Chapter 9: Compactification and Supersymmetry Breaking

1. We compactify the heterotic string along just one dimension, making it a compact circle of radius R with all 16 Wilson lines turned on.

   Each noncompact boson contributes
$$\frac{1}{\sqrt{\tau_2}\eta\bar{\eta}}$$
   The fermions on the supersymmetric side contribute
$$\sum_{a,b=0}^{1}(-1)^{a+b+ab}\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}^4}{\eta^4}$$
   The (p,p) compact bosons and 16 complex right-moving fermions that can be written as the pair $\psi^I(\bar{z}),\bar{\psi}^I(\bar{z})$ have the action as in **E.1** (setting $\ell_s=1$)
$$\frac{1}{4\pi}\int d^2\sigma\sqrt{\det g}g^{ab}G_{\alpha\beta}\partial_a X^\alpha\partial_b X^\beta+\frac{1}{4\pi}\int d^2\sigma\epsilon^{ab}B_{\alpha\beta}\partial_a X^\alpha\partial_b X^\beta+\frac{1}{4\pi}\int d^2\sigma\sqrt{-\det g}\sum_I\psi^I[\bar{\nabla}+Y_\alpha^I\bar{\partial}X^\alpha]\bar{\psi}^I$$
   Here α,β are the toral coordinates for the compact spacetime and Y_α^I is the Wilson line along torus cycle α. To evaluate the path integral, as we did in the purely bosonic case, we have a factor of
$$\frac{\sqrt{\det G}}{\tau_2^{p/2}(\eta\bar{\eta})^p}$$
   coming from evaluating the determinant $(\det\nabla^2)^{-1/2}$ of the bosons. This multiplies a sum over instanton contributions labelled by m^α,n^α taking values in a (p,p)-signature lattice with classical action
$$\sum_{m^\alpha,n^\alpha}e^{-\frac{\pi}{\tau}(G+B)_{\alpha\beta}(m+\tau n)^\alpha(m+\bar{\tau}n)^\beta}\times\text{fermions.}$$
   The fermion contribution depends via the Wilson lines on the configuration of the X^α. In each such instanton sector, the fermion path integral with a constant background Wilson line is equivalent to a free fermion with twisted boundary conditions. For simplicity, let's compactify just on S^1, and denote $\theta^I=Y^In,\phi^I=-Y^Im$. We get boundary conditions:
$$\psi^I(\sigma+1,\sigma_2)=-(-1)^a e^{2\pi i\theta^I}$$
$$\psi^I(\sigma,\sigma_2+1)=-(-1)^b e^{-2\pi i\phi^I}$$
   where $a,b=0,1$ denotes anti-periodic/periodic boundary conditions respectively. We know that (in the absence of Wilson lines) the determinant of ∂ acting on complex fermions is:
$$\det{}_{a,b}\,\partial=\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}}{\eta}$$
   Let us now investigate the twisted boundary conditions. For simplicity its enough to take $a=b=0$ (all antiperiodic). We have two different ways to write the partition function. As a product over modes, we have $\psi_m,\bar{\psi}_m$ modes, with respective weights $m-\frac{1}{2}-\theta,m-\frac{1}{2}+\theta$ and respective fermion numbers ±1 *relative to the ground state*. The fermion number of the ground state has no canonical value (as far as I can see). On the other hand, the ground state energy is given by the standard mneumonic to be $-\frac{1}{24}+\frac{1}{2}\theta^2$. This gives:
$$\mathrm{Tr}_\theta[e^{2i\pi\phi F}q^H]=q^{\frac{\theta^2}{2}-\frac{1}{24}}\prod_{m=1}^{\infty}(1+q^{m-1/2+\theta}e^{2\pi i\phi})(1+q^{m-1/2-\theta}e^{-2\pi i\phi})=q^{\theta^2/2}\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(\phi+\theta\tau|\tau)}{\eta}$$
   For other boundary conditions, we can apply the same logic to get
$$q^{\theta^2/2}\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(\phi+\theta\tau|\tau)}{\eta}$$

The overall phase is still a mystery. Writing $\theta\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix}$ as a new theta function, we can fix the phase by requiring modular invariance

$$\begin{aligned}
\theta\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix}(\tau+1) &= \theta\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}\theta\\\phi+\theta\end{bmatrix}(\tau) &\qquad \theta\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix}(\tau+1) &= \theta\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}\theta\\\phi+\theta\end{bmatrix}(\tau)\\
\theta\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix}(\tau+1) &= e^{i\pi/4}\theta\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}\theta\\\phi+\theta\end{bmatrix}(\tau) &\qquad \theta\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix}(\tau+1) &= e^{i\pi/4}\theta\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}\theta\\\phi+\theta\end{bmatrix}(\tau)
\end{aligned} \tag{1}$$

