Solutions to Kiritsis' *String Theory in a Nutshell*

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Chapter 2: Classical String Theory

1. I don't know what this question asks exactly given that 2.1.16 is an infinitesimal diffeomorphism.

   We are still allowed to assume WLOG that τ runs from 0 to 1. For ξ infinitesimal, we have $\delta e = \xi \dot{e} + \dot{\xi} e = \partial_\tau(\xi e)$. So for a general $e(\tau)$ define

   $$\tau_2(\tau) = \frac{\int_0^\tau d\tau' e(\tau')}{\int_0^1 d\tau' e(\tau')} \tag{1}$$

   Take $L = \int_0^1 d\tau' e(\tau')$. Then $e_2(\tau_2(\tau)) = \left(\frac{d\tau_2}{d\tau}\right)^{-1} e(\tau) = \left(\frac{e(\tau)}{L}\right)^{-1} e(\tau) = L$.

   Note that we cannot get rid of this L, since it is invariant $L = \int_0^1 d\tau e(\tau) = \int_0^1 d\tau_2 e(\tau_2)$

2. From analytic continuation, we have the functional equation for the Riemann zeta function:

   $$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s) \tag{2}$$

   It is worth knowing that near $s = 0$ we have $\zeta(1-s) = -\frac{1}{s} + \gamma$ and $\Gamma(1-s) = 1 + \gamma s$. Expanding the right hand side about $s = 0$ gives

   $$\zeta(\epsilon) = -\frac{1}{2} - \frac{1}{2}\sqrt{2\pi}\epsilon \tag{3}$$

   This gives $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\sqrt{2\pi}$. Further, $\zeta'(s) = -\sum_{n=1}^\infty \frac{\log n}{n^s}$.

   So we get $\prod_{n=1}^\infty \frac{1}{L^2} = L^{-2\sum_{n=1}^\infty 1} = L^{-2\zeta(0)} = L$ and $\prod_{n=1}^\infty n^2 = \exp\left(2\sum_{n=1}^\infty \log n\right) = 2\pi$.

3. For simplicity, we will work in the action with the einbein.

   $$\frac{1}{2}\int d\tau e(e^{-2} G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - m^2)$$

   The Euler-Lagrange equations for x^μ is:

   $$\begin{aligned}\frac{d}{d\tau}\frac{\partial}{\partial x^\mu}(e^{-1}G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) - \frac{\partial}{\partial x^\mu}\left(e^{-1}G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right) &= 2e^{-1}G_{\mu\nu}\ddot{x}^\nu + 2e^{-1}\partial_\gamma G_{\mu\nu}\dot{x}^\nu\dot{x}^\gamma - 2\frac{G_{\mu\nu}\dot{x}^\nu}{e^2}\dot{e} - e^{-1}\partial_\mu G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \\ &\to G_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}(\partial_\gamma G_{\mu\nu} + \partial_\nu G_{\mu\gamma} - \partial_\mu G_{\nu\gamma})\dot{x}^\nu\dot{x}^\gamma - \frac{1}{2}G_{\mu\nu}\dot{x}^\nu\partial_\tau \log e^2\end{aligned} \tag{4}$$

   This last term looks particularly annoying, and is ignored by other authors. We have total freedom in reparameterization of e, so we can WLOG set it equal to a (metric-dependent) constant by problem 1. Then the term drops out and we get exactly the geodesic equations.

   We could have done this explicitly as well:

   $$\begin{aligned}\frac{d}{d\tau}\frac{G_{\mu\nu}\dot{x}^\nu}{\sqrt{G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}} - \frac{\partial}{\partial x^\mu}\sqrt{G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} &= \frac{G_{\mu\nu}\ddot{x}^\nu + \partial_\lambda G_{\mu\nu}\dot{x}^\nu\dot{x}^\lambda - \frac{1}{2}\partial_\mu G_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda}{\sqrt{-G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}} + G_{\mu\nu}\dot{x}^\nu\frac{d}{d\tau}(-G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{-1/2} \\ &= G_{\mu\nu}(\ddot{x}^\nu + \Gamma^\nu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) - \frac{1}{2}G_{\mu\nu}\dot{x}^\nu\frac{d}{d\tau}\log(-G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)\end{aligned} \tag{5}$$

   And fix the parameterization so that $x^2 = \text{const}$ and the last term vanishes.

4. We get the same as before, but now cannot drop the last term. Now the dots represent time derivatives.

   $$G_{\mu\nu}(\ddot{x}^\nu + \Gamma^\nu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) - \frac{1}{2}G_{\mu\nu}\dot{x}^\nu\partial_{X^0}\log(-G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) \tag{6}$$

5. We get:
$$-mc\int d\tau\sqrt{-(G_{00}\dot{x}^0\dot{x}^0+2G_{0i}\dot{x}^0\dot{x}^i+G_{ij}\dot{x}^i\dot{x}^j)} \tag{7}$$
Taking $\tau = x^0 = ct$ gives us our result. Further, we write $G_{00} = -1 - \frac{2\phi}{c^2}$ where ϕ is the gravitational potential. To first order then we get:
$$-mc^2\int dt\sqrt{-(G_{00}+2c^{-1}G_{0i}\dot{x}^i+c^{-2}G_{ij}\dot{x}^i\dot{x}^j)} \approx \int dt\left(-mc^2-m\phi+mcG_{0i}v^i+mG_{ij}v^iv^j\right) \tag{8}$$
The last two terms in brackets are positive (kinetic) while the first two are negative (potential). **This explains why there is a - sign out front of the action**.

6. The Lagrangian for a special relativistic particle in an electromagnetic field is $-mc^2\sqrt{1-v^2/c^2}-e\phi+e\vec{v}\cdot\mathbf{A}$. This has the Lorentz invariant form: $-m\sqrt{-G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}+eA_\mu\dot{x}^\mu$. We get equations of motion as before: The additional term gives the equations of motion:
$$\frac{e}{m}(\dot{A}_\mu-\partial_\mu A_\nu\dot{x}^\nu)=\frac{e}{m}(\partial_\nu A_\mu-\partial_\mu A_\nu)\dot{x}^\nu=\frac{e}{m}F_{\mu\nu}\dot{x}^\nu \tag{9}$$
don't confuse e with the einbein.

If one coordinate is cyclic (neither the metric nor the vector potential depend on it), the corresponding momentum is
$$\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}=m\frac{G_{\mu\nu}\dot{x}^\nu}{\sqrt{-G_{\mu}\nu\dot{x}^\mu\dot{x}^\nu}}+eA_\mu \tag{10}$$

7. Ignoring the cosmological constant term (which is not reparameterization invariant), we note that any term that involves the metric $G_{\mu\nu}$ will require at least 2 x^μ variables for it to be contracted with. Also, reparameterization invariance requires that under $d\tau \to f'(\tau)d\tau$ we get $\mathcal{L}\to\mathcal{L}/\lambda$. The simplest such term is $\sqrt{-G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$. Terms with more than 2 x^μs will be suppressed by powers of $1/\ell_s^2$. Similarly, terms with more derivatives w.r.t. worldsheet coordinates will be less relevant in the IR.

8. Let's set $G_{i0}=0$ for simplicity. The Nambu-Goto action is:
$$-T\int d\tau d\sigma\sqrt{(\dot{X}\cdot X')^2-(\dot{X}^2)(X'^2)}$$
Take $\tau = ct$, $\sigma = x$ and note $T=\rho c^2$ with ρ the mass per unit length. Take $X^0=ct$ and $\vec{u}=X'^i, \vec{v}=\dot{X}^i$. Appreciate that v gives us how that point of the string is moving, while u gives the direction parallel to the string at that point (scaled according to σ's parameterization). Inside the radical:
$$\begin{aligned}&(G_{00}\dot{X}^0X'^0+G_{ij}\dot{X}^iX'^j)^2-(G_{00}\dot{X}^0\dot{X}^0+G_{ij}\dot{X}^i\dot{X}^j)(G_{00}X'^0X'^0+G_{ij}X'^iX'^j)\\&=c^{-2}(G_{ij}u^iv^j)^2-(G_{00}+c^{-2}G_{ij}v^iv^j)(G_{ij}u^iu^j)\end{aligned}$$
Take $G_{00}=-1-2\varphi/c^2$. Then the radical becomes:
$$\sqrt{u^2-c^{-2}2\phi u^2+c^{-2}(\vec{u}\cdot\vec{v})^2-c^{-2}v^2u^2}=|u|\sqrt{1-c^{-2}(2\phi+v^2-\frac{(\vec{u}\cdot\vec{v})^2}{u^2})}=|u|\left(1-c^{-2}\left(-\phi+\frac{1}{2}v^2-\frac{1}{2}\frac{(\vec{u}\cdot\vec{v})^2}{u^2}\right)\right)$$
But note that $v^2-\frac{(\vec{u}\cdot\vec{v})^2}{u^2}=(\vec{v}-\frac{u\cdot v}{u^2}\vec{u})^2$. This is exactly the part of v transverse to u (the string itself). So we can write this as $\vec{v}_T$, the transverse velocity.
$$-T\int dtd\sigma|u|(1+c^{-2}\phi-c^{-2}\frac{1}{2}v_T^2)=\int dtd\sigma|u|(-c^2-\phi+\frac{1}{2}v_T^2) \tag{11}$$
Note that $\rho\int d\sigma|u|=\rho L_s=M_s$. The first term is thus $-M_sc^2$. The second terms is exactly the mass density of the string interacting with the gravitational field, while the third (kinetic) is the motion of the transverse components of the string. Note that the longitudinal excitations do not contribute.

9. Let's work in lightcone gauge. We have $\partial_+\partial_- X = 0$ The vanishing of the stress-energy tensor gives us $\dot{X}^2 + X'^2 = 0$ and $\dot{X} \cdot X'^2 = 0$. But at the endpoints we get $X' = 0$ so that $\dot{X}^2 = 0$ and the endpoints with Neumann boundary conditions need to move at the speed of light.

10. The cosmological constant term gives the equation of motion $\frac{\delta S}{\delta g^{ab}} = -(\frac{T_{ab}}{4\pi} + \frac{\lambda_1}{2} g_{ab})\sqrt{-g}$. But by reparameterization invariance we *need* $T_{ab} = 0$ so that λ_1 must be 0.

11. It is quick to derive the current P^μ under $\delta X^\nu = \epsilon \delta^{\mu\nu}$:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} = -T\sqrt{-g} g^{\alpha\beta}\partial_\beta X_\mu \tag{12}$$

Similarly under $\delta X^\lambda = \epsilon M^{\lambda\delta}_{\mu\nu} X_\delta$ with $M^{\lambda\delta}_{\mu\nu} = (\delta^\lambda_\mu \delta^\delta_\nu - \delta^\delta_\mu \delta^\lambda_\nu)$. Then we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\lambda)}(\delta^\lambda_\mu \delta^\delta_\nu - \delta^\delta_\mu \delta^\lambda_\nu) X_\delta = -T\sqrt{-g} g^{\alpha\beta}(X_\mu \partial_\beta X_\nu - X_\nu \partial_\beta X_\mu) \tag{13}$$

12. Write:

$$\begin{aligned} X^\mu(\tau,\sigma) &= x^\mu + \ell_s^2 p^\mu \tau + \frac{i\ell_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}-\{0\}} \frac{1}{n}(\alpha_n e^{-in\sigma} + \bar{\alpha}_n e^{in\sigma})e^{-in\tau} \\ \dot{X}^\mu(\tau,\sigma) &= \ell_s^2 p^\mu + \frac{\ell_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}-\{0\}} (\alpha_n e^{-in\sigma} + \bar{\alpha}_n e^{in\sigma})e^{-in\tau} \end{aligned} \tag{14}$$

Now take the Fourier series (in σ) of the commutation relation:

$$\{X^\mu_n, \dot{X}^\mu_m\} = \frac{\delta_{n+m}}{2\pi}\frac{1}{T}\eta^{\mu\nu} \tag{15}$$

The only nonzero terms are those we get when we pair each mode with its negative (in σ). Also note that there is no τ dependence on the right-hand side, so we need to pair each τ mode with its negative. Let's look at x^μ, the zero mode of X. We can only pair this with the other mode p^μ and we necessarily have:

$$\{x^\mu, p^\nu\} = \frac{1}{2\pi\ell_s^2 T}\eta^{\mu\nu} = \eta^{\mu\nu} \tag{16}$$

Similarly, we can only pair α_n with α_{-n} giving:

$$\{\alpha^\mu_m, \alpha^\nu_n\} + \{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_m\} = \frac{2m\delta_{m+n}}{2\pi i \ell_s^2 T}\eta^{\mu\nu} \tag{17}$$

By parity symmetry, both of these brackets should be the same. We get then that:

$$\{\alpha^\mu_m, \alpha^\nu_n\} = \{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_m\} = -i\delta_{m+n}\eta^{\mu\nu} \tag{18}$$

13. For each coordinate on the n-torus, we have $X^i(\tau, \sigma + 2\pi) = X(\tau,\sigma) + 2\pi n_i R_i$. Then the corresponding momenta have difference $p - \bar{p} = \frac{2}{\ell_s^2} n_i R_i$ while the total momentum is quantized in multiples of $p + \bar{p} = \frac{2m_i}{R_i}$. Therefore we have:

$$\alpha^i_0 = \frac{1}{\sqrt{2}}\left(m_i \frac{\ell_s}{R_i} + n_i \frac{R_i}{\ell_s}\right) \tag{19}$$

14. We begin with a redefined $p^\mu \to 2p^\mu$ as in the book.

$$\begin{aligned} X'^\mu(\tau,\sigma)|_{\sigma=0} &= \ell_s^2(p^\mu - \bar{p}^\mu) + \frac{\ell_s}{\sqrt{2}}\sum_n (\alpha_n - \bar{\alpha}_n)e^{-in\tau} \\ \dot{X}^\mu(\tau,\sigma)|_{\sigma=0} &= \ell_s^2(p^\mu + \bar{p}^\mu) + \frac{\ell_s}{\sqrt{2}}\sum_n (\alpha_n + \bar{\alpha}_n)e^{-in\tau} \end{aligned} \tag{20}$$

So then

$$X' + \lambda\dot{X} = \ell_s^2\left((\lambda+1)p^\mu + (\lambda-1)\bar{p}^\mu\right) + \frac{\ell_s}{\sqrt{2}}\sum_n e^{-in\tau}\left((\lambda+1)\alpha_n + (\lambda-1)\bar{\alpha}_n\right) = 0$$

This gives $p^\mu = \frac{1-\lambda}{1+\lambda}\bar{p}^\mu$ and similarly $\alpha^\mu = \frac{1-\lambda}{1+\lambda}\bar{\alpha}^\mu_n$.

Further:

$$\begin{aligned} X'^\mu(\tau,\sigma)|_{\sigma=\pi} &= \ell_s^2(p^\mu - \bar{p}^\mu) + \frac{\ell_s}{\sqrt{2}}\sum_n(\alpha_n^\mu e^{-i\pi n} - \bar{\alpha}_n^\mu e^{i\pi n})e^{-in\tau} \to \sum_n \alpha_n^\mu(e^{-i\pi n} - \frac{1+\lambda}{1-\lambda}e^{i\pi n})e^{-in\tau} \\ \dot{X}^\mu(\tau,\sigma)|_{\sigma=\pi} &= \ell_s^2(p^\mu + \bar{p}^\mu) + \frac{\ell_s}{\sqrt{2}}\sum_n(\alpha_n^\mu e^{-i\pi n} + \bar{\alpha}_n^\mu e^{i\pi n})e^{-in\tau} \to \sum_n \alpha_n^\mu(e^{-i\pi n} + \frac{1+\lambda}{1-\lambda}e^{i\pi n})e^{-in\tau} \end{aligned} \tag{21}$$

This gives:

$$(1+\lambda)e^{-i\pi n} - (1-\lambda)\frac{1+\lambda}{1-\lambda}e^{i\pi n} = 0 \Rightarrow \sin(\pi n) = 0 \Rightarrow n \in \mathbb{Z}. \tag{22}$$

The full equation is then

$$X^\mu = x^\mu + \frac{2\ell_s^2 p^\mu}{1-\lambda} + \frac{i\sqrt{2}\ell_s}{(1-\lambda)}\sum_{n\in\mathbb{Z}-\{0\}}\frac{\alpha_n^\mu}{n}e^{-in\tau}(\cos(\sigma n) + i\lambda\sin(\sigma n)) \tag{23}$$

Clearly as $\lambda \to 0$ we recover Neumann boundary conditions. On the other hand as $\lambda \to \infty$ we see that the endpoint of the string is constrained to be unable to move and we indeed recover Dirichlet.

15. Looking at the DD solution:

$$X'^\mu(\tau,\sigma) = w^\mu + \sqrt{2}\ell_s\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in\tau}\cos(n\sigma) \tag{24}$$

At the endpoints the momentum flow is

$$w^\mu \pm \sqrt{2}\ell_s\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in\tau} \tag{25}$$

16. In conformal gauge we have $\mathcal{L} = 2T\,\partial_+X^\mu\partial_-X_\mu = \frac{T}{2}(\partial_\tau + \partial_\sigma)X^\mu(\partial_\tau - \partial_\sigma)X^\mu = \frac{T}{2}((\dot{X})^2 - (X')^2)$ so that $\Pi = T(\dot{X})$ and $\int d\sigma\,\Pi\dot{X} - \mathcal{L} = \frac{T}{2}\int d\sigma\,((\dot{X})^2 + (X')^2)\cdot\dot{X})$ as we needed.

For the closed string:

$$\dot{X} = \frac{\ell_s^2(p_\mu + \bar{p}_\mu)}{2} + \frac{\ell_s}{\sqrt{2}}\sum_{n\neq 0}(\alpha_n e^{-in\sigma} + \bar{\alpha}_n e^{in\sigma})e^{-in\tau} \qquad X' = \frac{\ell_s^2(p_\mu - \bar{p}_\mu)}{2} + \frac{\ell_s}{\sqrt{2}}\sum_{n\neq 0}(\alpha_n e^{-in\sigma} - \bar{\alpha}_n e^{in\sigma})e^{-in\tau}$$

Assuming no winding, we have $p = \bar{p}$. In the hamiltonian, the only contributions that will not vanish is when each $e^{in\sigma}$ is paired with $e^{-in\sigma}$. So we can look at this mode-by-mode. Between the two of these, the cross terms involving $\alpha_n\bar{\alpha}_n e^{-2in\tau}$ will cancel. We will get:

$$\frac{T}{2}\times\frac{\ell_s^2}{2}\times 2\pi\times\sum_{n\neq 0}(\alpha_n\alpha_{-n} + \bar{\alpha}_n\bar{\alpha}_{-n})\times 2 = \frac{1}{2}\sum_{n\neq 0}(\alpha_{-n}\alpha_n + \bar{\alpha}_{-n}\bar{\alpha}_n) = \sum_{n=1}^{\infty}(\alpha_{-n}\alpha_n + \bar{\alpha}_{-n}\bar{\alpha}_n)$$

The zero mode will contribute $\ell_s^4 p^2 \times 2\pi \times T/2 = \frac{1}{2}\ell_s^2 p^2$ as required.

For NN we again have $p = \bar{p}$

$$\dot{X} = 2\ell_s^2 p^\mu + \sqrt{2}\ell_s\sum_{n\neq 0}\alpha_n\cos(n\sigma)e^{-in\tau} \qquad X' = -i\sqrt{2}\ell_s\sum_{n\neq 0}\alpha_n\sin(n\sigma)e^{-in\tau}$$

The zero mode gives $4\ell_s^4 p^2 \times \pi$ After squaring this, we can only pair $\cos(n\sigma)$ either with itself or $\cos(-n\sigma)$. Pairing it with itself will give $\alpha_n^2\cos^2(n\sigma)e^{-in\tau}$ which will be cancelled by the $-\alpha_n^2\sin^2(n\sigma)e^{-in\tau}$ obtained from multiplying $\sin(n\sigma)$ with itself. On the other hand, pairing $\cos(n\sigma)$ and $\sin(n\sigma)$ with their negative frequency counterparts and integrating gives two factors of $\pi\alpha_n\alpha_{-n}$ so that in total we get:

$$\ell_s^2 p^2 + \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}\alpha_n = \ell_s^2 p^2 + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n \tag{26}$$

The exact same logic applies for DD except now only the difference term contributes. Instead of $2\ell_s^2 p^\mu$ we have $w^\mu = (y-x)^\mu/\pi$ which must thus give zero mode $(x-y)^2/(2\pi\ell_s)^2$.

Lastly, for DN we have no zero-modes at all, only:

$$X^\mu(\sigma,\tau) = x^\mu - \sqrt{2}\ell_s \sum_{k\in\mathbb{Z}+\frac{1}{2}} \frac{\alpha_k^\mu}{k} e^{-ik\tau}\sin(k\sigma) \tag{27}$$

$$\Rightarrow \dot{X}^\mu = i\sqrt{2}\ell_s \sum_k \alpha_k^\mu e^{-ik\tau}\sin(k\sigma), \quad X'^\mu = -\sqrt{2}\ell_s\sum_k \alpha_k^\mu e^{-ik\tau}\cos(k\sigma)$$

By the same reason as in DD and DN, the only terms that don't cancel is when we pair each $\sin(k\sigma)$ with its negative and similarly for cos. We get

$$\frac{T}{2}\ell_s^2 \times \pi \times 2 \times \sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha_{-n}\alpha_n = \frac{1}{2}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha_{-n}\alpha_n = \sum_{n=1}^{\infty}\alpha_{-n+\frac{1}{2}}\alpha_{n-\frac{1}{2}} \tag{28}$$

17. Immediately we have $\{L_m, \bar{L}_n\} = 0$. For $\{L_m, L_n\}$ we have:

$$\begin{aligned}
\{L_m, L_n\} &= \frac{1}{4}\sum_{k,l}\{\alpha_{m-k}\alpha_k, \alpha_{n-l}\alpha_l\} \\
&= \frac{1}{4}\sum_{k,l}\alpha_{n-l}\alpha_{m-k}\{\alpha_k,\alpha_l\} + \alpha_{m-k}\{\alpha_k,\alpha_{n-l}\}\alpha_l + \alpha_{n-l}\{\alpha_{m-k},\alpha_l\}\alpha_k + \{\alpha_{m-k},\alpha_{n-l}\}\alpha_k\alpha_l \\
&= -\frac{i}{4}\sum_{k,l}\alpha_{n-l}\alpha_{m-k}\,k\delta_{k+l} + \alpha_{m-k}\alpha_l\,k\delta_{k+n-l} + \alpha_{n-l}\alpha_k\,(m-k)\delta_{m-k+l} + \alpha_k\alpha_l(m-k)\delta_{m-k+n-l} \\
&= -\frac{i}{4}\sum_k \alpha_{n+k}\alpha_{m-k}k + \alpha_{m-k}\alpha_{n+k}k + \underbrace{\alpha_{n+m-k}\alpha_k\,(m-k) + \alpha_k\alpha_{n+m-k}(m-k)}_{k\to k+n} \\
&= -\frac{i}{2}\sum_k \alpha_{m-k}\alpha_{n+k}k + \alpha_{m-k}\alpha_{n+k}(m-n-k) \\
&= -i\frac{1}{2}\sum_k \alpha_{m-k}\alpha_{n+k}(m-n) \to -i(m-n)\frac{1}{2}\sum_{k'}\alpha_{m+n-k}\alpha_k = -i(m-n)L_{m+n}
\end{aligned}$$

The exact same logic applies to the conjugate charges.

Chapter 3: Quantization of Bosonic Strings

1. For simplicity we will ignore the μ index in our calculation first.

   First consider $[L_m, L_n]$ with $m+n \neq 0$. Then expanding in terms of commutators: This is the same as before, but now we must be careful about commutation:

$$[L_m, L_n] = \frac{1}{4}\sum_{k,l}[:\alpha_{m-k}\alpha_k :, :\alpha_{n-l}\alpha_l :]$$

   Note that the indices $m-k, k, n-l, l$ sum to $n+m$, if any pairwise sum of them is equal to zero (necessary for a nonvanishing commutator), then the other two will have sum equal to $n+m$. Then as long as $m+n \neq 0$ α_p, there will be no normal-ordering ambiguity and we will recover the standard commutation relations as before.

   So the remaining case to consider is $n = -m$. Take m positive WLOG. The logic of the question from last chapter applies, but now we must be careful about the ordering of the α_i outside of the commutator.

$$\begin{aligned}
[L_m, L_{-m}] &= \frac{1}{4}\sum_{k,l}[\alpha_{m-k}\alpha_k, \alpha_{-m-l}\alpha_l] \\
&= \frac{1}{4}\sum_k \alpha_{m-k}\alpha_l k\delta_{k-m-l} + \alpha_{-m-l}\alpha_k(m-k)\delta_{m-k+l} + \alpha_{-m-l}\alpha_{m-k}k\delta_{k+l} + \alpha_k\alpha_l(m-k)\delta_{k+l} \quad (29) \\
&= \frac{1}{4}\sum_k \alpha_{m-k}\alpha_{k-m}k + \alpha_{-k}\alpha_k(m-k) + \alpha_{-m+k}\alpha_{m-k}k + \alpha_k\alpha_{-k}(m-k)
\end{aligned}$$

   We can split this into $k \geqslant 1$ and $k \leqslant 1$. The $k=0$ term is already in normal order. When $k \geqslant 1$, the first, third, and fourth terms of the sum are out of normal order. The first term has only m terms out of normal order. Rearranging these gives the constant:

$$\frac{1}{4}\sum_{k=1}^{m} k(k-m) = \frac{1}{4}\frac{m(m^2-1)}{6}$$

   The fourth term has all terms out of normal order and gives the formally infinite sum

$$\sum_{k=1}^{\infty} k(m-k)$$

   The last term has all but the first m terms out of normal order, and so contributes the sum

$$\sum_{k=m+1}^{\infty}(-m+k)k = -\sum_{k=1}^{\infty}(m-k)k + \sum_{k=1}^{m}(k-m)k$$

   The first part of this exactly cancels with the third term's infinite contribution. The last part of this gives exactly the same contribution as the first term.

   Now, for $k \leqslant -1$ only the first two terms contribute. The first term contributes $\sum_k (m-k)k$ while the second term contributes $\sum_k(-k)(m-k)$ which cancel. Thus the term left behind is exactly:

$$2 \times \frac{1}{4}\frac{m(m^2-1)}{6} = \frac{m(m^2-1)}{12} \quad (30)$$

   Note however that in fact our oscillators carry with them a μ index which we have ignored. If we incorporate it, then each normal ordering of $\alpha_i^\mu \alpha_\nu^j$ will include a factor of $\eta^{\mu\nu}$ which would have to be summed over. This will add in a copy of D to our final result for the normal ordering term.

   Finally, we see that the normal ordering constant a must be equal to:

$$\frac{1}{2}\sum_k \alpha_{-k}^i\alpha_k^i \to \sum_{k=0}\alpha_{-k}^i\alpha_k^i + \frac{1}{2}\sum_{k>0}[\alpha_k^i, \alpha_{-k}^i] =: L_0 : + \frac{D-2}{2}\underbrace{\sum_k k}_{\zeta(-1)} = -\frac{D-2}{24} \quad (31)$$

2. I believe that the treatment of the prior derivation of the central term was sufficiently careful, as I did not need to use any zeta regularization to compute an infinite sum. I only used zeta regularization in calculating the normal-ordering constant

3. Given that the Witt algebra is already given as an associative algebra, the commutator directly satisfies the Jacobi identity, since $(a-(b-c))+(b-(c-a))+(c-(a-b)) = a+b+c = 0$. Adding a central term gives

$$[L_a,[L_b,L_c]]+[L_b,[L_c,L_a]]+[L_c,[L_a,L_b]] = \frac{1}{12}\delta_{a+b+c}(a(a^2-1)(b-c)+b(b^2-1)(c-a)-c(c^2-1)(a-b)) \tag{32}$$

This is zero by algebra.

4. For the closed string, we have:

$$\dot{X}^\mu(\tau,\sigma) = \ell_s^2 p^\mu + \frac{\ell_s}{\sqrt{2}}\sum_{n\neq 0}(\alpha_n^\mu e^{-in\sigma} + \bar{\alpha}_n^\mu e^{in\sigma})e^{-in\tau}$$
$$X'^\mu(\tau,\sigma) = \frac{\ell_s}{\sqrt{2}}\sum_{n\neq 0}(\alpha_n^\mu e^{-in\sigma} - \bar{\alpha}_n^\mu e^{in\sigma})e^{-in\tau}$$

Taking $X^+ = x^+ + \ell_s p^+\tau$ sets $\alpha_n^+, \bar{\alpha}_n^+ = 0$ for all $n \neq 0$.

$$\dot{X}^\mu + X'^\mu = \ell_s^2 p^\mu + \sqrt{2}\ell_s\sum_{n\neq 0}\alpha_n^\mu e^{-in\sigma}e^{-in\tau}$$
$$\dot{X}^\mu - X'^\mu = \ell_s^2 p^\mu + \sqrt{2}\ell_s\sum_{n\neq 0}\bar{\alpha}_n^\mu e^{in\sigma}e^{-in\tau}$$

Let's just look at the constraint $(\dot{X}+X')^2 = 0$ and then the other constraint will give the same result for the right-movers.

$$0 = \ell_s^4 p^2 + \sqrt{2}\ell_s^3\sum_{n\neq 0} p\cdot\alpha_n e^{-in(\sigma+\tau)} + 2\ell_s^2\sum_{n,m\neq 0}\alpha_n\cdot\alpha_m e^{-i(n+m)(\sigma+\tau)}$$

The zero mode gives $p^2 = 0$. Noting that $\alpha_n\cdot\alpha_m = -\alpha_n^+\alpha_m^- - \alpha_m^+\alpha_n^- + \alpha_n^i\alpha_m^i = \alpha_n^i\alpha_m^i$, we look at the remaining terms of each mode individually, so:

$$0 = \ell_s p\cdot\alpha_n + \sqrt{2}\sum_m \alpha_{m-n}^i\alpha_m^i = -\ell_s p^+\alpha^- + \underbrace{\cancel{\ell_s p^i\alpha^i}}_{\alpha^i \text{ is transverse}} + \sqrt{2}\sum_m \alpha_{m-n}\alpha_n$$
$$\Rightarrow \alpha^- = \frac{\sqrt{2}}{\ell_s p^+}\underbrace{\sum_m \alpha_{m-n}^i\alpha_m^i}_{2L_0} = \frac{\sqrt{2}}{\ell_s p^+}\left[:\sum_m \alpha_{m-n}^i\alpha_m^i: -2a\delta_n\right]$$

5. Firstly, we see that $L_0 - \bar{L}_0$ can only differ by an integer, otherwise there's no combination of $\alpha_{-n}\bar{\alpha}_{-m}$ acting on $|p^\mu\rangle$ that will give a physical state. Now lets say they differ by an integer n. Then $\alpha_{-n}^i\bar{\alpha}_{-1}^i$ will be the lowest-lying excitation at level $(n+1,1)$. We see there are 24 of these that transform under SO(24), so they must give us a massless particle. We note also that we have exactly 24 excitations at levels $(n+k,k)$ for $1 \leqslant k < n$, as the only way to get them is applying $\alpha_{-n-k}^i\bar{\alpha}_{-1-k}^i$. On the other hand, each of these has mass-shell condition:

$$0 = (L_0 - a)\alpha_{-n-k}\bar{\alpha}_{-k}|p^\mu\rangle \Rightarrow \ell_s^2 m^2 = 4(n+k-a)$$

However if this is massless for some value of k, it will be massive for $k+1$, breaking Lorentz invariance.

Note that $L - \bar{L}_0$ generates translations along σ so this shows that any state should be invariant under $\sigma \to \sigma + c$.

6. Note that SO(25) acts on 25×25 traceless symmetric tensors. Note that if we restrict to a subgroup SO(24) that leaves one of the spatial direction fixed, the SO(25) representation breaks down into two SO(24) representations: the symmetric tensor representation (including trace) on the 24 transverse directions, and the vector representation in those directions as well. This is exactly what we have at level two. So, we see we can arrange these two SO(24) rep'ns into the traceless symmetric $SO(25)$ tensor rep.

7. The generators (for the closed string) are:

$$J^{\mu\nu} = T\int_0^{2\pi} d\sigma(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}[\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n + \overline{(\dots)}]$$

Upon computing the commutator $[J^{\mu\nu}, J^{\rho\sigma}]$ the $x^\mu p^\nu - x^\nu p^\mu$ will give no problems, and there will be no cross terms between the right and left moving modes. So it is enough to look at the left movers. **I'm gonna pass on doing this computation...**

8. For NN boundary conditions, α_k^μ is associated to the wavefunction $\cos(k\sigma), \sigma \in [0, \pi]$. This has eigenvalue 1 under flip if k is even and -1 if k is odd. Thus this α_k must transform identically: $\Omega\alpha_k^\mu\Omega^{-1} = (-1)^k\alpha_k$. For DD boundary conditions, we have $\sin(k\sigma)$, which has opposite eigenvalues, so instead we get $(-1)^{k+1}$

9. This is a Lie algebra of dimension $n(n-1)/2$, which already looks promising. In the case of all θ_i equal, we can pick basis so that the R_{ij} are all 1. This is clearly $\mathfrak{so}(n)$. Now, take a diagonal unitary matrix γ (note $\gamma^T = \gamma$). It clear that $\tilde{\lambda}_{ij} := \gamma^{1/2}\lambda_{ij}\gamma^{-1/2}$ gives the right structure under transposition:

$$\tilde{\lambda}^T = \gamma^{-1/2}\lambda^T\gamma^{1/2} = -\gamma^{-1/2}\lambda\gamma^{1/2} = -\gamma\tilde{\lambda}\gamma$$

   But since $\tilde{\lambda}_{ij}$ is just a conjugation action on the λ_{ij}, we will still have that the Lie algebra structure is preserved, and maintain $\mathfrak{so}(n)$.

