1 Chapter 8: D-Branes

1. First, a simple magnetic monopole for a 1-form gauge field in D spacetime dimensions has a radial magnetic field that $B_r = \frac{\tilde{Q}_1}{\Omega_{D-2} r^{D-2}}$ where $\Omega_{D-2} = 2\pi^{d/2}\Gamma(d/2)$ is the volume of a unit $D-2$ sphere. This way, the flux of the solution over any $D-2$ sphere surrounding the (point) monopole will be $\tilde{Q}_1$.

   Upon taking the Hodge star we get the solution is $F = \tilde{Q}_1 \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$. We can write this as $A = \tilde{Q}_1(c - \cos\theta)\mathrm{d}\phi$. Taking $c = 1$ we get A vanishes at $\theta = 0$ (which we need since the ϕ coordinate degnerates there) while taking $c = -1$ we get A vanishes at $\theta = \pi$, which we also need.

   We cannot have *both* solutions, and so we realize we are dealing with two As, corresponding to local sections of a line bundle over S^2 on different hemispheres. Let A^+ be well-defined on all points on S^2 except $\theta = \pi$. Then A^+ is a section of a line bundle on the punctured sphere. The punctured sphere is contractible so any fiber bundle over it is trivial, so A^+ is just a *function* on the punctured sphere $S^2\backslash\{\theta = \pi\}$. So let's define $A^+ = \tilde{Q}_1(1 - \cos\theta)\mathrm{d}\phi$. Similarly, we define A^- to be the nonsingular A on the sphere with $\theta = 0$ removed, namely $A^- = \tilde{Q}_1(-1 - \cos\theta)\mathrm{d}\phi$.

   On the overlap, $A^+ - A^-$ differ by an integer, which labels the degree of "twisting" of this line bundle over S^2.

   For a p form, our monopole will now be spatially extended in $p-1$ directions. Label these (locally), by $x^1 \dots x^{p-1}$. Time is x^0. Locally transverse to these coordinates will be $r, \varphi^1 \dots, \varphi^{D-1-p}$, where φ^i parameterize a $D-1-p$ sphere enclosing the monopole. The field strength looks like:

$$F = \tilde{Q}_p \, \Omega_{D-p-1}$$

   where Ω is the canonical $D-p-1$-sphere area form:

$$\Omega = \sin^{D-p-2}(\varphi_1)\sin^{D-p-3}(\varphi_2)\dots\sin(\varphi_{D-p-2})\,\mathrm{d}\varphi_1 \wedge \dots \mathrm{d}\varphi_{D-p-1}$$

   This can be written (unfortunately unavoidably) in terms of a hypergeometric function:

$$A = {}_2F_1\left(\frac{1}{2}, \frac{D-p-1}{2}, \frac{D-p+1}{2}, \sin^2(\varphi_1)\right)\frac{\sin^{D-p-1}(\varphi_1)}{D-p-1}\mathrm{d}\varphi_2 \wedge \dots \wedge \mathrm{d}\varphi_{D-p-1}$$

   there is no need for an overall constant, as the function above vanishes at both $\varphi_1 = 0$ and π, *however* this is compensated by the hypergeometric function having a branch cut at $\varphi_1 = \pi/2$. Across this cut, it will have a discontinuity set by an integer depending on the convention of the arcsin function, and again we will have $A^+ - A^-$ differing by an integer. The same quantization condition follows.

   Again A^+ will be defined on the S^{D-p-1} sphere minus the south-pole (this is homeomorphic to the $D-p-1$ ball, and hence contractible, so again the line bundle trivializes and A^+ is a bona-fide function for any D, p) and A^- is similarly defined on the sphere with the excision of the north pole.