Even from the first of these conditions, we see that we need a term going as $e^{i\theta\phi}$ out front. After adding this in, all other transformations will hold automatically. The $\tau \to -1/\tau$ transformation will thus hold automatically. **Interpret this as an anomaly? Yes, Narain, Witten do this in Section 3 of their paper. It seems careful anomaly analysis is not enough and one must indeed impose modular invariance by hand.**

Altogether then the 16 complex antiholomorphic fermions contribute in each instanton sector:

$$e^{-i\pi\sum_I \theta^I(\phi^I+\bar\tau\theta^I)}\frac{1}{2}\sum_{a,b=0}^{1}\prod_{i=1}^{16}\frac{\bar\theta\begin{bmatrix}a\\b\end{bmatrix}(\phi+\bar\tau\theta|\bar\tau)}{\bar\eta}$$

Giving a total partition function as in the second (unnumbered) equation of **Appendix E**:

$$\left[\frac{R}{\sqrt{\tau_2}\eta\bar\eta^{17}}\sum_{m,n}e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\underline{e^{-i\pi\sum_I nY^I(m+n\bar\tau)Y^IY^I}\frac{1}{2}\sum_{a,b=0}^{1}\prod_{i=1}^{16}\bar\theta\begin{bmatrix}a\\b\end{bmatrix}(Y^I(m+\bar\tau n)|\bar\tau)}\right]\times\frac{1}{\tau_2^{7/2}\eta^7\bar\eta^7}\frac{1}{2}\sum_{a,b=0}^{1}\frac{\theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4}$$

From the properties of the theta functions in Equation (1), the underlined fermionic sum has the exact same transformation properties as a sum of θ^{16} terms and thus makes the full partition function modular invariant.

Each theta function can be written in sum form as:

$$\theta\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix} = e^{\pi i\theta\phi}q^{\theta^2/2}\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}(n-\frac{a}{2})^2}e^{2\pi i(n-\frac{a}{2})(\phi+\tau\theta-\frac{b}{2})} = \sum_{n\in\mathbb{Z}}q^{\frac{1}{2}(n+\theta-\frac{a}{2})^2}e^{2\pi i\phi(n+\frac{1}{2}\theta-\frac{a}{2})-\pi ib(n-\frac{a}{2})}$$

Then we get the following expression for the underlined fermionic term:

$$\begin{aligned}
&\frac{1}{2}\sum_{a,b=0}^{1}\prod_{I=1}^{16}\sum_{k\in\mathbb{Z}}\bar q^{\frac{1}{2}(k+nY^I-\frac{a}{2})^2}e^{-2\pi imY^I(k+\frac{1}{2}nY^I-\frac{a}{2})+\pi ib(k-\frac{a}{2})}\\
&=\frac{1}{2}\sum_{a,b=0}^{1}\sum_{q^I\in\mathbb{Z}^{16}}\bar q^{\frac{1}{2}(q^I+nY^I-\frac{a}{2})^2}e^{-2\pi imY^I(q^I+nY^I-\frac{a}{2})+\pi ib(k-\frac{a}{2})}\\
&=\frac{1}{2}\sum_{q^I\in\mathbb{Z}^{16}}\left[\bar q^{\frac{1}{2}(q^I+nY^I)^2}e^{-2\pi imY^I(q^I+\frac{1}{2}nY^I)}(1+(-1)^{\sum_I q^I})+\bar q^{\frac{1}{2}(q^I+nY^I-\frac{1}{2})^2}e^{-2\pi imY^I(q^I+\frac{1}{2}nY^I-\frac{1}{2})}(1+(-1)^{\sum_I(q^I-\frac{1}{2})})\right]\\
&=\sum_{q^I\in\Lambda^{16}}q^{(q^I+nY^I)^2}e^{-2\pi imY^I(q^I+\frac{1}{2}nY^I)}
\end{aligned}$$

We note that the second-to last line is indeed the sum over the roots of $O(32)$ augmented with one of the spinor weight lattices. Altogether the compact dimensions contribute:

$$\frac{R}{\sqrt{\tau_2}\eta\bar\eta^{17}}\sum_{m\in\mathbb{Z},n\in\mathbb{Z},q^I\in\Lambda^{16}}\exp\left[\frac{\pi R^2}{\tau_2}(m+n\tau)(m+n\bar\tau)+\pi i\tau(q^I+nY^I)^2-2\pi imY^I(k+\frac{1}{2}Y^I)\right]$$

