Solutions to Kiritsis' *String Theory in a Nutshell*

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Chapter 2: Classical String Theory

1. I don't know what this question asks exactly given that 2.1.16 is an infinitesimal diffeomorphism.

   We are still allowed to assume WLOG that τ runs from 0 to 1. For ξ infinitesimal, we have $\delta e = \xi \dot{e} + \dot{\xi} e = \partial_\tau(\xi e)$. So for a general $e(\tau)$ define

$$\tau_2(\tau) = \frac{\int_0^\tau d\tau' e(\tau')}{\int_0^1 d\tau' e(\tau')} \tag{1}$$

   Take $L = \int_0^1 d\tau' e(\tau')$. Then $e_2(\tau_2(\tau)) = \left(\frac{d\tau_2}{d\tau}\right)^{-1} e(\tau) = \left(\frac{e(\tau)}{L}\right)^{-1} e(\tau) = L$.

   Note that we cannot get rid of this L, since it is invariant $L = \int_0^1 d\tau e(\tau) = \int_0^1 d\tau_2 e(\tau_2)$

2. From analytic continuation, we have the functional equation for the Riemann zeta function:

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s)\zeta(1-s) \tag{2}$$

   It is worth knowing that near $s = 0$ we have $\zeta(1-s) = -\frac{1}{s} + \gamma$ and $\Gamma(1-s) = 1 + \gamma s$. Expanding the right hand side about $s = 0$ gives

$$\zeta(\epsilon) = -\frac{1}{2} - \frac{1}{2}\sqrt{2\pi}\epsilon \tag{3}$$

   This gives $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\sqrt{2\pi}$. Further, $\zeta'(s) = -\sum_{n=1}^\infty \frac{\log n}{n^s}$.

   So we get $\prod_{n=1}^\infty \frac{1}{L^2} = L^{-2\sum_{n=1}^\infty 1} = L^{-2\zeta(0)} = L$ and $\prod_{n=1}^\infty n^2 = \exp\left(2\sum_{n=1}^\infty \log n\right) = 2\pi$.

3. For simplicity, we will work in the action with the einbein.

$$\frac{1}{2}\int d\tau e (e^{-2} G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - m^2)$$

   The Euler-Lagrange equations for x^μ is:

$$\begin{aligned} \frac{d}{d\tau}\frac{\partial}{\partial x^\mu}(e^{-1}G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) - \frac{\partial}{\partial x^\mu}\left(e^{-1}G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right) &= 2e^{-1}G_{\mu\nu}\ddot{x}^\nu + 2e^{-1}\partial_\gamma G_{\mu\nu}\dot{x}^\nu\dot{x}^\gamma - 2\frac{G_{\mu\nu}\dot{x}^\nu}{e^2}\dot{e} - e^{-1}\partial_\mu G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \\ &\to G_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}(\partial_\gamma G_{\mu\nu} + \partial_\nu G_{\mu\gamma} - \partial_\mu G_{\nu\gamma})\dot{x}^\nu\dot{x}^\gamma - \frac{1}{2}G_{\mu\nu}\dot{x}^\nu\partial_\tau \log e^2 \end{aligned} \tag{4}$$

   This last term looks particularly annoying, and is ignored by other authors. We have total freedom in reparameterization of e, so we can WLOG set it equal to a (metric-dependent) constant by problem 1. Then the term drops out and we get exactly the geodesic equations.

   We could have done this explicitly as well:

$$\begin{aligned} \frac{d}{d\tau}\frac{G_{\mu\nu}\dot{x}^\nu}{\sqrt{G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}} - \frac{\partial}{\partial x^\mu}\sqrt{G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} &= \frac{G_{\mu\nu}\ddot{x}^\nu + \partial_\lambda G_{\mu\nu}\dot{x}^\nu\dot{x}^\lambda - \frac{1}{2}\partial_\mu G_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda}{\sqrt{-G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}} + G_{\mu\nu}\dot{x}^\nu\frac{d}{d\tau}(-G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{-1/2} \\ &= G_{\mu\nu}(\ddot{x}^\nu + \Gamma^\nu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) - \frac{1}{2}G_{\mu\nu}\dot{x}^\nu\frac{d}{d\tau}\log(-G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) \end{aligned} \tag{5}$$

   And fix the parameterization so that $x^2 = \text{const}$ and the last term vanishes.

4. We get the same as before, but now cannot drop the last term. Now the dots represent time derivatives.

$$G_{\mu\nu}(\ddot{x}^\nu + \Gamma^\nu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) - \frac{1}{2}G_{\mu\nu}\dot{x}^\nu\partial_{X^0}\log(-G_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) \tag{6}$$

5. We get:

$$- mc \int d\tau \sqrt{-(G_{00}\dot{x}^0\dot{x}^0 + 2G_{0i}\dot{x}^0\dot{x}^i + G_{ij}\dot{x}^i\dot{x}^j)} \tag{7}$$

Taking $\tau = x^0 = ct$ gives us our result. Further, we write $G_{00} = -1 - \frac{2\phi}{c^2}$ where ϕ is the gravitational potential. To first order then we get:

$$- mc^2 \int dt \sqrt{-(G_{00} + 2c^{-1}G_{0i}\dot{x}^i + c^{-2}G_{ij}\dot{x}^i\dot{x}^j)} \approx \int dt \left(-mc^2 - m\phi + mcG_{0i}v^i + mG_{ij}v^iv^j\right) \tag{8}$$

The last two terms in brackets are positive (kinetic) while the first two are negative (potential). **This explains why there is a - sign out front of the action**.

6. The Lagrangian for a special relativistic particle in an electromagnetic field is $-mc^2\sqrt{1 - v^2/c^2} - e\phi + e\vec{v} \cdot \mathbf{A}$. This has the Lorentz invariant form: $-m\sqrt{-G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + eA_\mu\dot{x}^\mu$. We get equations of motion as before: The additional term gives the equations of motion:

$$\frac{e}{m}(\dot{A}_\mu - \partial_\mu A_\nu \dot{x}^\nu) = \frac{e}{m}(\partial_\nu A_\mu - \partial_\mu A_\nu)\dot{x}^\nu = \frac{e}{m}F_{\mu\nu}\dot{x}^\nu \tag{9}$$

don't confuse e with the einbein.

If one coordinate is cyclic (neither the metric nor the vector potential depend on it), the corresponding momentum is

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = m\frac{G_{\mu\nu}\dot{x}^\nu}{\sqrt{-G_\mu\nu\dot{x}^\mu\dot{x}^\nu}} + eA_\mu \tag{10}$$

7. Ignoring the cosmological constant term (which is not reparameterization invariant), we note that any term that involves the metric $G_{\mu\nu}$ will require at least 2 x^μ variables for it to be contracted with. Also, reparameterization invariance requires that under $d\tau \to f'(\tau)d\tau$ we get $\mathcal{L} \to \mathcal{L}/\lambda$. The simplest such term is $\sqrt{-G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$. Terms with more than 2 x^μs will be suppressed by powers of $1/\ell_s^2$. Similarly, terms with more derivatives w.r.t. worldsheet coordinates will be less relevant in the IR.

8. Let's set $G_{i0} = 0$ for simplicity. The Nambu-Goto action is:

$$-T \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X}^2)(X'^2)}$$

Take $\tau = ct$, $\sigma = x$ and note $T = \rho c^2$ with ρ the mass per unit length. Take $X^0 = ct$ and $\vec{u} = X'^i, \vec{v} = \dot{X}^i$. Appreciate that v gives us how that point of the string is moving, while u gives the direction parallel to the string at that point (scaled according to σ's parameterization). Inside the radical:

$$\begin{aligned}(G_{00}\dot{X}^0X'^0 + G_{ij}\dot{X}^iX'^j)^2 - (G_{00}\dot{X}^0\dot{X}^0 + G_{ij}\dot{X}^i\dot{X}^j)(G_{00}X'^0X'^0 + G_{ij}X'^iX'^j)\\ = c^{-2}(G_{ij}u^iv^j)^2 - (G_{00} + c^{-2}G_{ij}v^iv^j)(G_{ij}u^iu^j)\end{aligned}$$

Take $G_{00} = -1 - 2\varphi/c^2$. Then the radical becomes:

$$\sqrt{u^2 - c^{-2}2\phi u^2 + c^{-2}(\vec{u}\cdot\vec{v})^2 - c^{-2}v^2u^2} = |u|\sqrt{1 - c^{-2}(2\phi + v^2 - \frac{(\vec{u}\cdot\vec{v})^2}{u^2})} = |u|\left(1 - c^{-2}\left(-\phi + \frac{1}{2}v^2 - \frac{1}{2}\frac{(\vec{u}\cdot\vec{v})^2}{u^2}\right)\right)$$

But note that $v^2 - \frac{(\vec{u}\cdot\vec{v})^2}{u^2} = (\vec{v} - \frac{u\cdot v}{u^2}\vec{u})^2$. This is exactly the part of v transverse to u (the string itself). So we can write this as $\vec{v}_T$, the transverse velocity.

$$- T \int dtd\sigma|u|(1 + c^{-2}\phi - c^{-2}\frac{1}{2}v_T^2) = \int dtd\sigma|u|(-c^2 - \phi + \frac{1}{2}v_T^2) \tag{11}$$

Note that $\rho \int d\sigma|u| = \rho L_s = M_s$. The first term is thus $-M_sc^2$. The second terms is exactly the mass density of the string interacting with the gravitational field, while the third (kinetic) is the motion of the transverse components of the string. Note that the longitudinal excitations do not contribute.

9. Let's work in lightcone gauge. We have $\partial_+\partial_- X = 0$ The vanishing of the stress-energy tensor gives us $\dot{X}^2 + X'^2 = 0$ and $\dot{X} \cdot X'^2 = 0$. But at the endpoints we get $X' = 0$ so that $\dot{X}^2 = 0$ and the endpoints with Neumann boundary conditions need to move at the speed of light.

10. The cosmological constant term gives the equation of motion $\frac{\delta S}{\delta g^{ab}} = -(\frac{T_{ab}}{4\pi} + \frac{\lambda_1}{2} g_{ab})\sqrt{-g}$. But by reparameterization invariance we *need* $T_{ab} = 0$ so that λ_1 must be 0.

11. It is quick to derive the current P^μ under $\delta X^\nu = \epsilon \delta^{\mu\nu}$:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} = -T\sqrt{-g} g^{\alpha\beta}\partial_\beta X_\mu \tag{12}$$

Similarly under $\delta X^\lambda = \epsilon M^{\lambda\delta}_{\mu\nu} X_\delta$ with $M^{\lambda\delta}_{\mu\nu} = (\delta^\lambda_\mu \delta^\delta_\nu - \delta^\delta_\mu \delta^\lambda_\nu)$. Then we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\lambda)}(\delta^\lambda_\mu \delta^\delta_\nu - \delta^\delta_\mu \delta^\lambda_\nu) X_\delta = -T\sqrt{-g} g^{\alpha\beta}(X_\mu \partial_\beta X_\nu - X_\nu \partial_\beta X_\mu) \tag{13}$$

12. Write:

$$\begin{aligned} X^\mu(\tau,\sigma) &= x^\mu + \ell_s^2 p^\mu \tau + \frac{i\ell_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}-\{0\}} \frac{1}{n}(\alpha_n e^{-in\sigma} + \bar{\alpha}_n e^{in\sigma})e^{-in\tau} \\ \dot{X}^\mu(\tau,\sigma) &= \ell_s^2 p^\mu + \frac{\ell_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}-\{0\}} (\alpha_n e^{-in\sigma} + \bar{\alpha}_n e^{in\sigma})e^{-in\tau} \end{aligned} \tag{14}$$

Now take the Fourier series (in σ) of the commutation relation:

$$\{X^\mu_n, \dot{X}^\mu_m\} = \frac{\delta_{n+m}}{2\pi}\frac{1}{T}\eta^{\mu\nu} \tag{15}$$

The only nonzero terms are those we get when we pair each mode with its negative (in σ). Also note that there is no τ dependence on the right-hand side, so we need to pair each τ mode with its negative. Let's look at x^μ, the zero mode of X. We can only pair this with the other mode p^μ and we necessarily have:

$$\{x^\mu, p^\nu\} = \frac{1}{2\pi\ell_s^2 T}\eta^{\mu\nu} = \eta^{\mu\nu} \tag{16}$$

Similarly, we can only pair α_n with α_{-n} giving:

$$\{\alpha^\mu_m, \alpha^\nu_n\} + \{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_m\} = \frac{2m\delta_{m+n}}{2\pi i \ell_s^2 T}\eta^{\mu\nu} \tag{17}$$

By parity symmetry, both of these brackets should be the same. We get then that:

$$\{\alpha^\mu_m, \alpha^\nu_n\} = \{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_m\} = -i\delta_{m+n}\eta^{\mu\nu} \tag{18}$$

13. For each coordinate on the n-torus, we have $X^i(\tau, \sigma + 2\pi) = X(\tau,\sigma) + 2\pi n_i R_i$. Then the corresponding momenta have difference $p - \bar{p} = \frac{2}{\ell_s^2} n_i R_i$ while the total momentum is quantized in multiples of $p + \bar{p} = \frac{2m_i}{R_i}$. Therefore we have:

$$\alpha^i_0 = \frac{1}{\sqrt{2}}\left(m_i \frac{\ell_s}{R_i} + n_i \frac{R_i}{\ell_s}\right) \tag{19}$$

14. We begin with a redefined $p^\mu \to 2p^\mu$ as in the book.

$$\begin{aligned} X'^\mu(\tau,\sigma)|_{\sigma=0} &= \ell_s^2(p^\mu - \bar{p}^\mu) + \frac{\ell_s}{\sqrt{2}}\sum_n (\alpha_n - \bar{\alpha}_n)e^{-in\tau} \\ \dot{X}^\mu(\tau,\sigma)|_{\sigma=0} &= \ell_s^2(p^\mu + \bar{p}^\mu) + \frac{\ell_s}{\sqrt{2}}\sum_n (\alpha_n + \bar{\alpha}_n)e^{-in\tau} \end{aligned} \tag{20}$$

So then

$$X' + \lambda \dot{X} = \ell_s^2\left((\lambda+1)p^\mu + (\lambda-1)\bar{p}^\mu\right) + \frac{\ell_s}{\sqrt{2}}\sum_n e^{-in\tau}\left((\lambda+1)\alpha_n + (\lambda-1)\bar{\alpha}_n\right) = 0$$

This gives $p^\mu = \frac{1-\lambda}{1+\lambda}\bar{p}^\mu$ and similarly $\alpha^\mu = \frac{1-\lambda}{1+\lambda}\bar{\alpha}_n^\mu$.

Further:

$$\begin{aligned} X'^\mu(\tau,\sigma)|_{\sigma=\pi} &= \ell_s^2(p^\mu - \bar{p}^\mu) + \frac{\ell_s}{\sqrt{2}}\sum_n(\alpha_n^\mu e^{-i\pi n} - \bar{\alpha}_n^\mu e^{i\pi n})e^{-in\tau} \to \sum_n \alpha_n^\mu(e^{-i\pi n} - \frac{1+\lambda}{1-\lambda}e^{i\pi n})e^{-in\tau} \\ \dot{X}^\mu(\tau,\sigma)|_{\sigma=\pi} &= \ell_s^2(p^\mu + \bar{p}^\mu) + \frac{\ell_s}{\sqrt{2}}\sum_n(\alpha_n^\mu e^{-i\pi n} + \bar{\alpha}_n^\mu e^{i\pi n})e^{-in\tau} \to \sum_n \alpha_n^\mu(e^{-i\pi n} + \frac{1+\lambda}{1-\lambda}e^{i\pi n})e^{-in\tau} \end{aligned} \tag{21}$$

This gives:

$$(1+\lambda)e^{-i\pi n} - (1-\lambda)\frac{1+\lambda}{1-\lambda}e^{i\pi n} = 0 \Rightarrow \sin(\pi n) = 0 \Rightarrow n \in \mathbb{Z}. \tag{22}$$

The full equation is then

$$X^\mu = x^\mu + \frac{2\ell_s^2 p^\mu}{1-\lambda} + \frac{i\sqrt{2}\ell_s}{(1-\lambda)}\sum_{n\in\mathbb{Z}-\{0\}}\frac{\alpha_n^\mu}{n}e^{-in\tau}(\cos(\sigma n) + i\lambda\sin(\sigma n)) \tag{23}$$

Clearly as $\lambda \to 0$ we recover Neumann boundary conditions. On the other hand as $\lambda \to \infty$ we see that the endpoint of the string is constrained to be unable to move and we indeed recover Dirichlet.

15. Looking at the DD solution:

$$X'^\mu(\tau,\sigma) = w^\mu + \sqrt{2}\ell_s\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in\tau}\cos(n\sigma) \tag{24}$$

At the endpoints the momentum flow is

$$w^\mu \pm \sqrt{2}\ell_s\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in\tau} \tag{25}$$

16. In conformal gauge we have $\mathcal{L} = 2T\,\partial_+ X^\mu \partial_- X_\mu = \frac{T}{2}(\partial_\tau + \partial_\sigma)X^\mu(\partial_\tau - \partial_\sigma)X^\mu = \frac{T}{2}((\dot{X})^2 - (X')^2)$ so that $\Pi = T(\dot{X})$ and $\int d\sigma\,\Pi\dot{X} - \mathcal{L} = \frac{T}{2}\int d\sigma\,((\dot{X})^2 + (X')^2)\cdot\dot{X})$ as we needed.

For the closed string:

$$\dot{X} = \frac{\ell_s^2(p_\mu + \bar{p}_\mu)}{2} + \frac{\ell_s}{\sqrt{2}}\sum_{n\neq 0}(\alpha_n e^{-in\sigma} + \bar{\alpha}_n e^{in\sigma})e^{-in\tau} \qquad X' = \frac{\ell_s^2(p_\mu - \bar{p}_\mu)}{2} + \frac{\ell_s}{\sqrt{2}}\sum_{n\neq 0}(\alpha_n e^{-in\sigma} - \bar{\alpha}_n e^{in\sigma})e^{-in\tau}$$

Assuming no winding, we have $p = \bar{p}$. In the hamiltonian, the only contributions that will not vanish is when each $e^{in\sigma}$ is paired with $e^{-in\sigma}$. So we can look at this mode-by-mode. Between the two of these, the cross terms involving $\alpha_n\bar{\alpha}_n e^{-2in\tau}$ will cancel. We will get:

$$\frac{T}{2}\times\frac{\ell_s^2}{2}\times 2\pi\times\sum_{n\neq 0}(\alpha_n\alpha_{-n} + \bar{\alpha}_n\bar{\alpha}_{-n})\times 2 = \frac{1}{2}\sum_{n\neq 0}(\alpha_{-n}\alpha_n + \bar{\alpha}_{-n}\bar{\alpha}_n) = \sum_{n=1}^{\infty}(\alpha_{-n}\alpha_n + \bar{\alpha}_{-n}\bar{\alpha}_n)$$

The zero mode will contribute $\ell_s^4 p^2 \times 2\pi \times T/2 = \frac{1}{2}\ell_s^2 p^2$ as required.

For NN we again have $p = \bar{p}$

$$\dot{X} = 2\ell_s^2 p^\mu + \sqrt{2}\ell_s\sum_{n\neq 0}\alpha_n\cos(n\sigma)e^{-in\tau} \qquad X' = -i\sqrt{2}\ell_s\sum_{n\neq 0}\alpha_n\sin(n\sigma)e^{-in\tau}$$

The zero mode gives $4\ell_s^4 p^2 \times \pi$ After squaring this, we can only pair $\cos(n\sigma)$ either with itself or $\cos(-n\sigma)$. Pairing it with itself will give $\alpha_n^2\cos^2(n\sigma)e^{-in\tau}$ which will be cancelled by the $-\alpha_n^2\sin^2(n\sigma)e^{-in\tau}$ obtained from multiplying $\sin(n\sigma)$ with itself. On the other hand, pairing $\cos(n\sigma)$ and $\sin(n\sigma)$ with their negative frequency counterparts and integrating gives two factors of $\pi\alpha_n\alpha_{-n}$ so that in total we get:

$$\ell_s^2 p^2 + \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}\alpha_n = \ell_s^2 p^2 + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n \tag{26}$$

The exact same logic applies for DD except now only the difference term contributes. Instead of $2\ell_s^2 p^\mu$ we have $w^\mu = (y-x)^\mu/\pi$ which must thus give zero mode $(x-y)^2/(2\pi\ell_s)^2$.

Lastly, for DN we have no zero-modes at all, only:

$$X^\mu(\sigma,\tau) = x^\mu - \sqrt{2}\ell_s \sum_{k\in\mathbb{Z}+\frac{1}{2}} \frac{\alpha_k^\mu}{k} e^{-ik\tau}\sin(k\sigma) \tag{27}$$

$$\Rightarrow \dot{X}^\mu = i\sqrt{2}\ell_s \sum_k \alpha_k^\mu e^{-ik\tau}\sin(k\sigma), \quad X'^\mu = -\sqrt{2}\ell_s\sum_k \alpha_k^\mu e^{-ik\tau}\cos(k\sigma)$$

By the same reason as in DD and DN, the only terms that don't cancel is when we pair each $\sin(k\sigma)$ with its negative and similarly for cos. We get

$$\frac{T}{2}\ell_s^2 \times \pi \times 2 \times \sum_{n\in\mathbb{Z}+\frac{1}{2}} \alpha_{-n}\alpha_n = \frac{1}{2}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha_{-n}\alpha_n = \sum_{n=1}^{\infty}\alpha_{-n+\frac{1}{2}}\alpha_{n-\frac{1}{2}} \tag{28}$$

17. Immediately we have $\{L_m, \bar{L}_n\} = 0$. For $\{L_m, L_n\}$ we have:

$$\begin{aligned}
\{L_m, L_n\} &= \frac{1}{4}\sum_{k,l}\{\alpha_{m-k}\alpha_k, \alpha_{n-l}\alpha_l\} \\
&= \frac{1}{4}\sum_{k,l}\alpha_{n-l}\alpha_{m-k}\{\alpha_k,\alpha_l\} + \alpha_{m-k}\{\alpha_k,\alpha_{n-l}\}\alpha_l + \alpha_{n-l}\{\alpha_{m-k},\alpha_l\}\alpha_k + \{\alpha_{m-k},\alpha_{n-l}\}\alpha_k\alpha_l \\
&= -\frac{i}{4}\sum_{k,l}\alpha_{n-l}\alpha_{m-k}\,k\delta_{k+l} + \alpha_{m-k}\alpha_l\,k\delta_{k+n-l} + \alpha_{n-l}\alpha_k\,(m-k)\delta_{m-k+l} + \alpha_k\alpha_l(m-k)\delta_{m-k+n-l} \\
&= -\frac{i}{4}\sum_k \alpha_{n+k}\alpha_{m-k}k + \alpha_{m-k}\alpha_{n+k}k + \underbrace{\alpha_{n+m-k}\alpha_k\,(m-k) + \alpha_k\alpha_{n+m-k}(m-k)}_{k\to k+n} \\
&= -\frac{i}{2}\sum_k \alpha_{m-k}\alpha_{n+k}k + \alpha_{m-k}\alpha_{n+k}(m-n-k) \\
&= -i\frac{1}{2}\sum_k \alpha_{m-k}\alpha_{n+k}(m-n) \to -i(m-n)\frac{1}{2}\sum_{k'}\alpha_{m+n-k}\alpha_k = -i(m-n)L_{m+n}
\end{aligned}$$

The exact same logic applies to the conjugate charges.

Chapter 3: Quantization of Bosonic Strings

1. For simplicity we will ignore the μ index in our calculation first.

   First consider $[L_m, L_n]$ with $m+n \neq 0$. Then expanding in terms of commutators: This is the same as before, but now we must be careful about commutation:

$$[L_m, L_n] = \frac{1}{4}\sum_{k,l}[:\alpha_{m-k}\alpha_k:, :\alpha_{n-l}\alpha_l:]$$

   Note that the indices $m-k, k, n-l, l$ sum to $n+m$, if any pairwise sum of them is equal to zero (necessary for a nonvanishing commutator), then the other two will have sum equal to $n+m$. Then as long as $m+n \neq 0$ α_p, there will be no normal-ordering ambiguity and we will recover the standard commutation relations as before.

   So the remaining case to consider is $n = -m$. Take m positive WLOG. The logic of the question from last chapter applies, but now we must be careful about the ordering of the α_i outside of the commutator.

$$\begin{aligned}[L_m, L_{-m}] &= \frac{1}{4}\sum_{k,l}[\alpha_{m-k}\alpha_k, \alpha_{-m-l}\alpha_l] \\ &= \frac{1}{4}\sum_k \alpha_{m-k}\alpha_l k\delta_{k-m-l} + \alpha_{-m-l}\alpha_k(m-k)\delta_{m-k+l} + \alpha_{-m-l}\alpha_{m-k}k\delta_{k+l} + \alpha_k\alpha_l(m-k)\delta_{k+l} \qquad (29) \\ &= \frac{1}{4}\sum_k \alpha_{m-k}\alpha_{k-m}k + \alpha_{-k}\alpha_k(m-k) + \alpha_{-m+k}\alpha_{m-k}k + \alpha_k\alpha_{-k}(m-k)\end{aligned}$$

   We can split this into $k \geqslant 1$ and $k \leqslant 1$. The $k=0$ term is already in normal order. When $k \geqslant 1$, the first, third, and fourth terms of the sum are out of normal order. The first term has only m terms out of normal order. Rearranging these gives the constant:

$$\frac{1}{4}\sum_{k=1}^{m} k(k-m) = \frac{1}{4}\frac{m(m^2-1)}{6}$$

   The fourth term has all terms out of normal order and gives the formally infinite sum

$$\sum_{k=1}^{\infty} k(m-k)$$

   The last term has all but the first m terms out of normal order, and so contributes the sum

$$\sum_{k=m+1}^{\infty}(-m+k)k = -\sum_{k=1}^{\infty}(m-k)k + \sum_{k=1}^{m}(k-m)k$$

   The first part of this exactly cancels with the third term's infinite contribution. The last part of this gives exactly the same contribution as the first term.

   Now, for $k \leqslant -1$ only the first two terms contribute. The first term contributes $\sum_k (m-k)k$ while the second term contributes $\sum_k(-k)(m-k)$ which cancel. Thus the term left behind is exactly:

$$2 \times \frac{1}{4}\frac{m(m^2-1)}{6} = \frac{m(m^2-1)}{12} \qquad (30)$$

   Note however that in fact our oscillators carry with them a μ index which we have ignored. If we incorporate it, then each normal ordering of $\alpha_i^\mu \alpha_\nu^j$ will include a factor of $\eta^{\mu\nu}$ which would have to be summed over. This will add in a copy of D to our final result for the normal ordering term.

   Finally, we see that the normal ordering constant a must be equal to:

$$\frac{1}{2}\sum_k \alpha_{-k}^i\alpha_k^i \to \sum_{k=0}\alpha_{-k}^i\alpha_k^i + \frac{1}{2}\sum_{k>0}[\alpha_k^i, \alpha_{-k}^i] =: L_0 : + \frac{D-2}{2}\underbrace{\sum_k k}_{\zeta(-1)} = -\frac{D-2}{24} \qquad (31)$$

2. I believe that the treatment of the prior derivation of the central term was sufficiently careful, as I did not need to use any zeta regularization to compute an infinite sum. I only used zeta regularization in calculating the normal-ordering constant

3. Given that the Witt algebra is already given as an associative algebra, the commutator directly satisfies the Jacobi identity, since $(a-(b-c))+(b-(c-a))+(c-(a-b)) = a+b+c = 0$. Adding a central term gives

$$[L_a,[L_b,L_c]]+[L_b,[L_c,L_a]]+[L_c,[L_a,L_b]] = \frac{1}{12}\delta_{a+b+c}(a(a^2-1)(b-c)+b(b^2-1)(c-a)-c(c^2-1)(a-b)) \quad (32)$$

This is zero by algebra.

4. For the closed string, we have:

$$\dot{X}^\mu(\tau,\sigma) = \ell_s^2 p^\mu + \frac{\ell_s}{\sqrt{2}}\sum_{n\neq 0}(\alpha_n^\mu e^{-in\sigma} + \bar{\alpha}_n^\mu e^{in\sigma})e^{-in\tau}$$
$$X'^\mu(\tau,\sigma) = \frac{\ell_s}{\sqrt{2}}\sum_{n\neq 0}(\alpha_n^\mu e^{-in\sigma} - \bar{\alpha}_n^\mu e^{in\sigma})e^{-in\tau}$$

Taking $X^+ = x^+ + \ell_s p^+\tau$ sets $\alpha_n^+, \bar{\alpha}_n^+ = 0$ for all $n \neq 0$.

$$\dot{X}^\mu + X'^\mu = \ell_s^2 p^\mu + \sqrt{2}\ell_s\sum_{n\neq 0}\alpha_n^\mu e^{-in\sigma}e^{-in\tau}$$
$$\dot{X}^\mu - X'^\mu = \ell_s^2 p^\mu + \sqrt{2}\ell_s\sum_{n\neq 0}\bar{\alpha}_n^\mu e^{in\sigma}e^{-in\tau}$$

Let's just look at the constraint $(\dot{X}+X')^2 = 0$ and then the other constraint will give the same result for the right-movers.

$$0 = \ell_s^4 p^2 + \sqrt{2}\ell_s^3\sum_{n\neq 0} p\cdot\alpha_n e^{-in(\sigma+\tau)} + 2\ell_s^2\sum_{n,m\neq 0}\alpha_n\cdot\alpha_m e^{-i(n+m)(\sigma+\tau)}$$

The zero mode gives $p^2 = 0$. Noting that $\alpha_n\cdot\alpha_m = -\alpha_n^+\alpha_m^- - \alpha_m^+\alpha_n^- + \alpha_n^i\alpha_m^i = \alpha_n^i\alpha_m^i$, we look at the remaining terms of each mode individually, so:

$$0 = \ell_s p\cdot\alpha_n + \sqrt{2}\sum_m \alpha^i_{m-n}\alpha^i_m = -\ell_s p^+\alpha^- + \underbrace{\cancel{\ell_s p^i\alpha^i}}_{\alpha^i \text{ is transverse}} + \sqrt{2}\sum_m \alpha_{m-n}\alpha_n$$
$$\Rightarrow \alpha^- = \frac{\sqrt{2}}{\ell_s p^+}\underbrace{\sum_m \alpha^i_{m-n}\alpha^i_m}_{2L_0} = \frac{\sqrt{2}}{\ell_s p^+}\left[:\sum_m \alpha^i_{m-n}\alpha^i_m : -2a\delta_n\right]$$

5. Firstly, we see that $L_0 - \bar{L}_0$ can only differ by an integer, otherwise there's no combination of $\alpha_{-n}\bar{\alpha}_{-m}$ acting on $|p^\mu\rangle$ that will give a physical state. Now lets say they differ by an integer n. Then $\alpha^i_{-n}\bar{\alpha}^i_{-1}$ will be the lowest-lying excitation at level $(n+1,1)$. We see there are 24 of these that transform under SO(24), so they must give us a massless particle. We note also that we have exactly 24 excitations at levels $(n+k,k)$ for $1 \leqslant k < n$, as the only way to get them is applying $\alpha^i_{-n-k}\bar{\alpha}^i_{-1-k}$. On the other hand, each of these has mass-shell condition:

$$0 = (L_0 - a)\alpha_{-n-k}\bar{\alpha}_{-k}|p^\mu\rangle \Rightarrow \ell_s^2 m^2 = 4(n+k-a)$$

However if this is massless for some value of k, it will be massive for $k+1$, breaking Lorentz invariance.

Note that $L - \bar{L}_0$ generates translations along σ so this shows that any state should be invariant under $\sigma \to \sigma + c$.

6. Note that SO(25) acts on 25×25 traceless symmetric tensors. Note that if we restrict to a subgroup SO(24) that leaves one of the spatial direction fixed, the SO(25) representation breaks down into two SO(24) representations: the symmetric tensor representation (including trace) on the 24 transverse directions, and the vector representation in those directions as well. This is exactly what we have at level two. So, we see we can arrange these two SO(24) rep'ns into the traceless symmetric $SO(25)$ tensor rep.

7. The generators (for the closed string) are:

$$J^{\mu\nu} = T\int_0^{2\pi} d\sigma(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}[\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n + \overline{(\dots)}]$$

Upon computing the commutator $[J^{\mu\nu}, J^{\rho\sigma}]$ the $x^\mu p^\nu - x^\nu p^\mu$ will give no problems, and there will be no cross terms between the right and left moving modes. So it is enough to look at the left movers. **I'm gonna pass on doing this computation...**

8. For NN boundary conditions, α^μ_k is associated to the wavefunction $\cos(k\sigma), \sigma \in [0, \pi]$. This has eigenvalue 1 under flip if k is even and -1 if k is odd. Thus this α_k must transform identically: $\Omega\alpha^\mu_k\Omega^{-1} = (-1)^k\alpha_k$. For DD boundary conditions, we have $\sin(k\sigma)$, which has opposite eigenvalues, so instead we get $(-1)^{k+1}$

9. This is a Lie algebra of dimension $n(n-1)/2$, which already looks promising. In the case of all θ_i equal, we can pick basis so that the R_{ij} are all 1. This is clearly $\mathfrak{so}(n)$. Now, take a diagonal unitary matrix γ (note $\gamma^T = \gamma$). It clear that $\tilde{\lambda}_{ij} := \gamma^{1/2}\lambda_{ij}\gamma^{-1/2}$ gives the right structure under transposition:

$$\tilde{\lambda}^T = \gamma^{-1/2}\lambda^T\gamma^{1/2} = -\gamma^{-1/2}\lambda\gamma^{1/2} = -\gamma\tilde{\lambda}\gamma$$

But since $\tilde{\lambda}_{ij}$ is just a conjugation action on the λ_{ij}, we will still have that the Lie algebra structure is preserved, and maintain $\mathfrak{so}(n)$.

For the second part, again when all the $\theta_i = 0$, this is just the definition of the symplectic group and we have $\lambda = -\omega\lambda^T\omega^{-1} = \omega\lambda^T\omega$ for ω the canonical symplectic written in the $(x_1, p_1, x_2, p_2, \dots)$ basis. Now note that the new symplectic form γ can be written as $\sigma^{1/2}\omega\sigma^{-1/2}$ with $\sigma = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_2}, \dots)$. Then define $\tilde{\lambda} = \sigma^{-1/2}\lambda\sigma^{1/2}$ and note that

$$\tilde{\lambda}^T = \sigma^{1/2}\lambda^T\sigma^{-1/2} = -\sigma^{1/2}\omega\lambda\omega\sigma^{-1/2} = -\gamma\tilde{\lambda}\gamma$$

as required. Again, conjugation action will preserve the Lie algebra structure, so this will remain $\mathfrak{sp}(2n)$.

10. In the symmetric case, we have $\lambda^T = \lambda$, so these are symmetric matrices of N indices. Naturally $\mathrm{SO}(N)$ acts on these, and we see that they can be written as $\mathrm{F}\otimes\mathrm{F}$ for F the fundamental representation. This can be decomposed as the trivial representation and the traceless symmetric representation.

    In the anti-symmetric case with N even, I know that the symplectic group acts on $\mathbb{R}^N$. I'll call this the fundamental rep, and then note that tensoring it with its dual again gives an antisymmetric $N \times N$ matrix on which $\mathrm{Sp}(N)$ can act. This can be decomposed into the singlet and the skew-traceless antisymmetric matrix.

11. Traceless means that any pair of indices contracted with $\eta^{\mu\nu}$ gives zero. Locally, we can pick the metric so that only $\eta_{+-} = \eta_{-+} = 1$ is nonzero. This means that $T_{i_1\dots i_n} = 0$ if any one i is set to $+$ with the other set to $-$. Thus we can have only $T_{+\dots+}$ and $T_{-\dots-}$ nonzero.

12. The round metric is

$$ds^2 = \frac{4dzd\bar{z}}{(1+z\bar{z})^2}$$

The Lie derivative is:

$$\mathcal{L}_X g_{ab} = X^c\partial_c g_{ab} + g_{ac}\partial_b X^c + g_{cb}\partial_a X^c \tag{33}$$

Working with $z, \bar{z}$ we get:

$$\begin{aligned} \mathcal{L}_X g_{zz} &= 2g_{z\bar{z}}\partial_z X^{\bar{z}} = 0 \\ \mathcal{L}_X g_{zz} &= 2g_{z\bar{z}}\partial_z X^{\bar{z}} = 0 \\ \mathcal{L}_X g_{z\bar{z}} &= (X^z\partial_z + X^{\bar{z}}\partial_{\bar{z}})g_{z\bar{z}} = \lambda(z,\bar{z})g_{z\bar{z}} \end{aligned} \tag{34}$$

The first two equation shows us that $X^z, X^{\bar{z}}$ must be holomorphic and anti-holomorphic respectively. We want the function λ to be well-defined on the entire Riemann sphere and so the last equation gives us:

$$-2\frac{(X^z\bar{z} + X^{\bar{z}}z)}{1+z\bar{z}} = \lambda(z,\bar{z}) \tag{35}$$

We see that $X^z, X^{\bar{z}}$ cannot have any poles. Further, they cannot grow faster than $z^2, \bar{z}^2$ respectively as $z \to \infty$ otherwise λ will blow up at the north pole. So our solutions space is spanned by $\partial_z, z\partial_z, z^2\partial_z$ and their conjugates.

Next, right away we can see that the only nonzero Christoffel symbols in the round metric are Γ^z_{zz} and $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$. Second, because T is traceless, by the previous problem we see it has only two components: T_{zz} and $T_{\bar{z}\bar{z}}$. Now looking at $\nabla^\beta T_{\alpha\beta}$ we see that this gives two equations:

$$\begin{aligned} g^{z\bar{z}}\nabla_{\bar{z}} T_{zz} &= \partial_{\bar{z}} T_{zz} \\ g^{z\bar{z}}\nabla_{z} T_{\bar{z}\bar{z}} &= \partial_{z} T_{\bar{z}\bar{z}} \end{aligned} \tag{36}$$

Note that there can be no Christoffel contribution. This simply asks for globally-defined holomorphic 2-forms. Let's look at T_{zz}. Around $z = 0$, it must be a polynomial to avoid poles. Transforming to $w = 1/z, dw = -dz/z^2 \Rightarrow \frac{dz}{dw} = -z^2 = -w^{-2}$ we get $T_{ww}(w) = (\frac{dz}{dw})^2 T_{zz}(w)$. Note that the right hand side will only have poles at least as bad as w^{-2} so we cannot have any global section of this vector bundle. Thus, there are no Teichmuller parameters.

13. We can think of the torus as $\mathbb{C}/\Lambda$. Note that scaling and rotation preserve the complex structure of the fundamental parallelogram so WLOG we can pick $\Lambda = \mathbb{Z}\text{-span}\{1, \tau\}$ with $\tau \in \mathbb{H}$. Thus we need vector fields on $\mathbb{C}$ that respect the translation-invariance under Λ. Any translation-invariant holomorphic function is zero, we can only have the constant vector fields $\partial_z, \partial_{\bar{z}}$.

    We now look for holomorphic and anti-holomorphic traceless tensors. Again, T_{zz} and $T_{\bar{z}\bar{z}}$ be translation-invariant w.r.t the lattice, so again they must be constants. We get $dz \otimes dz$ and $d\bar{z} \otimes d\bar{z}$ as our two Teichmuller deformations. As real tensors these are:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = dx \otimes dx - dy \otimes dy, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2dx \otimes dy,$$

14. Not sure exactly how they want us to calculate this. Let's assume they are OK with Gauss-Bonnet. For the disk with the flat metric, we have right away that the curvature R vanishes. The geodesic curvature at the boundary is a constant, and is easily seen to be 1. Integrating this over the boundary of the disk gives 2π so that $\chi = 1$.

    Using the round metric, it is quick to see that the only contribution to $R_{\mu\nu} = R^z_{\mu z\nu} + R^{\bar{z}}_{\mu\bar{z}\nu}$ is for $R^z_{zz\bar{z}} = -\partial_{\bar{z}}\Gamma^z_{zz}$ and $R^{\bar{z}}_{\bar{z}\bar{z}z} = \partial_z \Gamma^{\bar{z}}_{\bar{z}\bar{z}}$ giving $R_{z\bar{z}} = \frac{2}{(1+|z|^2)}$. Tracing this gives $R = 1$. Integrating this over half the sphere gives 2π. The geodesic curvature vanishes on the great circle by symmetry, and we get $\chi = 1$ again.