2. Our simply charged point particle with a Wilson line $A_9 = \chi/2\pi R$ turned on will have an action

$$S = \int d\tau \left(\underbrace{\frac{1}{2}\dot{x}^M\dot{x}_M - \frac{m^2}{2} + qA_9\dot{x}^9}_{\mathcal{L}}\right)$$

   The canonical momentum will be $p_i = \dot{x}^\mu$ for $\mu = 0\dots 8$ and $p_9 = \dot{x}^9 + \frac{q\chi}{2\pi R}$. Consequently, our hamiltonian is

$$\begin{aligned}
H = p_M\dot{x}^M - \mathcal{L} &= \frac{1}{2}p_\mu p^\mu + p_9(p_9 - \frac{q\chi}{2\pi R}) - \left[\frac{1}{2}(p_9 - \frac{q\chi}{2\pi R})^2 - \frac{m^2}{2} + (p_9 - \frac{q\chi}{2\pi R})\frac{q\chi}{2\pi R}\right] \\
&= \frac{1}{2}\left(p_\mu p^\mu + (p_9 - \frac{q\chi}{2\pi R})^2 + m^2\right) \\
&= \frac{1}{2}\left(p_\mu p^\mu + \left(\frac{2\pi n - q\chi}{2\pi R}\right)^2 + m^2\right)
\end{aligned}$$

3. For a string satisfying Dirichlet boundary conditions, the total momentum is not conserved (along the directions associated with the D boundary conditions). This is easily interpretable as momentum transfer to the brane that it is attached to.

4. For an open string of state $|ij\rangle$, A_9 will act as $\frac{\chi_i - \chi_j}{2\pi i}$. Since this is an open string with no winding, we can only have momentum contribution, and so we will get a mass formula

$$m_{ij}^2 = \frac{\hat{N} - \frac{1}{2}}{\ell_s^2} + \left(\frac{n}{R} - \frac{\chi_i - \chi_j}{2\pi R}\right)^2$$

In particular at the lowest (massless) level for a string without momentum we will get the desired spectrum

$$m_{ij}^2 = \left(\frac{\chi_i - \chi_j}{2\pi R}\right)^2$$

5. We will do this for the superstring. The massless vector and scalar modes come from the NS sector, and are associated with the gauge boson vertex operator

$$V_{-1}^{a,\mu} = g_o \lambda^a \psi^\mu e^{-\phi} c e^{ipX}, \qquad V_0^{a,\mu} = \frac{g_o}{\sqrt{2}\ell_s} \lambda^a (i\dot{X} + 2p \cdot \psi\, \psi^\mu) c e^{ipX}$$

We can explicitly scatter four such gauge bosons - two in the -1 picture and two in the 0 picture.

Recall the four-gaugini amplitude from problem **23** of the last chapter:

$$\begin{aligned} &- 8ig_{open}^2 \ell_s^2 \delta\!\!\!/^{10}(\Sigma k) K(u_1, u_2, u_3, u_4) \left(\frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 u)}{\Gamma(1 - \ell_s^2 s - \ell_s^2 u)}[1234] + 2\,\text{perms.}\right) \\ &K(u_1, u_2, u_3, u_4) = \frac{1}{8}(u\, \bar{u}_1 \Gamma^\mu u_2 \bar{u}_3 \Gamma_\mu u_4 - s\, \bar{u}_1 \Gamma^\mu u_4 \bar{u}_3 \Gamma_\mu u_2) \end{aligned}$$

For now I will just quote **Polchinski 12.4.25**. We can get the four-gauge boson amplitude simply by replacing K with

$$K(k_1, k_2, k_3, k_4, e_1, e_2, e_3, e_4) = \frac{1}{8}\Big(4\,\text{tr}(M^1 M^2 M^3 M^4) - \text{tr}(M_1 M_2)\text{tr}(M_3 M_4)\Big) + (1234 \to 1342) + (1234 \to 1423)$$

with $M_{\mu\nu}^i = k_\mu^i e_\nu^i - k_\nu^i e_\mu^i$. Now, the $9-p$ transverse momenta vanish. This gives a V_{9-p} from the $\delta\!\!\!/$ functions. Altogether we get an amplitude:

$$-8ig_{open}^2 \ell_s^2 V_{9-p} \delta\!\!\!/^{p+1}(\Sigma k) K(k_i, e_i) \left(\frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 u)}{\Gamma(1 - \ell_s^2 s - \ell_s^2 u)}[1234] + 2\,\text{perms.}\right)$$

6. I have done the previous problem in full generality, including CP indices.

7.