   For the second part, again when all the $\theta_i = 0$, this is just the definition of the symplectic group and we have $\lambda = -\omega\lambda^T\omega^{-1} = \omega\lambda^T\omega$ for ω the canonical symplectic written in the $(x_1, p_1, x_2, p_2, \dots)$ basis. Now note that the new symplectic form γ can be written as $\sigma^{1/2}\omega\sigma^{-1/2}$ with $\sigma = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_2}, \dots)$. Then define $\tilde{\lambda} = \sigma^{-1/2}\lambda\sigma^{1/2}$ and note that

$$\tilde{\lambda}^T = \sigma^{1/2}\lambda^T\sigma^{-1/2} = -\sigma^{1/2}\omega\lambda\omega\sigma^{-1/2} = -\gamma\tilde{\lambda}\gamma$$

   as required. Again, conjugation action will preserve the Lie algebra structure, so this will remain $\mathfrak{sp}(2n)$.

10. In the symmetric case, we have $\lambda^T = \lambda$, so these are symmetric matrices of N indices. Naturally $\mathrm{SO}(N)$ acts on these, and we see that they can be written as $\mathrm{F}\otimes\mathrm{F}$ for F the fundamental representation. This can be decomposed as the trivial representation and the traceless symmetric representation.

    In the anti-symmetric case with N even, I know that the symplectic group acts on $\mathbb{R}^N$. I'll call this the fundamental rep, and then note that tensoring it with its dual again gives an antisymmetric $N \times N$ matrix on which $\mathrm{Sp}(N)$ can act. This can be decomposed into the singlet and the skew-traceless antisymmetric matrix.

11. Traceless means that any pair of indices contracted with $\eta^{\mu\nu}$ gives zero. Locally, we can pick the metric so that only $\eta_{+-} = \eta_{-+} = 1$ is nonzero. This means that $T_{i_1\dots i_n} = 0$ if any one i is set to $+$ with the other set to $-$. Thus we can have only $T_{+\dots+}$ and $T_{-\dots-}$ nonzero.

12. The round metric is

$$ds^2 = \frac{4dzd\bar{z}}{(1+z\bar{z})^2}$$

    The Lie derivative is:

$$\mathcal{L}_X g_{ab} = X^c\partial_c g_{ab} + g_{ac}\partial_b X^c + g_{cb}\partial_a X^c \tag{33}$$

    Working with $z, \bar{z}$ we get:

$$\begin{aligned}\mathcal{L}_X g_{zz} &= 2g_{z\bar{z}}\partial_z X^{\bar{z}} = 0\\ \mathcal{L}_X g_{zz} &= 2g_{z\bar{z}}\partial_z X^{\bar{z}} = 0\\ \mathcal{L}_X g_{z\bar{z}} &= (X^z\partial_z + X^{\bar{z}}\partial_{\bar{z}})g_{z\bar{z}} = \lambda(z,\bar{z})g_{z\bar{z}}\end{aligned} \tag{34}$$

    The first two equation shows us that $X^z, X^{\bar{z}}$ must be holomorphic and anti-holomorphic respectively. We want the function λ to be well-defined on the entire Riemann sphere and so the last equation gives us:

$$-2\frac{(X^z\bar{z} + X^{\bar{z}}z)}{1+z\bar{z}} = \lambda(z,\bar{z}) \tag{35}$$

We see that $X^z, X^{\bar{z}}$ cannot have any poles. Further, they cannot grow faster than $z^2, \bar{z}^2$ respectively as $z \to \infty$ otherwise λ will blow up at the north pole. So our solutions space is spanned by $\partial_z, z\partial_z, z^2\partial_z$ and their conjugates.

Next, right away we can see that the only nonzero Christoffel symbols in the round metric are Γ^z_{zz} and $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$. Second, because T is traceless, by the previous problem we see it has only two components: T_{zz} and $T_{\bar{z}\bar{z}}$. Now looking at $\nabla^\beta T_{\alpha\beta}$ we see that this gives two equations:

$$\begin{aligned} g^{z\bar{z}}\nabla_{\bar{z}}T_{zz} &= \partial_{\bar{z}}T_{zz} \\ g^{z\bar{z}}\nabla_z T_{\bar{z}\bar{z}} &= \partial_z T_{\bar{z}\bar{z}} \end{aligned} \tag{36}$$

Note that there can be no Christoffel contribution. This simply asks for globally-defined holomorphic 2-forms. Let's look at T_{zz}. Around $z = 0$, it must be a polynomial to avoid poles. Transforming to $w = 1/z, dw = -dz/z^2 \Rightarrow \frac{dz}{dw} = -z^2 = -w^{-2}$ we get $T_{ww}(w) = (\frac{dz}{dw})^2 T_{zz}(w)$. Note that the right hand side will only have poles at least as bad as w^{-2} so we cannot have any global section of this vector bundle. Thus, there are no Teichmuller parameters.

13. We can think of the torus as $\mathbb{C}/\Lambda$. Note that scaling and rotation preserve the complex structure of the fundamental parallelogram so WLOG we can pick $\Lambda = \mathbb{Z}\text{-span}\{1, \tau\}$ with $\tau \in \mathbb{H}$. Thus we need vector fields on $\mathbb{C}$ that respect the translation-invariance under Λ. Any translation-invariant holomorphic function is zero, we can only have the constant vector fields $\partial_z, \partial_{\bar{z}}$.

    We now look for holomorphic and anti-holomorphic traceless tensors. Again, T_{zz} and $T_{\bar{z}\bar{z}}$ be translation-invariant w.r.t the lattice, so again they must be constants. We get $dz \otimes dz$ and $d\bar{z} \otimes d\bar{z}$ as our two Teichmuller deformations. As real tensors these are:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = dx \otimes dx - dy \otimes dy, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2dx \otimes dy,$$

14. Not sure exactly how they want us to calculate this. Let's assume they are OK with Gauss-Bonnet. For the disk with the flat metric, we have right away that the curvature R vanishes. The geodesic curvature at the boundary is a constant, and is easily seen to be 1. Integrating this over the boundary of the disk gives 2π so that $\chi = 1$.

    Using the round metric, it is quick to see that the only contribution to $R_{\mu\nu} = R^z_{\mu z\nu} + R^{\bar{z}}_{\mu\bar{z}\nu}$ is for $R^z_{zz\bar{z}} = -\partial_{\bar{z}}\Gamma^z_{zz}$ and $R^{\bar{z}}_{\bar{z}\bar{z}z} = \partial_z \Gamma^{\bar{z}}_{\bar{z}\bar{z}}$ giving $R_{z\bar{z}} = \frac{2}{(1+|z|^2)}$. Tracing this gives $R = 1$. Integrating this over half the sphere gives 2π. The geodesic curvature vanishes on the great circle by symmetry, and we get $\chi = 1$ again.

15. Any such surface can be decomposed as a sphere with $2n$ holes connected to n handles. Let's integrate the scalar curvature over each piece individually. First the curvature integrated on the sphere with $2n$ disks removed is equal to the curvature integrated on the Riemann sphere: 4π minus the curvature integrated on $2n$ disks. We have just done this in the previous problem, and we get $2\pi \times 2n$. Lastly, the curvature on the handles is just the same as the curvature on the sphere with two holes cut out, which we have just calculated is $4\pi - 2 \times 2\pi = 0$. Thus the total curvature is just $2\pi(2 - 2n)$, giving us $\chi = 2 - 2n$ as required.

16. Our point particle action is $S_0 = \int d\tau e(e^{-2}(\partial_\tau x)^2 - m^2)$. Let's look at:

$$\int \frac{\mathcal{D}X\mathcal{D}e}{V_{gauge}} e^{-S_0} \sim \int \mathcal{D}X\mathcal{D}e\mathcal{D}b\mathcal{D}c \; e^{-S_0 - \int b(\delta F)c - i\int BF(e)}$$

    Note we don't need an α index on B, b, c because they just parameterize the continuous symmetry with no discrete parameters:

$$(\delta_{\tau_1}\tau)(\tau) = \delta(\tau - \tau_1)$$

    Using Polchinski's convention for coordinate transformation (he also has $-B_A$ of Kiritsis),

$$\delta_{\tau_1}X(\tau) = -\delta(\tau - \tau_1)\partial_\tau X, \qquad \delta_{\tau_1}e(\tau) = -\partial_\tau(\delta(\tau - \tau_1)e(\tau))$$

and

$$[\delta_{\tau_1}\delta_{\tau_2}](\tau) = -(\delta(\tau-\tau_1)\partial_\tau\delta(\tau-\tau_2)-\delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_1))\,\partial_\tau \Rightarrow f^{\tau_3}_{\tau_1\tau_2} = \delta(\tau_3-\tau_1)\partial_{\tau_3}\delta(\tau_3-\tau_2)-\delta(\tau_3-\tau_2)\partial_{\tau_3}\delta(\tau_3-\tau_1)$$

Then the BRST transformation is given by:

$$\begin{aligned}
\delta_\epsilon X &= i\epsilon c\dot{X} \\
\delta_\epsilon e &= i\epsilon(\dot{cX}) \\
\delta_\epsilon b &= \epsilon B_A \\
\delta_\epsilon c &= i\epsilon c\dot{c} \\
\delta_\epsilon B &= 0
\end{aligned} \tag{37}$$

Now let's take $F(e) = e - 1$. Then we get a ghost action

$$\begin{aligned}
b_A c^\alpha \delta_\alpha F^\alpha &\to \int d\tau_1 d\tau_2 b(\tau_1)c(\tau_2)\delta_{\tau_2}\ (1 - e(\tau_2)) \\
&= \int d\tau_1 b(\tau_1)\partial_{\tau_1}\int d\tau_2 c(\tau_2)[\delta(\tau_1-\tau_2)e(\tau_1)] = -\int d\tau\dot{b}c
\end{aligned}$$

We enforce this constraint by integrating over B. Now we will have $\delta_\epsilon e = 0$ and $\delta_\epsilon b = i\epsilon(T^X + T^{gh})$. Because we are in Euclidean signature, we have $p = \partial_t X = i\dot{X}$ and similarly $p = -ic$ ($-i$ because the real time Lagrangian has term $-i\dot{b}c$). Then the BRST current (equal to charge because we're in 1D) is:

$$Q_B = p_X\delta_B X + p_b\delta_B b = -c\dot{X}^2 - ici(-\frac{1}{2}\dot{X} + \frac{1}{2}m^2 - \dot{b}c) = -c\frac{1}{2}\dot{X} + \frac{1}{2}m^2 = c\frac{1}{2}(p^2+m^2) = cH.$$

Clearly $Q_B^2 = 0$. As before, the ghosts generate a two-state system. Our set of states is given by $|k^\mu,\uparrow\rangle, |k^\mu,\downarrow\rangle$. Following convention, c raises and b lowers. $Q_B|k,\downarrow\rangle = \frac{1}{2}(k^2+m^2)|k,\uparrow\rangle$ so all states of the form $|k,\uparrow\rangle$ with $k^2+m^2 \neq 0$ are BRST exact. Similarly all states $|k,\uparrow\rangle$ are BRST closed along with all states of the form $|k,\downarrow\rangle$ with $k^2+m^2 = 0$. So the closed states that are not exact are $|k,\downarrow\rangle$, $|k,\uparrow\rangle$ with $k^2+m^2 = 0$. We take only the states with $b|\psi\rangle = 0$. The reason is that all states $|k,\uparrow\rangle$ are physical, and so we would need amplitudes between such states to be proportional to $\delta(k^2+m^2)$ in order for the states to decouple, but *amplitudes* cannot have such extreme singularities **Don't understand this. Appreciate it, and its relation to Siegel gauge.**.

17. I believe these variations have Kiritsis taking $c \to -ic$ in his formalism. They also follow directly from Polchinski's formalism. Under a diffeomorphism $\delta_{\xi,\bar{\xi}}X = -\xi\partial X - \bar{\xi}\bar{\partial}X$. These are two copies of the reparameterization algebra developed in the previous problem, and so the commutation relations are the same. We get, again in Polchinski's formalism (37)

$$\begin{aligned}
\delta_\epsilon X &= i\epsilon(c\partial X + \bar{c}\bar{d}X) \\
\delta_\epsilon c &= i\epsilon(c\partial c + \bar{c}\bar{\partial}\bar{c}) \\
\delta_\epsilon b &= i\epsilon(T^X + T^{gh})
\end{aligned} \tag{38}$$

I see no problem with Q_B^2 giving zero when acting on the X and c fields.

$$\delta_B(c\partial X + \bar{c}\bar{d}X) = i\epsilon(\cancel{c(\partial c)\partial X} + \cancel{\bar{c}(\bar{\partial}\bar{c})\bar{\partial}X} - c\partial(\cancel{c\partial X} + \bar{c}\bar{\partial}\bar{c}X) - \bar{c}\partial(\cancel{c\partial X} + \bar{c}\bar{\partial}\bar{c}X))$$

and the remaining terms die by the equations of motion. The c variation will always die because we've already shown the transformations satisfy a Lie algebra with Bianchi identity.

It looks like the b field will be nontrivial. If one of the equations of motion is the $T^X + T^{gh} = 0$ then this will be zero right away. Otherwise, we want to compute (WLOG in the holomorphic sector):

$$\delta_B(T^X + T^{gh}) = \delta_B(\frac{1}{\alpha}(\partial X)^2 + 2b\partial c + \partial bc) = i\epsilon[\frac{2}{\alpha}\partial X\partial(c\partial X) + 2(T^X+T^{gh})\partial c + 2b\,\partial(c\partial c) + \partial(T^X+T^{gh})\partial c + \partial b\, c\partial c]$$

The purely bc terms cancel. I'm left with

$$\frac{2}{\alpha}[(\partial X)^2 \partial c + \partial X \partial^2 X c]$$

I don't know how to get rid of this. I can write it as a total derivative less something proportional to $(\partial X)^2 \partial c$ and perhaps note that this is just ∂c times the stress tensor, which perhaps vanishes classically? At any rate, there is no need to use $d = 26$ here.

18. Integrating over σ will as usual pick out the zero mode. For T_{++} this gives us

$$\sum_n (2nb_{-n+m}c_n + n + mc_{-n}b_{n+m}) = \sum_n (m-n) : b_{n+m}c_{-n} :$$

and similarly for the right-movers. To get the central charge we'll have to proceed as before, noting that only $[L_m, L_{-m}]$ can give a nonzero central term. As before, we expect only a finite part of the infinite sum to contribute to this. We thus take out the only terms of the sum with m, n having the same sign:

$$\sum_{n=1}^{m} (m+n) b_{m-n} c_n, \qquad \sum_{n=1}^{m} (-2m+n) b_{-n} c_{-m+n}$$

Then our commutators of these finite terms give:

$$\begin{aligned}\sum_{k,k'} (m+k)(-2m+k')[b_{m-k}c_k, b_{-k'}c_{-m+k'}] &= \sum_{k,k'} (m+k)(-2m+k')(b_{m-k}c_{m+k'} - b_{-k'}c_k)\delta_{k-k'} \\ &= \sum_{k=1}^{m} (m+k)(-2m+k)(b_{m-k}c_{m+k} - b_{-k}c_k)\end{aligned}$$

Looking at the non-normal-ordered part, this leaves:

$$\sum_{k=1}^{m} b_{m-k}c_{-m+k}(m+k)(k-2m) \to \sum_{k=1}^{m} (m+k)(k-2m) = \frac{1}{6}(m - 13m^3)$$

19. We have

$$j_B = \frac{\partial \mathcal{L}}{\partial(\bar{\partial} X)} c \partial X + \frac{\partial \mathcal{L}}{\partial(\bar{\partial} c)} c \partial c = \frac{2}{2\pi\alpha'}(\partial X)^2 c + \frac{1}{\pi} bc\partial c \to cT^X + bc\partial c = cT^X + \frac{1}{2}cT^{gh}$$

I don't understand how other references include a $\frac{3}{2}\partial^2 c$.

20. Let's do this for the open string, so we are then just calculating the holomorphic sector. We have:

$$Q_B = \sum_{m=-\infty}^{\infty} : (L^X_{-m} + \frac{1}{2} L^{gh}_{-m} - a\delta_{m,0}) c_m :$$

Note that the a is just from the X component of the theory since by definition Q contains the term $: cT^{gh} :$ already in normal order.

We now need to consider the total BRST charge $Q + \bar{Q}$. Then:

$$Q_B^2 = \sum_{n,m} ([L^X_m, L^X_n] + [L^{gh}_m, L^{gh}_n] + (m-n)L^X_{m+n} + (m-n)L^{gh}_{m+n} + 2am\delta_{m+n}) c_{-m} c_m$$

This will vanish only if the commutators give no anomalous term. From previous exercises we see this is only if:

$$\frac{d(m^3 - m)}{12} + \frac{(m - 13m^3)}{6} + 2am = 0$$

This happens exactly when $d = 26$ and $a = 1$.

21. We now have $Q+\bar{Q}$ that we need to be zero on states. Again, each of $Q,\bar{Q}$ will no change the level, so their sum will not either, and we have a (double) grading on the space which they will preserve.

$$Q = Q_0 + Q_1, \qquad Q_0 = c_0(L_0^X - 1), \quad Q_1 = c_{-1}L_1^X + c_1 L_{-1}^X + c_0(b_{-1}c_1 + c_{-1}b_1)$$

and $\bar{Q}$ is the conjugate of this.

At level zero we will again have $Q^0 = c_0 L_0^X + \bar{c}_0 \bar{L}_0^X$. We now have two copies of the Clifford algebra and our Siegel gauge condition will make it so that we only consider states $|\downarrow,\bar{\downarrow},p,\bar{p}\rangle$. Now we need $((L_0-1)c_0 + (\bar{L}_0-1)\bar{c}_0)|\downarrow,\bar{\downarrow},p,\bar{p}\rangle = (L_0-1)|\uparrow,\bar{\downarrow},p,\bar{p}\rangle + (\bar{L}_0-1)|\downarrow,\bar{\uparrow},p,\bar{p}\rangle$ so we need $\frac{\ell_s^2 p^2}{4} = \frac{\ell_s^2\bar{p}^2}{4} = 1$ i.e. the total mass is $m^2 = -4$. We thus have the tachyon state.

Also because $b_0|\psi\rangle = 0$ for any physical state, we will have $\{Q,b_0\}|\psi\rangle = 0 = (L_0 - a)|\psi\rangle$ so we have $L_0 - 1 = \bar{L}_0 - 1 = 0$ and this gives us the level-matching condition. So the next level we can have a state is at $(1,1)$.

As in the open string treatment, the most general such state has nine terms:

$$\begin{aligned} |\psi_1\rangle = (&\zeta\cdot\alpha_{-1}\bar{\zeta}\cdot\bar{\alpha}_{-1} + \zeta_{ab}\cdot\alpha_{-1}\bar{b}_{-1} + \zeta_{ac}\cdot\alpha_{-1}\bar{c}_{-1} \\ &+ b_{-1}\zeta_{ba}\cdot\bar{\alpha}_{-1} + \xi_{bb}b_{-1}\bar{b}_{-1} + \xi_{bc}b_{-1}\bar{c}_{-1} \\ &+ c_{-1}\zeta_{ca}\cdot\bar{\alpha}_{-1} + \xi_{cb}c_{-1}\bar{b}_{-1} + \xi_{cc}c_{-1}\bar{c}_{-1})\,|\downarrow,\bar{\downarrow},p\rangle \end{aligned}$$

Let's act on this with $Q_0 + Q_1$. First lets look at Q_0. On the $\alpha\bar{\alpha}$ term it will give eigenvalue $c_0\frac{\alpha_0^2}{2} + \bar{c}_0\frac{\bar{\alpha}_0^2}{2}$ while something like the $\zeta_{ab}\alpha\bar{b}$ term it will give eigenvalue $c_0\frac{\alpha_0^2}{2} + \bar{c}_0(\frac{\bar{\alpha}_0^2}{2} - 1)$. This will be compensated by the action of the $\bar{c}_0(\bar{b}_{-1}\bar{c}_1)$ (from Q_1) on $\zeta_{ab}\alpha_{-1}b_{-1}$. The exact same argument can be applied to any of those four terms - there will always be one the four bc terms of $Q_1 + \bar{Q}_1$ that will give us the extra factor of 1 from its commutation relation with that term in $|\psi_1\rangle$ (it commutes with everything else).

So we get a term $c_0\frac{\ell_s^2 p^2}{4}|\psi_1\rangle$ which then gives the $p^2 = 0$ constraint. The remaining term comes from the $c_{-1}L_1^X + c_1 L_{-1}^X + c.c.$ action. The $c_{-1}L_1 + c_1 L_{-1}$ will each annihilate everything except six terms, giving

$$\begin{aligned} &\zeta\cdot p\, c_{-1}\bar{\zeta}\cdot\bar{\alpha}_{-1} + \zeta_{ab}\cdot p\, c_{-1}\bar{b}_{-1} + \zeta_{ac}\cdot p\, c_{-1}\bar{c}_{-1} \\ &+ p\cdot\alpha_{-1}\zeta_{ba}\cdot\bar{\alpha}_{-1} + \xi_{bb}p\cdot\alpha_{-1}\bar{b}_{-1} + \xi_{bc}p\cdot\alpha_{-1}\bar{c}_{-1} + c.c. \end{aligned} \tag{39}$$

and the conjugate of this will contribute the conjugate terms. For this to all be zero we need each of the $\zeta_i\cdot p = 0$ as well as their conjugates. We also need $\xi_{bb} = \xi_{bc} = \xi_{cb} = 0$. We also see that $\zeta_{ba} = \zeta_{ab} = 0$.

On the other hand the general form of an exact state is also given by (39) for the ζ_i and ξ_i arbitrary. Thus all the terms involving c and/or $\bar{c}$ are exact and so upon quotienting we get $\zeta_{ac} = \zeta_{bc} = \xi_{cc} = 0$. Lastly we get the relation that we should identify $\zeta_i\bar{\zeta}_j = \zeta_i\bar{\zeta}_j + p_i\zeta'_j + \zeta'_i p$ ie we project out any tensor of the form $p\otimes\zeta'$ or $\zeta'\otimes p$. This is equivalent to identifying $\zeta \cong \zeta' + \xi p$ and identically for $\bar{\zeta}$.

So we have eliminated everything except for $\zeta,\bar{\zeta}$, each of which must be transverse to p and we identify ζ differing by a longitudinal p component. This is 24×24 parameters, as required.

22. If I have the Clifford algebra $C\ell(2)$, any vector v will have an orbit generated by $1, b_0, c_0, b_0c_0$, so there can be no irreducible representation of dimension greater than 4. Further, there is a vector v_0 annihilated by b. Consider $v_1 = cv_0$ and assume it is distinct. Now $bv_1 = bcv_0 = v_0 - cbv_0 = v_0$. So v_1 and v_0 span the irreducible representation meaning that any irrep in fact has dimension 2. Thus, any higher dimensional generalization would only be (probably direct or semidirect) extensions of this and the trivial irrep, and give us no new information.

Chapter 4: Conformal Field Theory

1. We'll do this directly. First observe:

$$\begin{aligned}
\frac{d}{dt}|_{t=0} e^{-itP_\mu} f(x) &= -\partial_\mu f \\
\frac{d}{dt}|_{t=0} e^{-\frac{it}{2}\omega^{\mu\nu} J_{\mu\nu}} f(x) &= -\omega^\mu_\nu x^\nu \partial_\mu \\
\frac{d}{dt}|_{t=0} e^{-itD} f(x) &= x \cdot \partial f(x) \text{ annoying that there is no -} \\
\frac{d}{dt}|_{t=0} e^{-itK_\mu} f(x) &= -(x^2 \partial_\mu - 2x_\mu (x \cdot \delta)) f(x)
\end{aligned} \tag{40}$$

The last one is exactly the first-order expansion of $\frac{x^\mu + x^2 a^\mu}{1 + 2a \cdot x + a^2 x^2}$. Note the dilatation and special conformal generators are the negative of Di Francesco's (SO ANNOYING OGM).

Now let's do the commutator

$$\begin{aligned}
[J_{\mu\nu}, P_\rho] &= -\partial_\rho (x_\mu \partial_\nu - \partial_\nu \partial_\mu) = -(\eta_{\mu\rho}\partial_\nu - \eta_{\nu\rho}\partial_\mu) = -i(\eta_{\mu\rho}\partial_\nu - \eta_{\nu\rho}\partial_\mu) \\
[P_\mu, K_\nu] &= -\partial_\mu (x^2 \partial_\nu - 2x_\nu x \cdot \partial) = -(2x_\mu \partial_\nu - 2\eta_{\mu\nu} x^\lambda \partial_\lambda - 2x_\nu \delta^\lambda_\mu \partial_\lambda) = 2iJ_{\mu\nu} - 2i\eta_{\mu\nu} D \\
[J_{\mu\nu}, J_{\rho\sigma}] &= -i(\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\nu\sigma} J_{\mu\rho}) \leftarrow \textit{Everyone has done this one like 20 times} \\
[J_{\mu\nu}, K_\rho] &= -i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu) \\
[D, K_\mu] &= x^\nu \cdot \partial_\nu [x^2 \partial_\mu - 2x_\mu (x^\lambda \partial_\lambda)] - [x^2 \partial_\mu - 2x_\mu x \cdot \partial] x^\lambda \partial_\lambda \\
&= \cancel{2} x^\nu x_\nu \partial_\mu - 2x^\nu \eta_{\mu\nu} (x \cdot \partial) - \cancel{2x_\mu (x \cdot \partial)} - \cancel{x^2 \partial_\mu} + \cancel{2x_\mu x \cdot \partial} = iK_\mu \\
[D, P_\mu] &= -\partial_\mu x^\lambda \partial_\lambda = -\partial_\mu = -iP_\mu \\
[J_{\mu\nu}, D] &= 0
\end{aligned}$$

The way we did the $[J, K]$ commutator is by noting it should look the same as $[J, P]$, since P is just translation about the point at ∞. The $[J, D]$ commutator follows because rotation is scale invariant.

2. We see immediately that the $J_{\mu\nu}$ can be mapped to the $M_{\mu\nu}$ corresponding to a $\mathrm{SO}(p, q)$ subgroup of $\mathrm{SO}(p+1, q+1)$. The full group has:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) \tag{41}$$

Note the commutation relations of J with P and K gives us:

$$[J_{\mu\nu}, \frac{1}{2}(K_\rho \pm P_\rho)] = -i\left(\eta_{\mu\rho} \frac{1}{2}(K \pm P)_\nu - \eta_{\nu\rho} \frac{1}{2}(K \pm P)_\mu\right)$$

Writing these as $M_{\rho,d+1}$ and $M_{\rho,d}$ respectively, we see that we get the second and fourth terms nonzero and we get exactly (41). Note at this stage I didn't need to do such linear combinations of K and P. That is important for appreciating that we want:

$$[M_{\mu d}, M_{\nu d+1}] = -i\eta_{\mu\nu} M_{dd+1} = -i\eta_{\mu\nu} M_{d,d+1} = i\eta_{\mu\nu} D$$

and we get exactly this:

$$\frac{1}{4}[(K - P)_\mu, (K + P)_\nu] = \frac{1}{4}([K_\mu, P_\nu] - [P_\mu, K_\nu]) = i\eta_{\mu\nu} D$$

We needed that combination so that $J_{\mu\nu}$ wouldn't appear. As required $[J_{\mu\nu}, D] = [M_{\mu\nu}, M_{d,d+1}] = 0$ for $\mu \in 0 \ldots d-1$. **I'm getting the wrong sign. Perhaps our friend's convention is off.**

3. This comes from noting that for $f = z + \epsilon(z)$

$$\left(\frac{\partial f}{\partial z}\right)^{\Delta}\left(\frac{\partial f}{\partial \bar{z}}\right)^{\bar{\Delta}} - 1 = (1+\partial\epsilon)^{\Delta}(1+\bar{\partial}\epsilon)^{\bar{\Delta}} - 1 = \Delta\partial\epsilon + \bar{\Delta}\bar{\partial}\epsilon$$
$$\Rightarrow \Phi(z)(1 - (\Delta\partial\epsilon + \bar{\Delta}\bar{\partial}\epsilon)) = \Phi'(f(z), \bar{f}(\bar{z})) = (1 + \epsilon\partial + \bar{\epsilon}\bar{\partial})\Phi'(z)$$
$$\Rightarrow (1 - (\Delta\partial\epsilon + \bar{\Delta}\bar{\partial}\epsilon + \epsilon\partial + \bar{\epsilon}\bar{\partial}))\Phi(z) = \Phi'(z)$$
$$\Rightarrow \Phi(z) - \Phi'(z) = (\Delta\partial\epsilon + \epsilon\partial + \bar{\Delta}\bar{\partial}\epsilon + \bar{\epsilon}\bar{\partial})\Phi(z)$$

How weird... think about this in terms of active/passive. Contrast with Di Francesco.

4. As in the 2-point greens function case, note that:

$$\delta_\epsilon G^N = 0 \Rightarrow \left(\sum_{i=1}^{N} \epsilon(z_i)\partial_{z_i} + \Delta_i \partial\epsilon(z_i) + c.c.\right) G^N = 0$$

We can WLOG look at just the holomorphic sector (set $\bar{\epsilon} = 0$) Now first set $\epsilon(z) = 1$. This directly gives $\sum_i \partial_i G^N = 0$, as we wanted. Next, take $\epsilon(z) = z$. This gives $\sum_i (z_i\partial_i + \Delta_i)G^N = 0$. Finally, take $\epsilon = z^2$ to get $\sum_i (z_i^2\partial_i + 2z_i\Delta_i)G^N = 0$ as desired. Note in all these cases, we are exactly performing the global SL(2) transformations, so these Ward identities will always hold.

5. The first Ward identity tells us that the function can only depend on z_{12}, z_{23}. Then the next two can be written as:

$$(x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + \Sigma\Delta_i)f(x_{12}, x_{23}) = ((x_1 - x_2)\partial_{12} + (x_2 - x_3)\partial_{23} + \Sigma\Delta_i)\, f = 0$$
$$(x_1^2\partial_1 + x_2^2\partial_2 + x_3^2\partial_3 + \Sigma 2x_i\Delta_i)f(x_{12}, x_{23}) = \left((x_1^2 - x_2^2)\partial_{12} + (x_2^2 - x_3^2)\partial_{23} + \Sigma 2x_i\Delta_i\right) f = 0$$

We can subtract out ∂_{23} to get the differential equation:

$$0 = \left(\frac{x_1 + x_2}{x_2 + x_3} - 1\right) x_{12}\partial_{12} + \sum_i \left(\frac{2x_i}{x_2 + x_3} - 1\right)\Delta_i \to (x_{12} + x_{23})x_{12}\partial_{12} + (x_{12} + x_{12} + x_{23})\Delta_1 + x_{23}(\Delta_2 - \Delta_3)$$
$$\Rightarrow 0 = \left(x_{12}^2\partial_{12} + x_{23}x_{12}\partial_{12} + 2x_{12}\Delta_{12} + x_{23}(\Delta_1 + \Delta_2 - \Delta_3)f\right)$$

Now write $f(x_{12}, x_{23}) = e^g(u, x_{23})$ with $u = \log x_1 2$. This substitution gives the ODE:

$$(e^u + x_{23})g'(u) + 2\Delta_1 e^u + x_{23}(\Delta_1 + \Delta_2 - \Delta_3) = 0 \Rightarrow g(u) = \int_{-\infty}^{\log x_{12}} du \frac{2\Delta_1 e^u - x_{23}(\Delta_1 + \Delta_2 - \Delta_3)}{e^u + x_{23}}$$

This integral can be done and gives:

$$\frac{C}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}(x_{12} + x_{23})^{\Delta_1+\Delta_3-\Delta_2}} = \frac{C}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2}}$$

We can do the same for ∂_{23} and get the general form:

$$\frac{\lambda_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}} \times c.c.$$

for $\lambda_{123}, \bar{\lambda}_{123}$ undetermined constants (call their product C_{123}).

6. Again specialize to the holomorphic part. We see G^N depends only on relative positions x_{12}, x_{13}, x_{14}. We can WLOG take $G^{(4)}$ to have the form:

$$G^{(4)}(z_1, z_2, z_3, z_4) = \frac{f(z_1, z_2, z_3, z_4)}{z_{12}^{\Delta_{12}} z_{13}^{\Delta_{13}} z_{14}^{\Delta_{14}} z_{23}^{\Delta_{23}} z_{24}^{\Delta_{24}} z_{34}^{\Delta_{34}}}$$

Here, because f is arbitrary, we have not made any assumptions. The Ward identities imply the following:

- f depends only on the relative positions z_{ij}
- $\sum_{i<j} \Delta_{ij} = \Delta$ with $\Delta = \sum_i \Delta_i$ and $\sum_i z_i \partial_i f = 0$
- $\Delta_{23}+\Delta_{24}+\Delta_{34} = 2\Delta_1, \quad \Delta_{13}+\Delta_{14}+\Delta_{34} = 2\Delta_2, \quad \Delta_{12}+\Delta_{14}+\Delta_{24} = 2\Delta_3, \quad \Delta_{12}+\Delta_{13}+\Delta_{23} = 2\Delta_4$ and $\sum_i z_i^2 \partial_i f = 0$

These give 4 constraints for the 6 Δ_{ij}, so the system is underdetermined. The most symmetric solution is given by:

$$\Delta_{ij} = \Delta_i + \Delta_j - \frac{1}{3}\Delta$$

It remains to find the general form of f.

- The first ward identity gives us that it can only depend on the z_i through z_{ij}.
- Further, it must transform trivially under dilatation, so we see that it can only depend on ratios of the z_{ij} with an equal number of each z_{ij} in the numerator and denominator.
- Under special conformal transformations, each such ratio will transform as $\frac{z_{ij}}{z_{kl}} \to \frac{z_{ij}}{z_{kl}}(z_i + z_j - z_k - z_l)$, and more generally
$$\prod_a \frac{z_{i_a j_a}}{z_{k_a l_a}} \to \prod_a \frac{z_{i_a j_a}}{z_{k_a l_a}} \times \sum_a (z_{i_a} + z_{j_a} - z_{k_a} - z_{l_a})$$
The third Ward identity shows that f must transform trivially under these, and so f can only depend on ratios where each z_i appears an equal number of times in the numerator and denominator.

In total: we need ratios of z_{ij} with an equal number of z_{ij} in the numerator and denominator, and each z_i appears the same number of times in the numerator and denominator. All such ratios can be obtained as rational functions of:

$$x := \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad y := \frac{z_{14}z_{24}}{z_{13}z_{24}}$$

But we see that $y = 1-x$ so in fact the most general such function is any function of x alone, as was required.

7. With conformal invariance (rescaling in particular), an infinite cylinder has no moduli, so you can set its radius to be whatever you like and get the same theory.