15. Any such surface can be decomposed as a sphere with $2n$ holes connected to n handles. Let's integrate the scalar curvature over each piece individually. First the curvature integrated on the sphere with $2n$ disks removed is equal to the curvature integrated on the Riemann sphere: 4π minus the curvature integrated on $2n$ disks. We have just done this in the previous problem, and we get $2\pi \times 2n$. Lastly, the curvature on the handles is just the same as the curvature on the sphere with two holes cut out, which we have just calculated is $4\pi - 2 \times 2\pi = 0$. Thus the total curvature is just $2\pi(2 - 2n)$, giving us $\chi = 2 - 2n$ as required.

16. Our point particle action is $S_0 = \int d\tau e(e^{-2}(\partial_\tau x)^2 - m^2)$. Let's look at:

$$\int \frac{\mathcal{D}X\mathcal{D}e}{V_{gauge}} e^{-S_0} \sim \int \mathcal{D}X\mathcal{D}e\mathcal{D}b\mathcal{D}c\, e^{-S_0 - \int b(\delta F)c - i\int BF(e)}$$

    Note we don't need an α index on B, b, c because they just parameterize the continuous symmetry with no discrete parameters:

$$(\delta_{\tau_1}\tau)(\tau) = \delta(\tau - \tau_1)$$

    Using Polchinski's convention for coordinate transformation (he also has $-B_A$ of Kiritsis),

$$\delta_{\tau_1} X(\tau) = -\delta(\tau - \tau_1)\partial_\tau X, \qquad \delta_{\tau_1} e(\tau) = -\partial_\tau(\delta(\tau - \tau_1)e(\tau))$$

and

$$[\delta_{\tau_1}\delta_{\tau_2}](\tau) = -(\delta(\tau-\tau_1)\partial_\tau\delta(\tau-\tau_2)-\delta(\tau-\tau_2)\partial_\tau\delta(\tau-\tau_1))\,\partial_\tau \Rightarrow f^{\tau_3}_{\tau_1\tau_2} = \delta(\tau_3-\tau_1)\partial_{\tau_3}\delta(\tau_3-\tau_2)-\delta(\tau_3-\tau_2)\partial_{\tau_3}\delta(\tau_3-\tau_1)$$

Then the BRST transformation is given by:

$$\begin{aligned}\delta_\epsilon X &= i\epsilon c\dot{X}\\ \delta_\epsilon e &= i\epsilon(\dot{cX})\\ \delta_\epsilon b &= \epsilon B_A\\ \delta_\epsilon c &= i\epsilon c\dot{c}\\ \delta_\epsilon B &= 0\end{aligned} \tag{37}$$

Now let's take $F(e) = e - 1$. Then we get a ghost action

$$\begin{aligned} b_A c^\alpha \delta_\alpha F^\alpha \to & \int d\tau_1 d\tau_2 b(\tau_1)c(\tau_2)\delta_{\tau_2}\ (1 - e(\tau_2))\\ &= \int d\tau_1 b(\tau_1)\partial_{\tau_1}\int d\tau_2 c(\tau_2)[\delta(\tau_1-\tau_2)e(\tau_1)] = -\int d\tau \dot{b}c\end{aligned}$$

We enforce this constraint by integrating over B. Now we will have $\delta_\epsilon e = 0$ and $\delta_\epsilon b = i\epsilon(T^X + T^{gh})$. Because we are in Euclidean signature, we have $p = \partial_t X = i\dot{X}$ and similarly $p = -ic$ ($-i$ because the real time Lagrangian has term $-i\dot{b}c$). Then the BRST current (equal to charge because we're in 1D) is:

$$Q_B = p_X\delta_B X + p_b\delta_B b = -c\dot{X}^2 - ici(-\frac{1}{2}\dot{X} + \frac{1}{2}m^2 - \dot{b}c) = -c\frac{1}{2}\dot{X} + \frac{1}{2}m^2 = c\frac{1}{2}(p^2+m^2) = cH.$$

Clearly $Q_B^2 = 0$. As before, the ghosts generate a two-state system. Our set of states is given by $|k^\mu,\uparrow\rangle, |k^\mu,\downarrow\rangle$. Following convention, c raises and b lowers. $Q_B|k,\downarrow\rangle = \frac{1}{2}(k^2+m^2)|k,\uparrow\rangle$ so all states of the form $|k,\uparrow\rangle$ with $k^2+m^2 \neq 0$ are BRST exact. Similarly all states $|k,\uparrow\rangle$ are BRST closed along with all states of the form $|k,\downarrow\rangle$ with $k^2+m^2 = 0$. So the closed states that are not exact are $|k,\downarrow\rangle$, $|k,\uparrow\rangle$ with $k^2+m^2 = 0$. We take only the states with $b|\psi\rangle = 0$. The reason is that all states $|k,\uparrow\rangle$ are physical, and so we would need amplitudes between such states to be proportional to $\delta(k^2+m^2)$ in order for the states to decouple, but *amplitudes* cannot have such extreme singularities **Don't understand this. Appreciate it, and its relation to Siegel gauge.**.

17. I believe these variations have Kiritsis taking $c \to -ic$ in his formalism. They also follow directly from Polchinski's formalism. Under a diffeomorphism $\delta_{\xi,\bar{\xi}}X = -\xi\partial X - \bar{\xi}\bar{\partial}X$. These are two copies of the reparameterization algebra developed in the previous problem, and so the commutation relations are the same. We get, again in Polchinski's formalism (37)

$$\begin{aligned}\delta_\epsilon X &= i\epsilon(c\partial X + \bar{c}\bar{d}X)\\ \delta_\epsilon c &= i\epsilon(c\partial c + \bar{c}\bar{\partial}\bar{c})\\ \delta_\epsilon b &= i\epsilon(T^X + T^{gh})\end{aligned} \tag{38}$$

I see no problem with Q_B^2 giving zero when acting on the X and c fields.

$$\delta_B(c\partial X + \bar{c}\bar{d}X) = i\epsilon(\cancel{c(\partial c)\partial X} + \cancel{\bar{c}(\bar{\partial}\bar{c})\bar{\partial}X} - c\partial(\cancel{c\partial X} + \bar{c}\bar{\partial}\bar{c}X) - \bar{c}\partial(\cancel{c\partial X} + \bar{c}\bar{\partial}\bar{c}X))$$

and the remaining terms die by the equations of motion. The c variation will always die because we've already shown the transformations satisfy a Lie algebra with Bianchi identity.

It looks like the b field will be nontrivial. If one of the equations of motion is the $T^X + T^{gh} = 0$ then this will be zero right away. Otherwise, we want to compute (WLOG in the holomorphic sector):

$$\delta_B(T^X+T^{gh}) = \delta_B(\frac{1}{\alpha}(\partial X)^2+2b\partial c+\partial bc) = i\epsilon[\frac{2}{\alpha}\partial X\partial(c\partial X)+2(T^X+T^{gh})\partial c+2b\,\partial(c\partial c)+\partial(T^X+T^{gh})\partial c+\partial b\,c\partial c]$$

The purely bc terms cancel. I'm left with

$$\frac{2}{\alpha}[(\partial X)^2\partial c + \partial X\partial^2 Xc]$$

I don't know how to get rid of this. I can write it as a total derivative less something proportional to $(\partial X)^2\partial c$ and perhaps note that this is just ∂c times the stress tensor, which perhaps vanishes classically? At any rate, there is no need to use $d = 26$ here.

18. Integrating over σ will as usual pick out the zero mode. For T_{++} this gives us

$$\sum_n (2nb_{-n+m}c_n + n + mc_{-n}b_{n+m}) = \sum_n (m-n) : b_{n+m}c_{-n} :$$

and similarly for the right-movers. To get the central charge we'll have to proceed as before, noting that only $[L_m, L_{-m}]$ can give a nonzero central term. As before, we expect only a finite part of the infinite sum to contribute to this. We thus take out the only terms of the sum with m, n having the same sign:

$$\sum_{n=1}^{m}(m+n)b_{m-n}c_n, \qquad \sum_{n=1}^{m}(-2m+n)b_{-n}c_{-m+n}$$

Then our commutators of these finite terms give:

$$\begin{aligned}\sum_{k,k'}(m+k)(-2m+k')[b_{m-k}c_k, b_{-k'}c_{-m+k'}] &= \sum_{k,k'}(m+k)(-2m+k')(b_{m-k}c_{m+k'} - b_{-k'}c_k)\delta_{k-k'}\\ &= \sum_{k=1}^{m}(m+k)(-2m+k)(b_{m-k}c_{m+k} - b_{-k}c_k)\end{aligned}$$

Looking at the non-normal-ordered part, this leaves:

$$\sum_{k=1}^{m} b_{m-k}c_{-m+k}(m+k)(k-2m) \to \sum_{k=1}^{m}(m+k)(k-2m) = \frac{1}{6}(m-13m^3)$$

19. We have

$$j_B = \frac{\partial\mathcal{L}}{\partial(\bar\partial X)}c\partial X + \frac{\partial\mathcal{L}}{\partial(\bar\partial c)}c\partial c = \frac{2}{2\pi\alpha'}(\partial X)^2 c + \frac{1}{\pi}bc\partial c \to cT^X + bc\partial c = cT^X + \frac{1}{2}cT^{gh}$$

I don't understand how other references include a $\frac{3}{2}\partial^2 c$.

20. Let's do this for the open string, so we are then just calculating the holomorphic sector. We have:

$$Q_B = \sum_{m=-\infty}^{\infty} : (L^X_{-m} + \frac{1}{2}L^{gh}_{-m} - a\delta_{m,0})c_m :$$

Note that the a is just from the X component of the theory since by definition Q contains the term $: cT^{gh} :$ already in normal order.

We now need to consider the total BRST charge $Q + \bar{Q}$. Then:

$$Q_B^2 = \sum_{n,m}([L^X_m, L^X_n] + [L^{gh}_m, L^{gh}_n] + (m-n)L^X_{m+n} + (m-n)L^{gh}_{m+n} + 2am\delta_{m+n})c_{-m}c_m$$

This will vanish only if the commutators give no anomalous term. From previous exercises we see this is only if:

$$\frac{d(m^3-m)}{12} + \frac{(m-13m^3)}{6} + 2am = 0$$

This happens exactly when $d = 26$ and $a = 1$.

21. We now have $Q+\bar{Q}$ that we need to be zero on states. Again, each of $Q, \bar{Q}$ will no change the level, so their sum will not either, and we have a (double) grading on the space which they will preserve.

$$Q = Q_0 + Q_1, \qquad Q_0 = c_0(L_0^X - 1), \quad Q_1 = c_{-1}L_1^X + c_1 L_{-1}^X + c_0(b_{-1}c_1 + c_{-1}b_1)$$

and $\bar{Q}$ is the conjugate of this.

At level zero we will again have $Q^0 = c_0 L_0^X + \bar{c}_0 \bar{L}_0^X$. We now have two copies of the Clifford algebra and our Siegel gauge condition will make it so that we only consider states $|\downarrow, \bar{\downarrow}, p, \bar{p}\rangle$. Now we need $((L_0-1)c_0 + (\bar{L}_0 - 1)\bar{c}_0)\,|\downarrow, \bar{\downarrow}, p, \bar{p}\rangle = (L_0 - 1)\,|\uparrow, \bar{\downarrow}, p, \bar{p}\rangle + (\bar{L}_0 - 1)\,|\downarrow, \bar{\uparrow}, p, \bar{p}\rangle$ so we need $\frac{\ell_s^2 p^2}{4} = \frac{\ell_s^2 \bar{p}^2}{4} = 1$ i.e. the total mass is $m^2 = -4$. We thus have the tachyon state.

Also because $b_0\,|\psi\rangle = 0$ for any physical state, we will have $\{Q, b_0\}\,|\psi\rangle = 0 = (L_0 - a)\,|\psi\rangle$ so we have $L_0 - 1 = \bar{L}_0 - 1 = 0$ and this gives us the level-matching condition. So the next level we can have a state is at $(1,1)$.

As in the open string treatment, the most general such state has nine terms:

$$\begin{aligned}
|\psi_1\rangle = (&\zeta \cdot \alpha_{-1}\bar{\zeta} \cdot \bar{\alpha}_{-1} + \zeta_{ab} \cdot \alpha_{-1}\bar{b}_{-1} + \zeta_{ac} \cdot \alpha_{-1}\bar{c}_{-1} \\
&+ b_{-1}\zeta_{ba} \cdot \bar{\alpha}_{-1} + \xi_{bb} b_{-1}\bar{b}_{-1} + \xi_{bc} b_{-1}\bar{c}_{-1} \\
&+ c_{-1}\zeta_{ca} \cdot \bar{\alpha}_{-1} + \xi_{cb} c_{-1}\bar{b}_{-1} + \xi_{cc} c_{-1}\bar{c}_{-1})\,|\downarrow, \bar{\downarrow}, p\rangle
\end{aligned}$$

Let's act on this with $Q_0 + Q_1$. First lets look at Q_0. On the $\alpha\bar{\alpha}$ term it will give eigenvalue $c_0\frac{\alpha_0^2}{2} + \bar{c}_0\frac{\bar{\alpha}_0^2}{2}$ while something like the $\zeta_{ab}\alpha\bar{b}$ term it will give eigenvalue $c_0\frac{\alpha_0^2}{2} + \bar{c}_0(\frac{\bar{\alpha}_0^2}{2} - 1)$. This will be compensated by the action of the $\bar{c}_0(\bar{b}_{-1}\bar{c}_1)$ (from Q_1) on $\zeta_{ab}\alpha_{-1}b_{-1}$. The exact same argument can be applied to any of those four terms - there will always be one the four bc terms of $Q_1 + \bar{Q}_1$ that will give us the extra factor of 1 from its commutation relation with that term in $|\psi_1\rangle$ (it commutes with everything else).

So we get a term $c_0\frac{\ell_s^2 p^2}{4}\,|\psi_1\rangle$ which then gives the $p^2 = 0$ constraint. The remaining term comes from the $c_{-1}L_1^X + c_1 L_{-1}^X + c.c.$ action. The $c_{-1}L_1 + c_1 L_{-1}$ will each annihilate everything except six terms, giving

$$\begin{aligned}
&\zeta \cdot p\, c_{-1}\bar{\zeta} \cdot \bar{\alpha}_{-1} + \zeta_{ab} \cdot p\, c_{-1}\bar{b}_{-1} + \zeta_{ac} \cdot p\, c_{-1}\bar{c}_{-1} \\
&+ p \cdot \alpha_{-1}\zeta_{ba} \cdot \bar{\alpha}_{-1} + \xi_{bb} p \cdot \alpha_{-1}\bar{b}_{-1} + \xi_{bc} p \cdot \alpha_{-1}\bar{c}_{-1} + c.c.
\end{aligned} \tag{39}$$

and the conjugate of this will contribute the conjugate terms. For this to all be zero we need each of the $\zeta_i \cdot p = 0$ as well as their conjugates. We also need $\xi_{bb} = \xi_{bc} = \xi_{cb} = 0$. We also see that $\zeta_{ba} = \zeta_{ab} = 0$.

On the other hand the general form of an exact state is also given by (39) for the ζ_i and ξ_i arbitrary. Thus all the terms involving c and/or $\bar{c}$ are exact and so upon quotienting we get $\zeta_{ac} = \zeta_{bc} = \xi_{cc} = 0$. Lastly we get the relation that we should identify $\zeta_i\bar{\zeta}_j = \zeta_i\bar{\zeta}_j + p_i\zeta_j' + \zeta_i' p$ ie we project out any tensor of the form $p \otimes \zeta'$ or $\zeta' \otimes p$. This is equivalent to identifying $\zeta \cong \zeta' + \xi p$ and identically for $\bar{\zeta}$.

So we have eliminated everything except for $\zeta, \bar{\zeta}$, each of which must be transverse to p and we identify ζ differing by a longitudinal p component. This is 24×24 parameters, as required.

22. If I have the Clifford algebra $C\ell(2)$, any vector v will have an orbit generated by $1, b_0, c_0, b_0c_0$, so there can be no irreducible representation of dimension greater than 4. Further, there is a vector v_0 annihilated by b. Consider $v_1 = cv_0$ and assume it is distinct. Now $bv_1 = bcv_0 = v_0 - cbv_0 = v_0$. So v_1 and v_0 span the irreducible representation meaning that any irrep in fact has dimension 2. Thus, any higher dimensional generalization would only be (probably direct or semidirect) extensions of this and the trivial irrep, and give us no new information.

Chapter 4: Conformal Field Theory

1. We'll do this directly. First observe:

$$\begin{aligned}
\frac{d}{dt}|_{t=0} e^{-itP_\mu} f(x) &= -\partial_\mu f \\
\frac{d}{dt}|_{t=0} e^{-\frac{it}{2}\omega^{\mu\nu}J_{\mu\nu}} f(x) &= -\omega^\mu_\nu x^\nu \partial_\mu \\
\frac{d}{dt}|_{t=0} e^{-itD} f(x) &= x \cdot \partial f(x) \text{ annoying that there is no -} \\
\frac{d}{dt}|_{t=0} e^{-itK_\mu} f(x) &= -(x^2\partial_\mu - 2x_\mu(x\cdot\delta))f(x)
\end{aligned} \tag{40}$$

The last one is exactly the first-order expansion of $\frac{x^\mu + x^2 a^\mu}{1+2a\cdot x + a^2x^2}$. Note the dilatation and special conformal generators are the negative of Di Francesco's (SO ANNOYING OGM).

Now let's do the commutator

$$\begin{aligned}
[J_{\mu\nu}, P_\rho] &= -\partial_\rho(x_\mu\partial_\nu - \partial_\nu\partial_\mu) = -(\eta_{\mu\rho}\partial_\nu - \eta_{\nu\rho}\partial_\mu) = -i(\eta_{\mu\rho}\partial_\nu - \eta_{\nu\rho}\partial_\mu) \\
[P_\mu, K_\nu] &= -\partial_\mu(x^2\partial_\nu - 2x_\nu x\cdot\partial) = -(2x_\mu\partial_\nu - 2\eta_{\mu\nu}x^\lambda\partial_\lambda - 2x_\nu\delta^\lambda_\mu\partial_\lambda) = 2iJ_{\mu\nu} - 2i\eta_{\mu\nu}D \\
[J_{\mu\nu}, J_{\rho\sigma}] &= -i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho}) \leftarrow \textit{Everyone has done this one like 20 times} \\
[J_{\mu\nu}, K_\rho] &= -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \\
[D, K_\mu] &= x^\nu\cdot\partial_\nu[x^2\partial_\mu - 2x_\mu(x^\lambda\partial_\lambda)] - [x^2\partial_\mu - 2x_\mu x\cdot\partial]x^\lambda\partial_\lambda \\
&= \cancel{2}x^\nu x_\nu\partial_\mu - 2x^\nu\eta_{\mu\nu}(x\cdot\partial) - \cancel{2x_\mu(x\cdot\partial)} - \cancel{x^2\partial_\mu} + \cancel{2x_\mu x\cdot\partial} = iK_\mu \\
[D, P_\mu] &= -\partial_\mu x^\lambda\partial_\lambda = -\partial_\mu = -iP_\mu \\
[J_{\mu\nu}, D] &= 0
\end{aligned}$$

The way we did the $[J, K]$ commutator is by noting it should look the same as $[J, P]$, since P is just translation about the point at ∞. The $[J, D]$ commutator follows because rotation is scale invariant.

2. We see immediately that the $J_{\mu\nu}$ can be mapped to the $M_{\mu\nu}$ corresponding to a SO(p, q) subgroup of SO$(p+1, q+1)$. The full group has:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \tag{41}$$

Note the commutation relations of J with P and K gives us:

$$[J_{\mu\nu}, \frac{1}{2}(K_\rho \pm P_\rho)] = -i\left(\eta_{\mu\rho}\frac{1}{2}(K\pm P)_\nu - \eta_{\nu\rho}\frac{1}{2}(K\pm P)_\mu\right)$$

Writing these as $M_{\rho,d+1}$ and $M_{\rho,d}$ respectively, we see that we get the second and fourth terms nonzero and we get exactly (41). Note at this stage I didn't need to do such linear combinations of K and P. That is important for appreciating that we want:

$$[M_{\mu d}, M_{\nu d+1}] = -i\eta_{\mu\nu}M_{dd+1} = -i\eta_{\mu\nu}M_{d,d+1} = i\eta_{\mu\nu}D$$

and we get exactly this:

$$\frac{1}{4}[(K-P)_\mu, (K+P)_\nu] = \frac{1}{4}([K_\mu, P_\nu] - [P_\mu, K_\nu]) = i\eta_{\mu\nu}D$$

We needed that combination so that $J_{\mu\nu}$ wouldn't appear. As required $[J_{\mu\nu}, D] = [M_{\mu\nu}, M_{d,d+1}] = 0$ for $\mu \in 0\ldots d-1$. **I'm getting the wrong sign. Perhaps our friend's convention is off.**

3. This comes from noting that for $f = z + \epsilon(z)$

$$\left(\frac{\partial f}{\partial z}\right)^{\Delta}\left(\frac{\partial f}{\partial \bar{z}}\right)^{\bar{\Delta}} - 1 = (1+\partial\epsilon)^{\Delta}(1+\bar{\partial}\epsilon)^{\bar{\Delta}} - 1 = \Delta\partial\epsilon + \bar{\Delta}\bar{\partial}\epsilon$$
$$\Rightarrow \Phi(z)(1-(\Delta\partial\epsilon+\bar{\Delta}\bar{\partial}\epsilon)) = \Phi'(f(z),\bar{f}(\bar{z})) = (1+\epsilon\partial+\bar{\epsilon}\bar{\partial})\Phi'(z)$$
$$\Rightarrow (1-(\Delta\partial\epsilon+\bar{\Delta}\bar{\partial}\epsilon+\epsilon\partial+\bar{\epsilon}\bar{\partial}))\Phi(z) = \Phi'(z)$$
$$\Rightarrow \Phi(z)-\Phi'(z) = (\Delta\partial\epsilon+\epsilon\partial+\bar{\Delta}\bar{\partial}\epsilon+\bar{\epsilon}\bar{\partial})\Phi(z)$$

How weird... think about this in terms of active/passive. Contrast with Di Francesco.

4. As in the 2-point greens function case, note that:

$$\delta_\epsilon G^N = 0 \Rightarrow \left(\sum_{i=1}^{N}\epsilon(z_i)\partial_{z_i} + \Delta_i\partial\epsilon(z_i) + c.c.\right)G^N = 0$$

We can WLOG look at just the holomorphic sector (set $\bar{\epsilon}=0$) Now first set $\epsilon(z)=1$. This directly gives $\sum_i \partial_i G^N = 0$, as we wanted. Next, take $\epsilon(z)=z$. This gives $\sum_i(z_i\partial_i+\Delta_i)G^N=0$. Finally, take $\epsilon = z^2$ to get $\sum_i(z_i^2\partial_i + 2z_i\Delta_i)G^N = 0$ as desired. Note in all these cases, we are exactly performing the global SL(2) transformations, so these Ward identities will always hold.

5. The first Ward identity tells us that the function can only depend on z_{12}, z_{23}. Then the next two can be written as:

$$(x_1\partial_1+x_2\partial_2+x_3\partial_3+\Sigma\Delta_i)f(x_{12},x_{23}) = ((x_1-x_2)\partial_{12}+(x_2-x_3)\partial_{23}+\Sigma\Delta_i)\,f=0$$
$$(x_1^2\partial_1+x_2^2\partial_2+x_3^2\partial_3+\Sigma 2x_i\Delta_i)f(x_{12},x_{23}) = \left((x_1^2-x_2^2)\partial_{12}+(x_2^2-x_3^2)\partial_{23}+\Sigma 2x_i\Delta_i\right)f=0$$

We can subtract out ∂_{23} to get the differential equation:

$$0=\left(\frac{x_1+x_2}{x_2+x_3}-1\right)x_{12}\partial_{12}+\sum_i\left(\frac{2x_i}{x_2+x_3}-1\right)\Delta_i \to (x_{12}+x_{23})x_{12}\partial_{12}+(x_{12}+x_{12}+x_{23})\Delta_1+x_{23}(\Delta_2-\Delta_3)$$
$$\Rightarrow 0 = \left(x_{12}^2\partial_{12}+x_{23}x_{12}\partial_{12}+2x_{12}\Delta_{12}+x_{23}(\Delta_1+\Delta_2-\Delta_3)f\right)$$

Now write $f(x_{12},x_{23}) = e^g(u,x_{23})$ with $u=\log x_12$. This substitution gives the ODE:

$$(e^u+x_{23})g'(u)+2\Delta_1e^u+x_{23}(\Delta_1+\Delta_2-\Delta_3)=0 \Rightarrow g(u)=\int_{-\infty}^{\log x_{12}}du\frac{2\Delta_1e^u-x_{23}(\Delta_1+\Delta_2-\Delta_3)}{e^u+x_{23}}$$

This integral can be done and gives:

$$\frac{C}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}(x_{12}+x_{23})^{\Delta_1+\Delta_3-\Delta_2}} = \frac{C}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{13}^{\Delta_1+\Delta_3-\Delta_2}}$$

We can do the same for ∂_{23} and get the general form:

$$\frac{\lambda_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{13}^{\Delta_1+\Delta_3-\Delta_2}x_{23}^{\Delta_2+\Delta_3-\Delta_1}}\times c.c.$$

for $\lambda_{123},\bar{\lambda}_{123}$ undetermined constants (call their product C_{123}).

6. Again specialize to the holomorphic part. We see G^N depends only on relative positions x_{12}, x_{13}, x_{14}. We can WLOG take $G^{(4)}$ to have the form:

$$G^{(4)}(z_1,z_2,z_3,z_4) = \frac{f(z_1,z_2,z_3,z_4)}{z_{12}^{\Delta_{12}}z_{13}^{\Delta_{13}}z_{14}^{\Delta_{14}}z_{23}^{\Delta_{23}}z_{24}^{\Delta_{24}}z_{34}^{\Delta_{34}}}$$

Here, because f is arbitrary, we have not made any assumptions. The Ward identities imply the following:

- f depends only on the relative positions z_{ij}
- $\sum_{i<j} \Delta_{ij} = \Delta$ with $\Delta = \sum_i \Delta_i$ and $\sum_i z_i \partial_i f = 0$
- $\Delta_{23} + \Delta_{24} + \Delta_{34} = 2\Delta_1, \quad \Delta_{13} + \Delta_{14} + \Delta_{34} = 2\Delta_2, \quad \Delta_{12} + \Delta_{14} + \Delta_{24} = 2\Delta_3, \quad \Delta_{12} + \Delta_{13} + \Delta_{23} = 2\Delta_4$ and $\sum_i z_i^2 \partial_i f = 0$

These give 4 constraints for the 6 Δ_{ij}, so the system is underdetermined. The most symmetric solution is given by:

$$\Delta_{ij} = \Delta_i + \Delta_j - \frac{1}{3}\Delta$$

It remains to find the general form of f.

- The first ward identity gives us that it can only depend on the z_i through z_{ij}.
- Further, it must transform trivially under dilatation, so we see that it can only depend on ratios of the z_{ij} with an equal number of each z_{ij} in the numerator and denominator.
- Under special conformal transformations, each such ratio will transform as $\frac{z_{ij}}{z_{kl}} \to \frac{z_{ij}}{z_{kl}}(z_i + z_j - z_k - z_l)$, and more generally

  $$\prod_a \frac{z_{i_a j_a}}{z_{k_a l_a}} \to \prod_a \frac{z_{i_a j_a}}{z_{k_a l_a}} \times \sum_a (z_{i_a} + z_{j_a} - z_{k_a} - z_{l_a})$$

  The third Ward identity shows that f must transform trivially under these, and so f can only depend on ratios where each z_i appears an equal number of times in the numerator and denominator.

In total: we need ratios of z_{ij} with an equal number of z_{ij} in the numerator and denominator, and each z_i appears the same number of times in the numerator and denominator. All such ratios can be obtained as rational functions of:

$$x := \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad y := \frac{z_{14}z_{24}}{z_{13}z_{24}}$$

But we see that $y = 1 - x$ so in fact the most general such function is any function of x alone, as was required.

7. With conformal invariance (rescaling in particular), an infinite cylinder has no moduli, so you can set its radius to be whatever you like and get the same theory.

8. Let's perform the OPE within the correlator:

$$\langle \Phi_i(z_1)\Phi_j(z_2)\Phi_k(z_3)\rangle = \sum_\ell z_{12}^{\Delta_\ell - \Delta_i - \Delta_j} \bar{z}_{12}^{\bar{\Delta}_\ell - \bar{\Delta}_i - \bar{\Delta}_j} C_{ij\ell} \langle \Phi_\ell(z_2)\Phi_k(z_3)\rangle$$

By the orthonormality assumption of the OPE, we then have

$$\langle \Phi_\ell(z_2)\Phi_k(z_3)\rangle = \frac{\delta_{\ell k}}{z_{23}^{2\Delta_k}\bar{z}_{23}^{2\bar{\Delta}_k}} \Rightarrow \langle \Phi_i(z_1)\Phi_j(z_2)\Phi_k(z_3)\rangle = \frac{C_{ijk}(z_{12})}{z_{23}^{2\Delta_k}\bar{z}_{23}^{2\bar{\Delta}_k} z_{12}^{\Delta_i + \Delta_j - \Delta_k}\bar{z}_{12}^{\bar{\Delta}_i + \bar{\Delta}_j - \bar{\Delta}_k}}$$

9. We assume that $\mu \ll 1/r$. The integral is in fact real, and we can approximate it by

$$\int d^2p \frac{\cos(pr\cos(\theta))}{p^2 + m^2} = \int d\theta \int_0^\infty \frac{p\,dp\, e^{-\frac{1}{2}(pr\cos(\theta))^2}}{p^2 + \mu^2} = \int d\theta \frac{1}{2}\int_{\frac{1}{2}\mu^2 r^2 \cos^2(\theta)}^\infty \frac{du\, e^{-u}}{u} = \frac{1}{2}\int_0^{2\pi} d\theta \Gamma(0, \tilde{\mu}^2 r^2 \cos^2(\theta))$$

It is known that $\Gamma(0,\epsilon) = -\gamma - \log\epsilon$ so up to a constant (that can be absorbed into the redefinition of μ) we get;

$$-\frac{\ell_s^2}{2\pi}\frac{1}{2}(2\pi)\log(\mu^2|x-y|^2) = -\frac{\ell_s^2}{2}\log(\mu^2|x-y|^2)$$

10. By Stokes' theorem its clear. Let Ω be any disk enclosing the origin:

$$\int_\Omega d^2z \bar{\partial}\partial \log|z|^2 = i\int_\Omega dz \wedge d\bar{z}\, \bar{\partial}\partial \log|z|^2 = -i\oint_{\partial\Omega} dz \partial \log|z|^2 = -i\oint_{\partial\Omega}\frac{dz}{z} = 2\pi$$

Alternatively we could put in a regulator and evaluate this directly:

$$\int_\Omega d^2z\partial\bar{d}\log(|z|^2+\mu^2) = \int_\Omega d^2z\partial\frac{z}{|z|^2+\mu^2} = \int_\Omega d^2z\frac{\mu^2}{(|z|^2+\mu^2)^2}$$

As $\mu \to 0$ this approaches 0 everywhere except for the origin. Taking $|z| = r$ and integrating in polar coordinates (note $d^2z = 2dxdy = 2rdrd\theta$):

$$2\pi \times 2 \times \int_0^\infty \frac{\mu^2 r}{(r^2+\mu^2)^2} = 2\pi$$

as required.

11. We have:

$$\frac{1}{4\pi\ell_s^2}\int d^2\xi\,\sqrt{-g}\,g^{ab}\partial_a X\,\partial_b X \Rightarrow T_{ab} = -\frac{4\pi}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} - \frac{1}{\ell_s^2}\left(\partial_a X\partial_b X - \frac{1}{2}g_{ab}\partial_c X\partial^c X\right)$$

This is clearly traceless. Let's specialize to the holomorphic sector to get $T(z) = -\frac{1}{\ell_s^2} : \partial X\partial X :$ and of course this is the non-singular part of the $\partial X(z)\partial X(w)$ OPE as $z \to w$.

12. The scaling dimensions of conserved currents don't change.

For a current to be conserved, we must that the surface operator $\frac{1}{2\pi i}\oint dz J(z)$ is topological (independent of contour). Applying dilatation $z \to z/\lambda$ on this does not change the operator, so long as it does not pass any operator insertions. So we have:

$$\frac{1}{2\pi i}\oint dz J(z) + c.c. = \frac{1}{2\pi i}\oint d\frac{z}{\lambda}J'(z/\lambda, \bar{z}/\lambda) + c.c.$$

And thus we get $J(z,\bar{z}) = \lambda^{-1}J'(z/\lambda,\bar{z}/\lambda)$, and we get J has scaling dimension 1.

On the other hand for $T^{\mu\nu}$, we have the conserved charge:

$$P_\nu = \oint dn^\mu T_{\mu\nu}$$

Applying dilatation, we see from exponentiating the commutation relation for $[D, P_\nu]$ that $P_\nu = P'_\nu/\lambda$ so

$$P_\nu = \oint dn^\mu T_{\mu\nu} + c.c. = \frac{1}{\lambda}\underbrace{\oint \frac{dn^\mu}{\lambda}T'_{\mu\nu}(z/\lambda,\bar{z}/\lambda) + c.c.}_{=P'_\nu)} = P'_\nu/\lambda$$

giving us that

$$T_{\mu\nu}(z,\bar{z}) = \lambda^{-2}T'_{\mu\nu}(z/\lambda,\bar{z}/\lambda)$$

so T properly has scaling dimension 2.

13.

$$\langle\prod_{n=1}^N e^{ipX(z,\bar{z})}\rangle = \int \mathcal{D}X e^{-\frac{1}{2\pi\ell_s^2}\int d^2z\partial X\bar{\partial}X + i\int d^2z\,X(z)\sum_i p_i\delta^2(z-z_i)}$$

$$= 2\pi\delta(\Sigma p_i)e^{-\frac{1}{2}\int d^2\sigma d^2\sigma' J(\sigma)J(\sigma')G(\sigma,\sigma')} = 2\pi\delta(\Sigma p_i)e^{-\frac{1}{2}\sum_{i,j=1}^N p_i p_j\langle X(z_i)X(z_j)\rangle}$$

Appreciate both the UV divergence (from coincident points in the correlator) and the IR divergence (from the correlator going as a logarithm) will cancel (think Kosterlitz-Thouless/Mermin Wagner stuff here):

$$\mu^{2\frac{\ell_s^2}{4}(\Sigma p_i)^2}\epsilon^{2\frac{\ell^2}{4}\sum p_i^2}$$

Momentum conservation removes the IR, and if we normal-order the vertex operators within the product we will not get the UV divergence.

14. By explicit calculation:

$$T(z)[(\partial X)^4](w) \sim \frac{-3\alpha(\partial X)^2(w)}{(z-w)^4} + \frac{4(\partial X)^4}{(z-w)^2} + \dots$$
$$T(z)[(\partial^2 X)^2](w) \sim \frac{-2\alpha}{(z-w)^6} + \frac{4(\partial X \partial^2 X)(w)}{(z-w)^3} + \frac{4(\partial^2 X)^2(w)}{(z-w)^2}$$
$$T(z)[\partial^3 X \partial X](w) \sim \frac{-3\alpha}{(z-w)^6} + \frac{6(\partial X)^2(w)}{(z-w)^4} + \frac{6(\partial X \partial^2 X)(w)}{(z-w)^3} + \frac{6(\partial^2 X)^2(w)}{(z-w)^2} + \dots$$

where $+\dots$ are terms that are $O((z-w)^{-1})$ or higher powers, which will not affect the non-primary terms. We see that the combination:

$$(\partial X)^4 + \frac{\alpha}{2}\partial^3 X \partial X - \frac{3}{4\alpha}(\partial^2 X)^2$$

gives a primary operator of dimension 4. Along the way I noticed that there are no primary operators of dimension 2 or 3 that are finite sums of products of derivatives of ∂X.

I can't help but think that this might have *something* to do with the Schwarzian.

15. We look at:

$$: i\frac{\sqrt{2}}{\ell_s}\partial X(z) :: e^{ipX(w)} : = i\frac{\sqrt{2}}{\ell_s}\sum_{n=0}^{\infty} : \partial X(z) :: (X(w))^n : \frac{(ip)^n}{n!} + \text{finite}$$
$$= i\frac{\sqrt{2}}{\ell_s}\sum_{n=0}^{\infty}(-)\frac{\ell_s^2}{2}\frac{n}{z-w}\frac{(ip)^2 : X(w)^{n-1} :}{n!} = \frac{\ell_s p}{\sqrt{2}}\frac{1}{z-w}V_p(w) + \text{finite}$$

16. Directly:

$$\sum_{n,m}\frac{(ia)^n(ib)^m}{n!\,m!} : X^n(z) :: X^m(w) :$$

First lets look at when $n = m$ and say we contract everything. Then we need to contract all n $X(z)$ with all n $X(w)$. There are $n!$ ways to do this, and each produces a factor of $-\frac{\ell^2}{2}\log|z-w|^2$. The diagonal components thus give the sum:

$$\sum_n \frac{1}{n!}\left(\frac{ab\ell_s^2}{2}\log|z-w|^2\right)^n = |z-w|^{ab\ell_s^2/2}$$

Now a more general term, say $: X(z)^n :: X(w)^m :$ where we want to contract $k < n, m$ of them we must choose k $X(z)$ and $kX(w)$ to contract the $X(z)$ with and then figure out the order to contract those k amongst themselves ($k!$), so we have $\binom{n}{k} \times \binom{m}{k} \times k! = \frac{n!m!}{(m-k)!(n-k)!k!}$ ways to do this. The contraction again gives the $\log^k$ term as before, and now we have a remaining factor of $\frac{(ia)^{n-k}(ib)^{m-k}}{(n-k)!(m-k)!} : X(z)^{n-k} :: X(w)^{m-k} :$. For each k-contracted set which gives the $\log^k$ term, we should therefore multiply it by:

$$\sum_{m,n=k}^{\infty}\frac{(ia)^{n-k}(ib)^{m-k}}{(n-k)!(m-k)!} : X(z)^{n-k} :: X(w)^{m-k} := e^{iaX(z)+ibX(w)}$$

So the OPE is:

$$: e^{iaX(z)} :: e^{iaX(w)} := |z-w|^{ab\ell^2/2} e^{iaX(z)+ibX(w)}$$

17. Directly:

$$\partial_z J(z)\partial_w J(w) = \partial_z\partial_w\left(\frac{1}{(z-w)^2}\right) = -\frac{6}{(w-z)^4} + \text{finite}$$

We have no $\frac{2}{(z-w)^2}$ term, as would otherwise be required

18. The stress energy tensor is:

$$T(z) = -\frac{1}{2} :\psi(z)\partial\psi(z): \Rightarrow T(z)\psi(w) = -\frac{1}{2}\psi(z)(\frac{-1}{(z-w)^2}) + \frac{1}{2}\frac{\partial\psi(z)}{z-w} = \frac{1}{2}\frac{1}{(z-w)^2}\psi(w) + \frac{\partial\psi(w)}{(z-w)}$$

so this shows that ψ is primary with weight 1/2.

19. I'll instead have the notation $w = g \circ f(z)$. For $T(z) = (f')^2 T(f) + \{f, z\}$ consider $h = g \circ f$. Then we have:

$$T(z) = (f')^2 T(f) + C(f) = (f')^2((g')^2 T(g \circ f) + C(g \circ f)) + C(f) = (h')^2 T(h) + (f')^2 C(g) + C(f)$$

So we get the desired cocycle property:

$$C(h) = (f')^2 C(g) + C(f)$$

Now, we need $C(f)$ to have units of $[z]^{-2}$. The most naive guess is to let $C(h) = h''$, but this gives:

$$h'' = (f')^2 g'' + f'' g'$$

If that last factor of g' were not there, we would be done. Instead we must think more deeply. We also need the Schwarzian to include a term linear in the third derivative, and the only such term is a constant times f'''/f'. Let us look at how this transforms:

$$\frac{h'''}{h'} = (f')^2 \frac{g'''}{g'} + 3\frac{f'' g''}{g'} + \frac{f'''}{f'}$$

Now what stops us is the cross-term. The only terms that we can add to f'''/f' that involve less than third derivatives in ϵ are f'', $(f')^2$ $(f''/f')^2$.

There is one last term we could have built out of terms of order $\leqslant 3$ that would give units of $[z]^{-2}$: $(f'''/f'')^2$, however in the limit of an infinitesimal transformation $z + \epsilon(z)$, this would give $(\epsilon'''/\epsilon'')^2$ which is nonlinear in ϵ, so this term cannot contribute. .