To put this whole thing into Hamiltonian form, we proceed as in the bosonic case and perform a Poisson summation over m. The terms that contribute are:

$$
\begin{aligned}
&e^{-\frac{\pi R^2}{\tau_2}n^2\tau_1^2-n^2\pi R^2\tau_2}\sum_m e^{-\frac{\pi R^2}{\tau_2}m^2-2\pi i m Y^I(q^I+\frac{1}{2}nY^I)-i\frac{nR^2\tau_1}{\tau_2}} \\
&\qquad = e^{-\frac{\pi R^2}{\tau_2}n^2\tau_1^2-n^2\pi R^2\tau_2}\frac{\sqrt{\tau_2}}{R}\sum_m e^{-\frac{\pi\tau_2}{R^2}(m+Y^I(q^I+\frac{1}{2}nY^I)-in\frac{R^2\tau_1}{\tau_2})^2} \\
&\qquad = e^{-\frac{\pi R^2}{\tau_2}n^2\tau_1^2-n^2\pi R^2\tau_2}\frac{\sqrt{\tau_2}}{R}\sum_m e^{-\frac{\pi\tau_2}{R^2}(m+Y^I(q^I+\frac{1}{2}nY^I))^2+\pi R^2\frac{\tau_1^2}{\tau_2}n^2+2\pi i(m+q^I+\frac{1}{2}nY^I)n\tau_1} \\
&\qquad = e^{-n^2\pi R^2\tau_2}\frac{\sqrt{\tau_2}}{R}\sum_m e^{-\frac{\pi\tau_2}{R^2}(m+Y^I(q^I+\frac{1}{2}nY^I))^2+2\pi i(m+q^I+\frac{1}{2}nY^I)n\tau_1}
\end{aligned}
$$

Together with the other terms this gives us

$$
\begin{aligned}
&\frac{1}{\eta\bar{\eta}^{17}}\sum_{n,m,q^I} q^{\frac{1}{2}(q^I+nY^I)^2}e^{-n^2\pi R^2\tau_2}e^{-\frac{\pi\tau_2}{R^2}(m+Y^I(q^I+\frac{1}{2}nY^I))^2+2\pi i(m+q^I+\frac{1}{2}nY^I)n\tau_1} \\
&= \frac{1}{\eta\bar{\eta}^{17}}\sum_{n,m,q^I} q^{\frac{1}{2}(q^I+nY^I)^2}q^{\frac{1}{2}(\frac{1}{R}(m-Y^I(q^I+\frac{1}{2}nY^I)+nR)^2}\bar{q}^{\frac{1}{2}(\frac{1}{R}(m-Y^I(q^I+\frac{1}{2}nY^I)-nR)^2}
\end{aligned}
$$

where I've flipped $m \to -m$ at the end there. We get momenta

$$
\begin{aligned}
k_L &= \frac{1}{R}(m-q^IY^I-\frac{1}{2}nY^IY^I)+nR = \frac{m}{R}+n(R-\frac{1}{2}Y^IY^I)-q^IY^I \\
k_R &= \frac{1}{R}(m-q^IY^I-\frac{1}{2}nY^IY^I)-nR = \frac{m}{R}-n(R+\frac{1}{2}Y^IY^I)-q^IY^I \\
k_R^I &= q^I+nY^I
\end{aligned}
$$

consistent with Polchinski with $m \leftarrow n_m, n \leftarrow w^n, Y^I \leftarrow RA^I$ and $\alpha' = 0$ **(might be off by a factor of 2 for k_R^I rel. to Polchinski but I think I'm consistent with Ginsparg)**. We only care about the $SO(1,1,\mathbb{Z})$ T-duality group coming from the compact x^9. This does not act on the Y^I as far as I can see **CHECK**

The SO$(16,\mathbb{Z})$ on the other hand acts on the Y^I as in the standard vector representation.

2. I am going to re-do the computations of appendix F Hatted indices denote the 10D terms. Greek indices from the start of the alphabet denote compact 10-D-dimensional indices while greek indices from the middle of the alphabet denote noncompact D-dimensional indices.

   The 10D action is

$$
\int d^{10}x\sqrt{-\hat{G}_{10}}\,e^{-2\hat{\Phi}}[\hat{R}+4(\nabla\hat{\Phi})^2-\frac{1}{12}\hat{H}^2-\frac{1}{4}\mathrm{Tr}\hat{F}^2]+O(\ell_s^2)
$$

   with $\hat{F}^I_{\mu\nu} = \partial_\mu\hat{A}^I_\nu-\partial_\nu\hat{A}^I_\mu$ and $\hat{H}_{\mu\nu\rho} = \partial_\mu\hat{B}_{\nu\rho}-\frac{1}{2}\sum_I\hat{A}^I_\mu\hat{F}^I_{\nu\rho}+2\,\mathrm{perms.}$. Here I is the internal 16-dimensional index for the heterotic string.

