Chapter 11: Duality Connections and Nonperturbative Effects

1. Taking the expression for a toroidal heterotic compactification from exercise **9.1**

$$\left[\frac{R}{\sqrt{\tau_2}\eta\bar{\eta}^{17}}\sum_{m,n}e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}e^{-i\pi\sum_I nY^I(m+n\bar{\tau})Y^IY^I}\frac{1}{2}\sum_{a,b=0}^{1}\prod_{i=1}^{16}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}(Y^I(m+\bar{\tau}n)|\bar{\tau})\right]\times\frac{1}{\tau_2^{7/2}\eta^7\bar{\eta}^7}\frac{1}{2}\sum_{a,b=0}^{1}\frac{\theta^4\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}{\eta^4}$$

Using θ function identites as in the second equation in appendex **E**, we get

$$\Gamma_{1,17}(R,Y)=\frac{R}{\sqrt{\tau_2}}\sum_{m,n}e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\frac{1}{2}\sum_{a,b=0}^{1}e^{i\pi mY^IY^In-i\pi bnY^I}\bar{\theta}\begin{bmatrix}a-2nY^I\\b-2mY^I\end{bmatrix}$$

Now take $Y^I=0$ for $I=1\dots8$ and $Y^I=1/2$ for $I=1\dots16$. Then

$$\prod_I e^{i\pi mY^IY^In-i\pi bnY^I}=e^{i\pi m\sum_I(Y^I)^2-i\pi b\sum_I Y^I}=1$$

and we can ignore this term. Similarly because we are taking a product over 16 $\bar{\theta}$, no phases will interfere with us replacing $\theta\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]$ with $\theta\left[\begin{smallmatrix}-u\\-v\end{smallmatrix}\right]$ for integer u,v. This gives us the desired first step

$$\Gamma_{1,17}(R,Y)=R\sum_{m,n}e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\frac{1}{2}\sum_{a,b=0}^{1}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}^8\bar{\theta}\begin{bmatrix}a+n\\b+m\end{bmatrix}^8$$

Now again because we have enough $\theta\left[\begin{smallmatrix}a+n\\b+m\end{smallmatrix}\right]$ that phases do not interfere, we see that we only care about n,m modulo 2 in the fermion term. We know how to divide the partition function of the compact boson into parity odd and even blocks by doing the $\mathbb{Z}^2$ stratification corresponding to the πR translation orbifold of the circle. This gives our desired answer:

$$\frac{1}{2}\sum_{h,g}\Gamma_{1,1}(2R)\begin{bmatrix}h\\g\end{bmatrix}\frac{1}{2}\sum_{a,b}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}^8\bar{\theta}\begin{bmatrix}a+h\\b+g\end{bmatrix}^8$$

with

$$\Gamma_{1,1}(2R)=2R\sum_{m,n}\exp\left[\frac{-\pi R^2}{\tau_2}|2m+g+(2n+h)\tau|^2\right]$$

2. As before, take the ansatz

$$ds^2=e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu+e^{2B(r)}dx^i\cdot dx^i,\qquad A_{012}=\pm e^{C(r)}\Rightarrow G_{r012}=\pm C'(r)\,e^{C(r)}$$

The BPS states in 11D require only the gravitino variation to vanish:

$$\delta\psi_M=\partial_M\epsilon+\frac{1}{4}\omega_M^{PQ}\Gamma_{PQ}\epsilon+\frac{1}{2\cdot3!\cdot4!}G_{PQRS}\Gamma^{PQRS}\Gamma_M\epsilon-\frac{8}{2\cdot3!\cdot4!}G_{MQRS}\Gamma^{QRS}\epsilon$$

We have worked out ω in **8.43**.

$$\omega_{\hat{\mu}\hat{\nu}}=0,\quad \omega_{\hat{\mu}\hat{i}}=(-)^{\mu=0}\,\partial_iA\,e^{A-B}dx^\mu,\quad \omega_{\hat{i}\hat{j}}=\partial_jBdx^i-\partial_iBdx^j$$

Let's look first at $M=\mu$ parallel. Since ϵ is Killing we expect no longitudinal variation and we get

$$\begin{aligned}0&=\cancel{\partial_\mu\epsilon}+\frac{1}{2}A'\,e^{A-B}\Gamma^{\hat{\mu}\hat{r}}\epsilon\pm\frac{1}{2\cdot3!}C'(r)e^C\cancel{\Gamma^{r012}\Gamma_\mu}\epsilon\mp\frac{1}{3!}C'(r)e^C\Gamma_\mu\Gamma^{r012}\epsilon\\&=\frac{1}{2}A'\,e^{A-B}\Gamma^{\hat{\mu}\hat{r}}\epsilon\mp\frac{1}{3!}C'e^{C-B-2A}\Gamma^{\hat{\mu}\hat{r}\hat{0}\hat{1}\hat{2}}\epsilon\\&\Rightarrow 0=A'\epsilon\mp\frac{1}{3}C'e^{C-3A}\Gamma^{\hat{0}\hat{1}\hat{2}}\epsilon\end{aligned}$$

If we would like these two terms to be proportional, then we should take $C = 3A$, and we get the following condition for ϵ

$$(1 \mp \Gamma^{\hat{0}\hat{1}\hat{2}})\epsilon = 0$$

So half the dimension of the space of spinors satisfies this at any given point. We thus get

For $M = i$ transverse, we recall Γ_{ij} generates rotations, so assuming rotational invariance in the transverse space, we'll cancel this. We get

$$\begin{aligned}
&\partial_r \epsilon + \cancel{\frac{1}{4}\omega_r^{jk}\Gamma_{jk}\epsilon} + \cancel{\frac{1}{2\cdot 3!}G_{r012}\Gamma^{r012}}\Gamma_r\epsilon \mp \frac{1}{3!}G_{r012}\Gamma^{012}\epsilon = 0\\
\Rightarrow &\partial_r\epsilon \mp \frac{1}{3!}G_{r012}\Gamma^{012}\epsilon = 0\\
\Rightarrow &\partial_r\epsilon \mp \frac{e^{-3A}}{3!}C' e^C \Gamma^{\hat{0}\hat{1}\hat{2}}\epsilon
\end{aligned}$$

Solving this gives us that

$$\epsilon(r) = e^{C(r)/6}\epsilon_0$$

for ϵ_0 some constant spinor. We still do not have a relationship between C and B. This can be obtained by not assuming rotational invariance but rather imposing cancelation of the second and third terms above as follows:

$$\begin{aligned}
&\frac{1}{2}\partial_j B\,\Gamma^{\hat{i}\hat{j}}\epsilon \pm \frac{1}{2\cdot 3!}\partial_j C\, e^C\, \Gamma^{j012}\Gamma_i\epsilon\\
&= \frac{1}{2}\partial_j B\,\Gamma^{\hat{i}\hat{j}}\epsilon \pm \frac{1}{2\cdot 3!}\partial_j C\, e^{C-3A}\,\Gamma^{\hat{i}\hat{j}\hat{0}\hat{1}\hat{2}}\epsilon\\
&\Rightarrow \partial_j B + \frac{1}{3!}\partial_j C = 0
\end{aligned}$$

where we have used the condition on ϵ already obtained. Thus $C = 3A = -6B$. Finally Let's look at G's equation of motion:

$$\mathrm{d}G = 0, \qquad \frac{1}{3!}\mathrm{d}\star G + \frac{3}{(144)^2}\epsilon^{MNOPQRST}G_{MNOP}G_{QRST} = 0$$

By assumption, the term quadratic in G vanishes. What remains gives us:

$$0 = \partial_r(e^{3A+8B}e^{-6A-2B}C'(r)e^C) = \partial_r(e^{-3A+6B+C}C') = \partial_r(C'e^{-C}) \Rightarrow \partial_r^2 e^{-C} = 0$$

So we have that $e^{-C} = H(r)$ as required, where

$$H(r) = 1 + \frac{L^6}{r^6}$$

I'm happy with this. I could use Mathematica to show that the other EOM:

$$R_{MN} - \frac{1}{2}g_{MN}R = \kappa^2 T_{MN}, \quad \kappa^2 T_{MN} = \frac{1}{2\cdot 4!}\left(4G_{MPQR}G_N^{PQR} - \frac{1}{2}g_{MN}G^2\right)$$

is satisfied - but this is barely different from what I've done several times before for the D-branes and fundamental string solutions in chapter 8.

As before, this generalizes straightforwardly to multi-membrane configurations.

The charge of the M2 brane with $H = 1 + \frac{32\pi^2 N\ell_s^6}{r^6}$ is given by integrating $\frac{\star G}{2\kappa_{11}^2}$ on a seven-sphere at infinity. Here $2\kappa_{11}^2 = (2\pi)^8\ell_{11}^9$ Asymptotically we will get the field strength going as

$$\frac{32\times 6\pi^2 N\ell_{11}^6}{r^6}$$

Altogether, using $\Omega_7 = \frac{\pi^4}{3}$ this gives a total charge of

$$\frac{\pi^4}{3}\frac{32\times 6\pi^2 N\ell_{11}^6}{(2\pi)^8\ell_{11}^9} = \frac{N}{(2\pi)^2\ell_{11}^2}$$

This is exactly consistent with **11.4.10-13**, with $\mu = N = 1$ corresponding to a single M2 brane.

Calculating the Ricci scalar curvature in fact gives a *constant* as $r \to 0$ so we do *not* encounter a divergence. This signifies that this is just a coordinate singularity and we can extend past.

In[120]:= `R = RicciScalar[g, xx]`

Out[120]= $-\dfrac{6144\, \mathrm{ll}^{12}\, \mathrm{NN}^2\, \pi^4}{\left(1+\frac{32\, \mathrm{ll}^6\, \mathrm{NN}\, \pi^2}{r^6}\right)^{1/3} \left(32\, \mathrm{ll}^6\, \mathrm{NN}\, \pi^2\, r + r^7\right)^2}$

In[122]:= `Series[` $-\dfrac{6144\, \mathrm{ll}^{12}\, \mathrm{NN}^2\, \pi^4}{\left(1+\frac{32\, \mathrm{ll}^6\, \mathrm{NN}\, \pi^2}{r^6}\right)^{1/3} \left(32\, \mathrm{ll}^6\, \mathrm{NN}\, \pi^2\, r + r^7\right)^2}$ `, {r, 0, 0}]`

Out[122]= $-\dfrac{3}{(2\,\pi)^{2/3} \left(\frac{\mathrm{ll}^6\, \mathrm{NN}}{r^6}\right)^{1/3} r^2} + O[r]^1$

Finally, we can take the near-horizon limit and get

$$
\begin{aligned}
ds^2 &= \frac{r^4}{L^4}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dx^i \cdot dx^i \\
&= \frac{r^4}{L^4}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_7^2
\end{aligned}
$$

Take now $r = L/\sqrt{z}$ to get the first term to look like $1/z^2$ while not affecting the second term much:

$$
\frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + 4L^2 dz^2) + L^2 d\Omega_7^2
$$

We can rescale z, x^μ and see that this geometry is $\mathrm{AdS}_4 \times S^7$

3. The M5 brane is now magnetically charged under C_3. Now the equations of motion $\mathrm{d} \star \mathrm{d}C = 0$ are trivially satisfied but the Bianchi identity is nontrivial, giving

$$
\partial_r^2 H = 0 \Rightarrow H = 1 + \frac{L^3}{r^3}
$$

The metric form can be fixed by analyzing the gravitino variation similar to before. Longitudinally:

$$
\begin{aligned}
0 &= \frac{1}{2}A' e^{A-B}\Gamma^{\hat{\mu}\hat{r}} + \frac{1}{2\cdot 3!}C' e^{C+A-4B}\Gamma^{\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4\hat{\mu}} \\
&\Rightarrow A'\epsilon + \frac{1}{3!}C' e^{C-3B}\Gamma^{\hat{r}\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4}\epsilon
\end{aligned}
$$

We see that we must take $C = 3B$ and $A = -C/6$, and we get the half-BPS condition:

$$
(1 - \Gamma^{\hat{7}\hat{8}\hat{9}\hat{1}\hat{0}\hat{1}\hat{1}})\epsilon = 0
$$

The transverse components will give the profile for ϵ.

$$
\partial_r \epsilon + \frac{1}{2\cdot 3!}C' e^{C-3B}\Gamma^{\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4\hat{r}}\epsilon
$$

and this gives a profile

$$
\epsilon = e^{-C/12}\epsilon_0
$$

The membrane charge is given by integrating G on a 4-sphere whose area is given by $8\pi^2/3$, so we get

$$
\frac{8\pi^2}{3}\frac{3\pi N \ell_{11}^3}{(2\pi^8)\ell_{11}^9} = \frac{N}{(2\pi\ell_{11})^5 \ell_{11}}
$$

Again we get that the Ricci scalar tends to a constant as $r \to 0$, giving regularity at the horizon. Again, this signifies that this is just a coordinate singularity and we can extend past.