$(h')^2 = (f'g')^2$ has none of the properties we'd like, and adding it would break the term that $(f')^2$ multiplies being proportional to $C(g)$. Similarly, adding f'' would break the term that $(f')^2$ *doesn't* multiply being proportional to $C(f)$. What is left is $\left(\frac{f''}{f'}\right)$. This transforms as:

$$\left(\frac{h''}{h'}\right)^2 = (f')^2 \left(\frac{g''}{g'}\right)^2 + \left(\frac{f''}{f'}\right)^2 + \frac{2f''g''}{g'}$$

The cross term is exactly of the form of the cross term in f'''/f', and so by appropriately subtracting:

$$\frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 = (f')^2 \left(\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right) + \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

Another way to do this is to first look at the general nth derivative of the global conformal transformations (the Möbius transformations). Note that:

$$g = \frac{az+b}{cz+d}, \quad g'(z) = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2} \quad \Rightarrow \partial_z^n g = \frac{n!(-c)^{n-1}}{(cz+d)^{n+1}}$$

In particular:

$$g''(z) = \frac{-2c}{(cz+d)^3}, \quad g'''(z) = \frac{6c^2}{(cz+d)^4}$$

The simplest combination of g', g'', and g''' that can give zero is:

$$(g'')^2 - \frac{2}{3} g'''(z) g'(z)$$

We want this to have units of $[g]/[z]^2$ and to behave as $\epsilon'''(z)$ to leading order when $g = z + \epsilon(z)$. The only way to do this (which fixes overall normalization and all) is to divide through by $-2/3(g'(z))^2$ and get:

$$\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2.$$

It is easy to check that this satisfies the cocycle property for composition, namely:

$$\{z_3, z_1\} = \left(\frac{\partial z_2}{\partial z_1}\right)^2 \{z_3, z_2\} + \{z_2, z_1\} \tag{42}$$

Since for $h = g \circ f$ we get:

$$\frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 = \frac{f'''}{f'} + 3\frac{f''f'g''}{f'g'} + \frac{(f')^3 g'''}{f'g'} - \frac{3}{2}\left(\frac{f''g' + (f')^2 g''}{f'g'}\right)^2 = \{f, z\} + (f')^2\frac{g''}{g'} - \frac{3}{2}\frac{g''}{g'} = \{f, z\} + (f')^2\{g, f(z)\}$$

20. I will use shorthand $\binom{z'}{z}$ for $\frac{\partial z'}{\partial z}$ and $\binom{z'}{zz}$ for $\frac{\partial z'}{\partial^2 z}$, also I will just write $\Gamma^z_{zz}, g_{z\bar{z}}, g^{z\bar{z}}$ as Γ, g, g^{-1} respectively. Now

$$\Gamma = g^{-1}\partial g \quad \Rightarrow \quad \Gamma' = g^{-1\prime}\partial' g' = g^{-1}\partial\left(\binom{z}{z'} g\right) = \binom{z}{z'}\Gamma - \binom{z}{z'}^2\binom{z'}{zz}$$

So

$$(\Gamma')^2 = \binom{z}{z'}^2 \Gamma^2 - 2\Gamma\binom{z}{z'}^3\binom{z'}{zz} + \binom{z}{z'}^4\binom{z'}{zz}^2$$

$$\partial'\Gamma' = \binom{z}{z'}\partial\left[\binom{z}{z'}\Gamma - \binom{z}{z'}^2\binom{z'}{zz}\right] = \binom{z}{z'}^2\partial\Gamma - \Gamma\binom{z}{z'}^3\binom{z'}{zz} + 2\binom{z}{z'}^2\binom{z'}{zz}^2 - \binom{z}{z'}^3\binom{z'}{zzz}$$

To cancel out the Γ term we look at $2\partial\Gamma - \Gamma^2$. We see this transforms as:

$$2\partial'\Gamma' - \Gamma'^2 = \binom{z}{z'}^2(2\partial\Gamma - \Gamma^2) + 3\binom{z}{z'}^4\binom{z'}{zz}^2 - 2\binom{z}{z'}^3\binom{z'}{zzz} = \binom{z}{z'}^2(2\partial\Gamma - \Gamma^2 - 2\{z', z\})$$

So that

$$T_{zz} - \frac{c}{24}(2\partial\Gamma - \Gamma^2) = \binom{z'}{z}^2(T_{z'z'} - \frac{c}{24}(2\partial'\Gamma' - \Gamma'^2)) + \frac{c}{12}\{z', z\} - \frac{c}{24}2\{z', z\} = \binom{z'}{z}^2(T_{z'z'} - \frac{c}{24}(2\partial'\Gamma' - \Gamma'^2))$$

So indeed $\hat{T}_{zz} = T_{zz} - \frac{c}{24}(2\partial\Gamma - \Gamma^2)$ transforms as a tensor.

21. We have:

$$-\bar{\nabla}T_{z\bar{z}} = \nabla\hat{T}_{zz} = g^{z\bar{z}}\bar{\partial}\hat{T} = -\frac{c}{24}g^{z\bar{z}}\bar{\partial}[2\partial(g^{-1}\partial g) - (g^{-1}\partial g)^2] = -\frac{c}{24}g^{z\bar{z}}\left[2\partial\bar{\partial}(g^{-1}\partial g) - 2(g^{-1}\partial g)\bar{\partial}(g^{-1}\partial g)\right]$$

We can recognize this as:

$$\frac{c}{24}2g^{z\bar{z}}(\partial R_{\bar{z}z} - \Gamma^z_{zz}R_{\bar{z}z}) = \frac{c}{24}2g^{z\bar{z}}\nabla_z R_{\bar{z}\bar{z}} = \frac{c}{24}\nabla_z R = \frac{c}{24}\partial R = -\frac{A}{2}\partial R$$

so we have $A = -c/12$

22. The first part of the action is truly invariant. Let us look at how R changes under Weyl rescaling:

$$-2e^{-\chi}g^{-1}\bar{\partial}(e^{\chi}g^{-1}\partial(e^{\chi}g)) = e^{-\chi}(R - 2g^{-1}\partial\bar{\partial}\chi) = e^{-\chi}(R - 2\partial\bar{\partial}\chi)$$

Consequently: $\sqrt{-g}R \rightarrow \sqrt{-g}(R - 2\nabla^2\chi)$
So the action part will transform as:

$$S_L(g_{\alpha\beta}e^{\chi}, \phi) = S_L(g_{\alpha\beta}, \phi) - \frac{1}{48\pi}\int d^2\xi\sqrt{g}\phi\nabla^2\chi = S_L(g_{\alpha\beta}, \phi) + \frac{1}{24\pi}\int d^2\xi\sqrt{g}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\chi$$

23. The most general variation of the effective action is:

$$\delta \log Z = -\frac{1}{4\pi}\int_\Sigma d^2\xi\sqrt{g}(a_1 R + a_2)\delta\phi - \frac{1}{4\pi}\int_{\partial\Sigma} d\xi(a_3 + a_4 K + a_5 n^a \nabla_a)\delta\phi \tag{43}$$

The counterterms that we can introduce are:

$$\int_\Sigma d^2\xi\sqrt{g}b_1 + \int_{\partial\Sigma} d\xi(b_2 + kb_3) \tag{44}$$

and the variation of the counterterm action:

$$\int_\Sigma d^2\xi\sqrt{g}b_1\delta\omega + \frac{1}{2}\int_{\partial\Sigma} d\xi(b_2 + b_3 n^a\partial_a)\delta\omega$$

So we can use this to set $a_2, a_3, a_5 = 0$. Further, we know the bulk integral's variation is in fact:

$$\delta \log Z = -\delta S_{eff} = \frac{1}{4\pi}\int d^2\xi\sqrt{g}\,T_{\alpha\beta}\delta g^{\alpha\beta} = -\frac{1}{4\pi}\int d^2\xi\sqrt{g}\,T^\alpha_\alpha\delta\phi = \frac{c}{48\pi}\int d^2\xi\,\sqrt{g}R\delta\phi \Rightarrow a_1 = -\frac{c}{12}$$

Now let's start with a flat metric and do two changes:

$$\begin{aligned}\delta_{\phi_1}\delta_{\phi_2}\log Z &= -\frac{c}{24\pi}\int d^2\xi\sqrt{g}\,\delta\phi_2\nabla^2\delta\phi_1 - \frac{a_4}{4\pi}\int d\xi\sqrt{g}\,\delta\phi_2 n^a\partial_a\delta\phi_1\\ &= \frac{c}{24\pi}\int d^2\xi\sqrt{g}\,\partial^a\delta\phi_2\partial_a\delta\phi_1 + \left(\frac{c}{24\pi} - \frac{a_4}{4\pi}\right)\int d\xi\sqrt{g}\,\delta\phi_2 n^a\partial_a\delta\phi_1\end{aligned}$$

Note that the second term is *not* symmetric under $\delta_{\phi_1} \leftrightarrow \delta_{\phi_2}$, and so we must have the counterterm $\frac{a_4}{4\pi} = \frac{c}{24\pi}$. A variation of this argument can be used to show that c is truly a constant, independent of any worldsheet coordinates.

24. Take the map $\frac{L}{2\pi}\log z$, mapping the plane to the cylinder of circumference L. We get:

$$T^{cyl} = (\partial z')^{-2}(T^{plane} - \{z', z\}) = \left(\frac{2\pi}{L}\right)^2 z^2\left(T^{plane} - \frac{c}{12}\frac{1}{2z^2}\left(\frac{L}{2\pi}\right)^2\right) = \left(\frac{2\pi z}{L}\right)^2 T^{plane} - \frac{c}{24}$$

So the zero mode of T^{cyl} is modified. By T^{cyl} has the expansion $\sum_n L_n e^{-2\pi i n x}$ so we see L_0 gets modified by $-\frac{c}{24}$.

Because L_0 is a codimension 1 operator, it will get modified the same way, whether on the cylinder or torus.

25. Each raising operator L_{-n} acts by raising the level by n, and so assuming each one gives a unique state not expressible in terms of the action of the other L_{-k}, we get that it will contribute:

$$1 + q^n + q^{2n} + \cdots = \frac{1}{1-q^n}$$

to the partition function. All together these give

$$\frac{1}{\prod_{n=1}^\infty(1-q^n)} \Rightarrow \mathrm{Tr}[e^{2\pi i\tau(\Delta - c/24)}] = \frac{q^{\Delta - c/24}}{\prod_{n=1}^\infty(1-q^n)}.$$

This also shows that at level n there will generically be as many states as there are partitions of the number n.

26. Consider a nontrivial state $|h\rangle$ so that $L_n|h\rangle = 0$ for some n sufficiently positive. Then:

$$0 = \langle h| L_{-n}L_n |h\rangle = \langle h| (\frac{n(n^2-1)}{12}c - 2nh)|h\rangle$$

If $c = 0$ we get a contradiction unless either $|0\rangle$ is null (and thus decouples) or otherwise $h = 0$, and so we get a vacuum state.

I think we need to add the assumption of irreducibility to have a unique ground state (ie a counterexample would be TQFTs with multiple ground states).

27. It is enough to show that L_1 and L_2 acting on this state give zero, since then all other L_n can be obtained by commutators of these two. Indeed:

$$L_1(L_{-2} - \frac{3}{4}L_{-1}^2)\,|1/2\rangle = (3L_{-1} - \frac{3}{4}(2L_0L_{-1} + 2L_{-1}L_0))\,|1/2\rangle = (3L_{-1} - \frac{3}{4}(2L_{-1} + 4L_{-1}L_0))\,|1/2\rangle = 0$$

$$L_2(L_{-2} - \frac{3}{4}L_{-1}^2)\,|1/2\rangle = (4L_0 + \frac{2(2^2-1)}{12}c - \frac{3}{4}3(L_{-1}L_1 + L_1L_{-1}))\,|1/2\rangle = (4L_0 + \frac{1}{4} - \frac{9}{2}L_0))\,|1/2\rangle = 0$$

28. The null state's field must satisfy:

$$(\mathcal{L}_{-2} - \frac{3}{4}\mathcal{L}_{-1}^2)\,\langle\psi_w \prod_i \psi_{w_i}\rangle = \left[\sum_i \left(\frac{1/2}{(w_i - w)^2} - \frac{1}{w_i - w}\partial_i\right) - \frac{3}{4}\frac{\partial^2}{\partial^2 w}\right]\langle\psi_w \prod_i \psi_{w_i}\rangle \tag{45}$$

For the three-point function (holomorphic sector) this gives:

$$\left[\frac{1/2}{(w-w_1)^2} + \frac{1}{w-w_i}\partial_1 + \frac{1/2}{(w-w_2)^2} + \frac{1}{w-w_i}\partial_2 - \frac{3}{4}\partial_w^2\right]\frac{\lambda}{(w-w_1)^{1/2}(w_1-w_2)^{1/2}(w_2-w)^{1/2}} = 0$$

This gives:

$$\frac{7\lambda}{16}\frac{(w_1-w_2)^{3/2}}{(w-w_1)^{5/2}(w_2-w)^{5/2}} = 0$$

which gives $\lambda = 0$. We could have inferred this from fermion parity.

Next, for the four-point function, first note that all the ψ have the same scaling dimension, so WLOG we can write this as:

$$\langle\psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle = \frac{1}{z_{12}z_{34}}h\left(\frac{z_{12}z_{34}}{z_{13}z_{24}}\right)$$

plugging this into (45) gives a complicated-looking differential equation, but this can be simplified substantially by taking $z_1 = z, z_2 = 0, z_3 = \infty, z_4 = 0$. Notice then that z here is indeed the cross ratio. We then get the simpler differential equation:

$$2zg(z) + 2(1-z^2)g'(z) - 3z(1-z)^2g''(z) = 0$$

This can be solved in terms of known functions (we should more specifically give boundary conditions by specifying residues of $g(z)$ at $z = 0, 1, \infty$). All in all we get:

$$g(z) = \frac{z^2 - z + 1}{1 - z}$$

Thus

$$\langle\psi(z)\psi(z_1)\psi(z_2)\psi(z_3)\rangle = \frac{1}{z_{12}z_{34}} + \frac{1}{z_{14}z_{23}} - \frac{1}{z_{13}z_{24}}$$

exactly as we would get for Wick contraction.

29. Assume it is not primary - then it is a descendant. By positivity of scaling dimensions, it must be a descendant of a field of scaling dimension 0, but as we have shown two exercises ago, the only such field is the vacuum $|0\rangle$. The vacuum is translation invariant $\partial_z \mathbf{1} = 0$ and so it has no descendants of scaling dimension 1. (It *does* have T as a descendant of scaling dimension 2 under $\partial_z^2 \mathbf{1}$).

30. Assume $z > w$. On one hand,

$$:[J^a(z), J^b(w)] := J^a(z)J^b(w) = \sum_{m,n}[J_m^a, J_n^b]z^{-m-1}w^{-n-1}$$

On the other

$$J^a(z)J^b(z) = \frac{G^{ab}}{(z-w)^2} + \frac{if^{ab}_c J^c(w)}{z-w} + \ldots = \sum_m mG^{ab}z^{-2}\left(\frac{w}{z}\right)^{m-1} + \sum_{m,n} if^{ab}_c J^c_m w^{-m-1}z^{-1}\left(\frac{w}{z}\right)^n$$
$$= \sum_m mG^{ab}z^{-m-1}w^{m-1} + \sum_{m,n} if^{ab}_c J^c_m w^{-(m-n)-1}z^{-n-1}$$
$$= \sum_{m,n}\left(m\delta_{m+n}G^{ab}w^{-m-1}w^{-n-1} + if^{ab}_c J^c_{m+n}w^{-m-1}z^{-n-1}\right)$$

so we get:

$$[J^a_m, J^b_n] = m\delta_{m+n}G^{ab} + if^{ab}_c J^c_{m+n}$$

31. Rewrite the first part of the action as $-\frac{1}{4\lambda^2}\int d^2\xi\, \mathrm{Tr}[(g^{-1}\partial g)^2]$. Now note:

$$\delta(g^{-1}\partial g) = g^{-1}\partial\delta g - g^{-1}\,\delta g\, g^{-1}\partial g$$

Then we can write the variation of the action as:

$$-\frac{1}{2\lambda^2}\int d^2\xi \mathrm{Tr}\left[(g^{-1}\partial_\mu\delta g - g^{-1}\,\delta g\, g^{-1}\partial_\mu g)(g^{-1}\partial^\mu g)\right] = \frac{1}{2\lambda^2}\int d^2\xi \mathrm{Tr}\left[\delta g\left(\partial_\mu(g^{-1}\partial^\mu g\, g^{-1}) + \underbrace{g^{-1}\partial_\mu g g^{-1}\partial^\mu g g^{-1}}_{g^{-1}\partial_\mu g\,\partial^\mu(g^{-1})}\right)\right]$$
$$= \frac{1}{2\lambda^2}\int d^2\xi \mathrm{Tr}\left[g^{-1}\delta g\,\partial^\mu\left(g^{-1}\partial_\mu g\right)\right]$$

So we see that we must have $g^{-1}\partial_\mu g$ be a conserved current if we only had the first part of the action. In $z, \bar z$ cooredinates we have $\partial J^z + \bar\partial J^{\bar z} = 0$. We would like both $J = J^z$ and $\bar J = J^{\bar z}$ to be separately conserved $\bar\partial J = \partial \bar J = 0$. However, this is equivalent to also having $\varepsilon^{\mu\nu}J_\nu$ conserved. However $\partial_\mu J_\nu - \partial_\nu J_\mu = -[J_\mu, J_\nu]$ gives that $\partial_\mu \epsilon^{\mu\nu}J_\nu = -\epsilon^{\mu\nu}J_\mu J_\nu \neq 0$ for nonabelian algebras.

On the other hand, the second term has variation:

$$\frac{ik}{8\pi}\int_B d^3\xi\,\varepsilon_{\alpha\beta\gamma}\mathrm{Tr}\left[(g^{-1}\partial^\alpha\delta g - g^{-1}\delta g g^{-1}\partial^\alpha g)(g^{-1}\partial^\beta g)(g^{-1}\partial^\gamma g)\right] + \text{ perms.}$$

this will all vanish identically as an action on B, since $\mathrm{Tr}(A\wedge A\wedge A)$ is already closed for our 1-form $A = g^{-1}\mathrm{d}g$. On the other hand, the first term in parenthesis contributes a boundary term when α is transverse

$$\frac{ik}{8\pi}\int_{\partial B} d^2\xi\,\varepsilon_{\beta\gamma}\mathrm{Tr}(g^{-1}\delta g g^{-1}\partial^\beta g(g^{-1}\partial^\gamma g)) = -\frac{ik}{8\pi}\int_{\partial B} d^2\xi\,\varepsilon_{\beta\gamma}\mathrm{Tr}\left[g^{-1}\delta g\,\partial^\beta\left(g^{-1}\partial^\gamma g\right)\right]$$

Appreciate the difference between this and the factor of 3 *in Di Francesco. I believe we only account for 1 of the 3 terms, since only 1 of the 3 indices will give a transverse direction.*

This gives a total equation of motion of:

$$\frac{1}{2\lambda^2}\partial^\mu(g^{-1}\partial_\mu g) - \frac{ik}{8\pi}\varepsilon_{\mu\nu}\partial^\mu(g^{-1}\partial^\nu g) = 0 \tag{46}$$

Taking the basis $z, \bar z$, $\partial^z = 2\partial_{\bar z}$, $\varepsilon_{z\bar z} = i/2$, we get:

$$[\partial_{\bar z}(g^{-1}\partial_z g) + \partial_z(g^{-1}\partial_{\bar z}g)] - \frac{ik\lambda^2}{4\pi}\left[i\partial_{\bar z}(g^{-1}\partial_z g)g^{-1} - i\partial_z(g^{-1}\partial_{\bar z}g)\right] = \left(1+\frac{k\lambda^2}{4\pi}\right)\partial_{\bar z}(g^{-1}\partial_z g) + \left(1-\frac{k\lambda^2}{4\pi}\right)\partial_z(g^{-1}\partial_{\bar z}g)$$

When $\lambda^2 = 4\pi/k$ (meaning k must be positive) the second term goes away and we get the conservation law $\bar\partial J_z$. Taking the conjugate of this equation gives the other conservation law.

$$\bar d(g^{-1}\partial g) = 0 \to -\partial(\bar\partial g\, g^{-1}) = 0$$

In particular the classical solutions factorize into the form $g(z,\bar z) = f(z)\bar f(\bar z)$. It is also quick to show that $g(z,\bar z) \to \Omega(z)g(z,\bar z)\bar\Omega(\bar z)$ (for $\Omega, \bar\Omega$ two independent matrices valued in the same rep'n of G) keeps the action invariant, and so we see that the $G\times G$ classical invariance of the action is *enhanced* to a full $G(z)\times G(\bar z)$ invariance. This is the real power of WZW models, and should be appreciated.

32. Importantly, the 3D action does not have any metric dependence. For the 2D boundary we have:

$$\frac{1}{4\lambda^2}\int d^2\xi\sqrt{g}g^{\mu\nu}\mathrm{Tr}[g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g]$$

this gives a stress tensor:

$$T_{\mu\nu} = -\frac{\pi}{\lambda^2}\left(\mathrm{Tr}[g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g] - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\mathrm{Tr}[g^{-1}\partial_\alpha g\, g^{-1}\partial_\beta g]\right)$$

we see that this is traceless. The holomorphic part is:

$$-\frac{\pi}{\lambda^2}\mathrm{Tr}[J^2] = \frac{k}{4}J^a J^a$$

the constant out front can have a field strength renormalization from its classical value (because the J are not free fields), and so we would not expect it to agree with the one given in the definition of T.

Give another reason for this discrepancy. Try to account for it.

33. This one is direct. Take $z > w$

$$[J^a(z), R_i(w)] = \sum_m J^a_m z^{-m-1}R_i(w) = \sum_n z^{-1}\left(\frac{w}{z}\right)^n T_{ij}R_j(w)$$

and so we get:

$$J^a_m R_i(w) = w^m T^a_{ij}R_j(w)$$

34. We have:

$$\frac{1}{2(k+\bar{h})}\sum_{n,m} z^{-2-(n+m)} : J^a_m J^a_n := \sum_k L_m z^{-2+m}$$

Appropriately shifting, we see that $L_m = \frac{1}{2(k+\bar{h})} : J_{m+n}J_{-n} :$ as required. The only term here that doesn't give zero when acting on a WZW primary is $J^a_{-1}J^a_0$ which acts as $J^a_{-1}T^a_{ij}$ and this terms appears twice, so we get that

$$|\chi_i\rangle = (L_{-1}\delta_{ij} - \frac{1}{k+\bar{h}}T^a_{ij}J^a_{-1})\,|R_j\rangle$$

is null. But we also have that:

$$\begin{aligned}
\langle (J^a_{-1}R(z_1))R(z_2)\ldots R(z_N)\rangle &= \frac{1}{2\pi i}\oint_{z_1}\frac{dz}{z-z_1}J^a(z)R(z_1)R(z_2)\ldots R(z_N) \\
&= -\frac{1}{2\pi i}\sum_{i\neq 1}\oint_{z_i}\frac{dz}{z-z_1}R(z_1)R(z_2)\ldots J^a(z)R(z_k)\ldots R(z_N) \\
&= -\frac{1}{2\pi i}\sum_{k\neq 1}\oint_{z_k}\frac{dz}{z-z_1}\frac{1}{z-z_k}R(z_1)R(z_2)\ldots T^a_{ij}R_j(z_k)\ldots R(z_N) \\
&= \sum_{k\neq 1}\frac{T^a_{ij}R_j(z_k)}{z_1-z_k}
\end{aligned}$$

Here we chose to do this with $R(z_1)$, but we could have picked arbitrary z_i. This means that correlators must satisfy:

$$\left(\partial_{z_1} - \frac{1}{k+\bar{h}}\sum_{j\neq i}^{N}\frac{T^a_i\otimes T^a_j}{z_i-z_j}\right)\langle\prod_{k=1}^{N}R(z_k)\rangle = 0$$

where T^a_i acts on the ith primary field in the correlator.

35. I think its instructive to do this one out in detail. First let's take a look at just $T_G(z)$ acting on any current $J^a(w)$. We want the singular terms:

$$\begin{aligned}
\frac{1}{2(k+\bar{h})}(J^bJ^b)(z)J^a(w) &= \frac{1}{2(k+\bar{h})}\frac{1}{2\pi i}\oint_z \frac{dx}{x-z}(J^b(x)J^b(z)J^a(w) + J^b(x)J^b(z)J^a(w)) \\
&= \frac{1}{2(k+\bar{h})}\frac{1}{2\pi i}\oint_z \frac{dx}{x-z}\left[\left(\frac{G^{ba}}{(x-w)^2} + \frac{if^{ba}_c J^c(w)}{x-w}\right)J^b(z) + J^b(x)(z \leftrightarrow x)\right] \\
&= \frac{1}{k+\bar{h}}\left(\frac{G^{ab}J^b(z)}{(z-w)^2} + \frac{1}{2}f_{abc}\frac{J^c(w)J^b(z)+J^b(z)J^c(w)}{z-w}\right)
\end{aligned}$$

but note that last term will have

$$J^c(w)J^b(z) + J^b(z)J^c(w) = \frac{2G^{bc}}{(z-w)^2} + \frac{2if_{cbd}J^d(w)}{w-z} + (J^cJ^b)(w) + (J^bJ^c)(w)$$

The first term will cancel when multiplied by the anti-symmetric f_{abc}, as will the last (regular) term. The second term will give $f_{abc}f_{cbd} = -f_{abc}f_{dbc} = 2\bar{h}\delta_{ad}$, by the definition of dual coxeter number. On the other hand we have $G_{ab} = k\delta_{ab}$ so altogether we get:

$$\frac{1}{k+\bar{h}}\frac{(k+\bar{h})J^a(z)}{(z-w)^2} = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}$$

as we wished. *Note* we could have run this logic in reverse, and demanded that a stress tensor must its OPE make second term involving ∂J^a have coefficient 1, giving the required normalization of $(2(k+\bar{h}))^{-1}$. Now note that if we define $T^H(z) := \frac{1}{2(k+\bar{h}_H)}\sum_{a\in H}(J^aJ^a)(z)$, then as long as we are taking the OPE with J^a for $a \in H$, we see that the singular terms are *exactly* the same. Indeed, we get the same factor of $k\delta_{ab}$ from the quadratic OPE term, and the sum over $f_{abc}f_{dbc}$ restricts b and c to be in H by the subgroup property, so we get $\bar{h}_H$. Thus $(T_G - T_H)J^a = T_{G/H}J^a$ is regular for $a \in H$.

For the next step, again lets first just look at the singular terms in the T_GT_G OPE:

$$\begin{aligned}
T(z)T(w) &= \frac{1}{2(k+\bar{h})}\frac{1}{2\pi i}\oint \frac{dx}{x-w}T(z)J^a(x)J^a(w) \\
&= \frac{1}{2(k+\bar{h})}\frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\left(\frac{J^a(x)}{(z-x)^2} + \frac{\partial J^a(x)}{z-x}\right)J^a(w) + J^a(x)(w \leftrightarrow x)\right] \\
&= \frac{1}{2(k+\bar{h})}\frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\frac{k\dim G}{(z-x)^2(x-w)^2} + \frac{\partial J^a(x)\,J^a(w)}{z-x} + (w \leftrightarrow x)\right] \\
&= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}
\end{aligned}$$

here we have $c = \frac{k\dim G}{k+\bar{h}}$ as required. The same logic applies to the $T_H(z)T_H(w)$ OPE, where we would get:

$$\frac{c_H/2}{(z-w)^4} + \frac{2T_H(w)}{(z-w)^2} + \frac{\partial T_H(w)}{(z-w)}, \quad c_H = \frac{k\dim H}{k+\bar{h}_H}$$

Now it remains to evaluate:

$$\begin{aligned}
T_G(z)T_H(w) &= \frac{1}{2(k+\bar{h}_H)}\frac{1}{2\pi i}\oint \frac{dx}{x-w}\sum_{a\in H}T_G(z)J^a(x)J^a(w) \\
&= \frac{1}{2(k+\bar{h}_H)}\frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\left(\frac{J^a(x)}{(z-x)^2} + \frac{\partial J^a(x)}{z-x}\right)J^a(w) + J^a(x)(w \leftrightarrow x)\right] \\
&= \frac{1}{2(k+\bar{h}_H)}\frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\frac{k\dim H}{(z-x)^2(x-w)^2} + \frac{\partial J^a(x)\,J^a(w)}{z-x} + (w \leftrightarrow x)\right] \\
&= \frac{c_H/2}{(z-w)^4} + \frac{2T_H(w)}{(z-w)^2} + \frac{\partial T_H(w)}{(z-w)}
\end{aligned}$$

so indeed $T_G(z)T_H(w) - T_H(z)T_H(w) = T_{G/H}(z)T_H(w)$ has a regular OPE. This further gives us that $T_{G/H}(z)T_{G/H}(w)$ has singular part coming from $T_{G/H}(z)T_G(w) = T_G(z)T_G(w) - T_G(z)T_H(w)$, which gives:

$$\frac{(c_G - c_H)/2}{(z-w)^4} + \frac{2T_{G/H}(w)}{(z-w)^2} + \frac{\partial T_{G/H}(w)}{z-w}$$

So a G theory can be re-written as a set of "decoupled" CFTs with stress tensors T_H and $T_{G/H}$. Now take $G = \mathrm{SU}(2)_m \times \mathrm{SU}(2)_1$. This theory have total level $m+1$. So now take the diagonal subgroup $\mathrm{SU}(2)_{m+1}$. We see that the G/H theory has central charge:

$$c_G - c_H = \left(\frac{m \times 3}{m+2} + \frac{1 \times 3}{1+2}\right) - \frac{(m+1)\times 3}{m+1+2} = 1 + \frac{3m}{m+2} - \frac{3(m+1)}{m+3} = 1 - \frac{6}{(m+2)(m+3)}$$

exactly coincident with the prescribed formula for the minimal models. So, we expect at $m=1$ to get the Ising CFT.

36. We have

$$\begin{aligned}\psi^i(z) = \sum_n \psi^i_n z^{-n-1/2} \Rightarrow \langle \psi^i(z)\psi^j(w)\rangle &= \sum_{n,m\in\mathbb{Z}} \langle \psi^i_n \psi^j_m\rangle z^{-n-1/2} w^{-m-1/2} \\ &= \sum_{m=0}^{\infty} \langle \psi^i_m \psi^j_{-m}\rangle z^{-m-1/2} w^{m-1/2} \\ &= \frac{\delta^i}{\sqrt{zw}}\left[\sum_{m=0}^{\infty}\left(\frac{w}{z}\right)^m - \frac{1}{2}\right] \\ &= \frac{\delta_{ij}}{2\sqrt{zw}}\frac{z+w}{z-w}\end{aligned}$$

the 1/2 comes from the zero-mode Clifford algebra $\{\psi^i_0, \psi^j_0\} = \delta^{ij}$.

37. We can get this directly from the Ward identity:

$$\langle T(z_1)\phi(z_2)\phi(z_3)\rangle = \left(\frac{\partial_{z_2}}{z_1 - z_2} + \frac{\partial_{z_3}}{z_1 - z_3} + \frac{\Delta}{(z_1-z_2)^2} + \frac{\Delta}{(z_1-z_3)^2}\right)\frac{1}{(z_2-z_3)^{2\Delta}} = \frac{\Delta}{z_{12}^2 z_{13}^2 z_{23}^{2\Delta-2}}.$$

Next, we can write:

$$\langle X|\, T(z)\, |X\rangle = \lim_{w\to 0} \bar{w}^{-2\Delta}\, \langle 0|\, X(1/\bar{w})\, T(z)\, X(0)\, |0\rangle = \lim_{w\to 0}\frac{\bar{w}^{-2\Delta}\,\Delta}{z^2 \bar{w}^{-2\Delta}} = \frac{\Delta}{z^2}.$$

Finally, let's look at the $O(N)$ fermion. We have that $T(z) = -\frac{1}{2}\sum_{i=1}^N :\psi^i\partial\psi^i:$ so we get:

$$\langle S|\, T\, |S\rangle = -\frac{1}{2}\Sigma_{i=1}^N \lim_{z\to w}\left[\partial_w\left(\frac{z+w}{2\sqrt{zw}}\frac{1}{z-w}\right) - \underbrace{\partial_w \frac{1}{(z-w)}}_{\text{Normal ordering constant}}\right] = -\frac{N}{2}\left(-\frac{1}{8w^2}\right) = \frac{N/16}{w^2}$$

as required.

38. This is direct:

$$D_\theta \hat{X} = (\partial_\theta + \theta\partial_z)\left(X + i\theta\psi + i\bar{\theta}\bar{\psi} + \theta\bar{\theta}F\right) = i\psi + \theta\partial X + \bar{\theta}F + \theta\bar{\theta}\partial\bar{\psi}, \quad \bar{D}_{\bar{\theta}}\hat{X} = i\bar{\psi} + \bar{\theta}\partial X + \theta F + \bar{\theta}\theta\bar{\partial}\psi$$

Now we only want the $\theta\bar{\theta}$ terms of $(D_\theta\hat{X})(\bar{D}_{\bar{\theta}}\hat{X})$ as everything else will vanish in the Berezin integral. This gives:

$$S = \frac{1}{2\pi\ell_s^2}\int d^2z \int d\bar{\theta}d\theta\, \theta\bar{\theta}(\partial X\partial X - F^2 + i\bar{\psi}\partial\bar{\psi} + i\psi\bar{\partial}\psi) = \frac{1}{2\pi\ell_s^2}\int d^2z(\partial X\partial X + i\bar{\psi}\partial\bar{\psi} + i\psi\bar{\partial}\psi)$$

we have dropped F^2 because it has no dynamics or interactions with X, ψ whatsoever.

39. Expanding

$$e^{ip\cdot\hat{X}} = e^{ip_\mu(X^\mu + i\theta\psi^\mu + i\bar{\theta}\bar{\psi}^\mu + \theta\bar{\theta}F^\mu)} = (1 + i\theta p\cdot\psi)(1 + i\bar{\theta}p\cdot\bar{\psi})(1 + \theta\bar{\theta}p\cdot F)e^{ipX}$$

Imposting EOM's gives $F = 0$ right away. Now for the rest:

$$D_\theta \hat{X}^\mu\, D_{\bar{\theta}}\hat{X}^\nu e^{ipX}|_{\theta\bar{\theta}} = [(\partial X^\mu \partial X^\nu + i\bar{\psi}^\mu\partial\bar{\psi}^\nu + i\psi^\nu\bar{\partial}\psi^\mu) + (i\partial X^\nu\psi^\mu)p\cdot\psi + (i\partial X^\mu\bar{\psi}^\nu)p\cdot\bar{\psi}]e^{ipX}$$

again using the equations of motion we get rid of the $\partial\bar{\psi}, \bar{d}\psi$ terms. Now we get:

$$[\partial X^\mu\partial X^\nu + (i\partial X^\nu\psi^\mu)p\cdot\psi + (i\partial X^\mu\bar{\psi}^\nu)p\cdot\bar{\psi}]e^{ip\cdot X} = (\partial X^\mu + i(p\cdot\psi)\psi^\mu)(\partial X^\nu + i(p\cdot\psi)\psi^\nu)e^{ip\cdot X}$$

40. Following the same logic as the $\mathcal{N} = (2,0)$ case, we can now compute in the R sector:

$$\{G_0^\alpha, \bar{G}_0^\beta\} = \frac{4k}{2}\left(-\frac{1}{4}\right)\delta^{\alpha\beta} + 2L_0\delta^{\alpha\beta}$$

for this to be positive we need:

$$2(\Delta - k/4) \geqslant 0 \Rightarrow \Delta \geqslant k/4.$$

In the NS sector, we have a positivity condition on

$$\{G_{-1/2}^\alpha, \bar{G}_{1/2}^\beta\} = -2\sigma^a_{\alpha\beta}J_0^a + 2\delta^{\alpha\beta}L_0$$

The positivity condition on this operator translates to the matrix:

$$2\Delta\mathbf{1} - 2\sigma^a_{\alpha\beta}J^a$$

being positive semidefinite. But the determinant of this matrix is given by

$$\Delta^2 - |J|^2 = \Delta^2 - j^2$$

So for this to be $\geqslant 0$, given that $\Delta \geqslant 0$, we need $\Delta - j \geqslant 0$

41. This calculation is also direct:

$$\begin{aligned}
T(z)T(w) &= \frac{1/2}{(z-w)^4} + 2\frac{-\frac{1}{\ell_s^2}(\partial X)^2(w)}{(z-w)^2} + \frac{\partial\left(-\frac{1}{\ell_s^2}(\partial X)^2(w)\right)}{z-w} \\
&\quad + \frac{\ell_s}{2}\partial X(z)\frac{Q}{\sqrt{2}\ell_s^3}2\frac{2}{(z-w)^3} + \frac{\ell_s}{2}\partial X(w)\frac{Q}{\sqrt{2}\ell_s^3}2\frac{-2}{(z-w)^3} - \frac{\ell_s^2}{2}\frac{Q^2}{2\ell_s^2}\frac{-6}{(z-w)^4} \\
&= \frac{1/2(1+3Q^2)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}
\end{aligned}$$

we thus get a central charge equal to $1 + 3Q^2$ as required.

42. The integral over the zero mode will give no contribution from the $\partial X\bar{\partial}\bar{X}$ term in the action and instead will just:

$$\int \mathcal{D}X \exp\left(-\int d^2z\sqrt{g}\left(\frac{Q}{4\pi\ell_s\sqrt{2}}R^{(2)} - i\sum_i p_i\delta^2(z-z_i)\right)X(z)\right)$$

This is a δ-functional on the p_i. We have:

$$\boldsymbol{\delta}\left[\frac{Q}{4\pi\ell_s\sqrt{2}}R^{(2)} - i\sum_i p_i\delta^2(z-z_i)\right]$$

but this can only happen if, after integrating over z, we get:

$$\frac{Q}{\ell_s\sqrt{2}}\chi = i\sum_i p_i \Rightarrow i\sqrt{2}\ell_s\sum_i p_i = Q\chi.$$

Give an interpretation of the vertex operators as "contributing curvature".

43. Note that:

$$[L_{-m}, J_{-n}] = nJ_{-m-n} + \frac{A}{2}m(m-1)\delta_{m+n}$$

Take $(m+n) = 0$ and look at the central term. We see that we cannot simply identify $L_m^\dagger = L_{-m}, J_n^\dagger = J_{-n}$ because then the commutation relation above has a central term $-mA$ off from the correct central term. This can be corrected by redefining *just* the zero mode $J_0^\dagger = J_0 + A$. We see that then the hermitian conjugates satisfy the same algebra. We see sufficiency. Is this necessary?

We now show that we cannot change the algebra in any other way and keep the commutation relations. Firstly, we cannot add a (necessarily zero weight) central term to any other $J_m^\dagger = J_{-m}$ relation since only J_0 transforms with 0 weight. In fact we cannot form any linear combination $\tilde{J}_m$ of the J_m and expect the commutation relations to hold, since each J_m has different eigenvalue under L_0. We can thus only rescale the J_m- and applying $[L_1, J_m]$ shows that to keep the commutation relation, this rescaling must be the same for all J_m - but this would necessarily modify the central term by changing the charge A.

The same logic applies to the $L_m^\dagger$. We cannot mix L_m for different m to define $L_m^\dagger$ since they have different weight under L_0. There is also no consistent way to rescale all of them and keep the commutation relations the same. The only possibility is adding a central term to the relation $L_0^\dagger = L_0$, but any redefinition of this will modify the central charge of the conjugate theory.