   We take the 10-bein (r, a denote D and $10-D$ 10-bein indices, hatted indices $\hat{r},\hat{\mu}$ should not be confused for 10-bein indices!!)

$$
e^{\hat{r}}_{\hat{\mu}} = \begin{pmatrix} e^r_\mu & A^\beta_\mu E^a_\beta \\ 0 & E^a_\alpha \end{pmatrix} \qquad e^{\hat{\mu}}_{\hat{r}} = \begin{pmatrix} e^\mu_r & -e^\nu_r A^\alpha_\nu \\ 0 & E^\alpha_a \end{pmatrix}
$$

   This gives us the metric:

$$
G_{\hat{\mu},\hat{\nu}} = \begin{pmatrix} G_{\mu\nu}-A^\alpha_\mu G_{\alpha\beta}A^\beta_\nu & G_{\alpha\beta}A^\beta_\mu \\ G_{\alpha\beta}A^\beta_\nu & G_{\alpha\beta} \end{pmatrix}
$$

   As we've done before in chapter 7, we then define

$$
\phi = \Phi-\frac{1}{4}\log\det G_{\alpha\beta}, \qquad F^A_{\mu\nu} = \partial_\mu A_\nu-\partial_\nu A_\mu
$$

With this, the compactification of $R + 4(\nabla\phi)^2$ is clear:

$$\int d^D \sqrt{g} e^{-2\phi}[R + 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta} - \frac{1}{4}G_{\alpha\beta}F^{A\ \alpha}_{\mu\nu}F^{A\ \beta}_{\mu\nu}]$$

The first and second terms are clear. The third term makes up for the redefinition of Φ in terms of ϕ while the last term is the standard KK mechanism generating a gauge field strength from the compact dimensions.

Next, let's look $\hat{H}$. Because we have no sources for the H field, $\hat{H}$ is on the compact cycles. We can define the D-dimensional fields using the 10-bein as:

$$H_{\mu\alpha\beta} = e^r_\mu e^{\hat{\mu}}_r \hat{H}_{\hat{\mu}\alpha\beta} = \hat{H}_{\mu\alpha\beta} \tag{2}$$

$$H_{\mu\nu\alpha} = e^r_\mu e^s_\nu e^{\hat{\mu}}_r e^{\hat{\nu}}_s H_{\hat{\mu}\hat{\nu}\alpha} = \hat{H}_{\mu\nu\alpha} - A^\beta_\mu \hat{H}_{\nu\alpha\beta} + A^\beta_\nu \hat{H}_{\mu\alpha\beta} \tag{3}$$

$$H_{\mu\nu\rho} = e^r_\mu e^s_\nu e^t_\rho e^{\hat{\mu}}_r e^{\hat{\nu}}_s e^{\hat{\rho}}_t \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} + [-A^\alpha_\mu \hat{H}_{\alpha\nu\rho} + A^\alpha_\mu A^\beta_\nu \hat{H}_{\alpha\beta\rho} + 2\,\text{perms.}] \tag{4}$$

The point of defining these coordinates in terms of the 10-bein coordinate is that now, we can just directly separate the $\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}}$ sum into terms without worrying about the metric, and yield directly:

$$\int d^D \sqrt{-g} e^{-2\phi}[-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{3}{12}H_{\mu\nu\alpha}H^{\mu\nu\alpha} - \frac{3}{12}H_{\mu\alpha\beta}H^{\mu\alpha\beta}]$$

The method is the same for the F tensor. We define new Wilson lines and field strengths:

$$Y^I_\alpha = A^I_\alpha, \qquad A^I_\mu = e^r_\mu e^{\hat{\mu}}_r \hat{A}^I_{\hat{\mu}} = \hat{A}^I_\mu - Y^I_\alpha A^\alpha_\mu$$

I can define F in the standard $F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$, $\tilde{F}^I_{\mu\alpha} = \partial_\mu Y^I_\alpha$. This gives me $\hat{F}^I_{\mu\nu} = F^I_{\mu\nu} + \partial_\mu(Y^I_\alpha A^\alpha_\nu) - \partial_\nu(Y^I_\alpha A^\alpha_\nu)$. By redefining

$$\tilde{F}^I_{\mu\nu} = F^I_{\mu\nu} + Y^I_\alpha F^{A,\alpha}_{\mu\nu}$$

we can equate this with $\hat{F}^I_{\mu\nu}$. For the compact coordinates its more simple and I take $\tilde{F}_{\mu\alpha} = \partial_\mu Y^I_\alpha$. Again $\tilde{F}_{\alpha\beta}$ vanishes since we cannot have internal sources. This yields directly

$$\int d^D x \sqrt{-g} e^{-2\phi}[-\frac{1}{4}\sum_I^{16} \tilde{F}^I_{\mu\nu}\tilde{F}^{I,\mu\nu} - \frac{2}{4}\tilde{F}^I_{\mu\alpha}\tilde{F}^{I,\mu\alpha}]$$

Its not good enough for us to write everything in terms of an abstract H 3-form. We want to relate H to B and Y. From our relationship in 10D we can directly write:

$$H_{\mu\alpha\beta} = \partial_\mu B_{\alpha\beta} + \frac{1}{2}\sum_I (Y^I_\alpha \partial_\mu Y^I_\beta - Y^I_\beta \partial_\mu Y^I_\alpha)$$