8. Let's perform the OPE within the correlator:

$$\langle \Phi_i(z_1)\Phi_j(z_2)\Phi_k(z_3)\rangle = \sum_\ell z_{12}^{\Delta_\ell - \Delta_i - \Delta_j} \bar{z}_{12}^{\bar{\Delta}_\ell - \bar{\Delta}_i - \bar{\Delta}_j} C_{ij\ell} \langle \Phi_\ell(z_2)\Phi_k(z_3)\rangle$$

By the orthonormality assumption of the OPE, we then have

$$\langle \Phi_\ell(z_2)\Phi_k(z_3)\rangle = \frac{\delta_{\ell k}}{z_{23}^{2\Delta_k}\bar{z}_{23}^{2\bar{\Delta}_k}} \Rightarrow \langle \Phi_i(z_1)\Phi_j(z_2)\Phi_k(z_3)\rangle = \frac{C_{ijk}(z_{12})}{z_{23}^{2\Delta_k}\bar{z}_{23}^{2\bar{\Delta}_k} z_{12}^{\Delta_i+\Delta_j-\Delta_k}\bar{z}_{12}^{\bar{\Delta}_i+\bar{\Delta}_j-\bar{\Delta}_k}}$$

9. We assume that $\mu \ll 1/r$. The integral is in fact real, and we can approximate it by

$$\int d^2p \frac{\cos(pr\cos(\theta))}{p^2+m^2} = \int d\theta \int_0^\infty \frac{p\,dp\,e^{-\frac{1}{2}(pr\cos(\theta))^2}}{p^2+\mu^2} = \int d\theta \frac{1}{2}\int_{\frac{1}{2}\mu^2 r^2\cos^2(\theta)}^\infty \frac{du\,e^{-u}}{u} = \frac{1}{2}\int_0^{2\pi} d\theta \Gamma(0, \tilde{\mu}^2 r^2 \cos^2(\theta))$$

It is known that $\Gamma(0,\epsilon) = -\gamma - \log\epsilon$ so up to a constant (that can be absorbed into the redefinition of μ) we get;

$$-\frac{\ell_s^2}{2\pi}\frac{1}{2}(2\pi)\log(\mu^2|x-y|^2) = -\frac{\ell_s^2}{2}\log(\mu^2|x-y|)$$

10. By Stokes' theorem its clear. Let Ω be any disk enclosing the origin:

$$\int_\Omega d^2z \bar{\partial}\partial \log|z|^2 = i\int_\Omega dz \wedge d\bar{z}\,\bar{\partial}\partial \log|z|^2 = -i\oint_{\partial\Omega} dz \partial \log|z|^2 = -i\oint_{\partial\Omega}\frac{dz}{z} = 2\pi$$

Alternatively we could put in a regulator and evaluate this directly:

$$\int_\Omega d^2z\partial\bar{d}\log(|z|^2+\mu^2) = \int_\Omega d^2z\partial\frac{z}{|z|^2+\mu^2} = \int_\Omega d^2z\frac{\mu^2}{(|z|^2+\mu^2)^2}$$

As $\mu \to 0$ this approaches 0 everywhere except for the origin. Taking $|z| = r$ and integrating in polar coordinates (note $d^2z = 2dxdy = 2rdrd\theta$):

$$2\pi \times 2 \times \int_0^\infty \frac{\mu^2 r}{(r^2+\mu^2)^2} = 2\pi$$

as required.

11. We have:

$$\frac{1}{4\pi\ell_s^2}\int d^2\xi\,\sqrt{-g}\,g^{ab}\partial_a X\,\partial_b X \Rightarrow T_{ab} = -\frac{4\pi}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} - \frac{1}{\ell_s^2}\left(\partial_a X\partial_b X - \frac{1}{2}g_{ab}\partial_c X\partial^c X\right)$$

This is clearly traceless. Let's specialize to the holomorphic sector to get $T(z) = -\frac{1}{\ell_s^2} : \partial X\partial X :$ and of course this is the non-singular part of the $\partial X(z)\partial X(w)$ OPE as $z \to w$.

12. The scaling dimensions of conserved currents don't change.

For a current to be conserved, we must that the surface operator $\frac{1}{2\pi i}\oint dz J(z)$ is topological (independent of contour). Applying dilatation $z \to z/\lambda$ on this does not change the operator, so long as it does not pass any operator insertions. So we have:

$$\frac{1}{2\pi i}\oint dz J(z) + c.c. = \frac{1}{2\pi i}\oint d\frac{z}{\lambda}J'(z/\lambda, \bar{z}/\lambda) + c.c.$$

And thus we get $J(z,\bar{z}) = \lambda^{-1}J'(z/\lambda, \bar{z}/\lambda)$, and we get J has scaling dimension 1.

On the other hand for $T^{\mu\nu}$, we have the conserved charge:

$$P_\nu = \oint dn^\mu T_{\mu\nu}$$

Applying dilatation, we see from exponentiating the commutation relation for $[D, P_\nu]$ that $P_\nu = P'_\nu/\lambda$ so

$$P_\nu = \oint dn^\mu T_{\mu\nu} + c.c. = \frac{1}{\lambda}\underbrace{\oint\frac{dn^\mu}{\lambda}T'_{\mu\nu}(z/\lambda,\bar{z}/\lambda) + c.c.}_{=P'_\nu)} = P'_\nu/\lambda$$

giving us that

$$T_{\mu\nu}(z,\bar{z}) = \lambda^{-2}T'_{\mu\nu}(z/\lambda,\bar{z}/\lambda)$$

so T properly has scaling dimension 2.

13.

$$\langle\prod_{n=1}^N e^{ipX(z,\bar{z})}\rangle = \int \mathcal{D}X e^{-\frac{1}{2\pi\ell_s^2}\int d^2z\partial X\bar{\partial}X + i\int d^2z\,X(z)\sum_i p_i\delta^2(z-z_i)}$$

$$= 2\pi\delta(\Sigma p_i)e^{-\frac{1}{2}\int d^2\sigma d^2\sigma' J(\sigma)J(\sigma')G(\sigma,\sigma')} = 2\pi\delta(\Sigma p_i)e^{-\frac{1}{2}\sum_{i,j=1}^N p_i p_j\langle X(z_i)X(z_j)\rangle}$$

Appreciate both the UV divergence (from coincident points in the correlator) and the IR divergence (from the correlator going as a logarithm) will cancel (think Kosterlitz-Thouless/Mermin Wagner stuff here):

$$\mu^{2\frac{\ell_s^2}{4}(\Sigma p_i)^2}\epsilon^{2\frac{\ell^2}{4}\sum p_i^2}$$

Momentum conservation removes the IR, and if we normal-order the vertex operators within the product we will not get the UV divergence.

14. By explicit calculation:

$$T(z)[(\partial X)^4](w) \sim \frac{-3\alpha(\partial X)^2(w)}{(z-w)^4} + \frac{4(\partial X)^4}{(z-w)^2} + \dots$$
$$T(z)[(\partial^2 X)^2](w) \sim \frac{-2\alpha}{(z-w)^6} + \frac{4(\partial X \partial^2 X)(w)}{(z-w)^3} + \frac{4(\partial^2 X)^2(w)}{(z-w)^2}$$
$$T(z)[\partial^3 X \partial X](w) \sim \frac{-3\alpha}{(z-w)^6} + \frac{6(\partial X)^2(w)}{(z-w)^4} + \frac{6(\partial X \partial^2 X)(w)}{(z-w)^3} + \frac{6(\partial^2 X)^2(w)}{(z-w)^2} + \dots$$

where $+\dots$ are terms that are $O((z-w)^{-1})$ or higher powers, which will not affect the non-primary terms. We see that the combination:

$$(\partial X)^4 + \frac{\alpha}{2}\partial^3 X \partial X - \frac{3}{4\alpha}(\partial^2 X)^2$$

gives a primary operator of dimension 4. Along the way I noticed that there are no primary operators of dimension 2 or 3 that are finite sums of products of derivatives of ∂X.

I can't help but think that this might have *something* to do with the Schwarzian.

15. We look at:

$$: i\frac{\sqrt{2}}{\ell_s}\partial X(z) :: e^{ipX(w)} : = i\frac{\sqrt{2}}{\ell_s}\sum_{n=0}^{\infty} : \partial X(z) :: (X(w))^n : \frac{(ip)^n}{n!} + \text{finite}$$
$$= i\frac{\sqrt{2}}{\ell_s}\sum_{n=0}^{\infty}(-)\frac{\ell_s^2}{2}\frac{n}{z-w}\frac{(ip)^2 : X(w)^{n-1} :}{n!} = \frac{\ell_s p}{\sqrt{2}}\frac{1}{z-w}V_p(w) + \text{finite}$$

16. Directly:

$$\sum_{n,m}\frac{(ia)^n(ib)^m}{n!\,m!} : X^n(z) :: X^m(w) :$$

First lets look at when $n = m$ and say we contract everything. Then we need to contract all n $X(z)$ with all n $X(w)$. There are $n!$ ways to do this, and each produces a factor of $-\frac{\ell^2}{2}\log|z-w|^2$. The diagonal components thus give the sum:

$$\sum_n \frac{1}{n!}\left(\frac{ab\ell_s^2}{2}\log|z-w|^2\right)^n = |z-w|^{ab\ell_s^2/2}$$

Now a more general term, say $: X(z)^n :: X(w)^m :$ where we want to contract $k < n, m$ of them we must choose k $X(z)$ and $kX(w)$ to contract the $X(z)$ with and then figure out the order to contract those k amongst themselves ($k!$), so we have $\binom{n}{k} \times \binom{m}{k} \times k! = \frac{n!m!}{(m-k)!(n-k)!k!}$ ways to do this. The contraction again gives the $\log^k$ term as before, and now we have a remaining factor of $\frac{(ia)^{n-k}(ib)^{m-k}}{(n-k)!(m-k)!} : X(z)^{n-k} :: X(w)^{m-k} :$. For each k-contracted set which gives the $\log^k$ term, we should therefore multiply it by:

$$\sum_{m,n=k}^{\infty}\frac{(ia)^{n-k}(ib)^{m-k}}{(n-k)!(m-k)!} : X(z)^{n-k} :: X(w)^{m-k} := e^{iaX(z)+ibX(w)}$$

So the OPE is:

$$: e^{iaX(z)} :: e^{iaX(w)} := |z-w|^{ab\ell^2/2}e^{iaX(z)+ibX(w)}$$

17. Directly:

$$\partial_z J(z)\partial_w J(w) = \partial_z\partial_w\left(\frac{1}{(z-w)^2}\right) = -\frac{6}{(w-z)^4} + \text{finite}$$

We have no $\frac{2}{(z-w)^2}$ term, as would otherwise be required

18. The stress energy tensor is:

$$T(z) = -\frac{1}{2} : \psi(z)\partial\psi(z) :\Rightarrow T(z)\psi(w) = -\frac{1}{2}\psi(z)(\frac{-1}{(z-w)^2}) + \frac{1}{2}\frac{\partial\psi(z)}{z-w} = \frac{1}{2}\frac{1}{(z-w)^2}\psi(w) + \frac{\partial\psi(w)}{(z-w)}$$

so this shows that ψ is primary with weight 1/2.

19. I'll instead have the notation $w = g \circ f(z)$. For $T(z) = (f')^2 T(f) + \{f, z\}$ consider $h = g \circ f$. Then we have:

$$T(z) = (f')^2 T(f) + C(f) = (f')^2((g')^2 T(g \circ f) + C(g \circ f)) + C(f) = (h')^2 T(h) + (f')^2 C(g) + C(f)$$

So we get the desired cocycle property:

$$C(h) = (f')^2 C(g) + C(f)$$

Now, we need $C(f)$ to have units of $[z]^{-2}$. The most naive guess is to let $C(h) = h''$, but this gives:

$$h'' = (f')^2 g'' + f'' g'$$

If that last factor of g' were not there, we would be done. Instead we must think more deeply. We also need the Schwarzian to include a term linear in the third derivative, and the only such term is a constant times f'''/f'. Let us look at how this transforms:

$$\frac{h'''}{h'} = (f')^2 \frac{g'''}{g'} + 3\frac{f'' g''}{g'} + \frac{f'''}{f'}$$

Now what stops us is the cross-term. The only terms that we can add to f'''/f' that involve less than third derivatives in ϵ are f'', $(f')^2$ $(f''/f')^2$.

There is one last term we could have built out of terms of order $\leqslant 3$ that would give units of $[z]^{-2}$: $(f'''/f'')^2$, however in the limit of an infinitesimal transformation $z + \epsilon(z)$, this would give $(\epsilon'''/\epsilon'')^2$ which is nonlinear in ϵ, so this term cannot contribute. .

$(h')^2 = (f'g')^2$ has none of the properties we'd like, and adding it would break the term that $(f')^2$ multiplies being proportional to $C(g)$. Similarly, adding f'' would break the term that $(f')^2$ *doesn't* multiply being proportional to $C(f)$. What is left is $\left(\frac{f''}{f'}\right)$. This transforms as:

$$\left(\frac{h''}{h'}\right)^2 = (f')^2 \left(\frac{g''}{g'}\right)^2 + \left(\frac{f''}{f'}\right)^2 + \frac{2 f'' g''}{g'}$$

The cross term is exactly of the form of the cross term in f'''/f', and so by appropriately subtracting:

$$\frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 = (f')^2 \left(\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right) + \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

Another way to do this is to first look at the general nth derivative of the global conformal transformations (the Möbius transformations). Note that:

$$g = \frac{az+b}{cz+d}, \quad g'(z) = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2} \quad \Rightarrow \partial_z^n g = \frac{n!(-c)^{n-1}}{(cz+d)^{n+1}}$$

In particular:

$$g''(z) = \frac{-2c}{(cz+d)^3}, \quad g'''(z) = \frac{6c^2}{(cz+d)^4}$$

The simplest combination of g', g'', and g''' that can give zero is:

$$(g'')^2 - \frac{2}{3} g'''(z) g'(z)$$

We want this to have units of $[g]/[z]^2$ and to behave as $\epsilon'''(z)$ to leading order when $g = z + \epsilon(z)$. The only way to do this (which fixes overall normalization and all) is to divide through by $-2/3(g'(z))^2$ and get:

$$\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2.$$

It is easy to check that this satisfies the cocycle property for composition, namely:

$$\{z_3, z_1\} = \left(\frac{\partial z_2}{\partial z_1}\right)^2 \{z_3, z_2\} + \{z_2, z_1\} \tag{42}$$

Since for $h = g \circ f$ we get:

$$\frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 = \frac{f'''}{f'} + 3\frac{f''f'g''}{f'g'} + \frac{(f')^3 g'''}{f'g'} - \frac{3}{2}\left(\frac{f''g' + (f')^2 g''}{f'g'}\right)^2 = \{f,z\} + (f')^2\frac{g''}{g'} - \frac{3}{2}\frac{g''}{g'} = \{f,z\} + (f')^2\{g, f(z)\}$$

20. I will use shorthand $\binom{z'}{z}$ for $\frac{\partial z'}{\partial z}$ and $\binom{z'}{zz}$ for $\frac{\partial z'}{\partial^2 z}$, also I will just write $\Gamma^z_{zz}, g_{z\bar z}, g^{z\bar z}$ as Γ, g, g^{-1} respectively. Now

$$\Gamma = g^{-1}\partial g \quad \Rightarrow \quad \Gamma' = g^{-1'}\partial' g' = g^{-1}\partial\left(\binom{z}{z'}g\right) = \binom{z}{z'}\Gamma - \binom{z}{z'}^2\binom{z'}{zz}$$

So

$$(\Gamma')^2 = \binom{z}{z'}^2\Gamma^2 - 2\Gamma\binom{z}{z'}^3\binom{z'}{zz} + \binom{z}{z'}^4\binom{z'}{zz}^2$$

$$\partial'\Gamma' = \binom{z}{z'}\partial\left[\binom{z}{z'}\Gamma - \binom{z}{z'}^2\binom{z'}{zz}\right] = \binom{z}{z'}^2\partial\Gamma - \Gamma\binom{z}{z'}^3\binom{z'}{zz} + 2\binom{z}{z'}^2\binom{z'}{zz}^2 - \binom{z}{z'}^3\binom{z'}{zzz}$$

To cancel out the Γ term we look at $2\partial\Gamma - \Gamma^2$. We see this transforms as:

$$2\partial'\Gamma' - \Gamma'^2 = \binom{z}{z'}^2(2\partial\Gamma - \Gamma^2) + 3\binom{z}{z'}^4\binom{z'}{zz}^2 - 2\binom{z}{z'}^3\binom{z'}{zzz} = \binom{z}{z'}^2(2\partial\Gamma - \Gamma^2 - 2\{z', z\})$$

So that

$$T_{zz} - \frac{c}{24}(2\partial\Gamma - \Gamma^2) = \binom{z'}{z}^2(T_{z'z'} - \frac{c}{24}(2\partial'\Gamma' - \Gamma'^2)) + \frac{c}{12}\{z', z\} - \frac{c}{24}2\{z', z\} = \binom{z'}{z}^2(T_{z'z'} - \frac{c}{24}(2\partial'\Gamma' - \Gamma'^2))$$

So indeed $\hat T_{zz} = T_{zz} - \frac{c}{24}(2\partial\Gamma - \Gamma^2)$ transforms as a tensor.

21. We have:

$$-\bar\nabla T_{z\bar z} = \nabla \hat T_{zz} = g^{z\bar z}\bar\partial\hat T = -\frac{c}{24}g^{z\bar z}\bar\partial[2\partial(g^{-1}\partial g) - (g^{-1}\partial g)^2] = -\frac{c}{24}g^{z\bar z}\left[2\partial\bar\partial(g^{-1}\partial g) - 2(g^{-1}\partial g)\bar\partial(g^{-1}\partial g)\right]$$

We can recognize this as:

$$\frac{c}{24}2g^{z\bar z}(\partial R_{\bar z z} - \Gamma^z_{zz}R_{\bar z z}) = \frac{c}{24}2g^{z\bar z}\nabla_z R_{\bar z\bar z} = \frac{c}{24}\nabla_z R = \frac{c}{24}\partial R = -\frac{A}{2}\partial R$$

so we have $A = -c/12$

22. The first part of the action is truly invariant. Let us look at how R changes under Weyl rescaling:

$$-2e^{-\chi}g^{-1}\bar\partial(e^\chi g^{-1}\partial(e^\chi g)) = e^{-\chi}(R - 2g^{-1}\partial\bar\partial\chi) = e^{-\chi}(R - 2\partial\bar\partial\chi)$$

Consequently: $\sqrt{-g}R \to \sqrt{-g}(R - 2\nabla^2\chi)$

So the action part will transform as:

$$S_L(g_{\alpha\beta}e^\chi, \phi) = S_L(g_{\alpha\beta}, \phi) - \frac{1}{48\pi}\int d^2\xi\sqrt{g}\phi\nabla^2\chi = S_L(g_{\alpha\beta}, \phi) + \frac{1}{24\pi}\int d^2\xi\sqrt{g}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\chi$$

23. The most general variation of the effective action is:

$$\delta \log Z = -\frac{1}{4\pi}\int_\Sigma d^2\xi\sqrt{g}(a_1 R + a_2)\delta\phi - \frac{1}{4\pi}\int_{\partial\Sigma} d\xi(a_3 + a_4 K + a_5 n^a \nabla_a)\delta\phi \tag{43}$$

The counterterms that we can introduce are:

$$\int_\Sigma d^2\xi\sqrt{g}b_1 + \int_{\partial\Sigma} d\xi(b_2 + kb_3) \tag{44}$$

and the variation of the counterterm action:

$$\int_\Sigma d^2\xi\sqrt{g}b_1\delta\omega + \frac{1}{2}\int_{\partial\Sigma} d\xi(b_2 + b_3 n^a\partial_a)\delta\omega$$

So we can use this to set $a_2, a_3, a_5 = 0$. Further, we know the bulk integral's variation is in fact:

$$\delta \log Z = -\delta S_{eff} = \frac{1}{4\pi}\int d^2\xi\sqrt{g}\,T_{\alpha\beta}\delta g^{\alpha\beta} = -\frac{1}{4\pi}\int d^2\xi\sqrt{g}\,T^\alpha_\alpha\delta\phi = \frac{c}{48\pi}\int d^2\xi\,\sqrt{g}R\delta\phi \Rightarrow a_1 = -\frac{c}{12}$$

Now let's start with a flat metric and do two changes:

$$\begin{aligned}\delta_{\phi_1}\delta_{\phi_2}\log Z &= -\frac{c}{24\pi}\int d^2\xi\sqrt{g}\,\delta\phi_2\nabla^2\delta\phi_1 - \frac{a_4}{4\pi}\int d\xi\sqrt{g}\,\delta\phi_2 n^a\partial_a\delta\phi_1\\ &= \frac{c}{24\pi}\int d^2\xi\sqrt{g}\,\partial^a\delta\phi_2\partial_a\delta\phi_1 + \left(\frac{c}{24\pi} - \frac{a_4}{4\pi}\right)\int d\xi\sqrt{g}\,\delta\phi_2 n^a\partial_a\delta\phi_1\end{aligned}$$

Note that the second term is *not* symmetric under $\delta_{\phi_1} \leftrightarrow \delta_{\phi_2}$, and so we must have the counterterm $\frac{a_4}{4\pi} = \frac{c}{24\pi}$. A variation of this argument can be used to show that c is truly a constant, independent of any worldsheet coordinates.

24. Take the map $\frac{L}{2\pi}\log z$, mapping the plane to the cylinder of circumference L. We get:

$$T^{cyl} = (\partial z')^{-2}(T^{plane} - \{z', z\}) = \left(\frac{2\pi}{L}\right)^2 z^2\left(T^{plane} - \frac{c}{12}\frac{1}{2z^2}\left(\frac{L}{2\pi}\right)^2\right) = \left(\frac{2\pi z}{L}\right)^2 T^{plane} - \frac{c}{24}$$

So the zero mode of T^{cyl} is modified. By T^{cyl} has the expansion $\sum_n L_n e^{-2\pi i n x}$ so we see L_0 gets modified by $-\frac{c}{24}$.

Because L_0 is a codimension 1 operator, it will get modified the same way, whether on the cylinder or torus.

25. Each raising operator L_{-n} acts by raising the level by n, and so assuming each one gives a unique state not expressible in terms of the action of the other L_{-k}, we get that it will contribute:

$$1 + q^n + q^{2n} + \cdots = \frac{1}{1-q^n}$$

to the partition function. All together these give

$$\frac{1}{\prod_{n=1}^\infty(1-q^n)} \Rightarrow \mathrm{Tr}[e^{2\pi i\tau(\Delta - c/24)}] = \frac{q^{\Delta - c/24}}{\prod_{n=1}^\infty(1-q^n)}.$$

This also shows that at level n there will generically be as many states as there are partitions of the number n.

26. Consider a nontrivial state $|h\rangle$ so that $L_n|h\rangle = 0$ for some n sufficiently positive. Then:

$$0 = \langle h| L_{-n}L_n |h\rangle = \langle h| (\frac{n(n^2-1)}{12}c - 2nh)|h\rangle$$

If $c = 0$ we get a contradiction unless either $|0\rangle$ is null (and thus decouples) or otherwise $h = 0$, and so we get a vacuum state.

I think we need to add the assumption of irreducibility to have a unique ground state (ie a counterexample would be TQFTs with multiple ground states).

27. It is enough to show that L_1 and L_2 acting on this state give zero, since then all other L_n can be obtained by commutators of these two. Indeed:

$$L_1(L_{-2} - \frac{3}{4}L_{-1}^2)\,|1/2\rangle = (3L_{-1} - \frac{3}{4}(2L_0L_{-1} + 2L_{-1}L_0))\,|1/2\rangle = (3L_{-1} - \frac{3}{4}(2L_{-1} + 4L_{-1}L_0))\,|1/2\rangle = 0$$

$$L_2(L_{-2} - \frac{3}{4}L_{-1}^2)\,|1/2\rangle = (4L_0 + \frac{2(2^2-1)}{12}c - \frac{3}{4}3(L_{-1}L_1 + L_1L_{-1}))\,|1/2\rangle = (4L_0 + \frac{1}{4} - \frac{9}{2}L_0))\,|1/2\rangle = 0$$

28. The null state's field must satisfy:

$$(\mathcal{L}_{-2} - \frac{3}{4}\mathcal{L}_{-1}^2)\,\langle\psi_w \prod_i \psi_{w_i}\rangle = \left[\sum_i \left(\frac{1/2}{(w_i - w)^2} - \frac{1}{w_i - w}\partial_i\right) - \frac{3}{4}\frac{\partial^2}{\partial^2 w}\right]\langle\psi_w \prod_i \psi_{w_i}\rangle \tag{45}$$

For the three-point function (holomorphic sector) this gives:

$$\left[\frac{1/2}{(w-w_1)^2} + \frac{1}{w-w_i}\partial_1 + \frac{1/2}{(w-w_2)^2} + \frac{1}{w-w_i}\partial_2 - \frac{3}{4}\partial_w^2\right]\frac{\lambda}{(w-w_1)^{1/2}(w_1-w_2)^{1/2}(w_2-w)^{1/2}} = 0$$

This gives:

$$\frac{7\lambda}{16}\frac{(w_1-w_2)^{3/2}}{(w-w_1)^{5/2}(w_2-w)^{5/2}} = 0$$

which gives $\lambda = 0$. We could have inferred this from fermion parity.

Next, for the four-point function, first note that all the ψ have the same scaling dimension, so WLOG we can write this as:

$$\langle\psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle = \frac{1}{z_{12}z_{34}}h\left(\frac{z_{12}z_{34}}{z_{13}z_{24}}\right)$$

plugging this into (45) gives a complicated-looking differential equation, but this can be simplified substantially by taking $z_1 = z, z_2 = 0, z_3 = \infty, z_4 = 0$. Notice then that z here is indeed the cross ratio. We then get the simpler differential equation:

$$2zg(z) + 2(1-z^2)g'(z) - 3z(1-z)^2g''(z) = 0$$

This can be solved in terms of known functions (we should more specifically give boundary conditions by specifying residues of $g(z)$ at $z = 0, 1, \infty$). All in all we get:

$$g(z) = \frac{z^2 - z + 1}{1-z}$$

Thus

$$\langle\psi(z)\psi(z_1)\psi(z_2)\psi(z_3)\rangle = \frac{1}{z_{12}z_{34}} + \frac{1}{z_{14}z_{23}} - \frac{1}{z_{13}z_{24}}$$

exactly as we would get for Wick contraction.

29. Assume it is not primary - then it is a descendant. By positivity of scaling dimensions, it must be a descendant of a field of scaling dimension 0, but as we have shown two exercises ago, the only such field is the vacuum $|0\rangle$. The vacuum is translation invariant $\partial_z\mathbf{1} = 0$ and so it has no descendants of scaling dimension 1. (It *does* have T as a descendant of scaling dimension 2 under $\partial_z^2\mathbf{1}$).

30. Assume $z > w$. On one hand,

$$:[J^a(z), J^b(w)]:= J^a(z)J^b(w) = \sum_{m,n}[J_m^a, J_n^b]z^{-m-1}w^{-n-1}$$

On the other

$$J^a(z)J^b(z) = \frac{G^{ab}}{(z-w)^2} + \frac{if_c^{ab}J^c(w)}{z-w} + \ldots = \sum_m mG^{ab}z^{-2}\left(\frac{w}{z}\right)^{m-1} + \sum_{m,n} if_c^{ab}J_m^c w^{-m-1}z^{-1}\left(\frac{w}{z}\right)^n$$

$$= \sum_m mG^{ab}z^{-m-1}w^{m-1} + \sum_{m,n} if_c^{ab}J_m^c w^{-(m-n)-1}z^{-n-1}$$

$$= \sum_{m,n}\left(m\delta_{m+n}G^{ab}w^{-m-1}w^{-n-1} + if_c^{ab}J_{m+n}^c w^{-m-1}z^{-n-1}\right)$$

so we get:

$$[J_m^a, J_n^b] = m\delta_{m+n}G^{ab} + if_c^{ab}J_{m+n}^c$$

31. Rewrite the first part of the action as $-\frac{1}{4\lambda^2}\int d^2\xi\,\mathrm{Tr}[(g^{-1}\partial g)^2]$. Now note:

$$\delta(g^{-1}\partial g) = g^{-1}\partial\delta g - g^{-1}\,\delta g\, g^{-1}\partial g$$

Then we can write the variation of the action as:

$$-\frac{1}{2\lambda^2}\int d^2\xi \mathrm{Tr}\left[(g^{-1}\partial_\mu\delta g - g^{-1}\,\delta g\, g^{-1}\partial_\mu g)(g^{-1}\partial^\mu g)\right] = \frac{1}{2\lambda^2}\int d^2\xi \mathrm{Tr}\left[\delta g\left(\partial_\mu(g^{-1}\partial^\mu g\, g^{-1}) + \underbrace{g^{-1}\partial_\mu g g^{-1}\partial^\mu g g^{-1}}_{g^{-1}\partial_\mu g\,\partial^\mu(g^{-1})}\right)\right]$$

$$= \frac{1}{2\lambda^2}\int d^2\xi \mathrm{Tr}\left[g^{-1}\delta g\,\partial^\mu\left(g^{-1}\partial_\mu g\right)\right]$$

So we see that we must have $g^{-1}\partial_\mu g$ be a conserved current if we only had the first part of the action. In $z, \bar z$ cooredinates we have $\partial J^z + \bar\partial J^{\bar z} = 0$. We would like both $J = J^z$ and $\bar J = J^{\bar z}$ to be separately conserved $\bar\partial J = \partial \bar J = 0$. However, this is equivalent to also having $\varepsilon^{\mu\nu}J_\nu$ conserved. However $\partial_\mu J_\nu - \partial_\nu J_\mu = -[J_\mu, J_\nu]$ gives that $\partial_\mu\epsilon^{\mu\nu}J_\nu = -\epsilon^{\mu\nu}J_\mu J_\nu \neq 0$ for nonabelian algebras.

On the other hand, the second term has variation:

$$\frac{ik}{8\pi}\int_B d^3\xi\,\varepsilon_{\alpha\beta\gamma}\mathrm{Tr}\left[(g^{-1}\partial^\alpha\delta g - g^{-1}\delta g g^{-1}\partial^\alpha g)(g^{-1}\partial^\beta g)(g^{-1}\partial^\gamma g)\right] + \text{ perms.}$$

this will all vanish identically as an action on B, since $\mathrm{Tr}(A\wedge A\wedge A)$ is already closed for our 1-form $A = g^{-1}\mathrm{d}g$. On the other hand, the first term in parenthesis contributes a boundary term when α is transverse

$$\frac{ik}{8\pi}\int_{\partial B} d^2\xi\,\varepsilon_{\beta\gamma}\mathrm{Tr}(g^{-1}\delta g g^{-1}\partial^\beta g(g^{-1}\partial^\gamma g)) = -\frac{ik}{8\pi}\int_{\partial B} d^2\xi\,\varepsilon_{\beta\gamma}\mathrm{Tr}\left[g^{-1}\delta g\,\partial^\beta\left(g^{-1}\partial^\gamma g\right)\right]$$

Appreciate the difference between this and the factor of 3 *in Di Francesco. I believe we only account for 1 of the 3 terms, since only 1 of the 3 indices will give a transverse direction.*

This gives a total equation of motion of:

$$\frac{1}{2\lambda^2}\partial^\mu(g^{-1}\partial_\mu g) - \frac{ik}{8\pi}\varepsilon_{\mu\nu}\partial^\mu(g^{-1}\partial^\nu g) = 0 \tag{46}$$

Taking the basis $z, \bar z$, $\partial^z = 2\partial_{\bar z}$, $\varepsilon_{z\bar z} = i/2$, we get:

$$[\partial_{\bar z}(g^{-1}\partial_z g) + \partial_z(g^{-1}\partial_{\bar z}g)] - \frac{ik\lambda^2}{4\pi}\left[i\partial_{\bar z}(g^{-1}\partial_z g)g^{-1} - i\partial_z(g^{-1}\partial_{\bar z}g)\right] = \left(1 + \frac{k\lambda^2}{4\pi}\right)\partial_{\bar z}(g^{-1}\partial_z g) + \left(1 - \frac{k\lambda^2}{4\pi}\right)\partial_z(g^{-1}\partial_{\bar z}g)$$

When $\lambda^2 = 4\pi/k$ (meaning k must be positive) the second term goes away and we get the conservation law $\bar\partial J_z$. Taking the conjugate of this equation gives the other conservation law.

$$\bar d(g^{-1}\partial g) = 0 \to -\partial(\bar\partial g\, g^{-1}) = 0$$

In particular the classical solutions factorize into the form $g(z, \bar z) = f(z)\bar f(\bar z)$. It is also quick to show that $g(z, \bar z) \to \Omega(z)g(z, \bar z)\bar\Omega(\bar z)$ (for $\Omega, \bar\Omega$ two independent matrices valued in the same rep'n of G) keeps the action invariant, and so we see that the $G \times G$ classical invariance of the action is *enhanced* to a full $G(z) \times G(\bar z)$ invariance. This is the real power of WZW models, and should be appreciated.

32. Importantly, the 3D action does not have any metric dependence. For the 2D boundary we have:

$$\frac{1}{4\lambda^2}\int d^2\xi\sqrt{g}g^{\mu\nu}\mathrm{Tr}[g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g]$$

this gives a stress tensor:

$$T_{\mu\nu} = -\frac{\pi}{\lambda^2}\left(\mathrm{Tr}[g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g] - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\mathrm{Tr}[g^{-1}\partial_\alpha g\, g^{-1}\partial_\beta g]\right)$$

we see that this is traceless. The holomorphic part is:

$$-\frac{\pi}{\lambda^2}\mathrm{Tr}[J^2] = \frac{k}{4}J^a J^a$$

the constant out front can have a field strength renormalization from its classical value (because the J are not free fields), and so we would not expect it to agree with the one given in the definition of T.

Give another reason for this discrepancy. Try to account for it.