44. Noting that

$$\begin{aligned}
b(z)\partial c(w) = c(z)\partial b(w) &= \frac{1}{(z-w)^2}\\
\partial b(z) c(w) = \partial c(z) b(w) &= -\frac{1}{(z-w)^2}\\
\partial b(z)\partial c(w) = \partial c(z)\partial b(w) &= -\frac{2}{(z-w)^3}
\end{aligned}$$

we can just directly compute the TT OPE:

$$\begin{aligned}
T(z)T(w) &= (-\lambda b(z)\partial c(z) + (1-\lambda)\partial b(z)c(z))\,(-\lambda b(w)\partial c(w) + (1-\lambda)\partial b(w)c(w))\\
&= \lambda^2(b\partial c)(z)(b\partial c)(w) + \lambda(\lambda-1)\left[(b\partial c)(z)(\partial bc)(w) + (\partial bc)(z)(b\partial c)(w)\right] + (1-\lambda)^2(\partial bc)(z)(\partial bc)(w)\\
&= -\frac{\lambda^2 + (1-\lambda)^2 + 4\lambda(\lambda-1)}{(z-w)^4} + \frac{\lambda^2(-b(z)\partial c(w) + \partial c(z)b(w))}{(z-w)^2} + \frac{(1-\lambda)^2(\partial b(z)c(w) - c(z)\partial b(w))}{(z-w)^2}\\
&\quad + \lambda(\lambda-1)\frac{\partial c(z)\partial b(w) + \partial b(z)\partial c(w)}{z-w} - 2\lambda(\lambda-1)\frac{b(z)c(w) + c(z)b(w)}{(z-w)^3}
\end{aligned}$$

The first term on the last line will die since we can take $z \to w$ and ignore first-order terms capturing the differences. The second term in the last line will become:

$$-2\lambda(\lambda-1)\frac{\partial b(w)c(w) + \partial c(w)b(w)}{(z-w)^2} - \lambda(\lambda-1)\frac{\partial^2 b(w)c(w) + \partial^2 c(w)b(w)}{(z-w)} \tag{47}$$

the second two terms in the first line contribute a $(z-w)^{-2}$ term of:

$$\lambda^2(2\partial c(w)b(w)) + (1-\lambda)^2(2\partial b(w)c(w))$$

this will combine with the $(z-w)^{-2}$ terms in (47) to give:

$$2\left[\lambda\partial c(w)b(w) + (1-\lambda)\partial b(w)c(w)\right] = 2T(w)$$

as required. Finally, the $(z-w)^{-1}$ terms all collected give coefficient (dropping the w dependence, as it is understood):

$$\begin{aligned}
&\lambda^2(-\partial b\partial c + \partial^2 cb)) + (1-\lambda)^2(\partial^2 bc - \partial c\partial b) - \lambda(\lambda-1)(\partial^2 bc + \partial^2 cb)\\
&= -\lambda^2\cancel{(\partial b\partial c + \partial c\partial b)} - 2\lambda\partial b\partial bc + 1\partial b\partial c + [\lambda^2 + \lambda(1-\lambda)](\partial^2 cb) + [(1-\lambda)^2 + \lambda(1-\lambda)](\partial^2 bc)\\
&= \lambda\partial^2 cb + (1-\lambda)\partial^2 bc + (1-2\lambda)\partial b\partial c = \partial T
\end{aligned}$$

as required. So altogether we get exactly the stress tensor OPE needed to satisfy the Virasoro algebra with central charge:

$$-2(\lambda^2 + (1-\lambda)^2 + 4\lambda(\lambda-1)) = -2(6\lambda^2 - 6\lambda + 1) = 1 - 3Q^2, \quad Q = (1-2\lambda)$$

45. The BRST current is:

$$j_B(z) = c(z)T^X(z) + (bc\partial c)(z)$$

There are several OPEs to do. Let's start with the easier ones:

$$\begin{aligned}
(cT^X)(cT^X) &\sim c(z)c(w)\left[\frac{c^X/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right] \\
&= -\sum_{n=1}^{\infty} \frac{(z-w)^n}{n!} c(w)\partial^n c(w)\left[\frac{c^X/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right] \\
&\sim -\frac{\frac{1}{2}c^X\, c(w)\partial c(w)}{(z-w)^3} - \frac{1}{2}\frac{\frac{1}{2}c^X\, c(w)\partial^2 c(w)}{(z-w)^2} - \frac{1}{6}\frac{\frac{1}{2}c^X\, c(w)\partial^3 c(w)}{z-w} - \frac{2T(w)c(w)\partial c(w)}{z-w}
\end{aligned} \tag{48}$$

Next:

$$\begin{aligned}
(cT^X)(bc\partial c) + (bc\partial c)(cT^X) &\sim \frac{T^X(z)c(w)\partial(w)}{z-w} + \frac{c(z)\partial c(z)T^X(w)}{z-w} \\
&\sim \frac{2T(w)c(w)\partial c(w)}{z-w}
\end{aligned} \tag{49}$$

This exactly cancels the last term in the previous expression. Now the hard one. Being careful of fermion minus signs, I'll underline the contractions that will give them:

$$\begin{aligned}
(bc\partial c)(bc\partial c) = {} & (bc\partial c)(bc\partial c) + \underline{(bc\partial c)(bc\partial c)} + \underline{(bc\partial c)(bc\partial c)} + (bc\partial c)(bc\partial c) \\
& + (bc\partial c)(bc\partial c) + (bc\partial c)(bc\partial c) + \cancel{\underline{(bc\partial c)(bc\partial c)}} + \cancel{\underline{(bc\partial c)(bc\partial c)}}
\end{aligned} \tag{50}$$

the last two terms are canceled because they contribute only $(z-w)^{-1}$ singularities multiplying $c(z)\partial c(w)$ which is $O(z-w)$ and so only contributes finite terms. The remaining terms give:

$$-\frac{c(z)c(w)}{(z-w)^4} + \frac{c(z)\partial c(w)}{(z-w)^3} - \frac{\partial c(z)c(w)}{(z-w)^3} + \frac{\partial c(z)\partial c(w)}{(z-w)^2} + \frac{c(z)\partial c(z)b(w)c(w)}{(z-w)^2} + \frac{b(z)c(z)c(w)\partial c(w)}{(z-w)^2} \tag{51}$$

The last two terms will cancel, as they contribute a $(z-w)^{-1}$ singularity with numerator $c\partial^2 cbc + \partial c\partial cbc + b\partial cc\partial c + \partial bcc\partial c$. All of these terms are evaluated at w, so all are zero. Now we have (all evaluated at w)

$$\begin{aligned}
& \frac{-\partial cc + c\partial c - \partial cc}{(z-w)^3} + \frac{-\frac{1}{2}\partial^2 cc + \partial c\partial c - \partial^2 cc + \partial c\partial c}{(z-w)^2} + \frac{-\frac{1}{6}\partial^3 cc + \frac{1}{2}\partial^2 c\partial c - \frac{1}{2}\partial^3 cc + \partial^2 c\partial c}{z-w} \\
&= \frac{3c(w)\partial c(w)}{(z-w)^3} + \frac{\frac{3}{2}c(w)\partial^2 c(w)}{(z-w)^2} + \frac{\frac{2}{3}c(w)\partial^3 c(w) + \frac{3}{2}\partial^2 c(w)\partial c(w)}{z-w}
\end{aligned} \tag{52}$$

Combining Equations (48), (49) and (52) we get:

$$j_B(z)j_B(w) = \frac{(3 - \frac{1}{2}c^X)c(w)\partial c(w)}{(z-w)^3} + \frac{(\frac{3}{2} - \frac{1}{4}c^X)c(w)\partial^2 c(w)}{(z-w)^2} + \frac{(\frac{2}{3} - \frac{c^X}{12})c(w)\partial^3 c(w) + \frac{3}{2}\partial^2 c(w)\partial c(w)}{z-w} \tag{53}$$

Now for $Q_B^2 = 0$, we need to look at $j_B(z)j_B(w)$ residue as $z \to w$ as a function of w and ensure that this has no residue in w. First we just need to look at the $(z-w)^{-1}$ term and reduce it all to the integral:

$$Q_B^2 = \frac{1}{2\pi i}\oint dw\left[\left(\frac{2}{3} - \frac{c^X}{12}\right)c(w)\partial^3 c(w) + \frac{3}{2}\partial^2 c(w)\partial c(w)\right] = \frac{1}{2\pi i}\oint dw\left(\frac{13}{6} - \frac{c^X}{12}\right)c(w)\partial^3 c(w)$$

This will vanish exactly when $c^X = 26$ as required.

NB in Polchinski, there is an additional $c\partial^3 c$ term in the definition of j_B that contributes to this OPE (which makes Equation (53) look nicer), but the conclusion about $D = 26$ is still the same.

46. This is the type of question with a two-line answer that depends on a lot of conceptual build up. It is instructive to go through some of the details. Here I will set $\ell_s^2 = 2$. First lets start with the system of two Majorana-Weyl fermions ψ^1, ψ^2. This has central charge $c = 1$. Moreover, we can compute everything in terms of

$$\psi = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2), \quad \bar{\psi} = \frac{1}{\sqrt{2}}(\psi^1 - i\psi^2).$$

Note both $\psi(z)$ and $\bar{\psi}(z)$ are in the holomorphic sector. The anti-holomorphic fields, if we considered them, can be labeled as in polchinski by $\tilde{\psi}(\bar{z}), \tilde{\bar{\psi}}(\bar{z})$. These fields give OPE:

$$\psi(z)\psi(w) = O(z-w), \quad \bar{\psi}(z)\bar{\psi}(w) = O(z-w) \quad \psi(z)\bar{\psi}(w) = \frac{1}{z-w} + :\psi\bar{\psi}:(w) + O(z-w) \tag{54}$$

Now $J(z) =: \psi\bar{\psi} : (z)$ can be seen to have scaling dimension 1 by OPE, so it is a conserved current (and necessarily a primary operator in a unitary theory). Indeed $JJ = (z-w)^{-2}$ and $J\psi = \psi(z-w)^{-1}, J\bar{\psi} = \bar{\psi}(z-w)^{-1}$ so $\psi, \bar{\psi}$ have charge ± 1 under J. From extending equation (54) to terms of order $(z-w)$ the stress energy tensor $T = -\frac{1}{2} : \psi^i \partial \psi^i := \frac{1}{2} J^2$.

Now note that this shares everything in common with the free scalar theory. The central charge $c = 1$. The $\mathfrak{u}(1)$ currents there are $J = i\partial\phi$ and have the same OPE. The analogues of the fermions $\psi, \bar{\psi}$ are then the operators $e^{\pm i\phi(z)}$ respectively. Indeed these have charge ± 1 under J. But it would be surprising if these operators anti-commuted, being built out of bosons and all. In fact they do! By Baker-Campbell-Hausdorff:

$$e^{i\phi(z)} e^{i\phi(z')} = e^{-[\phi(z),\phi(z')]} e^{i\phi(z')} e^{i\phi(z)} = -e^{i\phi(z')} e^{i\phi(z)}$$

since $[\phi(z), \phi(w)] = -\log \frac{z-w}{w-z} = -i\pi$ The anti-commutation property comes out of the non-locality of the vertex operators in terms of ϕ. We can make the exact same argument for $e^{i\phi(z)} e^{-i\phi(w)}$ or any combination thereof. So all of these fields are in fact fermionic. They have the same OPEs as the fermions above:

$$e^{\pm i\phi(z)} e^{\pm i\phi(w)} = O(z-w), \quad e^{\pm i\phi(z)} e^{\mp i\phi(w)} \sim \frac{1}{z-w}$$

Note also the OPE

$$\begin{aligned}
:e^{i\phi(z)}::e^{i\phi(w)}: &= \exp\left[-\int dz' dw' \log(z'-w') \delta_{\phi(z')} \delta_{\phi(w')}\right] : e^{i\phi(z)} e^{-i\phi(w)} : \\
&= \frac{1}{z-w}(1 + i\partial\phi(w)(z-w) + O(z-w)^2) \\
&= \frac{1}{z-w} + i\partial\phi(w) + O(z-w)
\end{aligned}$$

as required.

We can actually perform this procedure to the bc ghosts as well, for any value of λ. The trick is to note that we have performed it for $\lambda = 1/2$, and now the stress-energy tensor changes to:

$$T^\lambda = T^{\lambda=1/2} - (\lambda - 1/2)\partial(:bc:)$$

If we still take $b = e^{i\phi}, c = e^{-i\phi}$ then $:bc := i\partial\phi$ and so the stress-energy tensor looks like:

$$T^\lambda = -\frac{1}{2}(\partial\phi)^2 - i(\lambda - 1/2)\partial^2\phi$$

which is just the Coloumb gas model with $Q = -i(2\lambda - 1)$. The central charge is $1 + 3Q^2 = 1 - 3(2\lambda - 1)^2$, exactly as we want. The conformal weights are $k^2/2 \pm iQk/2 \to \frac{1}{2} \pm (\lambda - 1/2)$ at the lowest level, and this is exactly λ and $1 - \lambda$ as desired. Note that b and c are hermitian, so we need ϕ to be anti-hermitian. Equivalently we can write $\phi = i\rho$ for ρ hermitian. Then

$$b = e^{-\rho}, \quad c = e^{\rho}, \qquad J = -\partial\rho.$$

Note ρ has opposite OPE from ϕ so that $\partial\rho(z)\,\partial\rho(w) \sim \frac{1}{(z-w)^2}$.

Now lets look at the *bosonic* $\beta\gamma$ theory. Can we bosonize this too? For one, the charge is $J = -\beta\gamma$ which has opposite sign OPE $J(z)J(w) = -\frac{1}{z-w}$, so we will now need ρ to have the regular-sign OPE (ie the same as ϕ). We'll just call this hermitian field ϕ. Let's take $\beta = e^{-\phi}, c = e^{\phi}$ as before and $J = -\partial\phi$. Already there is an issue. If ϕ satisfies the standard OPE then β and γ will be anticommute. Further, $\beta\gamma = e^{-\phi(w)}e^{\phi(z)} = O(z-w)$ while by the same logic $\beta\beta \sim \gamma\gamma \sim (z-w)^{-1}$. We want $\beta\beta = O((z-w)^0), \gamma\gamma = O((z-w)^0), \beta\gamma \sim -(z-w)^{-1}, \gamma\beta \sim (z-w)^{-1}$.

Another way to see that we are missing something: we can try to write a Coulomb gas model for the $\beta\gamma$ theory:

$$T^{\lambda} = T^{\lambda=1/2} - (\lambda - 1/2)\partial(\beta\gamma) = -\frac{1}{2}J^2 - \left(\frac{1}{2} - \lambda\right)\partial J = -\frac{1}{2}(\partial\phi)^2 + \frac{1-2\lambda}{2}\partial^2\phi$$

notice the $-$ sign in front of $\frac{1}{2}J^2$, as we want. We have a coulomb gas model with $Q = 1 - 2\lambda$. This gives a central charge $1 + 3Q^2 = 4 - 6\lambda + 12\lambda^2$. On the other hand, the $\beta\gamma$ theory should have central charge $-1 + 3Q^2$. We are off by 2.

All of this indicates that we need to add an uncoupled $c = -2$ including fermions–namely the bc fermi theory at $\lambda = 1$– and redefine $\beta\gamma$ in terms of ϕ to incorporate this. Take η, ξ of scaling dimensions $1, 0$ and charges ∓ 1 respectively. Then define

$$\beta = e^{-\phi}\partial\xi, \quad \gamma e^{\phi}\eta.$$

We now have the OPE:

$$\beta(z)\gamma(w) = (z-w) \times -\frac{1}{(z-w)^2} = -\frac{1}{z-w}, \quad \gamma(z)\beta(w) = \frac{1}{z-w}$$

This is **4.15.2**. Further because $\eta\eta = O(z-w)$ and $\partial\xi\partial\xi = O(z-w)$ we get $\beta\beta = O((z-w)^0)$ and likewise for $\gamma\gamma$ as needed. We also know how to interpolate between NS and R sectors by taking $\phi \to \phi/2$ etc.

The total current $-:\beta\gamma:$ stays the same because we look for the constant term in the expansion:

$$\beta(z)\gamma(w) = -\frac{1}{(z-w)^2}e^{-\phi(z)}e^{\phi(w)} = -\frac{1}{(z-w)^2}\left((z-w) - \partial\phi(w)(z-w)^2\right) \to \partial\phi(w) \Rightarrow J = -\partial\phi(w)$$

so we identify $:\beta\gamma:$ with $\partial\phi$, which are both $-J$. This is **14.15.10**. Writing out the full stress tensor now gives:

$$-\frac{1}{2}(\partial\phi)^2 + \frac{1-2\lambda}{2}\partial^2\phi - \eta\partial\xi = T^{\lambda=1/2} + (1/2-\lambda)\partial(\beta\gamma)$$

It remains to show that $T^{\lambda=1/2} = -\frac{1}{2}\beta\partial\gamma + \frac{1}{2}\partial\beta\gamma = \frac{1}{2}(2\partial\beta\,\gamma - \partial(\beta\gamma))$. Now looking at the $\beta\gamma$ OPE to order $z-w$ we get:

$$\begin{aligned} e^{-\phi(z)}\partial\xi(z)e^{\phi(w)}\eta(w) &= \partial\xi(z)\eta(w)e^{-\phi(z)}e^{\phi(w)} \\ &= \left(\frac{-1}{(z-w)^2} + :\partial\xi\eta:\right)\left((z-w) - (z-w)^2\partial\phi + \frac{1}{2}(z-w)^3((\partial\phi)^2 - \partial^2\phi)\right) \end{aligned}$$

NOTE I had to assume that while ξ, η and $e^{\phi}, e^{-\phi}$ separately anticommute with their partners, the $e^{\pm\phi}$ fields commute with the ξ, η fields. Give an interpretation/example in condensed matter of this.

The order $z-w$ term here is:

$$:\partial\xi\eta: -\frac{1}{2}((\partial\phi)^2 - \partial^2\phi)$$

So this is the normal ordered product of $\partial\beta\,\gamma$. The $\partial(\beta\gamma) = \partial^2\phi$ term will cancel the $\partial^2\phi$ term there and we'll get the stress tensor

$$-\eta\partial\xi - \frac{1}{2}(\partial\phi)^2 = T^{\lambda=1/2}$$

which is **4.15.8** as desired.

We can also bosonize the η, ξ theory in terms of an auxiliary bosonic field χ, but this was not necessary for the exercise.

47. We are looking at DN boundary conditions. Let us do this directly from definitions:

$$X(\tau,\sigma) = x - \sqrt{2}\ell_s \sum_{k\in\mathbb{Z}+1/2} \frac{\alpha_k}{k} e^{-ik\tau}\sin(k\sigma) = x + i\frac{\ell_s}{\sqrt{2}} \sum_{k\in\mathbb{Z}+1/2} \frac{\alpha_k}{k}(z^{-k}-\bar{z}^{-k})$$

$$\Rightarrow \langle X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{\ell_s^2}{2} \sum_{k,l\in\mathbb{Z}+1/2} \frac{\alpha_k\alpha_l}{kl}(z^{-k}-\bar{z}^{-k})(w^{-l}-\bar{w}^{-l})$$

$$= \frac{\ell_s^2}{2}\sum_{k=0}^{\infty}\frac{1}{k+1/2}\left[\left(\frac{w}{z}\right)^{k+1/2} - \left(\frac{\bar{w}}{z}\right)^{k+1/2} - \left(\frac{w}{\bar{z}}\right)^{k+1/2} + \left(\frac{\bar{w}}{\bar{z}}\right)^{k+1/2}\right]$$

Now we have

$$\sum_{k=0}^{\infty}\frac{x^{k+1/2}}{k+1/2} = 2\sum_{k=0}^{\infty}\frac{(\sqrt{x})^{2k+1}}{2k+1} = 2\operatorname{arctanh}(\sqrt{x}) = -(\log(1-\sqrt{x}) - \log(1+\sqrt{x})).$$

Our convention on the square root branch cut is along the negative real axis. We get:

$$-\frac{\ell_s^2}{2}\left[\log(1-\sqrt{w/z}) - \log(1+\sqrt{w/z}) - \log(1-\sqrt{\bar{w}/z}) + \log(1-\sqrt{\bar{w}/z}) + c.c.\right]$$

so the final result gives us:

$$-\frac{\ell_s^2}{2}\left[\log|1-\sqrt{w/z}|^2 - \log|1+\sqrt{w/z}|^2 - \log|1-\sqrt{\bar{w}/z}|^2 + \log|1+\sqrt{\bar{w}/z}|^2\right].$$

We can simplify this to:

$$-\frac{\ell_s^2}{2}\left[\log|\frac{\sqrt{z}-\sqrt{w}}{\sqrt{z}+\sqrt{w}}|^2 - \log|\frac{\sqrt{z}-\sqrt{\bar{w}}}{\sqrt{z}+\sqrt{\bar{w}}}|^2 + \log|\sqrt{z}+\sqrt{\bar{w}}|^2\right].$$

For ND boundary conditions, the $-$ between the two logs becomes a $+$.

Interpret this in terms of image charges

48. Firstly, $\partial X\bar{\partial}X$ requires no normal ordering constant to be added ordinarily, since it has a wick contraction of zero. Now to go from the plane from the disk we have $x = \frac{z-i}{z+i}$. Vice versa is $z = i\frac{1+x}{1-x}$. This gives

$$\log|z-w|^2 = \log|x-y|^2 + \log|\frac{2}{(1-x)(1-y)}|^2$$
$$\log|z-\bar{w}|^2 = \log|1-x\bar{y}|^2 + \log|\frac{2}{(1-x)(1-\bar{y})}|^2$$

So for NN and DD boundary conditions we get:

$$\langle X_{NN}(x,\bar{x})X_{NN}(y,\bar{y})\rangle = -\frac{\ell_s^2}{2}\left(\log|x-y|^2 + \log|1-x\bar{y}|^2 - 2\log|(1-x)(1-y)|^2 + 4\log 2\right)$$
$$\langle X_{DD}(x,\bar{x})X_{DD}(y,\bar{y})\rangle = -\frac{\ell_s^2}{2}\left(\log|x-y|^2 - \log|1-x\bar{y}|^2\right).$$

So NN boundary conditions correspond to putting an image charge of the same sign at $1/x^*$ while DD boundary conditions correspond to putting an image charge of opposite sign at $1/x^*$ as well as a *neutralizing* charge of the opposite sign at 1–corresponding to ∞ in the $\mathbb{H}$ setting. **Interpret this**.

Differentiating the above with $\partial_x\bar{\partial}_y$ shows that in either case only the $\log(1-x\bar{y})$ term contributes:

$$\langle\partial X_{NN}(x)\bar{\partial}X_{NN}(\bar{y})\rangle = \frac{\ell_s^2}{2}\frac{1}{(1-x\bar{y})^2}$$
$$\langle\partial X_{DD}(x)\bar{\partial}X_{DD}(\bar{y})\rangle = -\frac{\ell_s^2}{2}\frac{1}{(1-x\bar{y})^2}.$$

This will become singular only as z approaches the boundary of the unit circle. We encounter the divergence $\pm\frac{\ell_s^2}{2}\frac{1}{(1-x\bar{y})^2}$ in the NN and DD cases respectively and so we can define

$$ {}^\star_\star\partial X(z)\bar{\partial}X(\bar{w})^\star_\star = \partial X(z)\bar{\partial}X(\bar{w}) \mp \frac{\ell_s^2}{2}\frac{1}{(1-z\bar{w})^2} $$

On the other hand for $\partial X\,\partial X$ we get the normal ordering constant:

$$ {}^\star_\star\partial X(z)\partial X(w)^\star_\star = \partial X(z)\partial X(w) + \frac{\ell_s^2}{2}\frac{1}{(z-w)^2} $$

We have $\bar{X}(1/\bar{w}) = \pm X(w)$ so consequently $\partial X(w) = \pm\bar{\partial}_{1/\bar{w}}X(1/\bar{w})$. Now its a quick check (being careful to keep subscripts on $\bar{\partial}$ so we know what we're differentiating w.r.t.):

$$ \begin{aligned} {}^\star_\star\partial X(z)\bar{\partial}_{\bar{w}}X(1/\bar{w})^\star_\star &= \partial X(z)\bar{\partial}_{\bar{w}}X(1/\bar{w}) \mp \frac{\ell_s^2}{2}\frac{1}{(1-z/\bar{w})^2} \\ \Rightarrow {}^\star_\star\partial X(z)\bar{\partial}_{1/\bar{w}}X(1/\bar{w})^\star_\star &= \partial X(z)\bar{\partial}_{1/\bar{w}}X(1/\bar{w}) \mp (-\bar{w}^{-2})\frac{\ell_s^2}{2}\frac{1}{(1-z/\bar{w})^2} \\ \Rightarrow {}^\star_\star\partial X(z)\partial X(w)^\star_\star &= \partial X(z)\partial X(w) + \frac{\ell_s^2}{2}\frac{1}{(z-w)^2} \end{aligned} $$

where the extra minus sign in the Dirichlet boundary condition case removes any sign ambiguity in the last line. Thus, we see that indeed ${}^\star_\star\partial X(z)\partial X(w)^\star_\star = \pm{}^\star_\star\partial X(z)\bar{\partial}X(1/\bar{w})^\star_\star$ for Neumann and Dirichlet boundary conditions respectively.

49. Using the doubling trick we have $\bar{\psi}(\bar{z}) = \psi(z^*)$. So $z_i =$ We can compute the correlator by Wick contraction:

$$ \langle\prod_{i=1}^{m}\psi(z_i)\prod_{j=1}^{2n-m}\bar{\psi}(\bar{z}_j)\rangle = \langle\prod_{i=1}^{n}\psi(w_i)\rangle = \frac{1}{2^n n!}\sum_{\pi\in S_{2n}}\mathrm{sgn}(\pi)\prod_{i=1}^{n}\frac{1}{w_{\pi(2i-1)}-w_{\pi(2i)}} = \mathrm{Pf}\Big[\frac{1}{w_i-w_j}\Big] $$

where $w_i = z_i$ for $z = 1\ldots m$ and $w_{i+m} = z_i^*$ for $z = 1 \to 2n-m$

50. I feel that this has already been done in 2.3.31. Rotating to euclidean signature, the most general solution for X is

$$ X(\tau,\sigma) = x^\mu + \frac{\ell_s^2}{2}(p+\bar{p})\tau + \frac{\ell_s^2}{2}(p-\bar{p})\sigma + i\frac{\ell_s}{\sqrt{2}}\sum_{k\neq 0}\frac{e^{-k\tau}}{k}(\alpha_k e^{-ik\sigma} + \bar{\alpha}_k e^{ik\sigma}) $$

The first boundary condition $\dot{X} = 0$ at $\sigma = 0$ gives:

$$ \alpha_k = -\bar{\alpha}_k, \quad p+\bar{p} = 0 $$

while the second boundary condition $X' = 0$ at $\sigma = \pi$ gives:

$$ \sin(k\pi) = 0 \Rightarrow k \in \mathbb{Z}+1/2 \quad p-\bar{p} = 0 $$

Thus we have neither momentum nor winding-number. So for the mode expansion is:

$$ X(\tau,\sigma) = x - \sqrt{2}\ell_s\sum_{k\in\mathbb{Z}+1/2}\frac{\alpha_k}{k}e^{-k\tau}\sin(k\sigma) = x + i\frac{\ell_s}{\sqrt{2}}\sum_{k\in\mathbb{Z}+1/2}\frac{\alpha_k}{k}(z^{-k}-\bar{z}^{-k}) $$

as desired. This gives:

$$ \partial X = -i\frac{\ell_s}{\sqrt{2}}\sum_{k\in\mathbb{Z}+1/2}\alpha_k z^{-k-1}, \quad \bar{\partial}X = i\frac{\ell_s}{\sqrt{2}}\sum_{k\in\mathbb{Z}+1/2}\alpha_k\bar{z}^{-k-1} $$

51. We have N scalars with $\partial X^i(z) = O^{ij}\bar{\partial}X^j(\bar{z})$ on the real axis. Because the conformal group includes the translation group, O^{ij} must be translationally invariant, ie it cannot depend on z. Further because X^i is a scalar $\partial + \bar{\partial}$ and $\partial - \bar{\partial}$ both act on it in an invariant way. These are the two boundary conditions we can set on each X^i. So we see that O^{ij} can definitely be a diagonal matrix of ± 1s. However, because all the scalars are identical we can also transform $X'^j(z,\bar{z}) = R_i^j X^i(z,\bar{z})$, with R any orthogonal matrix (not just special orthogonal) and still get a valid boundary condition. So O is any orbit of the matrix of ± 1s under the conjugation action of the orthogonal group $O \to P^T O P$. This can be easily appreciated as boundary conditions for an open string along the various coordinate directions being either Neumann or Dirichlet.

Its surprising that O can't vary on the real axis - corresponding to the D-brane changing which X^i live on it. Think about this more.

52. Everything is in the NS sector. Let's first evaluate $\langle \psi_{NN}(z)\psi_{NN}(w)\rangle$. We have

$$\sum_{n,m} \underbrace{\langle 0|\, b_{n+1/2} b_{m+1/2}\, |0\rangle}_{\delta_{n=-m-1}} z^{-n-1} w^{-m-1} = \sum_{n=0}^{\infty} z^{-n-1} = \frac{1}{z-w}$$

For the NS sector we have the following cases:

- NN: $b_{n+1/2} + \bar{b}_{n+1/2} = 0$
- DD: $b_{n+1/2} - \bar{b}_{n+1/2} = 0$
- DN: $b_n + \bar{b}_n = 0$

so we see that $\langle \psi(z)\bar{\psi}(\bar{w})\rangle$ will add an extra minus sign in the NN case. It will not do so in the in the DD case. Collecting our results.

$$\langle \psi_{NN}(z)\psi_{NN}(w)\rangle = \frac{1}{z-w}, \quad \langle \psi_{NN}(z)\bar{\psi}_{NN}(\bar{w})\rangle = -\frac{1}{z-\bar{w}}$$
$$\langle \psi_{DD}(z)\psi_{DD}(w)\rangle = \frac{1}{z-w}, \quad \langle \psi_{DD}(z)\bar{\psi}_{DD}(\bar{w})\rangle = \frac{1}{z-\bar{w}}$$

Lastly, for the DN case, ψ now takes integer values and so:

$$\langle \psi_{DN}(z)\psi_{DN}(w)\rangle = \sum_{n,m} \underbrace{\langle 0|\, b_n b_m\, |0\rangle}_{\delta_{n=-m}} z^{-n-1/2} w^{-m-1/2} = \sum_{n=0}^{\infty} z^{-n-1/2} w^{n-1/2} - \underbrace{\frac{1}{2}}_{\text{zero mode}} z^{-1/2} w^{-1/2} = \frac{z+w}{2\sqrt{zw}(z-w)}.$$

Because $b_n = -\bar{b}_n$ we then also have

$$\langle \psi_{DN}(z)\bar{\psi}_{DN}(\bar{w})\rangle = -\frac{z+\bar{w}}{2\sqrt{z\bar{w}}(z-\bar{w})}.$$

53. On to the R sector.

- NN: $b_n - \bar{b}_n = 0$
- DD: $b_n + \bar{b}_n = 0$
- DN: $b_{n+1/2} - \bar{b}_{n+1/2} = 0$

Let's again evaluate $\langle \psi_{NN}(z)\psi_{NN}(w)\rangle$. The calculation is exactly the same as the DN calculation above. Using the above relations between the b and $\bar{b}$ in the different sectors we'll get:

$$\langle \psi_{NN}(z)\psi_{NN}(w)\rangle = \frac{z+w}{2\sqrt{zw}(z-w)}, \quad \langle \psi_{NN}(z)\bar{\psi}_{NN}(\bar{w})\rangle = \frac{z+\bar{w}}{2\sqrt{z\bar{w}}(z-\bar{w})}$$
$$\langle \psi_{DD}(z)\psi_{DD}(w)\rangle = \frac{z+w}{2\sqrt{zw}(z-w)}, \quad \langle \psi_{DD}(z)\bar{\psi}_{DD}(\bar{w})\rangle = -\frac{z+\bar{w}}{2\sqrt{z\bar{w}}(z-\bar{w})}$$
$$\langle \psi_{DN}(z)\psi_{DN}(w)\rangle = \frac{1}{z-w}, \quad \langle \psi_{DN}(z)\bar{\psi}_{DN}(\bar{w})\rangle = \frac{1}{z-\bar{w}}$$

54. There are several ways to do this. One way is directly by using the identity relating an expectation of an exponential to the exponential of an expectation:

$$\langle e^{iaX(z)}\rangle_{\mathbb{RP}^2} = \langle e^{iaX(z)}e^{-ia\bar{X}(\bar{z})}\rangle_{\mathbb{CP}^1} \propto \exp\left(\frac{a^2}{2}\times 2\langle X(z)\bar{X}(\bar{z})\rangle\right) = \exp\left(-\frac{a^2\ell_s^2}{2}\log(1+z\bar{z})\right) = \frac{1}{(1+|z|^2)^{a^2\ell_s^2/2}}.$$

It is not clear that we haven't omitted a proportionality constant. Another way to compute this is to note that $\langle : X(z,\bar{z})X(z,\bar{z}) :\rangle = -\frac{\ell_s^2}{2}\log|1+z\bar{z}|^2$ and so expanding out:

$$e^{iaX} = \sum_{n=0}^{\infty}\frac{(ia)^n}{n!}\langle X(z,\bar{z})^n\rangle.$$

Now we do wick contractions. For each even term we need to put $2n$ elements in to n pairs. There are $(2n-1)(2n-3)\dots(3)(1)$ ways to do this. Simplifying we get:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(a)^{2n}}{2^n n!}\left(-\frac{\ell_s^2}{2}\right)^n \log^n|1+z\bar{z}|^2 = \exp\left(\log|1+z\bar{z}|^{a^2\ell_s^2/2}\right) = (1+|z|^2)^{a^2\ell_s^2/2}$$

This doesn't look right. If instead we had:

$$e^{iaX(z)}e^{-ia\bar{X}(\bar{z})} = \sum_{n,m=0}^{\infty}\frac{(ia)^n}{n!}\frac{(-ia)^m}{m!}\langle : X(z)^n\bar{X}(\bar{z})^m :\rangle = \sum_n \frac{a^{2n}\cancel{n!}}{\cancel{n!}n!}\left(-\frac{\ell_s^2}{2}\log(1+z\bar{z})\right)^n = \frac{1}{(1+|z|^2)^{a^2\ell_s^2/2}}$$

as required.

In doing this problem, I needed to consider the $e^{iaX}e^{-ia\bar{X}}$ correlator rather than the $e^{ia(X+\bar{X})}$ correlator - otherwise I would get an ill-defined one-point function that blows up as $z\to\infty$ (ie is not a globally-defined differential). Perhaps this comes from boundary conditions in the case of $\mathbb{RP}^2$, since $H_1 = \mathbb{Z}_{\not{2}}$ and so we can enforce anti-periodic boundary conditions that would be consistent with a negative charge vertex operator being placed a $-1/\bar{z}$.

55. For the non-supersymmetric theory, we have the action (on the sphere, with $\sqrt{-g}R^2 = 1$):

$$S = \frac{1}{4\pi\ell_s^2}\int d^2z\sqrt{g}g^{\alpha\beta}\partial_\alpha X\partial_\beta X + \frac{Q}{4\pi\ell_s\sqrt{2}}\int d^2z\sqrt{g}R^{(2)}X = \frac{1}{2\pi\ell_s^2}\int d^2z\partial X\bar{\partial}X + \frac{Q}{4\pi\ell_s\sqrt{2}}\int d^2z\, X$$

this gives a stress-energy tensor:

$$T = -\frac{1}{\ell_s^2}\partial X^2 + \frac{Q}{\ell_s\sqrt{2}}\partial^2 X$$

Now for $\mathcal{N}=1$ we might expect an action of the form:

$$S = \frac{1}{4\pi\ell_s^2}\int d^2z\sqrt{g}g^{\alpha\beta}\partial_\alpha X\partial_\beta X + \frac{Q}{4\pi\ell_s\sqrt{2}}\int d^2z\sqrt{g}R^{(2)}X = \frac{1}{2\pi\ell_s^2}\int d^2z\partial X\bar{\partial}X + \frac{Q}{4\pi\ell_s\sqrt{2}}\int d^2z\, X$$

This gives:

$$T = -\frac{1}{\ell_s^2}\partial X\partial X + \frac{Q}{\ell_s\sqrt{2}}\partial^2 X - \frac{1}{2}\psi\partial\psi, \quad G = i\frac{\sqrt{2}}{\ell_s}\psi\partial X - iQ\partial\psi$$

The TT OPE will give central charge $\frac{3}{2}+3Q^2$. G remains primary, so we'll have $TG = \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}$. Finally, GG will give

$$\frac{1}{(z-w)^3} + \frac{2Q^2}{(z-w)^3} + \cancel{\frac{\frac{\sqrt{2}}{\ell_s}Q\partial X - \frac{\sqrt{2}}{\ell_s}Q\partial X}{(z-w)^2}} + \frac{2T}{z-w}$$

so we get $\hat{c} = 1+2Q^2$ as desired.

Now for $\mathcal{N}=2$, following the same example, we still get get same TT OPE and $G^{\pm}$ remains primary, so we have the $TG^{\pm}$ OPE staying the same. The GG OPE will have $\hat{c} = 1+2Q^2$ as before and J will have to be modified to include $\partial^2 X$ so as to remain primary under T.

56. For X a compact scalar valued in S^1 of radius R we have the solutions $X = 2\pi R(n\sigma_1 + m\sigma_2)$, which have vanishing Laplacian. The action of these instanton solutions is:

$$S = \frac{1}{4\pi\ell_s^2}\int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \frac{1}{\tau_2}|\tau\partial_1 X - \partial_2 X|^2 = \frac{\pi R^2}{\ell_s^2 \tau_2}|n\tau - m|^2$$

Expanding $X = X^{cl} + \chi$, we get no cross-terms in the action. We now do the path integral over the χ with periodic conditions around both cycles. χ separates into the zero mode $\chi_0 + \delta\chi$ and $\delta\chi$ can be expanded in terms of eigenfunctions of the laplacian on periodic functions. These are precisely $e^{2\pi i(m_1\sigma_1+m_2\sigma_2)}$ with eigenvalues $\frac{(2\pi)^2}{\tau_2}|m_1\tau - m_2|^2$. They form an orthonormal basis. The contribution to the action is then

$$\frac{1}{4\pi\ell_s^2}\sum_{m_1,m_2\in\mathbb{Z}^2}\lambda_{m_1m_2}|A_{m_1m_2}|^2$$

The measure on the space of functions comes from the norm of δX

$$||\delta_X||^2 = \frac{1}{\ell_s}\int d^2\sigma\sqrt{g}(d\chi) = \sum_{m_1,m_2}\frac{|A_{m_1m_2}|^2}{\ell_s^2} \Rightarrow \int \mathcal{D}\chi = \int_0^{2\pi R}\frac{d\chi_0}{\ell_s}\int_{-\infty}^{\infty}\prod_{m_1,m_2\neq\{0,0\}}\frac{dA_{m_1,m_2}}{\ell_s}.$$

Note the difference with Kiritsis. This is crucial to get the right factors of 2π in the end. This then gives:

$$\int\mathcal{D}\chi e^{-S(\chi)} = \frac{2\pi R}{\ell_s}\times\prod_{m_1,m_2\in\mathbb{Z}^2_{\geqslant 0}\backslash\{0,0\}}\int_{-\infty}^{\infty}dA_{m,n}\frac{e^{-\frac{\lambda_{m_1m_2}|A_{m_1m_2}|^2}{4\pi\ell_s^2}}}{(2\pi\ell_s)^2} = \frac{2\pi R}{\ell_s}\times{\prod_{m,n}}'\sqrt{\frac{2\pi}{\lambda_{m_1m_2}}} = \frac{2\pi R}{\ell_s}\times(\det{}'\frac{\nabla^2}{2\pi})^{-1/2}$$

Henceforth a primed sum or product means that we omit the origin 0 or $\{0,0\}$ and sum over the integers. It remains to evaluate

$$\prod\sqrt{\frac{2\pi}{\lambda_{n,m}}} = \exp\left(-\frac{1}{2}{\sum_{m,n}}'\log\left(\frac{2\pi}{\tau_2}|m+n\tau|^2\right)\right)$$

Notice that this sum can be obtained by explicitly calculating the Eisenstein series

$$G(s) = \left(\frac{\tau_2}{2\pi}\right)^s{\sum_{m,n}}'\frac{1}{|m+n\tau|^{2s}}$$

and evaluating $\frac{1}{2}G'(0)$. Let's do that. First note:

$${\sum_{m,n}}'\frac{1}{|m+n\tau|^{2s}} = 2\zeta(2s) + {\sum_n}'\sum_m\frac{1}{|m+n\tau|^{2s}}$$

The derivative of $2\zeta(2s)$ at $s = 0$ yields $-2\log(2\pi)$. On the other hand $2\zeta(0)$ is -1, which multiplies the order s factor in the expansion of $\left(\frac{\tau_2}{2\pi}\right)^s$ (none of the subsequent terms will have an $O(s^0)$ term to multiply this). This gives $\log(2\pi/\tau_2)$. Together these contribute

$$-\frac{1}{2}\log(2\pi\tau_2)$$

to $\frac{1}{2}G'(0)$.