Taking $C_{\alpha\beta} = \hat{B}_{\alpha\beta} - \frac{1}{2}\sum_I Y^I_\alpha Y^I_\beta$ we get

$$H_{\mu\alpha\beta} = \partial_\mu C_{\alpha\beta} + \sum_I Y^I_\alpha \partial_\mu Y^I_\beta$$

Next

$$H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} - \partial_\nu B_{\mu\alpha} + \frac{1}{2}\sum_I (\hat{A}^I_\nu \partial_\mu Y^I_\alpha - \hat{A}^I_\mu \partial_\nu Y^I_\alpha - Y^I_\alpha F^I_{\mu\nu})$$

We define the B field using not just the vielbein but also the gauge connection:

$$B_{\mu\alpha} := \hat{B}_{\mu\alpha} + B_{\alpha\beta}A^\beta_\mu + \frac{1}{2}\sum_I Y^I_\alpha A^I_\mu, \qquad F^B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

Then using (3) we get

$$H_{\mu\nu\alpha} = F^B_{\alpha\mu\nu} - C_{\alpha\beta}F^{A\ \beta}_{\mu\nu} - \sum_I Y^I_\alpha F^I_{\mu\nu}$$

Finally, using both vielbein and connection

$$B_{\mu\nu} = \hat{B}_{\mu\nu} + \frac{1}{2}[A^\alpha_\mu B_{\nu\alpha} + \sum_I A^I_\mu A^\alpha_\nu Y^I_\alpha - (\nu \leftrightarrow \mu)] - A^\alpha_\mu A^\beta_\nu B_{\alpha\beta}$$

And this gives us

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2} L_{ij} A^i_\mu F^j_{\nu\rho} + 2\,\text{perms.}$$

where L_{ij} is the $(10-D, 26-D)$-invariant metric and we have combined $A^\alpha_\mu, B_{\alpha\mu}, A^I_\mu$ into a length $36-2D$ vector.

Now the full action is:

$$\begin{aligned}\int d^D\sqrt{g}e^{-2\phi}[R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} \\ - \frac{1}{4}G^{\alpha\beta}H_{\mu\nu\alpha}H^{\mu\nu\beta} - \frac{1}{4}G_{\alpha\beta}F^{A\ \alpha}_{\mu\nu}F^{A\mu\nu\beta} - \frac{1}{4}\tilde{F}^I_{\mu\nu}\tilde{F}^{I,\mu\nu} \\ - \frac{1}{4}H_{\mu\alpha\beta}H^{\mu\alpha\beta} + \frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta} - \frac{1}{2}\tilde{F}^I_{\mu\alpha}\tilde{F}^{I,\mu\alpha}]\end{aligned}$$

Using our expressions for $H_{\mu\nu\alpha}$ and $\tilde{F}^A_{\mu\nu}$, the middle line can be combined into

$$-\frac{1}{4}\begin{pmatrix} G + C^TG^{-1}C + Y^TY & -C^TG^{-1} & C^TG^{-1}Y^T + Y^T \\ -G^{-1}C & G^{-1} & -G^{-1}Y^T \\ YG^{-1}C + Y & -YG^{-1} & 1 + YG^{-1}Y^T \end{pmatrix}_{ij} F^i_{\mu\nu}F^{\mu\nu\, j}$$

here $F^i = (F^{A^\alpha}, F^B{}_\alpha, F^I)$. Call the matrix M^{-1} and notice that $LML = M^{-1}$, and indeed we get M transforms in the adjoint of $\mathrm{SO}(26-D, 10-D)$.

Similar arguments would give that the last line becomes $\frac{1}{8}\mathrm{Tr}\partial_\mu M\partial^\mu M^{-1}$ (Too much algebra).

From this, its immediate that any $\mathrm{SO}(10-D, 26-D)$ transformation on the scalar matrix (adjoint rep) and array of vector bosons (vector rep) will preserve both of these last two terms. It will also preserve H since it depends on the invariant $B_{\nu\rho}$ and SO-invariant combination $L_{ij}A^i_\mu F^j_{\nu\rho}$.

3. The action for IIA in the string frame is

$$\frac{1}{2\kappa^2_{10}}\int d^{10}x\sqrt{-\hat{G}}\left[e^{-2\hat{\Phi}}[\hat{R} + 4(\nabla\hat{\Phi})^2 - \frac{1}{12}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}}] - \frac{1}{4}F_2^2 - \frac{1}{2\cdot 4!}F_4^2\right] + \frac{1}{4\kappa^2}\int B_2 \wedge \mathrm{d}C_3 \wedge \mathrm{d}C_3$$

