Chapter 7: Superstrings and Supersymmetry

1. We already know that TT will have the desired OPE, since the bosons and fermions are uncoupled and we already have shown their own respective stress tensor OPEs. Next

$$\begin{aligned}
G(z)G(w) &= -\frac{2}{\ell_s^4}\psi_\mu(z)\partial X^\mu(z)\psi_\nu(w)\partial X^\nu(w) \\
&= -\frac{2}{\ell_s^4}\left(\ell_s^2\frac{\eta_{\mu\nu}}{z-w} + (z-w) : \partial\psi_\mu\psi_\nu(w) :\right)\left(-\frac{\ell_s^2}{2}\frac{\eta_{\mu\nu}}{(z-w)^2} + : \partial X_\mu \partial X_\nu(w) :\right) \\
&= \frac{D}{(z-w)^3} + \frac{-\frac{2}{\ell_s^2}\partial X_\mu\partial X^\mu(w) - \frac{1}{\ell_s^2}\psi^\mu\partial\psi_\mu(w)}{z-w} \\
&= \frac{\hat{c}}{(z-w)^3} + \frac{2T(w)}{z-w}
\end{aligned}$$

Finally

$$\begin{aligned}
T(z)G(w) &= -\frac{1}{\ell_s^2}\left(: \partial X_\mu \partial X^\mu(z) : + \frac{1}{2}\psi^\mu\partial\psi_\mu(z)\right) i\frac{\sqrt{2}}{\ell_s^2}\psi_\nu\partial X^\nu(w) \\
&= -i\frac{\sqrt{2}}{\ell_s^4}\left(-\frac{\ell_s^2}{2}\frac{\psi_\mu\partial X^\mu(w) + \psi_\mu\partial^2 X^\mu(w)(z-w)}{(z-w)^2} - \frac{\ell_s^2}{2}\frac{\psi_\mu\partial X^\mu(w)}{(z-w)^2} + (-)\frac{\ell_s^2}{2}\frac{\partial_\mu\psi\partial X^\mu(w)}{(z-w)}\right) \\
&= \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}
\end{aligned}$$

2. We will take the OPE of $j_B(z)j_B(w)$, but just look at the $(z-w)^{-1}$ term as a function of w, as this, when integrated around the origin in w will give Q_B^2. This is an extension of exercise **4.45**, and there is nothing conceptually further, except for some $\beta\gamma$ manipulation. There are altogether 16 terms to consider, and we will get $c = 15$. The algebra is heavy, so I will skip this. An alternative is to do this as in **Polchinski 4.3**.

   To do it this way, note the following OPEs:

$$\begin{aligned}
j_B(z)b(w) &\sim \frac{T_{matter}(z)}{z-w} - \frac{1}{(z-w)^2}\left(bc(z) + \frac{3}{4}\beta\gamma(z)\right) + \frac{1}{z-w}\left(-b\partial c(z) + \frac{1}{4}\partial\beta\gamma(z) - \frac{3}{4}\beta\partial\gamma(z)\right) \\
&= \cdots + \frac{1}{z-w}\left[T_{matter}(z) - \partial b\, c(w) - 2b\partial c(w) - \frac{1}{2}\partial\beta\gamma(w) - \frac{3}{2}\beta\partial\gamma(w)\right] \\
&= \cdots + \frac{T_{matter}(w) + T_{gh}(w)}{z-w} \Rightarrow \{Q_B, b_n\} = L_n
\end{aligned}$$

   Similarly

$$j_B(z)\beta(w) = \cdots + \frac{G_{matter}(w) + G_{gh}(w)}{z-w} \Rightarrow [Q_B, \beta_n] = G_n$$

   Now note that the Jacobi identity on Q_B reads:

$$\{[Q_B, L_m], b_n\} - \{\overbrace{[L_m, b_n]}^{(m-n)b_{m+n}}, Q_B\} - [\overbrace{\{b_n, Q_B\}}^{L_n}, L_m] = 0 \Rightarrow \{[Q_B, L_m], b_n\} = (m-n)L_{m+n} - [L_m, L_n]$$

   So if the total central charge is zero we'll get $\{[Q_B, L_m], b_n\} = 0$, implying that $[Q_b, L_m]$ is independent of the c ghost. But on the other hand this operator has ghost number 1, so it must therefore vanish. Further, the Jacobi identity also yields

$$[\{Q_B, Q_B\}, b_n] = -2[\{b_n, Q_B\}, Q_B] = 2[Q_B, L_n]$$

   since we just showed that this last term vanishes, we must have Q_B, Q_B is also independent of c, but again since Q_B^2 has positive ghost number, we get that it is in fact zero. We can do the same argument with β and G and get that the superstring BRST operator is zero, as long as the total central charge vanishes. This was much cleaner than the OPE way.

