Chapter 11: Duality Connections and Nonperturbative Effects

1. Taking the expression for a toroidal heterotic compactification from exercise **9.1**

$$\left[\frac{R}{\sqrt{\tau_2}\eta\bar{\eta}^{17}}\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2} e^{-i\pi\sum_I nY^I(m+n\bar{\tau})Y^IY^I}\frac{1}{2}\sum_{a,b=0}^{1}\prod_{i=1}^{16}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}(Y^I(m+\bar{\tau}n)|\bar{\tau})\right]\times\frac{1}{\tau_2^{7/2}\eta^7\bar{\eta}^7}\frac{1}{2}\sum_{a,b=0}^{1}\frac{\theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4}$$

Using θ function identites as in the second equation in appendex **E**, we get

$$\Gamma_{1,17}(R,Y) = \frac{R}{\sqrt{\tau_2}}\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\frac{1}{2}\sum_{a,b=0}^{1} e^{i\pi mY^IY^In - i\pi bnY^I}\bar{\theta}\begin{bmatrix}a-2nY^I\\b-2mY^I\end{bmatrix}$$

Now take $Y^I = 0$ for $I = 1\ldots 8$ and $Y^I = 1/2$ for $I = 1\ldots 16$. Then

$$\prod_I e^{i\pi mY^IY^In - i\pi bnY^I} = e^{i\pi m\sum_I (Y^I)^2 - i\pi b\sum_I Y^I} = 1$$

and we can ignore this term. Similarly because we are taking a product over 16 $\bar{\theta}$, no phases will interfere with us replacing $\theta\begin{bmatrix}u\\v\end{bmatrix}$ with $\theta\begin{bmatrix}-u\\-v\end{bmatrix}$ for integer u, v. This gives us the desired first step

$$\Gamma_{1,17}(R,Y) = R\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\frac{1}{2}\sum_{a,b=0}^{1}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}^8\bar{\theta}\begin{bmatrix}a+n\\b+m\end{bmatrix}^8$$

Now again because we have enough $\theta\begin{bmatrix}a+n\\b+m\end{bmatrix}$ that phases do not interfere, we see that we only care about n, m modulo 2 in the fermion term. We know how to divide the partition function of the compact boson into parity odd and even blocks by doing the $\mathbb{Z}^2$ stratification corresponding to the πR translation orbifold of the circle. This gives our desired answer:

$$\frac{1}{2}\sum_{h,g}\Gamma_{1,1}(2R)\begin{bmatrix}h\\g\end{bmatrix}\frac{1}{2}\sum_{a,b}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}^8\bar{\theta}\begin{bmatrix}a+h\\b+g\end{bmatrix}^8$$

with

$$\Gamma_{1,1}(2R) = 2R\sum_{m,n}\exp\left[\frac{-\pi R^2}{\tau_2}|2m+g+(2n+h)\tau|^2\right]$$

2. As before, take the ansatz

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}dx^i\cdot dx^i, \qquad A_{012} = \pm e^{C(r)} \Rightarrow G_{r012} = \pm C'(r)\, e^{C(r)}$$

The BPS states in 11D require only the gravitino variation to vanish:

$$\delta\psi_M = \partial_M\epsilon + \frac{1}{4}\omega_M^{PQ}\Gamma_{PQ}\epsilon + \frac{1}{2\cdot 3!\cdot 4!}G_{PQRS}\Gamma^{PQRS}\Gamma_M\epsilon - \frac{8}{2\cdot 3!\cdot 4!}G_{MQRS}\Gamma^{QRS}\epsilon$$

We have worked out ω in **8.43**.

$$\omega_{\hat{\mu}\hat{\nu}} = 0, \quad \omega_{\hat{\mu}\hat{i}} = (-)^{\mu=0}\,\partial_i A\, e^{A-B}dx^\mu, \quad \omega_{\hat{i}\hat{j}} = \partial_j B dx^i - \partial_i B dx^j$$

