN} \quad \text{and finding } h_{MN} \text{ near } z \rightarrow 0$$

$$h_{\mu\nu} = \underbrace{\frac{a_{\mu\nu}}{z^2}}_{+ b_{\mu\nu}} z^{d-2} \quad \text{as } z \rightarrow 0$$

scaling argument gives $\delta = d$

More generally:

Any bulk field Φ with n indices

$$\Phi(x, z) = A(x) z^{d-\Delta-n} + B(x) z^{\Delta-n}$$

$$\text{boundary source} = \lim_{z \rightarrow 0} z^\alpha \Phi(z, x)$$

$$\alpha = \Delta + n - d$$

$\Delta = \underset{\text{corresp. bulk operator}}{\dim}$

3.1.3 Euclidean Correlation Functions

Basic observables of a CFT:
 correlation functions of local ops.
 large- N : single-trace ops.

Recall: $\langle \phi \phi \rangle_c \sim O(1) + \dots$

$$\langle \phi \phi \phi \rangle_c \sim O(1/N) + \dots$$

$$\langle \phi \phi \dots \phi_n \rangle \sim O(N^{2-n}) + \dots$$

leading behavior suggests a tree theory
 with coupling $1/N$.

convenient to consider Euclidean correlation function

$$+ = -i\tau$$

can consider generating functional:

$$Z_{\text{CFT}}[\phi] \underset{\substack{\text{(order} \\ \text{not mattering)}}}{=} \left\langle e^{\int d^d x \phi(x) \partial(x)} \right\rangle_c \quad (*)$$

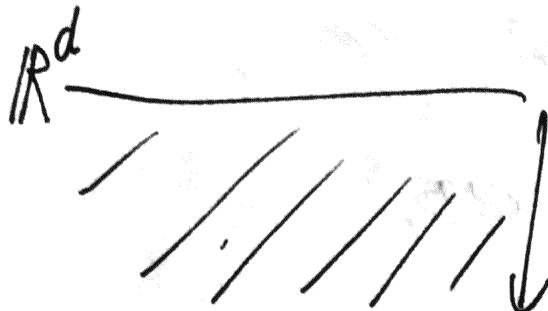
∂ : collections of all single-trace ops

ϕ : sources

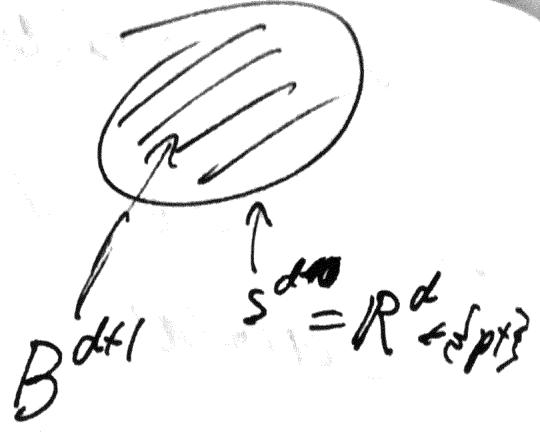
~ Analytically continue AdS to Euclidean space

$$ds^2 = \frac{R^2}{z^2} (dt^2 + d\vec{x}^2 + dz^2) \leftarrow \begin{matrix} \text{covers full} \\ \text{Euclidean AdS} \end{matrix}$$

$$ds^2 = \frac{R^2}{z^2} (dt^2 + dx^2 + dz^2) \rightarrow$$



$$z=0$$



$$+ \cdot z = \infty$$

Given that $\theta \leftrightarrow \varphi$
 $\rho(x) \leftrightarrow \varphi|_{\partial AdS}$

$$Z_{CFT}[\varphi] = Z_{bulk}[\varphi|_{\partial AdS}] = \rho$$

in the sense of

$$\lim_{\varepsilon \rightarrow 0} z^\alpha \varphi(z, x)$$

We know
how to define
this

not known
how to define
in general

In the low energy limit, $G_N \rightarrow 0, \alpha' \rightarrow 0$

$$Z_{bulk} = \int D\varphi e^{S_E[\varphi]}$$

and we can evaluate this perturbatively around pure (Euclidean) AdS

General n-point function:

$$\langle \theta_1(x_1) \cdots \theta_n(x_n) \rangle = \frac{\delta^n \log Z_{\text{bulk}}}{\delta \theta_1 \cdots \delta \theta_n(x_n)|_{\theta=0}}$$

Recall around AdS:

$$S_{\text{bulk}} = \int d^{d+1}x \sqrt{g} \left[-\frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} m^2 \Phi^2 - \chi \Phi^3 - \chi^2 \Phi^4 + \dots \right]$$

$(x \sim \sqrt{\epsilon_N} \sim 1/N)$

let Φ_i collectively denote all perturbations around AdS, including metric and all matter fields

$$\lambda \sim 1/N \quad X \sim \lambda^2 \sim 1/N^2$$

$$K \sim 1/N^3$$

$$\partial^2 \Phi_i - m^2 \Phi_i = \chi L_i^{\Phi^2}, L_i = \int d^d x' K(z, x; x') \Phi(x')$$

$$\Phi_i = \int dx' dz' G(x, z; x', z') K L_i^{\Phi^2}$$

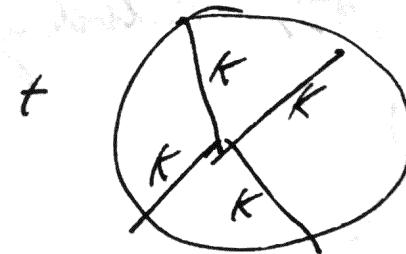
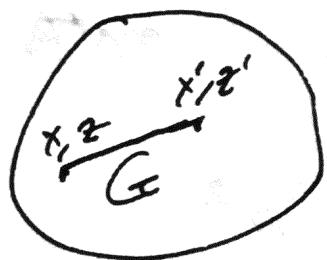
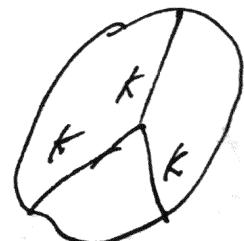
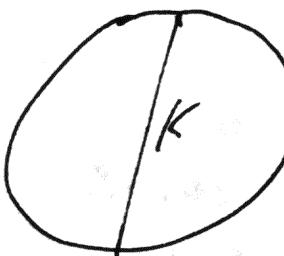
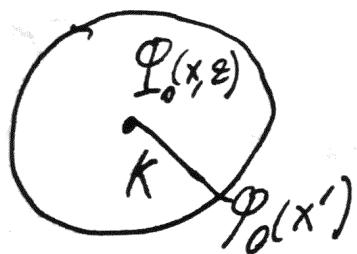
From this structure, we get that
the tree-level is

$$\log Z_{\text{tree}}[\phi] = \phi^2 + x\phi^3 + x^2\phi^4 + \dots$$

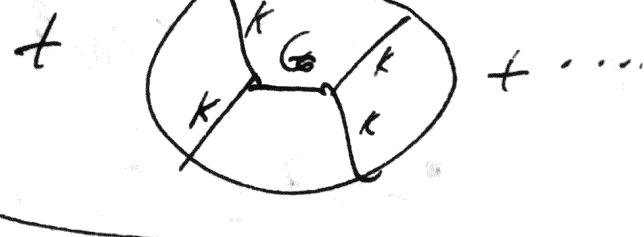
K : boundary to bulk propagator

G : bulk to bulk propagator

↑ demands fields fall
off at ∞



Witten
Diagrams



$$\left\langle \exp\left[\int \partial(x)\phi(x)\right] \right\rangle = Z_{\text{bulk}} \left[\frac{\partial}{\partial A_{\text{AdS}}} = \phi(x) \right] \quad (\star)$$