Doing the same reduction as before, the $\hat{R} + 4(\nabla\hat{\Phi})^2 - \frac{1}{12}H^2$ term becomes:

$$\begin{aligned}&\int d^4\sqrt{-g}e^{-2\phi}\Big[R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F^{A\ \alpha}_{\mu\nu}F^{A\mu\nu}_{\ \ \ \ \alpha} + \frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta} - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}H_{\mu\alpha\beta}H^{\mu\alpha\beta} - \frac{1}{4}G^{\alpha\beta}H_{\mu\nu\alpha}H^{\mu\nu\alpha}\Big] \\ &= \int d^4\sqrt{-g}e^{-2\phi}\Big[R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}M^{-1}_{ij}F^i_{\mu\nu}F^{\mu\nu\, j} + \frac{1}{8}\mathrm{Tr}[\partial_\mu M\partial^\mu M^{-1}]\Big]\end{aligned}$$

Here we used H as in the last problem and the matrix M consisting of the 21 $G_{\alpha\beta}$ and 15 $B_{\alpha\beta}$. The F^i are the field strengths of the $6+6$ $U(1)$ vectors coming from G and B compactification.

$$H_{\mu\nu\rho} = \partial_\mu B_{\mu\rho} - \frac{1}{2}L_{ij}A^i_\mu F^j_{\nu\rho} + 2\,\text{perms.} \qquad M^{-1} = \begin{pmatrix} G + B^TG^{-1}B & -B^TG^{-1} \\ -G^{-1}BG^{-1} & G \end{pmatrix}$$

The $H_{\mu\nu\rho}$ can be dualized to provide a *sixteenth* scalar coming from the B field. By analogy to **9.1.13**, in the string frame I would expect to write:

$$e^{-2\phi}H_{\mu\nu\rho} = E_{\mu\nu\rho\sigma}\nabla^\sigma a$$

The $B_{\mu\nu}$ equations $\nabla^\mu(e^{-2\phi}H_{\mu\nu\rho})$ are now automatically satisfied. The axion EOMs come from the Bianchi identity:

$$E^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma} = -\frac{1}{2}L_{ij}E^{\mu\nu\rho\sigma}F^i_{\rho\sigma}F^j_{\mu\nu} = -L_{ij}\tilde{F}^i_{\mu\nu}F^{j\,\mu\nu}, \qquad \tilde{F}^i_{\mu\nu} = \frac{1}{2}E^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

Here we have defined the dual 2-form as required. This can now be recast as the equation of motion for the axion (contracting the Es gives a 4):

$$\nabla^\mu(e^{2\phi}\nabla_\mu a) = -\frac{1}{4}L_{ij}F^i_{\mu\nu}\tilde{F}^{j\,\mu\nu}$$

With this, we can dualize the action in terms of the axion to yield:

$$\int d^4\sqrt{-g}e^{-2\phi}\Big[R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{4\phi}(\partial a)^2 + \frac{1}{4}e^{2\phi}aL_{ij}F^i_{\mu\nu}\tilde{F}^{j\,\mu\nu} - \frac{1}{4}M^{-1}_{ij}F^i_{\mu\nu}F^{\mu\nu\,j} + \frac{1}{8}\mathrm{Tr}[\partial_\mu M\partial^\mu M^{-1}]\Big]$$

We could also do this in the Einstein frame and get *exactly* the same action as in **9.1.15** with the M matrix as we have it (no sum over heterotic internals).

The only thing left is the RR fields. We follow Kiritis' treatment of the 4-form field strength. We use the 10-bein to get:

$$\begin{aligned}
C_{\alpha\beta\gamma} &= \hat{C}_{\alpha\beta\gamma} \\
C_{\mu\alpha\beta} &= \hat{C}_{\mu\alpha\beta} - C_{\alpha\beta\gamma}A^\gamma_\mu \\
C_{\mu\nu\alpha} &= \hat{C}_{\mu\nu\alpha} + \hat{C}_{\mu\alpha\beta}A^\beta_\nu - \hat{C}_{\nu\alpha\beta}A^\beta_\mu + C_{\alpha\beta\gamma}A^\beta_\mu A^\alpha_\nu \\
C_{\mu\nu\rho} &= \hat{C}_{\mu\nu\rho} - (A^\alpha_\mu\hat{C}_{\nu\rho\alpha} + A^\alpha_\mu A^\beta_\nu C_{\alpha\beta\rho} + 2\,\text{perms.}) - C_{\alpha\beta\gamma}A^\alpha_\mu A^\beta_\nu A^\gamma_\rho
\end{aligned}$$

Let's now define the field strengths. Now we must have $F_{\alpha\beta\gamma\delta} = 0$ since the internal dimensions do not contain sources for the field. What remains is

$$\begin{aligned}
F_{\mu\alpha\beta\gamma} &= \partial_\mu C_{\alpha\beta\gamma} \\
F_{\mu\nu\alpha\beta} &= \partial_\mu C_{\nu\alpha\beta} - \partial_\nu C_{\mu\alpha\beta} + C_{\alpha\beta\gamma}F^\gamma_{\mu\nu} \\
F_{\mu\nu\rho\alpha} &= \partial_\mu C_{\nu\rho\alpha} + C_{\mu\alpha\beta}F^\beta_{\nu\rho} + 2\,\text{perms.} \\
F_{\mu\nu\rho\sigma} &= (\partial_\mu C_{\alpha\beta\gamma} + 3\,\text{perms.}) + (C_{\sigma\rho\alpha}F^\alpha_{\mu\nu} + 5\,\text{perms.})
\end{aligned}$$