In[138]:= `R = RicciScalar[g, xx]`

Out[138]= $$\frac{3\,\mathtt{ll}^6\,\mathtt{NN}^2\,\pi^2}{2\left(1+\frac{\mathtt{ll}^3\,\mathtt{NN}\,\pi}{r^3}\right)^{2/3}\left(\mathtt{ll}^3\,\mathtt{NN}\,\pi\, r+r^4\right)^2}$$

In[139]:= `Series[R, {r, 0, 0}]`

Out[139]= $$\frac{3}{2\,\pi^{2/3}\left(\frac{\mathtt{ll}^3\,\mathtt{NN}}{r^3}\right)^{2/3} r^2}+O[r]^1$$

Taking the near-horizon limit we arrive at

$$ds^2 = \frac{r}{L}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dx^i \cdot dx^i = \frac{r}{L}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_4^2$$

Now take $r = L/z^2$ yielding

$$\frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + 4L^2 dr^2) + L^2 d\Omega_4^2$$

so again after rescaling the same was as before we get $\mathrm{AdS}_7 \times S^4$.

As before, a solution can consist of an arbitrary number of $M5$ branes at different places, in which case we get

$$H(r) = 1 + \sum_i \frac{L_i}{|r - r_i|^3}$$

This remains half-BPS.

4. First look at the field strengths. The general M5 brane solution For a uniform distribution of M5 charges, we know that in the transverse (3D) space the potential must now decay as

$$H = 1 + \int dx^{11} \frac{L}{|\vec{r} - x^{11}\hat{e}_{11}|^2} = 1 + \frac{2L}{r_{10D}^2}$$

where L depends on the density of the distribution. Then the 3-form field strength in 10D will just be

$$(dB)_{abc} = \epsilon_{abce}\partial_e H$$

Given this source in 10D, we have already worked out Einstein's equations in **Chapter 8**. Another way to see this is that we remain half-BPS after adding even an infinite number of parallel branes.

We have that $e^{4\Phi/3} = G_{11,11}$ so that $e^\Phi = H^{1/2}$ consistent with the NS5 solution.

Using the perscription of dimensional reduction in appendix **I.2**, we take $e^\sigma = e^{2\Phi/3} = H^{1/3}$. Using $g_{\mu\nu} = e^{-\sigma} g^S_{\mu\nu}$, we see that multiplying by $H^{1/3}$ takes us to the *string frame* NS5 metric solution.

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + H(r) dx^i \cdot dx^i$$

This is exactly the NS5 metric in string frame.

We can further take $g^S_{\mu\nu} = e^{\Phi/2} g^E_{\mu\nu}$ and multiply the string frame by $e^{-\Phi/2} = H^{-1/4}$ to get us to the Einstein frame.

5. Recall the BPS D6 brane in 10D is described by

$$H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2} d\vec{x}\cdot d\vec{x}, \quad H = 1 + \frac{L}{|x|}, L = g_s \ell_s N/2, \qquad F = L\,\mathrm{d}\Omega_2, \quad e^\Phi = g_s^2 H^{-3/4}$$

This means that $e^{-2\Phi/3} = H^{1/2}$. Multiplying ds^2_{string} by this factor, we the 10D part of 11D metric

$$\eta_{ab} d\gamma^a d\gamma^b + V d\vec{x}\cdot d\vec{x}$$

Here we've picked notation consistent with the problem so that $\gamma^{0\ldots 6} = x^{0\ldots 6}$, $H(r) = V(r)$, and x^i is the same.

Note also that
$$\frac{1}{2\kappa_{10}^2}\int_{S^2} F = \frac{L4\pi}{(2\pi)^7\ell_s^8 g_s^2} = nT_p \Rightarrow 2L = \ell_s n g_s$$
This should be supplemented by the metric component in the internal 11th dimension, given by $e^{4\Phi/3}(d\gamma + A_\mu \cdot d\vec{x})^2 = V^{-1}(d\gamma + A_\mu \cdot d\vec{x})^2$ where A_μ is the 10D gauge field generated by the monopole solution.

Now A cannot be globally defined because of the monopole. Given $L = 2N$, it takes the same form as A_μ does in 3D about a monopole of charge $n = N/\ell_s$.

We could have taken a more "active" approach, demonstrating that this metric ansatz does indeed solve Einstein's equations, and shown that for the field strength to satisfy the Bianchi identity in this geometry it needed to indeed be a harmonic function of the transverse coordinates taken with flat metric.

6. The DBI action for a two-brane *in flat space with vanishing B-field and constant dilaton* is given in euclidean signature as
$$-T_2\int d^3x\sqrt{\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + 2\pi\ell_s^2 F_{ab})} + i\int C^{(3)} \wedge \mathrm{Tr}[e^{\mathcal{F}}] \wedge \mathcal{G},$$
where the second integral consists of Chern-Simons terms that we will ignore in this argument. We can work with the field variable F rather than A by imposing the Bianchi identity "by hand", namely writing the (non-CS) part of the action as
$$-T_2\int d^3x\left[\sqrt{\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + 2\pi\ell_s^2 F_{ab})} + \frac{i}{2}\lambda\epsilon^{abc}\partial_a F_{bc}\right]$$
This last term can just as well be integrated by parts to give $\epsilon^{abc}\partial_a\lambda F_{bc}$.

We now introduce an auxiliary V variable to rewrite the action as
$$\begin{aligned}
&-T_2\int d^3x\left[\frac{1}{2}V\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + 2\pi\ell_s^2 F_{ab}) + \frac{1}{2}\frac{1}{V} + \frac{i}{2}\epsilon^{abc}\partial_a\lambda F_{bc}\right]\\
&= -T_2\int d^3x\left[\frac{1}{2}V(1 + \frac{1}{2}(2\pi\ell_s^2)^2 F_{ab}^2 + \dots) + \frac{1}{2}\frac{1}{V} + \frac{i}{2}\epsilon^{abc}\partial_a\lambda F_{bc}\right]
\end{aligned}$$
here $\dots$ involves terms depending on the $\partial_a X^\mu$. The equations of motion for F then give
$$F_{ab} = -i\frac{\epsilon^{abc}\partial_a\lambda}{(2\pi\ell_s^2)^2 V}$$
Substituting this back in gives
$$-T_2\int d^3x\left[\frac{1}{2}V(1 + (-\frac{1}{2} + 1)(2\pi\ell_s^2)^{-2}(\partial\lambda)^2 + \dots) + \frac{1}{2}\frac{1}{V}\right]$$
Integrating out V gives us the square root action again, but now with F replaced by $\partial\lambda$, a new coordinate
$$-T_2\int d^3x\sqrt{\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + (2\pi\ell_s^2)^{-2}\partial_a\lambda\partial_b\lambda)}$$
Taking $X = \lambda/2\pi\ell_s^2$ gives our desired result

I have only shown classical equivalence. How to I prove this is quantum-mechanically true as well?

7. We are looking at the transformation $\tau \to -1/\tau$. We see that
$$C_0 + ie^{-\Phi} \to \frac{-1}{C_0 + ie^{-\Phi}} = \frac{-C_0 + ie^{-\Phi}}{C_0^2 + e^{-2\Phi}}$$
So we see $C_0 \to -\frac{C_0}{C_0^2+e^{-2\phi}}$ and $e^{-\Phi} \to \frac{e^{-\Phi}}{C_0^2+e^{-2\Phi}}$. On the other hand, C_0 will not affect the C_2, B_2 transformations. Nor will it affect C_4, which remains invariant

In the Einstein frame the metric is invariant. That means that $e^{-\Phi/2}g_{string}$ is invariant, which means g_{string} transforms as $e^{-\Phi/2}$ times the Einstein frame metric. Consequently, in the string frame $g'_{string} = e^{-\Phi}g_{string}$ (I think Kiritsis is wrong here, and Polchinski agrees with this)

Am I missing anything with that last one?

8. There's effectively nothing to derive. Translating the Einstein frame means multiplying all lengths by $e^{-\Phi/4}$. At fixed dilaton this is $g_s^{-1/4}$. Given ℓ_s^2 in the denominator will then contribute a factor $\sqrt{g_s}$ overall, that's exactly what was done here.

9. We have that C_4 is invariant. That means that objects charged under C_4 remain charged under C_4, with the same charge. These are precisely the D3/anti-D3 branes. Now recall the DBI action has coupling constant

$$g_{YM}^2 = \frac{1}{(2\pi\ell_s^2)^2 T_3} = 2\pi g_s$$

note that this is dimensionless, as it should be for a gauge theory in 4D. At low energies, the closed strings decouple we can reliably trust the DBI action, considering the D-brane gauge theory on its own. In the absence of axion, the $\mathrm{SL}(2,\mathbb{Z})$ of IIB takes $g_s \to 1/g_s$. This corresponds to

$$g_{YM}^2 \to \frac{4\pi^2}{g_{YM}^2}$$

So this is the Weak-Strong Montonen-Olive duality of $\mathcal{N} = 4$ SYM.

The only subtlety is that one must take care to include the Chern-Simons terms in the DBI action in order to get the full duality, specifically

$$\int C_0 \mathrm{Tr}[F \wedge F].$$

At fixed $C_0 = \theta/2\pi$ this produces the instanton number. The duality $C_0 \to C_0 + 1$ is a bona-fide duality of the $\mathcal{N} = 4$ theory, a consequence of the fact that instanton charge is quantized.

Is there anything else that I can say that constitutes any form of "showing" that this fact is true? The only thing is I think I'm assuming that the D3 brane is the only object charged under C_3 at leading order in ℓ_s. Can I safely assume this?

10. I'll start from the F1 string rather than the D1, not that it matters. Let us look at the macroscopic solution in the Einstein frame, so we multiply the string frame solution obtained in the chapter 8 exercises by $H^{1/4}$. We get:

$$ds_E^2 = H^{-3/4}(-dt^2 + (dx^1)^2) + H^{1/4} d\vec{x} \cdot d\vec{x}, \qquad H = 1 + \frac{L^6}{r^6}$$

Here $L^6 = \frac{2\kappa_{10}^2 T_{F1}}{6\Omega_7} = 32\ell_s^6 g_s^2 \pi^2$ Note this is the same metric as the D1 solution, and indeed the metric will stay the same for all (p,q) strings.

The C_0 field has been set to zero. For F1 the dilaton and B-field have the profile

$$e^{\Phi} = g_s H^{-1/2}, \quad B_{01} = H^{-1}$$

and indeed the dilaton has the inverse of this for the D1 while B and C exchange. Indeed, consider the $\mathrm{SL}_2(\mathbb{Z})$ action

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here, we have $ad - bc = 1$, implying c, d are relatively prime. This will correspond to the fact that (p,q) bound states only exist for p, q relatively prime, since otherwise there is a decay process of marginal instability allowing the (p,q) system to separate into two or more sub-systems. Further $\mathcal{S} = C_0 + ie^{-\phi}$ and C_2, B_2 transform as

$$\mathcal{S} \to \frac{a\mathcal{S} + b}{c\mathcal{S} + d}, \qquad \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to (\Lambda^T)^{-1} \cdot \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}$$

There is a subtlety in the problem, which resolved the ambiguity in our choice of Λ. I learned of it from reading arXiv:hep-th/9508143. The subtlety is as follows: We need to fix the dilaton's asymptotic value as $r \to \infty$ so as to define the vacuum of our string theory. First, consider $\phi, C_0 = 0$ asymptotically, i.e. $\mathcal{S} \to i$. We then stay within the $\mathrm{SO}(2) \subset \mathrm{SL}_2(\mathbb{R})$ that fixes $\mathcal{S} = i$. We want to take $(1,0)$ to the string p, q. This is now uniquely determined:

$$\Lambda = \frac{1}{\sqrt{p^2+q^2}} \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$$

Applying this to B_2, C_2 given that we start with only NS charge $(1,0)$ gives

$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} = \frac{H^{-1}}{\sqrt{p^2+q^2}} \begin{pmatrix} p \\ q \end{pmatrix}$$

Upon doing this, the B_2, C_2 fluxes will have coefficients that get modified from just p, q by a factor of $\frac{1}{\sqrt{p^2+q^2}}$, so will no longer be integers satisfying the quantization condition. We can fix this by modifying $T \to T_{p,q} = \sqrt{p^2+q^2} T$. Since this only serves to modify L, which was an arbitrary parameter of the classical solution, this still remains a valid solution.