$$Z = \int D\Phi \exp[S_E[\Phi]]$$

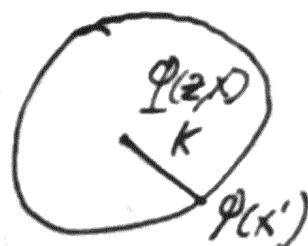
$$S_E[\Phi] = - \int d^d x \sqrt{g} \left[\frac{1}{2} (\partial\Phi)^2 + \frac{1}{2} m^2 \Phi^2 + \lambda \Phi^3 + \lambda^2 \Phi^4 + \dots \right]$$

$$x \sim G_N^{1/2} \sim 1/N$$

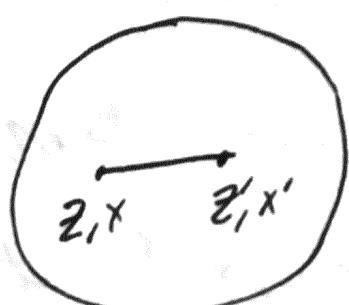
$$\log Z_{\text{bulk}}[\Phi] = \log Z_{\text{tree}}[\Phi] + \log Z_{\text{loop}}[\Phi] + \dots$$

$$\Phi_0(z, x) = \int d^d x' K(z, x; x') \phi(x')$$

$$\lim_{z \rightarrow 0} \Phi_0(z, x) = z^{d-\Delta} \phi(x)$$

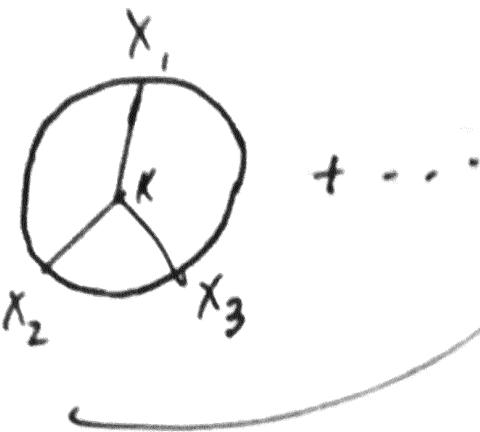
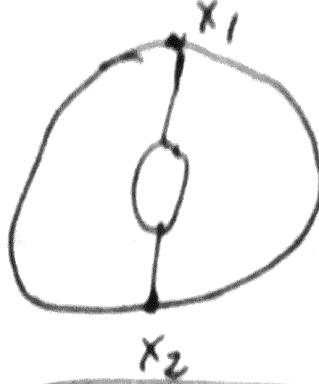
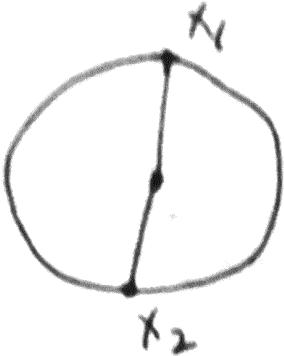


next:



$$(d^2 - m^2) G(z, x; z', x') \\ = \frac{1}{\sqrt{g}} \delta(z - z') S^{(d)}(x - x')$$

$$\langle \theta(x_1) \theta(x_2) \rangle = \frac{\delta \log Z_{\text{bulk}}}{\delta \theta(x_1) \delta \theta(x_2)} \Big|_{\rho=0}$$



Remarks :

- (1) Both sides of (*) are divergent, even at tree-level, in the bulk

LHS: Usual UV divergences of a QFT $\xrightarrow{\text{IR/UV}}$

RHS: Volume divergences + asymptotic behavior
of $\varPhi \rightarrow$ divergences as $z \rightarrow 0$

~ We need to renormalize them:

$$\text{If } \log Z_{\text{CFT}}^{(R)} = \underbrace{\log Z_{\text{CFT}}}_{\text{bare}} + S_{\text{ct}}[\varPhi] \quad \text{local}$$

$$\overline{[142]} \quad \log \tilde{Z}_{\text{bulk}}^{(R)} = \underbrace{\log Z_{\text{bulk}}}_{\text{bare}} + S_{\text{ct}}[\varPhi] \quad \text{local}$$

(2) One-point function

consider CM of 1 dof.

$$S[x_c] = \int_{t_0}^{t_1} dt L[x(t), \dot{x}(t)]$$

$$\begin{aligned}x_c(t_0) &= x_0 \\x_c(t_1) &= x_1\end{aligned}$$

$$\delta S = p_1 \delta X_1 - p_0 \delta X_1 \Rightarrow \frac{\delta S}{\delta X_1} = p_1$$

$$\langle \phi(x) \rangle = \frac{\delta \log Z_{\text{tree}}}{\delta \phi(x)} \quad \log Z_{\text{tree}} = S_E[\Phi_c]$$

$$\Phi_c \Big|_{\partial A \partial B} = \Phi(x)$$

$$\langle \phi(x) \rangle = \frac{\delta S_E^{(R)}[\Phi_c]}{\delta \phi(x)} = \lim_{z \rightarrow 0} z^{\Delta-d} \frac{\delta S_E^{(R)}[\Phi_c]}{\delta \Phi_c(zx)}$$
$$= \lim_{z \rightarrow 0} z^{\Delta-d} \overline{I^{(R)}(\Phi_c)}$$

Π : canonical momentum for Φ
treating z as "time"

- boundary-to-bulk prop.

$$K(z, x; x') : (\partial^2 - m^2) K(z, x; x') = 0$$

$$K(z \rightarrow 0, x; x') = z^{d-1} \delta^{(d)}(x - x')$$

- bulk-to-bulk prop.

Counterpart of standard flat space prop.

$G(z, x; z', x')$ is normalizable if either $z, z' \rightarrow 0$

$$G(z, x; z', x') \propto z^{\Delta} \quad z \rightarrow 0$$

Both G and K are found by going into momentum space

$$\text{e.g. } K(z, x; x') = \int \frac{d^d k}{(2\pi)^d} K(z, k) e^{ik(x-x')}$$

$$z^{d+1/2} (z^{1-d} \partial_z K) - (k^2 z^2 + m^2 R^2) K = 0$$

$$\text{with } K(z, k) = \dots \quad \text{as } z \rightarrow 0$$

Find $K(z, x; x')$ directly in coordinate space

$$\begin{array}{ccc} \hline & z=0 & \\ \downarrow & & \frac{R^2}{z^2} (dt^2 + d\vec{x}^2 + dz^2) \\ & \infty & P = (z=\infty) \end{array}$$

$$R(z) = K(z, x; P)$$

$$(z^2 \partial_z^2 + (1-d)z \partial_z + m^2 R^2) R = 0$$

$$R = a z^{d-\Delta} + b z^\Delta \Rightarrow \hat{R} = b z^\Delta$$

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2}$$

Inversion

$$z \rightarrow \frac{z}{z^2 + x^2}$$

$$x^\mu \rightarrow \frac{x^\mu}{z^2 + x^2}$$

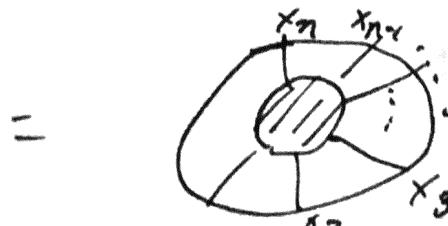
$$I: P \rightarrow \begin{matrix} z' = 0 \\ x' = 0 \end{matrix}, \quad K(z, x; x') = b \left(\frac{z}{z^2 + (x-x')^2} \right)^{\Delta}$$