Let's look first at $M = \mu$ parallel. Since ϵ is Killing we expect no longitudinal variation and we get

$$\begin{aligned}0 &= \cancel{\partial_\mu}\epsilon + \frac{1}{2}A'\, e^{A-B}\Gamma^{\hat{\mu}\hat{r}}\epsilon \pm \frac{1}{2\cdot 3!}C'(r)e^C\cancel{\Gamma^{r012}\Gamma_\mu}\epsilon \mp \frac{1}{3!}C'(r)e^C\Gamma_\mu\Gamma^{r012}\epsilon\\ &= \frac{1}{2}A'\, e^{A-B}\Gamma^{\hat{\mu}\hat{r}}\epsilon \mp \frac{1}{3!}C'e^{C-B-2A}\Gamma^{\hat{\mu}\hat{r}\hat{0}\hat{1}\hat{2}}\epsilon\\ &\Rightarrow 0 = A'\epsilon \mp \frac{1}{3}C'e^{C-3A}\Gamma^{\hat{0}\hat{1}\hat{2}}\epsilon\end{aligned}$$

If we would like these two terms to be proportional, then we should take $C = 3A$, and we get the following condition for ϵ

$$(1 \mp \Gamma^{\hat{0}\hat{1}\hat{2}})\epsilon = 0$$

So half the dimension of the space of spinors satisfies this at any given point. We thus get

For $M = i$ transverse, we recall Γ_{ij} generates rotations, so assuming rotational invariance in the transverse space, we'll cancel this. We get

$$\begin{aligned}
&\partial_r \epsilon + \cancel{\frac{1}{4}\omega_r^{jk}\Gamma_{jk}}\epsilon + \cancel{\frac{1}{2\cdot 3!}G_{r012}\Gamma^{r012}}\Gamma_r \epsilon \mp \frac{1}{3!}G_{r012}\Gamma^{012}\epsilon = 0\\
\Rightarrow &\partial_r \epsilon \mp \frac{1}{3!}G_{r012}\Gamma^{012}\epsilon = 0\\
\Rightarrow &\partial_r \epsilon \mp \frac{e^{-3A}}{3!}C' e^C \Gamma^{\hat{0}\hat{1}\hat{2}}\epsilon
\end{aligned}$$

Solving this gives us that

$$\epsilon(r) = e^{C(r)/6}\epsilon_0$$

for ϵ_0 some constant spinor. We still do not have a relationship between C and B. This can be obtained by not assuming rotational invariance but rather imposing cancelation of the second and third terms above as follows:

$$\begin{aligned}
&\frac{1}{2}\partial_j B\,\Gamma^{\hat{i}\hat{j}}\epsilon \pm \frac{1}{2\cdot 3!}\partial_j C\, e^C\, \Gamma^{j012}\Gamma_i \epsilon\\
&= \frac{1}{2}\partial_j B\,\Gamma^{\hat{i}\hat{j}}\epsilon \pm \frac{1}{2\cdot 3!}\partial_j C\, e^{C-3A}\, \Gamma^{\hat{i}\hat{j}\hat{0}\hat{1}\hat{2}}\epsilon\\
&\Rightarrow \partial_j B + \frac{1}{3!}\partial_j C = 0
\end{aligned}$$

where we have used the condition on ϵ already obtained. Thus $C = 3A = -6B$. Finally Let's look at G's equation of motion:

$$\mathrm{d}G = 0, \qquad \frac{1}{3!}\mathrm{d}\star G + \frac{3}{(144)^2}\epsilon^{MNOPQRST}G_{MNOP}G_{QRST} = 0$$

By assumption, the term quadratic in G vanishes. What remains gives us:

$$0 = \partial_r(e^{3A+8B}e^{-6A-2B}C'(r)e^C) = \partial_r(e^{-3A+6B+C}C') = \partial_r(C'e^{-C}) \Rightarrow \partial_r^2 e^{-C} = 0$$

So we have that $e^{-C} = H(r)$ as required, where

$$H(r) = 1 + \frac{L^6}{r^6}$$

I'm happy with this. I could use Mathematica to show that the other EOM:

$$R_{MN} - \frac{1}{2}g_{MN}R = \kappa^2 T_{MN}, \quad \kappa^2 T_{MN} = \frac{1}{2\cdot 4!}\left(4G_{MPQR}G_N^{PQR} - \frac{1}{2}g_{MN}G^2\right)$$

is satisfied - but this is barely different from what I've done several times before for the D-branes and fundamental string solutions in chapter 8.

As before, this generalizes straightforwardly to multi-membrane configurations.