This means: $H_{p,q} = 1 + \frac{L_{p,q}^6}{r^6}$, $L_{p,q}^6 = \frac{2\kappa^2 T_{p,q}}{6\Omega_7} = \sqrt{q_1^2+q_2^2}\,\frac{2\kappa^2 T_{1,0}}{6\Omega_7} = \sqrt{q_1^2+q_2^2} L^6$.

Our solution is now:

$$ds_E^2 = H_{p,q}^{-3/4}(-dt^2 + (dx^1)^2) + H_{p,q}^{1/4} d\vec{x}\cdot d\vec{x} \qquad \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} = \frac{H_{p,q}^{-1}}{\sqrt{p^2+q^2}} \begin{pmatrix} p \\ q \end{pmatrix} \qquad \mathcal{S} = \chi_0 + ie^{-\phi} = \frac{ipH_{p,q}^{1/2} - q}{iqH_{p,q}^{1/2} + p}$$

Note that as $r \to \infty, \mathcal{S} \to i$ as we expect. Now, let us generalize this for different asymptotic values of the dilaton and axion. After applying Λ, we can further apply

$$\Lambda' = \begin{pmatrix} e^{-\phi_0/2} & \chi_0 e^{\phi_0/2} \\ 0 & e^{\phi_0/2} \end{pmatrix}$$

$\mathcal{S}$ now asymptotes to

$$\frac{e^{-\phi_0/2} i + \chi_0 e^{\phi_0/2}}{0 + e^{\phi_0/2}} = \chi_0 + ie^{-\phi_0}$$

exactly as we want. To get the right final field strengths, take Λ initially arbitrary:

$$(\Lambda^T)^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Again, applying this will break our quantization condition. Now, the electric charges transform *contragradiently* from the fields strengths, which means that

$$\begin{pmatrix} Q_B \\ Q_C \end{pmatrix} = e^{\phi_0/2} \begin{pmatrix} e^{-\phi_0}\cos\theta + \chi_0 \sin\theta \\ \sin\theta \end{pmatrix} =: \frac{1}{\sqrt{\Delta_{p,q}}} \begin{pmatrix} p \\ q \end{pmatrix}$$

We can solve this to get

$$\sin\theta = \frac{e^{\phi_0/2}}{\sqrt{\Delta_{p,q}}} e^{-\phi_0} q \Rightarrow \cos\theta = \frac{e^{\phi_0/2}}{\sqrt{\Delta_{p,q}}}(p - \chi_0 q) \Rightarrow e^{i\theta} = \frac{e^{\phi_0/2}}{\sqrt{\Delta_{p,q}}}(p - \overline{\mathcal{S}} q)$$

The asymptotic value of the charges of B_2, C_2 is thus given by $(p,q)/\Delta_{p,q}^{1/2}$. Unimodularity gives:

$$1 = e^{i\theta} e^{-i\theta} \Rightarrow \Delta_{p,q} = e^{\phi_0}|p - q\mathcal{S}|^2 = e^{\phi_0}(p - q\chi_0)^2 + e^{-\phi_0} q^2$$

This coincides with the invariant

$$\begin{pmatrix} p & q \end{pmatrix} \mathcal{S}_2^{-2} \begin{pmatrix} |\mathcal{S}|^2 & \mathcal{S}_1 \\ \mathcal{S}_1 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = e^{\phi_0}(p - q\chi_0)^2 + e^{-\phi_0} q^2$$

So in full generality we get the tension:

$$T_{p,q} = \sqrt{e^{\phi_0}(p - q\chi_0)^2 + e^{-\phi_0}q^2}\, T_{F1}$$

Where $T_{F1} = \frac{1}{2\pi\ell_s^2}$ is the tension in the string frame.

Because (aside from redefining L) the metric is unchanged, the singularity structure of (p,q) strings is no different from $(1,0)$ or $(0,1)$ strings. Neither of these has a regular horizon. **Confirm**

11. First, by naive reasoning - there is no reason to write the full effective action to see what β should look like. From the perspective of IIA in the string frame, we have coupling $g_A = \tilde{R}_{11}/\ell_s = R_{11}\tau_2/\ell_s = R_{11}/\ell_s g_B$. Recognizing $R_B = \ell_s^2/R_11$ we can write this as $\frac{\ell_s}{R_B g_B}$. Because the translation of metrics between the 11-D frame and the standard string frame in IIA involves the factor $g_A^{2/3}$ we get $\beta = \left(\frac{\ell_s}{R_B g_B}\right)^{2/3}$. **What about conversion to the Einstein frame?**

    Now let's do it the long way. The 11-D SUGRA Lagrangian is

$$\mathcal{L}_{D=11} = \frac{1}{2\kappa_{11}^2}\left[R - \frac{1}{2}|G_4|^2 + G_4 \wedge G_4 \wedge \hat{C}_3\right]$$

    In this problem we'll ignore the Chern-Simons terms.

    Let's take M-theory to 9 dimensions. The metric takes the form

$$G_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + G_{\alpha\beta}A_\mu^\alpha A_\nu^\beta & G_{\alpha\beta}A_\mu^\beta \\ G_{\alpha\beta}A_\nu^\beta & G_{\alpha\beta}\end{pmatrix}$$

    Here

$$G_{\alpha\beta} = \frac{e^\sigma}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2\end{pmatrix}$$

    with $e^\sigma = \sqrt{\det G}$ the Kähler parameter (area) of the torus. The metric's R term becomes:

$$\frac{1}{2\kappa_{11}^2}e^\sigma\Big[R + \partial^\mu\sigma\partial_\mu\sigma + \frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta} - \frac{1}{4}G_{\alpha\beta}F^{A\,\alpha}_{\mu\nu}F^{A\,\mu\nu\beta}\Big].$$

    Here $F^A_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\mu\alpha$. We now have *two* $U(1)$ field strengths. Using the fact that we have the explicit form of the torus metric, we can further write this as:

$$\frac{(2\pi R_{11})^2\tau_2}{(2\pi)^8\ell_{11}^9}e^\sigma\left[R + \frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}\frac{(\partial\tau_1)^2}{\tau_2^2} - \frac{1}{2}\frac{(\partial\tau_2)^2}{\tau_2^2} - \frac{1}{2\cdot 2!}\frac{e^\sigma}{\tau_2}\begin{pmatrix}F^{A,1} & F^{A,2}\end{pmatrix}\begin{pmatrix}1 & \tau_1 \\ \tau_1 & |\tau|^2\end{pmatrix}\begin{pmatrix}F^{A,1} \\ F^{A,2}\end{pmatrix}\right] \tag{1}$$

    The kinetic 3-form potential yields a four-form, two three-form, and a two-form field strength:

$$\frac{R_{11}^2\tau_2}{(2\pi)^6\ell_{11}^9}e^\sigma\Big[-\frac{1}{2\cdot 4!}F^{(4)}_{\mu\nu\rho\sigma}F^{(4)\,\mu\nu\rho\sigma} - \frac{1}{2\cdot 3!}G^{\alpha\beta}F^{(3)}_{\mu\nu\rho\alpha}F^{(3)\,\mu\nu\rho}{}_\beta - \frac{1}{2\cdot 2!}G^{\alpha\beta}G^{\gamma\delta}F_{\mu\nu\alpha\gamma}F^{(2)\,\mu\nu}_{\beta\delta}\Big]$$

    Again, using the explicit form of the metric we can write this as

$$\begin{aligned}&\frac{R_{11}^2\tau_2}{(2\pi)^6\ell_{11}^9}e^\sigma\Big[-\frac{1}{2}|F_4|^2 - \frac{1}{2\cdot 3!}\frac{e^{-\sigma}}{\tau_2}\begin{pmatrix}F^{(3)} & H^{(3)}\end{pmatrix}\begin{pmatrix}1 & \tau_1 \\ \tau_1 & |\tau|^2\end{pmatrix}\begin{pmatrix}F^{(3)} \\ H^{(3)}\end{pmatrix} - \frac{1}{2\cdot 2!}\frac{e^{-2\sigma}}{\tau_2^2}(|\tau|^2 - \tau_1^2)F^{(2)}_{\mu\nu 12}F^{(2)\,\mu\nu}{}_{12}\Big] \\ &= \frac{R_{11}^2\tau_2}{(2\pi)^6\ell_{11}^9}\Big[-\frac{e^\sigma}{2}|F_4|^2 - \frac{1}{2\cdot 3!}\frac{1}{\tau_2}\begin{pmatrix}F^{(3)} & H^{(3)}\end{pmatrix}\begin{pmatrix}1 & \tau_1 \\ \tau_1 & |\tau|^2\end{pmatrix}\begin{pmatrix}F^{(3)} \\ H^{(3)}\end{pmatrix} - \frac{e^{-\sigma}}{2}|F_2|^2\Big]\end{aligned}$$

    Here the last 2-form field strength is defined as $F^{(2)}_{\mu\nu} := F^{(2)}_{\mu\nu 12}$.

This action not in any standard frame. Let's take it to the Einstein frame $g_{11} = e^{-2/7\sigma} g_E$:

$$\frac{R_{11}^2 \tau_2}{(2\pi)^6 \ell_{11}^9} \Big[R - \frac{3}{7}(\partial\sigma)^2 - \frac{1}{2}\frac{(\partial\tau_1)^2}{\tau_2^2} - \frac{1}{2}\frac{(\partial\tau_2)^2}{\tau_2^2} - \frac{1}{2\cdot 2!}\frac{e^{9\sigma/7}}{\tau_2}\begin{pmatrix} F^{A,1} & F^{A,2} \end{pmatrix}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}\begin{pmatrix} F^{A,1} \\ F^{A,2} \end{pmatrix}$$
$$- \frac{e^{-12\sigma/7}}{2}|F_2|^2 - \frac{1}{2\cdot 3!}\frac{e^{-3\sigma/7}}{\tau_2}\begin{pmatrix} F^{(3)} & H^{(3)} \end{pmatrix}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}\begin{pmatrix} F^{(3)} \\ H^{(3)} \end{pmatrix} - \frac{e^{6\sigma/7}}{2}|F_4|^2 \Big]$$

The IIB SUGRA Lagrangian **in the string frame** is

$$\frac{1}{2\kappa_{10}^2}\left[e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{2}|H_3|^2 \right] - \frac{1}{2}|F_1|^2 - \frac{1}{2}|F_3|^2 - \frac{1}{4}|F_5|^2 \right]$$

supplemented by $\star F_5 = F_5$. Taking this to 9 dimensions, the NSNS terms become

$$\frac{2\pi R_B}{(2\pi)^7 \ell_s^8 g_B^2} e^{-2\phi}\Big[R + 4(\nabla\phi)^2 - (\partial\rho)^2 - \frac{e^{2\rho}}{2}|F^A|^2 - \frac{1}{2}|H_3|^2 - \frac{e^{-2\rho}}{2}|H_2|^2 \Big]$$

with $G_{10,10} = e^{2\rho}$, $\phi = \Phi - \frac{1}{2}\rho$. The RR forms give

$$\frac{R_B}{(2\pi)^6 \ell_s^8 g_B^2} e^{\rho}\Big[-\frac{1}{2}F_1^2 - \frac{1}{2\cdot 3!}F_3^2 - \frac{e^{-2\rho}}{4}F_2^2 - \underbrace{\frac{1}{4\cdot 5!}F_5^2}_{\text{dualize}} - \frac{e^{-2\rho}}{4\cdot 5!}F_4^2 \Big].$$

Here F_2 comes from F_3 and F_4 from F_5. We can dualize the 9D F_5 to give the canonical normalization to the F_4 term.