3. First a lemma: An abelian p-form field A has $\binom{D-2}{p}$ on shell DOF. To prove this, note that we have a gauge symmetry of $A \to A + \partial\Lambda$ which has $\binom{D}{p-1}$ parameters. Next, the Euler-Lagrange equations give us that the components $A^{0i_1\ldots i_{p-1}}$ are non-propagating. We thus get $\binom{D-1}{p}$ massless propagating off-shell d.o.f. which have $\binom{D-2}{p-1}$ gauge symmetries left over. These can be used to enforce Coulomb gauge conditions which allow for there to be no polarizations along one of the spatial directions. We thus get $\binom{D-1}{p} - \binom{D-2}{p-1} = \binom{D-2}{p}$ massless on-shell degrees of freedom. For A_μ this is $D-2$ and for $B_{\mu\nu}$ this is $(D-2)(D-3)/2$.

   The metric has $\frac{1}{2}D(D-3)$ on-shell degrees of freedom. There are two ways to see this, first, that the dynamically allowed variation δg may on-shell be described by a symmetric traceless tensor in dimension $D-2$ which gives
$$\frac{(D-1)(D-2)}{2} - 1 = \frac{1}{2}D(D-3)$$
   or by noting that since we are gauging translation symmetry locally, each translation makes 2 polarizations unphysical and so we get:
$$\frac{D(D+1)}{2} - 2D = \frac{1}{2}D(D-3)$$
   as required.

   We now consider the R-R, R-NS, NS-R, NS-NS sectors together. For NS-NS we have the scalar $= 1$ both on-shell and off-shell, the antisymmetric two-form, which has only transverse degrees of freedom $= 8 * 7/2 = 28$ and the gravity, $= 10 * 7/2 = 35$ altogether we get 64 on-shell degrees of freedom.

   In both the R-NS and NS-R sector, we have a Weyl representation of dimension $2^{5-1} = 16$. There are however only 8 on-shell degrees of freedom. Similarly, we only consider the on-shell $\psi^\mu_{-1/2}$ acting on the NS part of the vacuum which gives another factor of 8. This gives 64 fermionic variables in each sector for a grand total of 128.

   In R-R for IIA we have a 0, 2, and *self-dual* 4-form. This gives:
$$1 + \binom{8}{2} + \frac{1}{2}\binom{8}{4} = 64$$
   For IIB we have a 1 and 3-form. This gives
$$\binom{8}{1} + \binom{8}{3} = 64$$
   so in either case we have 64 on-shell degrees of freedom here. This is consistent with each $|S\rangle$ state having 8 on-shell degrees of freedom giving $8 \times 8 = 64$. All together, we have the same number of on-shell fermionic and bosonic degrees of freedom.

   Now for the massive case. In the NS sector you might expect the next excitations come from the bosons α_{-1}, but this gets projected out by GSO, so in fact the next states come from $C_{ijk}\psi^i_{-1}\psi^j_{-1}\psi^k_{-1}$ and $C_{ij}\psi^i_{-1}\alpha^j_{-1}$. The physical state conditions will force them to transform as $\mathbf{8}^3$ and $\mathbf{8}^2$. In the R sector, it is quick to see that neither α_{-1} nor ψ_{-1} will satisfy $G_0 = 0$, so the massive state will come from the next level. This will then have mass higher than the NS sector, and so we can ignore it here.

   Consequently, the NS-NS sector will have a spin 6, spin 4, and two spin 5 massive particles. From R-NS and NS-R I can tensor the R vacuum $|S_\alpha\rangle$ or $|C_\alpha\rangle$ with the NS states and get two copies of $\mathbf{8}^4$ and $\mathbf{8}^3$. These will appropriately combine to give representations of $SO(9)$. **SHOW THIS PART**

   Finally, in the RR sector we get massive bosons of larger mass, which we thus disregard . I did not have to find the first massive state in the R sector to do this problem.

4. In terms of theta functions:

$$\chi_O = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_3(\nu_i)}{\eta} - \prod_{i=1}^{4}\frac{\theta_4(\nu_i)}{\eta}\right)$$
$$\chi_V = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_3(\nu_i)}{\eta} + \prod_{i=1}^{4}\frac{\theta_4(\nu_i)}{\eta}\right)$$
$$\chi_S = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_2(\nu_i)}{\eta} - \prod_{i=1}^{4}\frac{\theta_1(\nu_i)}{\eta}\right)$$
$$\chi_C = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_2(\nu_i)}{\eta} + \prod_{i=1}^{4}\frac{\theta_1(\nu_i)}{\eta}\right)$$

   We'll take $\nu_i = 0$ here **(I assume this is what I'm supposed to do)** and so $\theta_1 = 0 \Rightarrow \chi_S = \chi_C$.