The charge of the M2 brane with $H = 1 + \frac{32\pi^2 N\ell_s^6}{r^6}$ is given by integrating $\frac{\star G}{2\kappa_{11}^2}$ on a seven-sphere at infinity. Here $2\kappa_{11}^2 = (2\pi)^8\ell_{11}^9$ Asymptotically we will get the field strength going as

$$\frac{32\times 6\pi^2 N\ell_{11}^6}{r^6}$$

Altogether, using $\Omega_7 = \frac{\pi^4}{3}$ this gives a total charge of

$$\frac{\pi^4}{3}\frac{32\times 6\pi^2 N\ell_{11}^6}{(2\pi)^8\ell_{11}^9} = \frac{N}{(2\pi)^2\ell_{11}^2}$$

This is exactly consistent with **11.4.10-13**, with $\mu = N = 1$ corresponding to a single M2 brane.

Calculating the Ricci scalar curvature in fact gives a *constant* as $r \to 0$ so we do *not* encounter a divergence. This signifies that this is just a coordinate singularity and we can extend past.

In[120]:= `R = RicciScalar[g, xx]`

Out[120]= $-\dfrac{6144\,\mathtt{ll}^{12}\,\mathtt{NN}^2\,\pi^4}{\left(1+\frac{32\,\mathtt{ll}^6\,\mathtt{NN}\,\pi^2}{r^6}\right)^{1/3}\left(32\,\mathtt{ll}^6\,\mathtt{NN}\,\pi^2\,r+r^7\right)^2}$

In[122]:= `Series`$\left[-\dfrac{6144\,\mathtt{ll}^{12}\,\mathtt{NN}^2\,\pi^4}{\left(1+\frac{32\,\mathtt{ll}^6\,\mathtt{NN}\,\pi^2}{r^6}\right)^{1/3}\left(32\,\mathtt{ll}^6\,\mathtt{NN}\,\pi^2\,r+r^7\right)^2},\ \{r,0,0\}\right]$

Out[122]= $-\dfrac{3}{(2\pi)^{2/3}\left(\frac{\mathtt{ll}^6\,\mathtt{NN}}{r^6}\right)^{1/3} r^2} + O[r]^1$

Finally, we can take the near-horizon limit and get

$$\begin{aligned} ds^2 &= \frac{r^4}{L^4}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dx^i \cdot dx^i \\ &= \frac{r^4}{L^4}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_7^2 \end{aligned}$$

Take now $r = L/\sqrt{z}$ to get the first term to look like $1/z^2$ while not affecting the second term much:

$$\frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + 4L^2 dz^2) + L^2 d\Omega_7^2$$

We can rescale z, x^μ and see that this geometry is $\text{AdS}_4 \times S^7$

3. The M5 brane is now magnetically charged under C_3. Now the equations of motion $\mathrm{d} \star \mathrm{d}C = 0$ are trivially satisfied but the Bianchi identity is nontrivial, giving

$$\partial_r^2 H = 0 \Rightarrow H = 1 + \frac{L^3}{r^3}$$

The metric form can be fixed by analyzing the gravitino variation similar to before. Longitudinally:

$$\begin{aligned} 0 &= \frac{1}{2}A' e^{A-B}\Gamma^{\hat{\mu}\hat{r}} + \frac{1}{2\cdot 3!}C' e^{C+A-4B}\Gamma^{\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4\hat{\mu}} \\ &\Rightarrow A'\epsilon + \frac{1}{3!}C' e^{C-3B}\Gamma^{\hat{r}\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4}\epsilon \end{aligned}$$

We see that we must take $C = 3B$ and $A = -C/6$, and we get the half-BPS condition:

$$(1 - \Gamma^{\hat{7}\hat{8}\hat{9}\hat{1}\hat{0}\hat{1}\hat{1}})\epsilon = 0$$

The transverse components will give the profile for ϵ.

$$\partial_r \epsilon + \frac{1}{2\cdot 3!}C' e^{C-3B}\Gamma^{\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4\hat{r}}\epsilon$$

and this gives a profile

$$\epsilon = e^{-C/12}\epsilon_0$$

The membrane charge is given by integrating G on a 4-sphere whose area is given by $8\pi^2/3$, so we get

$$\frac{8\pi^2}{3}\frac{3\pi N \ell_{11}^3}{(2\pi^8)\ell_{11}^9} = \frac{N}{(2\pi\ell_{11})^5\ell_{11}}$$