$$\frac{R_B}{(2\pi)^6 \ell_s^8 g_B^2}\Big[-\frac{e^{\rho}}{2}|F_1|^2 - \frac{e^{\rho}}{2}|F_3|^2 - \frac{e^{-\rho}}{2}|F_2|^2 - \frac{e^{-\rho}}{2}|F_4|^2 \Big].$$

It is important to T-dualize this to get to IIA. This takes $\phi \to \phi, \rho \to -\rho, G^{(9)} \to G^{(9)}$ and also swaps H_2 and F^A. Lastly, we have $g_B^2/R_B = g_A^2/R_A$. We then get

$$\mathcal{L}_{IIA} = \frac{R_A}{(2\pi)^6 \ell_s^8 g_A^2}\Bigg[e^{-2\phi}\Big[R + 4(\nabla\phi)^2 - (\partial\rho)^2 - \frac{e^{2\rho}}{2}|F^A|^2 - \frac{1}{2}|H_3|^2 - \frac{e^{-2\rho}}{2}|H_2|^2 \Big]$$
$$- \frac{e^{-\rho}}{2}|F_1|^2 - \frac{e^{-\rho}}{2}|F_3|^2 - \frac{e^{\rho}}{2}|F_2|^2 - \frac{e^{\rho}}{2}|F_4|^2 \Bigg]$$

Now let's take this to the Einstein frame $g_S = e^{4/7\phi} g_E$:

$$\mathcal{L}_{IIA}^E = \frac{R_A}{(2\pi)^6 \ell_s^8 g_A^2}\Bigg[R - \frac{4}{7}(\nabla\phi)^2 - (\partial\rho)^2 - \frac{e^{2\rho - 4\phi/7}}{2}|F^A|^2 - \frac{e^{-8\phi/7}}{2}|H_3|^2 - \frac{e^{-2\rho-4\phi/7}}{2}|H_2|^2$$
$$- \frac{e^{-\rho+2\phi}}{2}|F_1|^2 - \frac{e^{\rho+10\phi/7}}{2}|F_2|^2 - \frac{e^{-\rho+6\phi/7}}{2}|F_3|^2 - \frac{e^{\rho+2\phi/7}}{2}|F_4|^2 \Bigg]$$

Comparing $|\tau_1|$ with $|F_1|^2$ since these are the only two scalars that aren't minimally coupled, we get $-\rho_A + 2\phi = -2\log(\tau_2)$. T-dualizing to get back to IIB gives $\rho_B + 2\phi = 2\Phi_B = -2\log\tau_2$ implying that $\tau_1 = C_0$ and $\tau_2 = e^{-\Phi}$ in IIB as required.

Comparing the F_4 coefficient gives $\rho_A + 2\phi/7 = 6\sigma/7$. This gives $\sigma = \frac{4}{3}\rho_A + \frac{1}{3}\Phi_B = -\frac{4}{3}\rho_B + \frac{1}{3}\Phi_B$. This gives $A^{3/2}g^{-1/2} \sim R_B^{-2}$, close to what is desired. Expressing the relevant quantities in terms of the fundamental units of their respective frames, this gives our desired relationship

$$\frac{\ell_s^2}{R_B^2} = \frac{A^{3/2}}{(2\pi\ell_{11})^3 g^{1/2}} \Rightarrow \frac{1}{R_B^2} = \frac{R_{11}^3}{\ell_s^5 g^{5/2}}$$

Off by a factor of $g^{1/2}$

Note that in the IIA action, F^A and F_2 have coefficients that differ by $-\rho_A + 2\phi = 2\Phi_B$. We should thus identify them with $e^{9\sigma/7 \pm \Phi_B}$ of the M theory action. This implies that $9\sigma/7 = \frac{3}{2}\rho_A + 3\phi/7$, exactly what we got from the F_4 coefficient. The same argument for the F_3, H_3 terms in both theories gives the same difference between them, and their average gives the same relationship. Finally, the lone H_2 term in IIA compared to the F_2 gives the same dependence as well, giving three nontrivial checks that what we've done is correct.

Finally let's get the conversion factor. To go from 11D to the string frame we must do $e^{4/7\phi}e^{2/7\sigma}$. We now understand $\sigma = -\frac{4}{3}\rho_B + \frac{1}{3}\Phi_B$ and $\phi = \Phi - \frac{\rho_B}{2}$ we get the relationship

$$\frac{2}{7}\sigma + \frac{4}{7}\phi = -\frac{2}{3}(\rho_B - \Phi_B) \Rightarrow \beta = \left(\frac{\ell_s}{R_B g_s}\right)^{2/3}$$

as required. **The dilaton dependence is flipped, fix!**

12. There is a subtlety in this problem involving the form of the metric. Recall that the Einstein frame metric g_E gets mapped to itself under S-duality $g_E = g_E'$. This implies that the string frame metric $g_S = e^{\Phi/2} g_S$ is related to its S-dual by:

$$g_S = e^{-\Phi'} g_S'$$

We can verify this at the level of the solutions to the string equations of motion:

$$\begin{aligned} &\mathbf{D5}: \; ds_E^2 = H^{-1/4} dx_\parallel + H^{3/4} dx_\perp, \quad ds_S^2 = H^{-1/2} dx_\parallel + H^{1/2} dx_\perp, \qquad e^\Phi = g_s H^{-1/2} \\ &\mathbf{NS5}: \; ds_E^2 = H^{-1/4} dx_\parallel + H^{3/4} dx_\perp, \quad ds_S^2 = dx_\parallel + H dx_\perp, \qquad e^\Phi = g_s H^{1/2} \end{aligned}$$

We see that the string frame metric are related in this way *except for the issue of rescaling by g_s*. This means we should redefine length so that ds_S^2 asymptotes to $g_s \eta_{\mu\nu}$ for the NS5 metric **why don't we modify D5 instead?**.

First, let's calculate the energy of the F1 string stretched between two $D5$ branes. Directly from the Nambu-Goto action, noting that the parallel X^μ will be along the τ direction while the transverse X^i will be along the σ direction we can write

$$\begin{aligned} S_{NG} = -T_{F1} \int d^2\xi \sqrt{\det(G_{ab} + B_{ab})} &= -\frac{1}{2\pi\ell_s^2} \int d^2\xi \sqrt{\begin{vmatrix} \partial_\tau X^\mu \partial_\tau X_\mu & \partial_\tau X^\mu \partial_\sigma X_i (G_{\mu i} + B_{\mu i}) \\ \partial_\tau X^\mu \partial_\sigma X_i (G_{\mu i} + B_{\mu i}) & \partial_\sigma X^i \partial_\sigma X_i \end{vmatrix}} \\ &= -\frac{1}{2\pi\ell_s^2} \int d\sigma d\tau \sqrt{H^{-1/2} \partial_\tau X^\mu \partial_\tau X^\mu} \sqrt{H^{1/2} \partial_\sigma X^i \partial_\sigma X^i} \\ &= -\underbrace{\frac{1}{2\pi\ell_s^2} \int_0^\pi d\sigma |\partial_\sigma X^i|}_{m_S} \int d\tau |\partial_\tau X^\mu| \end{aligned}$$

This gives the string and Einstein frame mass:

$$m_S = \frac{1}{2\pi\ell_s^2} \int_0^\pi |\partial_\sigma X^1| d\sigma \Rightarrow m_E = \frac{g^{1/4}}{2\pi\ell_s^2} \Delta x^1$$

For the D1 stretching the two NS5s, we apply the same logic to the DBI action:

$$\begin{aligned} S_{DBI} &= -\int d^2\xi \, T_{D1} \sqrt{-\det(G_{ab} + B_{ab})} = -\frac{1}{2\pi\ell_s^2} \int d\sigma e^{-\Phi(x^i)} \sqrt{\partial_\sigma X^i \partial_\sigma X_i} \int d\tau \sqrt{\partial_\tau X^\mu \partial_\tau X_\mu} \\ &= -\underbrace{\frac{\sqrt{g}}{2\pi\ell_s^2 g} \int d\sigma |\partial_\sigma X^i| \int d\tau}_{m_S} \sqrt{\partial_\tau X^\mu \partial_\tau X_\mu} \end{aligned}$$

Again we get string and Einstein frame mass:

$$m_S = \frac{1}{2\pi\ell_s^2\sqrt{g}} \int d\sigma |\partial_\sigma X^1| \Rightarrow m_E = \frac{1}{2\pi\ell_s^2 g^{1/4}} \Delta x^1$$

The masses agree under S duality: $g \to 1/g$.

13. The argument will go very similar to how it did for the string-like objects. Again, call $(p,q)=(1,0)$ the NS5 brane (magnetically charged under B_2) with $(p,q)=(0,1)$ the D5 brane (magnetically charged under C_2). Again, first take the axio-dilaton $\mathcal{S}$ to asymptote to i. The NS5 solution in the Einstein frame is:

$$ds_E^2 = H^{-1/4}\eta_{\mu\nu}dx^\mu dx^\nu + H^{3/4}d\vec{x}\cdot d\vec{x}, \qquad H = 1+\frac{L^2}{r^2}$$

Here $L^2 = Q\frac{2\kappa_{10}^2 T_{NS5}}{2\Omega_3} = Q\ell_s^2$. We also have

$$e^\Phi = g_s H^{1/2}, \qquad (dB)_{\theta\phi\psi} = -\partial_r H$$

This time, the magnetic charges transform in the same way as the field strengths (since they are associated with the Bianchi identity, not the EOMs), giving

$$\begin{pmatrix} Q_B \\ Q_C \end{pmatrix} = \begin{pmatrix} e^{\phi_0/2}\cos\theta \\ \chi_0 e^{\phi_0/2}\cos\theta + e^{-\phi_0/2}\chi_0\sin\theta \end{pmatrix} =: \frac{1}{\sqrt{\Delta_{p,q}}}\begin{pmatrix} p \\ q \end{pmatrix}$$

Solving this gives

$$\cos\theta = \frac{e^{-\phi_0/2}}{\sqrt{\Delta_{p,q}}}p \Rightarrow \sin\theta = \frac{e^{\phi_0/2}}{\sqrt{\Delta_{p,q}}}(q+p\chi_0) \Rightarrow e^{i\theta} = i\frac{e^{\phi_0/2}}{\sqrt{\Delta_{p,q}}}(q+\overline{\mathcal{S}}p)$$

Unimodularity gives

$$\Delta_{p,q} = e^{\phi_0}|q+p\mathcal{S}|^2 = e^{\phi_0}(q+p\chi_0)^2 + e^{-\phi_0}p^2$$

We thus get that the 5-brane tension in the Einstein frame satisfies a similar relation to the case of 1-branes:

$$T_{p,q} = \sqrt{e^{-\phi_0}p^2 + e^{\phi_0}(q+p\chi_0)^2}\; T$$

with $T = \frac{1}{(2\pi)^5\ell_s^6}$ the appropriate dimensionful constant.

For the general (p,q)-brane solution, we get $L_{p,q} = \sqrt{\Delta_{p,q}}L = \sqrt{\Delta_{p,q}}\,\ell_s^2$

14. We're going to work in the Einstein frame. After compactifying on T^2 we will get scalars not just from the ϕ and C_0 term but also from C_2, B_2, and the 3 metric components $G_{\alpha\beta}$.