Note also because this is a periodic function of τ of period one, we can represent it as a Fourier series in τ

$$\sum_m\frac{1}{|m+n\tau|^{2s}} = \sum_{p\in\mathbb{Z}}e^{2\pi ipn\tau_1}\int_0^1 dte^{-2\pi ipt}\sum_{m\in\mathbb{Z}}\frac{1}{((m+t)^2+n^2\tau_2^2)^s} = \sum_{p\in\mathbb{Z}}e^{2\pi ipn\tau_1}\underbrace{\int_{-\infty}^{\infty}dt\frac{1}{(t^2+n^2\tau_2^2)^s}}_{\text{combine } \int_0^1 \text{ with } \sum_{\mathbb{Z}}}$$

Using a clever Gamma function manipulation (following Di Francesco here):

$$\frac{1}{\Gamma(s)}\sum_p\int_{-\infty}^{\infty}dt\int_0^{\infty}dx\, e^{2\pi ip(n\tau_1-t)}x^{s-1}e^{-x(t^2+n^2\tau_2^2)} = \frac{\sqrt{\pi}}{\Gamma(s)}\sum_p\int_0^{\infty}dx\, x^{s-3/2}\, e^{-xn^2\tau_2^2-\pi^2p^2/x+2\pi ipn\tau_1}.$$

Now at $p=0$ this reduces to

$$\frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}|n\tau_2|^{1-2s}$$

Summing *this* over n gives $2\frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)$. We have explicit series formulae for these at $s=0$. Extracting the first-order term (this is in fact finite at $s=0$) gives $\frac{\pi\tau_2}{3}$.

Now let's evaluate the sum over $p\neq 0$. I'll directly take $s=3/2$ here. We get a sum over an integral that is now solvable:

$$\frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}\sum_{p>0}e^{-2\pi i p n\tau_1}\int_0^\infty x^{-3/2}e^{-xn^2\tau_2-\pi^2p^2/x}=\sqrt{\pi}s\sum_{p>0}\frac{\sqrt{\pi}}{\pi p}(e^{-2\pi i p n(\tau_1+i\tau_2)}+e^{-2\pi i p n(\tau_1-i\tau_2)})$$

We see that the contribution to $G'(0)$ from this will be:

$$\underbrace{2\sum_{n>0}}_{=\sum_n{}'}\sum_p\frac{1}{p}(q+\bar{q})=-2\sum_{n>0}\log(|1-q^n|^2)=-2\log(e^{\frac{\pi\tau_2}{6}}|\eta(\tau)|^2)=-2\log(|\eta(\tau)|^2)-\frac{\pi\tau_2}{3}$$

we see that the $p=0$ term cancels this last part and we are left with $\frac{1}{2}G'(0)=-\log(\sqrt{\tau}_2 2\pi)-\log(|\eta|^2)$, and so:

$$Z(R,\tau)=\frac{R}{\ell_s\sqrt{\tau_2}|\eta(\tau)|^2}\times\sum_{m,n}e^{-\frac{\pi R^2}{\tau_2\ell_s^2}|m-n\tau|^2}.$$

While we're at it, let's simplify this even further by applying Poisson summation. We have the 1D case for the Gaussian:

$$\sum_n e^{-\pi a n^2+\pi b n}=\frac{1}{\sqrt{a}}\sum_{\tilde{n}\in\mathbb{Z}}e^{-\frac{\pi}{a}(n+i\frac{b}{2})^2}.$$

Performing this over the m variable we get

$$\begin{aligned}\sum_{m,n}e^{-\frac{\pi R^2}{\ell_s^2\tau_2}(m^2-m\overbrace{(n\tau+n\bar{\tau})}^{2n\tau_1}+n^2|\tau|^2)}&=\frac{\ell_s\sqrt{\tau_2}}{R}\sum_{\tilde{m},n}e^{-\frac{\pi R^2}{\ell_s^2\tau_2}n^2|\tau|^2}e^{-\frac{\pi\ell_s^2\tau_2}{R^2}\left(\tilde{m}+i\frac{R^2 n\tau_1}{\ell_s\tau_2}\right)^2}\\&=\frac{\ell_s\sqrt{\tau_2}}{R}\sum_{\tilde{m},n}e^{-\pi\frac{R^2}{\ell_s^2}n^2\tau_2-\frac{\pi\ell_s^2}{R^2}\tilde{m}^2\tau_2-2\pi i\tilde{m}n\tau_1}\\&=\frac{\ell_s\sqrt{\tau_2}}{R}\sum_{\tilde{m},n}e^{\pi(i\tau_1-\tau_2)\frac{1}{2}\left(\frac{\ell_s}{R}\tilde{m}+\frac{R}{\ell_s}n\right)^2}e^{\pi(-i\tau_1-\tau_2)\left(\frac{\ell_s}{R}\tilde{m}-\frac{R}{\ell_s}n\right)^2}\\&=\frac{\ell_s\sqrt{\tau_2}}{R}\sum_{\tilde{m},n}q^{\frac{P_L^2}{2}}\bar{q}^{\frac{P_R^2}{2}}\end{aligned}$$

with $P_L=\frac{1}{\sqrt{2}}(m\ell_s/R+nR/\ell_s), P_R=\frac{1}{\sqrt{2}}(m\ell_s/R-nR/\ell_s)$. We then get a simple form for the partition function:

$$Z(R,\tau)=\sum_{\tilde{m},n}\frac{q^{\frac{P_L^2}{2}}\bar{q}^{\frac{P_R^2}{2}}}{|\eta(\tau)|^2}.$$

57. We follow Polchinski Vol 2 on advanced CFT. The following operator product arises when we calculate correlation functions of the energy-momentum tensor:

$$-T\mathcal{O}=-T_z z(z,\bar{z})\,g\int d^2w\,\phi_{\Delta,\Delta}(w,\bar{w})$$

We get:

$$\bar{\partial}_{\bar{z}}T(z,\bar{z})\phi(w,\bar{w})=\bar{\partial}_{\bar{z}}\left[\frac{\Delta}{(z-w)^2}+\frac{\partial_w}{z-w}\right]\phi(w,\bar{w})=(-2\pi\Delta\partial_z\delta(z-w)+2\pi\delta(z-w)\partial_w)\phi(w,\bar{w})$$

Where the last line was obtained using basic delta-function identities. Integrating over w gives:

$$-\bar{\partial}_{\bar{z}} T\mathcal{O} = 2\pi g(\Delta - 1)\partial\phi$$

Thus, unless $\Delta = 1$ we get that T gains an anti-holomorphic part. The exact same equation (with $z \to \bar{z}$) holds for $\bar{T}$. Further, the conservation equation $\bar{d}T_{zz} + \partial T_{z\bar{z}} = 0$ gives us that

$$T_{z\bar{z}} = 2\pi g(1 - \Delta)\phi.$$

There cannot be an overall constant, since this is zero when $\phi = 0$. Here we will *define* $\beta(g)$ by:

$$T_i^i(z, \bar{z}) = -2\pi {\sum_i}' \beta(g)\mathcal{O}_i(z, \bar{z})$$

where the sum runs over operators of dimension $\leqslant d$. The trace is $T_a^a = 2T_{z\bar{z}} = -4\pi g(1-\Delta)\phi$ so under this deformation $\beta = (2 - 2\Delta)g$. We now want to go to second order. The next contribution will come from:

$$-T\frac{1}{2}(\mathcal{O}\mathcal{O})^2 = -T_{zz}(z, \bar{)}\frac{g^2}{2!}\int d^2wd^2w'\phi(w, \bar{w})\phi(w', \bar{w}')$$

Doing an OPE we get to leading order:

$$\phi_{\Delta,\Delta}(w, \bar{w})\phi_{\Delta,\Delta}(w', \bar{w}') \sim \frac{C}{|z - w|^{2\Delta}}\phi_{\Delta,\Delta}(w', \bar{w}')$$

where here C is the coefficient of the $\phi_{\Delta,\Delta}$ 3-point function. We can now preform the w, w' integrals and get:

$$2\pi Cg^2 \int \frac{dr}{r^{2\Delta-1}} \times \int dw'\phi(w', \bar{w}')$$

Assuming $\Delta = 1$ we get a log term that must be regulated in the UV and IR. Regulation in the UV gives a scale that breaks conformal invariance. Rescaling by $1 + \epsilon$ increases the log by ϵ. Equivalently we get

$$\delta g = -2\pi C\epsilon g^2$$

This gives a second-order contribution to the beta function of Cg^2 as required. If the operator is not exactly marginal, the second order term will still have this form, plus higher-order corrections in $\Delta - 1$ and g.

58. Generalizing the preceding analysis to a deformation by a family of marginal operators $g_a\phi^a_{1,1}$, for the deformation to be marginal at second order in g we need the three-point function to satisfy $\lambda^c_{ab}g_ag_b = 0$ so that the second order term does not contribute the $1/r$ integral and thus does not break conformal invariance. In this case that means that we require

$$\lambda^c_{ab}\, g_{a\bar{a}}\, g_{b\bar{b}} = 0.$$

59. Again, we work from the same chapter of Polchinski. For a general 2D QFT with a stress tensor, we can define the quantities

$$\begin{aligned} F(r^2) &= z^4 \langle T_{zz}(z, \bar{z})T_{zz}(0, 0)\rangle \\ G(r^2) &= 4z^3\bar{z}\langle T_{zz}(z, \bar{z})T_{z\bar{z}}(0, 0)\rangle \\ H(r^2) &= 16z^2\bar{z}^2\langle T_{z\bar{z}}(z, \bar{z})T_{z\bar{z}}(0, 0)\rangle \end{aligned}$$

From rotational invariance, these can only depend on $r^2 = |z|^2$. The conservation law $\bar{\partial}T_{zz} + \partial T_{z\bar{z}} = 0$ gives us that:

$$4\dot{F} + \dot{G} - 3G = 0, \quad 4\dot{G} - 4G + \dot{H} - 2H = 0$$

where $\dot{F}, \dot{G}$ indicates the operator $\frac{1}{2}r\partial_r$ (ie differentiation wrt $\log r^2$). Note subtracting 3/4 of the second one from the first gives:

$$4\dot{F} - 2\dot{G} - \frac{3}{4}\dot{H} = -\frac{3}{2}H$$

Define $C = 2F - G - \frac{3}{8}H$. Note that in a CFT, where $G = H = 0$, C is exactly the central charge c. Further, from this definition we get that in the general setting $\dot{C} = -\frac{3}{4}H$. But note that an *exactly* marginal perturbation does not give the stress-energy tensor a trace, so $\dot{C} = 0$ and the central charge will remain fixed.

This technology wasn't developed in Kiritsis. I'm unsure how he would have wanted us to show this.

60. Note under $\tau \to \tau + 1$ the η function is invariant and we our constraint comes from:

$$\frac{1}{2}(P_L^2 - P_R^2) \in \mathbb{Z} \Rightarrow G^{ij} m_j G_{ik} G^{kl} n_l = m_k n^k \in \mathbb{Z}$$

as required. So in particular we have $P_L^2 - P_R^2 \in 2\mathbb{Z}$. We can interpret (P_L, P_R) as being a vector lying in an *even, Lorentzian* lattice, with signature (N, N). Note in the 1D case then get that

$$P^1 \cdot P^2 := P_L^1 P_L^2 - P_R^1 P_R^2 = \frac{1}{2}\left[(\frac{R}{\ell_s}n + \frac{\ell_s}{R}m)(\frac{R}{\ell_s}n' + \frac{\ell_s}{R}m') - (\frac{R}{\ell_s}n - \frac{\ell_s}{R}m)(\frac{R}{\ell_s}n' - \frac{\ell_s}{R}m')\right] = (mn' + nm') \in \mathbb{Z}$$

Going to higher dimensions and turning on G and B gives us the same result (take $\ell_s = 1$ for simplicity here). All terms will cancel except the ones given by the relative minus sign of G on the second term

$$\frac{1}{2}\left[m_i n'^i + n^i m'_i + \cancel{(n^i n'^j + n'^i n^j) B_{ij}}\right] \in \mathbb{Z}$$

The last term cancels by antisymmetry. Here $n^i, m_i \in \mathbb{Z}$ (note the index convention, different from Kiritsis). Under $\tau \to -1/\tau$ the η function is a modular form of weight 1/2, so $\eta(\tau)^N$ is a modular form of weight $N/2$ and $|\eta(\tau)|^{2N} = |\tau|^{-N}|\eta(-1/\tau)|^{2N}$. Let us now look at the remaining part

$$\Theta(\tau) := \sum_{P=(P_L,P_R)\in\Gamma} q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2}$$

is also a modular form of this weight. Let's show this. We can use the Poisson resummation formula to write:

$$\sum_{p' \in \Gamma} \delta(p - p') = \frac{1}{V_\Gamma} \sum_{p'' \in \Gamma*} e^{2\pi i p p''} \Rightarrow \sum_{p \in \Gamma} f(p) = \frac{1}{V_\Gamma} \sum_{q \in \Gamma*} \hat{f}(q)$$

here V_Γ^{-1} is the covolume of Γ. Taking $f = e^{i\pi\tau P_L^2 - i\pi\bar{\tau}P_R^2}$ and doing a $2N$-dimensional Fourier transform, we see that $\hat{f}(q) = \frac{1}{|\tau|^N} e^{-i\pi Q_L^2/\tau + i\pi Q_R^2/\bar{\tau}}$. We can use this to write:

$$\Theta(\tau) = \sum_{P \in \Gamma} \exp\left[\pi i(\tau P_L^2 - \bar{\tau}P_R^2)\right] = \frac{1}{|\tau|^N V_\Gamma} \sum_{Q \in \Gamma*} \exp\left[\pi i(-\frac{1}{\tau}Q_L^2 + \frac{1}{\bar{\tau}}Q_R^2)\right]$$

Now as long as $\Gamma = \Gamma*$, that is, Γ is an *even, Lorentzian, self-dual* lattice. Then $V_\Gamma = 1$ and the sum over $Q \in \Gamma*$ is the same as the sum over $P \in \Gamma$. So we get

$$\Theta(\tau) = |\tau|^{-N}\Theta(-1/\tau)$$

which is the exact same transformation law as the $|\eta|^{2N}$ in the denominator, and so we get that $Z(R)$ is indeed modular invariant.

61. We have in fact done this in the first part exercise 46.

62. Certainly this is an order 2 involution, just like $R \to 1/R$. Now we know $V_{m,n} \to V_{m,-n}$ under this involution, so

$$\cdot[H^{0\prime}] \sim \sum_{n,m} C^{2n,2m}[V_{2n,2m}] + C^{2n+1,2m}[V_{2n+1,2m}] = \frac{1}{2}([H^0]\cdot[H^0] + [H^\pi]\cdot[H^|\pi|]) + [H^0]\cdot[H^\pi]$$

$$[H^{\pi\prime}]\cdot[H^{\pi\prime}] \sim \sum_{n,m} C^{2n,2m}[V_{2n,2m}] - C^{2n+1,2m}[V_{2n+1,2m}] = \frac{1}{2}([H^0]\cdot[H^0] + [H^\pi]\cdot[H^|\pi|]) + [H^0]\cdot[H^\pi]$$

$$[H^{0\prime}]\cdot[H^0\pi'] \sim \sum_{n,m} C^{2n,2m+1}[V_{2n,2m+1}] = \frac{1}{2}([H^0]\cdot[H^0] - [H^\pi]\cdot[H^\pi])$$

the only consistent transformation with these OPEs is exactly:

$$\begin{pmatrix} H^{0\prime} \\ H^{\pi\prime} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} H^{0} \\ H^{\pi} \end{pmatrix}$$

63. Define the orbifold partition function as

$${}^{+}\underset{+}{\square}' = \frac{1}{2}\left({}^{+}\underset{+}{\square} + {}^{-}\underset{+}{\square} + {}^{+}\underset{-}{\square} + {}^{-}\underset{-}{\square} \right)$$

Note that the orbifolded theory itself has a $\mathbb{Z}_2$ symmetry obtained by taking all the states in the $\mathbb{Z}_2$ twisted sectors to minus themselves:

$${}^{\pm}\underset{+}{\square} \to {}^{\pm}\underset{+}{\square}, \qquad {}^{\pm}\underset{-}{\square} \to -{}^{\pm}\underset{-}{\square}$$

I can now *orbifold again* by this symmetry, defining (as before):

$${}^{\pm}\underset{+}{\square}' = \frac{1}{2}\left({}^{+}\underset{+}{\square} + {}^{-}\underset{+}{\square} \pm {}^{+}\underset{-}{\square} \pm {}^{-}\underset{-}{\square} \right)$$
$${}^{\pm}\underset{-}{\square}' = \frac{1}{2}\left({}^{+}\underset{+}{\square} - {}^{-}\underset{+}{\square} \pm {}^{+}\underset{-}{\square} \mp {}^{-}\underset{-}{\square} \right)$$

Then forming the new partition function of this double orbifold theory I see that almost everything cancels:

$$\frac{1}{2}\left({}^{+}\underset{+}{\square}' + {}^{-}\underset{+}{\square}' + {}^{+}\underset{-}{\square}' + {}^{-}\underset{-}{\square}' \right) = {}^{+}\underset{+}{\square}$$

64. Note first that at $R/\ell = 1/\sqrt{2}$ we get

$$P_L = m + \frac{n}{2}, P_R = m - \frac{n}{2}$$

So we are summing over these lattice values in the numerator Θ of $Z(R)$. On the other hand, we have:

$$\frac{1}{2}(|\theta_2|^2 + |\theta_3|^2 + |\theta_4|^2) = \sum_{n,m} \left(\frac{1}{2}(1 + (-1)^{n+m}) q^{\frac{1}{2}n^2} \bar{q}^{\frac{1}{2}m^2} + \frac{1}{2} q^{\frac{1}{2}(n-1/2)^2} \bar{q}^{\frac{1}{2}(m-1/2)^2} \right)$$

This is a sum over all lattice points whose sum is an even integer *union with* the set of all half-lattice points, but only *half* of the half-lattice points are counted in the sum. This agree exactly with the standard weighting for the lattice generated by $(1,1)$ and $\frac{1}{2}(1,-1)$ which is exactly the original theta function numerator in the untwisted $Z(R)$ at $R/\ell_s = 1/\sqrt{2}$.

Squaring the Ising model theta function then gives:

$$\frac{|\theta_2\theta_3| + |\theta_3\theta_4| + |\theta_2\theta_4|}{4|\eta|^2} + \underbrace{\frac{1}{4}\frac{1}{|\eta|^2}(|\theta_2|^2 + |\theta_3|^2 + |\theta_4|^2)}_{\frac{1}{2}Z(R)} = \frac{1}{2}Z(R) + \frac{1}{2}\left(\frac{|\eta|}{|\theta_2|} + \frac{|\eta|}{|\theta_3|} + \frac{|\eta|}{|\theta_4|} \right)$$

exactly as we wanted.

65. Take $\ell_s = 1$ here. The partition function will still have 1 twisted sector and a single projection. So we need to consider 4 terms. We have $Z\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] = Z(R_1, R_2) = Z(R_1)Z(R_2)$. Our vertex operators are labelled by (m_1, n_1, m_2, n_2), and g acts as $(m_1, n_1, m_2, n_2) \to (-1)^{m_2}(-m_1, -n_1, m_2, n_2)$. And so:

$$\frac{1}{2}Z\begin{bmatrix}0\\1\end{bmatrix} = \mathrm{Tr}_1[g\, q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - \bar{c}/24}] = \overbrace{\left|\frac{\eta}{\theta_2}\right|}^{X^1 \to -X^1} \underbrace{\frac{1}{\eta\bar{\eta}} \sum_{m,n} (-1)^m \exp\left[\frac{i\pi\tau}{2}\left(\frac{m}{R_2} + nR_2\right)^2 - \frac{i\pi\bar{\tau}}{2}\left(\frac{m}{R_2} - nR_2\right)^2 \right]}_{X^2 \to X^2 + \pi R_2}$$

$$\frac{1}{2}Z\begin{bmatrix}1\\0\end{bmatrix} = \mathrm{Tr}_g[g\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}] = \left|\frac{\eta}{\theta_4}\right| \quad \frac{1}{\eta\bar{\eta}}\sum_{m,n}\exp\left[\tfrac{i\pi\tau}{2}\left(\tfrac{m}{R_2}+(n+\tfrac{1}{2})R_2\right)^2 - \tfrac{i\pi\bar{\tau}}{2}\left(\tfrac{m}{R_2}-(n+\tfrac{1}{2})R_2\right)^2\right]$$

$$\frac{1}{2}Z\begin{bmatrix}1\\1\end{bmatrix} = \mathrm{Tr}_g[g\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}] = \left|\frac{\eta}{\theta_3}\right| \frac{1}{\eta\bar{\eta}}\sum_{m,n}(-1)^m\exp\left[\tfrac{i\pi\tau}{2}\left(\tfrac{m}{R_2}+(n+\tfrac{1}{2})R_2\right)^2 - \tfrac{i\pi\bar{\tau}}{2}\left(\tfrac{m}{R_2}-(n+\tfrac{1}{2})R_2\right)^2\right]$$

it is clear that the sum of all these is modular invariant. I am unsure if I should try to simplify this further. Certainly (unlike the freely-acting orbifold case) this doesn't look trivial. This is the CFT of fields *valued in the Klein bottle.*

66. Take $\ell_s = 1$ here. The symmetry interchanges $|m_1, n_1, m_2, n_2\rangle \to |m_2, n_2, m_1, n_1\rangle$ We have $Z[{0 \atop 0}] = Z(R)^2$. In the g-trace, we will need $m_1 = m_2, n_1 = n_2$. Then, excitations around this state must have equal mode number in m_1, m_2 and n_1, n_2 to contribute to the g-trace so for each factor of $q^{\frac{1}{2}P_L^2}\bar{q}^{\frac{1}{2}P_R^2}$ we have

$$Z\begin{bmatrix}0\\1\end{bmatrix} = (q\bar{q})^{-2/24}\sum_{m,n}\exp\left[\frac{i\pi\, 2\tau}{2}\left(\tfrac{m}{R}+nR\right)^2 - \frac{i\pi\, 2\bar{\tau}}{2}\left(\tfrac{m}{R}-nR\right)^2\right]\prod_{n'}\frac{1}{1-q^{2n'}}\prod_{m'}\frac{1}{1-\bar{q}^{2m'}}$$
$$= \frac{1}{|\eta(2\tau)|^2}\sum_{m,n}\exp\left[i\pi\tau\left(\tfrac{m}{R}+nR\right)^2 - i\pi\bar{\tau}\left(\tfrac{m}{R}-nR\right)^2\right] = \frac{2}{|\eta(\tau)||\theta[{1 \atop 0}](\tau)|}\sum\dots$$

On the other hand, the twisted sector we have boundary conditions $X^1(\sigma+2\pi) = X^2(\sigma), X^2(\sigma+2\pi) = X^1(\sigma)$. Applying $\tau \to -1/\tau$ on the preceding we get:

$$Z\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{|\eta(\tau)||\theta[{0 \atop 1}](\tau)|}\sum_{m,n}\exp\left[\frac{i\pi\tau}{4}\left(\tfrac{m}{R}+nR\right)^2 - \frac{i\pi\bar{\tau}}{4}\left(\tfrac{m}{R}-nR\right)^2\right]$$

Taking $\tau \to \tau+1$ gives

$$Z\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{|\eta(\tau)||\theta[{0 \atop 0}](\tau)|}\sum_{m,n}(-1)^{mn}\exp\left[\frac{i\pi\tau}{4}\left(\tfrac{m}{R}+nR\right)^2 - \frac{i\pi\bar{\tau}}{4}\left(\tfrac{m}{R}-nR\right)^2\right].$$

Let us check if this is modular invariant. Clearly $Z[{0 \atop 0}]$ maps to itself under both S and T. Under T, $Z[{0 \atop 1}]$ maps to itself, and $Z[{1 \atop 0}]$ and $Z[{1 \atop 1}]$ get exchanged by the properties of theta functions. Further, under $\tau \to -1/\tau$ $Z[{1 \atop 0}]$ and $Z[{0 \atop 1}]$ map to one another. However, $Z[{1 \atop 1}]$ does not map to itself under S, and we are led to conclude that this $\mathbb{Z}_2$ symmetry is anomalous.

67. If we orbifold the single free scalar by acting as $|m,n\rangle \to (-1)^{m+n}|m,n\rangle$ we have $Z[{0 \atop 0}] = Z(R)$ as before, but now:

$$Z\begin{bmatrix}0\\1\end{bmatrix} = \sum_{m,n}(-1)^{m+n}\exp\left[\tfrac{i\pi\tau}{2}\left(\tfrac{m}{R}+nR\right)^2 - \tfrac{i\pi\bar{\tau}}{2}\left(\tfrac{m}{R}-nR\right)^2\right]$$

Taking $\tau \to -1/\tau$ gives that both m and n shift by 1/2

$$Z\begin{bmatrix}1\\0\end{bmatrix} = \sum_{m,n}\exp\left[\tfrac{i\pi\tau}{2}\left(\tfrac{m-\frac{1}{2}}{R}+(n-\tfrac{1}{2})R\right)^2 - \tfrac{i\pi\bar{\tau}}{2}\left(\tfrac{m-\frac{1}{2}}{R}-(n-\tfrac{1}{2})R\right)^2\right]$$

Then doing $\tau \to \tau+1$ gives:

$$Z\begin{bmatrix}1\\1\end{bmatrix} = \sum_{m,n}(-1)^{m+n+\frac{1}{2}}\exp\left[\tfrac{i\pi\tau}{2}\left(\tfrac{m-\frac{1}{2}}{R}+(n-\tfrac{1}{2})R\right)^2 - \tfrac{i\pi\bar{\tau}}{2}\left(\tfrac{m-\frac{1}{2}}{R}-(n-\tfrac{1}{2})R\right)^2\right]$$

this already looks a little weird. Out front we don't necessarily have a ±1. Further, doing $\tau \to \tau+1$ again does not get us back to $Z[{1 \atop 0}]$, we need $\tau \to \tau+3$.

68. In the untwisted sector we have our vacuum state $|0\rangle$, with $\Delta = \bar{\Delta} = 0$ as required. Now consider the kth twisted sector. We have creation and annihilation operators $\alpha_{n+k/N}$ satisfying the same commutation relations $[\alpha_r, \alpha_s] = r\delta_{r+s}$. However as X is a *complex* boson, the α_r are complex numbers and so we have

two sets of them (which we can call $\alpha_r, \bar{\alpha}_r$ following previous convention). From commuting them across, we get:

$$\langle X(z)\partial X(w)\rangle = 2 \times \frac{1}{w} \sum_{r=\min(1,k/N)}^{\infty} \left(\frac{w}{z}\right)^r = 2 \times \frac{\frac{w}{z}\left(\frac{z}{w}\right)^{k/L}}{z-w}$$

Then, differentiating with respect to z gives:

$$\langle \partial X(z)\partial X(w)\rangle = -\frac{2}{(w-z)^2}\left(\frac{w}{z}\right)^{k/N}(1 - \tfrac{k}{L}(1-\tfrac{z}{w}))$$

Taking the finite part of this $-\frac{1}{2}$ of expression as $w \to z$ gives us:

$$\langle T\rangle = \frac{k(L-k)}{2L^2}$$

as required.

69. We have the scalar propagator written in terms of the eigen-modes as:

$$\langle X(z)X(0)\rangle = -\frac{\ell_s^2}{2}\sum_{m,n}{}' \frac{1}{|m+n\tau|^2} e^{2\pi i(m\sigma_1 + n\sigma_2)}$$

Rather than trying to massage this into our appropriate logarithm of theta functions, let's appreciate what properties we want our correlator to have. For $z \to 0$, the small-distance behavior of the correlator should reproduce the $\mathbb{CP}^1$ result, so we namely need it to go as:

$$-\frac{\ell_s^2}{2}\log|z|^2 + O(z)$$

Further, the *only* singularity on the torus is at $z \to 0$, nowhere else. Thus we should be able to write our correlator as

$$-\frac{\ell_s^2}{2}\log G(z,\bar{z})$$

where G must be a doubly-periodic harmonic function with a *single* zero at $z = 0$ on the torus and no poles. There are no such holomorphic functions since all non-constant elliptic functions need to have an equal number of zeros and poles (and also more than one zero, since the coefficients of all zeros must sum to 0). In other words, instead of looking at an elliptic function we should be looking at a section of a line bundle over the torus with a single zero.

We see that the theta functions give us exactly this- and moreover rational functions of the theta functions generate all such sections. The constraint of a *single* zero at $z = 0$ together with *modular invariance* singles out $\theta\begin{bmatrix}1\\1\end{bmatrix}$ uniquely. To give it the appropriate coefficient of the zero, we must have:

$$G(z) = \left|\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(z,\tau)}{\partial_z\theta\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)}\right|^2 \times (1 + O(z))$$

The problem is that this is a *quasi-periodic* foundtion in z. Under $z \to z+\tau$ we get that $\log G \to \log G + 2\pi\tau_2 + 4\pi\mathrm{Im}(z)$. This can be remedied by adding $e^{-2\pi\frac{z_2^2}{\tau_2}}$ to G.

Also, under $\tau \to 1/\tau$, $z \to z/\tau$ from the ratio we pick up a factor of $|\exp(i\pi z^2/\tau)|^2 = e^{-2\pi \mathrm{Im}(z^2/\tau)}$. But this is exactly the same factor as is picked up by $e^{-2\pi\frac{\mathrm{Im}z^2}{\tau_2}}$, so adding this term fixes modular invariance as well. Our final result is then:

$$G(z) = \left|\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(z,\tau)}{\partial_z\theta\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)}\right|^2 e^{-2\pi\frac{(\mathrm{Im}z)^2}{\tau_2}}.$$

So we now have an explicit formula for $\Delta(z-w,\tau)$ on the torus. The Klein bottle is given by identifying $z \cong -\bar{z} + \tau/2$. Then we expect the propagator to be

$$\Delta_{K_2}(z-w) = \Delta(z-w, 2it) + \Delta(z+\bar{w}+it, 2it)$$

Next, for the cylinder we have the involution $z \cong -1/\bar{z}$ so we have the propagator:

$$\Delta_{C_2}(z-w) = \Delta(z-w, it) + \Delta(z+\bar{w}, it)$$

Finally, for the Möbius strip, we have two involutions and get

$$\Delta_{M_2}(z-w) = \Delta(z-w, 2it) + \Delta(z+\bar{w}, 2it) + \Delta(z-w-2\pi(it+\tfrac{1}{2}), 2it) + \Delta(z+\bar{w}+2\pi(-it+\tfrac{1}{2}), 2it)$$

70. We already know how to calculate $\mathrm{Tr}_{NS/R}((\pm 1)^F q^{L_0^{cyl}})$ for the free fermion. Ω acts by sending a left-moving state to a right-moving one and vice-versa. Only states that are left-right symmetric survive. First lets do the NS sector. There is a single vacuum and we get:

$$\mathrm{Tr}_{\mathrm{NS}}[\Omega e^{-2\pi t(L_0+\bar{L}_0-c/12)}] = e^{2\pi t/24}\prod_{n=1}^{\infty}(1+e^{-2\pi t\times 2(n-1/2)}) = \sqrt{\frac{\theta_3(2it)}{\eta(2it)}}$$
$$\mathrm{Tr}_{\mathrm{NS}}[\Omega(-1)^F e^{-2\pi t(L_0+\bar{L}_0-c/12)}] = e^{2\pi t/24}\prod_{n=1}^{\infty}(1+e^{-2\pi t\times 2(n-1/2)}) = \sqrt{\frac{\theta_3(2it)}{\eta(2it)}}$$

Note that these two are the same, since only sectors with an equal number of left movers and right-movers contribute, and this necessarily forces F to be even. Then, for the Ramond sector we have

$$\mathrm{Tr}_{\mathrm{R}}[\Omega e^{-2\pi t(L_0+\bar{L}_0-c/12)}] = \sqrt{2}e^{-2\pi t(1/16-1/48)}\prod_{n=1}^{\infty}(1+e^{-2\pi t\times 2n}) = \sqrt{\frac{\theta_2(2it)}{\eta(2it)}}$$
$$\mathrm{Tr}_{\mathrm{R}}[\Omega(-1)^F q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}] = 0$$

where the last one is zero as before, since for any state, there is a corresponding one with opposite $(-1)^F$ eigenvalue, related by zero-modes.

Chapter 5: Scattering Amplitudes and Vertex Operators

0. A worthwhile exercise (that is not in the book) is to show that we have the correct Regge behavior of the Virasoro-Shapiro amplitude at large s, fixed t. From Stirling's approximation for large s, we have $\frac{\Gamma(a+s)}{\Gamma(b+s)} \sim s^{a-b}$, so:

$$\begin{aligned}
S_4(s,t,u) &\sim \frac{\Gamma(-1-\ell_s^2 t/4)\,\Gamma(-1-\ell_s^2 s/4)\Gamma(3+\ell_s^2 t/4+\ell_s^2 s/4)}{\Gamma(2+\ell_s^2 t/4)\,\Gamma(2+\ell_s^2 s/4)\Gamma(-2-\ell_s^2 t/4-\ell_s^2 s/4)} \sim \frac{\Gamma(-1-\ell_s^2 t/4)}{\Gamma(2+\ell_s^2 t/4)} s^{1+\ell_s^2 t/4} \frac{\Gamma(-1-\ell_s^2 s/4)}{\Gamma(-2-\ell_s^2 t/4-\ell_s^2 s/4)} \\
&= \frac{\Gamma(-1-\ell_s^2 t/4)}{\Gamma(2+\ell_s^2 t/4)} s^{1+\ell_s^2 t/4} \frac{\Gamma(3+\ell_s^2 t/4+\ell_s^2 s/4)}{\Gamma(2+\ell_s^2 s/4)} \frac{\sin(\ell_s^2(s+t)/4)}{\sin(\ell_s^2 s/4)} \\
&\to \frac{\Gamma(-1-\ell_s^2 t/4)}{\Gamma(2+\ell_s^2 t/4)} \frac{\sin(\ell_s^2 u/4)}{\sin(\ell_s^2 s/4)} s^{2+\ell_s^2 t/2}
\end{aligned}$$

Using the same argument, in the large s,t,u limit, we get the following soft behavior:

$$S_4(s,t,u) \sim \frac{s^{-1-\ell_s^2 s/4} t^{-1-\ell_s^2 t/4} u^{-1-\ell_s^2 u/4}}{s^{2+\ell_s^2 s/4} t^{2+\ell_s^2 t/4} u^{2+\ell_s^2 u/4}} \sim e^{-\frac{\ell_s^2}{2}((s+3)\log s+(t+3)\log t+(u+3)\log u)} \to e^{-\frac{\ell_s^2}{2}(s\log s+t\log t+u\log u)}$$

1. Note that we need 3 more c ghosts than b ghosts since the difference of the zero modes must be three. Now, c has scaling dimension 1 and b has scaling dimension -2 so the total scaling of the correlator $\langle\prod_{i=1}^{n+3} c(z_i)\prod_{j=1}^{n} b(w_j)\rangle$ will be $3-n$. Thus, viewed in the complex plane, we expect it to be a homogenous rational function of degree exactly $3-n$.

   We will have n contractions of the bs and cs with 3 cs left over. This gives:

$$\langle \prod_{i=1}^{n+3} c(z_i) \prod_{j=1}^{n} b(w_j)\rangle = \frac{(z_{n+1}-z_{n+2})(z_{n+1}-z_{n+3})(z_{n+2}-z_{n+3})}{(z_1-w_1)\dots(z_n-w_n)} \times c.c. + \text{perms.}$$

   where each permutations will pick up a sign for every odd combined permutation of the z_i, w_j. Another way to do it is as follows:

   As stated before, the correlator when viewed in the complex plane will be a homogenous rational function of degree exactly $3-n$. That way, it will be finite at infinity. We also know that this function is antisymmetric upon swapping any of the z_i, any of the w_i, or any of the z_i with the w_i. Further, if any of the $z_i = z_j$ or $w_i = w_j$, this function will vanish. On the other hand, if $z_i = w_j$, we expect a contribution of a pole $\frac{1}{z_i - w_j}$. There is only one such homogenous rational function:

$$\frac{\prod_{i<j}^{n+3}(z_i-z_j)\prod_{i<j}^{n}(w_i-w_j)}{\prod_{i=1}^{n+3}\prod_{j=1}^{n}(z_i-w_j)}.$$

   This is indeed of degree $3-n$, as desired.

2. It is clear from plugging things in that when $z_1\to 0, z_2\to 1, z_3\to\infty$, the 4-point tachyon amplitude becomes:

$$\lim_{z_3\to\infty}\frac{8\pi i}{\ell_s^2} g_c^2\,\delta\!\!\!/^{26}(\Sigma p)|z_3|^2|z_3-1|^2\int d^2 z_4 |z_4|^{\ell_s^2 p_1\cdot p_4}|1-z_4|^{\ell_s^2 p_2\cdot p_4}|z_4-z_3|^{\ell_s^2 p_3\cdot p_4} 1^{\ell_s^2 p_1\cdot p_2}|z_3|^{\ell_s^2 p_1\cdot p_3}|z_3-1|^{\ell_s^2 p_2\cdot p_3}$$

   here $\delta\!\!\!/ = 2\pi\delta$. Note all the terms that go to infinity cancel, since $\ell_s^2 p_3\cdot(p_1+p_2+p_3) = -\ell_s^2 p_3^2 = -4$ which cancels with the two powers of two outside the integral. Next, $\ell_s^2 p_1\cdot p_4 = \frac{1}{2}(p_1+p_4)^2 - \frac{1}{2}(\ell_s^2 p_1^2 - \ell_s^2 p_4^2) = -\ell_s^2 t/2 - 4$ etc so we get:

$$\frac{8\pi i}{\ell_s^2} g_c^2 \delta\!\!\!/^{26}(\Sigma p)\int d^2 z_4 |z_4|^{-\ell_s^2 t/2-4}|1-z_4|^{-\ell_s^2 u/2-4}$$

   as required.

3. For a conformal transformation we have $|x'_{ij}|^2 = \Omega(x_i)\Omega(x_j)|x_{ij}|^2$ where $\Omega(x_i)$ is the local scale factor $\det \partial x'/\partial x$ evaluated at x_i. Then, the N-point tachyon amplitude will pick up $\Omega(x_1)^2\Omega(x_2)^2\Omega(x_3)^2$ from the three terms outside of the integral. The terms inside the integral can be written as:

$$\prod_{i<j}(|z_{ij}|^2)^{\ell_s^2 p_i\cdot p_j/2}$$

so z_i in this term will pick up a power of $\sum_{j\neq i}\ell_s^2 p_i\cdot p_j/2 = -\ell_s^2 p_j^2/2 = -2$ on its scale factor. This exactly cancels for z_1, z_2, z_3. For the other z_i, we note that $d^2 z_i$ will pick up the factor $\Omega(z_i)^2$ upon transformation.

Another way to do this is directly from noting that each $\int d^2 z_i V_{p_i}(z_i, \bar{z}_i)$ for $i > 3$ is invariant under conformal transformation, and $c(z_i)\bar{c}(\bar{z}_i)V_{p_i}(z_i, \bar{z}_i)$ has scaling dimension zero, so transforms trivially under $\mathrm{SL}_2(\mathbb{C})$ transformations.

4. Note that the three-point tachyon amplitude is very simple and independent of momenta aside from a delta function: $S(k_1, k_2, k_3) = \frac{8\pi i}{\ell_s^2} g_c \delta\!\!\!/^{26}(\Sigma k)$.

Let's now consider the limit of a nearly on-shell particle of momenta k. From elementary field theory we get:

$$S(k_1, k_2, k_3, k)4) \sim i\int \frac{d^{26}k}{(2\pi)^{26}} \frac{S_{S^2}(k_1, k_2, k)S_{S_2}(-k, k_3, k_4)}{-k^2 + 4/\ell_s^2 + i\epsilon} = i\left(\frac{8\pi i}{\ell_s^2}\right)^2 g_c^2 \delta\!\!\!/^{26}(\Sigma k_i)\frac{1}{s + 4\ell_s^2 + i\epsilon}$$

This has a pole when $-(k_1 + k_2)^2 = s = -4/\ell_s^2$. We see that (ignoring the $\delta\!\!\!/$ term) this gives a residue of $-i\frac{64\pi^2}{\ell_s^4} g_c^2$

On the other hand we have from **5.2.5** a residue of:

$$\frac{8\pi i}{\ell_s^2} g_c^2 \times 2\pi \times -\frac{4}{\ell_s^2} = -i\frac{64\pi^2}{\ell_s^2} g_c^2$$

exactly consistent with unitarity. Note we needed every constant to be as it was so that we could get such agreement.