   For IIB we look at

$$\frac{|\chi_V - \chi_C|^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8} = \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4} \times \frac{1}{2}\sum_{\bar{a},\bar{b}=0}^{1}(-1)^{\bar{a}+\bar{b}}\frac{\bar{\theta}^4\left[{\bar{a} \atop \bar{b}}\right]}{\bar{\eta}^4}$$

   Under modular transformations $\tau \to \tau + 1$ $\theta^4\left[{0 \atop 1}\right] \leftrightarrow \theta^4\left[{0 \atop 0}\right]$, $\theta^4\left[{1 \atop 0}\right] \to -\theta^4\left[{1 \atop 0}\right]$ while $\eta^{12} \to -\eta^{12}$. In the holomorphic and anti-holomorphic parts separately, each term in the sum picks up a minus sign that is cancelled by the minus sign in the η^4.

   Under $\tau \to -1/\tau$, the $\frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}$ out front is invariant. On the other hand, the θ functions transform as $\theta^4\left[{0 \atop 0}\right] \to (-i\tau)^2\theta^4\left[{0 \atop 0}\right]$, $\theta^4\left[{0 \atop 1}\right] \to (-i\tau)^2\theta^4\left[{1 \atop 0}\right]$, $\theta^4\left[{1 \atop 0}\right] \to (-i\tau)^2\theta^4\left[{0 \atop 1}\right]$. These are exactly compensated by the η transformations in the denominator, and no overall sign is picked up

   For IIA we have similarly

$$\frac{(\chi_V - \chi_C)(\bar{\chi}_V - \bar{\chi}_S)}{(\sqrt{\tau_2}\eta\bar{\eta})^8} = \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4} \times \frac{1}{2}\sum_{\bar{a},\bar{b}=0}^{1}(-1)^{\bar{a}+\bar{b}+\bar{a}\bar{b}}\frac{\bar{\theta}^4\left[{\bar{a} \atop \bar{b}}\right]}{\bar{\eta}^4}$$

   Again, the holomorphic part transforms as before and as we have set the ν_i to zero, we have the same partition function. Using **D.18**, we see that each of the four above sums are zero since they are equal to a product of $\theta_1 = 0$.

5. Again, these are identical if I set the $\nu_i = 0$ (am I not supposed to be doing this? What do the ν_i represent physically?). They are equal to

$$\frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8 4\eta^4\bar{\eta}^4}(\cancel{|\theta_1^4|^2} + |\theta_2^4|^2 + |\theta_3^4|^3 + |\theta_4^4|^2)$$

   We have θ_3 and θ_4 swapping under $\tau \to \tau + 1$, generating no signs in this case, while the denominator looks like $|\eta|^{24}$ and also doesn't generate a sign. Then, under $\tau \to -1/\tau$ we have θ_2 and θ_4 swapping generating a $|\tau|^4$, identical to what is generated by the $(\eta\bar{\eta})^4$.

6. We can write this partition function as:

$$\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{\sqrt{\tau}^8\eta^{12}\bar{\eta}^{24}}\sum_{h,g}\sum_{\gamma,\delta,\gamma',\delta'}(-1)^{(\gamma+\gamma')g+(\delta+\delta')h}\bar{\theta}^8\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\bar{\theta}^8\begin{bmatrix}\gamma'\\ \delta'\end{bmatrix}\sum_{a,b}(-1)^{a+b+ab+ag+bh+gh}\theta^4\begin{bmatrix}a\\ b\end{bmatrix}$$

   under $\tau \to \tau + 1$ we have $\theta^4\left[{0 \atop 1}\right] \leftrightarrow \theta^4\left[{0 \atop 0}\right]$, $\theta^4\left[{1 \atop 0}\right] \to -\theta^4\left[{1 \atop 0}\right]$ while $\eta^{12} \to -\eta^{12}, \bar{\eta}^{24} \to \bar{\eta}^{24}$. And swapping $\theta^4\left[{0 \atop 0}\right]$ and $\theta^4\left[{0 \atop 1}\right]$ as well as $\bar{\theta}^4\left[{0 \atop 0}\right]\bar{\theta}^4\left[{0 \atop 1}\right]$ will give us $(-1)^{1+h}$.

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