$$\Rightarrow b = \frac{\Gamma(\Delta)}{\Gamma(\nu)} \pi^{-\frac{d-2}{2}}$$

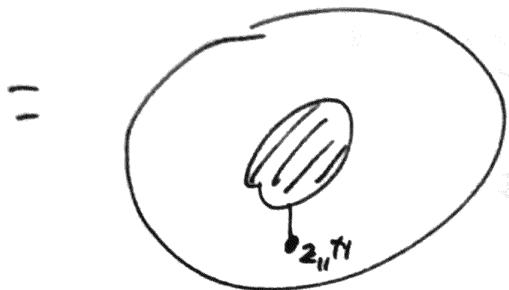
Can also show (without solving \mathbb{G})

$$K(z, x; x') = \lim_{z' \rightarrow 0} 2\nu z'^{-\Delta} G(z, x; z', x')$$

$\langle \mathcal{O}(x_1) \dots \mathcal{O}_n(x_n) \rangle = \text{sum over all Feynman diagrams with } n \text{ boundary endpoints}$



$$\langle \Phi_1(z_1, x_1) \cdots \Phi_n(z_n, x_n) \rangle$$



$$\Rightarrow \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle = \lim_{\substack{z_i \rightarrow 0 \\ z_n \rightarrow 0}} (2v, z_i^{-\Delta_1}) \cdots (2v, z_n^{-\Delta_n}) \times \langle \Phi_1(x_1, z_1) \cdots \Phi_n(x_n, z_n) \rangle \quad (\star\star)$$

Since Lorentzian corr. func. can be obtained from Euclidean ones using the same analytic continuation procedure, $(\star\star)$ must also apply to Lorentz corr.

Recall that: $|0\rangle_{AdS} \leftrightarrow |0\rangle_{CFT}$

$$\left\{ \Phi(z, x) \cdots |0\rangle_{AdS} \right\} \nleftrightarrow \left\{ \phi(x) \cdots |0\rangle_{CFT} \right\}$$

we can identify

$$\phi(x) = 2v \lim_{z \rightarrow 0} z^{-\Delta} \Phi(x/z)$$

$$\langle \psi | \theta(x) |\psi \rangle = 2V \lim_{z \rightarrow 0} z^{-\Delta} \underbrace{\langle \psi | \Phi(x, z) |\psi \rangle}_{= B(x) z^\Delta + \dots} = 2V B(x)$$

$$\begin{aligned} \langle 0 | \theta(x_1) \theta(x_2) | 0 \rangle &= \lim_{\substack{z_1 \rightarrow 0 \\ z_2 \rightarrow 0}} (2V z_1^{-\Delta}) (2V z_2^{-\Delta}) G(z_1, x_1; z_2, x_2) \\ &= \lim_{z_1 \rightarrow 0} 2V z_1^{-\Delta} K(z_1, x_1; x_2) \\ &= \frac{2V b}{|x_1 - x_2|^{2\Delta}} \end{aligned}$$

3.1.4 Wilson loops

non-local operators

recall:

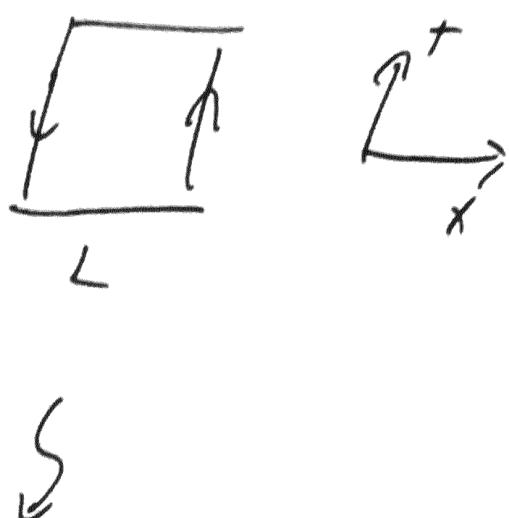
$$W[C] = \text{Tr}_P \text{Pe}^{i \oint_C A_\mu dx^\mu}$$

→ phase factor associated with transporting
an "external" particle in a given repin P

along C

$$\text{e.g. } \langle 0 | W_a(C) | 0 \rangle, \langle 0 | W_a(C_1) W_b(C_2) \dots | 0 \rangle$$

often-used C:



$$T \gg L,$$

expect $(W(C))_{\text{loc}} e^{-iET}$

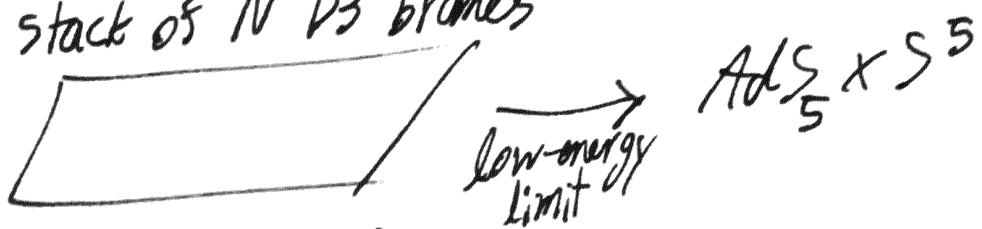
$E = \text{energy of quark-antiquark system}$

at $N \rightarrow \infty$

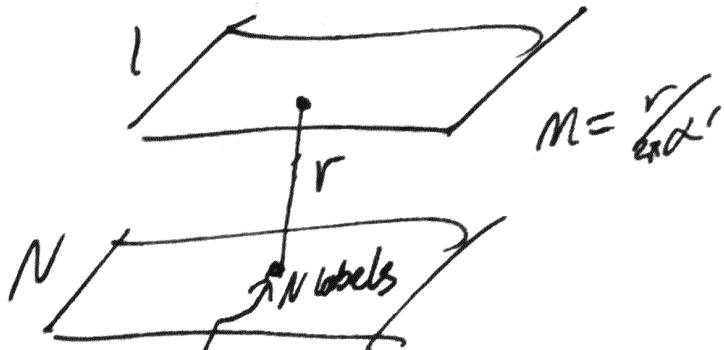
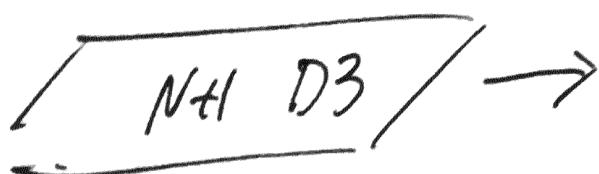
so they
don't move

First, we need to understand how
to introduce "external quarks"
into $M=4$ SYM

stack of N D3 branes

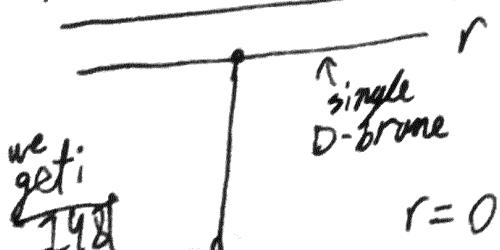


consider now:

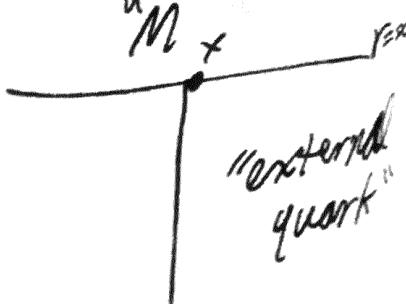


like a "quark" in fundamental rep'n

Now take low-energy limit, $r \rightarrow 0$, $\alpha' \rightarrow 0$ with $\sqrt{\alpha'}/r$ fixed



$$\frac{r=R^2/z}{M=r/\sqrt{\alpha'}} \quad \text{taking } r \rightarrow 0, \text{ we get}$$



Dec 5th, 2018 (Notes from Sam Lanthuisser)

parallel transport of such an "external quark" along a closed curve C gives a Wilson loop.