33. This one is direct. Take $z > w$

$$[J^a(z), R_i(w)] = \sum_m J^a_m z^{-m-1}R_i(w) = \sum_n z^{-1}\left(\frac{w}{z}\right)^n T_{ij}R_j(w)$$

and so we get:

$$J^a_m R_i(w) = w^m T^a_{ij}R_j(w)$$

34. We have:

$$\frac{1}{2(k+\bar{h})}\sum_{n,m} z^{-2-(n+m)} : J^a_m J^a_n := \sum_k L_m z^{-2+m}$$

Appropriately shifting, we see that $L_m = \frac{1}{2(k+\bar{h})} : J_{m+n}J_{-n} :$ as required. The only term here that doesn't give zero when acting on a WZW primary is $J^a_{-1}J^a_0$ which acts as $J^a_{-1}T^a_{ij}$ and this terms appears twice, so we get that

$$|\chi_i\rangle = (L_{-1}\delta_{ij} - \frac{1}{k+\bar{h}}T^a_{ij}J^a_{-1})\,|R_j\rangle$$

is null. But we also have that:

$$\begin{aligned}
\langle(J^a_{-1}R(z_1))R(z_2)\ldots R(z_N)\rangle &= \frac{1}{2\pi i}\oint_{z_1}\frac{dz}{z-z_1}J^a(z)R(z_1)R(z_2)\ldots R(z_N)\\
&= -\frac{1}{2\pi i}\sum_{i\neq 1}\oint_{z_i}\frac{dz}{z-z_1}R(z_1)R(z_2)\ldots J^a(z)R(z_k)\ldots R(z_N)\\
&= -\frac{1}{2\pi i}\sum_{k\neq 1}\oint_{z_k}\frac{dz}{z-z_1}\frac{1}{z-z_k}R(z_1)R(z_2)\ldots T^a_{ij}R_j(z_k)\ldots R(z_N)\\
&= \sum_{k\neq 1}\frac{T^a_{ij}R_j(z_k)}{z_1-z_k}
\end{aligned}$$

Here we chose to do this with $R(z_1)$, but we could have picked arbitrary z_i. This means that correlators must satisfy:

$$\left(\partial_{z_1} - \frac{1}{k+\bar{h}}\sum_{j\neq i}^{N}\frac{T^a_i\otimes T^a_j}{z_i-z_j}\right)\langle\prod_{k=1}^{N}R(z_k)\rangle = 0$$

where T^a_i acts on the ith primary field in the correlator.

35. I think its instructive to do this one out in detail. First let's take a look at just $T_G(z)$ acting on any current $J^a(w)$. We want the singular terms:

$$
\begin{aligned}
\frac{1}{2(k+\bar{h})}(J^bJ^b)(z)J^a(w) &= \frac{1}{2(k+\bar{h})}\frac{1}{2\pi i}\oint_z \frac{dx}{x-z}\left(J^b(x)J^b(z)J^a(w) + J^b(x)J^b(z)J^a(w)\right)\\
&= \frac{1}{2(k+\bar{h})}\frac{1}{2\pi i}\oint_z \frac{dx}{x-z}\left[\left(\frac{G^{ba}}{(x-w)^2} + \frac{if^{ba}_{c}J^c(w)}{x-w}\right)J^b(z) + J^b(x)(z\leftrightarrow x)\right]\\
&= \frac{1}{k+\bar{h}}\left(\frac{G^{ab}J^b(z)}{(z-w)^2} + \frac{1}{2}f_{abc}\frac{J^c(w)J^b(z)+J^b(z)J^c(w)}{z-w}\right)
\end{aligned}
$$

but note that last term will have

$$
J^c(w)J^b(z) + J^b(z)J^c(w) = \frac{2G^{bc}}{(z-w)^2} + \frac{2if_{cbd}J^d(w)}{w-z} + (J^cJ^b)(w) + (J^bJ^c)(w)
$$

The first term will cancel when multiplied by the anti-symmetric f_{abc}, as will the last (regular) term. The second term will give $f_{abc}f_{cbd} = -f_{abc}f_{dbc} = 2\bar{h}\delta_{ad}$, by the definition of dual coxeter number. On the other hand we have $G_{ab} = k\delta_{ab}$ so altogether we get:

$$
\frac{1}{k+\bar{h}}\frac{(k+\bar{h})J^a(z)}{(z-w)^2} = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}
$$

as we wished. *Note* we could have run this logic in reverse, and demanded that a stress tensor must its OPE make second term involving ∂J^a have coefficient 1, giving the required normalization of $(2(k+\bar{h}))^{-1}$. Now note that if we define $T^H(z) := \frac{1}{2(k+\bar{h}_H)}\sum_{a\in H}(J^aJ^a)(z)$, then as long as we are taking the OPE with J^a for $a \in H$, we see that the singular terms are *exactly* the same. Indeed, we get the same factor of $k\delta_{ab}$ from the quadratic OPE term, and the sum over $f_{abc}f_{dbc}$ restricts b and c to be in H by the subgroup property, so we get $\bar{h}_H$. Thus $(T_G - T_H)J^a = T_{G/H}J^a$ is regular for $a \in H$.

For the next step, again lets first just look at the singular terms in the T_GT_G OPE:

$$
\begin{aligned}
T(z)T(w) &= \frac{1}{2(k+\bar{h})}\frac{1}{2\pi i}\oint \frac{dx}{x-w}T(z)J^a(x)J^a(w)\\
&= \frac{1}{2(k+\bar{h})}\frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\left(\frac{J^a(x)}{(z-x)^2} + \frac{\partial J^a(x)}{z-x}\right)J^a(w) + J^a(x)(w\leftrightarrow x)\right]\\
&= \frac{1}{2(k+\bar{h})}\frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\frac{k\dim G}{(z-x)^2(x-w)^2} + \frac{\partial J^a(x)\,J^a(w)}{z-x} + (w\leftrightarrow x)\right]\\
&= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}
\end{aligned}
$$

here we have $c = \frac{k\dim G}{k+\bar{h}}$ as required. The same logic applies to the $T_H(z)T_H(w)$ OPE, where we would get:

$$
\frac{c_H/2}{(z-w)^4} + \frac{2T_H(w)}{(z-w)^2} + \frac{\partial T_H(w)}{(z-w)}, \quad c_H = \frac{k\dim H}{k+\bar{h}_H}
$$

Now it remains to evaluate:

$$
\begin{aligned}
T_G(z)T_H(w) &= \frac{1}{2(k+\bar{h}_H)}\frac{1}{2\pi i}\oint \frac{dx}{x-w}\sum_{a\in H}T_G(z)J^a(x)J^a(w)\\
&= \frac{1}{2(k+\bar{h}_H)}\frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\left(\frac{J^a(x)}{(z-x)^2} + \frac{\partial J^a(x)}{z-x}\right)J^a(w) + J^a(x)(w\leftrightarrow x)\right]\\
&= \frac{1}{2(k+\bar{h}_H)}\frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\frac{k\dim H}{(z-x)^2(x-w)^2} + \frac{\partial J^a(x)\,J^a(w)}{z-x} + (w\leftrightarrow x)\right]\\
&= \frac{c_H/2}{(z-w)^4} + \frac{2T_H(w)}{(z-w)^2} + \frac{\partial T_H(w)}{(z-w)}
\end{aligned}
$$

so indeed $T_G(z)T_H(w) - T_H(z)T_H(w) = T_{G/H}(z)T_H(w)$ has a regular OPE. This further gives us that $T_{G/H}(z)T_{G/H}(w)$ has singular part coming from $T_{G/H}(z)T_G(w) = T_G(z)T_G(w) - T_G(z)T_H(w)$, which gives:

$$\frac{(c_G - c_H)/2}{(z-w)^4} + \frac{2T_{G/H}(w)}{(z-w)^2} + \frac{\partial T_{G/H}(w)}{z-w}$$

So a G theory can be re-written as a set of "decoupled" CFTs with stress tensors T_H and $T_{G/H}$. Now take $G = \mathrm{SU}(2)_m \times \mathrm{SU}(2)_1$. This theory have total level $m+1$. So now take the diagonal subgroup $\mathrm{SU}(2)_{m+1}$. We see that the G/H theory has central charge:

$$c_G - c_H = \left(\frac{m \times 3}{m+2} + \frac{1 \times 3}{1+2}\right) - \frac{(m+1) \times 3}{m+1+2} = 1 + \frac{3m}{m+2} - \frac{3(m+1)}{m+3} = 1 - \frac{6}{(m+2)(m+3)}$$

exactly coincident with the prescribed formula for the minimal models. So, we expect at $m=1$ to get the Ising CFT.

36. We have

$$\begin{aligned}\psi^i(z) = \sum_n \psi^i_n z^{-n-1/2} \Rightarrow \langle \psi^i(z)\psi^j(w)\rangle &= \sum_{n,m\in\mathbb{Z}} \langle \psi^i_n \psi^j_m\rangle z^{-n-1/2} w^{-m-1/2} \\ &= \sum_{m=0}^{\infty} \langle \psi^i_m \psi^j_{-m}\rangle z^{-m-1/2} w^{m-1/2} \\ &= \frac{\delta^i}{\sqrt{zw}}\left[\sum_{m=0}^{\infty}\left(\frac{w}{z}\right)^m - \frac{1}{2}\right] \\ &= \frac{\delta_{ij}}{2\sqrt{zw}}\frac{z+w}{z-w}\end{aligned}$$

the $1/2$ comes from the zero-mode Clifford algebra $\{\psi^i_0, \psi^j_0\} = \delta^{ij}$.

37. We can get this directly from the Ward identity:

$$\langle T(z_1)\phi(z_2)\phi(z_3)\rangle = \left(\frac{\partial_{z_2}}{z_1 - z_2} + \frac{\partial_{z_3}}{z_1 - z_3} + \frac{\Delta}{(z_1-z_2)^2} + \frac{\Delta}{(z_1-z_3)^2}\right)\frac{1}{(z_2-z_3)^{2\Delta}} = \frac{\Delta}{z_{12}^2 z_{13}^2 z_{23}^{2\Delta-2}}.$$

Next, we can write:

$$\langle X|\, T(z)\, |X\rangle = \lim_{w\to 0} \bar{w}^{-2\Delta} \langle 0|\, X(1/\bar{w})\, T(z)\, X(0)\, |0\rangle = \lim_{w\to 0} \frac{\bar{w}^{-2\Delta}\,\Delta}{z^2 \bar{w}^{-2\Delta}} = \frac{\Delta}{z^2}.$$

Finally, let's look at the $O(N)$ fermion. We have that $T(z) = -\frac{1}{2}\sum_{i=1}^N : \psi^i \partial\psi^i :$ so we get:

$$\langle S|\, T\, |S\rangle = -\frac{1}{2}\Sigma_{i=1}^N \lim_{z\to w}\left[\partial_w\left(\frac{z+w}{2\sqrt{zw}}\frac{1}{z-w}\right) - \underbrace{\partial_w \frac{1}{(z-w)}}_{\text{Normal ordering constant}}\right] = -\frac{N}{2}\left(-\frac{1}{8w^2}\right) = \frac{N/16}{w^2}$$

as required.

38. This is direct:

$$D_\theta \hat{X} = (\partial_\theta + \theta\partial_z)\left(X + i\theta\psi + i\bar{\theta}\bar{\psi} + \theta\bar{\theta}F\right) = i\psi + \theta\partial X + \bar{\theta}F + \theta\bar{\theta}\partial\bar{\psi}, \quad \bar{D}_{\bar{\theta}}\hat{X} = i\bar{\psi} + \bar{\theta}\partial X + \theta F + \bar{\theta}\theta\bar{\partial}\psi$$

Now we only want the $\theta\bar{\theta}$ terms of $(D_\theta\hat{X})(\bar{D}_{\bar{\theta}}\hat{X})$ as everything else will vanish in the Berezin integral. This gives:

$$S = \frac{1}{2\pi\ell_s^2}\int d^2z \int d\bar{\theta}d\theta\, \theta\bar{\theta}(\partial X\partial X - F^2 + i\bar{\psi}\partial\bar{\psi} + i\psi\bar{\partial}\psi) = \frac{1}{2\pi\ell_s^2}\int d^2z(\partial X\partial X + i\bar{\psi}\partial\bar{\psi} + i\psi\bar{\partial}\psi)$$

we have dropped F^2 because it has no dynamics or interactions with X, ψ whatsoever.

39. Expanding

$$e^{ip\cdot\hat{X}} = e^{ip_\mu(X^\mu + i\theta\psi^\mu + i\bar{\theta}\bar{\psi}^\mu + \theta\bar{\theta}F^\mu)} = (1 + i\theta p\cdot\psi)(1 + i\bar{\theta}p\cdot\bar{\psi})(1 + \theta\bar{\theta}p\cdot F)e^{ipX}$$

Imposting EOM's gives $F = 0$ right away. Now for the rest:

$$D_\theta \hat{X}^\mu D_{\bar{\theta}}\hat{X}^\nu e^{ipX}|_{\theta\bar{\theta}} = [(\partial X^\mu \partial X^\nu + i\bar{\psi}^\mu \partial\bar{\psi}^\nu + i\psi^\nu\bar{\partial}\psi^\mu) + (i\partial X^\nu\psi^\mu)p\cdot\psi + (i\partial X^\mu\bar{\psi}^\nu)p\cdot\bar{\psi}]e^{ipX}$$

again using the equations of motion we get rid of the $\partial\bar{\psi}, \bar{d}\psi$ terms. Now we get:

$$[\partial X^\mu \partial X^\nu + (i\partial X^\nu\psi^\mu)p\cdot\psi + (i\partial X^\mu\bar{\psi}^\nu)p\cdot\bar{\psi}]e^{ip\cdot X} = (\partial X^\mu + i(p\cdot\psi)\psi^\mu)(\partial X^\nu + i(p\cdot\psi)\psi^\nu)e^{ip\cdot X}$$

40. Following the same logic as the $\mathcal{N} = (2,0)$ case, we can now compute in the R sector:

$$\{G_0^\alpha, \bar{G}_0^\beta\} = \frac{4k}{2}\left(-\frac{1}{4}\right)\delta^{\alpha\beta} + 2L_0\delta^{\alpha\beta}$$

for this to be positive we need:

$$2(\Delta - k/4) \geqslant 0 \Rightarrow \Delta \geqslant k/4.$$

In the NS sector, we have a positivity condition on

$$\{G_{-1/2}^\alpha, \bar{G}_{1/2}^\beta\} = -2\sigma^a_{\alpha\beta}J_0^a + 2\delta^{\alpha\beta}L_0$$

The positivity condition on this operator translates to the matrix:

$$2\Delta\mathbf{1} - 2\sigma^a_{\alpha\beta}J^a$$

being positive semidefinite. But the determinant of this matrix is given by

$$\Delta^2 - |J|^2 = \Delta^2 - j^2$$

So for this to be $\geqslant 0$, given that $\Delta \geqslant 0$, we need $\Delta - j \geqslant 0$

41. This calculation is also direct:

$$\begin{aligned}
T(z)T(w) &= \frac{1/2}{(z-w)^4} + 2\frac{-\frac{1}{\ell_s^2}(\partial X)^2(w)}{(z-w)^2} + \frac{\partial\left(-\frac{1}{\ell_s^2}(\partial X)^2(w)\right)}{z-w} \\
&\quad + \frac{\ell_s}{2}\partial X(z)\frac{Q}{\sqrt{2}\ell_s^3}2\frac{2}{(z-w)^3} + \frac{\ell_s}{2}\partial X(w)\frac{Q}{\sqrt{2}\ell_s^3}2\frac{-2}{(z-w)^3} - \frac{\ell_s^2}{2}\frac{Q^2}{2\ell_s^2}\frac{-6}{(z-w)^4} \\
&= \frac{1/2(1+3Q^2)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}
\end{aligned}$$

we thus get a central charge equal to $1 + 3Q^2$ as required.

42. The integral over the zero mode will give no contribution from the $\partial X\partial\bar{X}$ term in the action and instead will just:

$$\int \mathcal{D}X \exp\left(-\int d^2z\sqrt{g}\left(\frac{Q}{4\pi\ell_s\sqrt{2}}R^{(2)} - i\sum_i p_i\delta^2(z-z_i)\right)X(z)\right)$$

This is a δ-functional on the p_i. We have:

$$\boldsymbol{\delta}\left[\frac{Q}{4\pi\ell_s\sqrt{2}}R^{(2)} - i\sum_i p_i\delta^2(z-z_i)\right]$$

but this can only happen if, after integrating over z, we get:

$$\frac{Q}{\ell_s\sqrt{2}}\chi = i\sum_i p_i \Rightarrow i\sqrt{2}\ell_s\sum_i p_i = Q\chi.$$

Give an interpretation of the vertex operators as "contributing curvature".

43. Note that:

$$[L_{-m}, J_{-n}] = nJ_{-m-n} + \frac{A}{2}m(m-1)\delta_{m+n} = nJ_{-(m+n)} + \left(\frac{A}{2}m(m+1) - mA\right)\delta_{m+n}$$

so this will only have the same form of the central term if $J_0^\dagger = J_0 + A$, ie $J_{-m}^\dagger + A\delta_{m,0}$. This shows that it is necessary. By the above commutation relation, we cannot mix J_m labeled by different mode number in defining $J_m^\dagger$, as they would transform differently under L_0, so can only have J_{-m} and a central term on J_0. Similarly, we cannot mix L_n of different n in defining $L_0^\dagger$. Do we add a central term (necessarily to the definition of $L_0^\dagger$, since only this one transforms as a scalar under L_0)? We already have, since this is fixed by the $[L_m, L_{-m}]$ and $[L_m^\dagger, L_{-m}^\dagger]$ commutations that give the central charge.

44. Noting that

$$\begin{aligned}
b(z)\partial c(w) = c(z)\partial b(w) &= \frac{1}{(z-w)^2} \\
\partial b(z)c(w) = \partial c(z)b(w) &= -\frac{1}{(z-w)^2} \\
\partial b(z)\partial c(w) = \partial c(z)\partial b(w) &= -\frac{2}{(z-w)^3}
\end{aligned}$$

we can just directly compute the TT OPE:

$$\begin{aligned}
T(z)T(w) &= (-\lambda b(z)\partial c(z) + (1-\lambda)\partial b(z)c(z))\,(-\lambda b(w)\partial c(w) + (1-\lambda)\partial b(w)c(w)) \\
&= \lambda^2(b\partial c)(z)(b\partial c)(w) + \lambda(\lambda-1)\left[(b\partial c)(z)(\partial bc)(w) + (\partial bc)(z)(b\partial c)(w)\right] + (1-\lambda)^2(\partial bc)(z)(\partial bc)(w) \\
&= -\frac{\lambda^2 + (1-\lambda)^2 + 4\lambda(\lambda-1)}{(z-w)^4} + \frac{\lambda^2(-b(z)\partial c(w) + \partial c(z)b(w))}{(z-w)^2} + \frac{(1-\lambda)^2(\partial b(z)c(w) - c(z)\partial b(w))}{(z-w)^2} \\
&\quad + \lambda(\lambda-1)\frac{\partial c(z)\partial b(w) + \partial b(z)\partial c(w)}{z-w} - 2\lambda(\lambda-1)\frac{b(z)c(w) + c(z)b(w)}{(z-w)^3}
\end{aligned}$$

The first term on the last line will die since we can take $z \to w$ and ignore first-order terms capturing the differences. The second term in the last line will become:

$$-2\lambda(\lambda-1)\frac{\partial b(w)c(w) + \partial c(w)b(w)}{(z-w)^2} - \lambda(\lambda-1)\frac{\partial^2 b(w)c(w) + \partial^2 c(w)b(w)}{(z-w)} \tag{47}$$

the second two terms in the first line contribute a $(z-w)^{-2}$ term of:

$$\lambda^2(2\partial c(w)b(w)) + (1-\lambda)^2(2\partial b(w)c(w))$$

this will combine with the $(z-w)^{-2}$ terms in (47) to give:

$$2\left[\lambda\partial c(w)b(w) + (1-\lambda)\partial b(w)c(w)\right] = 2T(w)$$

as required. Finally, the $(z-w)^{-1}$ terms all collected give coefficient (dropping the w dependence, as it is understood):

$$\begin{aligned}
&\lambda^2(-\partial b\partial c + \partial^2 cb)) + (1-\lambda)^2(\partial^2 bc - \partial c\partial b) - \lambda(\lambda-1)(\partial^2 bc + \partial^2 cb) \\
&= -\lambda^2\cancel{(\partial b\partial c + \partial c\partial b)} - 2\lambda\partial b\partial bc + 1\partial b\partial c + [\lambda^2 + \lambda(1-\lambda)](\partial^2 cb) + [(1-\lambda)^2 + \lambda(1-\lambda)](\partial^2 bc) \\
&= \lambda\partial^2 cb + (1-\lambda)\partial^2 bc + (1-2\lambda)\partial b\partial c = \partial T
\end{aligned}$$

as required. So altogether we get exactly the stress tensor OPE needed to satisfy the Virasoro algebra with central charge:

$$-2(\lambda^2 + (1-\lambda)^2 + 4\lambda(\lambda-1)) = -2(6\lambda^2 - 6\lambda + 1) = 1 - 3Q^2, \quad Q = (1-2\lambda)$$

45. The BRST current is:

$$j_B(z) = c(z)T^X(z) + (bc\partial c)(z)$$

There are several OPEs to do. Let's start with the easier ones:

$$\begin{aligned}
(cT^X)(cT^X) &\sim c(z)c(w)\left[\frac{c^X/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right] \\
&= -\sum_{n=1}^{\infty}\frac{(z-w)^n}{n!}c(w)\partial^n c(w)\left[\frac{c^X/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right] \\
&\sim -\frac{\frac{1}{2}c^X\, c(w)\partial c(w)}{(z-w)^3} - \frac{1}{2}\frac{\frac{1}{2}c^X\, c(w)\partial^2 c(w)}{(z-w)^2} - \frac{1}{6}\frac{\frac{1}{2}c^X\, c(w)\partial^3 c(w)}{z-w} - \frac{2T(w)c(w)\partial c(w)}{z-w}
\end{aligned} \tag{48}$$

Next:

$$\begin{aligned}
(cT^X)(bc\partial c) + (bc\partial c)(cT^X) &\sim \frac{T^X(z)c(w)\partial(w)}{z-w} + \frac{c(z)\partial c(z)T^X(w)}{z-w} \\
&\sim \frac{2T(w)c(w)\partial c(w)}{z-w}
\end{aligned} \tag{49}$$

This exactly cancels the last term in the previous expression. Now the hard one. Being careful of fermion minus signs, I'll underline the contractions that will give them:

$$\begin{aligned}
(bc\partial c)(bc\partial c) = &(bc\partial c)(bc\partial c) + \underline{(bc\partial c)(bc\partial c)} + \underline{(bc\partial c)(bc\partial c)} + (bc\partial c)(bc\partial c) \\
&+ (bc\partial c)(bc\partial c) + (bc\partial c)(bc\partial c) + \underline{\cancel{(bc\partial c)(bc\partial c)}} + \underline{\cancel{(bc\partial c)(bc\partial c)}}
\end{aligned} \tag{50}$$

the last two terms are canceled because they contribute only $(z-w)^{-1}$ singularities multiplying $c(z)\partial c(w)$ which is $O(z-w)$ and so only contributes finite terms. The remaining terms give:

$$-\frac{c(z)c(w)}{(z-w)^4} + \frac{c(z)\partial c(w)}{(z-w)^3} - \frac{\partial c(z)c(w)}{(z-w)^3} + \frac{\partial c(z)\partial c(w)}{(z-w)^2} + \frac{c(z)\partial c(z)b(w)c(w)}{(z-w)^2} + \frac{b(z)c(z)c(w)\partial c(w)}{(z-w)^2} \tag{51}$$

The last two terms will cancel, as they contribute a $(z-w)^{-1}$ singularity with numerator $c\partial^2 cbc + \partial c\partial cbc + b\partial cc\partial c + \partial bcc\partial c$. All of these terms are evaluated at w, so all are zero. Now we have (all evaluated at w)

$$\begin{aligned}
&\frac{-\partial cc + c\partial c - \partial cc}{(z-w)^3} + \frac{-\frac{1}{2}\partial^2 cc + \partial c\partial c - \partial^2 cc + \partial c\partial c}{(z-w)^2} + \frac{-\frac{1}{6}\partial^3 cc + \frac{1}{2}\partial^2 c\partial c - \frac{1}{2}\partial^3 cc + \partial^2 c\partial c}{z-w} \\
&= \frac{3c(w)\partial c(w)}{(z-w)^3} + \frac{\frac{3}{2}c(w)\partial^2 c(w)}{(z-w)^2} + \frac{\frac{2}{3}c(w)\partial^3 c(w) + \frac{3}{2}\partial^2 c(w)\partial c(w)}{z-w}
\end{aligned} \tag{52}$$

Combining Equations (48), (49) and (52) we get:

$$j_B(z)j_B(w) = \frac{(3 - \frac{1}{2}c^X)c(w)\partial c(w)}{(z-w)^3} + \frac{(\frac{3}{2} - \frac{1}{4}c^X)c(w)\partial^2 c(w)}{(z-w)^2} + \frac{(\frac{2}{3} - \frac{c^X}{12})c(w)\partial^3 c(w) + \frac{3}{2}\partial^2 c(w)\partial c(w)}{z-w} \tag{53}$$

Now for $Q_B^2 = 0$, we need to look at $j_B(z)j_B(w)$ residue as $z \to w$ as a function of w and ensure that this has no residue in w. First we just need to look at the $(z-w)^{-1}$ term and reduce it all to the integral:

$$Q_B^2 = \frac{1}{2\pi i}\oint dw\left[\left(\frac{2}{3} - \frac{c^X}{12}\right)c(w)\partial^3 c(w) + \frac{3}{2}\partial^2 c(w)\partial c(w)\right] = \frac{1}{2\pi i}\oint dw\left(\frac{13}{6} - \frac{c^X}{12}\right)c(w)\partial^3 c(w)$$

This will vanish exactly when $c^X = 26$ as required.

NB in Polchinski, there is an additional $c\partial^3 c$ term in the definition of j_B that contributes to this OPE (which makes Equation (53) look nicer), but the conclusion about $D = 26$ is still the same.

46. This is the type of question with a two-line answer that depends on a lot of conceptual build up. It is instructive to go through some of the details. Here I will set $\ell_s^2 = 2$. First lets start with the system of two Majorana-Weyl fermions ψ^1, ψ^2. This has central charge $c = 1$. Moreover, we can compute everything in terms of

$$\psi = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2), \quad \bar{\psi} = \frac{1}{\sqrt{2}}(\psi^1 - i\psi^2).$$

Note both $\psi(z)$ and $\bar{\psi}(z)$ are in the holomorphic sector. The anti-holomorphic fields, if we considered them, can be labeled as in polchinski by $\tilde{\psi}(\bar{z}), \tilde{\bar{\psi}}(\bar{z})$. These fields give OPE:

$$\psi(z)\psi(w) = O(z-w), \quad \bar{\psi}(z)\bar{\psi}(w) = O(z-w) \quad \psi(z)\bar{\psi}(w) = \frac{1}{z-w} + :\psi\bar{\psi}:(w) + O(z-w) \tag{54}$$

Now $J(z) =: \psi\bar{\psi}:(z)$ can be seen to have scaling dimension 1 by OPE, so it is a conserved current (and necessarily a primary operator in a unitary theory). Indeed $JJ = (z-w)^{-2}$ and $J\psi = \psi(z-w)^{-1}, J\bar{\psi} = \bar{\psi}(z-w)^{-1}$ so $\psi, \bar{\psi}$ have charge ± 1 under J. From extending equation (54) to terms of order $(z-w)$ the stress energy tensor $T = -\frac{1}{2} :\psi^i \partial \psi^i: = \frac{1}{2} J^2$.

Now note that this shares everything in common with the free scalar theory. The central charge $c = 1$. The $\mathfrak{u}(1)$ currents there are $J = i\partial\phi$ and have the same OPE. The analogues of the fermions $\psi, \bar{\psi}$ are then the operators $e^{\pm i\phi(z)}$ respectively. Indeed these have charge ± 1 under J. But it would be surprising if these operators anti-commuted, being built out of bosons and all. In fact they do! By Baker-Campbell-Hausdorff:

$$e^{i\phi(z)} e^{i\phi(z')} = e^{-[\phi(z),\phi(z')]} e^{i\phi(z')} e^{i\phi(z)} = -e^{i\phi(z')} e^{i\phi(z)}$$

since $[\phi(z), \phi(w)] = -\log \frac{z-w}{w-z} = -i\pi$ The anti-commutation property comes out of the non-locality of the vertex operators in terms of ϕ. We can make the exact same argument for $e^{i\phi(z)} e^{-i\phi(w)}$ or any combination thereof. So all of these fields are in fact fermionic. They have the same OPEs as the fermions above:

$$e^{\pm i\phi(z)} e^{\pm i\phi(w)} = O(z-w), \quad e^{\pm i\phi(z)} e^{\mp i\phi(w)} \sim \frac{1}{z-w}$$

Note also the OPE

$$\begin{aligned}
:e^{i\phi(z)}::e^{i\phi(w)}: &= \exp\left[-\int dz' dw' \log(z'-w') \delta_{\phi(z')} \delta_{\phi(w')}\right] :e^{i\phi(z)} e^{-i\phi(w)}: \\
&= \frac{1}{z-w}(1 + i\partial\phi(w)(z-w) + O(z-w)^2) \\
&= \frac{1}{z-w} + i\partial\phi(w) + O(z-w)
\end{aligned}$$

as required.

We can actually perform this procedure to the bc ghosts as well, for any value of λ. The trick is to note that we have performed it for $\lambda = 1/2$, and now the stress-energy tensor changes to:

$$T^\lambda = T^{\lambda=1/2} - (\lambda - 1/2)\partial(:bc:)$$

If we still take $b = e^{i\phi}, c = e^{-i\phi}$ then $:bc: = i\partial\phi$ and so the stress-energy tensor looks like:

$$T^\lambda = -\frac{1}{2}(\partial\phi)^2 - i(\lambda - 1/2)\partial^2\phi$$

which is just the Coloumb gas model with $Q = -i(2\lambda - 1)$. The central charge is $1 + 3Q^2 = 1 - 3(2\lambda - 1)^2$, exactly as we want. The conformal weights are $k^2/2 \pm iQk/2 \to \frac{1}{2} \pm (\lambda - 1/2)$ at the lowest level, and this is exactly λ and $1 - \lambda$ as desired. Note that b and c are hermitian, so we need ϕ to be anti-hermitian. Equivalently we can write $\phi = i\rho$ for ρ hermitian. Then

$$b = e^{-\rho}, \quad c = e^{\rho}, \quad J = -\partial\rho.$$

Note ρ has opposite OPE from ϕ so that $\partial\rho(z)\,\partial\rho(w) \sim \frac{1}{(z-w)^2}$.

Now lets look at the *bosonic* $\beta\gamma$ theory. Can we bosonize this too? For one, the charge is $J = -\beta\gamma$ which has opposite sign OPE $J(z)J(w) = -\frac{1}{z-w}$, so we will now need ρ to have the regular-sign OPE (ie the same as ϕ). We'll just call this hermitian field ϕ. Let's take $\beta = e^{-\phi}, c = e^{\phi}$ as before and $J = -\partial\phi$. Already there is an issue. If ϕ satisfies the standard OPE then β and γ will be anticommute. Further, $\beta\gamma = e^{-\phi(w)}e^{\phi(z)} = O(z-w)$ while by the same logic $\beta\beta \sim \gamma\gamma \sim (z-w)^{-1}$. We want $\beta\beta = O((z-w)^0), \gamma\gamma = O((z-w)^0), \beta\gamma \sim -(z-w)^{-1}, \gamma\beta \sim (z-w)^{-1}$.

Another way to see that we are missing something: we can try to write a Coulomb gas model for the $\beta\gamma$ theory:

$$T^{\lambda} = T^{\lambda=1/2} - (\lambda - 1/2)\partial(\beta\gamma) = -\frac{1}{2}J^2 - \left(\frac{1}{2} - \lambda\right)\partial J = -\frac{1}{2}(\partial\phi)^2 + \frac{1-2\lambda}{2}\partial^2\phi$$

notice the $-$ sign in front of $\frac{1}{2}J^2$, as we want. We have a coulomb gas model with $Q = 1 - 2\lambda$. This gives a central charge $1 + 3Q^2 = 4 - 6\lambda + 12\lambda^2$. On the other hand, the $\beta\gamma$ theory should have central charge $-1 + 3Q^2$. We are off by 2.

All of this indicates that we need to add an uncoupled $c = -2$ including fermions–namely the bc fermi theory at $\lambda = 1$– and redefine $\beta\gamma$ in terms of ϕ to incorporate this. Take η, ξ of scaling dimensions $1, 0$ and charges ∓ 1 respectively. Then define

$$\beta = e^{-\phi}\partial\xi, \quad \gamma e^{\phi}\eta.$$

We now have the OPE:

$$\beta(z)\gamma(w) = (z-w) \times -\frac{1}{(z-w)^2} = -\frac{1}{z-w}, \quad \gamma(z)\beta(w) = \frac{1}{z-w}$$

This is **4.15.2**. Further because $\eta\eta = O(z-w)$ and $\partial\xi\partial\xi = O(z-w)$ we get $\beta\beta = O((z-w)^0)$ and likewise for $\gamma\gamma$ as needed. We also know how to interpolate between NS and R sectors by taking $\phi \to \phi/2$ etc.

The total current $- : \beta\gamma :$ stays the same because we look for the constant term in the expansion:

$$\beta(z)\gamma(w) = -\frac{1}{(z-w)^2}e^{-\phi(z)}e^{\phi(w)} = -\frac{1}{(z-w)^2}\left((z-w) - \partial\phi(w)(z-w)^2\right) \to \partial\phi(w) \Rightarrow J = -\partial\phi(w)$$

so we identify $: \beta\gamma :$ with $\partial\phi$, which are both $-J$. This is **14.15.10**. Writing out the full stress tensor now gives:

$$-\frac{1}{2}(\partial\phi)^2 + \frac{1-2\lambda}{2}\partial^2\phi - \eta\partial\xi = T^{\lambda=1/2} + (1/2 - \lambda)\partial(\beta\gamma)$$

It remains to show that $T^{\lambda=1/2} = -\frac{1}{2}\beta\partial\gamma + \frac{1}{2}\partial\beta\gamma = \frac{1}{2}(2\partial\beta\,\gamma - \partial(\beta\gamma))$. Now looking at the $\beta\gamma$ OPE to order $z - w$ we get:

$$\begin{aligned} e^{-\phi(z)}\partial\xi(z)e^{\phi(w)}\eta(w) &= \partial\xi(z)\eta(w)e^{-\phi(z)}e^{\phi(w)} \\ &= \left(\frac{-1}{(z-w)^2} + :\partial\xi\eta:\right)\left((z-w) - (z-w)^2\partial\phi + \frac{1}{2}(z-w)^3((\partial\phi)^2 - \partial^2\phi)\right) \end{aligned}$$

NOTE I had to assume that while ξ, η and $e^{\phi}, e^{-\phi}$ separately anticommute with their partners, the $e^{\pm\phi}$ fields commute with the ξ, η fields. Give an interpretation/example in condensed matter of this.