The torus moduli in $\frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta}$ will take the same form as in Equation (1), namely

$$\frac{(\partial T)^2}{T^2} + \frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta} = -\frac{1}{2}\frac{(\partial\tau_1^2)}{\tau_2^2} - \frac{1}{2}\frac{(\partial\tau_1^2)}{\tau_2^2} + \frac{1}{2}\frac{(\partial T)^2}{T^2}$$

Here $T = \sqrt{\det G_{\alpha\beta}}$ is the Kähler modulus and does not belong to the $\mathrm{SL}(2,\mathbb{R})/U(1)$ coset. We have that the axio-dilaton is $-\frac{1}{2}\frac{|\partial\mathcal{S}|^2}{\mathcal{S}_2^2}$. The scalars coming from B_2, C_2 give kinetic terms:

$$-\frac{1}{2}\frac{G^{11}G^{22}-(G^{12})^2}{\mathcal{S}_2}(\mathcal{S}_2\partial_\mu B_{12})^2 = -\frac{1}{2}\frac{\mathcal{S}_2}{T^2}(\partial_\mu B_{12})^2, \qquad -\frac{1}{2}\frac{(\partial_\mu C_{12})^2}{T^2\mathcal{S}_2}$$

Altogether the scalars have appear as:

$$\int d^8x\sqrt{-g}\,T\Big[R \underbrace{-\frac{1}{2}\frac{(\partial\tau_1^2)}{\tau_2^2} - \frac{1}{2}\frac{(\partial\tau_1^2)}{\tau_2^2}}_{\mathrm{SL}(2,\mathbb{R})/U(1)} + \underbrace{\frac{1}{2}\frac{(\partial T)^2}{T^2} - \frac{1}{2}\frac{|\partial\mathcal{S}|^2}{\mathcal{S}_2^2} - \frac{1}{2}\frac{\mathcal{S}_2}{T^2}(\partial_\mu B_{12})^2 - \frac{1}{2}\frac{(\partial_\mu C_{12}+C_0H_3)^2}{T^2\mathcal{S}_2}}_{\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})}\Big]$$

Taking things to the new Einstein frame will get rid of the T out front, and modify $\frac{1}{2}\frac{(\partial T)^2}{T^2} \to -\frac{2}{3}\frac{(\partial T)^2}{T^2}$

It remains to find the metric for $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. Because $\mathrm{SO}(3)$ is maximally compact, we can write the metric on this space in a set of global coordinates known as *Borel gauge*. This is given by taking the Einbein on T^3 symmetric space to be the exponentiation of the $\mathrm{SL}(3,\mathbb{Z})$ Borel sub-algebra: $L = \exp[\chi^i E_i]\exp[\phi^i H_i]$. From this, the T^3 metric is $\mathcal{M} = LL^T$, and the kinetic terms are then $\mathrm{Tr}[\partial_\mu\mathcal{M}\,\partial^\mu\mathcal{M}^{-1}]$. By choosing the χ_i and ϕ_i judiciously we see

```
In[366]:= L = {{1, -C0[r], F3[r]}, {0, 1, H3[r]}, {0, 0, 1}}.{{Exp[t[r]/3 - Φ[r]/2], 0, 0}, {0, Exp[t[r]/3 + Φ[r]/2], 0}, {0, 0, Exp[-2 t[r]/3]}};
M = L.Transpose[L] // FullSimplify;
M // MatrixForm
1/4 Tr[D[M, r].D[Inverse[M], r]] // Expand
```

Out[368]//MatrixForm=

$$\begin{pmatrix} e^{-\frac{4t[r]}{3}}\left(e^{2t[r]-\Phi[r]}+e^{2t[r]+\Phi[r]}\,C0[r]^2+F3[r]^2\right) & e^{-\frac{4t[r]}{3}}\left(-e^{2t[r]+\Phi[r]}\,C0[r]+F3[r]\times H3[r]\right) & e^{-\frac{4t[r]}{3}}\,F3[r] \\ e^{-\frac{4t[r]}{3}}\left(-e^{2t[r]+\Phi[r]}\,C0[r]+F3[r]\times H3[r]\right) & e^{-\frac{4t[r]}{3}}\left(e^{2t[r]+\Phi[r]}+H3[r]^2\right) & e^{-\frac{4t[r]}{3}}\,H3[r] \\ e^{-\frac{4t[r]}{3}}\,F3[r] & e^{-\frac{4t[r]}{3}}\,H3[r] & e^{-\frac{4t[r]}{3}} \end{pmatrix}$$

Out[369]=

$$-\frac{1}{2}e^{2\Phi[r]}\,C0'[r]^2-\frac{1}{2}e^{-2t[r]+\Phi[r]}\,F3'[r]^2-e^{-2t[r]+\Phi[r]}\,C0[r]\,F3'[r]\,H3'[r]-\frac{1}{2}e^{-2t[r]-\Phi[r]}\,H3'[r]^2-\frac{1}{2}e^{-2t[r]+\Phi[r]}\,C0[r]^2\,H3'[r]^2-\frac{2}{3}t'[r]^2-\frac{1}{2}\Phi'[r]^2$$

here $e^t = T, e^\Phi = \mathcal{S}_2^{-1}$. It is worth stressing that this *exactly* recovers our kinetic terms. Everything matches perfectly.

15. The the field strengths coming from the two-forms yield the following terms in the Einstein frame Lagrangian

$$\int d^8x\sqrt{-g}\,T^{2/3}\Big[-\frac{1}{2}\frac{|F_3+C_0H_3|^2}{\mathcal{S}_2}-\frac{1}{2}\mathcal{S}_2|H_3|^2-\frac{1}{2}\frac{|F_{3\leftarrow 5}|^2}{T^2}\Big]$$

The two-form field strengths F_3, H_3 are unaffected by dualities of the torus. F_5 can be dualized to an F_3 as well, and we also get a further $F_{3\leftarrow 5}$ by wrapping the D3 around the torus, which will combine with the $F_{3\leftarrow 5}^2$ to give a single (canonically normalized) field strength invariant under symmetries of the torus. Thus, the $F_3, F_{3\leftarrow 5}, H_3$ are invariant under the $\mathrm{SL}(2,\mathbb{Z})$ part of the U-duality group involving τ_1, τ_2.

We can indeed write these terms in a manifestly $\mathrm{SL}(3,\mathbb{R})$-invariant form, namely as

$$-\frac{1}{2}\begin{pmatrix}H_3 & F_3 & F'_{3\leftarrow 5}\end{pmatrix}\mathcal{M}\begin{pmatrix}H_3\\ F_3\\ F'_{3\leftarrow 5}\end{pmatrix}$$

Here though, we should take care that it is really $F_{3\leftarrow 5}+B_{12}F_3+C_{12}H_3$ that forms the kinetic term of the action. **Understand this, as well as the C_0H_3 in the Einstein frame generally.**

```
In[458]:= ({{H3, F3, F5}}.M.{{H3}, {F3}, {F5}})[[1, 1]] // FullSimplify
```

Out[458]=

$$e^{-\frac{4t[r]}{3}}\left(e^{2t[r]-\Phi[r]}\,H3^2+e^{2t[r]+\Phi[r]}\,(F3-H3\,C0[r])^2+(F5+H3\,F0[r]+F3\,H0[r])^2\right)$$

16. The metric will contribute 6 scalars while the 3-form C_3 will contribute a seventh. We understand how to generally build T^3 metrics from the last problem. Indeed, L there is the einbein not on the symmetric space itself but on the *torus* T^3. Given a Borel subgroup of $\mathrm{SL}(3,\mathbb{R})$, the einbein for the *unit* torus is specified by three twist "axion" parameters χ_1, χ_2, χ_3 and two dilaton parameters ϕ_1, ϕ_2 as:

$$L=\exp[\chi^iE_i]\exp[\phi^iH_i]=\begin{pmatrix}1 & \chi_1 & \chi_2\\ 0 & 1 & \chi_3\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}e^{\phi_1/3-\phi_2/2} & 0 & 0\\ 0 & e^{\phi_1/3+\phi_2/2} & 0\\ 0 & 0 & e^{-\phi_2/2}\end{pmatrix}$$

We see directly that the parameters of this three-torus coincide exactly with the scalars $C_0, C_{12}, B_{12}, T, \Phi$ in IIB compactified on T^2 from the prior problem.

The 3-torus volume parameter, which we will call T (not to be confused with T) in the prior problem, together will have kinetic terms

$$\int d^8\sqrt{-g}\,T[R+\frac{(\partial T)^2}{T^2}-\frac{1}{3}\frac{(\partial T)^2}{T^2}-\frac{1}{2}\frac{|C_{0\leftarrow 3}|^2}{T^2}]=\int d^8\sqrt{-g}\,T[R+\frac{2}{3}\frac{(\partial T)^2}{T^2}-\frac{1}{2}\frac{|C_{0\leftarrow 3}|^2}{T^2}]$$

Taking this to the Einstein frame:

$$\int d^8\sqrt{-g}[R - \frac{1}{2}\frac{(\partial T)^2}{T^2} - \frac{1}{2}\frac{|C_0|^2}{T^2}]$$

This is exactly the $\mathrm{SL}(2,\mathbb{Z})$-invariant action, which came from the perturbative T-duality in the earlier problem. We see they are neutral under $\mathrm{SL}(3,\mathbb{Z})$, while the other 5 belonging to the $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ coset are neutral under this $\mathrm{SL}(2,\mathbb{Z})$. This re-derives the results for scalars of **Section 11.6**.

From the M-theory perspective, the three distinct 2-form potentials come from wrapping the C_3 around different T^3 cycles from 11D.

17. From M-theory, the 3-form $C^{(3)}$ descends directly down to a 3-form in the 8D picture. This has a field strength G_4 with kinetic term

$$\int d^8x\sqrt{g}\,T\left[-\frac{1}{2}|G_4|^2\right]$$

which is the same in *both* the original *and* the Einstein frames. We could have started with $\star G_4$ in 11D, giving the 8D action:

$$\int d^8x\sqrt{g}\,T^{-1}\left[-\frac{1}{2}|G_4|^2\right]$$

The Chern-Simons term further contributes a further topological piece:

$$\int d^8x\sqrt{g}\,C_0T\,G_4\wedge G_4$$

Summing these all together gives the standard $\mathrm{SL}(2,\mathbb{R})$ invariant bilinear form. Thus, $\mathrm{SL}(2,\mathbb{R})$ acts by electric magnetic duality, transforming the tuple $(G,\star G)$ in the **2** representation.

Slightly incomplete, understand the origin of the action better.

18. Taking IIB down to 5D and looking at conserved vectors (coupling to point-like objects). First, note we can wind any of the (p,q) strings around any of the 5 cycles of T^5, giving 10 vector currents. We also get 5 KK currents from the dimensional reduction that are T-dual to the string modes and together forms a 10 of $\mathrm{SO}(5,5)$. We also have the D3 brane winding around any $\binom{5}{3}=10$ cycles. Finally, the D5-brane *and* NS5 can wrap the torus giving an additional 2 charges. The NS5 is a singlet of $\mathrm{SO}(5,5)$. The D-branes are all T-dual and give a 16-dimensional representation, which is either the spinor or conjugate spinor depending on whether we start from IIA or IIB.

Altogether we get **1+10+16**. This is exactly the **27** representation of E_6 under U-duality. This gives a total of 27 point-like charges, which are the 27 different electric charges than can be carried by black holes in 5D.

19. Note that rescaling the string length by e^{γ} will correspond rescaling the metric by $e^{-2\gamma}$. So relationships between the string lengths are inverse-square-root proportional to the relationships between the metrics.

At the level of the supergravity theory, we have $g^I = e^{-\Phi^{het}}g^{het}$ is a symmetry of the theory. Then the string length scales must obey

$$\ell_s^I = \ell_s^{het}\sqrt{g_s^{het}} \Rightarrow M_s^I = \frac{M_s^{het}}{\sqrt{g_s^{het}}}$$

20. The effective action will look like

$$\frac{V}{(2\pi)^7\ell_s^8g_s^2}\int d^4x\sqrt{g}\,e^{6\sigma}e^{-2\phi}[R + \cdots - \frac{e^{-2\sigma}}{2}|H_2|^2 - \frac{1}{4}\mathrm{Tr}[F^2]]$$

Taking this to the proper string frame requires $g\to e^{-6\sigma}g$. This gives

$$\frac{V}{(2\pi)^7\ell_s^8g_s^2}\int d^4x\sqrt{g}e^{-2\phi}[R + \cdots - \frac{e^{4\sigma}}{2}|H_2|^2 - \frac{e^{6\sigma}}{4}\mathrm{Tr}[F^2]]$$

Because σ is such that $Ve^{6\sigma}$ is strictly larger than the order of ℓ_s, then both of the gauge fields will have coupling constants that go as $O(\ell_s^8 g_s^2/Ve^{4\sigma})$ or $O(\ell_s^8 g_s^2/Ve^{6\sigma})$. This is only going to be $O(1)$ if $g_s \gg 1$. In this case, we can (after a possible T-duality that doesn't change the coupling substantially, esp. if one of the dimensions is reasonably close to the order of the string length already) apply the type I - heterotic O duality to get a weakly coupled type I description.