5. The massless state corresponds to $\zeta_{\mu\nu}\partial X^\mu \partial X^\nu e^{ip\cdot X}$. We don't have to integrate. Let's calculate the correlator

$$\langle \partial X(z_1)\bar{\partial}X(z_1)e^{ik_1 X(z_1)}\, e^{ik_2 X(z_2)}\, e^{ik_3 X(z_3)}\rangle = iC^X_{S^2}\delta\!\!\!/^{26}(\Sigma p)\prod_{i<j}|z_{ij}|^{\alpha' k_i\cdot k_j}\left(-\tfrac{i\ell_s^2}{2}\right)\left(\tfrac{k_2}{z_{12}} + \tfrac{k_3}{z_{13}}\right)\left(-\tfrac{i\ell_s^2}{2}\right)\left(\tfrac{k_2}{\bar{z}_{12}} + \tfrac{k_3}{\bar{z}_{13}}\right)$$

with the ghost correlator this gives:

$$iC^X_{S^2}C^{gh}_{S^2}\tfrac{-\ell_s^4}{4}\delta\!\!\!/^{26}(\Sigma p)\prod_{i<j}|z_{ij}|^{\alpha' k_i\cdot k_j + 2}\left(\tfrac{k_2}{z_{12}} + \tfrac{k_3}{z_{13}}\right)\left(\tfrac{k_2}{\bar{z}_{12}} + \tfrac{k_3}{\bar{z}_{13}}\right)$$

Now $k_1^2 = 0 = k_1\cdot k_2 + k_1\cdot k_3$. On the other hand $-4/\ell_s^2 = -k_2^2 = k_2\cdot k_3 + k_1\cdot k_2 = -k_3^2 = k_2\cdot k_3 + k_1\cdot k_3$. Solving this gives $k_1\cdot k_2 = k_1\cdot k_3 = 0$ while $k_2\cdot k_3 = -4/\ell_s^2$. Then, taking $z_1\to 0, z_2\to 1, z_3\to\infty$ gives:

$$-i\frac{\ell_s^2}{4}C^X_{S^2}C^{gh}_{S^2}\delta\!\!\!/^{26}(\Sigma p)\zeta_{\mu\nu}k_2^\mu k_2^\nu$$

Further, we have that $\zeta_{\mu\nu}k_1^\mu = \zeta_{\mu\nu}(k_2 + k_3)^\mu = 0$ so we can rewrite this symmetrically as

$$-i\frac{\ell_s^4}{16}\underbrace{C^X_{S^2}C^{gh}_{S^2}}_{:=8\pi g_c'/\ell_s^2}\delta\!\!\!/^{26}(\Sigma p)\zeta_{\mu\nu}k_{23}^\mu k_{23}^\nu = -\frac{i\pi\ell_s^2}{2}g_c'\delta\!\!\!/^{26}(\Sigma p)\zeta_{\mu\nu}k_{23}^\mu k_{23}^\nu.$$

The overall constants can be determined from unitarity. The pole of the Veneziano amplitude at $s = 0$ has residue (using that $s = 0, s + t + u = -16/\ell_s^2$) that is a delta function times:

$$\frac{8\pi i}{\ell_s^2}g_c^2 \times 2\pi \times \frac{4}{\ell_s^2 s}\frac{\Gamma(-1 - \ell_s^2 t/4)\Gamma(3 + \ell_s^2 t/4)}{\Gamma(-2 - \ell_s^2 t/4)\Gamma(2 + \ell_s^2 t/4)} = -i\frac{(4\pi)^2}{\ell_s^2}g_c^2 \times \frac{4}{\ell_s^2 s}\overbrace{(2 + \ell_s^2 t/4)^2}^{(\frac{\ell_s^2}{8}(t-u))^2} = -i\pi^2 g_c^2\frac{(t-u)^2}{s} \tag{55}$$

On the other hand, factorization of this into amplitudes with massless states yields a delta function times:

$$iC_{3pt}^2\sum_\zeta \zeta_{\mu\nu}\zeta_{\sigma\rho}k_{12}^\mu k_{12}^\nu k_{34}^\sigma k_{34}^\rho \times \frac{1}{(k_1+k_2)^2+i\epsilon} = iC_{3pt}^2(k_{12}\cdot k_{34})^2\times\frac{1}{s} = iC_{3pt}^2\frac{(u-t)^2}{s} \tag{56}$$

where we have used that, just as the sum over intermediate photon polarizations $\epsilon_\mu\epsilon_\nu^*$ can be replaced by just $\eta_{\mu\nu}$, the sum over intermediate polarizations $\zeta_{\mu\nu}\zeta_{\rho\sigma}$ be replaced by $\frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho})$. Comparing equations (55) and (56) We the get $C_{3pt}=-\pi i g_c$. Equivalently, $g_c'=2g_c/\ell_s^2$.

6. This problem is so nasty - I'm pretty sure Kiritsis meant for us to just look at scattering 4 *open* string states - which in and of itself is nasty enough.

   We have already determined the normalization in the previous question. It is also simple to check that it is correct to attach g_c' to each vertex operator in the 3-point and 4-point functions by considering first the 2 tachyon $\to$ 2 massless state scattering in the t and u channels, which relates the 3-point scatterings of tachyons and massless states to one another, and then use the 2 $\to$ 2 tachyon to tachyon scattering to express its normalization in terms of the 3-point tachyon amplitude. All of this equates to taking $g_c'=2g_c/\ell_s^2$.

   As a warm-up lets do the three-point massless amplitude. We compute the correlator

$$\langle :\partial X^\alpha(z_1)e^{ip_iX(z_1)}::\partial X^\beta(z_2)e^{ip_iX(z_2)}::\partial X^\gamma(z_3)e^{ip_iX(z_3)}:\times c.c.\rangle$$

   In the holomorphic part, there are two types of contribution: One where each ∂X contracts with an exponential and one where two of the ∂X contract with one another and the last one contracts with an exponential. Further, we see that $p_i\cdot p_j=0$, so the $\prod_{i<j}|z_{ij}|^{\ell_s^2 p_i\cdot p_j}$ is unity. The first contribution gives:

$$i\left(\frac{\ell_s^2}{2}\right)^3\left(\frac{k_2^\alpha}{z_{12}}+\frac{k_3^\alpha}{z_{13}}\right)\left(\frac{k_1^\alpha}{z_{21}}+\frac{k_3^\alpha}{z_{23}}\right)\left(\frac{k_1^\alpha}{z_{31}}+\frac{k_2^\alpha}{z_{32}}\right)\to i\left(\frac{\ell_s^2}{2}\right)^3\frac{1}{2^2}(k_1-k_2)^\gamma(k_2-k_3)^\alpha(k_3-k_1)^\beta$$

   The second contribution gives

$$i\left(\frac{\ell_s^2}{2}\right)^2\left[\frac{\eta^{\alpha\beta}}{z_{12}^2}\left(\frac{k_1^\gamma}{z_{31}}+\frac{k_2^\gamma}{z_{32}}\right)+\frac{\eta^{\beta\gamma}}{z_{23}^2}\left(\frac{k_2^\alpha}{z_{12}}+\frac{k_3^\alpha}{z_{13}}\right)+\frac{\eta^{\alpha\gamma}}{z_{13}^2}\left(\frac{k_1^\beta}{z_{21}}+\frac{k_3^\beta}{z_{23}}\right)\right]$$

   Multiplying this by the c contribution $z_{12}z_{23}z_{13}\times c.c.$ and setting $z_1=0,z_2=1,z_3=\infty$ we get the 3-point amplitude:

$$\pi i g_c\,\zeta_{1,\alpha\bar\alpha}\zeta_{2,\beta\bar\beta}\zeta_{3,\gamma\bar\gamma}T^{\alpha\beta\gamma}T^{\bar\alpha\bar\beta\bar\gamma},\quad T^{\alpha\beta\gamma}=\eta^{\alpha\beta}k_{12}^\gamma+\eta^{\beta\gamma}k_{23}^\alpha+\eta^{\alpha\gamma}k_{31}^\beta+\frac{\ell_s^2}{8}k_{12}^\gamma k_{23}^\alpha k_{31}^\beta. \tag{57}$$

   Now let's do the four-point amplitude. *First, I will work with the open string* (no CP indices, so $U(1)$ gauge symmetry) and use some tricks at the end to get the closed string amplitude. For the open string, there are six possible orderings of the y_1,y_2,y_3,y_4. In three of these cases we can send $y_1\to 0, y_2\to 1, y_3\to\infty$ and vary y_4. In the other three, cases we switch y_2 and y_3. This amounts to swapping $s\leftrightarrow t$. **HOWEVER** for Polchinski's trick, I only need to consider *one of these six*. WLOG I set y_4 to be between y_1,y_2 in $0,1$. I'll also absorb ℓ_s^2 in the definition of s,t,u. So we have,

$$\prod_{i<j}|y_{ij}|^{2k_i\cdot k_j}\to|y|^{-u}|1-y|^{-t}\leftrightarrow|y|^{-u}|1-y|^{-s}$$

   We now get three types of contributions: If all the ∂X^α contract with each other (3 terms), if two of the ∂X^α contract with each other (6 terms) and the remaining two contract with one of the $e^{ik_i\cdot X}$, or if they all contract with the $e^{ik_i\cdot X}$ (1 term).

   In the first case we get:

$$(-2\ell_s^2)^2\left(\frac{1}{y_{12}^2y_{34}^2}+\frac{1}{y_{13}^2y_{24}^2}+\frac{1}{y_{14}^2y_{23}^2}\right)\to(2\ell_s^2)^2\left(\eta^{\alpha\beta}\eta^{\gamma\delta}+\frac{\eta^{\alpha\gamma}\eta^{\beta\delta}}{(1-y)^2}+\frac{\eta^{\alpha\delta}\eta^{\beta\gamma}}{y^2}\right)$$

Integrating y from 0 to 1 gives

$$\frac{ig_o^2 \not{\delta}^{26}}{4\ell_s^4} \times (2\ell_s^2)^2 \left(\frac{\Gamma(1-t)\Gamma(1-u)}{\Gamma(2+s)} + \frac{\Gamma(1-t)\Gamma(-1-u)}{\Gamma(s)} + \frac{\Gamma(-1-t)\Gamma(1-u)}{\Gamma(s)} \right) \tag{58}$$

Now the annoying one[1]. Define $K_i = \sum_{j \neq i} \frac{k_j}{y_{ij}}$. Note:

$$K_1 = -k_2^\alpha - \frac{k_3^\alpha}{y}, \quad K_2 = k_1^\beta + \frac{k_4^\beta}{1-y}, \quad K_3 \to -(1+y)k_1^\gamma - yk_2^\gamma - k_4^\gamma, \quad K_4 = \frac{k_1^\delta}{y} + \frac{k_2^\delta}{y-1}.$$

We can now write the $(\alpha')^3$ contribution as $\frac{ig_o^2 \not{\delta}^{26}}{4\ell_s^4}(2\ell_s^2)^3$ times:

$$\begin{aligned}
&\left(\frac{K_3K_4}{y_{12}^2}\eta^{\alpha\beta} + \frac{K_1K_2}{y_{34}^2}\eta^{\gamma\delta} + \frac{K_1K_4}{y_{23}^2}\eta^{\beta\gamma} + \frac{K_2K_3}{y_{14}^2}\eta^{\alpha\delta} + \frac{K_2K_4}{y_{13}^2}\eta^{\alpha\gamma} + \frac{K_1K_3}{y_{24}^2}\eta^{\beta\delta} \right) \\
\to &\Big[-((1+y)k_1^\gamma + yk_2^\gamma + k_4^\gamma)(\tfrac{k_1^\delta}{y} + \tfrac{k_2^\delta}{y-1})\eta^{\alpha\beta} - (k_2^\alpha + \tfrac{k_4^\alpha}{y})(k_1^\beta + \tfrac{k_4^\beta}{1-y})\eta^{\gamma\delta} - (k_2^\alpha + \tfrac{k_4^\alpha}{y})(\tfrac{k_1^\delta}{y} + \tfrac{k_2^\delta}{y-1})\eta^{\beta\gamma} \\
&- (k_1^\beta + \tfrac{k_4^\beta}{1-y})((1+y)k_1^\gamma + yk_2^\gamma + k_4^\gamma)\tfrac{\eta^{\alpha\delta}}{y^2} + (k_1^\beta + \tfrac{k_4^\beta}{1-y})(\tfrac{k_1^\delta}{y} + \tfrac{k_2^\delta}{y-1})\eta^{\alpha\gamma} + (k_2^\alpha + \tfrac{k_4^\alpha}{y})((1+y)k_1^\gamma + yk_2^\gamma + k_4^\gamma)\tfrac{\eta^{\beta\delta}}{(1-y)^2} \Big]
\end{aligned}$$

Now we use shorthand $k_{i+j} = k_i + k_j$. Looking at the first of the six terms above, **we get the order** $(\alpha')^3$ **to our scattering amplitude to be** $\frac{ig_o^2 \not{\delta}^{26}}{4\ell_s^6}(2\ell_s^2)^3$ **multiplying**:

$$\begin{aligned}
&-\eta^{\alpha\beta} \int_0^1 dy \left(k_1^\delta [y^{-1}k_{1+4}^\gamma + k_{1+2}^\gamma] + k_2^\delta[(y-1)^{-1}k_{1+4}^\gamma + y(y-1)^{-1}k_{1+2}^\gamma] \right) |y|^{-u}|1-y|^{-t} \\
&= \eta^{\alpha\beta} \left(k_1^\delta k_{1+4}^\gamma \frac{\Gamma(1-t)\Gamma(-u)}{\Gamma(1+s)} + k_1^\delta k_{1+2}^\gamma \frac{\Gamma(1-t)\Gamma(1-u)}{\Gamma(2+s)} - k_2^\delta k_{1+4}^\gamma \frac{\Gamma(-t)\Gamma(1-u)}{\Gamma(1+s)} - k_2^\delta k_{1+2}^\gamma \frac{\Gamma(-t)\Gamma(-u)}{\Gamma(s)} \right) + 5\,\text{perms.}
\end{aligned} \tag{59}$$

Now for the order $(\alpha')^4$ term. This is given by contracting each ∂X against an exponential, yielding $K_1^\alpha K_2^\beta K_3^\gamma K_4^\delta$. Again we have y in $[0,1]$

$$i(g_o')^4 C_{D^2} \not{\delta}^{26} \, (2\ell_s^2)^4 \int_0^1 dy \, \left(k_2^\alpha + \frac{k_4^\alpha}{y} \right) \left(k_1^\beta + \frac{k_4^\beta}{1-y} \right) \left(k_{1+4}^\gamma + yk_{1+2}^\gamma \right) \left(\frac{k_1^\delta}{y} + \frac{k_2^\delta}{y-1} \right) y^{-u}(1-y)^{-t}$$

This gives a $2^4 = 16$ terms. Its not terrible. **The** $(\alpha')^4$ **term is** $\frac{ig_o^2 \not{\delta}^{26}}{4\ell_s^6}(2\ell_s^2)^4$ **times:**

$$\begin{aligned}
&k_2^\alpha k_1^\beta k_{1+4}^\gamma k_1^\delta B(-u, 1-t) + k_2^\alpha k_1^\beta k_{1+4}^\gamma k_2^\delta B(1-u, -t) + k_2^\alpha k_1^\beta k_{1+2}^\gamma k_1^\delta B(1-u, 1-t) + k_2^\alpha k_1^\beta k_{1+2}^\gamma k_2^\delta B(2-u, -t) \\
&+ k_2^\alpha k_4^\beta k_{1+4}^\gamma k_1^\delta B(-u, -t) + k_2^\alpha k_4^\beta k_{1+4}^\gamma k_2^\delta B(1-u, -1-t) + k_2^\alpha k_4^\beta k_{1+2}^\gamma k_1^\delta B(1-u, -t) + k_2^\alpha k_4^\beta k_{1+2}^\gamma k_2^\delta B(2-u, -1-t) \\
&+ k_4^\alpha k_1^\beta k_{1+4}^\gamma k_1^\delta B(-1-u, 1-t) + k_4^\alpha k_1^\beta k_{1+4}^\gamma k_2^\delta B(-u, -t) + k_4^\alpha k_1^\beta k_{1+2}^\gamma k_1^\delta B(-u, 1-t) + k_4^\alpha k_1^\beta k_{1+2}^\gamma k_2^\delta B(1-u, -t) \\
&+ k_4^\alpha k_4^\beta k_{1+4}^\gamma k_1^\delta B(-1-u, -t) + k_4^\alpha k_4^\beta k_{1+4}^\gamma k_2^\delta B(-u, -1-t) + k_4^\alpha k_4^\beta k_{1+2}^\gamma k_1^\delta B(-u, -t) + k_4^\alpha k_4^\beta k_{1+2}^\gamma k_2^\delta B(1-u, -1-t)
\end{aligned} \tag{60}$$

The open string amplitude is then given by summing equations (58), (59) and (60) and multiplying that result by $\frac{ig_o^2 \not{\delta}^{26}}{4\ell_s^6}$. Call this $A_o^{\alpha\beta\gamma\delta}(s, t, u, \ell_s, g_o)$. Using **Polchinski 6.6.23** we can write the closed string amplitude as:

$$A_c(s, t, u, \ell_s, g_c) = \zeta_{1,\alpha\bar{\alpha}} \zeta_{2,\beta\bar{\beta}} \zeta_{3,\gamma\bar{\gamma}} \zeta_{4,\delta\bar{\delta}} \frac{\pi i g_c^2 \ell_s^2}{g_o^4} g_o^4 \sin(\pi \ell_s^2 t) A_o^{\alpha\beta\gamma\delta}(s, t, u, \ell_s/2, g_o)[A_o^{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}(t, u, s, \ell_s/2, g_o)]^*$$

where ζ are our 24^2 closed string polarization vectors.

If we had not determined the relationship between g_c' and g_c from the prior problem, we could have determined it by using the KLT relation of the above formula from Polchsinski and specialized to relating g_o' and g_o.

[1]Wasted all of 1/17/20 on this. Not worth it

Then, we would only have needed to look at the (nice) *leading order* $(\alpha')^2$ term in this calculation and observed the pole structure at $s = 0$ corresponding to massless exchange. Making this agree with the square of the 3-point amplitude would then be sufficient. We illustrate the open string case with CP factors in **exercise 11**

7. There are three types of propagators to consider: bulk-bulk, bulk-boundary, and boundary-boundary. Using shorthand $X_i = X(z_i, \bar{z}_i), X_I = X(w_I)$, from **4.7.9** we have:

$$\langle \prod_{i=1}^{m} e^{ip_i X_i} \prod_{I=1}^{n} e^{iq_I X(w_I)} \rangle = \delta\!\!\!/^{26}(\Sigma p + \Sigma q) \exp\Big[- \sum_{i<j} p_i p_j \langle X_i X_j \rangle - \frac{1}{2} \sum_{i,I} p_i q_I \langle X_i X_I \rangle - \sum_{I<J} q_I q_J \langle X_I X_J \rangle \Big]$$

Using the form of the propagators

$$\begin{aligned} \langle X_i X_j \rangle &= -\frac{\ell_s^2}{2} (\log |z_i - z_j|^2 + \log |z_i - \bar{z}_j|^2) \\ \langle X_i X_I \rangle &= -\frac{\ell_s^2}{2} (\log |w_I - z_i|^2 + \log |w_I - \bar{z}_i|^2) \\ \langle X_I X_J \rangle &= -\ell_s^2 \log |w_I - w_J|^2 \end{aligned}$$

we get

$$\delta\!\!\!/^{26}(\Sigma p + \Sigma q) \prod_i |z_i - \bar{z}_i|^{\ell_s^2 p_i^2/2} \prod_{i<j}^{m} |(z_i - z_j)(z_i - \bar{z}_j)|^{\ell_s^2 p_i \cdot p_j} \prod_{I<J} |w_I - w_J|^{2\ell_s^2 q_I q_J} \prod_{I,i} |(w_I - z_i)(w_I - \bar{z}_i)|^{\ell_s^2 p_i \cdot q_I}$$

Note an additional term which I believe Kiritsis dropped. The extension to $\mathbb{RP}^2$ is no more difficult. We now have no boundary and the $\langle X_i X_j \rangle$ propagator is $-\frac{\ell_s^2}{2}(\log(z_i - z_j) + \log(1 + z_i \bar{z}_j))$ so we get:

$$\delta\!\!\!/^{26}(\Sigma p + \Sigma q) \prod_i |1 + z_i \bar{z}_i|^{\ell_s^2 p_i^2/2} \prod_{i<j} |(z_i - z_j)(1 + z_i \bar{z}_j)|^{\ell_s^2 p_i^2/2}$$

8. Forgetting c ghosts here, I can just integrate over all of $\mathbb{H}$. The massless closed-string state of zero momentum is given by $\partial X(z) \bar{\partial} X(z)$. Note that $\mathbb{H} = \mathrm{PSL}_2(\mathbb{R})/SO(2)$, so that:

$$-\frac{\ell_s^2 g_c}{2 g_o^2} \frac{1}{\mathrm{Vol}(\mathrm{PSL}_2(\mathbb{R}))} \int_{\mathbb{H}} dz \frac{1}{|z - \bar{z}|^2} = -\frac{\ell_s^2}{8} \frac{1}{\mathrm{Vol}(\mathrm{PSL}_2(\mathbb{R}))} \int_{\mathbb{H}} \frac{dx dy}{y^2} = -\frac{\ell_s^2}{8} \frac{\mathrm{Vol}(\mathbb{H})}{\mathrm{Vol}(\mathrm{PSL}_2(\mathbb{R}))} = -\frac{\pi \ell_s^2}{2}$$

Note that this answer is finite and invariant under conformal transformation. This gives an amplitude of $-\frac{\pi i}{2} \delta\!\!\!/^{26}(0)$.

9. Let p_1 be the momentum of the closed-string tachyon, and p_2, p_3 the the momenta of the open string tachyons. We get $2p_2 \cdot p_3 = p_1^2 - p_2^2 - p_3^2 = 2/\ell_s^2 \Rightarrow p_2 \cdot p_3 = 1/\ell_s^2, \quad 2p_1 \cdot p_2 = p_3^2 - p_2^2 - p_1^2 = -4/\ell_s^2 \Rightarrow p_1 \cdot p_2 = -2/\ell_s^2$

   I no longer have enough freedom to fix all three points. I can send one to ∞ on the real line, and fix the position of the closed string to be $i \in \mathbb{H}$. The remaining open string insertion can be anywhere on the real line, so we must integrate over this. The ghost and vertex operator correlator gives:

$$(z_1 - \bar{z}_1)(z_1 - w_3)(\bar{z}_1 - w_3) \, |z_1 - \bar{z}_1|^{\ell_s^2 p_1^2/2} |z_1 - w_3|^{2\ell_s^2 p_1 \cdot p_3} \int_{\mathbb{R}} dw_2 \, |w_2 - w_3|^{2\ell_s^2 p_2 \cdot p_3} |w_2 - z_1|^{2\ell_s^2 p_1 \cdot p_2} \, \delta\!\!\!/(\Sigma p)$$

   Setting $z_1 = i, w_3 \to \infty$ has momentum conservation and $p_3^2 = 1/\ell_s^2, p_1^2 = 4/\ell_s^2$ getting the w_3 factors to drop out. We are left with

$$2i \, 2^{\ell_s^2 p_1^2/2} \int_{\mathbb{R}} dw \, (w^2 + 1)^{\ell_s^2 p_1 \cdot p_2} \, \delta\!\!\!/(\Sigma p) = 8i\sqrt{\pi} \, \frac{\Gamma(-\frac{1}{2} + 2)}{\Gamma(2)} \delta\!\!\!/(\Sigma p) = 4\pi i \delta\!\!\!/(\Sigma p)$$

   This gives a scattering amplitude of:

$$-\frac{4\pi g_o^2}{\ell_s^2} \delta\!\!\!/^{26}(\Sigma p).$$

10. The conformal Killing group is now SO(3). Again, we can fix one operator to be at $z = 0$, but the other one can be at any value of $|z| \in [0, 1]$ (we have control over the phase). So we must integrate over the modulus. We do this on the disk using the $\mathbb{RP}^2$ propagator. We insert one vertex operator at 0 and the other z. The integral gives a delta function times:

$$\int_0^1 d|z_2| c(z_1)\tilde{c}(\bar{z}_1)c(z_2)(1+|z_1|^2)^{\ell_s^2 p^2/2}(1+|z_2|^2)^{\ell_s^2 p^2/2}|(z_1-z_2)(1+z_1\bar{z}_2)|^{-\ell_s^2 p^2} \to \int_0^1 r dr\, r^{-\ell_s^2 p^2}(1+r^2)^{\ell_s^2 p^2/2}$$

For the closed string tachyon, we have $p^2 = 4/\ell_s^2$. The integral is divergent, coming from the $(z-w)^{-4}$ singularity as the two tachyons approach one another. If we had the milder $(z-w)^{-1}$ singularity of the open-string tachyon, this could be fixed. **REVISIT**

11. To simplify this problem, as Polchinski asks in his problem 6.9, I will look at the terms that contribute to the $e_1 \cdot e_2\, e_3 \cdot e_4$ amplitude, which comes from contracting $\partial X^\alpha(y_1)\partial X^\beta(y_2)$ and $\partial X^\beta(y_3)\partial X^\delta(y_4)$. There are six possible orderings for the trace in the 4-point amplitude. We get $\frac{ig_o'^4}{g_o^2\ell_s^2}\delta\!\!\!/^{26}(\Sigma p)\times\left(2\ell_s^2\right)^2$ multiplying a sum of six integrals. Using $s := -\ell_s^2(p_1+p_2)^2 = -2p_1\cdot p_2, t := -\ell_s^2(p_1+p_3)^2 = -2p_1\cdot p_3, u := -\ell_s^2(p_1+p_4)^2 = -2p_1\cdot p_4$ and the shorthand [1234] for $\mathrm{Tr}(\lambda^{\mu_1}\lambda^{\mu_2}\lambda^{\mu_3}\lambda^{\mu_4})$, we get:

$$\left[[1234]\int_{-\infty}^0 +[1423]\int_0^1 +[1243]\int_1^\infty\right](|w|^{-u}|1-w|^{-t})dw$$
$$+\left[[1324]\int_{-\infty}^0 +[1432]\int_0^1 +[1342]\int_1^\infty\right](|w|^{-u}|1-w|^{-s})dw$$

Note the second triplet of integrals swaps 2 with 3 so equivalently swaps s and t. We get the amplitude

$$\begin{aligned}\frac{ig_o^2}{2\ell_s^2}e_1\cdot e_2\, e_3\cdot e_4\delta\!\!\!/^{26}(\Sigma p)\Big[\quad &([1234]+[1432])B(1-u,-1-s)\\ &+([1423]+[1324])B(1-t,1-u)\\ &+([1243]+[1342])B(1-t,-1-s)\Big]\end{aligned}$$

Now in the s channel, the first and third Beta functions give us poles at $s = 0$ with residues $-t$ and $-u = t$ respectively. This gives:

$$-\frac{ig_o^2}{2\ell_s^2}\delta\!\!\!/^{26}(\Sigma p)e_1\cdot e_2\, e_3\cdot e_4([1234]+[2143]-[1243]-[2134])\times\frac{t-u}{s} \tag{61}$$

On the other hand, the 3-point vertex (again just the leading order of the two terms, compare with (57)) for massless bosons comes from the correlator

$$\begin{aligned}&\frac{i(g_o')^3}{g_o^2\ell_s^2}|w_{12}w_{13}w_{23}|\langle :\partial X^{\mu_1}(w_1)e^{ik_1X(w_1}::\partial X^{\mu_2}(w_2)e^{ik_2X(w_2)}::\partial X^{\mu_3}(w_3)e^{ik_3X(w_3)}:\rangle\\ \to\ &\frac{i(g_o')^3}{g_o^2\ell_s^2}(-i2\ell_s^2)(-2\ell_s^2)\left(\frac{p_1^{\mu_3}}{w_{12}^2w_{13}}+\frac{p_2^{\mu_3}}{w_{12}^2w_{23}}+2\,\text{perms.}\right)|w_{12}|^{2\ell_s^2p_1\cdot p_2-1}|w_{13}|^{2\ell_s^2p_1\cdot p_3-1}|w_{23}|^{2\ell_s^2p_2\cdot p_3-1}\\ =\ &-ig_o\frac{\sqrt{2}}{\ell_s}(\eta^{\mu_1\mu_2}\frac{1}{2}p_{12}^{\mu_3}+2\,\text{perms.})\end{aligned}$$

using $g_o' = g_o/(\sqrt{2}\ell_s)$. Adding CP factors gives:

$$-\frac{ig_o}{\sqrt{2}\ell_s}(\eta^{\mu_1\mu_2}p_{12}^{\mu_3}+\eta^{\mu_1\mu_3}p_{13}^{\mu_2}+\eta^{\mu_2\mu_3}p_{23}^{\mu_1})\underbrace{([123]-[321])}_{f^{123}}$$

We care about the $e_1\cdot e_2\, e_3\cdot e_4$ term which means we only look at the $p_{12}\cdot p_{34} = t-u$ contribution in the s channel.

$$i\int\frac{d^{26}k}{(2\pi)^{26}}\frac{S(k_1,k_2,k)S(-k,k_3,k_4)}{-k^2+i\epsilon}\to -i\frac{g_o^2}{2\ell_s^2}\delta\!\!\!/^{26}(\Sigma p)\frac{t-u}{s}\times\sum_5\left(f^{125}f^{534}\right)$$

Lastly, note that the factors in equation (61) give $\mathrm{Tr}(f^{12a}\lambda_a f^{34b}\lambda_b)$, and with suitable normalization, this gives $\sum_5 f^{125} f^{534}$, exactly as desired.

We thus see that the amplitude indeed factorizes, respecting the structure of the $U(N)$ gauge group.

12. We have $p^2 + m^2 = \frac{1}{\ell_s^2} L_0$ for the open string. From **5.3.1** (and consequently **5.3.3**) this gives:

$$\frac{i}{2}\frac{V_{26}}{(4\pi)^{26}}\int_0^\infty \frac{dt}{t^{13+1}}\overbrace{\mathrm{Tr}'[e^{-2\pi t m^2}]}^{\text{transverse only}} = \frac{i}{2}\frac{V_{26}}{(16\pi^2\ell_s^2)^{13}}\int_0^\infty \frac{dt}{t^{13+1}}\mathrm{Tr}'[e^{-2\pi t L_0^{\text{cyl}}}] = \frac{i}{2}\frac{V_{26}}{(16\pi^2\ell_s^2)^{13}}\int_0^\infty \frac{dt N_1 N_2 \eta(it)^2}{t^{13+1}\eta(it)^{26}}$$

All together this gives:

$$iN_1N_2V_{26}\int_0^\infty \frac{dt}{2t}\frac{1}{(8\pi^2\ell_s^2 t)^{13}\eta(it)^{24}}$$

as required.

13. We already know the form of our propagators on the torus from exercise **4.69**. Take

$$G(z,w) = \left|\frac{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](z-w,\tau)}{\partial_z\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0,\tau)}\right|^2 e^{-2\pi(\mathrm{Im} z)^2/\tau_2}.$$

This gives us

$$\langle\prod_i :e^{ik_i X(z_i,\bar{z}_i)}:\rangle = iZ_{T^2}\delta\!\!\!/^{26}(\Sigma k)\prod_{i<j}|G(z_i,z_j)|^{\ell_s^2 k_i\cdot k_j/2}$$

where Z_{T^2} which is equal to the partition function of the torus $Z(\tau)$ that we have also computed in the last chapter. The amplitude is then:

$$i\delta\!\!\!/^{26}(\Sigma k)\frac{g_c^n}{(2\pi\ell_s)^{26}}\int\frac{d^2\tau}{\tau_2^2}\frac{1}{\tau_2^{12}|\eta|^{48}}\prod_{i=1}^n\int dz_i\prod_{i<j}|G(z_i,z_j)|^{\ell_s^2 k_i\cdot k_j/2}$$

14. We need to calculate the form of the propagators $\langle X^\mu(z)X^\nu(w)\rangle$ on the cylinder with NN boundary conditions. Let's use the image charge method. The finite cylinder can be thought of as the fundamental domain of the quotient of the upper half plane by the action $z\to\lambda z$ for λ a real number corresponding to the modulus of the cylinder. For X at z where $1<|z|<\lambda$ we place images at each $\lambda^n z$ in the upper half plane as well as at $\lambda^n\bar{z}$ on the lower half plane.

$$\langle X(z)X(w)\rangle = -\frac{\ell_s^2}{2}\sum_{n\in\mathbb{Z}}\left(\log|\lambda^{-n/2}z-\lambda^{n/2}w|^2+\log|\lambda^{-n/2}z-\lambda^{n/2}\bar{w}|^2\right)$$

This gives

$$\langle\prod_i :e^{ip_iX}:\rangle = \delta\!\!\!/^D(\Sigma p)\prod_n\prod_{i<j}|(\lambda^{-n/2}z_i-\lambda^{n/2}z_j)(\lambda^{-n/2}z_i-\lambda^{n/2}\bar{z}_j)|^{\ell_s^2 p_i\cdot p_j}$$

For open strings (operators inserted at the boundary) we must apply boundary normal ordering. We'll get:

$$\langle\prod_i {}^\star_\star e^{iq_iX}{}^\star_\star\rangle = \delta\!\!\!/^D(\Sigma q)\prod_n\prod_{I<J}|(\lambda^{-n/2}w_I-\lambda^{n/2}w_J)|^{2\ell_s^2 q_I\cdot q_J}$$

Lastly, for the correlations between boundary and bulk operators we'll get:

$$\prod_n\prod_{i,I}|\lambda^{-n/2}w_i-\lambda^{n/2}z_i|^{2\ell_s^2 p_i\cdot q_I}$$

Taking the product of the above three equations (with only a single momentum-conserving delta function) gives us the X correlator on the cylinder. The CKG here is simply the compact $SO(2)$ so it is best to ignore ghosts, integrate the insertions over the whole cylinder and divide at the end by the volume of the $SO(2)$ action: λ.

There is a cleaner way to do this. From exercise **4.69** we know the cylinder propagator can be written in terms of the torus propagator as an involution:

$$\Delta_{C_2}(z-w) = \Delta(z-w, it) + \Delta(z+\bar{w}, it)$$

Here $\Delta = -\frac{\ell_s^2}{2}\log G(z,\bar{z})$ from the problem above. This will then give us for m closed string and n open string tachyons:

$$\begin{aligned}\langle\prod_i :e^{ip_i X(z_i)}: \prod_I {}^\star_\star e^{iq_I X(w_I)}{}^\star_\star\rangle =& i\delta^D(\Sigma p + \Sigma q)\frac{g_c^m g_o^n}{(2\pi\ell_s)^{26}}\int_0^\infty \frac{dt}{2t}\frac{1}{(2t)^{13}\eta(it)^{24}}\prod_{i=1}^m\int_C dz_i \prod_{I=1}^n\int_{\partial C} dw_I \\ &\times \prod_{i<j}[G(z_i-z_j;\tau=it)G(z_i+\bar{z}_j;\tau=it)]^{\ell_s^2 p_i\cdot p_j/2}\prod_{I<J}^n G(w_I-w_J;\tau=it)^{\ell_s^2 q_I\cdot q_j} \\ &\times \prod_{i,I}[G(w_I-z_i;\tau=it)G(w_I+\bar{z}_i;\tau=it)]^{\ell_s^2 p_i\cdot q_I/2}\end{aligned}$$

15. Here I assume Kiritsis meant $\epsilon_c = 1$, since equation **3.4.3** refers specifically to closed string ground states. The open string constraint $\epsilon_o = 1$ comes from consistency of interactions stemming from the Jacobi identity for Lie algebras. The one-loop contribution for the unoriented closed string comes from the cylinder + Klein bottle + Möbius strip amplitude. As before, the only nonzero contributions come from states with an equal number of left and right movers. All that this gives is an overall factor of ϵ_c in this amplitude:

$$Z_{K_2} := \frac{1}{2}\mathrm{Tr}[\Omega e^{-2\pi t(L_0+\bar{L}_0-c/12)}] = \frac{iV_{26}}{2}\int\frac{d^{26}p}{(2\pi)^{26}}\epsilon_c\frac{e^{-\pi\ell_s^2 tp^2}}{\eta(2it)^{24}} \Rightarrow \Lambda_{K_2} = i\frac{V_{26}\epsilon_c}{(2\pi\ell_s)^{26}}\int_0^\infty\frac{dt}{4t^{1+13}\eta(2it)^{24}}$$

And working in the transverse channel $t = \pi/2\ell$ gives massless contribution:

$$\epsilon_c 2^{26}\times 24i\frac{V_{26}}{4\pi(2\sqrt{2}\pi\ell_s)^{26}}\int_0^\infty d\ell$$

Similarly, the Möbius strip amplitude is given by

$$Z_{M_2} = \frac{1}{2}\mathrm{Tr}_o[\Omega e^{-2\pi t(L_0-c/24)}] = \frac{iV_{26}}{2}\int\frac{d^{26}p}{(2\pi)^{13}}\frac{\zeta N e^{2\pi t\ell_s^2 p^2}}{(\eta(2it)\theta_3(2it))^{12}} \Rightarrow \Lambda_{M_2} = i\frac{V_{26}\zeta N}{(2\pi\ell_s)^{26}}\int_0^\infty\frac{dt}{2(2t)^{1+13}(\theta(2it)\eta(2it))^{12}}$$

In the transverse channel with $2t = \pi/2\ell$ now:

$$-\zeta N 2^{1+13}\times 24i\frac{V_{26}}{4\pi(2\sqrt{2}\pi\ell_s)^{26}}\int_0^\infty d\ell$$

This gives a tadpole cancelation condition of:

$$\epsilon_c 2^{26} - 2^{14}\zeta N + N^2 = 0$$

We have N is a positive integer. Further, we have that ζ is a *sign.* If $\zeta = -1$ then ϵ must be negative, and so by unitarity it is -1, but there are no integer solutions N to $2^{26} = 2^{14}N + N^2$. Thus we need $\zeta = 1$ and consequently $\epsilon = -1, N = 2^{13}$.

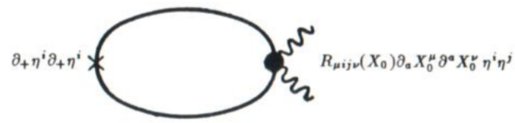
Chapter 6: Strings in Background Fields

Note this chapter is specific to *closed oriented strings.* As such, we will not consider the effects of the boundary.

0. This is not a required problem but it certainly should be [2]. Let's *calculate the β-functions of the nonlinear sigma model.* Here, I will borrow diagrams from the very nice set of TASI lecture notes of Callan and Thorlacius

   First, it is worth using a normal coordinate system for the X^μ (one in which all of the Γ symbols vanish and all higher symmetrized Γ symbols also vanish). We want to look at radiative corrections to $\langle T_{++}\rangle$, since they have integrals that are easier to handle than those for $\langle T_{+-}\rangle$. From conservation this will give us the trace anomaly for $\langle T_{+-}\rangle$. We will first look at how G, B affect the trace on a flat worldsheet.

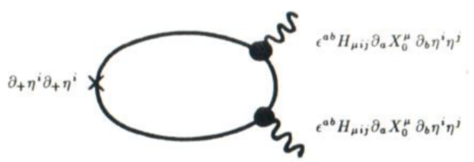
   For the graviton contribution to β^G, we have only only one diagram

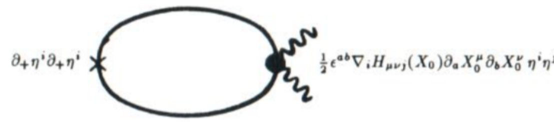


   This contributes an anomalous trace of

$$\langle T_{+-}\rangle = \frac{1}{4} R_{\mu\nu}\, \partial_a X_0^\mu \partial^a X_0^\nu$$

   For the B contribution to β^B, we have two such diagrams:





   These contribute anomalous traces of:

$$-\frac{1}{16} H_{\mu\rho\sigma} H_{\nu\rho\sigma}\, \partial_a X_0^\mu \partial^a X_0^\nu, \quad \frac{1}{8}\nabla^\lambda H_{\mu\nu\lambda}\epsilon^{ab}\partial_a X_0^\mu \partial_b X_0^\nu$$

   respectively.