Then this gives the contribution (here all two-lower one-upper index $F^\alpha_{\mu\nu}$ are taken to mean F^A):

$$S^{(4)}_{RR} = -\frac{1}{2\cdot 4!}\int d^4\sqrt{-g}\sqrt{\det G_{\alpha\beta}}[F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} + 4F_{\mu\nu\rho\alpha}F^{\mu\nu\rho\alpha} + 6F_{\mu\nu\alpha\beta}F^{\mu\nu\alpha\beta} + 4F_{\mu\alpha\beta\gamma}F^{\mu\alpha\beta\gamma}]$$

It is important to realize that in 4-D the 4-form field strength coming from the 3-form has *no* dynamical degrees of freedom. It plays the role of a cosmological constant **Check w/ Alek**.

The two-spacetime-index term can be directly dualized. It corresponds to $6\times 5/3 = 15$ vectors. The three-spacetime-index term can be dualized to become the kinetic term for 6 scalar axions a_α with no interaction term.

The $F_{\mu\alpha\beta\gamma}$ correspond to kinetic terms of the $6\times 5\times 4/3! = 20$ scalars $C^{(4)}_{\alpha\beta\gamma}$.

Let's do a similar thing for the 2-form field strength. There, we get $C_\alpha = \hat{C}_\alpha, C_\mu = \hat{C}_\mu - C_\alpha A^\alpha_\mu$. The corresponding field strength is $F_{\alpha\beta} = 0, F_{\mu\alpha} = \partial_\mu C_\alpha$ and $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu + C_\alpha F^\alpha_{\mu\nu}$. We then get contribution

$$S^{(2)}_{RR} = -\frac{1}{4}\int d^4\sqrt{-g}\sqrt{\det G_{\alpha\beta}}[F_{\mu\nu}F^{\mu\nu} + 2F_{\mu\alpha}F^{\mu\alpha}]$$

Again $F_{\mu\nu}$ can be written in terms of dual fields $\tilde{F}^{(2)}_{\mu\nu} = E_{\mu\nu\rho\sigma}F^{(2)\,\rho\sigma}$. This is one gauge fields and six further scalars.

Return and think about the effect of the CS terms. I bet they make the RR field equations non-free.

4. First note that using the OPE

$$\Sigma^I(z)\bar{\Sigma}^J(w) = \frac{\delta^{IJ}}{(z-w)^{3/4}} + (z-w)^{1/4} J^{IJ}(w)$$

the $\langle J^{II}\Sigma^J\bar{\Sigma}^J\rangle$ correlator can be evaluated as

$$\langle J^{II}(z_1)\Sigma^J(z_2)\bar{\Sigma}^J(z_3)\rangle = (\delta^{IJ} - \tfrac{1}{4})\frac{z_{23}^{1/4}}{z_{12}z_{13}}$$

Taking $z_1 \to z_2$ we see a singularity going as $\frac{(\delta^{IJ}-\frac{1}{4})}{z_{12}} z_{23}^{-3/4}$. Meanwhile taking the $J\Sigma$ OPE gives

$$q\frac{\langle \Sigma(z_2)\bar{\Sigma}(z_3)\rangle}{z_{12}} = \frac{q}{z_{12}} z_{23}^{-3/4}$$

So we see that under J^I the charge of Σ^J is $3/4$ if $I = J$ and $-1/4$ otherwise. We have 4 J^{II}, and notice that the total charge under all four of each Σ^I is always zero. Consider the following combination of charges, which provides a basis for the Σ^I charge space

$$\begin{aligned}
\tilde{J}^1 &= J^{11} + J^{22} - J^{33} - J^{44} \\
\tilde{J}^2 &= J^{11} - J^{22} + J^{33} - J^{44} \\
\tilde{J}^3 &= J^{11} - J^{22} - J^{33} + J^{44}
\end{aligned}$$

Under each of $\tilde{J}^i$ we have the following charges

$$\begin{array}{llll}
\Sigma^1 \to (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & \Sigma^2 \to (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), & \Sigma^3 \to (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), & \Sigma^4 \to (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) \\
\bar{\Sigma}^1 \to (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), & \bar{\Sigma}^2 \to (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & \bar{\Sigma}^3 \to (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & \bar{\Sigma}^4 \to (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})
\end{array}$$

These are exactly all combinations, and we can define the three bosonic fields ϕ_i with $T = \sum_i \frac{1}{2}(\partial\phi_i)^2$ so that

$$\Sigma^1 = \exp\left[i(\tfrac{1}{2}\phi_1 + \tfrac{1}{2}\phi_2 + \tfrac{1}{2}\phi_3)\right], \quad \Sigma^2 = \exp\left[i(\tfrac{1}{2}\phi_1 - \tfrac{1}{2}\phi_2 - \tfrac{1}{2}\phi_3)\right], \quad \text{etc.}$$

Each of these $\Sigma^I, \bar{\Sigma}^I$ has dimension $3/8$ as required.