The order $z - w$ term here is:

$$:\partial\xi\eta: -\frac{1}{2}((\partial\phi)^2 - \partial^2\phi)$$

So this is the normal ordered product of $\partial\beta\,\gamma$. The $\partial(\beta\gamma) = \partial^2\phi$ term will cancel the $\partial^2\phi$ term there and we'll get the stress tensor

$$-\eta\partial\xi - \frac{1}{2}(\partial\phi)^2 = T^{\lambda=1/2}$$

which is **4.15.8** as desired.

We can also bosonize the η, ξ theory in terms of an auxiliary bosonic field χ, but this was not necessary for the exercise.

47. Let us do this directly from definitions:

$$X(\tau,\sigma) = x - \sqrt{2}\ell_s \sum_{k\in\mathbb{Z}+1/2} \frac{\alpha_k}{k} e^{-ik\tau}\sin(k\sigma) = x + i\frac{\ell_s}{\sqrt{2}} \sum_{k\in\mathbb{Z}+1/2} \frac{\alpha_k}{k}(z^{-k} - \bar{z}^{-k})$$

$$\Rightarrow \langle X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{\ell_s^2}{2} \sum_{k,l\in\mathbb{Z}+1/2} \frac{\alpha_k\alpha_l}{kl}(z^{-k} - \bar{z}^{-k})(w^{-l} - \bar{w}^{-l})$$

$$= \frac{\ell_s^2}{2}\sum_{k=0}^{\infty} \frac{1}{k+1/2}\left[\left(\frac{w}{z}\right)^{k+1/2} - \left(\frac{\bar{w}}{z}\right)^{k+1/2} - \left(\frac{w}{\bar{z}}\right)^{k+1/2} + \left(\frac{\bar{w}}{\bar{z}}\right)^{k+1/2}\right]$$

Now we have

$$\sum_{k=0}^{\infty} \frac{x^{k+1/2}}{k+1/2} = 2\sum_{k=0}^{\infty}\frac{(\sqrt{x})^{2k+1}}{2k+1} = 2\,\mathrm{arctanh}(\sqrt{x}) = -(\log(1-\sqrt{x}) - \log(1+\sqrt{x})).$$

Our convention on the square root branch cut is along the negative real axis. We get:

$$-\frac{\ell_s^2}{2}\left[\log(1-\sqrt{w/z}) - \log(1+\sqrt{w/z}) - \log(1-\sqrt{\bar{w}/z}) + \log(1-\sqrt{\bar{w}/z}) + c.c.\right]$$

so the final result gives us:

$$-\frac{\ell_s^2}{2}\left[\log|1-\sqrt{w/z}|^2 - \log|1+\sqrt{w/z}|^2 - \log|1-\sqrt{\bar{w}/z}|^2 + \log|1+\sqrt{\bar{w}/z}|^2\right].$$

we can add and subtract $\log z$ to get:

$$-\frac{\ell_s^2}{2}\left[\log|\sqrt{z}-\sqrt{w}|^2 - \log|\sqrt{z}+\sqrt{w}|^2 - \log|\sqrt{z}-\sqrt{\bar{w}}|^2 + \log|\sqrt{z}+\sqrt{\bar{w}}|^2\right].$$

Interpret this in terms of image charges

48. Firstly, $\partial X\bar{\partial}X$ requires no normal ordering constant to be added ordinarily, since it has a wick contraction of zero. Now to go from the plane from the disk we have $x = \frac{z-i}{z+i}$. Vice versa is $z = i\frac{1+x}{1-x}$. This gives

$$\log|z-w|^2 = \log|x-y|^2 + \log|\frac{2}{(1-x)(1-y)}|^2$$
$$\log|z-\bar{w}|^2 = \log|1-x\bar{y}|^2 + \log|\frac{2}{(1-x)(1-\bar{y})}|^2$$

So for NN and DD boundary conditions we get:

$$\langle X_{NN}(x,\bar{x})X_{NN}(y,\bar{y})\rangle = -\frac{\ell_s^2}{2}\left(\log|x-y|^2 + \log|1-x\bar{y}|^2 - 2\log|(1-x)(1-y)|^2 + 4\log 2\right)$$
$$\langle X_{DD}(x,\bar{x})X_{DD}(y,\bar{y})\rangle = -\frac{\ell_s^2}{2}\left(\log|x-y|^2 - \log|1-x\bar{y}|^2\right).$$

So NN boundary conditions correspond to putting an image charge of the same sign at $1/x^*$ while DD boundary conditions correspond to putting an image charge of opposite sign at $1/x^*$ as well as a *neutralizing* charge of the opposite sign at 1–corresponding to ∞ in the $\mathbb{H}$ setting. **Interpret this**.

Differentiating the above with $\partial_x\bar{\partial}_y$ shows that in either case only the $\log(1-x\bar{y})$ term contributes:

$$\langle \partial X_{NN}(x)\bar{\partial}X_{NN}(\bar{y})\rangle = \frac{\ell_s^2}{2}\frac{1}{(1-x\bar{y})^2}$$
$$\langle \partial X_{DD}(x)\bar{\partial}X_{DD}(\bar{y})\rangle = -\frac{\ell_s^2}{2}\frac{1}{(1-x\bar{y})^2}.$$

This will become singular only as z approaches the boundary of the unit circle. We encounter the divergence $\pm\frac{\ell_s^2}{2}\frac{1}{(1-x\bar{y})^2}$ in the NN and DD cases respectively and so we can define

$$
{}^{\star}_{\star}\partial X(z)\bar{\partial}X(\bar{w})^{\star}_{\star} = \partial X(z)\bar{\partial}X(\bar{w}) \mp \frac{\ell_s^2}{2}\frac{1}{(1-z\bar{w})^2}
$$

On the other hand for $\partial X\,\partial X$ we get the normal ordering constant:

$$
{}^{\star}_{\star}\partial X(z)\partial X(w)^{\star}_{\star} = \partial X(z)\partial X(w) + \frac{\ell_s^2}{2}\frac{1}{(z-w)^2}
$$

We have $\bar{X}(1/\bar{w}) = \pm X(w)$ so consequently $\partial X(w) = \pm\bar{\partial}_{1/\bar{w}}X(1/\bar{w})$. Now its a quick check (being careful to keep subscripts on $\bar{\partial}$ so we know what we're differentiating w.r.t.):

$$
\begin{aligned}
{}^{\star}_{\star}\partial X(z)\bar{\partial}_{\bar{w}}X(1/\bar{w})^{\star}_{\star} &= \partial X(z)\bar{\partial}_{\bar{w}}X(1/\bar{w}) \mp \frac{\ell_s^2}{2}\frac{1}{(1-z/\bar{w})^2} \\
\Rightarrow {}^{\star}_{\star}\partial X(z)\bar{\partial}_{1/\bar{w}}X(1/\bar{w})^{\star}_{\star} &= \partial X(z)\bar{\partial}_{1/\bar{w}}X(1/\bar{w}) \mp (-\bar{w}^{-2})\frac{\ell_s^2}{2}\frac{1}{(1-z/\bar{w})^2} \\
\Rightarrow {}^{\star}_{\star}\partial X(z)\partial X(w)^{\star}_{\star} &= \partial X(z)\partial X(w) + \frac{\ell_s^2}{2}\frac{1}{(z-w)^2}
\end{aligned}
$$

where the extra minus sign in the Dirichlet boundary condition case removes any sign ambiguity in the last line. Thus, we see that indeed ${}^{\star}_{\star}\partial X(z)\partial X(w)^{\star}_{\star} = \pm\,{}^{\star}_{\star}\partial X(z)\bar{\partial}X(1/\bar{w})^{\star}_{\star}$ for Neumann and Dirichlet boundary conditions respectively.

49. We do this by Wick contraction:

$$
\langle\prod_{i=1}^{m}\psi(z_i)\prod_{j=1}^{2n-m}\bar{\psi}(\bar{z}_j)\rangle = \dots
$$

Unsure what I learn from the combinatorics.

50. I feel that this has already been done in 2.3.31. Rotating to euclidean signature, the most general solution for X is

$$
X(\tau,\sigma) = x^\mu + \frac{\ell_s^2}{2}(p+\bar{p})\tau + \frac{\ell_s^2}{2}(p-\bar{p})\sigma + i\frac{\ell_s}{\sqrt{2}}\sum_{k\neq 0}\frac{e^{-k\tau}}{k}(\alpha_k e^{-ik\sigma} + \bar{\alpha}_k e^{ik\sigma})
$$

The first boundary condition $\dot{X} = 0$ at $\sigma = 0$ gives:

$$
\alpha_k = -\bar{\alpha}_k, \quad p+\bar{p} = 0
$$

while the second boundary condition $X' = 0$ at $\sigma = \pi$ gives:

$$
\sin(k\pi) = 0 \Rightarrow k \in \mathbb{Z}+1/2 \quad p-\bar{p} = 0
$$

Thus we have neither momentum nor winding-number. So for the mode expansion is:

$$
X(\tau,\sigma) = x - \sqrt{2}\ell_s\sum_{k\in\mathbb{Z}+1/2}\frac{\alpha_k}{k}e^{-k\tau}\sin(k\sigma) = x + i\frac{\ell_s}{\sqrt{2}}\sum_{k\in\mathbb{Z}+1/2}\frac{\alpha_k}{k}(z^{-k} - \bar{z}^{-k})
$$

as desired. This gives:

$$
\partial X = -i\frac{\ell_s}{\sqrt{2}}\sum_{k\in\mathbb{Z}+1/2}\alpha_k z^{-k-1}, \quad \bar{\partial}X = i\frac{\ell_s}{\sqrt{2}}\sum_{k\in\mathbb{Z}+1/2}\alpha_k\bar{z}^{-k-1}
$$

51. We have N scalars with $\partial X^i(z) = O^{ij}\bar{\partial}X^j(\bar{z})$ on the real axis. Because the conformal group includes the translation group, O^{ij} must be translationally invariant, ie it cannot depend on z. Further because X^i is a scalar $\partial + \bar{\partial}$ and $\partial - \bar{\partial}$ both act on it in an invariant way. These are the two boundary conditions we can set on each X^i. So we see that O^{ij} can definitely be a diagonal matrix of ± 1s. However, because all the scalars are identical we can also transform $X'^j(z,\bar{z}) = R_i^j X^i(z,\bar{z})$, with R any orthogonal matrix (not just special orthogonal) and still get a valid boundary condition. So O is any orbit of the matrix of ± 1s under the conjugation action of the orthogonal group $O \to P^T O P$. This can be easily appreciated as boundary conditions for an open string along the various coordinate directions being either Neumann or Dirichlet.

Its surprising that O can't vary on the real axis - corresponding to the D-brane changing which X^i live on it. Think about this more.

52. Everything is in the NS sector. Let's first evaluate $\langle \psi_{NN}(z)\psi_{NN}(w)\rangle$. We have

$$\sum_{n,m} \underbrace{\langle 0|\, b_{n+1/2} b_{m+1/2}\, |0\rangle}_{\delta_{n=-m-1}} z^{-n-1} w^{-m-1} = \sum_{n=0}^{\infty} z^{-n-1} = \frac{1}{z-w}$$

For the NS sector we have the following cases:

- NN: $b_{n+1/2} + \bar{b}_{n+1/2} = 0$
- DD: $b_{n+1/2} - \bar{b}_{n+1/2} = 0$
- DN: $b_n + \bar{b}_n = 0$

so we see that $\langle \psi(z)\bar{\psi}(\bar{w})\rangle$ will add an extra minus sign in the NN case. It will not do so in the in the DD case. Collecting our results.

$$\langle \psi_{NN}(z)\psi_{NN}(w)\rangle = \frac{1}{z-w}, \quad \langle \psi_{NN}(z)\bar{\psi}_{NN}(\bar{w})\rangle = -\frac{1}{z-\bar{w}}$$
$$\langle \psi_{DD}(z)\psi_{DD}(w)\rangle = \frac{1}{z-w}, \quad \langle \psi_{DD}(z)\bar{\psi}_{DD}(\bar{w})\rangle = \frac{1}{z-\bar{w}}$$

Lastly, for the DN case, ψ now takes integer values and so:

$$\langle \psi_{DN}(z)\psi_{DN}(w)\rangle = \sum_{n,m} \underbrace{\langle 0|\, b_n b_m\, |0\rangle}_{\delta_{n=-m}} z^{-n-1/2} w^{-m-1/2} = \sum_{n=0}^{\infty} z^{-n-1/2} w^{n-1/2} - \underbrace{\frac{1}{2}}_{\text{zero mode}} z^{-1/2} w^{-1/2} = \frac{z+w}{2\sqrt{zw}(z-w)}.$$

Because $b_n = -\bar{b}_n$ we then also have

$$\langle \psi_{DN}(z)\bar{\psi}_{DN}(\bar{w})\rangle = -\frac{z+\bar{w}}{2\sqrt{z\bar{w}}(z-\bar{w})}.$$

53. On to the R sector.

- NN: $b_n - \bar{b}_n = 0$
- DD: $b_n + \bar{b}_n = 0$
- DN: $b_{n+1/2} - \bar{b}_{n+1/2} = 0$

Let's again evaluate $\langle \psi_{NN}(z)\psi_{NN}(w)\rangle$. The calculation is exactly the same as the DN calculation above. Using the above relations between the b and $\bar{b}$ in the different sectors we'll get:

$$\langle \psi_{NN}(z)\psi_{NN}(w)\rangle = \frac{z+w}{2\sqrt{zw}(z-w)}, \quad \langle \psi_{NN}(z)\bar{\psi}_{NN}(\bar{w})\rangle = \frac{z+\bar{w}}{2\sqrt{z\bar{w}}(z-\bar{w})}$$
$$\langle \psi_{DD}(z)\psi_{DD}(w)\rangle = \frac{z+w}{2\sqrt{zw}(z-w)}, \quad \langle \psi_{DD}(z)\bar{\psi}_{DD}(\bar{w})\rangle = -\frac{z+\bar{w}}{2\sqrt{z\bar{w}}(z-\bar{w})}$$
$$\langle \psi_{DN}(z)\psi_{DN}(w)\rangle = \frac{1}{z-w}, \quad \langle \psi_{DN}(z)\bar{\psi}_{DN}(\bar{w})\rangle = \frac{1}{z-\bar{w}}$$

54. There are several ways to do this. One way is directly by using the identity relating an expectation of an exponential to the exponential of an expectation:

$$\langle e^{iaX(z)}\rangle_{\mathbb{RP}^2} = \langle e^{iaX(z)}e^{-ia\bar{X}(\bar{z})}\rangle_{\mathbb{CP}^1} \propto \exp\left(\frac{a^2}{2}\times 2\langle X(z)\bar{X}(\bar{z})\rangle\right) = \exp\left(-\frac{a^2\ell_s^2}{2}\log(1+z\bar{z})\right) = \frac{1}{(1+|z|^2)^{a^2\ell_s^2/2}}.$$

It is not clear that we haven't omitted a proportionality constant. Another way to compute this is to note that $\langle : X(z,\bar{z})X(z,\bar{z}) :\rangle = -\frac{\ell_s^2}{2}\log|1+z\bar{z}|^2$ and so expanding out:

$$e^{iaX} = \sum_{n=0}^{\infty}\frac{(ia)^n}{n!}\langle X(z,\bar{z})^n\rangle.$$

Now we do wick contractions. For each even term we need to put $2n$ elements in to n pairs. There are $(2n-1)(2n-3)\ldots(3)(1)$ ways to do this. Simplifying we get:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(a)^{2n}}{2^n n!}\left(-\frac{\ell_s^2}{2}\right)^n\log^n|1+z\bar{z}|^2 = \exp\left(\log|1+z\bar{z}|^{a^2\ell_s^2/2}\right) = (1+|z|^2)^{a^2\ell_s^2/2}$$

This doesn't look right. If instead we had:

$$e^{iaX(z)}e^{-ia\bar{X}(\bar{z})} = \sum_{n,m=0}^{\infty}\frac{(ia)^n}{n!}\frac{(-ia)^m}{m!}\langle : X(z)^n\bar{X}(\bar{z})^m :\rangle = \sum_n\frac{a^{2n}\cancel{n!}}{\cancel{n!}n!}\left(-\frac{\ell_s^2}{2}\log(1+z\bar{z})\right)^n = \frac{1}{(1+|z|^2)^{a^2\ell_s^2/2}}$$

as required.

In doing this problem, I needed to consider the $e^{iaX}e^{-ia\bar{X}}$ correlator rather than the $e^{ia(X+\bar{X})}$ correlator - otherwise I would get an ill-defined one-point function that blows up as $z\to\infty$ (ie is not a globally-defined differential). Perhaps this comes from boundary conditions in the case of $\mathbb{RP}^2$, since $H_1 = \mathbb{Z}_{\cancel{2}}$ and so we can enforce anti-periodic boundary conditions that would be consistent with a negative charge vertex operator being placed a $-1/\bar{z}$.

55. For the non-supersymmetric theory, we have the action (on the sphere, with $\sqrt{-g}R^2 = 1$):

$$S = \frac{1}{4\pi\ell_s^2}\int d^2z\sqrt{g}g^{\alpha\beta}\partial_\alpha X\partial_\beta X + \frac{Q}{4\pi\ell_s\sqrt{2}}\int d^2z\sqrt{g}R^{(2)}X = \frac{1}{2\pi\ell_s^2}\int d^2z\partial X\bar{\partial}X + \frac{Q}{4\pi\ell_s\sqrt{2}}\int d^2z\, X$$

this gives a stress-energy tensor:

$$T = -\frac{1}{\ell_s^2}\partial X^2 + \frac{Q}{\ell_s\sqrt{2}}\partial^2 X$$

Now for $\mathcal{N}=1$ we might expect an action of the form:

$$S = \frac{1}{4\pi\ell_s^2}\int d^2z\sqrt{g}g^{\alpha\beta}\partial_\alpha X\partial_\beta X + \frac{Q}{4\pi\ell_s\sqrt{2}}\int d^2z\sqrt{g}R^{(2)}X = \frac{1}{2\pi\ell_s^2}\int d^2z\partial X\bar{\partial}X + \frac{Q}{4\pi\ell_s\sqrt{2}}\int d^2z\, X$$

This gives:

$$T = -\frac{1}{\ell_s^2}\partial X\partial X + \frac{Q}{\ell_s\sqrt{2}}\partial^2 X - \frac{1}{2}\psi\partial\psi, \quad G = i\frac{\sqrt{2}}{\ell_s}\psi\partial X - iQ\partial\psi$$

The TT OPE will give central charge $\frac{3}{2}+3Q^2$. G remains primary, so we'll have $TG = \frac{3}{2}\frac{G(w)}{(z-w)^2}+\frac{\partial G(w)}{z-w}$. Finally, GG will give

$$\frac{1}{(z-w)^3}+\frac{2Q^2}{(z-w)^3}+\cancel{\frac{\frac{\sqrt{2}}{\ell_s}Q\partial X-\frac{\sqrt{2}}{\ell_s}Q\partial X}{(z-w)^2}}+\frac{2T}{z-w}$$

so we get $\hat{c} = 1+2Q^2$ as desired.

Now for $\mathcal{N}=2$, following the same example, we still get get same TT OPE and $G^\pm$ remains primary, so we have the $TG^\pm$ OPE staying the same. The GG OPE will have $\hat{c}=1+2Q^2$ as before and J will have to be modified to include $\partial^2 X$ so as to remain primary under T.

56. For X a compact scalar valued in S^1 of radius R we have the solutions $X = 2\pi R(n\sigma_1 + m\sigma_2)$, which have vanishing Laplacian. The action of these instanton solutions is:

$$S = \frac{1}{4\pi\ell_s^2}\int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \frac{1}{\tau_2}|\tau\partial_1 X - \partial_2 X|^2 = \frac{\pi R^2}{\ell_s^2\tau_2}|n\tau - m|^2$$

Expanding $X = X^{cl} + \chi$, we get no cross-terms in the action. We now do the path integral over the χ with periodic conditions around both cycles. χ separates into the zero mode $\chi_0 + \delta\chi$ and $\delta\chi$ can be expanded in terms of eigenfunctions of the laplacian on periodic functions. These are precisely $e^{2\pi i(m_1\sigma_1 + m_2\sigma_2)}$ with eigenvalues $\frac{(2\pi)^2}{\tau_2}|m_1\tau - m_2|^2$. They form an orthonormal basis. The contribution to the action is then

$$\frac{1}{4\pi\ell_s^2}\sum_{m_1,m_2\in\mathbb{Z}^2}\lambda_{m_1m_2}|A_{m_1m_2}|^2$$

The measure on the space of functions comes from the norm of δX

$$||\delta_X||^2 = \frac{1}{\ell_s}\int d^2\sigma\sqrt{g}(d\chi) = \sum_{m_1,m_2}\frac{|A_{m_1m_2}|^2}{\ell_s^2} \Rightarrow \int \mathcal{D}\chi = \int_0^{2\pi R}\frac{d\chi_0}{\ell_s}\int_{-\infty}^{\infty}\prod_{m_1,m_2\neq\{0,0\}}\frac{dA_{m_1,m_2}}{\ell_s}.$$

Note the difference with Kiritsis. This is crucial to get the right factors of 2π in the end. This then gives:

$$\int\mathcal{D}\chi e^{-S(\chi)} = \frac{2\pi R}{\ell_s}\times\prod_{m_1,m_2\in\mathbb{Z}^2_{\geqslant 0}\backslash\{0,0\}}\int_{-\infty}^{\infty} dA_{m,n}\frac{e^{-\frac{\lambda_{m_1m_2}|A_{m_1m_2}|^2}{4\pi\ell_s^2}}}{(2\pi\ell_s)^2} = \frac{2\pi R}{\ell_s}\times\prod_{m,n}{}'\sqrt{\frac{2\pi}{\lambda_{m_1m_2}}} = \frac{2\pi R}{\ell_s}\times(\det{}'\frac{\nabla^2}{2\pi})^{-1/2}$$

Henceforth a primed sum or product means that we omit the origin 0 or $\{0,0\}$ and sum over the integers. It remains to evaluate

$$\prod\sqrt{\frac{2\pi}{\lambda_{n,m}}} = \exp\left(-\frac{1}{2}\sum_{m,n}{}'\log\left(\frac{2\pi}{\tau_2}|m+n\tau|^2\right)\right)$$

Notice that this sum can be obtained by explicitly calculating the Eisenstein series

$$G(s) = \left(\frac{\tau_2}{2\pi}\right)^s\sum_{m,n}{}'\frac{1}{|m+n\tau|^{2s}}$$

and evaluating $\frac{1}{2}G'(0)$. Let's do that. First note:

$$\sum_{m,n}{}'\frac{1}{|m+n\tau|^{2s}} = 2\zeta(2s) + \sum_n{}'\sum_m\frac{1}{|m+n\tau|^{2s}}$$

The derivative of $2\zeta(2s)$ at $s=0$ yields $-2\log(2\pi)$. On the other hand $2\zeta(0)$ is -1, which multiplies the order s factor in the expansion of $\left(\frac{\tau_2}{2\pi}\right)^s$ (none of the subsequent terms will have an $O(s^0)$ term to multiply this). This gives $\log(2\pi/\tau_2)$. Together these contribute

$$-\frac{1}{2}\log(2\pi\tau_2)$$

to $\frac{1}{2}G'(0)$.

Note also because this is a periodic function of τ of period one, we can represent it as a Fourier series in τ

$$\sum_m\frac{1}{|m+n\tau|^{2s}} = \sum_{p\in\mathbb{Z}}e^{2\pi ipn\tau_1}\int_0^1 dte^{-2\pi ipt}\sum_{m\in\mathbb{Z}}\frac{1}{((m+t)^2+n^2\tau_2^2)^s} = \sum_{p\in\mathbb{Z}}e^{2\pi ipn\tau_1}\underbrace{\int_{-\infty}^{\infty}dt\frac{1}{(t^2+n^2\tau_2^2)^s}}_{\text{combine }\int_0^1\text{ with }\sum_{\mathbb{Z}}}$$

Using a clever Gamma function manipulation (following Di Francesco here):

$$\frac{1}{\Gamma(s)}\sum_p\int_{-\infty}^{\infty}dt\int_0^{\infty}dx\, e^{2\pi ip(n\tau_1-t)}x^{s-1}e^{-x(t^2+n^2\tau_2^2)} = \frac{\sqrt{\pi}}{\Gamma(s)}\sum_p\int_0^{\infty}dx\, x^{s-3/2}\, e^{-xn^2\tau_2^2-\pi^2p^2/x+2\pi ipn\tau_1}.$$

Now at $p = 0$ this reduces to

$$\frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}|n\tau_2|^{1-2s}$$

Summing *this* over n gives $2\frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)$. We have explicit series formulae for these at $s = 0$. Extracting the first-order term (this is in fact finite at $s = 0$) gives $\frac{\pi\tau_2}{3}$.

Now let's evaluate the sum over $p \neq 0$. I'll directly take $s = 3/2$ here. We get a sum over an integral that is now solvable:

$$\frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}\sum_{p>0} e^{-2\pi ipn\tau_1}\int_0^\infty x^{-3/2}e^{-xn^2\tau_2-\pi^2p^2/x} = \sqrt{\pi}s\sum_{p>0}\frac{\sqrt{\pi}}{\pi p}(e^{-2\pi ipn(\tau_1+i\tau_2)}+e^{-2\pi ipn(\tau_1-i\tau_2)})$$

We see that the contribution to $G'(0)$ from this will be:

$$\underbrace{2\sum_{n>0}}_{=\sum_n{}'}\sum_p\frac{1}{p}(q+\bar{q}) = -2\sum_{n>0}\log(|1-q^n|^2) = -2\log(e^{\frac{\pi\tau_2}{6}}|\eta(\tau)|^2) = -2\log(|\eta(\tau)|^2) - \frac{\pi\tau_2}{3}$$

we see that the $p = 0$ term cancels this last part and we are left with $\frac{1}{2}G'(0) = -\log(\sqrt{\tau}_2 2\pi) - \log(|\eta|^2)$, and so:

$$Z(R,\tau) = \frac{R}{\ell_s\sqrt{\tau_2}|\eta(\tau)|^2}\times\sum_{m,n}e^{-\frac{\pi R^2}{\tau_2\ell_s^2}|m-n\tau|^2}.$$

While we're at it, let's simplify this even further by applying Poisson summation. We have the 1D case for the Gaussian:

$$\sum_n e^{-\pi an^2+\pi bn} = \frac{1}{\sqrt{a}}\sum_{\tilde{n}\in\mathbb{Z}}e^{-\frac{\pi}{a}(n+i\frac{b}{2})^2}.$$

Performing this over the m variable we get

$$\begin{aligned}
\sum_{m,n}e^{-\frac{\pi R^2}{\ell_s^2\tau_2}(m^2-m\overbrace{(n\tau+n\bar{\tau})}^{2n\tau_1}+n^2|\tau|^2)} &= \frac{\ell_s\sqrt{\tau_2}}{R}\sum_{\tilde{m},n}e^{-\frac{\pi R^2}{\ell_s^2\tau_2}n^2|\tau|^2}e^{-\frac{\pi\ell_s^2\tau_2}{R^2}\left(\tilde{m}+i\frac{R^2n\tau_1}{\ell_s\tau_2}\right)^2}\\
&= \frac{\ell_s\sqrt{\tau_2}}{R}\sum_{\tilde{m},n}e^{-\pi\frac{R^2}{\ell_s^2}n^2\tau_2-\frac{\pi\ell_s^2}{R^2}\tilde{m}^2\tau_2-2\pi i\tilde{m}n\tau_1}\\
&= \frac{\ell_s\sqrt{\tau_2}}{R}\sum_{\tilde{m},n}e^{\pi(i\tau_1-\tau_2)\frac{1}{2}\left(\frac{\ell_s}{R}\tilde{m}+\frac{R}{\ell_s}n\right)^2}e^{\pi(-i\tau_1-\tau_2)\left(\frac{\ell_s}{R}\tilde{m}-\frac{R}{\ell_s}n\right)^2}\\
&= \frac{\ell_s\sqrt{\tau_2}}{R}\sum_{\tilde{m},n}q^{\frac{P_L^2}{2}}\bar{q}^{\frac{P_R^2}{2}}
\end{aligned}$$

with $P_L = \frac{1}{\sqrt{2}}(m\ell_s/R + nR/\ell_s), P_R = \frac{1}{\sqrt{2}}(m\ell_s/R - nR/\ell_s)$. We then get a simple form for the partition function:

$$Z(R,\tau) = \sum_{\tilde{m},n}\frac{q^{\frac{P_L^2}{2}}\bar{q}^{\frac{P_R^2}{2}}}{|\eta(\tau)|^2}.$$

57. We follow Polchinski Vol 2 on advanced CFT. The following operator product arises when we calculate correlation functions of the energy-momentum tensor:

$$-T\mathcal{O} = -T_zz(z,\bar{z})\,g\int d^2w\phi_{\Delta,\Delta}(w,\bar{w})$$

We get:

$$\bar{\partial}_{\bar{z}}T(z,\bar{z})\phi(w,\bar{w}) = \bar{\partial}_{\bar{z}}\left[\frac{\Delta}{(z-w)^2}+\frac{\partial_w}{z-w}\right]\phi(w,\bar{w}) = (-2\pi\Delta\partial_z\delta(z-w)+2\pi\delta(z-w)\partial_w)\phi(w,\bar{w})$$

Where the last line was obtained using basic delta-function identities. Integrating over w gives:

$$-\bar{\partial}_{\bar{z}} T\mathcal{O} = 2\pi g(\Delta - 1)\partial\phi$$

Thus, unless $\Delta = 1$ we get that T gains an anti-holomorphic part. The exact same equation (with $z \to \bar{z}$) holds for $\bar{T}$. Further, the conservation equation $\bar{d}T_{zz} + \partial T_{z\bar{z}} = 0$ gives us that

$$T_{z\bar{z}} = 2\pi g(1 - \Delta)\phi.$$

There cannot be an overall constant, since this is zero when $\phi = 0$. Here we will *define* $\beta(g)$ by:

$$T^i_i(z, \bar{z}) = -2\pi \sum_i{}' \beta(g)\mathcal{O}_i(z, \bar{z})$$

where the sum runs over operators of dimension $\leqslant d$. The trace is $T^a_a = 2T_{z\bar{z}} = -4\pi g(1-\Delta)\phi$ so under this deformation $\beta = (2 - 2\Delta)g$. We now want to go to second order. The next contribution will come from:

$$-T\,\frac{1}{2}(\mathcal{O}\mathcal{O})^2 = -T_{zz}(z,\bar{)}\frac{g^2}{2!}\int d^2w d^2w' \phi(w, \bar{w})\phi(w', \bar{w}')$$

Doing an OPE we get to leading order:

$$\phi_{\Delta,\Delta}(w, \bar{w})\phi_{\Delta,\Delta}(w', \bar{w}') \sim \frac{C}{|z - w|^{2\Delta}}\phi_{\Delta,\Delta}(w', \bar{w}')$$

where here C is the coefficient of the $\phi_{\Delta,\Delta}$ 3-point function. We can now preform the w, w' integrals and get:

$$2\pi C g^2 \int \frac{dr}{r^{2\Delta-1}} \times \int dw' \phi(w', \bar{w}')$$

Assuming $\Delta = 1$ we get a log term that must be regulated in the UV and IR. Regulation in the UV gives a scale that breaks conformal invariance. Rescaling by $1 + \epsilon$ increases the log by ϵ. Equivalently we get

$$\delta g = -2\pi C\epsilon g^2$$

This gives a second-order contribution to the beta function of Cg^2 as required. If the operator is not exactly marginal, the second order term will still have this form, plus higher-order corrections in $\Delta - 1$ and g.

58. Generalizing the preceding analysis to a deformation by a family of marginal operators $g_a\phi^a_{1,1}$, for the deformation to be marginal at second order in g we need the three-point function to satisfy $\lambda^c_{ab} g_a g_b = 0$ so that the second order term does not contribute the $1/r$ integral and thus does not break conformal invariance. In this case that means that we require

$$\lambda^c_{ab}\, g_{a\bar{a}}\, g_{b\bar{b}} = 0.$$

59. Again, we work from the same chapter of Polchinski. For a general 2D QFT with a stress tensor, we can define the quantities

$$F(r^2) = z^4 \langle T_{zz}(z, \bar{z})T_{zz}(0, 0)\rangle$$
$$G(r^2) = 4z^3\bar{z} \langle T_{zz}(z, \bar{z})T_{z\bar{z}}(0, 0)\rangle$$
$$H(r^2) = 16z^2\bar{z}^2 \langle T_{z\bar{z}}(z, \bar{z})T_{z\bar{z}}(0, 0)\rangle$$

From rotational invariance, these can only depend on $r^2 = |z|^2$. The conservation law $\bar{\partial}T_{zz} + \partial T_{z\bar{z}} = 0$ gives us that:

$$4\dot{F} + \dot{G} - 3G = 0, \quad 4\dot{G} - 4G + \dot{H} - 2H = 0$$

where $\dot{F}, \dot{G}$ indicates the operator $\frac{1}{2}r\partial_r$ (ie differentiation wrt $\log r^2$). Note subtracting 3/4 of the second one from the first gives:

$$4\dot{F} - 2\dot{G} - \frac{3}{4}\dot{H} = -\frac{3}{2}H$$

Define $C = 2F - G - \frac{3}{8}H$. Note that in a CFT, where $G = H = 0$, C is exactly the central charge c. Further, from this definition we get that in the general setting $\dot{C} = -\frac{3}{4}H$. But note that an *exactly* marginal perturbation does not give the stress-energy tensor a trace, so $\dot{C} = 0$ and the central charge will remain fixed.

This technology wasn't developed in Kiritsis. I'm unsure how he would have wanted us to show this.