21. Since the B-field strength H^{het} gets mapped directly to the RR field strength H^I, we expect that the objects electrically charged between them should get mapped to one another. This means the heterotic fundamental string gets mapped to the D1 brane in type I. Their magnetic cousins should also be swapped, which will interchange the heterotic NS5 with the type I D5 brane. At the classical level this is easy to see, since the two branes have the same supergravity solution. Clearly this is not enough, eg in IIA vs IIB the worldvolume theories of the NS5 are radically different.

    To understand the quantum mechanical equivalence, we need to understand the origin of the Sp(2) on the D5 in type I and the NS5 in the heterotic picture. This question is answered (using nontrivial arguments involving ADHM) first in Witten "Small Instantons in String Theory". **Return and understand this when you know more $\mathcal{N} = 2$ SUSY**.

22. Certainly we see that heterotic-type I together with T-duality will relate both heterotic strings together, and connect this with type I which, after T-dualizing and moving the orientifold plane appropriately, will connect with the other type II string theories.

    It remains to look at the self-duality of type IIB. For this, we took a leaf from Sen's paper. Let's look at IIB on a $\mathbb{Z}_2$ orientifold $T^2/(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}$ where $\mathcal{I}$ is the inversion $z \to -z$ on the torus and Ω is worldsheet parity inversion. This manifold has 4 singular points that each carry -4 RR charge **why**. Since it is compact, we must cancel this by placing 4 D6 branes at each of the 4 points for a total of 16. In this case, the geometry of the tetrahedron is flat everywhere except for the 4 deficit angles of π at each vertex. The singularities at the verticies are of $D_4 = SO(8)$ type, so this theory has an unbroken $\mathrm{SO}(8)^4$ gauge symmetry. The torus has moduli T, τ together with axiodilaton $\mathcal{S}$. There is no B field in the orientifold.

    Now, let's T-dualize *both* cycles of the torus. This keeps us in IIB, but takes us to $(T^2)'/\Omega$, undoing the effects of $(-1)^{F_L}\mathcal{I}$. Type IIB on this space is just Type I on $(T^2)'$, but with SO(32) broken down to $\mathrm{SO}(8)^4$. Now it is time to dualize to heterotic O theory. We see that we have heterotic string theory on $(T^2)'$ with gauge group broken down to $\mathrm{SO}(8)^4$.

    Let's match the moduli:

    - The τ modulus is the same in IIB and the heterotic theory.
    - The torus in the heterotic picture now has a B_{89} scalar that gets mapped to the axion C_0 in IIA. B_{89} can combine with the heterotic torus volume to provide another modular parameter $\rho = B + iV_{het}$.
    - The standard parameter in compactification on a torus is $\Psi_{het} = \Phi_{het} - \frac{1}{4}\log\det G^{het}_{\alpha\beta} = \Phi_{het} - \frac{1}{2}\log V_{het}$. This will be mapped to $-\frac{1}{2}\Phi_{IIB} + \log V_{IIB}$ where V_{IIB} is the original T^2 radius.

    We know that heterotic on T^2 has T-duality $\mathcal{O}(18, 2; \mathbb{Z})$. This has a subgroup $\mathrm{SO}(2,2) \sim \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})'$ that does not affect the Wilson lines but acts only on the torus parameters. Both τ and ρ transform under fractional linear transformations of the two $\mathrm{SL}(2,\mathbb{R})$ separately, while Ψ_{het} remains unaffected.

    Now, taking $V_{IIB} \to \infty$, the two $\mathrm{SL}(2,\mathbb{Z})$ symmetries remain unbroken. One of these can be identified with large diffeomorphisms of the torus, and so combines with spacetime diffeomorphisms in the large V limit. The remaining $\mathrm{SL}(2,\mathbb{Z})$ then becomes the S-duality group. The SO(8) gauge theory living at each of the vertices is not seen, since the singularities and accompanying D7 branes have "flown off" to infinity.

    That Ψ_{het} remains unaffected means that $G_{IIB}e^{-\Phi_{IIB}/2}$ is an invariant under $\mathrm{SL}(2,\mathbb{Z})$. So the volume as measured in the frame of that modified metric is an invariant. This is exactly the Einstein frame metric.

    We have also seen in the chapter that the M theory - heterotic E duality can be obtained through a chain of dualities involving heterotic O - type I together with the M theory - type IIA. We are only asked to reproduce dualities between *string theories* in this question however.

23. The D9 brane is orthogonally projected, as we know from tadpole conditions on it from chapter 7, and the same argument with the cylinder gives a $\frac{1}{\sqrt{2}}$ reduction of tension relative to type II.

For a D1 brane interacting with itself, the gravitational contribution in the cylinder amplitude also has a extra $\frac{1}{2}$ factor due to the orientation-projection. Thus, the total tension of the D1 brane is lowered by a factor of $\frac{1}{\sqrt{2}}$ relative to type II as required.

Naively we could apply the same argument to D5 branes, which would then violate the D1-D5 Dirac quantization by a factor of 2.

However, from an analysis of the cylinder amplitude for 59 and 95 strings with orientation projection, we get the constraint $\epsilon_{59}^2\zeta_5\zeta_9 = 1$. By consistency of interactions of 59 strings with 55 and 99 strings, we get $\epsilon_{59}^2 = \epsilon_{55}^2 = \epsilon_{99}^2 = -1$. Consequently, the D5 brane will have opposite orientation projection than the D9 brane, namely the symplectic one. Taking the determinant of $\gamma = \zeta\gamma^T$ however gives $\zeta^N = 1$, so $\zeta = -1$ will only work for N even. Another way to say this is: "sympletically projected branes must move in pairs".

Thus, the "fundamental" D5 brane should be thought of as a D5 with Sp(2) index $a = 1, 2$. Repeating the cylinder amplitude calculation gives a factor of 2^2, which translates to a tension of $2 \times \frac{T_5^{II}}{\sqrt{2}} = \sqrt{2}T_5^{II}$.

24. The crucial component of this is to note that at Dp worldvolume theory contains a CP-odd term coupling to the lower-dimensional forms going as:

$$iT_p \int d^{p+1}x C \wedge \mathrm{Tr}[e^{\mathcal{F}}] \wedge \mathcal{G} \supset iT_p(2\pi\ell_s^2)^2 \int d^{p+1}x\, C_{p-3}\mathrm{Tr}[F \wedge F]$$

in the absence of an NS-NS background.

Consider the 9-brane with an instanton background in the 5678 directions with instanton number obtained from integrating over $x^{5,6,7,8}$

$$\int d^4x \frac{\mathrm{Tr}[F \wedge F]}{(2\pi)^2} = k$$

For the case of $k = 1$, the CP-odd term simplifies to

$$iT_9(2\pi)^2(2\pi\ell_s^2)^2 \int d^6x\, C_6 = iT_5 \int d^6x\, C_6$$

This is exactly the CP-odd term for a D5 brane. In the limit of vanishing instanton size, this sources RR fields in the same way with the exact same RR charge. It is also a BPS state, so has the same mass as a D5 brane. This satisfies all the criteria to qualify as a D5 brane.

We can extend this to k localized D-branes and see the exact same coupling

$$\sum_{i=1}^{k} iT_5 \int_{x^{5\ldots8}=x_i} d^6x\, C_6$$

as k distinct D5 branes. For nonvanishing instanton size, this describes D5 branes "dissolved" in the D9.

This argument can be carried over for an arbitrary pair $(p, p-4)$.

25. We have already seen by general arguments that we need the number of Newman-Dirichlet conditions to be a multiple of 4 so that the NS and R sectors have a chance of having degeneracy. I will repeat the argument here.

In the R sector, the zero-point energy is always zero because of the equal number of periodic fermions and bosons. The excitations above this will have integer or half-integer weights.

In the NS sector, the NN and DD fermions and bosons contribute zero point energies $-\frac{1}{24}$ and $-\frac{1}{48}$, so $-\frac{1}{16}$ total. The ND sector bosons and fermions contribute $\frac{1}{48}$ and $\frac{1}{24}$, ie the opposite. Altogether for ν ND boundary conditions we get:

$$-\frac{(8-\nu)}{16} + \frac{\nu}{16} = -\frac{1}{2} + \frac{\nu}{8}$$

This ground state and its excitations above it will have half-integer weight when $\nu = 0$ mod 4.

Since type I string theory necessitates 32 D9 branes to cancel out the O9 tension, we are only allowed $\nu = 8, 4, 0$ giving D1, D5, and D9 brane configurations preserving supersymmetry in the theory. In the text, we have seen that D1, D5, D9 all lead to consistent worldvolume excitations that respect GSO and Ω-projection

26. Let's review the logic so far. For supersymmetric open strings in the NS sector, we are principally interested in the ψ_r states. Orientation projection acts as on the NN string as $\Omega\psi_r = i^{2r}\psi_r$ (in both NS and R sectors) and on the DD string as $\Omega\psi_r = -i^{2r}\psi_r$. For the R sector ground states, supersymmetry requires that for all directions NN (D9 brane) $\epsilon_R = -1$: that is, $\Omega |R\rangle = -|R\rangle$.

    When we add indices, writing the NS state as $|p, ij\rangle$, for NN strings the massless levels are given by $\psi^\mu_{-1/2}\lambda_{ij} |p, ij\rangle$. We get the constraint $\lambda = -i\epsilon_{NS}\gamma\lambda^T\gamma$. WLOG we can either have $\gamma = 1$ for SO(N) with $\zeta = 1$ or $\gamma = i\omega$ for Sp(N) with N even, $\zeta = -1$. In *either case* the Jacobi identity require $\epsilon_{NS} = -i$. This gives that $\lambda = -\gamma^T\lambda\gamma^{-1}$ for the massless level. In both cases this corresponds to the *adjoint representation*. In the DD case, we get an extra minus sign, giving $\lambda = \gamma^T\lambda\gamma^{-1}$. This corresponds to the *symmetric traceless* representation plus a *singlet*.

    For the D1 brane, the above discussion already shows us that in the 1-1 NS sector, we get the 8 DD scalars transforming the symmetric traceless plus single representation of SO(N) together with the 2 NN scalars transforming in the adjoint.

    For the 1-1 R sector, before orientation projection we have the 16_+ ground state from GSO. The orientation projection acts as

    $$\Omega |S_\alpha, i, j\rangle = -e^{i\pi(s_1+s_2+s_3+s_4)}\gamma |S_\alpha, i, j\rangle \gamma^{-1}$$

    What Kiritsis writes can't be the adjoint for $N = 1$. We need it to have dim 1 in that case, but it would have dim 0. I believe that the correct thing is that we have 8 fermions forming the 8_- (ie left-moving) and in the *symmetric representation* of SO(N) while we have 8 forming the 8_+ (ie right-moving) but in the *adjoint* of SO(N) (these disappear for $N = 1$).

    In the 1-9 sector, we have 2 NN and 8 DN boundary conditions. The NS ground state energy is positive, so this will not contribute. The massless states come from the R ground state in the DN part combined with the $O(1, 1)$ spinor from the R sector of the NN part. The fermions are right-moving (chirality +) as before. We get 32 indices from the D9 brane and N from the D1 brane. The orthogonal projection guarantees that these transform in the $(N, 32)$ bi-fundamental representation. Orientation projection disallows for the second copy of this spectrum (ie the 1-9 string with the orientation reversed).

27. To get to the D5-brane from the D9-brane we T-dualize four times. In this problem we *focus only on the 5-5 strings*. Again, the R-sector contains the GSO-projected 16_+ spinor before orientation projection. We must decompose under SO$(5, 1)_\parallel \times$ SO$(4)_\perp$. The projection condition in the R sector reads:

    $$\Omega\lambda_{ij} |S_\alpha, ij\rangle = -e^{i\pi(s_1+s_2)}\gamma\lambda_{ij}\gamma^{-1} |S_\alpha, ij\rangle$$

    Here $\gamma = i\omega$, since we have the symplectic Sp(2) projection for the D5. Then, for $s_1 + s_2$ odd, the 6D fermion is negative chirality, and we require $\lambda = -\gamma\lambda^T\gamma^{-1}$. This gives a negative-chirality fermion in the adjoint representation of Sp(2), completing the vector multiplet.

    For $s_1 + s_2$ even, the 6D fermion is positive chirality and we require $\lambda = \gamma\lambda^T\gamma^{-1}$ which will leave the skew-traceless antisymmetric representation plus a singlet. For Sp(2) this is just the singlet, so we get a single positive chirality fermion, completing the hypermultiplet.