   The dilaton contribution *also* affects the trace on the flat world sheet (even though it does not couple at $R = 0$), by affecting the stress energy tensor as it is defined by varying the action w.r.t. the metric. Kiritsis has worked this out before and shown that the dilaton contributes $(\partial_a\partial_b - g_{ab}\Box)\Phi$ to the stress energy tensor, from which we get a dilaton contribution of $\Box_\xi \Phi(X(\xi))$ to the trace. Using covariant expressions for the D'alambertian we arrive at a contribution

$$\nabla_\mu\nabla_\nu\Phi(X_0)\, \partial_a X_0^\mu \partial^a X_0^\nu - \frac{1}{2}\nabla^\lambda\Phi(X_0) H_{\mu\nu\lambda}(X_0)\partial_a X_0 \partial_b X_0 \epsilon^{ab}$$

   Combining all of this together, we see that we will get the β-functions:

$$\beta^G = R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_\nu^{\rho\sigma} + 4\nabla_\mu\nabla_\nu\Phi, \quad \beta^B = -\frac{1}{2}\nabla^\lambda H_{\lambda\mu\nu} - 2\nabla^\lambda\Phi H_{\lambda\mu\nu}.$$

   As pointed out, these are not quite that RG beta functions (for example compare β^B to the correct form in Kiritsis), but around the fixed point, they capture the correct first order behavior. In particular their vanishing will mean that we have no Weyl anomaly.

[2] After seeing the details of this calculation, I can understand why it was omitted.

Now we need to account for the effects of a curved worldsheet geometry. We can account for this by looking at a $\langle T_{+-}T_{+-}\rangle$ correlator:

$$\frac{\delta}{\delta\phi(\xi)}\langle T_{+-}(0)\rangle_{e^{\phi}\delta_{ab}} = -\frac{1}{4\pi}\langle T_{+-}(\xi)T_{+-}(0)\rangle_{\delta_{ab}} \tag{62}$$

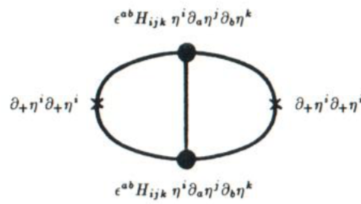
Again we can get this by first looking at $\langle T_{++}T_{++}\rangle$ and appealing to conservation. The Weyl anomaly comes from this diagram:

This gives $\langle T_{+-}T_{+-}\rangle = \frac{\pi D}{12}\Box\delta^{(2)}(\xi)$. Here we have a factor of D coming from each degree of freedom. This can be used to integrate equation (62) to yield:

$$\langle T_{+-}\rangle = -\frac{D}{48}\Box\phi = \frac{D}{24}\sqrt{\gamma}\,R$$

Note that the ghosts (which are otherwise decoupled) will here contribute their factor of -26.

We also now need to consider *two-loop* contributions of G, B to the TT correlator. The following diagrams contribute:



The calculations here are very involved, but will precisely give us

$$\frac{\alpha}{8}(-R + \frac{H^2}{12})$$

Finally, the dilaton both modifies the energy-momentum tensor, giving rise to a tree-level propagator contribution to the two-point function:

This contributes $\langle T_{+-}^{dil}T_{+-}^{dil}\rangle = \pi\alpha'(\nabla\Phi)^2\Box\delta^{(2)}(\xi)$ which will integrate to give a factor of $\frac{\alpha'}{2}(\nabla\Phi)^2\sqrt{\gamma}R$.

Also, the dilaton gives a loop-contribution to the unmodified energy-momentum tensor:

Which contributes the term $\langle T^{dil}_{+-}T^{dil}_{+-}\rangle = -\pi\alpha'\Box\Phi\Box\delta^{(2)}(\xi)$ which will integrate to give a factor of $-\frac{\alpha'}{2}\Box\Phi\sqrt{\gamma}R$. Altogether this gives:

$$\beta^{\Phi} = D - 26 + \frac{3}{2}\alpha'\left[4(\nabla\Phi)^2 - 4\Box\Phi - R + \frac{1}{12}H^2\right].$$

as required.

1. Each β-function of a coupling constant G, B, Φ as given in **6.1.5, 6.1.6, 6.1.7** is $\frac{\delta}{\delta\phi}$ of that coupling constant, since our scaling $\mu = e^{\phi} \Rightarrow \log\mu = \phi$. Since

$$T^a_a = \frac{\beta^{\Phi}}{12}R^{(2)} + \frac{1}{2\ell_s^2}(\beta^G_{\mu\nu}g^{\alpha\beta} + \beta^B_{\mu\nu}\varepsilon^{\alpha\beta})\partial_\alpha X\partial_\beta X$$

The change in effective action under an infinitesimal Weyl transformation $\delta g^{\alpha\beta} = -g^{\alpha\beta}\delta\phi$ is

$$\delta\log Z = -\delta S = \frac{1}{4\pi}\int d^2\xi\sqrt{g}T^a_a\delta\phi = \frac{1}{4\pi}\int d^2\xi\left[\frac{\beta^{\Phi}}{12}\sqrt{g}R^{(2)} + \frac{1}{2\ell_s^2}\left(\beta^G_{\mu\nu}\sqrt{g}\,g^{ab} + \beta^B_{\mu\nu}\varepsilon^{ab}\right)\partial_a X\partial_b X\right]\delta\phi$$

We can integrate this to get the change after a finite conformal transformation:

$$\frac{1}{4\pi}\int d^2\xi\left[\sqrt{g}\beta^{\Phi}\left(R^{(2)}\phi - \frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi\right) + \frac{\phi}{2\ell_s^2}\left(\beta^G_{\mu\nu} + \beta^B_{\mu\nu}\epsilon^{ab}\right)\partial_a X\partial_b X\right]$$

this vanishes, of course, when all beta functions are zero. When β^G, β^B are zero we can show (exercise 3) that β^{Φ} is a constant, and we recover the Liouville action from before.

2. First write G explicitly in the action:

$$S = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-\det G}e^{-2\Phi}\left[R + 4G^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi - \frac{1}{12}G^{\alpha\delta}G^{\beta\epsilon}G^{\gamma\zeta}H_{\alpha\beta\gamma}H_{\delta\epsilon\zeta} + 2\frac{26-D}{3\ell_s^2}\right]$$

The classical equations of motion from varying the action with respect to G give

$$\begin{aligned}0 &= \overbrace{R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \cancel{4\nabla_\mu\Phi\nabla_\nu\Phi} - 2G_{\mu\nu}\Box\Phi + 4G_{\mu\nu}(\nabla\Phi)^2}^{R\text{ variation}} + \overbrace{\cancel{4\nabla_\mu\Phi\nabla_\nu\Phi}}^{(\nabla\Phi)^2\text{ variation}} \\ &\quad \underbrace{-\frac{1}{4}H_{\mu\rho\sigma}H_\nu^{\rho\sigma}}_{H^2\text{ variation}} \underbrace{-\frac{1}{2}G_{\mu\nu}\left(R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2 + \frac{2}{3\ell_s^2}(26-D)\right)}_{\sqrt{-\det G}\text{ variation}} \\ &= \underbrace{R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\rho\sigma}H_\nu^{\rho\sigma}}_{:=\beta^G_{\mu\nu}} -\frac{1}{2}G_{\mu\nu}\left(R - 4(\nabla\Phi)^2 + 4\Box\Phi - \frac{1}{12}H^2 + 2\frac{26-D}{3\ell_s^2}\right)\end{aligned} \tag{63}$$

With respect to B we get:

$$-\frac{1}{12}e^{-2\Phi}(2(\delta_{B^{\mu\nu}}(\partial_\alpha B_{\beta\gamma} + 2\,\text{perms.}))H^{\alpha\beta\gamma}) \overset{IBP}{\longrightarrow} \frac{2\times 3}{12}e^{-2\Phi}(\nabla^\alpha H_{\alpha\mu\nu}) \overset{IBP}{\longrightarrow} \underbrace{-\frac{1}{4}\nabla^\alpha(e^{-2\Phi}H_{\alpha\mu\nu})}_{:=\beta^B_{\mu\nu}} = 0$$

Finally, with respect to Φ we get:

$$0 = -2\left(R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2 + 2\frac{26-D}{3\ell_s^2}\right) - 8\Box\Phi - 16(\nabla\Phi)^2 = -2\underbrace{\left(R - 4(\nabla\Phi)^2 + 4\Box\Phi - \frac{1}{12}H^2 + 2\frac{26-D}{3\ell_s^2}\right)}_{:=-\frac{2}{3}\beta^{\Phi}}$$

The term in parentheses is the same as the term in parentheses the bottom line of (63). This agrees with **Polchinski 3.7.21** (with appropriate conventions adopted)

$$\delta S = -\frac{1}{2\kappa^2}\int d^Dx\sqrt{-\det G}e^{-2\Phi}\left[\delta G^{\mu\nu}\left(\beta^G_{\mu\nu} - \frac{1}{2}G_{\mu\nu}\frac{2}{3}\beta^{\Phi}\right) + \delta B^{\mu\nu}\beta^B_{\mu\nu} + 2\delta\Phi\frac{2}{3}\beta^{\Phi}\right]$$

3. Let's look at $\frac{2}{3\ell_s^2}\nabla\beta^\Phi$. We get:

$$8\nabla_\nu\Phi\nabla_\mu\nabla^\nu\Phi - 4\Box\nabla_\mu\Phi - \nabla_\mu R + \frac{1}{6}(\nabla_\mu H_{\alpha\beta\gamma})H^{\alpha\beta\gamma}$$

The contracted Bianchi identity $\nabla_\mu R = 2\nabla^\nu R_{\nu\mu}$ together with the vanishing of $\beta^G_{\mu\nu}$ gives:

$$\nabla_\mu R = 2\nabla^\nu R_{\mu\nu} = \frac{1}{2}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) - 4\Box\nabla_\mu\Phi$$

which in turn gives

$$8\nabla_\nu\Phi\nabla_\mu\nabla^\nu\Phi - \frac{1}{2}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) + \frac{1}{6}(\nabla_\mu H_{\alpha\beta\gamma})H^{\alpha\beta\gamma}$$

The fact that H is exact gives us $\mathrm{d}H = 0$ so $\partial_{[\alpha}H_{\beta\gamma\delta]} = 0$. The symmetry properties of H imply that summing over the four cyclic permutations of this gives zero. Contracting with the metric then implies a contracted Bianchi-type identity for H, namely that $\nabla^\alpha H_{\alpha\beta\gamma} = 0$.

Using $\beta^B = 0$ together with the Bianchi identity, we have $0 = \nabla^\rho H_{\mu\nu\rho} = 2\nabla^\rho\Phi\, H_{\mu\nu\rho}$. So we have that H is divergence-free, and $\nabla^\rho\Phi$ dotted with any component of H is zero. This lets us rewrite:

$$\begin{aligned}
&-\frac{1}{2}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) = -\frac{1}{2}H^{\nu\rho\sigma}\nabla_\nu H_{\mu\rho\sigma} \\
&\frac{1}{6}\nabla_\mu(H_{\alpha\beta\gamma})H^{\alpha\beta\gamma} = -\frac{1}{6}H^{\alpha\beta\gamma}\left(\nabla_\alpha H_{\beta\gamma\mu} - \nabla_\beta H_{\gamma\alpha\mu} + \nabla_\gamma H_{\alpha\beta\mu}\right) = -\frac{1}{6}H^{\nu\rho\sigma}\nabla_\nu H_{\mu\rho\sigma} \\
&\Rightarrow \frac{1}{12\ell_s^2}\nabla_\mu\beta^\Phi = \nabla_\nu\Phi\nabla_\mu\nabla^\nu\Phi - \frac{1}{12}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) = -\frac{1}{2}\nabla^\nu\Phi R_{\mu\nu} - \frac{1}{12}\nabla^\nu H_{\mu\nu}
\end{aligned}$$

One last step. I am missing something.

This gives that $\nabla_\mu\beta^\Phi = 0$ as required. So $\beta^\Phi = c$ is a constant.

4. We get a linear dilaton giving rise to a Liouville action with $Q = 0$. This is our familiar free massless boson in $2D$ with $1D$ target space. So we get a string propagating in a single dimension.

5. Note that the only relevant parameters are ℓ_s, with units of length, and whatever length scales there are on the manifold, all of which depend on its volume (since its compact) as $V^{1/D}$. In particular $c = \beta^\Phi$ depends on ℓ_s as

$$c = D + O(\ell_s^2/V^{2/D}).$$

I think this is correct, though it is different from Kiristis' equation.

6. Note that a nonzero total flux of H over any closed 3-manifold is incompatible with $H = dB$ for a single-valued B. We can write:

$$e^{\frac{i}{2\pi\ell_s^2}\int_M B} = e^{\frac{i}{2\pi\ell_s^2}\int_N H}$$

where M is the 2D manifold corresponding to the embedding of the world-sheet into the target space and N is any manifold whose boundary is M. We need this to be independent of N, so for any three-cycle M_3 we need:

$$\frac{1}{2\pi\ell_s^2}\int_{M_3} H \in 2\pi\mathbb{Z} \Rightarrow \frac{1}{4\pi^2\ell_s^2}\int_{M_3} H \in \mathbb{Z}$$

7. (a) We have

$$H = 2R^2 \sin^2\psi \sin\theta \, \mathrm{d}\psi \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi \Rightarrow \int_{S^3} H = \frac{(2\pi R)^2}{4\pi^2 \ell_s^2} = \frac{R^2}{\ell_s^2} \in \mathbb{Z}$$

(b) The dilaton is $\Phi = 0$. Using Mathematica, the Ricci tensor is:

$$R_{\mu\nu} = \mathrm{diag}(2, 2\sin^2\psi, 2\sin^2\psi\sin^2\theta)$$

Which gives a Ricci scalar of $6/R^2$. From the previous part, $H_{123} = 2R^2 \sin^2\psi \sin\theta$. From the metric being diagonal, we get that $H^2_{\mu\nu} := H_{\mu\rho\sigma} H_\nu^{\rho\sigma}$ is diagonal. We have

$$H^2_{\mu\nu} = \mathrm{diag}(8, 8\sin^2\psi, 8\sin^2\psi\sin^2\theta) \Rightarrow \beta^G = R_{\mu\nu} - \frac{1}{4} H^2_{\mu\nu} = 0$$

as desired. Next, $\beta^B_{\mu\nu} = -\frac{1}{2}\nabla^\alpha(H_{\mu\nu\alpha})$. To take a contravariant divergence we divide by the volume element and differentiate, but the volume element is $\sin^2\psi\sin\theta$ which will give $H/\sqrt{g}$ is a constant, so $\beta_{\mu\nu}$ will vanish.

Lastly, $H^2 = (2R^2)^2/R^6 = 2/R^6$ so that $-R + \frac{1}{12}H^2 = -\frac{4}{R^2}$. Ignoring ghosts, this gives a central charge of:

$$D - 6\frac{\ell_s^2}{R^2} + O(\ell_s^4) = D - \frac{6}{k} + O(\ell_s^4)$$

as desired.

(c) Without using coordinates, the isometry of S^3 is $G = \mathrm{SO}(4) = [\mathrm{SU}(2) \times \mathrm{SU}(2)]/\mathbb{Z}_2$. TO see that equivalence, think of of S^3 as the unit quaternions, and take $\mathrm{SU}(2) \times \mathrm{SU}(2)$ act as unit quaternions on the left and right. We get a right G-action by: $x \to a^{-1}xb$. Note the kernel is the set of $(a, b) \in G$ $ax = xb$ for all x. In particular, for $x = 1$ we get $a = b$ so the kernel lies in the diagonal subgroup. To act trivially on all quaternions, a must be in the center, and for the unit quaternions this is exactly ± 1. So this is an injection $\varphi : [\mathrm{SU}(2) \times \mathrm{SU}(2)]/\mathbb{Z}_2 \to \mathrm{SO}(4)$. Since $\mathrm{SO}(4)$ is compact and connected, it is generated by the image of exponentiating $\mathfrak{so}(4)$, and so surjectivity of φ at the level of the Lie algebras (which is true by dimension-counting) implies surjectivity and hence equivalence at the level of Lie groups.

So we see that $\mathfrak{so}(4)$ acting on S^3 is just a simultaneous left and right copt of $\mathfrak{su}(2)$ acting on $\mathrm{SU}(2)$. Thus, we view this as the CFT of a nonlinear sigma model with target space $G = \mathrm{SU}(2)$ and the left, right copies of the $\mathfrak{su}(2)$ action correspond to currents $J = g^{-1}\partial g$ and $J = \bar{\partial} g \, g^{-1}$

We indeed get the central charge $c = \frac{3k}{k+2}$ which has the large k expansion $3 - 6/k + O(1/k^2)$. Since k in a non-negative integer in WZW models, except for the case $k = 0$ corresponding to the trivial CFT, we must have $k \geqslant 1$, where we get $R \geqslant \ell_s$.

8. Here the metric has three degrees of freedom and $B_{\mu\nu}$, Φ both have only one degree of freedom (which can be spatially varying). H, being a 3-index antisymmetric tensor, must vanish in $1+1$D, and so we will always have $\beta^B = 0$. The other two constraints become:

$$0 = \beta^G_{\mu\nu} = \frac{1}{2} R g_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi, \qquad 0 = \beta^\Phi = -24 + \frac{3}{2}\ell_s^2 \left[4(\nabla\Phi)^2 - 4\Box\Phi - R\right]$$

Translational isometry implies that R, g depend on only the time variable t. The x variable can therefore parameterize either S^1 or $\mathbb{R}$ endowed with constant metric.

Now taking the trace of the first equation implies $R(t) = -2\Box\Phi(x, t)$. Then the second equation will give:

$$\frac{16}{\ell_s^2} = 4(\nabla\Phi(x,t))^2 - 2(\Box\Phi)(t)$$

The only way for this to work is for $R = \Box\Phi = 0$ so that $\nabla\Phi$ can be a constant. We then have $\Phi = \alpha x + \beta t$ so that $\alpha^2 + \beta^2 = 4/\ell_s^2$, and g is Ricci flat everywhere (so we can pick it to be constant). In the case of either $\alpha, \beta = 0$, we can also safely take x, t respectively to be periodic without having Φ be multi-valued.

9. We still have $\beta^B = 0$, but $\beta^G = R_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi$ while $\beta^\Phi = D - 26 + \frac{3}{2}\ell_s^2(4(\nabla\Phi)^2 - 4\Box\Phi - R)$
   This can be recast in terms of a new 4D *Ricci flat* metric $ds^2 = F(\phi)d\phi^2 + \phi R^2 d\Omega_3^2$.
   Using Mathematica again to take the trace of this gives R_{ij} for $i = j \geqslant 1$ proportional to $R^2\phi F'(\phi) + 8\phi F(\phi)^2 - R^2 F(\phi)$. Solving this differential equation for F gives
   $$F(\phi) = \frac{R^2\phi}{4\phi^2 + R^2 c_1}$$
   Setting $c_1 = 0, F(\phi) = R^2/4\phi$ will also make R_{00} vanish. Then we can take the dilaton to be zero $\Phi(\phi) = 0$.

10. As stated in the problem, upon gauging the adapted compact $U(1) : \theta \to \theta + \epsilon$, which has radius 2π, we modify our derivative operator to act as $\partial_\alpha \theta \to \partial_\alpha \theta + A_\alpha$, where A_α gives our connection on the $U(1)$ principal bundle associated with gauging the Killing symmetry. The action gets modified:
   $$S \supseteq \frac{R^2}{4\pi\ell_s} \int |\partial\theta|^2 \to \frac{R^2}{4\pi\ell_s^2} \int |\partial\theta + A|^2$$
   This is a new theory, but we can *return to the old one* by enforcing that A be pure gauge as follows: introduce an auxiliary field ϕ and add to S the term
   $$\frac{i}{2\pi} \int \phi\, \epsilon^{\alpha\beta} \partial_\alpha A_\beta = -\frac{i}{2\pi} \int \mathrm{d}\phi \wedge A.$$
   Integrating out ϕ gives exactly a δ-function enforcing $\epsilon^{\alpha\beta}\partial_\alpha A_\beta = 0$. This gives that A is closed, but it need not be exact if our manifold has nontrivial topology. Going around any cycle, $\int A$ can pick up a factor of $2\pi n$.
   For a closed, genus g Riemann surface, there are $2g$ cycles labeled by a_i, b_i, $1 \leqslant i \leqslant g$ coming from viewing it as a $2g$-gon. we have *Riemann's bilinear identity*, namely for two closed 1-forms ω_1, ω_2,
   $$\int_\Sigma \omega_1 \wedge \omega_2 = \sum_{i=1}^{g} \left(\int_{a_i} \omega_1 \int_{b_i} \omega_2 - \int_{a_i} \omega_2 \int_{b_i} \omega_1 \right) \tag{64}$$
   Now take $\omega_1 = A$, $\omega_2 = \mathrm{d}\phi$. Now (64) gives us that $\frac{1}{2\pi}\int \mathrm{d}\phi \wedge A$ will not be zero in general, but in the path integral, it suffices to have it be an integral multiple of 2π, since then the nontrivial holonomies will have no contribution to the action. We have that A can have winding $2\pi\mathbb{Z}$, so the only solution is to have ϕ have winding $2\pi\mathbb{Z}$. This will exactly leave over a factor of $2\pi\mathbb{Z}$. So we return to our original action by introducing the field ϕ of period 2π. (NB if I had kept the fields dimensionful, then ϕ would have period $2\pi/R$ when θ has period $2\pi R$)
   In this new, equivalent action, we can gauge-fix $\theta = 0$ (do I need ghosts? No because this is abelian $U(1)$) and integrate out A. We get:
   $$\frac{\ell_s^4/R^2}{4\pi\ell_s^2} \int d^2\xi\, (\partial\phi)^2$$
   so we have obtained the same action but now on a circle of radius ℓ_s^2/R instead of R.
   In doing this path integral we get a determinant factor of $\sqrt{4\pi^2\ell_s^2/R^2} = 2\pi\ell_s/R$ for each mode. Using zeta function regularization this is equal to $\sqrt{R/2\pi\ell_s}$ which we can understand as adding a $-\frac{1}{2}\log(R/2\pi\ell_s)$ term to the action that will couple to the curvature R **(Show why)**, this shifting the dilaton as required.

11. We can simplify things by using the conventions of the next problem to do this one. Here, we have a *single* compact coordinate θ. In our convention:
   $$\hat{G}_{\mu\nu} = \begin{pmatrix} G_{00} & G_{00}A_j \\ G_{00}A_i & g_{ij} + G_{00}A_iA_j \end{pmatrix}, \quad B_{\mu\nu} = B_j d\theta \wedge dx^i + A_i B_j b_{ij} dx_i \wedge dx_j, \quad \phi = \Phi - \frac{1}{4}\log\det G_{00}$$

From formula **F.3** specialized to this case, we get that the metric and dilaton terms become

$$\int d^D x\sqrt{-\det \hat{G}_{\mu\nu}}e^{-2\Phi}\left[\hat{R}+4(\partial_\mu\Phi)^2\right]=\int d^{D-1}x\sqrt{-\det g}\,e^{-2\phi}\left[R+4(\partial_\mu\phi)^2+\frac{1}{4}\partial_\mu G_{00}\partial^\mu G^{00}-\frac{1}{4}G_{00}(F^A_{\mu\nu})^2\right] \tag{65}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $\hat{R}$ corresponds to the original $\hat{G}_{\mu\nu}$ while R corresponds to g_{ij}. Further $G^{00}=$

From **F.6-F.9**, the antisymmetric tensor changes as:

$$-\frac{1}{12}\int d^D\sqrt{-\det \hat{G}}\,e^{-2\Phi}\hat{H}_{ijk}\hat{H}^{ijk}=-\int d^{D-1}x\sqrt{-\det g}\,e^{-2\phi}\left[\frac{1}{12}H_{ijk}H^{ijk}+\frac{1}{4}\hat{H}_{ij0}\hat{H}^{ij0}\right] \tag{66}$$

Here where $H_{ij0}=\hat{H}_{ij0}$ and $H_{ijk}=\hat{H}_{ijk}-(A_iH_{0jk}+3\,\text{perms.})$. Here H_{ijk} is defined so that it is invariant under T-duality **(TYSM Kiritsis for pre-organizing these terms for me)** . Further, under T-duality

$$\begin{aligned}
G_{00}&\to G_{00}^{-1}=G^{00}\Rightarrow \partial_\mu G_{00}\partial^\mu G^{00}\text{ invariant}\\
g_{ij}&\to g_{ij}\Rightarrow R\text{ invariant}\\
A_i&\to B_i\\
B_i&\to A_i\\
\Phi&\to\Phi-\frac{1}{2}\log G_{00}\Rightarrow\phi\to\phi\Rightarrow(\partial_\mu\phi)\text{ invariant.}
\end{aligned} \tag{67}$$

We see that the $\sqrt{-\det g}\,e^{-2\phi}$ as well as first three terms of equation (65). We have that $F^A_{\mu\nu}\to\partial_\mu B_\nu-\partial_\nu B_\mu=:F^B_{\mu\nu}$ and $F^B_{ij}=H_{ij0}$. The last term of (65) will therefore become swap with the last term of (66) and we are done.

12. This one is quick. We have

$$ds^2=G_{00}d\theta^2+2G_{00}A_idx^idx^0+G_{ij}dx^idx^j,\quad B=B_jd\theta\wedge dx^j+(b_{ij}+A_iB_j)dx^i\wedge dx^j$$

Certainly we have $\tilde{G}_{00}=1/G_{00},\tilde{B}_i=G_{00}A_i/G_{00}$. Then $\tilde{A}_i=B_i$ is consistent both for the $i,0$ components of the line element and the $dx^i\wedge dx^j$ components of the B-field as long as we keep $\tilde{b}_{ij}=b_{ij}$ and $\tilde{g}_{ij}=g_{ij}$. Finally, the dilaton must be shifted by $\Phi=\Phi-\frac{1}{2}\log G_{00}$.

13. The N commuting isometries correspond to a fibration by N-dimensional tori over each point in the base space. As we have seen before (for strings valued in a N-dimensional torus target space), we have that modes are described by two momenta p_L,p_R that Lie on an integral lattice. Naively, we can rotate p_L,p_R by any $\mathrm{GL}(N)$ transformation, but the integrality condition restricts us to $\mathrm{GL}(N,\mathbb{Z})$. Now $\mathrm{GL}(N)$ acts separately on the left and the right momenta, but we are allowed to exchange between these two by applying T-duality, which still preserves our Lorentzian norm, so the T-duality group gets enhanced to $O(N,N,\mathbb{Z})$.

14. This is clear, since orientation reversal acts trivially on $g^{ab}G_{\mu\nu}\partial_aX^\mu\partial_bX^\nu$ while it acts with a minus sign on $\epsilon^{ab}B_{\mu\nu}\partial_aX^\mu\partial_bX^\nu$. The corresponding vertex operators are:

$$:\partial X^\mu\bar{\partial}X_\mu e^{ikX}:,\quad :G_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu:,\quad R:e^{ikX}:$$

If we assume the tachyon $:e^{ikX}:$ is negative under parity then so are the dilaton and graviton.

This is incompatible with **6.1.10**, as then parity will flip the sign of the dilaton in the exponential, substantially changing the action of the theory.

Chapter 7: Superstrings and Supersymmetry

1. We already know that TT will have the desired OPE, since the bosons and fermions are uncoupled and we already have shown their own respective stress tensor OPEs. Next

$$\begin{aligned}
G(z)G(w) &= -\frac{2}{\ell_s^4}\psi_\mu(z)\partial X^\mu(z)\psi_\nu(w)\partial X^\nu(w) \\
&= -\frac{2}{\ell_s^4}\left(\ell_s^2\frac{\eta_{\mu\nu}}{z-w} + (z-w) : \partial\psi_\mu\psi_\nu(w) :\right)\left(-\frac{\ell_s^2}{2}\frac{\eta_{\mu\nu}}{(z-w)^2} + : \partial X_\mu \partial X_\nu(w) :\right) \\
&= \frac{D}{(z-w)^3} + \frac{-\frac{2}{\ell_s^2}\partial X_\mu \partial X^\mu(w) - \frac{1}{\ell_s^2}\psi^\mu\partial\psi_\mu(w)}{z-w} \\
&= \frac{\hat{c}}{(z-w)^3} + \frac{2T(w)}{z-w}
\end{aligned}$$

Finally

$$\begin{aligned}
T(z)G(w) &= -\frac{1}{\ell_s^2}\left(: \partial X_\mu \partial X^\mu(z) : + \frac{1}{2}\psi^\mu\partial\psi_\mu(z)\right) i\frac{\sqrt{2}}{\ell_s^2}\psi_\nu \partial X^\nu(w) \\
&= -i\frac{\sqrt{2}}{\ell_s^4}\left(-\frac{\ell_s^2}{2}\frac{\psi_\mu\partial X^\mu(w) + \psi_\mu\partial^2 X^\mu(w)(z-w)}{(z-w)^2} - \frac{\ell_s^2}{2}\frac{\psi_\mu \partial X^\mu(w)}{(z-w)^2} + (-)\frac{\ell_s^2}{2}\frac{\partial_\mu\psi\partial X^\mu(w)}{(z-w)}\right) \\
&= \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}
\end{aligned}$$

2. We will take the OPE of $j_B(z)j_B(w)$, but just look at the $(z-w)^{-1}$ term as a function of w, as this, when integrated around the origin in w will give Q_B^2. This is an extension of exercise **4.45**, and there is nothing conceptually further, except for some $\beta\gamma$ manipulation. There are altogether 16 terms to consider, and we will get $c = 15$. The algebra is heavy, so I will skip this. An alternative is to do this as in **Polchinski 4.3**.

   To do it this way, note the following OPEs:

$$\begin{aligned}
j_B(z)b(w) &\sim \frac{T_{matter}(z)}{z-w} - \frac{1}{(z-w)^2}\left(bc(z) + \frac{3}{4}\beta\gamma(z)\right) + \frac{1}{z-w}\left(-b\partial c(z) + \frac{1}{4}\partial\beta\gamma(z) - \frac{3}{4}\beta\partial\gamma(z)\right) \\
&= \cdots + \frac{1}{z-w}\left[T_{matter}(z) - \partial b\, c(w) - 2b\partial c(w) - \frac{1}{2}\partial\beta\gamma(w) - \frac{3}{2}\beta\partial\gamma(w)\right] \\
&= \cdots + \frac{T_{matter}(w) + T_{gh}(w)}{z-w} \Rightarrow \{Q_B, b_n\} = L_n
\end{aligned}$$

   Similarly

$$j_B(z)\beta(w) = \cdots + \frac{G_{matter}(w) + G_{gh}(w)}{z-w} \Rightarrow [Q_B, \beta_n] = G_n$$

   Now note that the Jacobi identity on Q_B reads:

$$\{[Q_B, L_m], b_n\} - \{\overbrace{[L_m, b_n]}^{(m-n)b_{m+n}}, Q_B\} - [\overbrace{\{b_n, Q_B\}}^{L_n}, L_m] = 0 \Rightarrow \{[Q_B, L_m], b_n\} = (m-n)L_{m+n} - [L_m, L_n]$$

   So if the total central charge is zero we'll get $\{[Q_B, L_m], b_n\} = 0$, implying that $[Q_b, L_m]$ is independent of the c ghost. But on the other hand this operator has ghost number 1, so it must therefore vanish. Further, the Jacobi identity also yields

$$[\{Q_B, Q_B\}, b_n] = -2[\{b_n, Q_B\}, Q_B] = 2[Q_B, L_n]$$

   since we just showed that this last term vanishes, we must have Q_B, Q_B is also independent of c, but again since Q_B^2 has positive ghost number, we get that it is in fact zero. We can do the same argument with β and G and get that the superstring BRST operator is zero, as long as the total central charge vanishes. This was much cleaner than the OPE way.

3. First a lemma: An abelian p-form field A has $\binom{D-2}{p}$ on shell DOF. To prove this, note that we have a gauge symmetry of $A \to A + \partial\Lambda$ which has $\binom{D}{p-1}$ parameters. Next, the Euler-Lagrange equations give us that the components $A^{0i_1\dots i_{p-1}}$ are non-propagating. We thus get $\binom{D-1}{p}$ massless propagating off-shell d.o.f. which have $\binom{D-2}{p-1}$ gauge symmetries left over. These can be used to enforce Coulomb gauge conditions which allow for there to be no polarizations along one of the spatial directions. We thus get $\binom{D-1}{p} - \binom{D-2}{p-1} = \binom{D-2}{p}$ massless on-shell degrees of freedom. For A_μ this is $D-2$ and for $B_{\mu\nu}$ this is $(D-2)(D-3)/2$.

   The metric has $\frac{1}{2}D(D-3)$ on-shell degrees of freedom. There are two ways to see this, first, that the dynamically allowed variation δg may on-shell be described by a symmetric traceless tensor in dimension $D-2$ which gives
$$\frac{(D-1)(D-2)}{2} - 1 = \frac{1}{2}D(D-3)$$
   or by noting that since we are gauging translation symmetry locally, each translation makes 2 polarizations unphysical and so we get:
$$\frac{D(D+1)}{2} - 2D = \frac{1}{2}D(D-3)$$
   as required.

   We now consider the R-R, R-NS, NS-R, NS-NS sectors together. For NS-NS we have the scalar $=1$ both on-shell and off-shell, the antisymmetric two-form, which has only transverse degrees of freedom $= 8*7/2 = 28$ and the gravity, $= 10*7/2 = 35$ altogether we get 64 on-shell degrees of freedom.

   In both the R-NS and NS-R sector, we have a Weyl representation of dimension $2^{5-1} = 16$. There are however only 8 on-shell degrees of freedom. Similarly, we only consider the on-shell $\psi^\mu_{-1/2}$ acting on the NS part of the vacuum which gives another factor of 8. This gives 64 fermionic variables in each sector for a grand total of 128.

   In R-R for IIA we have a 0, 2, and *self-dual* 4-form. This gives:
$$1 + \binom{8}{2} + \frac{1}{2}\binom{8}{4} = 64$$
   For IIB we have a 1 and 3-form. This gives
$$\binom{8}{1} + \binom{8}{3} = 64$$
   so in either case we have 64 on-shell degrees of freedom here. This is consistent with each $|S\rangle$ state having 8 on-shell degrees of freedom giving $8 \times 8 = 64$. All together, we have the same number of on-shell fermionic and bosonic degrees of freedom.

   Now for the massive case. In the NS sector you might expect the next excitations come from the bosons α_{-1}, but this gets projected out by GSO, so in fact the next states come from $\psi^i_{-3/2}$, $C_{ijk}\psi^i_{-1/2}\psi^j_{-1/2}\psi^k_{-1/2}$ and $C_{ij}\psi^i_{-1/2}\alpha^j_{-1}$. These have dimensions $8 + 56 + 64 = 128$, which decomposes as the traceless symmetric **44** and three-index antisymmetric **84** representation of SO(9). In the R sector, we must look at $\alpha^i_{-1}|S_\alpha\rangle$ and $\psi^i_{-1}|C_\alpha\rangle$ for S_α, C_α suitably chosen so that the state satisfies $G_0 = 0$. This constraint gives a factor of two reduction for the dimension of the space of candidate S_α. Consequently, we get $\mathbf{8}_v \otimes \mathbf{8}_s \oplus \mathbf{8}_v \otimes \mathbf{8}_{s'}$ which has dimension 128. This indeed turns out to be a spinor representation of SO(9), and it comes from looking at the tensor product of *the* fundamental spinor representation with the vector representation $\mathbf{16}_s \otimes \mathbf{9}_v$. This turns must decompose as a sum of two spinor representations $\mathbf{16}_s \oplus \mathbf{128}_s$. One is again the fundamental, while the other is the required **128**.

   For the massive states in the type IIA and type IIB, we must tensor we wish to look at the lowest-level masses. Note we must match massive states with massive states. In this case, we match $2/\alpha$ on both sides to get massive states of mass $4/\alpha$. Since the particles already organize into representations of SO(9) on each side, the closed string massive spectrum will again clearly organize intro representations of SO(9). Also since fermionic and bosonic degrees of freedom already were equal on each side, they will be equal in the closed string as well. We will have $2 \times 128^2 = 32768$ bosonic and fermionic degrees of freedom.

4. In terms of theta functions:

$$\chi_O = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_3(\nu_i)}{\eta} - \prod_{i=1}^{4}\frac{\theta_4(\nu_i)}{\eta}\right)$$
$$\chi_V = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_3(\nu_i)}{\eta} + \prod_{i=1}^{4}\frac{\theta_4(\nu_i)}{\eta}\right)$$
$$\chi_S = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_2(\nu_i)}{\eta} - \prod_{i=1}^{4}\frac{\theta_1(\nu_i)}{\eta}\right)$$
$$\chi_C = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_2(\nu_i)}{\eta} + \prod_{i=1}^{4}\frac{\theta_1(\nu_i)}{\eta}\right)$$

We'll take $\nu_i = 0$ here **(I assume this is what I'm supposed to do)** and so $\theta_1 = 0 \Rightarrow \chi_S = \chi_C$.

For IIB we look at

$$\frac{|\chi_V - \chi_C|^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8} = \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4} \times \frac{1}{2}\sum_{\bar{a},\bar{b}=0}^{1}(-1)^{\bar{a}+\bar{b}}\frac{\bar{\theta}^4\left[{\bar{a} \atop \bar{b}}\right]}{\bar{\eta}^4}$$

Under modular transformations $\tau \to \tau+1$ $\theta^4\left[{0 \atop 1}\right] \leftrightarrow \theta^4\left[{0 \atop 0}\right]$, $\theta^4\left[{1 \atop 0}\right] \to -\theta^4\left[{1 \atop 0}\right]$ while $\eta^{12} \to -\eta^{12}$. In the holomorphic and anti-holomorphic parts separately, each term in the sum picks up a minus sign that is cancelled by the minus sign in the η^4.

Under $\tau \to -1/\tau$, the $\frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}$ out front is invariant. On the other hand, the θ functions transform as $\theta^4\left[{0 \atop 0}\right] \to (-i\tau)^2\theta^4\left[{0 \atop 0}\right]$, $\theta^4\left[{0 \atop 1}\right] \to (-i\tau)^2\theta^4\left[{1 \atop 0}\right]$, $\theta^4\left[{1 \atop 0}\right] \to (-i\tau)^2\theta^4\left[{0 \atop 1}\right]$. These are exactly compensated by the η transformations in the denominator, and no overall sign is picked up

For IIA we have similarly

$$\frac{(\chi_V - \chi_C)(\bar{\chi}_V - \bar{\chi}_S)}{(\sqrt{\tau_2}\eta\bar{\eta})^8} = \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4} \times \frac{1}{2}\sum_{\bar{a},\bar{b}=0}^{1}(-1)^{\bar{a}+\bar{b}+\bar{a}\bar{b}}\frac{\bar{\theta}^4\left[{\bar{a} \atop \bar{b}}\right]}{\bar{\eta}^4}$$

Again, the holomorphic part transforms as before and as we have set the ν_i to zero, we have the same partition function. Using **D.18**, we see that each of the four above sums are zero since they are equal to a product of $\theta_1 = 0$.

5. Again, these are identical if I set the $\nu_i = 0$ (am I not supposed to be doing this? What do the ν_i represent physically?). They are equal to

$$\frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8 4\eta^4\bar{\eta}^4}(\cancel{|\theta_1^4|^2} + |\theta_2^4|^2 + |\theta_3^4|^3 + |\theta_4^4|^2)$$

We have θ_3 and θ_4 swapping under $\tau \to \tau+1$, generating no signs in this case, while the denominator looks like $|\eta|^{24}$ and also doesn't generate a sign. Then, under $\tau \to -1/\tau$ we have θ_2 and θ_4 swapping generating a $|\tau|^4$, identical to what is generated by the $(\eta\bar{\eta})^4$.