60. Note under $\tau \to \tau + 1$ the η function is invariant and we our constraint comes from:

$$\frac{1}{2}(P_L^2 - P_R^2) \in \mathbb{Z} \Rightarrow G^{ij} m_j G_{ik} G^{kl} n_l = m_k n^k \in \mathbb{Z}$$

as required. So in particular we have $P_L^2 - P_R^2 \in 2\mathbb{Z}$. We can interpret (P_L, P_R) as being a vector lying in an *even, Lorentzian* lattice, with signature (N, N). Note in the 1D case then get that

$$P^1 \cdot P^2 := P_L^1 P_L^2 - P_R^1 P_R^2 = \frac{1}{2}\left[(\frac{R}{\ell_s}n + \frac{\ell_s}{R}m)(\frac{R}{\ell_s}n' + \frac{\ell_s}{R}m') - (\frac{R}{\ell_s}n - \frac{\ell_s}{R}m)(\frac{R}{\ell_s}n' - \frac{\ell_s}{R}m')\right] = (mn' + nm') \in \mathbb{Z}$$

Going to higher dimensions and turning on G and B gives us the same result (take $\ell_s = 1$ for simplicity here). All terms will cancel except the ones given by the relative minus sign of G on the second term

$$\frac{1}{2}\left[m_i n'^i + n^i m'_i + \cancel{(n^i n'^j + n'^i n^j)B_{ij}}\right] \in \mathbb{Z}$$

The last term cancels by antisymmetry. Here $n^i, m_i \in \mathbb{Z}$ (note the index convention, different from Kiritsis). Under $\tau \to -1/\tau$ the η function is a modular form of weight 1/2, so $\eta(\tau)^N$ is a modular form of weight $N/2$ and $|\eta(\tau)|^{2N} = |\tau|^{-N}|\eta(-1/\tau)|^{2N}$. Let us now look at the remaining part

$$\Theta(\tau) := \sum_{P=(P_L,P_R)\in\Gamma} q^{\frac{1}{2}P_L^2}\, \bar{q}^{\frac{1}{2}P_R^2}$$

is also a modular form of this weight. Let's show this. We can use the Poisson resummation formula to write:

$$\sum_{p'\in\Gamma} \delta(p - p') = \frac{1}{V_\Gamma} \sum_{p''\in\Gamma^*} e^{2\pi i p p''} \Rightarrow \sum_{p\in\Gamma} f(p) = \frac{1}{V_\Gamma} \sum_{q\in\Gamma^*} \hat{f}(q)$$

here V_Γ^{-1} is the covolume of Γ. Taking $f = e^{i\pi\tau P_L^2 - i\pi\bar{\tau}P_R^2}$ and doing a $2N$-dimensional Fourier transform, we see that $\hat{f}(q) = \frac{1}{|\tau|^N} e^{-i\pi Q_L^2/\tau + i\pi Q_R^2/\bar{\tau}}$. We can use this to write:

$$\Theta(\tau) = \sum_{P\in\Gamma} \exp\left[\pi i(\tau P_L^2 - \bar{\tau}P_R^2)\right] = \frac{1}{|\tau|^N V_\Gamma} \sum_{Q\in\Gamma^*} \exp\left[\pi i(-\frac{1}{\tau}Q_L^2 + \frac{1}{\bar{\tau}}Q_R^2)\right]$$

Now as long as $\Gamma = \Gamma^*$, that is, Γ is an *even, Lorentzian, self-dual* lattice. Then $V_\Gamma = 1$ and the sum over $Q \in \Gamma^*$ is the same as the sum over $P \in \Gamma$. So we get

$$\Theta(\tau) = |\tau|^{-N}\Theta(-1/\tau)$$

which is the exact same transformation law as the $|\eta|^{2N}$ in the denominator, and so we get that $Z(R)$ is indeed modular invariant.

61. We have in fact done this in the first part exercise 46.

62. Certainly this is an order 2 involution, just like $R \to 1/R$. Now we know $V_{m,n} \to V_{m,-n}$ under this involution, so

$$\cdot[H^{0\prime}] \sim \sum_{n,m} C^{2n,2m}[V_{2n,2m}] + C^{2n+1,2m}[V_{2n+1,2m}] = \frac{1}{2}([H^0]\cdot[H^0] + [H^\pi]\cdot[H^{|}\pi|]) + [H^0]\cdot[H^\pi]$$

$$[H^{\pi\prime}]\cdot[H^{\pi\prime}] \sim \sum_{n,m} C^{2n,2m}[V_{2n,2m}] - C^{2n+1,2m}[V_{2n+1,2m}] = \frac{1}{2}([H^0]\cdot[H^0] + [H^\pi]\cdot[H^{|}\pi|]) + [H^0]\cdot[H^\pi]$$

$$[H^{0\prime}]\cdot[H^0\pi'] \sim \sum_{n,m} C^{2n,2m+1}[V_{2n,2m+1}] = \frac{1}{2}([H^0]\cdot[H^0] - [H^\pi]\cdot[H^\pi])$$

the only consistent transformation with these OPEs is exactly:

$$\begin{pmatrix} H^{0\prime} \\ H^{\pi\prime} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} H^{0} \\ H^{\pi} \end{pmatrix}$$

63. Note first that at $R/\ell = 1/\sqrt{2}$ we get

$$P_L = m + \frac{n}{2}, P_R = m - \frac{n}{2}$$

So we are summing over these lattice values in the numerator Θ of $Z(R)$. On the other hand, we have:

$$\frac{1}{2}(|\theta_2|^2 + |\theta_3|^2 + |\theta_4|^2) = \sum_{n,m} \left(\frac{1}{2}(1 + (-1)^{n+m}) q^{\frac{1}{2}n^2} \bar{q}^{\frac{1}{2}m^2} + \frac{1}{2} q^{\frac{1}{2}(n-1/2)^2} \bar{q}^{\frac{1}{2}(m-1/2)^2} \right)$$

This is a sum over all lattice points whose sum is an even integer *union with* the set of all half-lattice points, but only *half* of the half-lattice points are counted in the sum. This agree exactly with the standard weighting for the lattice generated by $(1,1)$ and $\frac{1}{2}(1,-1)$ which is exactly the original theta function numerator.

Squaring the Ising model theta function then gives:

$$\frac{|\theta_2\theta_3| + |\theta_3\theta_4| + |\theta_2\theta_4|}{4|\eta|^2} + \underbrace{\frac{1}{4}\frac{1}{2}(|\theta_2|^2 + |\theta_3|^2 + |\theta_4|^2)}_{\frac{1}{2}Z(R)} = \frac{1}{2}Z(R) + \frac{1}{2}\left(\frac{|\eta|}{|\theta_2|} + \frac{|\eta|}{|\theta_3|} + \frac{|\eta|}{|\theta_4|} \right)$$

exactly as we wanted.

64. Define the orbifold partition function as

$${}^{+}\underset{+}{\square}' = \frac{1}{2}\left({}^{+}\underset{+}{\square} + {}^{-}\underset{+}{\square} + {}^{+}\underset{-}{\square} + {}^{-}\underset{-}{\square} \right)$$

Note that the orbifolded theory itself has a $\mathbb{Z}_2$ symmetry obtained by taking all the states in the $\mathbb{Z}_2$ twisted sectors to minus themselves:

$${}^{\pm}\underset{+}{\square} \to {}^{\pm}\underset{+}{\square}, \qquad {}^{\pm}\underset{-}{\square} \to -{}^{\pm}\underset{-}{\square}$$

I can now *orbifold again* by this symmetry, defining (as before):

$${}^{\pm}\underset{+}{\square}' = \frac{1}{2}\left({}^{+}\underset{+}{\square} + {}^{-}\underset{+}{\square} \pm {}^{+}\underset{-}{\square} \pm {}^{-}\underset{-}{\square} \right)$$
$${}^{\pm}\underset{-}{\square}' = \frac{1}{2}\left({}^{+}\underset{+}{\square} - {}^{-}\underset{+}{\square} \pm {}^{+}\underset{-}{\square} \mp {}^{-}\underset{-}{\square} \right)$$

Then forming the new partition function of this double orbifold theory I see that almost everything cancels:

$$\frac{1}{2}\left({}^{+}\underset{+}{\square}' + {}^{-}\underset{+}{\square}' + {}^{+}\underset{-}{\square}' + {}^{-}\underset{-}{\square}' \right) = {}^{+}\underset{+}{\square}$$

65. Take $\ell_s = 1$ here. The partition function will still have 1 twisted sector and a single projection. So we need to consider 4 terms. We have $Z\begin{bmatrix}0\\0\end{bmatrix} = Z(R_1, R_2) = Z(R_1)Z(R_2)$. Our vertex operators are labelled by (m_1, n_1, m_2, n_2), and g acts as $(m_1, n_1, m_2, n_2) \to (-1)^{m_2}(-m_1, -n_1, m_2, n_2)$. And so:

$$\frac{1}{2}Z\begin{bmatrix}0\\1\end{bmatrix} = \mathrm{Tr}_1[g\, q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - \bar{c}/24}] = \overbrace{\left|\frac{\eta}{\theta_2}\right|}^{X^1 \to -X^1} \underbrace{\frac{1}{\eta\bar{\eta}} \sum_{m,n} (-1)^m \exp\left[\frac{i\pi\tau}{2}\left(\frac{m}{R_2} + nR_2\right)^2 - \frac{i\pi\bar{\tau}}{2}\left(\frac{m}{R_2} - nR_2\right)^2 \right]}_{X^2 \to X^2 + \pi R_2}$$

$$\frac{1}{2}Z\begin{bmatrix}1\\0\end{bmatrix} = \mathrm{Tr}_g[g\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}] = \left|\frac{\eta}{\theta_4}\right| \quad \frac{1}{\eta\bar{\eta}}\sum_{m,n}\exp\left[\tfrac{i\pi\tau}{2}\left(\tfrac{m}{R_2}+(n+\tfrac{1}{2})R_2\right)^2 - \tfrac{i\pi\bar{\tau}}{2}\left(\tfrac{m}{R_2}-(n+\tfrac{1}{2})R_2\right)^2\right]$$

$$\frac{1}{2}Z\begin{bmatrix}1\\1\end{bmatrix} = \mathrm{Tr}_g[g\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}] = \left|\frac{\eta}{\theta_3}\right| \frac{1}{\eta\bar{\eta}}\sum_{m,n}(-1)^m\exp\left[\tfrac{i\pi\tau}{2}\left(\tfrac{m}{R_2}+(n+\tfrac{1}{2})R_2\right)^2 - \tfrac{i\pi\bar{\tau}}{2}\left(\tfrac{m}{R_2}-(n+\tfrac{1}{2})R_2\right)^2\right]$$

it is clear that the sum of all these is modular invariant. I am unsure if I should try to simplify this further. Certainly (unlike the freely-acting orbifold case) this doesn't look trivial. This is the CFT of fields *valued in the Klein bottle*.

66. Take $\ell_s = 1$ here. The symmetry interchanges $|m_1, n_1, m_2, n_2\rangle \to |m_2, n_2, m_1, n_1\rangle$ We have $Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = Z(R)^2$. In the g-trace, we will need $m_1 = m_2, n_1 = n_2$. Then, excitations around this state must have equal mode number in m_1, m_2 and n_1, n_2 to contribute to the g-trace so for each factor of $q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2}$ we have

$$\begin{aligned} Z\begin{bmatrix}0\\1\end{bmatrix} &= q^{-2/24}\bar{q}^{2/24}\sum_{m,n}\exp\left[i\pi\tau\left(\tfrac{m}{R}+nR\right)^2 - i\pi\bar{\tau}\left(\tfrac{m}{R}-nR\right)^2\right]\prod\frac{1}{1-q^{2n}}\prod\frac{1}{1-\bar{q}^{2m}} \\ &= \frac{1}{|\eta(2\tau)|^2}\sum_{m,n}\exp\left[i\pi\tau\left(\tfrac{m}{R}+nR\right)^2 - i\pi\bar{\tau}\left(\tfrac{m}{R}-nR\right)^2\right] = \frac{2}{|\eta(\tau)||\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](\tau)|}\sum\dots \end{aligned}$$

On the other hand, the twisted sector we have boundary conditions $X^1(\sigma+2\pi) = X^2(\sigma), X^2(\sigma+2\pi) = X^1(\sigma)$. Applying $\tau \to -1/\tau$ on the preceding we get:

$$Z\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{|\eta(\tau)||\theta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](\tau)|}\sum_{m,n}\exp\left[\frac{i\pi\tau}{4}\left(\tfrac{m}{R}+nR\right)^2 - \frac{i\pi\bar{\tau}}{4}\left(\tfrac{m}{R}-nR\right)^2\right]$$

Taking $\tau \to \tau+1$ gives

$$Z\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{|\eta(\tau)||\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](\tau)|}\sum_{m,n}(-1)^{mn}\exp\left[\frac{i\pi\tau}{4}\left(\tfrac{m}{R}+nR\right)^2 - \frac{i\pi\bar{\tau}}{4}\left(\tfrac{m}{R}-nR\right)^2\right].$$

Let us check if this is modular invariant. Clearly $Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ maps to itself under both S and T. Under T, $Z\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ maps to itself, and $Z\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $Z\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ get exchanged by the properties of theta functions. Further, under $\tau \to -1/\tau$ $Z\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $Z\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ map to one another. However, $Z\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ does not seem to map to itself. Doing Poisson resummation seems to spoil the determinant from being τ. **I expect it should be preserved though. Find my mistake.**

67. If we orbifold the single free scalar by acting as $|m,n\rangle \to (-1)^{m+n}|m,n\rangle$ we have $Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = Z(R)$ as before, but now:

$$Z\begin{bmatrix}0\\1\end{bmatrix} = \sum_{m,n}(-1)^{m+n}\exp\left[\tfrac{i\pi\tau}{2}\left(\tfrac{m}{R}+nR\right)^2 - \tfrac{i\pi\bar{\tau}}{2}\left(\tfrac{m}{R}-nR\right)^2\right]$$

Taking $\tau \to -1/\tau$ gives that both m and n shift by 1/2

$$Z\begin{bmatrix}1\\0\end{bmatrix} = \sum_{m,n}\exp\left[\tfrac{i\pi\tau}{2}\left(\tfrac{m-\frac{1}{2}}{R}+(n-\tfrac{1}{2})R\right)^2 - \tfrac{i\pi\bar{\tau}}{2}\left(\tfrac{m-\frac{1}{2}}{R}-(n-\tfrac{1}{2})R\right)^2\right]$$

Then doing $\tau \to \tau+1$ gives:

$$Z\begin{bmatrix}1\\1\end{bmatrix} = \sum_{m,n}(-1)^{m+n+\frac{1}{2}}\exp\left[\tfrac{i\pi\tau}{2}\left(\tfrac{m-\frac{1}{2}}{R}+(n-\tfrac{1}{2})R\right)^2 - \tfrac{i\pi\bar{\tau}}{2}\left(\tfrac{m-\frac{1}{2}}{R}-(n-\tfrac{1}{2})R\right)^2\right]$$

this already looks a little weird. Out front we don't necessarily have a ± 1. Further, doing $\tau \to \tau+1$ again does not get us back to $Z\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, we need $\tau \to \tau+4$.

68. In the untwisted sector we have our vacuum state $|0\rangle$, with $\Delta = \bar{\Delta} = 0$ as required. Now consider the kth twisted sector. We have creation and annihilation operators $\alpha_{n+k/N}$ satisfying the same commutation relations $[\alpha_r, \alpha_s] = r\delta_{r+s}$. However as X is a *complex* boson, the α_r are complex numbers and so we have *two* sets of them (which we can call $\alpha_r, \bar{\alpha}_r$ following previous convention). From commuting them across, we get:

$$\langle X(z)\partial X(w)\rangle = 2 \times \frac{1}{w} \sum_{r=\min(1,k/N)}^{\infty} \left(\frac{w}{z}\right)^r = 2 \times \frac{\frac{w}{z}\left(\frac{z}{w}\right)^{k/L}}{z-w}$$

Then, differentiating with respect to z gives:

$$\langle \partial X(z)\partial X(w)\rangle = -\frac{2}{(w-z)^2}\left(\frac{w}{z}\right)^{k/N} \left(1 - \tfrac{k}{L}(1 - \tfrac{z}{w})\right)$$

Taking the finite part of this $-\frac{1}{2}$ of expression as $w \to z$ gives us:

$$\langle T \rangle = \frac{k(L-k)}{2L^2}$$

as required.

69. We have the scalar propagator written in terms of the eigen-modes as:

$$\langle X(z)X(0)\rangle = -\frac{\ell_s^2}{2}\sum_{m,n}{}' \frac{1}{|m+n\tau|^2} e^{2\pi i(m\sigma_1 + n\sigma_2)}$$

Rather than trying to massage this into our appropriate logarithm of theta functions, let's appreciate what properties we want our correlator to have. For $z \to 0$, the small-distance behavior of the correlator should reproduce the $\mathbb{CP}^1$ result, so we namely need it to go as:

$$-\frac{\ell_s^2}{2}\log|z|^2 + O(z)$$

Further, the *only* singularity on the torus is at $z \to 0$, nowhere else. Thus we should be able to write our correlator as

$$-\frac{\ell_s^2}{2}\log G(z, \bar{z})$$

where G must be a doubly-periodic harmonic function with a *single* zero at $z = 0$ on the torus and no poles. There are no such holomorphic functions since all non-constant elliptic functions need to have an equal number of zeros and poles (and also more than one zero, since the coefficients of all zeros must sum to 0). In other words, instead of looking at an elliptic function we should be looking at a section of a line bundle over the torus with a single zero.

We see that the theta functions give us exactly this- and moreover rational functions of the theta functions generate all such sections. The constraint of a *single* zero at $z = 0$ together with *modular invariance* singles out $\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ uniquely. To give it the appropriate coefficient of the zero, we must have:

$$G(z) = \left|\frac{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](z,\tau)}{\partial_z\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,\tau)}\right|^2 \times (1 + O(z))$$

The problem is that this is a *quasi-periodic* foundtion in z. Under $z \to z + \tau$ we get that $\log G \to \log G + 2\pi\tau_2 + 4\pi \mathrm{Im}(z)$. This can be remedied by adding $e^{-2\pi\frac{z_2^2}{\tau_2}}$ to G.
Also, under $\tau \to 1/\tau$, $z \to z/\tau$ from the ratio we pick up a factor of $|\exp(i\pi z^2/\tau)|^2 = e^{-2\pi \mathrm{Im}(z^2/\tau)}$. But this is exactly the same factor as is picked up by $e^{-2\pi\frac{\mathrm{Im} z^2}{\tau_2}}$, so adding this term fixes modular invariance as well. Our final result is then:

$$G(z) = \left|\frac{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](z,\tau)}{\partial_z\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,\tau)}\right|^2 e^{-2\pi\frac{(\mathrm{Im} z)^2}{\tau_2}}.$$

So we now have an explicit formula for $\Delta(z-w,\tau)$ on the torus. The Klein bottle is given by identifying $z \cong -\bar{z}+\tau/2$. Then we expect the propagator to be

$$\Delta_{K_2}(z-w) = \Delta(z-w, 2it) + \Delta(z+\bar{w}+it, 2it)$$

Next, for the cylinder we have the involution $z \cong -1/\bar{z}$ so we have the propagator:

$$\Delta_{C_2}(z-w) = \Delta(z-w, it) + \Delta(z+\bar{w}, it)$$

Finally, for the Möbius strip, we have two involutions and get

$$\Delta_{M_2}(z-w) = \Delta(z-w, 2it) + \Delta(z+\bar{w}, 2it) + \Delta(z-w-2\pi(it+\tfrac{1}{2}), 2it) + \Delta(z+\bar{w}+2\pi(-it+\tfrac{1}{2}), 2it)$$

70. We already know how to calculate $\mathrm{Tr}_{NS/R}((\pm 1)^F q^{L_0^{cyl}})$ for the free fermion. Ω acts by sending a left-moving state to a right-moving one and vice-versa. Only states that are left-right symmetric survive. First lets do the NS sector. There is a single vacuum and we get:

$$\mathrm{Tr}_{\mathrm{NS}}[\Omega e^{-2\pi t(L_0+\bar{L}_0-c/12)}] = e^{2\pi t/24}\prod_{n=1}^{\infty}(1+e^{-2\pi t\times 2(n-1/2)}) = \sqrt{\frac{\theta_3(2it)}{\eta(2it)}}$$
$$\mathrm{Tr}_{\mathrm{NS}}[\Omega(-1)^F e^{-2\pi t(L_0+\bar{L}_0-c/12)}] = e^{2\pi t/24}\prod_{n=1}^{\infty}(1+e^{-2\pi t\times 2(n-1/2)}) = \sqrt{\frac{\theta_3(2it)}{\eta(2it)}}$$

Note that these two are the same, since only sectors with an equal number of left movers and right-movers contribute, and this necessarily forces F to be even. Then, for the Ramond sector we have

$$\mathrm{Tr}_{\mathrm{R}}[\Omega e^{-2\pi t(L_0+\bar{L}_0-c/12)}] = \sqrt{2}e^{-2\pi t(1/16-1/48)}\prod_{n=1}^{\infty}(1+e^{-2\pi t\times 2n}) = \sqrt{\frac{\theta_2(2it)}{\eta(2it)}}$$
$$\mathrm{Tr}_{\mathrm{R}}[\Omega(-1)^F q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}] = 0$$

where the last one is zero as before, since for any state, there is a corresponding one with opposite $(-1)^F$ eigenvalue, related by zero-modes.

Chapter 5: Scattering Amplitudes and Vertex Operators

1. Note that we need 3 more c ghosts than be ghosts since the difference of the zero modes must be three. Now, c has scaling dimension 1 and b has scaling dimension -2 so the total scaling of the correlator $\langle\prod_{i=1}^{n+3} c(z_i)\prod_{j=1}^{n} b(w_j)\rangle$ will be $3-n$. Thus, viewed in the complex plane, we expect it to be a homogenous rational function of degree exactly $3-n$.

   We will have n contractions of the bs and cs with 3 cs left over. This gives:

$$\langle\prod_{i=1}^{n+3} c(z_i)\prod_{j=1}^{n} b(w_j)\rangle = \frac{(z_{n+1}-z_{n+2})(z_{n+1}-z_{n+3})(z_{n+2}-z_{n+3})}{(z_1-w_1)\dots(z_n-w_n)} \times c.c. + \text{perms.}$$

   where each permutations will pick up a sign for every odd combined permutation of the z_i, w_j. Another way to do it is as follows:

   As stated before, the correlator when viewed in the complex plane will be a homogenous rational function of degree exactly $3-n$. That way, it will be finite at infinity. We also know that this function is antisymmetric upon swapping any of the z_i, any of the w_i, or any of the z_i with the w_i. Further, if any of the $z_i = z_j$ or $w_i = w_j$, this function will vanish. On the other hand, if $z_i = w_j$, we expect a contribution of a pole $\frac{1}{z_i - w_j}$. There is only one such homogenous rational function:

$$\frac{\prod_{i<j}^{n+3}(z_i-z_j)\prod_{i<j}^{n}(w_i-w_j)}{\prod_{i=1}^{n+3}\prod_{j=1}^{n}(z_i-w_j)}.$$

   This is indeed of degree $3-n$, as desired.

2. It is clear from plugging things in that when $z_1 \to 0, z_2 \to 1, z_3 \to \infty$, the 4-point tachyon amplitude becomes:

$$\lim_{z_3\to\infty}\frac{8\pi i}{\ell_s^2} g_c^2 \delta\!\!\!{}^-{}^{26}(\Sigma p)|z_3|^2|z_3-1|^2\int d^2 z_4 |z_4|^{\ell_s^2 p_1\cdot p_4}|1-z_4|^{\ell_s^2 p_2\cdot p_4}|z_4-z_3|^{\ell_s^2 p_3\cdot p_4} 1^{\ell_s^2 p_1\cdot p_2}|z_3|^{\ell_s^2 p_1\cdot p_3}|z_3-1|^{\ell_s^2 p_2\cdot p_3}$$

   here $\delta\!\!\!{}^-= 2\pi\delta$. Note all the terms that go to infinity cancel, since $\ell_s^2 p_3\cdot(p_1+p_2+p_3) = -\ell_s^2 p_3^2 = -4$ which cancels with the two powers of two outside the integral. Next, $\ell_s^2 p_1\cdot p_4 = \frac{1}{2}(p_1+p_4)^2 - \frac{1}{2}(\ell_s^2 p_1^2 - \ell_s^2 p_4^2) = -\ell_s^2 t/2 - 4$ etc so we get:

$$\frac{8\pi i}{\ell_s^2} g_c^2 \delta\!\!\!{}^-{}^{26}(\Sigma p)\int d^2 z_4 |z_4|^{-\ell_s^2 t/2-4}|1-z_4|^{-\ell_s^2 u/2-4}$$

   as required.

3. For a conformal transformation we have $|x'_{ij}|^2 = \Omega(x_i)\Omega(x_j)|x_{ij}|^2$ where $\Omega(x_i)$ is the local scale factor $\det \partial x'/\partial x$ evaluated at x_i. Then, the N-point tachyon amplitude will pick up $\Omega(x_1)^2\Omega(x_2)^2\Omega(x_3)^2$ from the three terms outside of the integral. The terms inside the integral can be written as:

$$\prod_{i<j}(|z_{ij}|^2)^{\ell_s^2 p_i\cdot p_j/2}$$

   so z_i in this term will pick up a power of $\sum_{j\neq i}\ell_s^2 p_i\cdot p_j/2 = -\ell_s^2 p_j^2/2 = -2$ on its scale factor. This exactly cancels for z_1, z_2, z_3. For the other z_i, we note that $d^2 z_i$ will pick up the factor $\Omega(z_i)^2$ upon transformation.

   Another way to do this is directly from noting that each $\int d^2 z_i V_{p_i}(z_i,\bar z_i)$ for $i > 3$ is invariant under conformal transformation, and $c(z_i)\bar c(\bar z_i)V_{p_i}(z_i,\bar z_i)$ has scaling dimension zero, so transforms trivially under $\mathrm{SL}_2(\mathbb{C})$ transformations.

4. Note that the three-point tachyon amplitude is very simple and independent of momenta aside from a delta function: $S(k_1,k_2,k_3) = \frac{8\pi i}{\ell_s^2} g_c \delta\!\!\!{}^-{}^{26}(\Sigma k)$.

   Let's now consider the limit of a nearly on-shell particle of momenta k. From elementary field theory we get:

$$S(k_1,k_2,k_3,k)4) \sim i\int\frac{d^{26}k}{(2\pi)^{26}}\frac{S_{S^2}(k_1,k_2,k)S_{S_2}(-k,k_3,k_4)}{-k^2+4/\ell_s^2+i\epsilon} = i\left(\frac{8\pi i}{\ell_s^2}\right)^2 g_c^2 \delta\!\!\!{}^-{}^{26}(\Sigma k_i)\frac{1}{s+4\ell_s^2+i\epsilon}$$

This has a pole when $-(k_1+k_2)^2 = s = -4/\ell_s^2$. We see that (ignoring the $\not\delta$ term) this gives a residue of $-i\frac{64\pi^2}{\ell_s^4}g_c^2$

On the other hand we have from **5.2.5** a residue of:

$$\frac{8\pi i}{\ell_s^2}g_c^2 \times 2\pi \times -\frac{4}{\ell_s^2} = -i\frac{64\pi^2}{\ell_s^2}g_c^2$$

exactly consistent with unitarity. Note we needed every constant to be as it was so that we could get such agreement.

5. The massless state corresponds to $\zeta_{\mu\nu}\partial X^\mu \partial X^\nu e^{ip\cdot X}$. We don't have to integrate. Let's calculate the correlator

$$\langle \partial X(z_1)\bar\partial X(z_1)e^{ik_1 X(z_1)}\, e^{ik_2 X(z_2)}\, e^{ik_3 X(z_3)}\rangle = iC_{S^2}^X \not\delta^{26}(\Sigma p)\prod_{i<j}|z_{ij}|^{\alpha' k_i\cdot k_j}\left(-\tfrac{i\ell_s^2}{2}\right)\left(\tfrac{k_2}{z_{12}}+\tfrac{k_3}{z_{13}}\right)\left(-\tfrac{i\ell_s^2}{2}\right)\left(\tfrac{k_2}{\bar z_{12}}+\tfrac{k_3}{\bar z_{13}}\right)$$

with the ghost correlator this gives:

$$iC_{S^2}^X C_{S^2}^{gh}\tfrac{-\ell_s^4}{4}\not\delta^{26}(\Sigma p)\prod_{i<j}|z_{ij}|^{\alpha' k_i\cdot k_j+2}\left(\tfrac{k_2}{z_{12}}+\tfrac{k_3}{z_{13}}\right)\left(\tfrac{k_2}{\bar z_{12}}+\tfrac{k_3}{\bar z_{13}}\right)$$

Now $k_1^2 = 0 = k_1\cdot k_2 + k_1\cdot k_3$. On the other hand $-4/\ell_s^2 = -k_2^2 = k_2\cdot k_3 + k_1\cdot k_2 = -k_3^2 = k_2\cdot k_3 + k_1\cdot k_3$. Solving this gives $k_1\cdot k_2 = k_1\cdot k_3 = 0$ while $k_2\cdot k_3 = -4/\ell_s^2$. Then, taking $z_1\to 0, z_2\to 1, z_3\to\infty$ gives:

$$-i\frac{\ell_s^2}{4}C_{S^2}^X C_{S^2}^{gh}\not\delta^{26}(\Sigma p)\zeta_{\mu\nu}k_2^\mu k_2^\nu$$

Further, we have that $\zeta_{\mu\nu}k_1^\mu = \zeta_{\mu\nu}(k_2+k_3)^\mu = 0$ so we can rewrite this symmetrically as

$$-i\frac{\ell_s^4}{16}\underbrace{C_{S^2}^X C_{S^2}^{gh}}_{:=8\pi g_c'/\ell_s^2}\not\delta^{26}(\Sigma p)\zeta_{\mu\nu}k_{23}^\mu k_{23}^\nu = -\frac{i\pi\ell_s^2}{2}g_c'\not\delta^{26}(\Sigma p)\zeta_{\mu\nu}k_{23}^\mu k_{23}^\nu.$$

The overall constants can be determined from unitarity. The pole of the Veneziano amplitude at $s=0$ has residue (using that $s=0, s+t+u=-16/\ell_s^2$) that is a delta function times:

$$\frac{8\pi i}{\ell_s^2}g_c^2 \times 2\pi \times \frac{4}{\ell_s^2 s}\frac{\Gamma(-1-\ell_s^2 t/4)\Gamma(3+\ell_s^2 t/4)}{\Gamma(-2-\ell_s^2 t/4)\Gamma(2+\ell_s^2 t/4)} = -i\frac{(4\pi)^2}{\ell_s^2}g_c^2\times\frac{4}{\ell_s^2 s}\overbrace{(2+\ell_s^2 t/4)^2}^{(\frac{\ell_s^2}{8}(t-u))^2} = -i\pi^2 g_c^2\frac{(t-u)^2}{s} \tag{55}$$

On the other hand, factorization of this into amplitudes with massless states yields a delta function times:

$$iC_{3pt}^2\sum_\zeta \zeta_{\mu\nu}\zeta_{\sigma\rho}k_{12}^\mu k_{12}^\nu k_{34}^\sigma k_{34}^\rho \times \frac{1}{(k_1+k_2)^2+i\epsilon} = iC_{3pt}^2(k_{12}\cdot k_{34})^2\times\frac{1}{s} = iC_{3pt}^2\frac{(u-t)^2}{s} \tag{56}$$

where we have used that, just as the sum over intermediate photon polarizations $\epsilon_\mu\epsilon_\nu^*$ can be replaced by just $\eta_{\mu\nu}$, the sum over intermediate polarizations $\zeta_{\mu\nu}\zeta_{\rho\sigma}$ be replaced by $\frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho})$. Comparing equations (55) and (56) We the get $C_{3pt} = -\pi i g_c$. Equivalently, $g_c' = 2g_c/\ell_s^2$.

6. We already determined the normalization in the previous question. It is also simple to check that it is correct to attach g_c' to each vertex operator in the 3-point and 4-point functions by considering first the 2 tachyon $\to$ 2 massless state scattering in the t and u channels, which relates the 3-point scatterings of tachyons and massless states to one another, and then use the $2\to 2$ tachyon to tachyon scattering to express its normalization in terms of the 3-point tachyon amplitude. All of this equates to taking $g_c' = 2g_c/\ell_s^2$.