$$W(C) = \text{Tr } P \exp[i \oint_C A_\mu dx^\mu]$$

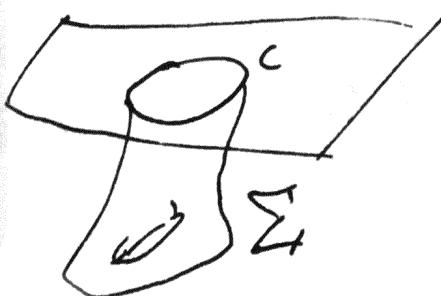
$$\tilde{W}(C) = \text{Tr } P \exp[i \oint_C (A_\mu ds + \vec{n} \cdot \vec{\Phi} \sqrt{s}) ds]$$

$\vec{\Phi}$ = six scalars in $M=4$ SYM

\vec{n} a unit vector on S^5

Since:

- 1) "quark" is endpoint of a string in AdS,
 C must be the boundary of a string worldsheet, Σ , i.e. $C = \partial \Sigma$
 - 2) $\langle W(C) \rangle$ is the "partition function of this quark"
- Guess: $\langle W(C) \rangle = Z_{\text{string}}(\partial \Sigma = C)$



single string partition function

$$Z_{\text{string}} = \int_{\partial\Sigma=C} D\bar{X}^{\alpha} e^{iS_{\text{string}}}$$

$$S_{\text{string}} \propto \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-h}$$

where $h = \det(h_{\alpha\beta})$, $h_{\alpha\beta} = g_{\mu\nu} \partial X^{\alpha} / \partial x^{\mu} \partial X^{\beta} / \partial x^{\nu}$

$(g_5 \rightarrow 0, \alpha' \rightarrow 0)$
 ↳ neglected other topologies beyond genus 0
 (large N)

saddle point approx.

(large λ)

$$\Rightarrow \langle W(C) \rangle = e^{iS[X_c]/\partial\Sigma-C}$$

$X_{\text{classical}}$: solution to worldsheet EOMs

action evaluated at classical string solution

Examples

(1) A static quark

$C = \begin{array}{c} \uparrow \\ \text{in } \mathbb{R}^3 \end{array}$ field theory
 length T

Field theory: $\langle W(C) \rangle = e^{-iMT}$, M is quark mass

$$\text{Gravity: } ds^2 = \frac{R^2}{z^2} (-dt^2 + d\vec{x}^2 + dz^2)$$

$$= \frac{r^2}{R^2} (-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2} dr^2 \quad z = \frac{R}{r}$$

ISO

$\sigma^\alpha = (\sigma, \tau)$, $\sigma = r$ $\tau = t \leftarrow$ worldsheet coordinate choice
 $X^i = \text{const.}$ (trivial solution)

$$ds_{ws}^2 = h_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = -\frac{r^2}{R^2} d\tau^2 + \frac{r^2}{R^2} d\sigma^2 \leftarrow D\text{-brane at } r_0$$

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma \sqrt{h} = -\frac{1}{2\pi\alpha'} \int dt \int_0^{r_0} d\sigma = -\frac{1}{2\pi\alpha'} T r_0$$

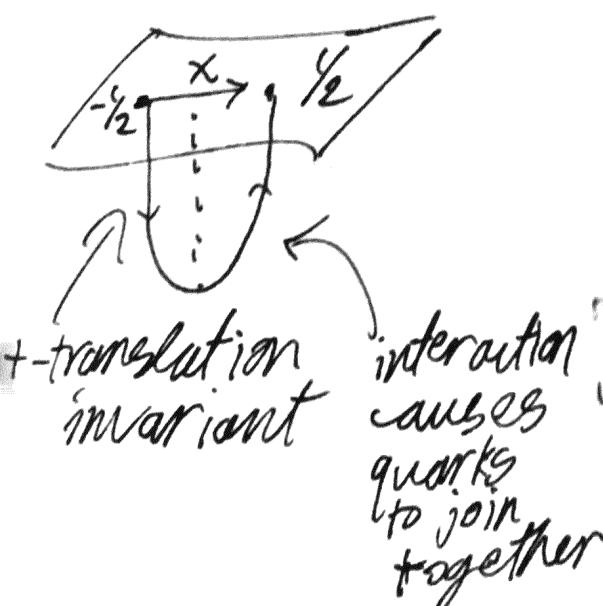
$$\text{so } S_{NG} = -M T, \quad M = \frac{r_0}{2\pi\alpha'}$$

External quark: $r_0 \rightarrow \infty \Rightarrow M \rightarrow \infty$

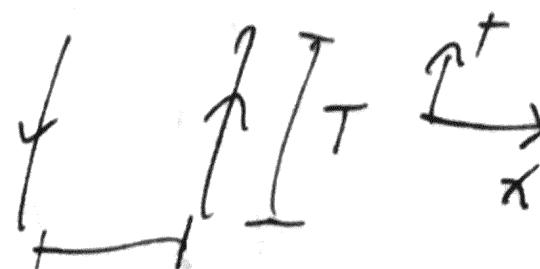
Take $r_0 = \Lambda$, $M = \frac{\Lambda}{2\pi\alpha'}$ - Λ : UV energy cutoff

$$\Lambda = \frac{R^2}{\epsilon}, \quad \epsilon = \varepsilon \Rightarrow \Lambda = \frac{R^2}{2\pi\alpha'} \frac{1}{\epsilon} = \frac{\sqrt{\Lambda}}{2\pi} \frac{1}{\epsilon}$$

2) Static Potential between quark/antiquark



Field theory:



$$\langle W(c) \rangle = e^{-iE_{tot}T}$$

$$E_{tot} = 2M + VL$$

choose: $t = \tau$, $\sigma = x$

$X^i = \text{const}$ for $i \neq X$, $Z = Z(\sigma) = Z(X)$
with boundary conditions
 $Z(\pm \gamma_2) = 0$

$$ds_{\text{WS}}^2 = \frac{R^2}{z^2} (-d\tau^2 + ((t(z'))^2) d\phi^2)$$

\uparrow "dx"
 \uparrow "dz"

$$S_{\text{NGE}} = \frac{2R^2}{2\pi\alpha'} T \int_0^{L/2} \frac{d\sigma}{z^2} \sqrt{(t(z'))^2} \quad z \text{ indep.}$$

$x+X \text{ sym}$
 $z(\sigma) = z(-\sigma)$
cutoff as UV

$$S_{\text{NGE}} V(L) = \frac{\sqrt{\lambda}}{\pi} \int_L^{L/2} d\sigma \sqrt{t(z)^2} - \frac{2\sqrt{\lambda}}{2\pi} \frac{1}{\epsilon}$$

Z is extremized by $z' \Pi_z - Z = \text{const}$, $\Pi_z = \frac{\partial L}{\partial z'}$

at $\sigma=0$, $Z(0) = Z_0$

$$\Rightarrow (z')^2 = \frac{z_0^4 - z^4}{z^4}, \quad Z_0 = L \sqrt{\pi} \frac{\Gamma(y_4)}{2 \Gamma(3y)}$$