    I think I'm off by a sign?

28. From the D5-D5 analysis of the previous problem, we immediately see the generalization to general Sp$(2N)$. The R sector yields fermions in the Sp(2) adjoint combining with the vectors $\psi^\mu_{-1/2}$ in the adjoint, yielding the vector multiplet. The DD boundary conditions reverse the projection sign for $\psi^i_{-1/2}\lambda_{ij} |p, ij\rangle$ yielding a sum of the skew-traceless antisymmetric representation plus a singlet. I assume this is the same as the

two-index symmetric rep, by analogy to SO(N), where a similar thing happens. We also know that the R sector also provides (positive chirality) fermions to combine with this to form the hypermultiplet.

Finally, we must look at the D5-D9 spectrum. We have 4 ND boundary conditions and 6 NN ones. For 4 ND boundary conditions, the NS sector ground states *also* contribute to the massless spectrum. The ND conditions these consists of ground states transforming in the **4** of SO(4), combining with the singlet NS ground state of the 6 NN coordinates. This yields 4 scalars.

In the R sector, the massless states come from the bosonic ND ground state combining with one of the 4 NN R sector states giving an SO(5, 1) spinor. After GSO projection, this gives a chirality + fermion, completing the hypermultiplet. **This part is a bit shifty, thing about it**

Each of these states has 32 labels from the D9 brane, and 2N labels from the D5 brane. Therefore, we get that this hypermultiplet in fact transforms in the $(2N, 32)$ bi-fundamental. Again, orientation projection simply restricts us to not have a second copy of this spectrum from 5-9 strings of opposite orientation.

Say we pull apart m D5 branes. Because the D5 branes move in pairs in type I, we must have m an even integer. The 5-9 strings now all have positive zero-point energy and will not contribute to the massless spectrum. The 5-5 strings remain the same, but transforming in Sp($2N - m$) instead of Sp($2N$).

29. We can focus on the purely chiral left-moving CFT, since this is the only part that the orbifold acts on nontrivially. Immediately, we see that the untwisted sector corresponds to the NS states, which are the same between IIA and IIB.

    In the twisted sector, we again have NS and R fermions. Because the NS fermions are taken to minus themselves, they are now *integrally modded* while the R fermions become half-integral. Again, the R fermions will be projected out by the $(-1)^{\mathbf{F}_L}$. The 8 NS fermions will give two (unprojected) ground states $8 + \bar{8}$ of fermion numbers $1, -1$ respectively. In Polchinski's convention, the original $|0\rangle$ NS ground state has fermion number -1, so the only the C operator on top of this will give something that is unprojected. In Kiritsis' convention, the NS ground state has fermion number 1 but we we take $(-1)^F = -1$ for GSO. In either case, we can only keep the C operator. In the original IIA we kept the S on the left and the C on the right. Now we keep C on both sides giving IIB (we could have done the same with $(-1)^{\mathbf{F}_R}$, and C, C or S, S both yield IIB, since they are related by parity).

    Orbifolding IIB by this symmetry is the same as orbifolding twice. This necessarily must return us back to IIA.

    The M theory parity orbifold differs from this $(-1)^{F_L}$ orbifold primarily in that it includes fixed points, on which the twisted sectors localize.

30. Start with the heterotic E theory and compactify on a circle. n units of KK momentum on this circle will be T-dualized to n units NS flux in the O(32) theory, ie a string wrapping the circle n times. Upon S-duality, this will correspond to a D1 brane wrapping the circle in type I n times. We T-dualize again to get a D0 brane in the type I' theory carrying n units of charge. In the strong coupling limit, this is understood as n units of momentum in the eleventh direction.

31. The coupling to a single boundary is given by

$$-\frac{1}{4\lambda^2}\int d^{10}x\sqrt{-g_{10}}\,\mathrm{Tr}[F^2]$$

    At first glance, λ would appear arbitrary. Anomaly cancelation will yield an exact value for it in terms of the eleven-dimensional gravitational coupling.

    In the presence of a boundary, the 11D Chern-Simons terms spoil gauge invariance.

32. Here M is a 20×20 matrix. It is quick to see that $MLM = L$ for the 20×20 matrix

$$L = \begin{pmatrix} 0 & 1_4 & 0 \\ 1_4 & 0 & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}.$$

This means that M is an element of $O(4,20)$. We can act on it as a bi-fundamental representation (on left and right). **This is more subtle, because not all $O(4,20)$ matrices have the form of M. Showing that M keeps the same form would take too much time**. This ensures that the last term is invariant.

The $4+4+16=24$ gauge fields from the compactification can be directly seen to transform in the contragradient representation of $O(4,20)$. This ensures that the second-to-last term is invariant. All other terms are invariant.

33. The heterotic action

$$\int d^6x\sqrt{-G}[R-\partial^\mu\Phi\partial_\mu\Phi-\frac{e^{-2\Phi}}{2}|H|^2-\frac{e^{-\Phi}}{4}M^{-1}_{ij}F^i_{\mu\nu}F^{j\,\mu\nu}+\frac{1}{8}\mathrm{Tr}[\partial_\mu M\partial^\mu M^{-1}]]$$
$$H_{\mu\nu\rho}=\partial_\mu B_{\nu\rho}-\frac{1}{2}L_{ij}A^i_\mu F^j_{\nu\rho}+2\,\text{perms.}$$

and the IIA action:

$$\int d^6x\sqrt{-G}[R-\partial^\mu\Phi\partial_\mu\Phi-\frac{e^{-2\Phi}}{2}|H|^2-\frac{e^{\Phi}}{4}M^{-1}_{ij}F^i_{\mu\nu}F^{j\,\mu\nu}+\frac{1}{8}\mathrm{Tr}[\partial_\mu M\partial^\mu M^{-1}]]+\frac{1}{2}\int d^6x\,L_{ij}\,B\wedge F^i\wedge F^j$$
$$H_{\mu\nu\rho}=\partial_\mu B_{\nu\rho}+2\,\text{perms.}$$

I will take the shorthand $\frac{1}{2}M^{-1}_{ij}F^i_{\mu\nu}F^{j\,\mu\nu}=|F|^2$

The EOMs for G give respectively

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R-2(\nabla_\mu\Phi\nabla_\nu\Phi-g_{\mu\nu}(\nabla\Phi)^2)-\frac{1}{2}Rg_{\mu\nu}-e^{-2\Phi}(H^2_{\mu\nu}-\frac{1}{2}g_{\mu\nu}|H|^2)-e^{-\Phi}(F_{\mu\nu}-\frac{1}{2}g_{\mu\nu}|F|^2)+(M\text{ terms})$$
$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R-2(\nabla_\mu\Phi\nabla_\nu\Phi-g_{\mu\nu}(\nabla\Phi)^2)-\frac{1}{2}Rg_{\mu\nu}-e^{-2\Phi}(H^2_{\mu\nu}-\frac{1}{2}g_{\mu\nu}|H|^2)-e^{\Phi}(F_{\mu\nu}-\frac{1}{2}g_{\mu\nu}|F|^2)+(\text{same }M\text{ terms})$$

here $H^2_{\mu\nu}=\frac{1}{4}H_{\mu\rho\sigma}H^{\nu}_{\rho\sigma}$ etc. All terms are invariant under $\Phi\to-\Phi$, including the terms involving H^2, since we will have

$$e^{-2\Phi'}H'^2=e^{2\Phi}(e^{-2\Phi}\star H)^2=e^{-2\Phi}|H|^2$$

and the same for $H_{\mu\nu}$.

The EOMs for Φ give respectively:

$$\nabla^2\Phi+e^{-2\Phi}|H|^2+\frac{e^{-\Phi}}{2}|F|^2$$
$$\nabla^2\Phi+e^{-2\Phi}|H|^2-\frac{e^{\Phi}}{2}|F|^2$$

The EOMs for the A_μ give respectively:

$$e^{-2\Phi}(\star H)\wedge F-\mathrm{d}(e^{-\Phi}M_{ij}\star F^j)=0$$
$$-\mathrm{d}(e^{\Phi}M_{ij}\star F^j)+H\wedge F=0$$

This is equivalent under $\Phi\to-\Phi, e^{-2\Phi}H=\star H'$.

The EOMs for the $B_{\mu\nu}$ give respectively

$$-\mathrm{d}(e^{-2\Phi}\star H)=0,\qquad \mathrm{d}H-F\wedge F=0$$
$$-\mathrm{d}(e^{-2\Phi}\star H)+F\wedge F=0,\qquad \mathrm{d}H=0$$

The fact that the H duality exchanges the Bianchi identity and the EOMs speaks to the fact that it is an *electric-magnetic duality of strings.*

The EOMs for the M terms are Φ,A,B independent. Consequently, the M matrices can be directly identified between the two theories.

34. Consider just the 3D space of the x^i. Note that V is harmonic, and consequently $F := - \star dV$ is a closed 2-form on that space. We can view F as a curvature 2-form on a principal $U(1)$ bundle, and can thus write (upon picking a trivialization of the $U(1)$) a potential A giving $dA = F$. Call the the $U(1)$ bundle X. We will write the connection as $\mathcal{A} = A + d\gamma$.

    For each of the three x^i there is a symplectic form on the 4D $U(1)$ bundle given by:

    $$\omega_i := \mathcal{A} \wedge dx^i + V \star dx^i \Rightarrow d\omega = - \star dV \wedge dx^i + dV \wedge \star dx^i = 0$$

    Here we have used that all the dx^i forms are (canonically) pulled back from $\mathbb{R}^3$ and in $\mathbb{R}^3$, $\star\alpha \wedge \beta = \alpha \wedge \star\beta$. Now, define a different basis of symplectic forms on X by

    $$\Omega_1 = \omega_2 + i\omega_3 = \mathcal{A} \wedge (dx^2 + idx^3) + iVdx^1 \wedge (dx^2 + idx^3)$$

    defining $z^1 = x^2 + ix^3$ this gives:

    $$\Omega_i = A \wedge dz^i + iVdx^i \wedge dz^i = V(\underbrace{V^{-1}\mathcal{A} + idx^1}_{\alpha_1}) \wedge dz^i.$$

    The kernel of this form on the $U(1)$-bundle is 2D. For Ω_1 it is spanned by

    $$\tilde{\partial}_{x^2} + i\tilde{\partial}_{x^3}, \quad V\partial_\gamma + i\tilde{\partial}_{x^1}$$

    Here each ∂_{x^i} is lifted to the $U(1)$ tangent space by using the connection A to identify the appropriate horizontal subspace. We identify this as a *holomorphic tangent space*. Similarly $\bar{\Omega}_1$ would complete the basis of T_pM and give the anti-holomorphic tangent space. Thus, each Ω_i gives a distinct stratification into holomorphic and anti-holomorphic tangent spaces. The closedness of Ω guarantees integrability. Defining 3 separate complex structures I_j to act as $+i$ on the jth holomorphic tangent space and as $-i$ on the jth anti-holomorphic tangent space, we can easily check that pointwise they reproduce the quaternion algebra. This makes the manifold hyper-Kähler, with metric given by:

    $$\begin{aligned} ds^2 &= V(\Re(\alpha_1)^2 + \Im(\alpha_1)^2) + V(\Re(dz_1)^2 + \Im(dz_1)^2) \\ &= V^{-1}(\mathcal{A})^2 + V|d\vec{x}|^2 \\ &= V^{-1}(Adx + d\gamma)^2 + V|d\vec{x}|^2 \end{aligned}$$

    In particular, V can take the form of the multi-center potential in the problem.

    Could we not have just exhibited a 3 Killing spinors? Are there such? In any case, this was more instructive

    Lastly, to see the asymptotic limit, we can take the x_i to collide. At a distance, V will look like $\frac{N}{r}$. This corresponds to an F with N units of flux asymptotically. The circle bundle over the $\mathbb{R}^3$ will asymptotically looks like an S^1 fibration over S^2. For $N = 1$, this is simply the Hopf fibration. For higher N, the connection is N times larger, which makes the $U(1)$ circle N times smaller, and corresponds to a fiberwise quotient of $S^3 \to S^2$ by $\mathbb{Z}_N$.

    Did Kiritsis mean to write N?

35.