As a warm-up lets do the three-point massless amplitude. We compute the correlator

$$\langle :\partial X^\alpha(z_1)e^{ip_i X(z_1)} :: \partial X^\beta(z_2)e^{ip_i X(z_2)} :: \partial X^\gamma(z_3)e^{ip_i X(z_3)} : \times c.c.\rangle$$

In the holomorphic part, there are two types of contribution: One where each ∂X contracts with an exponential and one where two of the ∂X contract with one another and the last one contracts with an exponential. Further, we see that $p_i \cdot p_j = 0$, so the $\prod_{i<j} |z_{ij}|^{\ell_s^2 p_i \cdot p_j}$ is unity. The first contribution gives:

$$i\left(\frac{\ell_s^2}{2}\right)^3 \left(\frac{k_2^\alpha}{z_{12}} + \frac{k_3^\alpha}{z_{13}}\right)\left(\frac{k_1^\alpha}{z_{21}} + \frac{k_3^\alpha}{z_{23}}\right)\left(\frac{k_1^\alpha}{z_{31}} + \frac{k_2^\alpha}{z_{32}}\right) \to i\left(\frac{\ell_s^2}{2}\right)^3 \frac{1}{2^2}(k_1 - k_2)^\gamma (k_2 - k_3)^\alpha (k_3 - k_1)^\beta$$

The second contribution gives

$$i\left(\frac{\ell_s^2}{2}\right)^2 \left[\frac{\eta^{\alpha\beta}}{z_{12}^2}\left(\frac{k_1^\gamma}{z_{31}} + \frac{k_2^\gamma}{z_{32}}\right) + \frac{\eta^{\beta\gamma}}{z_{23}^2}\left(\frac{k_2^\alpha}{z_{12}} + \frac{k_3^\alpha}{z_{13}}\right) + \frac{\eta^{\alpha\gamma}}{z_{13}^2}\left(\frac{k_1^\beta}{z_{21}} + \frac{k_3^\beta}{z_{23}}\right)\right]$$

Multiplying this by the c contribution $z_{12}z_{23}z_{13} \times c.c.$ and setting $z_1 = 0, z_2 = 1, z_3 = \infty$ we get the 3-point amplitude:

$$\pi i g_c \,\zeta_{1,\alpha\bar{\alpha}}\zeta_{2,\beta\bar{\beta}}\zeta_{3,\gamma\bar{\gamma}} T^{\alpha\beta\gamma} T^{\bar{\alpha}\bar{\beta}\bar{\gamma}}, \quad T^{\alpha\beta\gamma} = \eta^{\alpha\beta} k_{12}^\gamma + \eta^{\beta\gamma} k_{23}^\alpha + \eta^{\alpha\gamma} k_{31}^\beta + \frac{\ell_s^2}{8} k_{12}^\gamma k_{23}^\alpha k_{31}^\beta. \tag{57}$$

Now let's do the four-point amplitude. *First, I will work with the open string* (no CP indices, so $U(1)$ gauge symmetry) and use some tricks at the end to get the closed string amplitude. For the open string, there are six possible orderings of the y_1, y_2, y_3, y_4. In three of these cases we can send $y_1 \to 0, y_2 \to 1, y_3 \to \infty$ and vary y_4. In the other three, cases we switch y_2 and y_3. This amounts to swapping $s \leftrightarrow t$. **HOWEVER** for Polchinski's trick, I only need to consider *one of these six*. WLOG I set y_4 to be between y_1, y_2 in $0, 1$. I'll also absorb ℓ_s^2 in the definition of s, t, u. So we have,

$$\prod_{i<j} |y_{ij}|^{2k_i \cdot k_j} \to |y|^{-u}|1-y|^{-t} \leftrightarrow |y|^{-u}|1-y|^{-s}$$

We now get three types of contributions: If all the ∂X^α contract with each other (3 terms), if two of the ∂X^α contract with each other (6 terms) and the remaining two contract with one of the $e^{ik_i \cdot X}$, or if they all contract with the $e^{ik_i \cdot X}$ (1 term).

In the first case we get:

$$(-2\ell_s^2)^2 \left(\frac{1}{y_{12}^2 y_{34}^2} + \frac{1}{y_{13}^2 y_{24}^2} + \frac{1}{y_{14}^2 y_{23}^2}\right) \to 4\left(\eta^{\alpha\beta}\eta^{\gamma\delta} + \frac{\eta^{\alpha\gamma}\eta^{\beta\delta}}{(1-y)^2} + \frac{\eta^{\alpha\delta}\eta^{\beta\gamma}}{y^2}\right)$$

Integrating y from 0 to 1 gives

$$\frac{\Gamma(1-t)\Gamma(1-u)}{\Gamma(2+s)} + \frac{\Gamma(1-t)\Gamma(-1-u)}{\Gamma(s)} + \frac{\Gamma(-1-t)\Gamma(1-u)}{\Gamma(s)} \tag{58}$$

Now the annoying one[1]. Define $K_i = \sum_{j \neq i} \frac{k_j}{y_{ij}}$. Note:

$$K_1 \to -(k_2^\alpha + \frac{k_3^\alpha}{y}), \quad K_2 \to k_1^\beta + \frac{k_4^\beta}{1-y}, \quad K_3 \to (1+y)k_1^\gamma + yk_2^\gamma + k_4^\gamma, \quad K_4 \to \frac{k_1^\delta}{y} + \frac{k_2^\delta}{y-1}.$$

We can write the second case as:

$$\begin{aligned}
&(2\ell_s^2)^3 \left(\frac{K_3K_4}{y_{12}^2} + \frac{K_1K_2}{y_{34}^2} + \frac{K_1K_4}{y_{23}^2} + \frac{K_2K_3}{y_{14}^2} + \frac{K_2K_4}{y_{13}^2} + \frac{K_1K_3}{y_{24}^2}\right) \\
&\to (2\ell_s^2)^3 \Big[((1+y)k_1^\gamma + yk_2^\gamma + k_4^\gamma)(\tfrac{k_1^\delta}{y} + \tfrac{k_2^\delta}{y-1})\eta^{\alpha\beta} - (k_2^\alpha + \tfrac{k_4^\alpha}{y})(k_1^\beta + \tfrac{k_4^\beta}{1-y})\eta^{\gamma\delta} - (k_2^\alpha + \tfrac{k_4^\alpha}{y})(\tfrac{k_1^\delta}{y} + \tfrac{k_2^\delta}{y-1})\eta^{\beta\gamma} \\
&\qquad + \tfrac{1}{y^2}(k_1^\beta + \tfrac{k_4^\beta}{1-y})((1+y)k_1^\gamma + yk_2^\gamma + k_4^\gamma)\eta^{\alpha\delta} + (k_1^\beta + \tfrac{k_4^\beta}{1-y})(\tfrac{k_1^\delta}{y} + \tfrac{k_2^\delta}{y-1})\eta^{\alpha\gamma} \\
&\qquad - \tfrac{1}{(1-y)^2}(k_2^\alpha + \tfrac{k_4^\alpha}{y})((1+y)k_1^\gamma + yk_2^\gamma + k_4^\gamma)\eta^{\beta\delta}\Big]
\end{aligned}$$

[1]Wasted all of 1/17/20 on this. Not worth it

We can integrate this out to get (looking at just the first term):

$$(2\ell_s)^3\eta^{\alpha\beta}\int_0^1 dy\,\left(k_1^\delta[y^{-1}k_{1+4}^\gamma + k_{1+2}^\gamma] + k_2^\delta[(y-1)^{-1}k_{1+4}^\gamma + y(y-1)^{-1}k_{1+2}^\gamma]\right)|y|^{-u}|1-y|^{-t} + 5\,\text{perms.}$$

$$\begin{aligned}&= (2\ell_s)^3\eta^{\alpha\beta}\Big[k_{14}^\delta(k_{13}+k_{43})^\gamma\frac{\Gamma(1-t)\Gamma(1-u)}{\Gamma(2-s)} + k_{14}^\delta(k_{13}+k_{23})^\gamma\frac{\Gamma(2-u)\Gamma(1-t)}{\Gamma(3+s)} \\ &\qquad - k_{24}^\delta(k_{13}+k_{43})^\gamma\frac{\Gamma(2-t)\Gamma(1-u)}{\Gamma(3+s)} - k_{24}^\delta(k_{13}+k_{23})^\gamma - \frac{\Gamma(2-t)\Gamma(2-u)}{\Gamma(4+s)}\Big] + 5\,\text{perms.}\end{aligned} \tag{59}$$

The last term, given by contracting each ∂X against an exponential is $K_1^\alpha K_2^\beta K_3^\gamma K_4^\delta$ so we'll get

$$-(2\ell_s^2)^4\int_0^1 dy\,\left(k_2^\alpha + \frac{k_4^\alpha}{y}\right)\left(k_1^\beta + \frac{k_4^\beta}{1-y}\right)\left((k_1+k_4)^\gamma + y(k_1+k_2)^\gamma\right)\left(\frac{k_1^\delta}{y} + \frac{k_2^\delta}{y-1}\right)y^{-u}(1-y)^{-t} \tag{60}$$

This gives a lot of terms all multiplying gamma functions of some appropriate type. Although each term has a quick computation, I don't want to write them all out.

The open string amplitude is then given by summing equations (58), (59) and (60) and multiplying that result by $\frac{ig_o^2}{\ell_s^2}\delta\!\!\!/^{26}\Sigma p$. Call this $A_o^{\alpha\beta\gamma\delta}(s,t,u,\ell_s,g_o)$. Using **Polchinski 6.6.23** we can write the closed string amplitude as:

$$A_c(s,t,u,\ell_s,g_c) = \zeta_{1,\alpha\bar\alpha}\zeta_{2,\beta\bar\beta}\zeta_{3,\gamma\bar\gamma}\zeta_{4,\delta\bar\delta}\frac{\pi i g_c^2\ell_s^2}{g_o^4}g_o^4\sin(\pi\ell_s^2 t)A_o^{\alpha\beta\gamma\delta}[A_o^{\bar\alpha\bar\beta\bar\gamma\bar\delta}(t,u,s,\ell_s/4,g_o)]^*$$

where ζ are our 24^2 closed string polarization vectors.

7. There are three types of propagators to consider: bulk-bulk, bulk-boundary, and boundary-boundary. Using shorthand $X_i = X(z_i,\bar z_i), X_I = X(w_I)$, from **4.7.9** we have:

$$\langle\prod_{i=1}^m e^{ip_iX_i}\prod_{I=1}^n e^{iq_IX(w_I)}\rangle = \delta\!\!\!/^{26}(\Sigma p+\Sigma q)\exp\Big[-\sum_{i<j}p_ip_j\langle X_iX_j\rangle - \frac{1}{2}\sum_{i,I}p_iq_I\langle X_iX_I\rangle - \sum_{I<J}q_Iq_J\langle X_IX_J\rangle\Big]$$

Using the form of the propagators

$$\begin{aligned}\langle X_iX_j\rangle &= -\frac{\ell_s^2}{2}(\log|z_i-z_j|^2 + \log|z_i-\bar z_j|^2)\\ \langle X_iX_I\rangle &= -\frac{\ell_s^2}{2}(\log|w_I-z_i|^2 + \log|w_I-\bar z_i|^2)\\ \langle X_IX_J\rangle &= -\ell_s^2\log|w_I-w_J|^2\end{aligned}$$

we get

$$\delta\!\!\!/^{26}(\Sigma p+\Sigma q)\prod_i|z_i-\bar z_i|^{\ell_s^2p_i^2/2}\prod_{i<j}^m|(z_i-z_j)(z_i-\bar z_j)|^{\ell_s^2p_i\cdot p_j}\prod_{I<J}|w_I-w_J|^{2\ell_s^2q_Iq_J}\prod_{I,i}|(w_I-z_i)(w_I-\bar z_i)|^{\ell_s^2p_i\cdot q_I}$$

Note an additional term which I believe Kiritsis dropped. The extension to $\mathbb{RP}^2$ is no more difficult. We now have no boundary and the $\langle X_iX_j\rangle$ propagator is $-\frac{\ell_s^2}{2}(\log(z_i-z_j)+\log(1+z_i\bar z_j))$ so we get:

$$\delta\!\!\!/^{26}(\Sigma p+\Sigma q)\prod_i|1+z_i\bar z_i|^{\ell_s^2p_i^2/2}\prod_{i<j}|(z_i-z_j)(1+z_i\bar z_j)|^{\ell_s^2p_i^2/2}$$

8. Forgetting c ghosts here, I can just integrate over all of $\mathbb{H}$. The massless closed-string state of zero momentum is given by $\partial X(z)\bar\partial X(z)$. Note that $\mathbb{H} = \mathrm{PSL}_2(\mathbb{R})/SO(2)$, so that:

$$-\frac{\ell_s^2}{2}\frac{1}{\mathrm{Vol}(\mathrm{PSL}_2(\mathbb{R}))}\int_{\mathbb{H}}dz\frac{1}{|z-\bar z|^2} = -\frac{\ell_s^2}{8}\frac{1}{\mathrm{Vol}(\mathrm{PSL}_2(\mathbb{R}))}\int_{\mathbb{H}}\frac{dxdy}{y^2} = -\frac{\ell_s^2}{8}\frac{\mathrm{Vol}(\mathbb{H})}{\mathrm{Vol}(\mathrm{PSL}_2(\mathbb{R}))} = -\frac{\ell_s^2}{16\pi}$$

Note that this answer is finite and invariant under conformal transformation. This gives an amplitude of $-\frac{i}{16}\delta\!\!\!/^{26}(0)$.

9. Let p_1 be the momentum of the closed-string tachyon, and p_2, p_3 the the momenta of the open string tachyons. We get $2p_2 \cdot p_3 = p_1^2 - p_2^2 - p_3^2 = 2/\ell_s^2 \Rightarrow p_2 \cdot p_3 = 1/\ell_s^2, \quad 2p_1 \cdot p_2 = p_3^2 - p_2^2 - p_1^2 = -4/\ell_s^2 \Rightarrow p_1 \cdot p_2 = -2/\ell_s^2$

   I no longer have enough freedom to fix all three points. I can send one to ∞ on the real line, and fix the position of the closed string to be $i \in \mathbb{H}$. The remaining open string insertion can be anywhere on the real line, so we must integrate over this. The ghost and vertex operator correlator gives:

$$(z_1 - \bar{z}_1)(z_1 - w_3)(\bar{z}_1 - w_3)\,|z_1 - \bar{z}_1|^{\ell_s^2 p_1^2/2}|z_1 - w_3|^{2\ell_s^2 p_1 \cdot p_3} \int_{\mathbb{R}} dw_2\, |w_2 - w_3|^{2\ell_s^2 p_2 \cdot p_3}|w_2 - z_1|^{2\ell_s^2 p_1 \cdot p_2}\, \delta\!\!\!/(\Sigma p)$$

   Setting $z_1 = i, w_3 \to \infty$ has momentum conservation and $p_3^2 = 1/\ell_s^2, p_1^2 = 4/\ell_s^2$ getting the w_3 factors to drop out. We are left with

$$2i\, 2^{\ell_s^2 p_1^2/2} \int_{\mathbb{R}} dw\, (w^2+1)^{\ell_s^2 p_1 \cdot p_2}\, \delta\!\!\!/(\Sigma p) = 8i\sqrt{\pi}\, \frac{\Gamma(-\frac{1}{2}+2)}{\Gamma(2)} \delta\!\!\!/(\Sigma p) = 4\pi i \delta\!\!\!/(\Sigma p)$$

   This gives a scattering amplitude of:

$$-\frac{4\pi g_o^2}{\ell_s^2} \delta\!\!\!/^{26}(\Sigma p).$$

10. The conformal Killing group is now SO(3). Again, we can fix one operator to be at $z = 0$, but the other one can be at any value of $|z| \in [0, 1]$ (we have control over the phase). So we must integrate over the modulus. We do this on the disk using the $\mathbb{RP}^2$ propagator. We insert one vertex operator at 0 and the other z. The integral gives a delta function times:

$$\int_0^1 d|z_2| c(z_1)\bar{c}(\bar{z}_1)c(z_2)(1+|z_1|^2)^{\ell_s^2 p^2/2}(1+|z_2|^2)^{\ell_s^2 p^2/2}|(z_1 - z_2)(1 + z_1\bar{z}_2)|^{-\ell_s^2 p^2} \to \int_0^1 r dr\, r^{-\ell_s^2 p^2}(1+r^2)^{\ell_s^2 p^2/2}$$

   For the closed string tachyon, we have $p^2 = 4/\ell_s^2$. The integral is divergent, coming from the $(z-w)^{-4}$ singularity as the two tachyons approach one another. If we had the milder $(z-w)^{-1}$ singularity of the open-string tachyon, this could be fixed. **REVISIT**

11. To simplify this problem, as Polchinski asks in his problem 6.9, I will look at the terms that contribute to the $e_1 \cdot e_2\, e_3 \cdot e_4$ amplitude, which comes from contracting $\partial X^\alpha(y_1)\partial X^\beta(y_2)$ and $\partial X^\beta(y_3)\partial X^\delta(y_4)$. There are six possible orderings for the trace in the 4-point amplitude. We get $\frac{ig_o'^4}{g_o^2 \ell_s^2} \delta\!\!\!/^{26}(\Sigma p) \times (2\ell_s^2)^2$ multiplying a sum of six integrals. Using $s := -\ell_s^2(p_1+p_2)^2 = -2p_1 \cdot p_2, t := -\ell_s^2(p_1+p_3)^2 = -2p_1 \cdot p_3, u := -\ell_s^2(p_1+p_4)^2 = -2p_1 \cdot p_4$ and the shorthand [1234] for $\mathrm{Tr}(\lambda^{\mu_1}\lambda^{\mu_2}\lambda^{\mu_3}\lambda^{\mu_4})$, we get:

$$\left[[1234]\int_{-\infty}^0 + [1423]\int_0^1 + [1243]\int_1^\infty\right](|w|^{-u}|1-w|^{-t})dw$$
$$+\left[[1324]\int_{-\infty}^0 + [1432]\int_0^1 + [1342]\int_1^\infty\right](|w|^{-u}|1-w|^{-s})dw$$

   Note the second triplet of integrals swaps 2 with 3 so equivalently swaps s and t. We get the amplitude

$$\frac{ig_o^2}{2\ell_s^2} e_1 \cdot e_2\, e_3 \cdot e_4 \delta\!\!\!/^{26}(\Sigma p)\Big[\quad ([1234]+[1432])B(1-u, -1-s)$$
$$+ ([1423]+[1324])B(1-t, 1-u)$$
$$+ ([1243]+[1342])B(1-t, -1-s)\Big]$$

   Now in the s channel, the first and third Beta functions give us poles at $s = 0$ with residues $-t$ and $-u = t$ respectively. This gives:

$$-\frac{ig_o^2}{2\ell_s^2} \delta\!\!\!/^{26}(\Sigma p) e_1 \cdot e_2\, e_3 \cdot e_4([1234]+[2143]-[1243]-[2134]) \times \frac{t-u}{s} \tag{61}$$

On the other hand, the 3-point vertex (again just the leading order of the two terms, compare with (57)) for massless bosons comes from the correlator

$$\begin{aligned}
&\frac{i(g_o')^3}{g_o^2\ell_s^2}|w_{12}w_{13}w_{23}|\langle: \partial X^{\mu_1}(w_1)e^{ik_1X(w_1}::\partial X^{\mu_2}(w_2)e^{ik_2X(w_2)}::\partial X^{\mu_3}(w_3)e^{ik_3X(w_3)}:\rangle\\
&\to \frac{i(g_o')^3}{g_o^2\ell_s^2}(-i2\ell_s^2)(-2\ell_s^2)\left(\frac{p_1^{\mu_3}}{w_{12}^2w_{13}}+\frac{p_2^{\mu_3}}{w_{12}^2w_{23}}+2\,\text{perms.}\right)|w_{12}|^{2\ell_s^2p_1\cdot p_2-1}|w_{13}|^{2\ell_s^2p_1\cdot p_3-1}|w_{23}|^{2\ell_s^2p_2\cdot p_3-1}\\
&= -ig_o\frac{\sqrt{2}}{\ell_s}(\eta^{\mu_1\mu_2}\frac{1}{2}p_{12}^{\mu_3}+2\,\text{perms.})
\end{aligned}$$

using $g_o' = g_o/(\sqrt{2}\ell_s)$. Adding CP factors gives:

$$-\frac{ig_o}{\sqrt{2}\ell_s}(\eta^{\mu_1\mu_2}p_{12}^{\mu_3}+\eta^{\mu_1\mu_3}p_{13}^{\mu_2}+\eta^{\mu_2\mu_3}p_{23}^{\mu_1})\underbrace{([123]-[321])}_{f^{123}}$$

We care about the $e_1\cdot e_2\, e_3\cdot e_4$ term which means we only look at the $p_{12}\cdot p_{34} = t-u$ contribution in the s channel.

$$i\int\frac{d^{26}k}{(2\pi)^{26}}\frac{S(k_1,k_2,k)S(-k,k_3,k_4)}{-k^2+i\epsilon}\to -i\frac{g_o^2}{2\ell_s^2}\delta\!\!\!/^{26}(\Sigma p)\frac{t-u}{s}\times\sum_5\left(f^{125}f^{534}\right)$$

Lastly, note that the factors in equation (61) give $\mathrm{Tr}(f^{12a}\lambda_a f^{34b}\lambda_b)$, and with suitable normalization, this gives $\sum_5 f^{125}f^{534}$, exactly as desired.

We thus see that the amplitude indeed factorizes, respecting the structure of the $U(N)$ gauge group.

12. We have $p^2+m^2 = \frac{1}{\ell_s^2}L_0$ for the open string. From **5.3.1** (and consequently **5.3.3**) this gives:

$$\frac{i}{2}\frac{V_{26}}{(4\pi)^{26}}\int_0^\infty\frac{dt}{t^{13+1}}\overbrace{\mathrm{Tr}'[e^{-2\pi tm^2}]}^{\text{transverse only}} = \frac{i}{2}\frac{V_{26}}{(16\pi^2\ell_s^2)^{13}}\int_0^\infty\frac{dt}{t^{13+1}}\mathrm{Tr}'[e^{-2\pi tL_0^{\text{cyl}}}] = \frac{i}{2}\frac{V_{26}}{(16\pi^2\ell_s^2)^{13}}\int_0^\infty\frac{dtN_1N_2\eta(it)^2}{t^{13+1}\eta(it)^{26}}$$

All together this gives:

$$iN_1N_2V_{26}\int_0^\infty\frac{dt}{2t}\frac{1}{(8\pi^2\ell_s^2t)^{13}\eta(it)^{24}}$$

as required.

13. We already know the form of our propagators on the torus from exercise **4.69**. Take

$$G(z,w) = \left|\frac{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](z-w,\tau)}{\partial_z\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,\tau)}\right|^2 e^{-2\pi(\mathrm{Im}z)^2/\tau_2}.$$

This gives us

$$\langle\prod_i :e^{ik_iX(z_i,\bar z_i)}:\rangle = iC^X_{T^2}\delta\!\!\!/^{26}(\Sigma k)\prod_{i<j}|G(z_i,z_j)|^{\ell_s^2k_i\cdot k_j/2}$$

where C_X which is equal to the partition function of the torus $Z(\tau)$ that we have also computed in the last chapter.

14. We need to calculate the form of the propagators $\langle X^\mu(z)X^\nu(w)\rangle$ on the cylinder with NN boundary conditions. Let's use the image charge method. The finite cylinder can be thought of as the fundamental domain of the quotient of the upper half plane by the action $z\to\lambda z$ for λ a real number corresponding to the modulus of the cylinder. For X at z where $1<|z|<\lambda$ we place images at each $\lambda^n z$ in the upper half plane as well as at $\lambda^n\bar z$ on the lower half plane.

$$\langle X(z)X(w)\rangle = -\frac{\ell_s^2}{2}\sum_{n\in\mathbb{Z}}\left(\log|\lambda^{-n/2}z-\lambda^{n/2}w|^2+\log|\lambda^{-n/2}z-\lambda^{n/2}\bar w|^2\right)$$

This gives
$$\langle\prod_i :e^{ip_iX}:\rangle = \not{\delta}^D(\Sigma p)\prod_n\prod_{i<j}|(\lambda^{-n/2}z_i-\lambda^{n/2}z_j)(\lambda^{-n/2}z_i-\lambda^{n/2}\bar{z}_j)|^{\ell_s^2p_i\cdot p_j}$$

For open strings (operators inserted at the boundary) we must apply boundary normal ordering. We'll get:

$$\langle\prod_i {}^\star_\star e^{iq_iX}{}^\star_\star\rangle = \not{\delta}^D(\Sigma q)\prod_n\prod_{I<J}|(\lambda^{-n/2}w_I-\lambda^{n/2}w_J)|^{2\ell_s^2q_I\cdot q_J}$$

Lastly, for the correlations between boundary and bulk operators we'll get:

$$\prod_n\prod_{i,I}|\lambda^{-n/2}w_i-\lambda^{n/2}z_i|^{2\ell_s^2p_i\cdot q_I}$$

Taking the product of the above three equations (with only a single momentum-conserving delta function) gives us the X correlator on the cylinder. The CKG here is simply the compact $SO(2)$ so it is best to ignore ghosts, integrate the insertions over the whole cylinder and divide at the end by the volume of the $SO(2)$ action: λ.

There is a cleaner way to do this. From exercise **4.69** we know the cylinder propagator can be written in terms of the torus propagator as an involution:

$$\Delta_{C_2}(z-w) = \Delta(z-w,it)+\Delta(z+\bar{w},it)$$

Here $\Delta = -\frac{\ell_s^2}{2}\log G(z,\bar{z})$ from the problem above. This will then give us for m closed string and n open string tachyons:

$$\begin{aligned}\langle\prod_i :e^{ip_iX(z_i)}:\prod_I {}^\star_\star e^{iq_IX(w_I)}{}^\star_\star\rangle =& i\not{\delta}^D g_c^m g_o^n\int_0^\infty \frac{dt}{2t(8\pi^2\ell_s^2t)^{13}\eta(it)^{24}}\prod_{i=1}^m\int_C dz_i\prod_{I=1}^n\int_{\partial C}dw_I\\ &\times\prod_{i<j}[G(z_i-z_j;\tau=it)G(z_i+\bar{z}_j;\tau=it)]^{\ell_s^2p_i\cdot p_j/2}\prod_{I<J}^n G(w_I-w_J;\tau=it)^{\ell_s^2q_I\cdot q_j}\\ &\times\prod_{i,I}[G(w_I-z_i;\tau=it)G(w_I+\bar{z}_i;\tau=it)]^{\ell_s^2p_i\cdot q_I/2}\end{aligned}$$

15. Here I assume Kiritsis meant $\epsilon_c=1$, since equation **3.4.3** refers specifically to closed string ground states. The one-loop contribution for the unoriented closed string comes from the Klein bottle amplitude. As before, the only nonzero contributions come from states with an equal number of left and right movers. All that this gives is an overall factor of ϵ_c in this amplitude:

$$Z_{K_2} := \frac{1}{2}\mathrm{Tr}[\Omega e^{-2\pi t(L_0+\bar{L}_0-c/12)}] = \frac{V_{26}}{2}\int\frac{d^{26}p}{(2\pi)^{26}}\epsilon_c\frac{e^{-\pi\ell_s^2tp^2}}{\eta(2it)}\Rightarrow\Lambda_{K_2}=i\frac{V_{26}}{(2\pi\ell_s)^{26}}\int_0^\infty\frac{dt}{4t^{1+13}\eta(2it)^{24}}$$

And working in the transverse channel gives

$$24i\epsilon_c\frac{2^{26}V_{26}}{4\pi(8\pi^2\ell_s^2)^{13}}\int_0^\infty d\ell$$

This gives a total tadpole term given by:

$$\epsilon_c 2^{26}-2^{14}\zeta N+N^2$$

We have N is a positive integer. Further, we have that ζ is a *sign*. If $\zeta=-1$ then ϵ must be negative, and so by unitarity it is -1, but there are no integer solutions N to $2^{26}=2^{14}N+N^2$. Thus we need $\zeta=1$ and consequently $\epsilon=-1, N=2^{13}$.

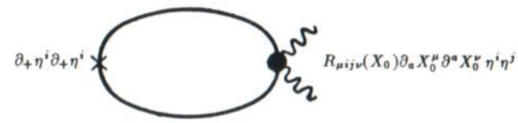
Chapter 6: Strings in Background Fields

Note this chapter is specific to *closed oriented strings*. As such, we will not consider the effects of the boundary.

0. This is not a required problem but it certainly should be [2]. Let's *calculate the β-functions of the nonlinear sigma model*. Here, I will borrow diagrams from the very nice set of TASI lecture notes of Callan and Thorlacius

   First, it is worth using a normal coordinate system for the X^μ (one in which all of the Γ symbols vanish and all higher symmetrized Γ symbols also vanish). We want to look at radiative corrections to $\langle T_{++}\rangle$, since they have integrals that are easier to handle than those for $\langle T_{+-}\rangle$. From conservation this will give us the trace anomaly for $\langle T_{+-}\rangle$. We will first look at how G, B affect the trace on a flat worldsheet.

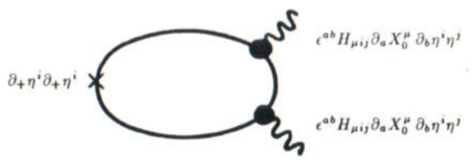
   For the graviton contribution to β^G, we have only only one diagram

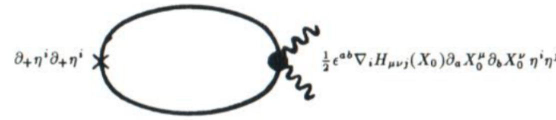


   This contributes an anomalous trace of

$$\langle T_{+-}\rangle = \frac{1}{4}R_{\mu\nu}\,\partial_a X_0^\mu \partial^a X_0^\nu$$

   For the B contribution to β^B, we have two such diagrams:





   These contribute anomalous traces of:

$$-\frac{1}{16}H_{\mu\rho\sigma}H_{\nu\rho\sigma}\,\partial_a X_0^\mu \partial^a X_0^\nu, \quad \frac{1}{8}\nabla^\lambda H_{\mu\nu\lambda}\epsilon^{ab}\partial_a X_0^\mu \partial_b X_0^\nu$$

   respectively.

   The dilaton contribution *also* affects the trace on the flat world sheet (even though it does not couple at $R = 0$), by affecting the stress energy tensor as it is defined by varying the action w.r.t. the metric. Kiritsis has worked this out before and shown that the dilaton contributes $(\partial_a\partial_b - g_{ab}\Box)\Phi$ to the stress energy tensor, from which we get a dilaton contribution of $\Box_\xi\Phi(X(\xi))$ to the trace. Using covariant expressions for the D'alambertian we arrive at a contribution

$$\nabla_\mu\nabla_\nu\Phi(X_0)\,\partial_a X_0^\mu \partial^a X_0^\nu - \frac{1}{2}\nabla^\lambda\Phi(X_0)H_{\mu\nu\lambda}(X_0)\partial_a X_0 \partial_b X_0 \epsilon^{ab}$$

   Combining all of this together, we see that we will get the β-functions:

$$\beta^G = R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_\nu^{\rho\sigma} + 4\nabla_\mu\nabla_\nu\Phi, \quad \beta^B = -\frac{1}{2}\nabla^\lambda H_{\lambda\mu\nu} - 2\nabla^\lambda\Phi H_{\lambda\mu\nu}.$$

   As pointed out, these are not quite that RG beta functions (for example compare β^B to the correct form in Kiritsis), but around the fixed point, they capture the correct first order behavior. In particular their vanishing will mean that we have no Weyl anomaly.

[2] After seeing the details of this calculation, I can understand why it was omitted.

Now we need to account for the effects of a curved worldsheet geometry. We can account for this by looking at a $\langle T_{+-}T_{+-}\rangle$ correlator:

$$\frac{\delta}{\delta\phi(\xi)}\langle T_{+-}(0)\rangle_{e^{\phi}\delta_{ab}} = -\frac{1}{4\pi}\langle T_{+-}(\xi)T_{+-}(0)\rangle_{\delta_{ab}} \tag{62}$$

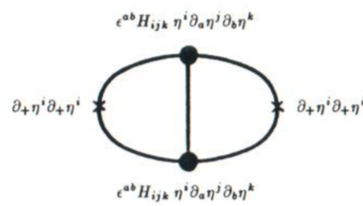
Again we can get this by first looking at $\langle T_{++}T_{++}\rangle$ and appealing to conservation. The Weyl anomaly comes from this diagram:

This gives $\langle T_{+-}T_{+-}\rangle = \frac{\pi D}{12}\Box\delta^{(2)}(\xi)$. Here we have a factor of D coming from each degree of freedom. This can be used to integrate equation (62) to yield:

$$\langle T_{+-}\rangle = -\frac{D}{48}\Box\phi = \frac{D}{24}\sqrt{\gamma}\,R$$

Note that the ghosts (which are otherwise decoupled) will here contribute their factor of -26.

We also now need to consider *two-loop* contributions of G, B to the TT correlator. The following diagrams contribute:



The calculations here are very involved, but will precisely give us

$$\frac{\alpha}{8}(-R + \frac{H^2}{12})$$

Finally, the dilaton both modifies the energy-momentum tensor, giving rise to a tree-level propagator contribution to the two-point function:

This contributes $\langle T_{+-}^{dil}T_{+-}^{dil}\rangle = \pi\alpha'(\nabla\Phi)^2\Box\delta^{(2)}(\xi)$ which will integrate to give a factor of $\frac{\alpha'}{2}(\nabla\Phi)^2\sqrt{\gamma}R$.

Also, the dilaton gives a loop-contribution to the unmodified energy-momentum tensor:

Which contributes the term $\langle T^{dil}_{+-} T^{dil}_{+-}\rangle = -\pi\alpha'\Box\Phi\Box\delta^{(2)}(\xi)$ which will integrate to give a factor of $-\frac{\alpha'}{2}\Box\Phi\sqrt{\gamma}R$. Altogether this gives:

$$\beta^{\Phi} = D - 26 + \frac{3}{2}\alpha'\left[4(\nabla\Phi)^2 - 4\Box\Phi - R + \frac{1}{12}H^2\right].$$

as required.

1. Each β-function of a coupling constant G, B, Φ as given in **6.1.5, 6.1.6, 6.1.7** is $\frac{\delta}{\delta\phi}$ of that coupling constant, since our scaling $\mu = e^{\phi} \Rightarrow \log\mu = \phi$. Since

$$T^a_a = \frac{\beta^{\Phi}}{12}R^{(2)} + \frac{1}{2\ell_s^2}(\beta^G_{\mu\nu}g^{\alpha\beta} + \beta^B_{\mu\nu}\varepsilon^{\alpha\beta})\partial_\alpha X\partial_\beta X$$

The change in effective action under an infinitesimal Weyl transformation $\delta g^{\alpha\beta} = -g^{\alpha\beta}\delta\phi$ is

$$\delta\log Z = -\delta S = \frac{1}{4\pi}\int d^2\xi\sqrt{g}T^a_a\delta\phi = \frac{1}{4\pi}\int d^2\xi\left[\frac{\beta^{\Phi}}{12}\sqrt{g}R^{(2)} + \frac{1}{2\ell_s^2}\left(\beta^G_{\mu\nu}\sqrt{g}\,g^{ab} + \beta^B_{\mu\nu}\varepsilon^{ab}\right)\partial_a X\partial_b X\right]\delta\phi$$

We can integrate this to get the change after a finite conformal transformation:

$$\frac{1}{4\pi}\int d^2\xi\left[\sqrt{g}\beta^{\Phi}\left(R^{(2)}\phi - \frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi\right) + \frac{\phi}{2\ell_s^2}\left(\beta^G_{\mu\nu} + \beta^B_{\mu\nu}\epsilon^{ab}\right)\partial_a X\partial_b X\right]$$

this vanishes, of course, when all beta functions are zero. When β^G, β^B are zero we can show (exercise 3) that β^{Φ} is a constant, and we recover the Liouville action from before.