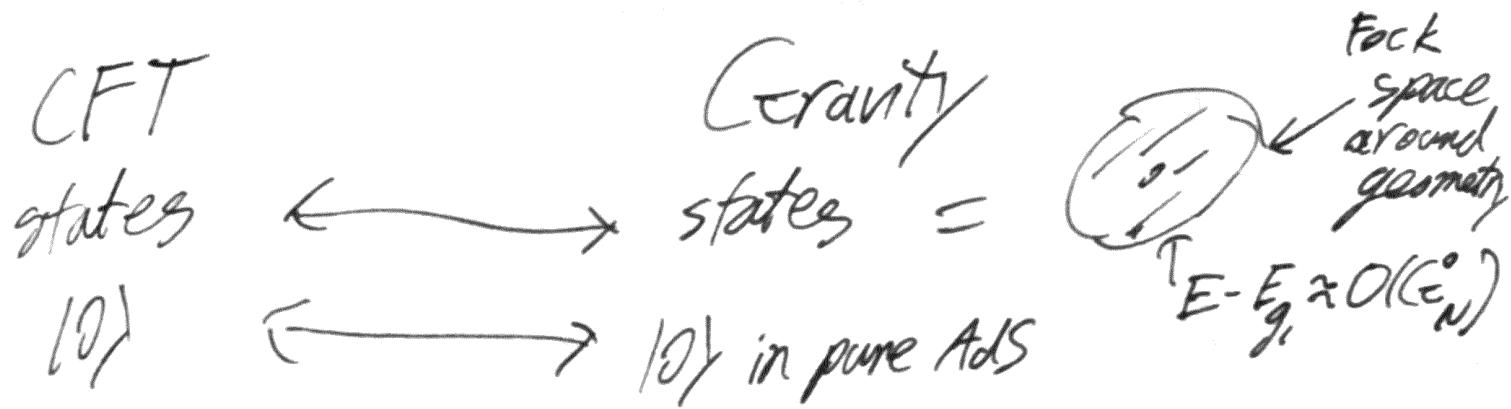
$$\Rightarrow V(L) = \frac{\sqrt{\lambda}}{\pi} \left[Z_0^2 \int_L^{Z_0} \frac{dz}{z^2} \frac{1}{\sqrt{z_0^4 - z^4}} - \frac{1}{\epsilon} \right] \Rightarrow V(L) = -\frac{\sqrt{\lambda}}{L} \frac{1}{4\pi^2 \Gamma^4(y_4)}$$

Remarks:

- (a) $V(L)$ is finite, negative \Rightarrow attraction of quark-antiquark
- (b) $V(L) \propto 1/L$ (scale invariance, only energy is $1/L$)
- (c) $V(L) \propto \sqrt{\lambda}$ (strong coupling result, at weak coupling $V(L) \propto e^{-L}$)
- (d) $Z_0 \propto L \Rightarrow$ deeper in well \leftrightarrow larger L (IR/UV connection)

Dec 10th, 2018 (Notes from Sam Leutherusser)

3.2 Finite Temperature



Finite Temperature:

SU(N) gauge theory in flat space

$$\rightarrow q \propto N^2, S \propto N^2, \dots$$

$$\Rightarrow \mathcal{E} \sim 1/G_N$$

Backreaction: $G_N \mathcal{E} \sim O(1)$

gravity backreaction
so this is a new geometry!

Q: What does the thermal state in CFT correspond to?
Criteria it should satisfy:

- (1) asymptotic AdS (normalizable, since finite T gives some theory, different state)
- (2) satisfy all laws of thermodynamics
- (3) translationally and rotationally invariant along boundary directions

Candidates:

1. Thermal AdS: $ds^2 = \frac{R^2}{z^2} (-dt^2 + dx^2 + dz^2)$

geometry is singular as $z \rightarrow \infty$ $\downarrow dt^2$, with top circle of t goes to zero size \Rightarrow singularities

2. A Black hole with event horizon that is topologically R^{d-1}

Ansatz: $ds^2 = \frac{R^2}{z^2} (-f(z) dt^2 + dx^2 + g(z) dz^2)$

flat horizon

flat horizon
due to

\Rightarrow Einstein equations: $f = g = -\frac{z^d}{z_0^d}$ (in AdS_{d+1})
 $(\Lambda < 0)$ Poincaré symmetry

$z_0 = \text{const}$ \Rightarrow horizon at $z = z_0$

standard trick of going to Euclidean time

\Rightarrow require Euclidean sol'n smooth

$$\Rightarrow \beta = \frac{1}{T} = \frac{4\pi}{d} z_0 \Rightarrow T = \frac{d}{4\pi} \frac{1}{z_0} \quad (\text{measured in units of } \tau)$$

$z=0$

{ Higher T horizon
 $\Rightarrow z=0$ probes higher energy

$z \sim z=z_0$ - - - - - horizon } Lower T horizon
 $\Rightarrow z=\infty$ probes low energy

Thermodynamics

$$S_{BH} = \frac{A_{hor}}{4G_N}$$

For AdS_5 , $d=4$ entropy density

$$A_{hor} = \frac{R^3}{2^3} \int_{\text{boundary}} d\vec{x} \Rightarrow S = \frac{R^3}{4Z^3 G_N} = \frac{\pi^2}{2} N^2 T^3 (x)$$

$V = \text{spatial volume}$
 of boundary

with $G_N/R^3 = \pi/2N^2$

(*) is also a prediction of entropy density on $N=4$ SYM in $N \rightarrow \infty, x \rightarrow \infty$ limit.

Read $\langle T_{\mu\nu} \rangle_B$ from metric $\Rightarrow \langle T_{\mu\nu} \rangle_B \propto \frac{1}{2^d} a^{-d}$

Can also use thermodynamic relations:

$$S = -\frac{\partial F}{\partial T} \Rightarrow F = -\frac{\pi^2}{8} N^2 T^4 \Rightarrow E = F + TS = \frac{3\pi^2}{8} N^2 T^4$$

Compare with free theory

$$S_{x=0} = (8 + 8 \cdot \frac{2}{8}) \frac{2\pi^2}{45} T^3 N^2 = \frac{2}{3} \pi^2 N^2 T^3$$

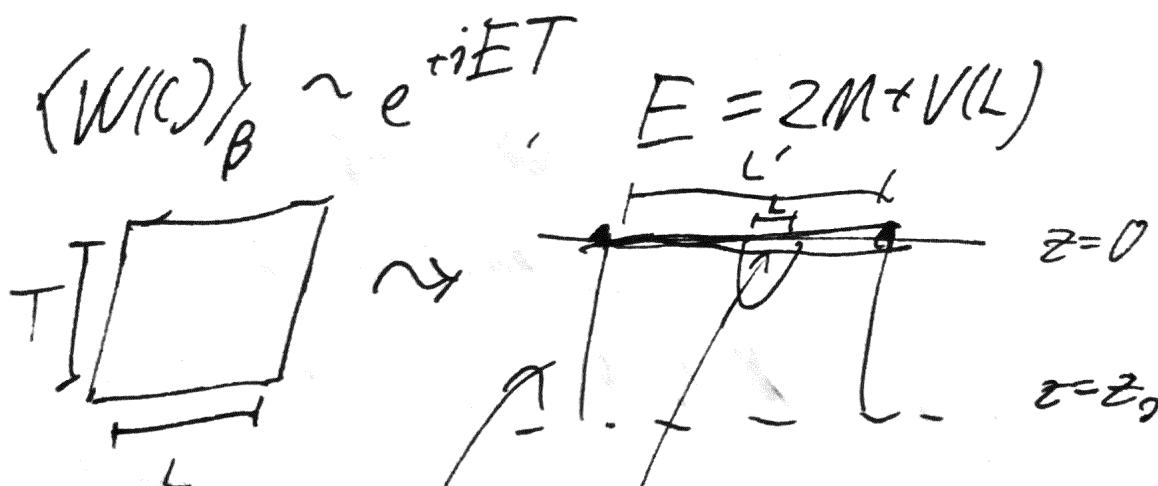
$\underbrace{2 \text{ on-shell } \lambda_\mu}_{+ 6 \text{ scalars}}$ $\underbrace{8 \text{ fermions}}_{\text{fermion contrib.}}$

$$\Rightarrow \left[\frac{S_{x=\infty}}{S_{x=0}} = \frac{3}{4} \right]$$

Many examples of CFT duals are known for $d=4$. In all known cases we get

$$\frac{S_{\text{strong}}}{S_{\text{free}}} = \frac{3}{4} h, \text{ with } \frac{8}{9} < h < 1.09$$

\approx ratio is always $\approx 3/4$ for CFT duals
(in ∞ of theories) \Leftarrow no idea why



small L' , short distance physics
doesn't feel temp. in CFT
(doesn't see z_0 in the bulk)

at large L' , minimal surface ends
on horizon \Rightarrow screened quarks,
"finite T plasma screens"

so far, CFT is on \mathbb{R}^d , dual to black brane

~~what happens in~~ scale inv. theory,

T is only scale, so all T
are the same, related by scaling
 $\Rightarrow T$ sets units

What happens if we consider global AdS?