Again we get that the Ricci scalar tends to a constant as $r \to 0$, giving regularity at the horizon. Again, this signifies that this is just a coordinate singularity and we can extend past.

```
In[138]:= R = RicciScalar[g, xx]
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Out[138]=
$$\frac{3\, ll^6\, NN^2\, \pi^2}{2\left(1+\frac{ll^3\, NN\, \pi}{r^3}\right)^{2/3}\left(ll^3\, NN\, \pi\, r + r^4\right)^2}$$

```
In[139]:= Series[R, {r, 0, 0}]
```

Out[139]=
$$\frac{3}{2\,\pi^{2/3}\left(\frac{ll^3\, NN}{r^3}\right)^{2/3} r^2} + O[r]^1$$

Taking the near-horizon limit we arrive at

$$ds^2 = \frac{r}{L}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dx^i \cdot dx^i = \frac{r}{L}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_4^2$$

Now take $r = L/z^2$ yielding

$$\frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + 4L^2 dr^2) + L^2 d\Omega_4^2$$

so again after rescaling the same was as before we get $\text{AdS}_7 \times S^4$.

As before, a solution can consist of an arbitrary number of $M5$ branes at different places, in which case we get

$$H(r) = 1 + \sum_i \frac{L_i}{|r - r_i|^3}$$

This remains half-BPS.

4. First look at the field strengths. The general M5 brane solution For a uniform distribution of M5 charges, we know that in the transverse (3D) space the potential must now decay as

$$H = 1 + \int dx^{11} \frac{L}{|\vec{r} - x^{11}\hat{e}_{11}|^2} = 1 + \frac{2L}{r_{10D}^2}$$

where L depends on the density of the distribution. Then the 3-form field strength in 10D will just be

$$(dB)_{abc} = \epsilon_{abce}\partial_e H$$

Given this source in 10D, we have already worked out Einstein's equations in **Chapter 8**. Another way to see this is that we remain half-BPS after adding even an infinite number of parallel branes.

We have that $e^{4\Phi/3} = G_{11,11}$ so that $e^\Phi = H^{1/2}$ consistent with the NS5 solution.

Using the perscription of dimensional reduction in appendix **I.2**, we take $e^\sigma = e^{2\Phi/3} = H^{1/3}$. Using $g_{\mu\nu} = e^{-\sigma} g^S_{\mu\nu}$, we see that multiplying by $H^{1/3}$ takes us to the *string frame* NS5 metric solution.

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + H(r)dx^i \cdot dx^i$$

This is exactly the NS5 metric in string frame.

We can further take $g^S_{\mu\nu} = e^{\Phi/2} g^E_{\mu\nu}$ and multiply the string frame by $e^{-\Phi/2} = H^{-1/4}$ to get us to the Einstein frame.

5. Recall the BPS D6 brane in 10D is described by

$$H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}d\vec{x}\cdot d\vec{x}, \quad H = 1 + \frac{L}{|x|}, L = g_s \ell_s N/2, \qquad F = L\,\mathrm{d}\Omega_2, \quad e^\Phi = g_s^2 H^{-3/4}$$

This means that $e^{-2\Phi/3} = H^{1/2}$. Multiplying ds^2_{string} by this factor, we the 10D part of 11D metric

$$\eta_{ab}d\gamma^a d\gamma^b + V d\vec{x}\cdot d\vec{x}$$

Here we've picked notation consistent with the problem so that $\gamma^{0\ldots6} = x^{0\ldots6}$, $H(r) = V(r)$, and x^i is the same.

Note also that

$$\frac{1}{2\kappa_{10}^2}\int_{S^2} F = \frac{L4\pi}{(2\pi)^7 \ell_s^8 g_s^2} = nT_p \Rightarrow 2L = \ell_s n g_s$$

This should be supplemented by $e^{4\Phi/3}(d\gamma + A_\mu \cdot d\vec{x})^2 = V^{-1}(d\gamma + A_\mu \cdot d\vec{x})^2$ where A_μ is the 10D gauge field generated by the monopole solution.

Now A cannot be globally defined because of the monopole. Given $L = 2N$, it takes the same form as A_μ does in 3D about a monopole of charge $n = N/\ell_s$.