36. In our case, wrapping 3-branes around 2-cycles give rise to two-forms. As one 2-cycle shrinks B to zero size, we get a tensionless string, of tension approximately $|\mathrm{Vol}(B)|/g_s$. For each isolated singularity of K3 (type ADE) there is such a tensionless string theory. Note that this is not yet the (2,0) SCFT, since we have not taken any sort of IR limit that would lead us to expect that the theory is conformal. We still have mass scales. This is an interacting QFT of light strings.

    Upon compactifying on an S^1, we can T-dualize to type IIA, where now we have the familiar appearance of massless states associated to a 3-cycle shrinking in K3. The IIA theory sees massless particles emerge at this transition, corresponding to the tensionless strings of IIB wrapping the S^1.

37. Here, our cycle is $C = n_i B^i$. Take a *euclidean* D2 brane wrapping this cycle. The total volume (counting orientation) will be $|n_i \int_{B^i} \Omega| = |n_i Z^i|$.

    Because of the BPS property of the 3-cycle, we will still have $M = T_p|Z|$, giving us

$$S_{inst} = \frac{1}{(2\pi)^2 \ell_s^3 g} |n_i Z^i|.$$

    It is worth remarking that we get contributions from all winding numbers of D instantons in this case, while in the IIB case, it looks like only the singly-wrapped D3 brane is stable.

    Is there anything else I should say? Reproduce Vafa+Ooguri's calculation?

38. In IIB, we have seen that as a three-cycle shrinks, a (BPS) D-brane wrapping this cycle contributes a hypermultiplet that becomes massless as the volume goes to zero. At the conifold point, we get a new massless multiplet. Resolving this singularity by expanding the 2-cycle corresponds to giving an expectation value to the massless hypermultiplet from the D-brane. In general, these hypermultiplets will have a potential. See the discussion on page **378**.

    From this POV, condensation of D-branes has the interpretation of topology change! For IIA the (instantonic) D2 branes instead serve to smooth out the singularity, which corresponds to the hypermultiplet moduli space receiving quantum corrections.

    This does not answer the question, though - which was about the resolution of the two-cycles. However, using the tool of mirror symmetry, we can posit a guess. A two-cycle shrinking in IIA causes a singularity in the vector multiplet, and maps to the familiar three-cycle shrinking in IIB. In IIA, then, we expect a wrapped D2 brane to contribute to a massless hypermultiplet. On the other hand, we expect quantum effects in IIB to smooth out this singularity.

    Check against literature.

39. I'll start from the Einstein frame in 6D. Taking $\sigma = \frac{1}{2} \det G_{\alpha\beta}$ we get for heterotic:

$$\int d^4x \sqrt{-G} e^{\sigma} \left(R + |\nabla\sigma|^2 + \frac{1}{4} \partial_\mu G_{\alpha\beta} \partial^\mu G^{\alpha\beta} - |\nabla\Phi|^2 - \frac{e^{-2\Phi}}{2}|H_3|^2 - \frac{e^{-2\Phi}}{2} G^{\alpha\beta} F^B_\alpha F^B_\beta - \frac{e^{-\Phi}}{2}|F_2|^2 + \frac{1}{8}\mathrm{Tr}[\partial M \partial M^{-1}] \right)$$

    The M matrix will combine with the $G_{\alpha\beta}$ to parameterize a SO(6, 22) coset. Rescaling $G \to e^{-\sigma/2} G$ we get:

$$\int d^4x \sqrt{-G} \left(R - |\nabla\Phi|^2 - \frac{e^{-2\Phi+\sigma}}{2}|H_3|^2 - \frac{e^{-\Phi+\sigma}}{2}|F_2|^2 + \frac{1}{8}\mathrm{Tr}[\partial M \partial M^{-1}] \right)$$

    For the IIA side we get something similar:

$$\int d^4x \sqrt{-G} e^{\sigma} \left(R + |\nabla\sigma|^2 + \frac{1}{4} \partial_\mu G_{\alpha\beta} \partial^\mu G^{\alpha\beta} - |\nabla\Phi|^2 - \frac{e^{-2\Phi}}{2}|H_3|^2 - \frac{e^{-\Phi}}{2}|F_2|^2 + \frac{1}{8}\mathrm{Tr}[\partial M \partial M^{-1}] \right)$$
$$=$$

40. I will just demonstrate this on the Bosonic sector. $(-1)^{\mathbf{F}_L}$ sends the RR fields to minus themselves (ie C_0, C_2, C_4), while S swaps B_2, C_2 and flips the axion part of the axio-dilaton $\tau \to -\bar{\tau}$. Conjugating $(-1)^{\mathbf{F}_L}$ by S will flip the sign of C_4 and B_2 and also take $\tau \to -\bar{\tau}$. The untwisted sector will thus be without B_2, C_0, C_4 leaving only G, ϕ, C_2. This is the closed-string sector of type I.

    On the other hand, we have shown that orbifolding IIB by just $(-1)^{\mathbf{F}_L}$ yields just IIA. At the level of bosonic fields, we already see that these operations do not commute.

41. It is worth appreciating that this duality was known before the Horava-Witten construction.

    First note that the moduli space of the heterotic string on T^3 is given by the coset space

$$\mathbb{R}^+ \times \mathrm{SO}(19,3;\mathbb{Z}) \backslash \mathrm{SO}(19,3)/(\mathrm{SO}(19) \times \mathrm{SO}(3))$$

with $\mathbb{R}^+$ parameterizing the dilaton. Now, in string compactifications K3 has a moduli space coming from cosets of $\mathrm{SO}(4,20)/\mathrm{SO}(4)\times\mathrm{SO}(20)$. This includes the *complexified* Kähler modulus, which takes into account the NSNS B-field. M theory lacks this parameter, and consequently the Kähler component of moduli space involves only *real* moduli (ie metrics). This gives $\mathrm{SO}(3,19)/\mathrm{SO}(3)\times\mathrm{SO}(19)$. The volume gives another factor of $\mathbb{R}$.

The low-energy effective actions also match. Wrapping the M-theory A_3 on K3 gives one 3-form C_3 and 22 1-forms A_1^i. We get an action:

$$\frac{1}{2\kappa_{11}^2}\int d^{11}\sqrt{G_{11}}(R+\frac{1}{2}|dA_3|^2) \to \int d^7\sqrt{G_7}[e^{4\sigma}(R+(\partial\sigma)^2 - \sum_i \frac{1}{2}|dA_1^i|^2 + \text{moduli}) - \frac{1}{2}|dC_3|^2]$$

Upon rescaling $g \to e^{-4\sigma}g$ and taking $\phi = 3\sigma$ we arrive at:

$$\int d^7\sqrt{G_7}[e^{-2\phi}(R+(\partial\sigma)^2 - \sum_i \frac{1}{2}|dA_1^i|^2 + \text{moduli} - \frac{1}{2}|dC_3|^2)]$$

This exactly matches with the heterotic theory. We go to strong heterotic coupling by taking $\sigma \to \infty$, ie taking the volume of the K3 to be large.

42. Because A_3 is odd under the $\mathbb{Z}_2$ transformation, we must wrap it on either a 1-cycle or a 3-cycle to have things survive. There are 5 1-cycles giving 5 vectors and $\binom{5}{2} = 10$ 2-cycles giving 10 0-forms in 6D. Further, the internal metric has $5\times 6/2 = 15$ even terms that survive. Altogether we get 5 2-forms, 25 scalars, and 10 vectors.

    This is $N = (2,0)$ (chiral) supergravity consisting of the supergravity multiplet and five tensor multiplets, each of which contains an anti-self-dual two-form field (the 5 self-dual parts are part of the SUGRA multiplet). Cancelation of anomalies **prove and understand compared to the $N = (1,0)$ case** require $N_T = 21$. We are missing *sixteen* tensor multiplets.

    The orbifold has $2^5 = 32$ fixed points which we expect will lead to twisted sectors. We shouldn't be too sure of how things go, though, because we don't know how to deal with twisted sectors of M-theory. It initially looks paradoxical that we need 16 extra tensor multiplets but have 32 fixed points. This is resolved in Witten 9512219. The solution is to recognize the fixed points as 32 magnetic sources of charge $-1/2$ for the G_4 field. The constraint that the total charges should vanish is satisfied when 16 of these sources have a five-brane on top of them of $+1$ magnetic charge. Each fivebrane can be seen to support a single tensor multiplet, giving our desired 16. We interpret the five scalar in each multiplet as describing the position of the fivebrane inside $T^5/\mathbb{Z}_2$.

    This now gives the massless spectrum of type IIB on K3.

    The question is how to arrange the fivebranes in such a way that we can see the duality to IIB on K3. The equivalence implies that when *any* circle shrinks **show more rigorously**, we would expect to recover weakly coupled IIB on K3. The naive guess I had is to arrange them in an alternating "checkerboard" pattern. Witten confirms this. Now, in the limit where any circle shrinks to zero size, the opposite charge sources cancel, giving zero 4-form field strength in the 6D spacetime, consistent with the fact that the 3-form has been projected out on the M-theory side and doesn't exist on the IIB side.

    As a second check, we can further compactify on S^1. We get IIA on $T^5/\mathbb{Z}_2$ vs IIB on $S^1\times K^3$. T-dualizing the latter along S^1 (the only 1-cycle!) gives IIA on $S^1\times K^3$ which is equivalent to heterotic on S^5. S-dualizing this gives type I on S^5 which is T-dual to IIA on $T^5/\mathbb{Z}_2$ as an *orientifold* **(why do we need to act with Ω too?)**

43. I will work with covariant derivatives and take the axiodilaton fields in terms of the $\mathcal{S},\bar{\mathcal{S}}$ basis. I will write the equations of motion for the axiodilaton as:

$$\bar\nabla\left(\frac{\partial\mathcal{S}}{(\mathcal{S}-\bar{\mathcal{S}})^2}\right) - 2\frac{\partial\mathcal{S}\bar\partial\bar{\mathcal{S}}}{(S-\bar{\mathcal{S}})^3} = \frac{\partial\bar\partial\mathcal{S}}{(S-\bar{\mathcal{S}})^2} - 2\frac{\partial\mathcal{S}\bar\partial\bar{\mathcal{S}}}{(S-\bar{\mathcal{S}})^3} + 2\frac{\partial\mathcal{S}\bar\partial\bar{\mathcal{S}} - \partial\mathcal{S}\bar\partial\mathcal{S}}{(S-\bar{\mathcal{S}})^3} \Rightarrow \partial\bar\partial\mathcal{S} + 2\frac{\partial\mathcal{S}\bar\partial\mathcal{S}}{\bar{\mathcal{S}}-\mathcal{S}} = 0$$

    Where it is important to note that we can write $\partial\bar\partial\mathcal{S}$ for the laplacian in complex 2D coordinates instead of $\nabla\nabla\mathcal{S}$. We thus get our desired EOM.

44. Recall that as a holomorphic function of z, $\mathcal{S}$ should have positive imaginary part, and have its image restricted to the fundamental domain $\mathcal{F}$. This mapping should be finite energy density. From the effective action we compute the energy density by pulling back as:

$$\mathcal{E} = -\frac{i}{\kappa_{10}^2}\int d^2z \frac{\partial\mathcal{S}\bar{\partial}\bar{\mathcal{S}}}{(\mathcal{S}-\bar{\mathcal{S}})^2} = \frac{i}{\kappa_{10}^2}\int_{\mathcal{F}} d^2\mathcal{S}\,\partial\bar{\partial}\log(\mathcal{S}-\bar{\mathcal{S}})$$

At this point we apply Stokes' theorem to get a boundary integral:

$$\frac{i}{\kappa_{10}^2}\int_{\partial\mathcal{F}} d\mathcal{S}\partial\log(\mathcal{S}-\bar{\mathcal{S}}) = \frac{i}{\kappa_{10}^2}\int_{\partial\mathcal{F}}\frac{d\mathcal{S}}{S-\bar{\mathcal{S}}}$$

The vertical lines of the fundamental domain have the same values but are traversed in opposite orientation **picture**. Therefore, only the semicircle counts. This integral is readily evaulated:

$$\int_{2\pi/3}^{\pi/3}\frac{d\theta\, ie^{i\theta}}{e^{i\theta}-e^{-i\theta}} = -\frac{i\pi}{6}$$

This gives our desired final answer of $\frac{\pi}{6\kappa_{10}^2}$: the angular defect due to a D7 brane.