Chapter 11: Duality Connections and Nonperturbative Effects

1. Taking the expression for a toroidal heterotic compactification from exercise **9.1**

$$\left[\frac{R}{\sqrt{\tau_2}\eta\bar{\eta}^{17}}\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2} e^{-i\pi\sum_I nY^I(m+n\bar{\tau})Y^IY^I}\frac{1}{2}\sum_{a,b=0}^{1}\prod_{i=1}^{16}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}(Y^I(m+\bar{\tau}n)|\bar{\tau})\right]\times\frac{1}{\tau_2^{7/2}\eta^7\bar{\eta}^7}\frac{1}{2}\sum_{a,b=0}^{1}\frac{\theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4}$$

Using θ function identites as in the second equation in appendex **E**, we get

$$\Gamma_{1,17}(R,Y) = \frac{R}{\sqrt{\tau_2}}\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\frac{1}{2}\sum_{a,b=0}^{1} e^{i\pi mY^IY^In - i\pi bnY^I}\bar{\theta}\begin{bmatrix}a-2nY^I\\b-2mY^I\end{bmatrix}$$

Now take $Y^I = 0$ for $I = 1\ldots 8$ and $Y^I = 1/2$ for $I = 1\ldots 16$. Then

$$\prod_I e^{i\pi mY^IY^In - i\pi bnY^I} = e^{i\pi m\sum_I (Y^I)^2 - i\pi b\sum_I Y^I} = 1$$

and we can ignore this term. Similarly because we are taking a product over 16 $\bar{\theta}$, no phases will interfere with us replacing $\theta\begin{bmatrix}u\\v\end{bmatrix}$ with $\theta\begin{bmatrix}-u\\-v\end{bmatrix}$ for integer u, v. This gives us the desired first step

$$\Gamma_{1,17}(R,Y) = R\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\frac{1}{2}\sum_{a,b=0}^{1}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}^8\bar{\theta}\begin{bmatrix}a+n\\b+m\end{bmatrix}^8$$

Now again because we have enough $\theta\begin{bmatrix}a+n\\b+m\end{bmatrix}$ that phases do not interfere, we see that we only care about n, m modulo 2 in the fermion term. We know how to divide the partition function of the compact boson into parity odd and even blocks by doing the $\mathbb{Z}^2$ stratification corresponding to the πR translation orbifold of the circle. This gives our desired answer:

$$\frac{1}{2}\sum_{h,g}\Gamma_{1,1}(2R)\begin{bmatrix}h\\g\end{bmatrix}\frac{1}{2}\sum_{a,b}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}^8\bar{\theta}\begin{bmatrix}a+h\\b+g\end{bmatrix}^8$$

with

$$\Gamma_{1,1}(2R) = 2R\sum_{m,n}\exp\left[\frac{-\pi R^2}{\tau_2}|2m+g+(2n+h)\tau|^2\right]$$

2. As before, take the ansatz

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}dx^i\cdot dx^i, \qquad A_{012} = \pm e^{C(r)} \Rightarrow G_{r012} = \pm C'(r)\,e^{C(r)}$$

Let's look at G's equation of motion:

$$\mathrm{d}G = 0, \qquad \frac{1}{3!}\mathrm{d}\star G + \frac{3}{(144)^2}\epsilon^{MNOPQRST}G_{MNOP}G_{QRST} = 0$$

By assumption, the term quadratic in G vanishes. What remains gives us:

$$\partial_r(e^{3A+8B}e^{-3A-B}C'(r)e^C) = 0$$

The BPS states in 11D require only the gravitino variation to vanish:

$$\delta\psi_M = \partial_M\epsilon + \frac{1}{4}\omega_M^{PQ}\Gamma_{PQ}\epsilon + \frac{1}{2\cdot 3!\cdot 4!}G_{PQRS}\Gamma^{PQRS}\Gamma_M\epsilon - \frac{8}{2\cdot 3!\cdot 4!}G_{MQRS}\Gamma^{QRS}\epsilon$$

We have worked out ω in **8.43**.

$$\omega_{\hat{\mu}\hat{\nu}} = 0, \quad \omega_{\hat{\mu}\hat{i}} = (-)^{\mu=0}\partial_i A e^{A-B}dx^\mu, \quad \omega_{\hat{i}\hat{j}} = \partial_j B dx^i - \partial_i B dx^j$$

Let's look first at $M = \mu$ parallel. Since ϵ is killing we expect no longitudinal variation and we get

$$\begin{aligned}0 &= \cancel{\partial_\mu \epsilon} + \frac{1}{2}A'\, e^{A-B}\Gamma^{\hat{\mu}\hat{r}} + \frac{1}{2\cdot 3!}C'(r)e^{C}\Gamma^{r012}\Gamma_\mu - \frac{1}{3!}C'(r)e^{C}\Gamma_\mu\Gamma^{r012}\\ &= \frac{1}{2}A'\, e^{A-B}\Gamma^{\hat{\mu}\hat{r}} - \frac{1}{2\cdot 3!}C' e^{C-B-2A}\Gamma^{\hat{\mu}\hat{r}\hat{0}\hat{1}\hat{2}}\\ &= \frac{1}{2}A'\, e^{A-B} - \frac{1}{2\cdot 3!}C' e^{C-B-2A}\Gamma^{\hat{0}\hat{1}\hat{2}}\end{aligned}$$

For $M = i$ transverse, we recall Γ_{ij} generates rotations, so assuming rotational invariance in the transverse space, we'll cancel this. We get

$$\partial_i \epsilon + \cancel{\frac{1}{4}\omega_r^{jk}\Gamma_{jk}}\epsilon + \frac{1}{2\cdot 3!}G_{r012}\Gamma^{r012}\Gamma_r\epsilon - \frac{1}{3!}G_{r012}\Gamma^{012}\epsilon = 0$$

I'm happy with this. I could use Mathematica to show that the EOMs:

$$R_{MN} - \frac{1}{2}g_{MN}R = \kappa^2 T_{MN}, \quad \kappa^2 T_{MN} = \frac{1}{2\cdot 4!}\left(4G_{MPQR}G_N^{PQR} - \frac{1}{2}g_{MN}G^2\right)$$

$$\mathrm{d}G = 0, \qquad \frac{1}{3!}\mathrm{d}\star G + \frac{3}{(144)^2}\epsilon^{MNOPQRST}G_{MNOP}G_{QRST} = 0$$

are satisfied - but this is barely different from what I've done several times before for the D-branes and fundamental string solutions in chapter 8.

As before, this generalizes straightforwardly to multi-membrane configurations