Let's look at the supercurrent G^{int}. It can be written in terms of an eigenbasis of the commuting $\tilde{J}^i$. In particular look at $\tilde{J}^1$.

$$G^{int} = \sum_q e^{iq\phi_1} T^{(q)}$$

Now consider the OPEs $G^{int} \cdot \Sigma^1$ and $G^{int} \cdot \bar{\Sigma}^1$. As observed in the chapter, both of these have only the singular term going as $(z-w)^{-1/2}$. Together both of these require that q in G can only be ± 1. We can repeat this argument for $\tilde{J}^2, \tilde{J}^3$ to see that G^{int} must be a sum of 6 terms:

$$e^{iq_1\phi_1} Z_1 + e^{-iq_1\phi_1}\bar{Z}_1 + e^{iq_2\phi_2} Z_2 + e^{-iq_2\phi_2}\bar{Z}_2 + e^{iq_3\phi_3} Z_3 + e^{-iq_3\phi_3}\bar{Z}_3$$

Each $Z_i, \bar{Z}_i$ must be dimension one operators, so they are themselves bosonic fields $i\partial X^i_\pm$. We thus have that $G^{int} = \sum_{i=1,\pm}^3 \psi_i^\pm \partial X^i_\pm$. This is exactly the supercurrent for six free boson-fermion systems and will give (under anticommutator) the stress tensor of a six free boson-fermion systems. This is exactly a toroidal CFT.

5. The relevant partition function is not difficult to compute, as we can follow 9.4's example but not do the twist on the internal $(0,16)$ part. Firstly the fermions on the left-moving (SUSY) side have orbifold blocks under the shifts as before:

$$Z_\psi\begin{bmatrix} h \\ g \end{bmatrix} = \frac{1}{2}\sum_{a,b=0}^{1} (-1)^{a+b+ab} \frac{\theta^2\left[{a \atop b}\right]\theta\left[{a+h \atop b+g}\right]\theta\left[{a-h \atop b-g}\right]}{\eta^4}$$

Similarly we've already constructed the bosonic blocks before. They are given by **4.12.10** as:

$$Z_{4,4}\begin{bmatrix}0\\0\end{bmatrix} = \frac{\Gamma_{2,2}}{\eta^4\bar{\eta}^4}, \quad Z_{4,4}\begin{bmatrix}h\\g\end{bmatrix} = 2^4\frac{\eta^2\bar{\eta}^2}{\theta^2\left[\begin{smallmatrix}1-h\\1-g\end{smallmatrix}\right]\bar{\theta}^2\left[\begin{smallmatrix}1-h\\1-g\end{smallmatrix}\right]}$$

Then the $(2,2)$ part is untouched, yielding $\frac{\Gamma_{2,2}}{\eta^2\bar{\eta}^2}$ as is the $(0,16)$ part. We get the partition function

$$Z^{het} = \frac{1}{2}\sum_{h,g=0}^{1}\frac{\Gamma_{2,2}Z_{4,4}\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]}{\tau_2\eta^4\bar{\eta}^4} \times \frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\frac{\theta^2\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a+h\\b+g\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a-h\\b-g\end{smallmatrix}\right]}{\eta^4} \times \frac{\left(\frac{1}{2}\sum_{a,b=0}^{1}\bar{\theta}\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^8\right)^2}{\bar{\eta}^{16}}$$

Under $\tau \to -1/\tau$

Under $\tau \to \tau + 1$

6. Now the partition function is given by

$$Z^{het}_{N=2} = \frac{1}{2}\sum_{h,g=0}^{1}\frac{\Gamma_{2,2}Z_{4,4}\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]}{\tau_2\eta^4\bar{\eta}^4}\times\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\frac{\theta^2\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a+h\\b+g\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a-h\\b-g\end{smallmatrix}\right]}{\eta^4}\times\frac{1}{2}\sum_{\gamma,\delta=0}^{1}\frac{\bar{\theta}^6\left[\begin{smallmatrix}\gamma\\\delta\end{smallmatrix}\right]\bar{\theta}\left[\begin{smallmatrix}\gamma+h\\\delta+g\end{smallmatrix}\right]\bar{\theta}\left[\begin{smallmatrix}\gamma-h\\\delta-g\end{smallmatrix}\right]}{\bar{\eta}^8}\times\frac{\frac{1}{2}\sum_{a,b=0}^{1}\bar{\theta}\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^8}{\bar{\eta}^8}$$

Let's check under $\tau \to -1/\tau$

Next under $\tau \to \tau + 1$

7.