2. First write G explicitly in the action:

$$S = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-\det G}e^{-2\Phi}\left[R + 4G^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi - \frac{1}{12}G^{\alpha\delta}G^{\beta\epsilon}G^{\gamma\zeta}H_{\alpha\beta\gamma}H_{\delta\epsilon\zeta} + 2\frac{26-D}{3\ell_s^2}\right]$$

The classical equations of motion from varying the action with respect to G give

$$\begin{aligned}0 &= \overbrace{R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \cancel{4\nabla_\mu\Phi\nabla_\nu\Phi} - 2G_{\mu\nu}\Box\Phi + 4G_{\mu\nu}(\nabla\Phi)^2}^{R\text{ variation}} + \overbrace{\cancel{4\nabla_\mu\Phi\nabla_\nu\Phi}}^{(\nabla\Phi)^2\text{ variation}}\\ &\quad \underbrace{-\frac{1}{4}H_{\mu\rho\sigma}H_\nu^{\rho\sigma}}_{H^2\text{ variation}} \underbrace{-\frac{1}{2}G_{\mu\nu}\left(R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2 + \frac{2}{3\ell_s^2}(26-D)\right)}_{\sqrt{-\det G}\text{ variation}}\\ &= \underbrace{R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\rho\sigma}H_\nu^{\rho\sigma}}_{:=\beta^G_{\mu\nu}} - \frac{1}{2}G_{\mu\nu}\left(R - 4(\nabla\Phi)^2 + 4\Box\Phi - \frac{1}{12}H^2 + 2\frac{26-D}{3\ell_s^2}\right)\end{aligned} \tag{63}$$

With respect to B we get:

$$-\frac{1}{12}e^{-2\Phi}(2(\delta_{B^{\mu\nu}}(\partial_\alpha B_{\beta\gamma} + 2\,\text{perms.}))H^{\alpha\beta\gamma}) \overset{IBP}{\longrightarrow} \frac{2\times 3}{12}e^{-2\Phi}(\nabla^\alpha H_{\alpha\mu\nu}) \overset{IBP}{\longrightarrow} \underbrace{-\frac{1}{4}\nabla^\alpha(e^{-2\Phi}H_{\alpha\mu\nu})}_{:=\beta^B_{\mu\nu}} = 0$$

Finally, with respect to Φ we get:

$$0 = -2\left(R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2 + 2\frac{26-D}{3\ell_s^2}\right) - 8\Box\Phi - 16(\nabla\Phi)^2 = -2\underbrace{\left(R - 4(\nabla\Phi)^2 + 4\Box\Phi - \frac{1}{12}H^2 + 2\frac{26-D}{3\ell_s^2}\right)}_{:=-\frac{2}{3}\beta^{\Phi}}$$

The term in parentheses is the same as the term in parentheses the bottom line of (63). This agrees with **Polchinski 3.7.21** (with appropriate conventions adopted)

$$\delta S = -\frac{1}{2\kappa^2}\int d^Dx\sqrt{-\det G}e^{-2\Phi}\left[\delta G^{\mu\nu}\left(\beta^G_{\mu\nu} - \frac{1}{2}G_{\mu\nu}\frac{2}{3}\beta^{\Phi}\right) + \delta B^{\mu\nu}\beta^B_{\mu\nu} + 2\delta\Phi\frac{2}{3}\beta^{\Phi}\right]$$

3. Let's look at $\frac{2}{3\ell_s^2}\nabla\beta^\Phi$. We get:

$$8\nabla_\nu\Phi\nabla_\mu\nabla^\nu\Phi - 4\Box\nabla_\mu\Phi - \nabla_\mu R + \frac{1}{6}(\nabla_\mu H_{\alpha\beta\gamma})H^{\alpha\beta\gamma}$$

   The contracted Bianchi identity $\nabla_\mu R = 2\nabla^\nu R_{\nu\mu}$ together with the vanishing of $\beta^G_{\mu\nu}$ gives:

$$\nabla_\mu R = 2\nabla^\nu R_{\mu\nu} = \frac{1}{2}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) - 4\Box\nabla_\mu\Phi$$

   which in turn gives

$$8\nabla_\nu\Phi\nabla_\mu\nabla^\nu\Phi - \frac{1}{2}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) + \frac{1}{6}(\nabla_\mu H_{\alpha\beta\gamma})H^{\alpha\beta\gamma}$$

   The fact that H is exact gives us $\mathrm{d}H = 0$ so $\partial_{[\alpha}H_{\beta\gamma\delta]} = 0$. The symmetry properties of H imply that summing over the four cyclic permutations of this gives zero. Contracting with the metric then implies a contracted Bianchi-type identity for H, namely that $\nabla^\alpha H_{\alpha\beta\gamma} = 0$.

   Using $\beta^B = 0$ together with the Bianchi identity, we have $0 = \nabla^\rho H_{\mu\nu\rho} = 2\nabla^\rho\Phi\, H_{\mu\nu\rho}$. So we have that H is divergence-free, and $\nabla^\rho\Phi$ dotted with any component of H is zero. This lets us rewrite:

$$-\frac{1}{2}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) = -\frac{1}{2}H^{\nu\rho\sigma}\nabla_\nu H_{\mu\rho\sigma}$$

$$\frac{1}{6}\nabla_\mu(H_{\alpha\beta\gamma})H^{\alpha\beta\gamma} = -\frac{1}{6}H^{\alpha\beta\gamma}\left(\nabla_\alpha H_{\beta\gamma\mu} - \nabla_\beta H_{\gamma\alpha\mu} + \nabla_\gamma H_{\alpha\beta\mu}\right) = -\frac{1}{6}H^{\nu\rho\sigma}\nabla_\nu H_{\mu\rho\sigma}$$

$$\Rightarrow \frac{1}{12\ell_s^2}\nabla_\mu\beta^\Phi = \nabla_\nu\Phi\nabla_\mu\nabla^\nu\Phi - \frac{1}{12}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) = -\frac{1}{2}\nabla^\nu\Phi R_{\mu\nu} - \frac{1}{12}\nabla^\nu H_{\mu\nu}$$

   One last step. I am missing something.

   This gives that $\nabla_\mu\beta^\Phi = 0$ as required. So $\beta^\Phi = c$ is a constant.

4. We get a linear dilaton giving rise to a Liouville action with $Q = 0$. This is our familiar free massless boson in $2D$ with $1D$ target space. So we get a string propagating in a single dimension.

5. Note that the only relevant parameters are ℓ_s, with units of length, and whatever length scales there are on the manifold, all of which depend on its volume (since its compact) as $V^{1/D}$. In particular $c = \beta^\Phi$ depends on ℓ_s as

$$c = D + O(\ell_s^2/V^{2/D}).$$

   I think this is correct, though it is different from Kiristis' equation.

6. Note that a nonzero total flux of H over any closed 3-manifold is incompatible with $H = dB$ for a single-valued B. We can write:

$$e^{\frac{i}{2\pi\ell_s^2}\int_M B} = e^{\frac{i}{2\pi\ell_s^2}\int_N H}$$

   where M is the 2D manifold corresponding to the embedding of the world-sheet into the target space and N is any manifold whose boundary is M. We need this to be independent of N, so for any three-cycle M_3 we need:

$$\frac{1}{2\pi\ell_s^2}\int_{M_3} H \in 2\pi\mathbb{Z} \Rightarrow \frac{1}{4\pi^2\ell_s^2}\int_{M_3} H \in \mathbb{Z}$$

7. (a) We have

$$H = 2R^2 \sin^2\psi \sin\theta \,\mathrm{d}\psi \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi \Rightarrow \int_{S^3} H = \frac{(2\pi R)^2}{4\pi^2 \ell_s^2} = \frac{R^2}{\ell_s^2} \in \mathbb{Z}$$

(b) The dilaton is $\Phi = 0$. Using Mathematica, the Ricci tensor is:

$$R_{\mu\nu} = \mathrm{diag}(2, 2\sin^2\psi, 2\sin^2\psi \sin^2\theta)$$

Which gives a Ricci scalar of $6/R^2$. From the previous part, $H_{123} = 2R^2 \sin^2\psi \sin\theta$. From the metric being diagonal, we get that $H^2_{\mu\nu} := H_{\mu\rho\sigma} H_\nu^{\rho\sigma}$ is diagonal. We have

$$H^2_{\mu\nu} = \mathrm{diag}(8, 8\sin^2\psi, 8\sin^2\psi \sin^2\theta) \Rightarrow \beta^G = R_{\mu\nu} - \frac{1}{4} H^2_{\mu\nu} = 0$$

as desired. Next, $\beta^B_{\mu\nu} = -\frac{1}{2}\nabla^\alpha(H_{\mu\nu\alpha})$. To take a contravariant divergence we divide by the volume element and differentiate, but the volume element is $\sin^2\psi \sin\theta$ which will give $H/\sqrt{g}$ is a constant, so $\beta_{\mu\nu}$ will vanish.

Lastly, $H^2 = (2R^2)^2/R^6 = 2/R^6$ so that $-R + \frac{1}{12}H^2 = -\frac{4}{R^2}$. Ignoring ghosts, this gives a central charge of:

$$D - 6\frac{\ell_s^2}{R^2} + O(\ell_s^4) = D - \frac{6}{k} + O(\ell_s^4)$$

as desired.

(c) Without using coordinates, the isometry of S^3 is $G = \mathrm{SO}(4) = [\mathrm{SU}(2) \times \mathrm{SU}(2)]/\mathbb{Z}_2$. TO see that equivalence, think of of S^3 as the unit quaternions, and take $\mathrm{SU}(2) \times \mathrm{SU}(2)$ act as unit quaternions on the left and right. We get a right G-action by: $x \to a^{-1}xb$. Note the kernel is the set of $(a, b) \in G$ $ax = xb$ for all x. In particular, for $x = 1$ we get $a = b$ so the kernel lies in the diagonal subgroup. To act trivially on all quaternions, a must be in the center, and for the unit quaternions this is exactly ± 1. So this is an injection $\varphi : [\mathrm{SU}(2) \times \mathrm{SU}(2)]/\mathbb{Z}_2 \to \mathrm{SO}(4)$. Since $\mathrm{SO}(4)$ is compact and connected, it is generated by the image of exponentiating $\mathfrak{so}(4)$, and so surjectivity of φ at the level of the Lie algebras (which is true by dimension-counting) implies surjectivity and hence equivalence at the level of Lie groups.

So we see that $\mathfrak{so}(4)$ acting on S^3 is just a simultaneous left and right copt of $\mathfrak{su}(2)$ acting on $\mathrm{SU}(2)$. Thus, we view this as the CFT of a nonlinear sigma model with target space $G = \mathrm{SU}(2)$ and the left, right copies of the $\mathfrak{su}(2)$ action correspond to currents $J = g^{-1}\partial g$ and $J = \bar{\partial} g\, g^{-1}$

We indeed get the central charge $c = \frac{3k}{k+2}$ which has the large k expansion $3 - 6/k + O(1/k^2)$. Since k in a non-negative integer in WZW models, except for the case $k = 0$ corresponding to the trivial CFT, we must have $k \geqslant 1$, where we get $R \geqslant \ell_s$.

8. Here the metric has three degrees of freedom and $B_{\mu\nu}$, Φ both have only one degree of freedom (which can be spatially varying). H, being a 3-index antisymmetric tensor, must vanish in $1+1$D, and so we will always have $\beta^B = 0$. The other two constraints become:

$$0 = \beta^G_{\mu\nu} = \frac{1}{2} R g_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi, \qquad 0 = \beta^\Phi = -24 + \frac{3}{2}\ell_s^2 \left[4(\nabla\Phi)^2 - 4\Box\Phi - R\right]$$

Translational isometry implies that R, g depend on only the time variable t. The x variable can therefore parameterize either S^1 or $\mathbb{R}$ endowed with constant metric.

Now taking the trace of the first equation implies $R(t) = -2\Box\Phi(x, t)$. Then the second equation will give:

$$\frac{16}{\ell_s^2} = 4(\nabla\Phi(x, t))^2 - 2(\Box\Phi)(t)$$

The only way for this to work is for $R = \Box\Phi = 0$ so that $\nabla\Phi$ can be a constant. We then have $\Phi = \alpha x + \beta t$ so that $\alpha^2 + \beta^2 = 4/\ell_s^2$, and g is Ricci flat everywhere (so we can pick it to be constant). In the case of either $\alpha, \beta = 0$, we can also safely take x, t respectively to be periodic without having Φ be multi-valued.

9. We still have $\beta^B = 0$, but $\beta^G = R_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi$ while $\beta^\Phi = D - 26 + \frac{3}{2}\ell_s^2(4(\nabla\Phi)^2 - 4\Box\Phi - R)$

   This can be recast in terms of a new 4D *Ricci flat* metric $ds^2 = F(\phi)d\phi^2 + \phi R^2 d\Omega_3^2$.

   Using Mathematica again to take the trace of this gives R_{ij} for $i = j \geqslant 1$ proportional to $R^2\phi F'(\phi) + 8\phi F(\phi)^2 - R^2 F(\phi)$. Solving this differential equation for F gives

$$F(\phi) = \frac{R^2\phi}{4\phi^2 + R^2 c_1}$$

   Setting $c_1 = 0, F(\phi) = R^2/4\phi$ will also make R_{00} vanish. Then we can take the dilaton to be zero $\Phi(\phi) = 0$.

10. As stated in the problem, upon gauging the adapted compact $U(1) : \theta \to \theta + \epsilon$, which has radius 2π, we modify our derivative operator to act as $\partial_\alpha\theta \to \partial_\alpha\theta + A_\alpha$, where A_α gives our connection on the $U(1)$ principal bundle associated with gauging the Killing symmetry. The action gets modified:

$$S \supseteq \frac{R^2}{4\pi\ell_s}\int |\partial\theta|^2 \to \frac{R^2}{4\pi\ell_s^2}\int |\partial\theta + A|^2$$

    This is a new theory, but we can *return to the old one* by enforcing that A be pure gauge as follows: introduce an auxiliary field ϕ and add to S the term

$$\frac{i}{2\pi}\int \phi\, \epsilon^{\alpha\beta}\partial_\alpha A_\beta = -\frac{i}{2\pi}\int \mathrm{d}\phi \wedge A.$$

    Integrating out ϕ gives exactly a δ-function enforcing $\epsilon^{\alpha\beta}\partial_\alpha A_\beta = 0$. This gives that A is closed, but it need not be exact if our manifold has nontrivial topology. Going around any cycle, $\int A$ can pick up a factor of $2\pi n$.

    For a closed, genus g Riemann surface, there are $2g$ cycles labeled by a_i, b_i, $1 \leqslant i \leqslant g$ coming from viewing it as a $2g$-gon. we have *Riemann's bilinear identity*, namely for two closed 1-forms ω_1, ω_2,

$$\int_\Sigma \omega_1 \wedge \omega_2 = \sum_{i=1}^{g}\left(\int_{a_i}\omega_1 \int_{b_i}\omega_2 - \int_{a_i}\omega_2 \int_{b_i}\omega_1\right) \tag{64}$$

    Now take $\omega_1 = A$, $\omega_2 = \mathrm{d}\phi$. Now (64) gives us that $\frac{1}{2\pi}\int \mathrm{d}\phi \wedge A$ will not be zero in general, but in the path integral, it suffices to have it be an integral multiple of 2π, since then the nontrivial holonomies will have no contribution to the action. We have that A can have winding $2\pi\mathbb{Z}$, so the only solution is to have ϕ have winding $2\pi\mathbb{Z}$. This will exactly leave over a factor of $2\pi\mathbb{Z}$. So we return to our original action by introducing the field ϕ of period 2π. (NB if I had kept the fields dimensionful, then ϕ would have period $2\pi/R$ when θ has period $2\pi R$)

    In this new, equivalent action, we can gauge-fix $\theta = 0$ (do I need ghosts? No because this is abelian $U(1)$) and integrate out A. We get:

$$\frac{\ell_s^4/R^2}{4\pi\ell_s^2}\int d^2\xi\,(\partial\phi)^2$$

    so we have obtained the same action but now on a circle of radius ℓ_s^2/R instead of R.

    In doing this path integral we get a determinant factor of $\sqrt{4\pi^2\ell_s^2/R^2} = 2\pi\ell_s/R$ for each mode. Using zeta function regularization this is equal to $\sqrt{R/2\pi\ell_s}$ which we can understand as adding a $-\frac{1}{2}\log(R/2\pi\ell_s)$ term to the action that will couple to the curvature R **(Show why)**, this shifting the dilaton as required.

11. We can simplify things by using the conventions of the next problem to do this one. Here, we have a *single* compact coordinate θ. In our convention:

$$\hat{G}_{\mu\nu} = \begin{pmatrix} G_{00} & G_{00}A_j \\ G_{00}A_i & g_{ij} + G_{00}A_iA_j \end{pmatrix}, \quad B_{\mu\nu} = B_j d\theta \wedge dx^i + A_iB_jb_{ij}dx_i \wedge dx_j, \quad \phi = \Phi - \frac{1}{4}\log\det G_{00}$$

From formula **F.3** specialized to this case, we get that the metric and dilaton terms become

$$\int d^D x\sqrt{-\det \hat{G}_{\mu\nu}}e^{-2\Phi}\left[\hat{R}+4(\partial_\mu\Phi)^2\right]=\int d^{D-1}x\sqrt{-\det g}\,e^{-2\phi}\left[R+4(\partial_\mu\phi)^2+\frac{1}{4}\partial_\mu G_{00}\partial^\mu G^{00}-\frac{1}{4}G_{00}(F^A_{\mu\nu})^2\right] \tag{65}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $\hat{R}$ corresponds to the original $\hat{G}_{\mu\nu}$ while R corresponds to g_{ij}. Further $G^{00}=$ From **F.6-F.9**, the antisymmetric tensor changes as:

$$-\frac{1}{12}\int d^D\sqrt{-\det\hat{G}}\,e^{-2\Phi}\hat{H}_{ijk}\hat{H}^{ijk}=-\int d^{D-1}x\sqrt{-\det g}\,e^{-2\phi}\left[\frac{1}{12}H_{ijk}H^{ijk}+\frac{1}{4}\hat{H}_{ij0}\hat{H}^{ij0}\right] \tag{66}$$

Here where $H_{ij0}=\hat{H}_{ij0}$ and $H_{ijk}=\hat{H}_{ijk}-(A_iH_{0jk}+3\,\text{perms.})$. Here H_{ijk} is defined so that it is invariant under T-duality **(TYSM Kiritsis for pre-organizing these terms for me)** . Further, under T-duality

$$\begin{aligned}
G_{00}&\to G_{00}^{-1}=G^{00}\Rightarrow\partial_\mu G_{00}\partial^\mu G^{00}\text{ invariant}\\
g_{ij}&\to g_{ij}\Rightarrow R\text{ invariant}\\
A_i&\to B_i\\
B_i&\to A_i\\
\Phi&\to\Phi-\frac{1}{2}\log G_{00}\Rightarrow\phi\to\phi\Rightarrow(\partial_\mu\phi)\text{ invariant.}
\end{aligned} \tag{67}$$

We see that the $\sqrt{-\det g}\,e^{-2\phi}$ as well as first three terms of equation (65). We have that $F^A_{\mu\nu}\to\partial_\mu B_\nu-\partial_\nu B_\mu=:F^B_{\mu\nu}$ and $F^B_{ij}=H_{ij0}$. The last term of (65) will therefore become swap with the last term of (66) and we are done.

12. This one is quick. We have

$$ds^2=G_{00}d\theta^2+2G_{00}A_idx^idx^0+G_{ij}dx^idx^j,\quad B=B_jd\theta\wedge dx^j+(b_{ij}+A_iB_j)dx^i\wedge dx^j$$

Certainly we have $\tilde{G}_{00}=1/G_{00},\tilde{B}_i=G_{00}A_i/G_{00}$. Then $\tilde{A}_i=B_i$ is consistent both for the $i,0$ components of the line element and the $dx^i\wedge dx^j$ components of the B-field as long as we keep $\tilde{b}_{ij}=b_{ij}$ and $\tilde{g}_{ij}=g_{ij}$. Finally, the dilaton must be shifted by $\Phi=\Phi-\frac{1}{2}\log G_{00}$.

13. The N commuting isometries correspond to a fibration by N-dimensional tori over each point in the base space. As we have seen before (for strings valued in a N-dimensional torus target space), we have that modes are described by two momenta p_L,p_R that Lie on an integral lattice. Naively, we can rotate p_L,p_R by any $\mathrm{GL}(N)$ transformation, but the integrality condition restricts us to $\mathrm{GL}(N,\mathbb{Z})$. Now $\mathrm{GL}(N)$ acts separately on the left and the right momenta, but we are allowed to exchange between these two by applying T-duality, which still preserves our Lorentzian norm, so the T-duality group gets enhanced to $O(N,N,\mathbb{Z})$.

14. This is clear, since orientation reversal acts trivially on $g^{ab}G_{\mu\nu}\partial_aX^\mu\partial_bX^\nu$ while it acts with a minus sign on $\epsilon^{ab}B_{\mu\nu}\partial_aX^\mu\partial_bX^\nu$. The corresponding vertex operators are:

$$:\partial X^\mu\bar{\partial}X_\mu e^{ikX}:,\quad :G_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu:,\quad R:e^{ikX}:$$

If we assume the tachyon $:e^{ikX}:$ is negative under parity then so are the dilaton and graviton.

This is incompatible with **6.1.10**, as then parity will flip the sign of the dilaton in the exponential, substantially changing the action of the theory.

Chapter 7: Superstrings and Supersymmetry

1. We already know that TT will have the desired OPE, since the bosons and fermions are uncoupled and we already have shown their own respective stress tensor OPEs. Next

$$
\begin{aligned}
G(z)G(w) &= -\frac{2}{\ell_s^4}\psi_\mu(z)\partial X^\mu(z)\psi_\nu(w)\partial X^\nu(w) \\
&= -\frac{2}{\ell_s^4}\left(\ell_s^2\frac{\eta_{\mu\nu}}{z-w} + (z-w) :\partial\psi_\mu\psi_\nu(w):\right)\left(-\frac{\ell_s^2}{2}\frac{\eta_{\mu\nu}}{(z-w)^2} + :\partial X_\mu\partial X_\nu(w):\right) \\
&= \frac{D}{(z-w)^3} + \frac{-\frac{2}{\ell_s^2}\partial X_\mu\partial X^\mu(w) - \frac{1}{\ell_s^2}\psi^\mu\partial\psi_\mu(w)}{z-w} \\
&= \frac{\hat{c}}{(z-w)^3} + \frac{2T(w)}{z-w}
\end{aligned}
$$

Finally

$$
\begin{aligned}
T(z)G(w) &= -\frac{1}{\ell_s^2}\left(:\partial X_\mu\partial X^\mu(z): + \frac{1}{2}\psi^\mu\partial\psi_\mu(z)\right) i\frac{\sqrt{2}}{\ell_s^2}\psi_\nu\partial X^\nu(w) \\
&= -i\frac{\sqrt{2}}{\ell_s^4}\left(-\frac{\ell_s^2}{2}\frac{\psi_\mu\partial X^\mu(w) + \psi_\mu\partial^2 X^\mu(w)(z-w)}{(z-w)^2} - \frac{\ell_s^2}{2}\frac{\psi_\mu\partial X^\mu(w)}{(z-w)^2} + (-)\frac{\ell_s^2}{2}\frac{\partial_\mu\psi\partial X^\mu(w)}{(z-w)}\right) \\
&= \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}
\end{aligned}
$$

2. We will take the OPE of $j_B(z)j_B(w)$, but just look at the $(z-w)^{-1}$ term as a function of w, as this, when integrated around the origin in w will give Q_B^2. This is an extension of exercise **4.45**, and there is nothing conceptually further, except for some $\beta\gamma$ manipulation. There are altogether 16 terms to consider, and we will get $c = 15$. The algebra is heavy, so I will skip this. An alternative is to do this as in **Polchinski 4.3**.

   To do it this way, note the following OPEs:

$$
\begin{aligned}
j_B(z)b(w) &\sim \frac{T_{matter}(z)}{z-w} - \frac{1}{(z-w)^2}\left(bc(z) + \frac{3}{4}\beta\gamma(z)\right) + \frac{1}{z-w}\left(-b\partial c(z) + \frac{1}{4}\partial\beta\gamma(z) - \frac{3}{4}\beta\partial\gamma(z)\right) \\
&= \cdots + \frac{1}{z-w}\left[T_{matter}(z) - \partial b\, c(w) - 2b\partial c(w) - \frac{1}{2}\partial\beta\gamma(w) - \frac{3}{2}\beta\partial\gamma(w)\right] \\
&= \cdots + \frac{T_{matter}(w) + T_{gh}(w)}{z-w} \Rightarrow \{Q_B, b_n\} = L_n
\end{aligned}
$$

   Similarly

$$
j_B(z)\beta(w) = \cdots + \frac{G_{matter}(w) + G_{gh}(w)}{z-w} \Rightarrow [Q_B, \beta_n] = G_n
$$

   Now note that the Jacobi identity on Q_B reads:

$$
\{[Q_B, L_m], b_n\} - \{\overbrace{[L_m, b_n]}^{(m-n)b_{m+n}}, Q_B\} - [\overbrace{\{b_n, Q_B\}}^{L_n}, L_m] = 0 \Rightarrow \{[Q_B, L_m], b_n\} = (m-n)L_{m+n} - [L_m, L_n]
$$

   So if the total central charge is zero we'll get $\{[Q_B, L_m], b_n\} = 0$, implying that $[Q_b, L_m]$ is independent of the c ghost. But on the other hand this operator has ghost number 1, so it must therefore vanish. Further, the Jacobi identity also yields

$$
[\{Q_B, Q_B\}, b_n] = -2[\{b_n, Q_B\}, Q_B] = 2[Q_B, L_n]
$$

   since we just showed that this last term vanishes, we must have Q_B, Q_B is also independent of c, but again since Q_B^2 has positive ghost number, we get that it is in fact zero. We can do the same argument with β and G and get that the superstring BRST operator is zero, as long as the total central charge vanishes. This was much cleaner than the OPE way.

3. First a lemma: An abelian p-form field A has $\binom{D-2}{p}$ on shell DOF. To prove this, note that we have a gauge symmetry of $A \to A + \partial\Lambda$ which has $\binom{D}{p-1}$ parameters. Next, the Euler-Lagrange equations give us that the components $A^{0i_1 \dots i_{p-1}}$ are non-propagating. We thus get $\binom{D-1}{p}$ massless propagating off-shell d.o.f. which have $\binom{D-2}{p-1}$ gauge symmetries left over. These can be used to enforce Coulomb gauge conditions which allow for there to be no polarizations along one of the spatial directions. We thus get $\binom{D-1}{p} - \binom{D-2}{p-1} = \binom{D-2}{p}$ massless on-shell degrees of freedom. For A_μ this is $D-2$ and for $B_{\mu\nu}$ this is $(D-2)(D-3)/2$.

   The metric has $\frac{1}{2}D(D-3)$ on-shell degrees of freedom. There are two ways to see this, first, that the dynamically allowed variation δg may on-shell be described by a symmetric traceless tensor in dimension $D-2$ which gives
$$\frac{(D-1)(D-2)}{2} - 1 = \frac{1}{2}D(D-3)$$
   or by noting that since we are gauging translation symmetry locally, each translation makes 2 polarizations unphysical and so we get:
$$\frac{D(D+1)}{2} - 2D = \frac{1}{2}D(D-3)$$
   as required.

   We now consider the R-R, R-NS, NS-R, NS-NS sectors together. For NS-NS we have the scalar = 1 both on-shell and off-shell, the antisymmetric two-form, which has only transverse degrees of freedom $= 8 * 7/2 = 28$ and the gravity, $= 10 * 7/2 = 35$ altogether we get 64 on-shell degrees of freedom.

   In both the R-NS and NS-R sector, we have a Weyl representation of dimension $2^{5-1} = 16$. There are however only 8 on-shell degrees of freedom. Similarly, we only consider the on-shell $\psi^\mu_{-1/2}$ acting on the NS part of the vacuum which gives another factor of 8. This gives 64 fermionic variables in each sector for a grand total of 128.

   In R-R for IIA we have a 0, 2, and *self-dual* 4-form. This gives:
$$1 + \binom{8}{2} + \frac{1}{2}\binom{8}{4} = 64$$
   For IIB we have a 1 and 3-form. This gives
$$\binom{8}{1} + \binom{8}{3} = 64$$
   so in either case we have 64 on-shell degrees of freedom here. This is consistent with each $|S\rangle$ state having 8 on-shell degrees of freedom giving $8 \times 8 = 64$. All together, we have the same number of on-shell fermionic and bosonic degrees of freedom.

   Now for the massive case. In the NS sector you might expect the next excitations come from the bosons α_{-1}, but this gets projected out by GSO, so in fact the next states come from $C_{ijk}\psi^i_{-1}\psi^j_{-1}\psi^k_{-1}$ and $C_{ij}\psi^i_{-1}\alpha^j_{-1}$. The physical state conditions will force them to transform as $\mathbf{8}^3$ and $\mathbf{8}^2$. In the R sector, it is quick to see that neither α_{-1} nor ψ_{-1} will satisfy $G_0 = 0$, so the massive state will come from the next level. This will then have mass higher than the NS sector, and so we can ignore it here.

   Consequently, the NS-NS sector will have a spin 6, spin 4, and two spin 5 massive particles. From R-NS and NS-R I can tensor the R vacuum $|S_\alpha\rangle$ or $|C_\alpha\rangle$ with the NS states and get two copies of $\mathbf{8}^4$ and $\mathbf{8}^3$. These will appropriately combine to give representations of $SO(9)$. **SHOW THIS PART**

   Finally, in the RR sector we get massive bosons of larger mass, which we thus disregard . I did not have to find the first massive state in the R sector to do this problem.

4. In terms of theta functions:

$$\chi_O = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_3(\nu_i)}{\eta} - \prod_{i=1}^{4}\frac{\theta_4(\nu_i)}{\eta}\right)$$
$$\chi_V = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_3(\nu_i)}{\eta} + \prod_{i=1}^{4}\frac{\theta_4(\nu_i)}{\eta}\right)$$
$$\chi_S = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_2(\nu_i)}{\eta} - \prod_{i=1}^{4}\frac{\theta_1(\nu_i)}{\eta}\right)$$
$$\chi_C = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_2(\nu_i)}{\eta} + \prod_{i=1}^{4}\frac{\theta_1(\nu_i)}{\eta}\right)$$

We'll take $\nu_i = 0$ here **(I assume this is what I'm supposed to do)** and so $\theta_1 = 0 \Rightarrow \chi_S = \chi_C$.

For IIB we look at

$$\frac{|\chi_V - \chi_C|^2}{(\sqrt{\tau_2}\eta\bar\eta)^8} = \frac{1}{(\sqrt{\tau_2}\eta\bar\eta)^8}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b}\frac{\theta^4\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}{\eta^4} \times \frac{1}{2}\sum_{\bar a,\bar b=0}^{1}(-1)^{\bar a+\bar b}\frac{\bar\theta^4\left[\begin{smallmatrix}\bar a\\\bar b\end{smallmatrix}\right]}{\bar\eta^4}$$

Under modular transformations $\tau \to \tau + 1$ $\theta^4\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \leftrightarrow \theta^4\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\theta^4\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \to -\theta^4\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ while $\eta^{12} \to -\eta^{12}$. In the holomorphic and anti-holomorphic parts separately, each term in the sum picks up a minus sign that is cancelled by the minus sign in the η^4.

Under $\tau \to -1/\tau$, the $\frac{1}{(\sqrt{\tau_2}\eta\bar\eta)^8}$ out front is invariant. On the other hand, the θ functions transform as $\theta^4\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \to (-i\tau)^2\theta^4\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\theta^4\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \to (-i\tau)^2\theta^4\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\theta^4\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \to (-i\tau)^2\theta^4\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. These are exactly compensated by the η transformations in the denominator, and no overall sign is picked up

For IIA we have similarly

$$\frac{(\chi_V - \chi_C)(\bar\chi_V - \bar\chi_S)}{(\sqrt{\tau_2}\eta\bar\eta)^8} = \frac{1}{(\sqrt{\tau_2}\eta\bar\eta)^8}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b}\frac{\theta^4\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}{\eta^4} \times \frac{1}{2}\sum_{\bar a,\bar b=0}^{1}(-1)^{\bar a+\bar b+\bar a\bar b}\frac{\bar\theta^4\left[\begin{smallmatrix}\bar a\\\bar b\end{smallmatrix}\right]}{\bar\eta^4}$$

Again, the holomorphic part transforms as before and as we have set the ν_i to zero, we have the same partition function. Using **D.18**, we see that each of the four above sums are zero since they are equal to a product of $\theta_1 = 0$.

5. Again, these are identical if I set the $\nu_i = 0$ (am I not supposed to be doing this? What do the ν_i represent physically?). They are equal to

$$\frac{1}{(\sqrt{\tau_2}\eta\bar\eta)^8 4\eta^4\bar\eta^4}\left(\cancel{|\theta_1^4|^2} + |\theta_2^4|^2 + |\theta_3^4|^3 + |\theta_4^4|^2\right)$$

We have θ_3 and θ_4 swapping under $\tau \to \tau + 1$, generating no signs in this case, while the denominator looks like $|\eta|^{24}$ and also doesn't generate a sign. Then, under $\tau \to -1/\tau$ we have θ_2 and θ_4 swapping generating a $|\tau|^4$, identical to what is generated by the $(\eta\bar\eta)^4$.

6. We can write this partition function as:

$$\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{\sqrt{\tau}^8\eta^{12}\bar\eta^{24}}\sum_{h,g}\sum_{\gamma,\delta,\gamma',\delta'}(-1)^{(\gamma+\gamma')g+(\delta+\delta')h}\bar\theta^8\begin{bmatrix}\gamma\\\delta\end{bmatrix}\bar\theta^8\begin{bmatrix}\gamma'\\\delta'\end{bmatrix}\sum_{a,b}(-1)^{a+b+ab+ag+bh+gh}\theta^4\begin{bmatrix}a\\b\end{bmatrix}$$

under $\tau \to \tau + 1$ we have $\theta^4\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \leftrightarrow \theta^4\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\theta^4\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \to -\theta^4\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ while $\eta^{12} \to -\eta^{12}, \bar\eta^{24} \to \bar\eta^{24}$. And swapping $\theta^4\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and $\theta^4\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ as well as $\bar\theta^4\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\bar\theta^4\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ will give us $(-1)^{1+h}$.

7.

8.

9.

10.

11.