We could have taken a more "active" approach, demonstrating that this metric ansatz does indeed solve Einstein's equations, and shown that for the field strength to satisfy the Bianchi identity in this geometry it needed to indeed be a harmonic function of the transverse coordinates taken with flat metric.

6. The DBI action for a two-brane *in flat space with vanishing B-field and constant dilaton* is given in euclidean signature as

$$-T_2 \int d^3x \sqrt{\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + 2\pi\ell_s^2 F_{ab})} + i\int C^{(3)} \wedge \mathrm{Tr}[e^{\mathcal{F}}] \wedge \mathcal{G},$$

where the second integral consists of Chern-Simons terms that we will ignore in this argument. We can work with the field variable F rather than A by imposing the Bianchi identity "by hand", namely writing the (non-CS) part of the action as

$$-T_2 \int d^3x \left[\sqrt{\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + 2\pi\ell_s^2 F_{ab})} + \frac{i}{2}\lambda \epsilon^{abc}\partial_a F_{bc}\right]$$

This last term can just as well be integrated by parts to give $\epsilon^{abc}\partial_a \lambda F_{bc}$.

We now introduce an auxiliary V variable to rewrite the action as

$$\begin{aligned} &-T_2 \int d^3x \left[\frac{1}{2}V \det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + 2\pi\ell_s^2 F_{ab}) + \frac{1}{2}\frac{1}{V} + \frac{i}{2}\epsilon^{abc}\partial_a \lambda F_{bc}\right] \\ &= -T_2 \int d^3x \left[\frac{1}{2}V(1 + \frac{1}{2}(2\pi\ell_s^2)^2 F_{ab}^2 + \dots) + \frac{1}{2}\frac{1}{V} + \frac{i}{2}\epsilon^{abc}\partial_a \lambda F_{bc}\right] \end{aligned}$$

here $\dots$ involves terms depending on the $\partial_a X^\mu$. The equations of motion for F then give

$$F_{ab} = -i\frac{\epsilon^{abc}\partial_a \lambda}{(2\pi\ell_s^2)^2 V}$$

Substituting this back in gives

$$-T_2 \int d^3x \left[\frac{1}{2}V(1 + (-\frac{1}{2} + 1)(2\pi\ell_s^2)^{-2}(\partial\lambda)^2 + \dots) + \frac{1}{2}\frac{1}{V}\right]$$

Integrating out V gives us the square root action again, but now with F replaced by $\partial\lambda$, a new coordinate

$$-T_2 \int d^3x \sqrt{\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + (2\pi\ell_s^2)^{-2}\partial_a \lambda \partial_b \lambda)}$$

Taking $X = \lambda/2\pi\ell_s^2$ gives our desired result

I have only shown classical equivalence. How to I prove this is quantum-mechanically true as well?

7. We are looking at the transformation $\tau \to -1/\tau$. We see taht

$$C_0 + ie^{-\Phi} \to \frac{-1}{C_0 + ie^{-\Phi}} = \frac{-C_0 + ie^{-\Phi}}{C_0^2 + e^{-2\Phi}}$$

So we see $C_0 \to -\frac{C_0}{C_0^2 + e^{-2\phi}}$ and $e^{-\Phi} \to \frac{e^{-\Phi}}{C_0^2 + e^{-2\Phi}}$. On the other hand, C_0 will not affect the C_2, B_2 transformations. Nor will it affect C_4, which remains invariant

In the Einstein frame the metric is invariant. That means that $e^{\Phi/2} g_{string}$ is invariant, which means g_{string} transforms as $e^{-\Phi/2}$ times the Einstein frame metric. Consequently, in the string frame $g'_{string} = e^{-\Phi} g_{string}$ (I think Kiritsis is wrong here, and Polchinski agrees with this)

Am I missing anything with that last one?

8. There's effectively nothing to derive. Translating the the Einstein frame means multiplying all lengths by $e^{-\Phi/4}$. At fixed dilaton this is $g_s^{-1/4}$. Given ℓ_s^2 in the denominator will then contribute a factor $\sqrt{g_s}$ overall, that's exactly what was done here.