\Rightarrow CFT is on $\mathbb{R} \times S^{d-1}$ at finite T

Take the sphere radius to be R (just sets scale)
 At finite T , there is dimensionless param. RT

Some important features:

(1) Thermal Ads is now allowed

$$\text{global AdS}_5 \quad ds^2 = -\left(1 + \frac{r^2}{R^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} + r^2 d\Omega_4^2$$

$$r \rightarrow \infty : ds^2 = \frac{r^2}{R^2} \left(-dt^2 + R^2 d\Omega_{d+1}^2 \right)$$

boundary metric

$t \rightarrow -it$, $\tau \sim \tau + \beta$ (no singularity at $r \rightarrow 0$ since g_{eff} finite)

$$(2) ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega_{d-1}^2$$

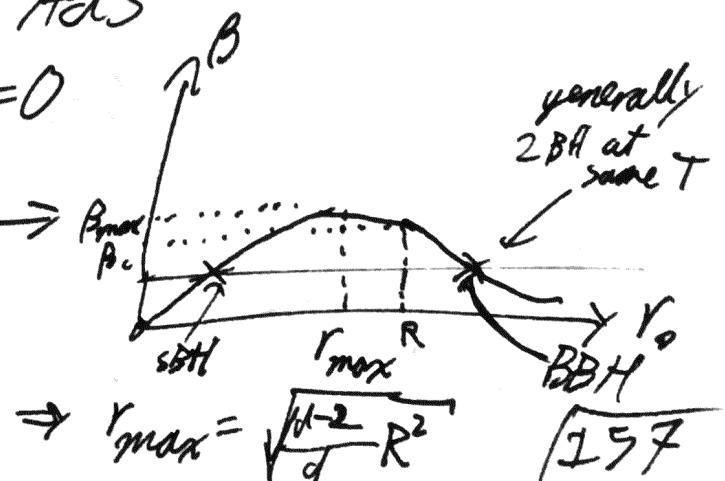
$$f = 1 - \frac{M}{r^{d-2}} + \frac{e^2}{R^2}$$

Schwarzschild
BH of mass M

AdS

horizon at $r=r_0$, $f(r_0)=0$

$$\beta = \frac{4\pi}{5(r_0)} = \frac{4\pi r_0 R}{d \cdot r_0^2 + (d-2)R^2}$$



- (i) $\beta_{\max} \rightarrow T_{\min}$
- (ii) two BH solutions at given β

(3) We find

- (i) For $T < T_{\min} \rightarrow$ no BH, only thermal AdS (TAdS)
- (ii) For $T > T_{\min} \rightarrow$ three possibilities:

TAdS, SBH, BBH

$$e^{-\beta F_{\text{eff}}} = Z_{\text{eff}}(\beta) = Z_{\text{grav}}(\beta) = \int D\Phi e^{S_E[\Phi]} \\ = \sum_{\text{saddles}} e^{S_E[\Phi_c]}$$

Dominant saddle has largest $S_E[\Phi_c]$

this solution
dominates

$$S_E = \frac{1}{16\pi G_N} \int [(R - 2\Lambda) + 2\text{matter}] \propto N^2$$

pure AdS: $S_E = 0$ (have to renormalize s.t. this holds)

TAdS: $S_E = 0 \cdot N^2 + O(N)$ (b.c. this only differs by global $T \sim \bar{c} + \beta$ so curvature terms are locally the same)

BBH: $\propto N^2$

SBH: sign of \star determines if these are dominant
→ complicated calculation

~~Start of BH~~

short cut: $W_{d-1} = \text{Vol of } S^{d-1}$

$$S = \frac{W_{d-1} r_0^{d-1}}{4 G_N} \xrightarrow{\text{integrate}} S = -\frac{\partial F}{\partial T} = -\frac{\partial F}{\partial r_0} \frac{\partial r_0}{\partial T}$$

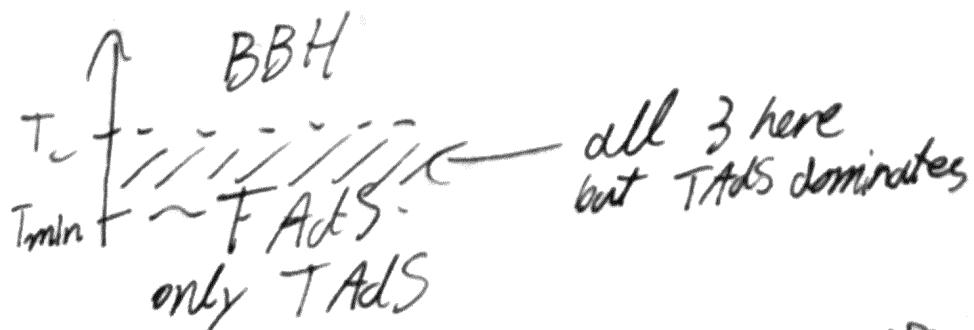
$$\Rightarrow F = \frac{W_{d-1}}{16\pi G_N} \left(r_0^{d-2} - \frac{r_0^d}{R^2} \right)$$

$F_{BH} > 0$ when $r_s < R$ $\leftarrow \beta_c = \beta(r_s=R) = \frac{1}{T_c}$

$F_{BH} > 0$ when $r_s > R$

$T_c > T_{\min}$, for $T < T_c$ thermal AdS dominates
for $T > T_c$ BBH dominates

always $S_E(\text{BBH}) > S_E(\text{SBH})$ and $S_E(\text{SBH}) < 0$



(5) BBH has positive specific heat
SBH has negative specific heat



\rightarrow stable



\rightarrow SBH unstable (doesn't know)
 \rightarrow evaps it's in box!

159

(6) Since physics only depends on RT
 keep RT fixed, $T \rightarrow \infty = \underbrace{\text{keep } RT \text{ fixed}, R \rightarrow \infty}_{\text{flat space limit}}$

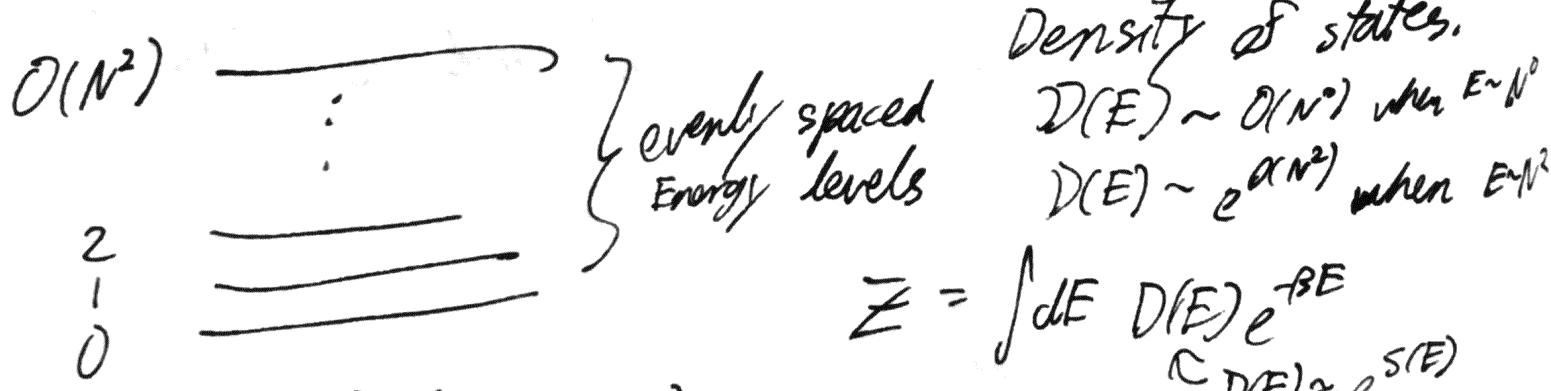
(7) Hawking-Page transition
 (sometimes called "deconfinement transition")

$$T < T_c \Rightarrow F_{CFT} \sim O(N^0) \quad \begin{cases} 1^{\text{st}} \text{ order} \end{cases}$$

$$T > T_c \Rightarrow F_{CFT} \sim O(N^2) \quad \begin{cases} \text{phase transition} \end{cases}$$

(8) Physics underlying HP transition:

A free theory of two matrices A, B ($N \times N$)
 $\approx 2N^2$ harmonic oscillators w/ frequency $\omega = 1$



Take $\beta \sim O(N^0)$ (indep of N)

$$\text{suppose } E = \varepsilon N^2 \Rightarrow S(E) = F(\varepsilon) N^2$$

$$\approx e^{-\beta E} D(E) = e^{(F(\varepsilon) - \beta \varepsilon) N^2}$$

\approx For $F(\varepsilon) < \beta \varepsilon$ highly excited ($E \sim N^2$) states won't contribute
 $\Rightarrow Z$ receives dominant contributions from $\Rightarrow F = O(N^0)$

IGOT IF $F(\varepsilon) \geq \beta \varepsilon$, $F = O(N^2)$

3.3 Holographic Entanglement Entropy

• Entanglement entropy:

divide system $A+B$

Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$|\psi\rangle = \sum_n \chi_n(A) \chi_n(B)$$

A and B in state $|\psi\rangle$ are entangled if $|\psi\rangle$ cannot be written as a simple product of states of A, B

EE: a measure to quantify how much A and B are entangled

$$\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$$

$$\Rightarrow \boxed{S_A = -\text{Tr}_A \rho_A \log \rho_A \geq 0}$$

$S_A = 0 \Leftrightarrow \rho_A$ is a pure state

$\Leftrightarrow |\psi\rangle$ can be written as a simple product

For a pure state: $S_A = S_B$ in general

IF AB is in a mixed state,
we do not in general have $S_A = S_B$

Example: 2 spin system $\begin{matrix} \uparrow & \uparrow \\ A & B \end{matrix}$

$$a) |\psi\rangle = \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$$

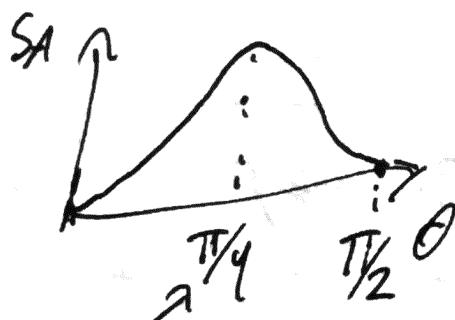
$$= \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$$

\rightarrow not entangled

$$b) |\psi\rangle = \cos\theta|\uparrow\downarrow\rangle + \sin\theta|\downarrow\uparrow\rangle$$

$$P_A = \cos^2\theta|\uparrow\rangle\langle\uparrow| + \sin^2\theta|\downarrow\rangle\langle\downarrow|$$

$$\rightarrow S_A = -\cos^2\theta \log \cos^2\theta - \sin^2\theta \log \sin^2\theta$$



maximally
entangled

Some important properties:

(1) Subadditivity

$$|S(A) - S(B)| \leq S(AB) \leq S(A) + S(B)$$

(2) Strong subadditivity:

$$S(ABC) \leq S(AC) + S(BC) \geq S(ABC) + S(C)$$

$$S(AC) + S(BC) \geq S(A) + S(B)$$

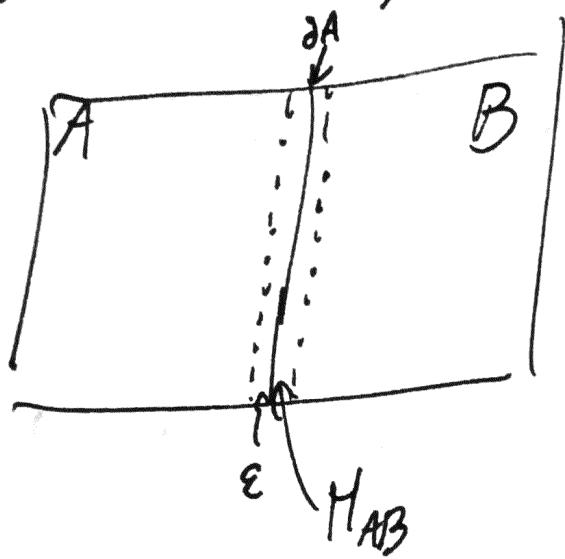
Entanglement Entropy in many-body systems:

IF $H = H_A + H_B$ \Rightarrow ground state is unentangled
~ start with unentangled initial state
then the system remains unentangled
 \Rightarrow interactions are crucial for generating entanglement

Now consider $H = H_A + H_B + H_{AB}$ ~ ground state is entangled

~ entanglement will be generated from time evolution

In realistic condensed matter systems and QFTs, H and H_{AB} are local



take ϵ here to be the lattice spacing

$\Rightarrow H_{AB}$ only involves d.o.f. near $dA = -dB$

e.g. take $H = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$

$$\text{QFT: } \mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\phi^4$$

One finds: in general in the ground state for a local H ,

$$S_A = \# \frac{\text{Area}(dA)}{\epsilon^{d-2}} + \dots \quad (*)$$

i.e. entanglement between A and B is dominated by short-range entanglement near dA , where H_{AB} is supported

Area law is universal, $\#$ is not universal, and depends on details of UV theory.

Sub-leading terms in $\langle \chi \rangle$, which encode long-range entanglement, can provide important characterization of a system

Example:

in $(2+1)$ -dimensions

(1) characterize topological order

realized by

X.G. Wen

M. Levin

and independently by

J. Preskill

A. Kitaev

~~typical gapped systems:~~ $\longrightarrow 1$
 $\longrightarrow 0$

\rightsquigarrow contains only short-range entanglement
but in topologically ordered systems:

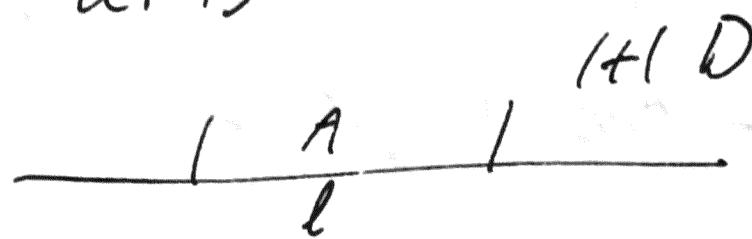
ground state can have subtle long-range correlations

$$\rightsquigarrow S_A = \# \frac{L}{\varepsilon} - \gamma \leftarrow \begin{array}{l} \text{topological entanglement} \\ \text{entropy} \end{array}$$

index. of
shape and size
of A

(2) characterize # d.o.f. of a system of relativistic QFTs

entanglement entropy in 1+1D \sim CFT



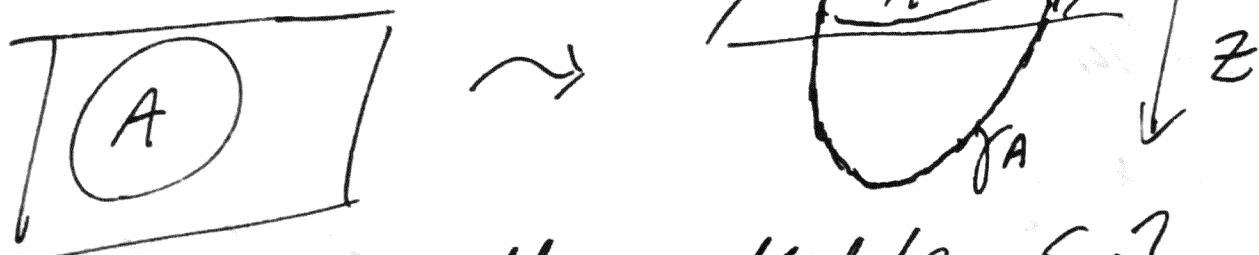
$$S_A = \frac{C}{3} \log \frac{l}{\epsilon}$$

where C is the CFT's central charge

in (2+1)-D CFT:

$$S_A = \# \underbrace{\int \partial A}_{\epsilon} - \gamma \quad \begin{matrix} \text{depends on} \\ \text{shape of } A \\ \text{minimized at } A = \text{circle} \end{matrix}$$