9. We have that C_4 is invariant. That means that objects charged under C_4 remain charged under C_4, with the same charge. These are precisely the D3/anti-D3 branes. Now recall the DBI action has coupling constant

$$g_{YM}^2 = \frac{1}{(2\pi\ell_s^2)^2 T_3} = 2\pi g_s$$

note that this is dimensionless, as it should be for a gauge theory in 4D. At low energies, the closed strings decouple we can reliably trust the DBI action, considering the D-brane gauge theory on its own. In the absence of axion, the SL$(2,\mathbb{Z})$ of IIB takes $g_s \to 1/g_s$. This corresponds to

$$g_{YM}^2 \to \frac{4\pi^2}{g_{YM}^2}$$

So this is the Weak-Strong Montonen-Olive duality of $\mathcal{N} = 4$ SYM.

The only subtlety is that one must take care to include the Chern-Simons terms in the DBI action in order to get the full duality, specifically

$$\int C_0 \mathrm{Tr}[F \wedge F].$$

At fixed $C_0 = \theta/2\pi$ this produces the instanton number. The duality $C_0 \to C_0 + 1$ is a bona-fide duality of the $\mathcal{N} = 4$ theory, a consequence of the fact that instanton charge is quantized.

Is there anything else that I can say that constitutes any form of "showing" that this fact is true?

The only thing is I think I'm assuming that the D3 brane is the only object charged under C_3 at leading order in ℓ_s. Can I safely assume this?

10. We should go to the Einstein frame, ie multiply the D1 solution by $H^{-1/4}$. The D_1 solution is:

$$ds_E^2 = H^{-3/4} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/4} d\vec{x} \cdot d\vec{x}, \qquad H = 1 + \frac{L^6}{r^6}$$

Note this is the same metric as the F_1 solution, and indeed the metric will stay the same for all (p,q) strings. The C_0 field has been set to zero. For D_1 the dilaton is

$$e^\Phi = g_s H^{1/2}$$

and indeed it is the inverse of this for the F_1. Now note in $\mathrm{SL}_2(\mathbb{Z})$ we have $ad - bc$, implying d, c are relatively prime.

$$S = \frac{p - iqH^{1/2}}{iqH^{1/2} + p}$$

Finish

11. The 11-D SUGRA Lagrangian is

$$L_{D=11} = \frac{1}{2\kappa_{11}^2} \left[R - \frac{1}{2}|G_4|^2 + G_4 \wedge G_4 \wedge \hat{C}_3 \right]$$

Let's take M-theory to 9 dimensions. The R term becomes:

$$\frac{1}{2\kappa_{11}^2}e^{-2\phi}\Big[R+4\partial^\mu\phi\partial_\mu\phi+\frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta}-\frac{1}{4}G_{\alpha\beta}F^{A\ \alpha}_{\mu\nu}F^{A\,\mu\nu\beta}\Big]$$

with $\phi=-\frac{1}{4}\det G_{\alpha\beta}, F^A_{\mu\nu}=\partial_\mu A^\alpha_\nu-\partial_\nu A^\mu\alpha$. The kinetic 3-form potential yields

$$\frac{1}{2\kappa_{11}^2}e^{-2\phi}\Big[-\frac{1}{2}|F_4|^2-4\times\frac{1}{2}|F_3|^2-6\times\frac{1}{2}|F_2|^2\Big]$$

The IIB SUGRA Lagrangian is

$$e^{-2\Phi}\left[R+4(\nabla\Phi)^2-\frac{1}{2}|H_3|^2\right]-\frac{1}{2}|F_1|^2-\frac{1}{2}|F_3|^2-\frac{1}{4}|F_5|^2$$

supplemented by $\star F_5=F_5$. Taking this to 9 dimensions, the $R+4(\nabla\Phi)^2$ term becomes

$$\frac{1}{2\kappa_{10}^2}e^{-2(\Phi)+\sigma}\Big[R+4(\nabla\Phi)^2+(\partial_\mu\sigma)^2-\frac{1}{4}e^{-2\sigma}F^A_{\mu\nu}F^{A\,\mu\nu}-\frac{1}{2}|H_3|^2-\frac{1}{2}e^{-4\rho}|H_2|^2\Big]$$

with $G_{10,10}=e^{2\sigma}$. The RR forms give

$$e^\sigma\Big[-\frac{1}{2}(\partial_\mu C_0)^2-\frac{1}{2}|F_3|^2-\frac{1}{2}e^{-2\sigma}|F_2|^2-\underbrace{\frac{1}{4}|F_5|^2}_{\text{dualize}}-\frac{1}{4}e^{-2\sigma}|F_4|^2\Big]$$

Here F_2 comes from F_3 and we can dualize the 9D F_5 to an F_4.

12.

13. Again, we know the $(1,0)$ and $(0,1)$ 5-brane, namely the D5 and NS5.

14.