Holographic entanglement entropy



How would we calculate S_A ?

Proposal: Find the minimal area surface γ_A which extends into the bulk with ∂A as its boundary

Ryu-Takanagi:

Then

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$

(**) This diverges as ϵ^{d-2} , where ϵ is the ϵ -cutoff

γ_A : d-2 dim

A, γ_A : d-1 dim in AdS_{d+1}

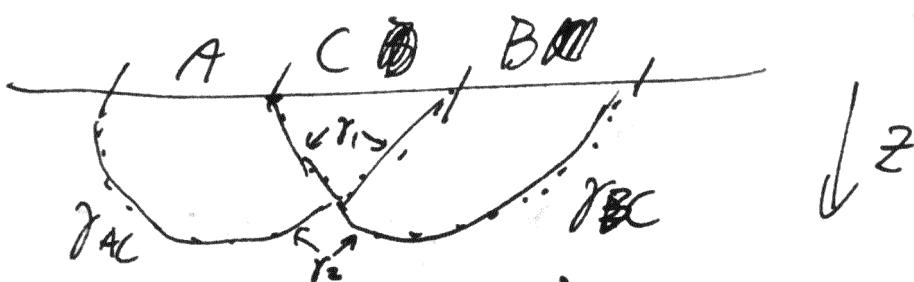
S_A : dimensionless

This formula is very valuable, as entanglement entropy is extremely difficult to calculate even in non-interacting QFTs, but this minimal surface area is relatively easy to calculate.

Things to check:

• strong subadditivity:

$$S(AC) + S(BC) \geq S(ABC) + S(C)$$

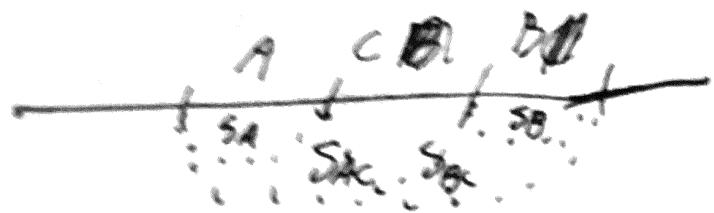


$$\begin{aligned} S(AC) + S(BC) &= A(\gamma_{AC}) + B(\gamma_{BC}) \\ &= A(\gamma_1) + A(\gamma_2) \end{aligned}$$

$$A(\gamma_2) \geq A(\gamma_C) \quad \text{QED}$$

$$A(\gamma_2) \geq A(\gamma_{ABC}) \quad \boxed{167}$$

Can also see: $S(AC) + S(BC) > S_{AB} + S(B)$



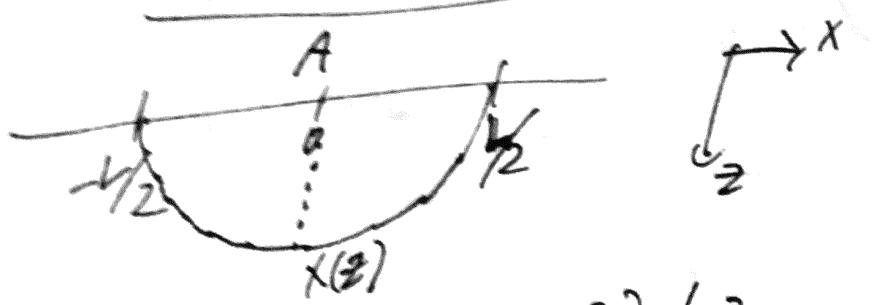
- Can get entanglement entropy of (1+1)-CFT

We've seen $\mathcal{CFT}_4 \leftrightarrow \text{AdS}_5$ $N^2 \sim 1/\epsilon_n$
can also construct $\mathcal{CFT}_2 \leftrightarrow \text{AdS}_3$

$$ds^2 = \frac{R^2}{z^2} (dz^2 - dt^2 + dx^2)$$

each \mathcal{CFT}_2 is characterized by a central charge c

$$\boxed{c = \frac{3R}{2G\epsilon_n}}$$



$$dl^2 = \frac{R^2}{z^2} ((1+x'(z)^2) dz^2) \quad x(z=0) = \frac{\pi}{2} \Rightarrow S_A = \frac{1}{4\pi} \cdot 2 \cdot \int_0^{\frac{\pi}{2}} dz \frac{R}{z} \sqrt{1+x'^2}$$

[IG8] (From high school, know geodesic in hyperbolic space is semicircle)

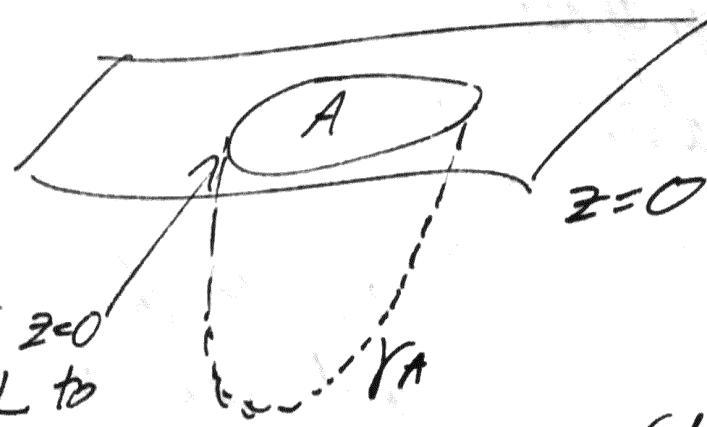
half-circle: $x = \sqrt{L^2 - z^2}$

$$S_A = \frac{1}{4G_N} 2R \frac{L}{2} \int_{-L}^{L/2} \frac{dz}{z} \frac{1}{\sqrt{\frac{L^2}{4} - z^2}}$$

logarithmic
UV divergence
change
to ϵ

$$\rightarrow \frac{1}{3} \cdot \frac{3R}{2G_N} \log \frac{L}{\epsilon}$$

- At finite temperature,
(**) is compatible with Beckenstein-Hawking formula for black hole entropy
- Area law for general dimensions:



near $z=0$
 ∂A is \perp to
boundary

$$\Rightarrow S_A = \frac{\text{Area}(dA)}{\epsilon^{d-2}} + \dots$$

Ryu-Takayanagi lets us understand this leading order term 169

Final words

why should we expect entropy, the quantum information of a system, be related to the area of a region in some spacetime?

Ryu-Takayanagi formula implies:

spacetime \longleftrightarrow geometrization of quantum entanglement

geometry \longleftrightarrow quantum informal

Quantum Information

Quantum Field Theory

Holographic Duality

Condensed Matter Theory

Black Holes and Quantum Gravity