6. The partition function is

$$Z^{\text{het}}_{\text{SO}(16)\times\text{SO}(16)} = \frac{1}{2}\sum_{h,g}\frac{\bar{Z}_{E_8}\left[{h \atop g}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b}(-1)^{a+b+ab+ag+bh+gh}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4}, \quad \bar{Z}_{E_8}\begin{bmatrix} h \\ g \end{bmatrix} = \frac{1}{2}\sum_{\gamma,\delta}(-1)^{\gamma g+\delta h}\frac{\bar{\theta}^8\left[{\gamma \atop \delta}\right]}{\bar{\eta}^8}$$

First look at $\bar{Z}_{E_8}$. Under modular transformations $\tau \to -1/\tau$ we get $\bar{Z}_{E_8}\left[{h \atop g}\right] \to \bar{Z}_{E_8}\left[{g \atop h}\right]$. Under $\tau \to \tau+1$, we get $\bar{Z}_{E_8}\left[{h \atop g}\right] \to (-1)^{h-2/3}\bar{Z}_{E_8}\left[{h \atop g+h}\right]$. With this, we can look at $Z^{\text{het}}_{\text{SO}(16)\times\text{SO}(16)}$ under $\tau \to -1/\tau$

$$\frac{1}{2}\sum_{h,g}\frac{\bar{Z}_{E_8}\left[{g \atop h}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b}(-1)^{a+b+ab+ag+bh+gh}\frac{\theta^4\left[{b \atop a}\right]}{\eta^4}$$

Under relabeling of $a \leftrightarrow b, g \leftrightarrow h$, this is the same. Next, under $\tau \to \tau + 1$:

$$
\begin{aligned}
&\frac{1}{2}\sum_{h,g} \frac{(-1)^{-4/3}\bar{Z}_{E_8}\left[{h \atop g+h}\right]^2}{(-1)^{4/3}(\sqrt{\tau_2}\eta\bar{\eta})^8} \frac{1}{2}\sum_{a,b}(-1)^{a+b+ab+ag+bh+gh} \frac{(-1)^a\theta^4\left[{a \atop a+b-1}\right]}{(-1)^{1/3}\eta^4} \\
&= \frac{1}{2}\sum_{h,g} -\frac{\bar{Z}_{E_8}\left[{h \atop g+h}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8} \frac{1}{2}\sum_{a,b}(-1)^{b+ab+ag+bh+gh} \frac{\theta^4\left[{a \atop a+b-1}\right]}{\eta^4} \\
&= \frac{1}{2}\sum_{h,g'} \frac{\bar{Z}_{E_8}\left[{h \atop g'}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8} \frac{1}{2}\sum_{a,b}(-1)^{1+b+ab+ag'+(a+b)h+g'h+h} \frac{\theta^4\left[{a \atop a+b-1}\right]}{\eta^4} \\
&= \frac{1}{2}\sum_{h,g'} \frac{\bar{Z}_{E_8}\left[{h \atop g'}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8} \frac{1}{2}\sum_{a,b}(-1)^{\cancel{1}+(b'+a+\cancel{1})+(ab'+\cancel{a}-\cancel{a})+ag'+(b'h+\cancel{h})+g'h+\cancel{h}} \frac{\theta^4\left[{a \atop b'}\right]}{\eta^4} \\
&= \frac{1}{2}\sum_{h,g'} \frac{\bar{Z}_{E_8}\left[{h \atop g'}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8} \frac{1}{2}\sum_{a,b}(-1)^{a+b'+ab'+ag'+b'h+g'h} \frac{\theta^4\left[{a \atop b'}\right]}{\eta^4}
\end{aligned}
$$

Keep in mind that $x^2 = x \bmod 2$.

Before we do the next part, let's elaborate on why $Z_{E_8} = \frac{1}{2}\sum_{a,b}\theta^8\left[{a \atop b}\right]$ is the partition function of the E_8 lattice. From the sixteen fermion picture, this is just the $(-1)^F = 1$ in the NS sector (corresponding to the $\chi_O = \frac{1}{2}(\theta^8\left[{0 \atop 0}\right] + \theta^8\left[{0 \atop 1}\right])$ character) together with the R sector $\chi_S = \frac{1}{2}\theta^8\left[{1 \atop 0}\right]$ giving the spinor representation.

Indeed, the roots of E_8 consist of the roots of O(16) as well as the spinor weights of O(16). Note that the spinor representation comes from the half-integral points, corresponding to $\theta\left[{1 \atop 0}\right]$ in the sum, while the adjoint representation comes from $\theta\left[{0 \atop 1}\right]$ and $\theta\left[{0 \atop 0}\right]$. Consequently the action of $\mathcal{S}_i$ that fixes the adjoint vectors but flips the sign of the spinor acts on our partition function as $\mathcal{S}_i Z_{E_8} = \frac{1}{2}\sum_{a,b}(-1)^a\,\theta^8\left[{a \atop b}\right]$. It of course also gives rise to a twisted sector, so altogether we get the four twisted blocks $\bar{Z}_{E_8}\left[{h \atop g}\right]$ as required.

Since we have projected out the spinor representation, the current algebra only contains the NS currents $\bar{J}^{ij}$ corresponding to the adjoint of SO(16), and we have two copies of this for each group of 16 fermions.

From the factor of $(\sqrt{\tau_2}\eta\bar{\eta})^{-8}$ we see that we have 8 on-shell noncompact massless bosonic excitations as well as all of their descendants (on both left and right moving sides). We also see on the left-moving side we get a theta-function corresponding to $N = 8$ fermions transforming under a spacetime SO(8), forming the superpartners of the bosons. On the right side instead of the superpartner fermions, we have the 16 internal fermions that transform in the adjoint representations.

Let's see what massless states we can build. In the NS sector of the left-movers, we have $L_0 = 1/2, \bar{L}_0 = 1$ and so we get $\psi^i_{-1/2}\alpha^j_{-1}|p\rangle$ which gives us our usual graviton, two-form field, and dilaton. We also have $\psi^i_{-1/2}\bar{J}^a_{-1}|p\rangle$ for the O(16) × O(16) currents. This gives us vectors corresponding to gauge bosons valued in the adjoint of O(16) × O(16) as required.

In the R sector we have $G_0 = 0, \bar{L}_0 = 1$ we'll get a gravitino, fermion, and gaugino as before, but again this time valued in O(16) × O(16).

7. Because we have seen that T-duality flips the antichiral $U(1)$ $\bar{\partial}X \to -\bar{\partial}X$, and we want to preserve the (1,1) supersymmetry G in the type II string (and so must keep it as a periodic variable **Why is this absolutely necessary. Can we not work with double covers in some clever way when defining supercurrents?**), we must consequently flip $\bar{\psi}$. This corresponds to inserting $(-1)^{F_R}$. For the right-moving R sector, this changes the chirality of the R spinor, taking $S_\alpha \to \Gamma^9\Gamma^{11}S_\alpha$ (there can be no phase, by reality conditions of Γ). We thus flip IIA to IIB and vice versa.

   From this we get that

$$
F_{\alpha\beta} = S_\alpha(\Gamma^0)_{\beta\gamma}\tilde{S}_\gamma \to S_\alpha(\Gamma^0\Gamma^9\Gamma^{11})_{\beta\gamma}\tilde{S}_\gamma = -\xi\, S_\alpha(\Gamma^9\Gamma^0)_{\beta\gamma}\tilde{S}_\gamma = -\xi F\Gamma^9
$$

Expanding in terms of the $F_{\mu_1\ldots\mu_k}$ gives the action:

$$F_{\alpha\beta} \to -\xi \sum_{k=0}^{10} \frac{(-1)^k}{k!} F_{\mu_1\ldots\mu_k} \Gamma^{\mu_1\ldots\mu_k} \Gamma^9$$

This gives that

$$\tilde{F}_{\mu_1\ldots\mu_k,9} = -\xi F_{\mu_1\ldots\mu_k}, \quad \tilde{F}_{\mu_1\ldots\mu_k} = F_{\mu_1\ldots\mu_k,9}$$

Then

$$\partial_{\mu_1}\tilde{C}_{\mu_2\ldots\mu_k 9} = -\xi \partial_{\mu_1} C_{\mu_2\ldots\mu_k}, \quad \partial_{\mu_1}\tilde{C}_{\mu_2\ldots\mu_k} = \partial_{\mu_1}\tilde{C}_{\mu_2\ldots\mu_k 9}$$

so that (up to a closed term)

$$\tilde{C}^{(p)}_{\mu_1\ldots\mu_{p-1}9} = -\xi C^{p-1}_{\mu_1\ldots\mu_{p-1}}, \quad \tilde{C}^{(p)}_{\mu_1\ldots\mu_p} = C^{(p+1)}_{\mu_1\ldots\mu_p 9}$$

Get rid of the ξ factor

8. We have that $\Omega\,|S_\alpha\tilde{S}_\beta\rangle = \varepsilon_R\,|S_\beta\tilde{S}_\alpha\rangle$. Further, it acts trivially on Γ^0 **(you sure?)**. Now, in the operator language we will have $\Omega S_\alpha \Omega^{-1} = \epsilon_1 \tilde{S}_\alpha$ and $\Omega\tilde{S}_\beta\Omega^{-1} = \epsilon_2 S_\beta$. In any case, we must have for the bi-spinor that $\Omega S_\alpha \tilde{S}_\beta \Omega^{-1} = \epsilon_R S_\beta \tilde{S}_\alpha$, which gives that $\epsilon_1\epsilon_2 = -\epsilon_R$ Thus, we have:

$$\Omega F_{\alpha\beta}\Omega^{-1} = \Omega S_\alpha \Gamma^0_{\beta\gamma}\tilde{S}_\gamma\Omega^{-1} = -\epsilon_R \Gamma^0_{\beta\gamma} S_\gamma \tilde{S}_\alpha = -\epsilon_R \Gamma^0_{\beta\gamma} F_{\gamma\delta}\Gamma^0_{\delta\alpha} = -\epsilon_R(\Gamma^0 F \Gamma^0)_{\beta\alpha} = -\epsilon_R(\Gamma^0 F^T \Gamma^0)_{\beta\alpha}$$

I think 7.3.3 of Kiritsis has the derivation wrong. Ask Nathan/Xi.

9. When we take $\epsilon_R = -1$m the scalar and four-index self-dual tensor survive. In this case, we will *not* have consistent interactions. Since the graviton survives, there must be an equal number of massless bosonic and fermionic excitations. The fermions come just from the NS-R sector (there is no R-NS now), giving 64 on-shell fermionic excitations. From the NS-NS sector, the dilaton and gravity will give $1 + 35 = 36$ on-shell bosonic degrees of freedom. We are missing 28 bosonic degrees of freedom.

   The scalar and four-index self dual tensor contribute $1 + \frac{1}{2}\frac{8\times7\times6\times5}{4!} = 36$ on-shell bosonic degrees of freedom. This is too much. The two-form, on the other hand, contributes the requisite $8 \times 7/2 = 28$. Consistency of interaction thus *demands* we keep only the 2-form and drop the 0 and self-dual 4-form. This necessitates $\epsilon_R = 1$.

10. We are just looking at the *open* superstrings here. Any open string that consistently couples to type I or type II string theory must have a GSO projection as well. We have already seen how the oriented open strings look like in exercise **7.3**. In the NS sector we have at $-p^2 = m^2 = 2/\ell_s^2$

$$\begin{aligned} &\psi^i_{-3/2}\lambda_{ab}\,|p;ab\rangle_{NS} \\ &C_{ijk}\psi^i_{-1/2}\psi^j_{-1/2}\psi^k_{-1/2}\lambda_{ab}\,|p;ab\rangle_{NS} \\ &C_{ij}\psi^i_{-1/2}\alpha^j_{-1}\lambda_{ab}\,|p;ab\rangle_{NS} \end{aligned} \tag{68}$$

In the R sector we have (for S_α suitably chosen so that the state satisfies $G_0 = 0$):

$$\begin{aligned} &\alpha^i_{-1}\lambda_{ab}\,|S_\alpha;ab\rangle_R \\ &\psi^i_{-1}\lambda_{ab}\,|C_\alpha;ab\rangle_R \end{aligned} \tag{69}$$

I will assume NN boundary conditions. In this case

$$\begin{aligned} \Omega\alpha_{-1}\Omega^{-1} &= -\alpha_{-1} \\ \Omega\psi_{-1}\Omega^{-1} &= -\psi_{-1} \\ \Omega\psi_{-\frac{1}{2}}\Omega^{-1} &= -i\psi_{-\frac{1}{2}} \\ \Omega\psi_{-\frac{3}{2}}\Omega^{-1} &= i\psi_{-\frac{3}{2}} \end{aligned}$$

So all of the terms in (68) are terms of the form $\mathcal{A}\lambda_{ab}\,|p;ab\rangle_{NS}$ with the operator $\mathcal{A}$ transforming as $\mathcal{A}\to i\mathcal{A}$ under parity. Doing parity twice therefore will generate a $-\epsilon_{NS}^2\mathcal{A}(\gamma\gamma^{T^{-1}})_{ii'}\,|p;a'b'\rangle\,(\gamma^T\gamma^{-1})_{j'j}$. This is exactly the same as in **7.3.10**. Demanding that Ω act on the state with eigenvalue $+1$ will make it so that $\lambda = i\epsilon_{NS}\gamma\lambda^T\gamma^{-1}$. We already have $\epsilon_{NS}=-i$ so $\lambda=\gamma\lambda^T\gamma^{-1}$ here. Imposing the tadpole cancelation condition $\zeta=1$ and we get gauge group SO(32). So we get that states at this level will transform in the *the traceless symmetric tensor + singlet representation* of $SO(32)$.

All of the terms in (69) will transform under parity twice as as $\epsilon_R^2\mathcal{A}(\gamma\gamma^{T^{-1}})_{ii'}\,|S_\alpha;a'b'\rangle\,(\gamma^T\gamma^{-1})_{j'j}$. We will have the same γ matrix as in the NS sector, as required for consistency of interactions. Here, though, we will get $\epsilon_R=-1\Rightarrow\epsilon_R^2=1$ and we will get $\lambda=-\gamma\lambda^T\gamma^{-1}$ (this is what we got from the massless sector with an extra minus sign since ψ_{-1},α_{-1} now transform with minus signs). Again we will have that these states will transform in the symmetric representation of SO(32).

Again we get 128 bosonic states that will transform as the $\mathbf{44}\oplus\mathbf{84}$ representation of SO(9). We will also get fermions transforming in the **128** spinor representation as in exercise **3**. All of these states will transform in the traceless symmetric representation of SO(32). **Confirm**

11. Certainly in the untwisted sector, the theory we get corresponds to tracing over the projection operator $\frac{1}{2}(1+g)$ where g is orientation-reversal. Now in the twisted sector, we still have X^μ satisfies the Laplace equation $\partial_+\partial_-X=0$ so we can write

$$X(\sigma,\tau)=x^\mu+\tau\ell_s^2\frac{p^\mu+\bar{p}^\mu}{2}+\sigma\ell_s^2\frac{p^\mu-\bar{p}^\mu}{2}+\frac{i\ell_s}{\sqrt{2}}\sum_n\left(\frac{\alpha_n}{n}e^{-in(\tau+\sigma)}+\frac{\tilde{\alpha}_n}{n}e^{-in(\tau+\sigma)}\right)$$

The condition that $X(\sigma+2\pi)=X(2\pi-\sigma)$ give that $p^\mu=\bar{p}^\mu$ and the σ term vanishes. We must have n is a half integer. For integer modding we have $e^{-in(\tau\pm\sigma)}\to e^{-in(\tau\mp\sigma)}$. For half-integer modding we have $e^{-in(\tau\pm\sigma)}=(-1)^n e^{-in(\tau\mp\sigma)}$. We should thus have $\alpha_n=\tilde{\alpha}_n$ for n integral and $\alpha_n=-\tilde{\alpha}_n$ We thus get

$$X(\sigma,\tau)=x^\mu+2\ell_s^2p^\mu\tau+\sigma i\sqrt{2}\ell_s\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\alpha_n}{n}\cos(n\sigma)e^{-in\tau}-\sqrt{2}\ell_s\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{\alpha_n}{n}\sin(n\sigma)e^{-in\tau}$$

This is the twisted sector. The last sum picks up a minus sign under orientation reversal, and so will be projected out. We are left with the equations of motion for the open string.

12. In NS we have (up to an overall irrelevant factor of $i^{-1/2}$)

$$\psi(\sigma,\tau)=\sum_{n\in\mathbb{Z}}\psi_{n+1/2}e^{(n+1/2)(\tau+i\sigma)},\quad \bar{\psi}(\sigma,\tau)=\sum_{n\in\mathbb{Z}}\bar{\psi}_{n+1/2}e^{(n+1/2)(\tau-i\sigma)}$$

In the closed string case have that $\Omega\psi_{n+1/2}\Omega^{-1}=\bar{\psi}_{n+1/2}$. Given that $\Omega\psi(\sigma,\tau)\Omega^{-1}=\bar{\psi}(\pi-\sigma,\tau)$, we directly get $\Omega\psi_{n+1/2}\Omega^{-1}=i(-1)^n\bar{\psi}_{n+1/2}$. For DD boundary conditions we get an extra minus sign to this, since there $\Omega\psi(\sigma,\tau)\Omega^{-1}=-\bar{\psi}(\pi-\sigma,\tau)$.

In the R sector we have

$$\psi(\sigma,\tau)=\sum_{n\in\mathbb{Z}}b_ne^{n(\tau+i\sigma)},\quad \bar{\psi}(\sigma,\tau)=\sum_{n\in\mathbb{Z}}\bar{b}_ne^{n(\tau-i\sigma)}$$

Following the same logic we get that $\Omega\psi_n\Omega^{-1}=(-1)^n\psi_n$ for NN and $\Omega\psi_n\Omega^{-1}=-(-1)^n\psi_n$ for DD.

All of these cases can be summarized by

$$\begin{aligned}\text{NN:}\quad &\Omega\psi_r\Omega^{-1}=(-1)^r\psi_r\\ \text{DD:}\quad &\Omega\psi_r\Omega^{-1}=-(-1)^r\psi_r.\end{aligned}$$

13. Let's clarify a bit of terminology before we begin. We are looking at just the fermions of the left moving and right moving sides of the heterotic string theory. On the left-hand (supersymmetric) side, in the light-cone gauge these form an $\widehat{O(8)}$ current algebra at *level 1*. On the right-hand side the form a $\widehat{O(32)}$ current algebra at level 1 again (**why must we always have level 1? Ask Xi.**).

The characters of $\widehat{O(N)}_1$ for N even correspond to the integrable representations labeled by O, V, S, C corresponding to the trivial, vector, spinor, and conjugate spinor. For our purposes (ie the heterotic string), we do not need to distinguish between S and C, which will have the same character. The characters can be written in terms of θ functions as

$$\chi_O = \frac{1}{2}\left[\left(\frac{\theta_3}{\eta}\right)^{N/2} + \left(\frac{\theta_v}{\eta}\right)^{N/2}\right], \quad \chi_V = \frac{1}{2}\left[\left(\frac{\theta_3}{\eta}\right)^{N/2} - \left(\frac{\theta_4}{\eta}\right)^{N/2}\right], \quad \chi_S = \frac{1}{2}\left(\frac{\theta_2}{\eta}\right)^{N/2}$$

(a) Now let us first look at $O(32)$. The $\widehat{O(8)}_1$ characters transform under $\tau \to \tau + 1$ as

$$\chi_O^8 \to (-1)^{-1/6}\chi_O, \quad \chi_V^8 \to -(-1)^{-1/6}\chi_V, \quad \chi_S^8 \to -(-1)^{-1/6}\chi_S$$

And under $\tau \to -1/\tau$ they transform as

$$\chi_O^8 \to \frac{1}{2}(\chi_O^8 + \chi_V^8) + \chi_S^8, \quad \chi_V^8 \to \frac{1}{2}(\chi_O^8 + \chi_V^8) - \chi_S^8, \quad \chi_S^8 \to \frac{1}{2}(\chi_O^8 - \chi_V^8)$$

The $\widehat{O(32)}_1$ characters *depending on* $\bar{q}$ transform the same way under $\tau \to -1/\tau$ but under $\tau \to \tau + 1$ transform as

$$\chi_O^{32} \to (-1)^{2/3}\chi_O^{32}, \quad \chi_V^{32} \to -(-1)^{2/3}\chi_V^{32}, \quad \chi_S^{32} \to (-1)^{2/3}\chi_S^{32}$$

Our partition functions in question can be constructed from a linear combination of products of exactly one $\widehat{O(8)}_1$ and one $\widehat{O(32)}_1$ character. This gives 9 possible terms $\chi_i^8\chi_j^{32*}$. I label these in the table below. I cancel all terms that are not invariant under $\tau \to \tau + 1$.

~~O O~~	O V	~~O S~~
V O	~~V V~~	V S
S O	~~S V~~	S S

But we are not done. It is easy to see that while that χ_V^8, χ_S^8 blocks have Taylor series $O(q^{1/3})$, the χ_O block contains a singular term going as $q^{-1/6}$. Similarly, χ_O^{32} contains a singular term going as $O(q^{-2/3})$ while $\chi_V^{32} = O(\bar{q}^{-1/6})$ and $\chi_S^{32} = O(\bar{q}^{4/3})$. The tachyon can come exactly (and only!) from combining $\chi_O^8\chi_V^{32*}$ to get $1/|q|^{1/6}$ that will be singular and satisfy level-matching. Thus we must drop OV above as well. We are left with four possible terms that can work.

Modular invariance under $\tau \to 1/\tau$ further constrains this to take a form proportional to

$$(\chi_V^8 - \chi_S^8)(\chi_1^{32} + \chi_S^{32})$$

The normalization of the identity to 1 fixes this entirely. Note that we get spin statistics for *free*, as the only character combinations appearing with a minus sign are precisely those containing χ_S^8, associated with the spacetime fermions.

(b) Having $\widehat{O(32)}_1$ out of the way, let's move on to $O(16) \times E_8$. $\widehat{E_8}_1$ has only one integrable representation and thus one corresponding character, χ^{E_8}. As pointed out in the text, it is related to the characters of $\widehat{O(16)}_1$ by $\chi^{E_8} = \chi_O^{16} + \chi_S^{16}$. Thus we have trilinear combinations $\chi_i^8(q)\chi_j^{16}(\bar{q})\chi^{E_8}(\bar{q})$. Upon noting that the characters of $\widehat{O(16)}_1$ multiplied by χ_{E_8} transform the *same way* under modular transformations as $\widehat{O(32)}_1$ and the *same* combination $\chi_1^8\chi_V^{16}\chi^{E_8}$ uniquely gives the tachyon, we see that *again* the argument goes as before and the only viable character we can have is

$$(\chi_V^8 - \chi_S^8)(\chi_1^{32} + \chi_S^{32})\chi^{E_8} = (\chi_V^8 - \chi_S^8)\chi^{E_8 \times E_8}.$$

This is exactly the heterotic E string theory.

(c) Finally we get to the hard one: $O(16) \times O(16)$. Here we have 27 trilinear terms that can contribute. I will write them out, and again cross out the ones that are not invariant under $\tau \to \tau + 1$ as well as double crossing out the tachyons. Here, though, the notation OO, OV, SV etc will represent *just* the right-moving characters $\chi_O^{16}(\bar{q})\chi_O^{16}(\bar{q}), \chi_O^{16}(\bar{q})\chi_V^{16}(\bar{q}), \chi_S^{16}(\bar{q})\chi_V^{16}(\bar{q})$ respectively.

$$\chi_O^8 \times \begin{matrix} \cancel{\mathrm{O\,O}} & \cancel{\mathrm{O\,V}} & \cancel{\mathrm{O\,S}} \\ \cancel{\mathrm{V\,O}} & \cancel{\mathrm{V\,V}} & \mathrm{V\,S} \\ \cancel{\mathrm{S\,O}} & \mathrm{SV} & \cancel{\mathrm{S\,S}} \end{matrix}\,, \qquad \chi_V^8 \times \begin{matrix} \mathrm{O\,O} & \cancel{\mathrm{O\,V}} & \mathrm{O\,S} \\ \cancel{\mathrm{V\,O}} & \mathrm{V\,V} & \cancel{\mathrm{V\,S}} \\ \mathrm{S\,O} & \cancel{\mathrm{S\,V}} & \mathrm{S\,S} \end{matrix}\,, \quad \chi_S^8 \times \begin{matrix} \mathrm{O\,O} & \cancel{\mathrm{O\,V}} & \mathrm{O\,S} \\ \cancel{\mathrm{V\,O}} & \mathrm{V\,V} & \cancel{\mathrm{V\,S}} \\ \mathrm{S\,O} & \cancel{\mathrm{S\,V}} & \mathrm{S\,S} \end{matrix}$$

It may look that 12 independent terms remain. The fact that the characters are symmetric under exchange of the last two labels mean that there are in fact only 12.

Let us look at two cases. First, assume $\chi_O^8 \chi_V^{16*} \chi_S^{16*}$ does *not* contribute (ie its coefficient vanishes). Then the first 9 terms are all zero. The remaining constraint of modular invariance under $\tau \to \tau + 1$ constrains the partition function to take the form

$$(\chi_V - \chi_S)\left[\chi_O^{16}\chi_O^{16} + 2\alpha\chi_O^{16}\chi_S^{16} + (1-\alpha)\chi_V^{16}\chi_V^{16} + (2-\alpha)\chi_S^{16}\chi_S^{16}\right]$$

for any value of α. Spin statistics requires all these contributions to come in with positive coefficient, so $0 \leqslant \alpha \leqslant 1$. Moreover, if α is non-integral we will have coefficients that are not integers in the character expansion, which would lack a Hilbert space interpretation **Think more about the integrality condition**. Thus we can have only $\alpha = 0$ and $\alpha = 1$ corresponding exactly to the $O(32)$ and $E_8 \times E_8$ superstrings.

So our remaining possibility is that $\chi_O^8 \chi_V^{16*} \chi_S^{16*}$ does *not* have vanishing coefficient. WLOG set this coefficient to 1. Invariance under $\tau \to -1/\tau$ constrains us to:

$$\begin{aligned} 2\chi_O^8\chi_V^{16}\chi_S^{16} &+ \chi_V^8\left[\alpha\chi_O^{16}\chi_O^{16} + 2\beta\chi_O^{16}\chi_S^{16} + (-1+\alpha-\beta)\chi_V^{16}\chi_V^{16} + (-1+2\alpha-\beta)\chi_S^{16}\chi_S^{16}\right] \\ &+ \chi_S^8\left[(1-\alpha)\chi_O^{16}\chi_O^{16} - 2(1+\beta)\alpha\chi_O^{16}\chi_S^{16} + (-\alpha+\beta)\chi_V^{16}\chi_V^{16} + (1-2\alpha+\beta)\chi_S^{16}\chi_S^{16}\right] \end{aligned}$$

Again, spin-statistics requires the coefficient of all the characters involving χ_V to have positive sign and all the characters involving χ_S to have negative sign. This makes $1 \leqslant \alpha, 0 \leqslant \beta \leqslant \alpha - 1$. Integrality then forces $\alpha = 1, \beta = 0$. **More general solution? We need to impose that $\chi_i^8\chi_O^{16}\chi_O^{16}$ has coefficient 1 or 0**

Of all these theories, the first two theories have vanishing partition function - an indicator of spacetime supersymmetry, but not necessarily an identifier. Of course, we can identify them as the heterotic string theories, which indeed have space time SUSY. The last theory has nonvanishing partition function and thus cannot have spacetime SUSY as the fermions and bosons do not cancel at one loop.

14. I think this problem is backwards. For 32 fermions *all* with the same boundary conditions, its immediate to see that they will reproduce the partition function for the Spin$(32)/\mathbb{Z}_2$ string:

$$\frac{1}{2}\sum_{a,b}\theta^{16}\begin{bmatrix} a \\ b \end{bmatrix}$$

Just by considering the $O(N)$ fermion at $N = 32$. On the other hand, if we split the fermions into $16 + 16$, and consider separately boundary conditions for each of *those*, then our partition function is the square of the 16-fermion system. We then get the $E_8 \times E_8$ lattice theta-function, as required

$$\left[\frac{1}{2}\sum_{a,b}\theta^{8}\begin{bmatrix} a \\ b \end{bmatrix}\right]^2$$

15. Note this was a Lorentzian lattice of signature (n, n). The norm was thus $P_L^2 - P_R^2 = 2mn \in 2\mathbb{Z}$. It is also self dual, since it is already integral, and there is no integral sublattice.

16. We have

$$\gamma G_{ghost} = -c\gamma\partial\beta - \frac{3}{2}\partial c\gamma\beta - 2\gamma^2 b, \quad cT_{ghost} = 2bc\partial c - \frac{1}{2}c\gamma\partial\beta - \frac{3}{2}c\partial\gamma\beta$$

Here Kitisis' conventions are different than Polchinski. Recall upon bosonization $\beta(z) = e^{-\phi(z)}\partial\xi(z), \gamma = e^{\phi(z)}\eta(z)$. Although we can solve this problem very quickly since we already know what the stress tensor looks like in the bosonized variables, I think it's way more instructive to explicitly compute OPEs to $O(z-w)$. First let's look at the η, ξ theory, which is a fermoinic bc theory of weights $1, 0$. We get

$$\xi(z)\eta(w) = \frac{1}{z-w} + :\xi\eta:(w) + O(z-w)$$

We can bosonize this theory in terms of hermitian χ field so that $\eta = e^{-\chi}, \xi = e^{-\chi}$. Using these coordinates

$$\begin{aligned} \xi(z)\eta(w) = e^{\chi(z)}e^{-\chi(w)} &= \frac{1}{z-w}\left[1 + (z-w)\partial\chi + \frac{1}{2}(z-w)^2(\partial^2\chi + (\partial\chi)^2) + \dots\right] \\ \Rightarrow \partial\xi(z)\eta(w) &= -\frac{1}{(z-w)^2} + \frac{1}{2}(\partial^2\chi + (\partial\chi)^2) \end{aligned}$$

Using this we can write

$$\begin{aligned} \beta(z)\gamma(w) &= e^{-\phi(z)}\partial\xi(z)\, e^{\phi(w)}\eta(w) \\ &= (z-w)\left[1 - (z-w)\partial\phi(w) + \frac{1}{2}(z-w)^2((\partial\phi)^2 - \partial^2\phi)\right]\left[-\frac{1}{(z-w)^2} + \frac{1}{2}(\partial^2\chi + (\partial\chi)^2)\right] \end{aligned}$$

The constant term gives $:\beta\gamma:= \partial\phi \Rightarrow :\partial(\beta\gamma):= \partial^2\phi$. The $(z-w)$ term gives exactly the stress tensor of the $\beta\gamma$ theory at $\lambda = 0$, which makes sense since this is exactly $\partial\beta\gamma$

$$\begin{aligned} :\partial\beta\gamma: &= -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\partial^2\phi + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\partial^2\chi \\ \Rightarrow T_{\beta\gamma} = \partial\beta\gamma - \lambda\partial(\beta\gamma) &= -\frac{1}{2}(\partial\phi)^2 + \left(\frac{1}{2} - \lambda\right)\partial^2\phi + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\partial^2\chi. \end{aligned}$$

In our case we have $\lambda = 3/2$.

$$\begin{aligned} \gamma G_{ghost} &= -c\left(-\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\partial^2\phi + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\partial^2\chi\right) - \frac{3}{2}\partial\phi\,\partial c - 2\gamma^2 b \\ cT_{ghost} &= 2bc\partial c + c\left(-\frac{1}{2}(\partial\phi)^2 - \partial^2\phi + \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\partial^2\chi\right) \end{aligned}$$

Altogether this gives a BRST current:

$$\begin{aligned} j_B &= cT_X + \gamma G_X + \frac{1}{2}\left(cT_{gh} + \gamma G_{gh}\right) \\ &= cT_X + \gamma G_X + bc\partial c - \frac{3}{4}\partial\phi\partial c - \frac{3}{4}c\partial^2\phi - \gamma^2 b \end{aligned}$$

17. We are looking at $[Q_B, \xi e^{-\phi/2}S_\alpha e^{ipX}]$. Therefore we should look at the $1/(z-w)$ pole in the OPE of j_B with $\xi e^{-\phi/2}S_\alpha e^{ipX}$. The terms that contribute to this pole must involve pairing ξ with its conjugate η. η appears in j_B wherever $\gamma = e^\phi\eta$ appears. From the previous exercise, we see that we need only look at the terms γG_X and $-\gamma^2 b$.

    These two terms contribute poles:

$$-\left[\frac{:e^\phi G_X::e^{-\phi/2}S_\alpha e^{ipX}:}{z-w} - \frac{:e^{3\phi/2}\eta b S_\alpha e^{ipX}:}{z-w}\right]$$

The overall minus sign comes from commuting across an odd number of fermions for the Wick-contraction. We will need to recall two things:

$$\psi^\mu(z)\cdot S_\alpha(w) \sim \frac{\ell_s}{\sqrt{2}\sqrt{z-w}}\left(\Gamma^\mu_{\alpha\beta}S^\beta(w) + \frac{1}{\ell_s^2(\frac{D}{2}-1)}\Gamma^\nu_{\alpha\beta}S_\beta\psi_\nu\psi^\mu\,(z-w)\right), \qquad e^{\phi(z)}e^{-\phi(w)/2} \sim \sqrt{z-w}e^{\phi(w)/2}$$

The subleading term in the first expansion is taken from *Blumenhagen 13.81*. That means that first term is:

$$
\begin{aligned}
&e^{\phi(z)} i\frac{\sqrt{2}}{\ell_s^2}\psi^\mu(z)\partial X_\mu(z)\cdot e^{-\phi(w)/2}S_\alpha(w)e^{ipX(z)} \\
&\quad\sim i\frac{\sqrt{2}}{\ell_s^2}\sqrt{z-w}\,e^{\phi(w)/2}\frac{\ell_s}{\sqrt{2}}\left(\frac{\Gamma^\mu_{\alpha\beta}S^\beta\partial X_\mu e^{ip\cdot X}}{\sqrt{z-w}}+\frac{-i\ell_s^2 p_\mu e^{ip\cdot X}}{2(z-w)}\frac{1}{4\ell_s^2}\Gamma^\nu_{\alpha\beta}S^\beta\psi_\nu\psi^\mu\sqrt{z-w}\right) \\
&\quad=-\frac{e^{\phi/2}}{\ell_s}\left(\Gamma^\mu_{\alpha\beta}S^\beta\partial X_\mu - i\Gamma^\nu_{\alpha\beta}S^\beta\psi_\nu p\cdot\psi\right)e^{ip\cdot X}
\end{aligned}
$$

Note this OPE has no singularity, so we exactly got the normal ordered term we required: $:e^{\phi/2}G_X S_\alpha e^{ip\cdot X}:$. Altogether this gives us:

$$
V^{(1/2)}_{\text{fermion}}(u,p) = u^\alpha(p)\left[\frac{e^{\phi/2}}{\ell_s}\Gamma^\mu_{\alpha\beta}S^\beta\partial X^\mu - \frac{i}{8}\frac{e^{\phi/2}}{\ell_s}\Gamma^\nu_{\alpha\beta}S^\beta\psi_\nu p\cdot\psi + e^{3\phi/2}\eta b S_\alpha\right]e^{ipX}.
$$

I believe this is right, and moreover that the inclusion of the ℓ_s^{-1} factor is necessary for the dimensional analysis to make sense.

18. Here, I followed the discussion of Polchinski **12.5**. The picture changing operator is:

$$
\mathrm{X}(z) := Q_B\cdot\xi(z)
$$

Over the sphere, the $\beta\gamma$ path integral is equivalent to the ϕ,η,ξ path integral *plus* an additional insertion of ξ to make up for the fact that it pics up a zero mode due to to the vacuum degeneracy it produces. Because the expectation value is *just* proportional to the zero-mode of ξ, which depends on global information rather than the specific local insertion point, $\langle\chi(z)\rangle$ is independent of position and we can normalize ξ so that this is 1.

Say we have a null state. This means it is BRST exact. This means that we can rewrite its pointlike insertion as a local operator surrounded by a BRST contour (direct, from the definition of exact). For that null state to decouple, we need to be able to contract the BRST contour off the sphere (i.e. by pulling it off to the north pole). The fact that ξ is inserted will seem to obstruct this. What happens now as we pull the BRST charge to infinity is that it will circle ξ, creating the PCO $\mathrm{X}(z)$. However, when the ξ insertion is replaced by X, the path integral will *vanish* since there is now no ξ insertion to avoid the zero-mode.

Now consider a path integral with a PCO insertion as well as additional BRST-invariant operators (meaning. the contour integral around them of j_B is zero). Then we can write $\mathrm{X}(z_1)\xi(z_2) = Q_B\xi(z_1)\xi(z_2) = (-)^2\xi(z_1)Q^B\xi(z_2) = \xi(z_1)\mathrm{X}(z_2)$ where I have pulled the Q_B contour around the sphere (there two minus signs, one from commuting Q_B across a fermionic variable and one from reversing the orientation of the contour.)

This is interesting: although X is null, it does *not* vanish in the path integral, since pulling Q_B off of it will make Q_B encircle $\xi(z_2)$ but leave behind $\mathrm{X}(z_1)$'s $\xi(z_1)$, so the ξ zero-mode will remain saturated and we won't get zero.

The X can be brought near any of the local BRST closed operators to change their picture (the OPE is nonsingular). Ie note that the main term we look at is $\gamma G_X = e^\phi\eta G_X$ in j_B so that $\mathrm{X}\mathcal{O}^{(-1)}(z) = zG_X(z)\mathcal{O}(0)\to G_{-1/2}\mathcal{O}(0)$. We can move X to any other point on the sphere - since the exact position of X does not matter any more than the position of ξ.

19. It is enough to look at the $1/(z-w)$ term in the OPE

$$
:e^{-\phi(z)/2}S_\alpha(z): V^{(-1/2)}_{\text{fermion}}(w) = e^{-\phi(z)/2}S_\alpha(z)u^\beta(p)e^{-\phi(w)/2}S_\beta(w)e^{ip\cdot X(w)}
$$

We will use the fact of **4.12.42**:

$$
S_\alpha(z)S_\beta(w) = \frac{C_{\alpha\beta}}{(z-w)^{N/8}} + \frac{\Gamma^\mu_{\alpha\beta}\psi_\mu(w)}{\sqrt{2}\ell_s(z-w)^{N/8-1/2}}
$$

where $C_{\alpha\beta}$ is the charge conjugation matrix and here $N=10$. We also have $e^{-\phi/2}e^{-\phi/2}=(z-w)^{-1/4}e^{-\phi}$. This leaves the $(z-w)^{-1}$ term to be the requisite

$$e^{-\phi}u^{\beta}(p)\frac{\Gamma^{\mu}_{\alpha\beta}}{\sqrt{2}\ell_s}\psi_{\mu}e^{ip\cdot X}=V^{(-1)}_{\text{boson}}(\epsilon=\frac{\Gamma^{\mu}_{\alpha\beta}u^{\beta}}{\sqrt{2}\ell_s},p,z)$$

For the second example, we will look at the $(z-w)^{-1}$ term in the OPE

$$e^{-\phi(z)/2}S_{\alpha}(z)\epsilon_{\mu}\left(\partial X^{\mu}-\frac{i}{2}p_{\mu}\psi^{\mu}\,\psi^{\nu}\right)e^{ip\cdot X}.$$

The first term in parentheses will not contribute to the singular term. Also the $e^{-\phi/2}$ and $e^{ip\cdot X}$ contract with nothing. Here, we use **4.12.41** to evaluate

$$S_{\alpha}(z)\cdot\underbrace{\psi^{\mu}\psi^{\nu}}_{-iJ^{\mu\nu}}(w)\sim-\frac{\ell_s^2(\Gamma_{\mu\nu})_{\alpha}^{\beta}S_{\beta}(w)}{2(z-w)}$$

The $-$ sign comes from the fact that the fermion current is coming from the *right* this time so z and w are swapped. This gives a variation

$$e^{-\phi/2}ip^{\mu}\epsilon^{\nu}\frac{\ell_s^2}{4}\epsilon^{\nu}(\Gamma_{\mu\nu})_{\alpha}^{\beta}S_{\beta}e^{ip\cdot X}=V^{(-1/2)}_{\text{fermion}}(u^{\beta}=\frac{ip^{\mu}\epsilon^{\nu}\ell_s^2(\Gamma_{\mu\nu})_{\alpha}^{\beta}}{4},p,z)$$

20. We are in type I. We have

$$\frac{1}{\ell_s^4g_o^2}\langle:cV^{(-1)}(w_1)::cV^{-1}(w_2)::cV^{0}(w_3):\rangle+1\leftrightarrow2,\quad x_1>x_2>x_3$$

That constant out front is not obvious from Kiritsis, c.f. the discussion in Polchinski 12.4 and allow for another factor of ℓ_s^2 since the fermions are dimensionful The relevant expectation values are

$$\langle c(w_1)c(w_2)c(w_3)\rangle=|w_{12}w_{13}w_{23}|,\quad\langle e^{-\phi(w_1)}e^{-\phi(w_2)}\rangle=w_{12}^{-1},$$
$$\langle\psi^{\mu}(w_1)\psi^{\nu}(w_2)\rangle=\ell_s^2\eta^{\mu\nu}w_{12}^{-1},\qquad\langle\dot{X}^{\mu}(w_1)e^{ik_1X}(w_2)\rangle=-2i\ell_s^2k_1^{\mu}e^{ik_1X}w_{12}^{-1}$$

In the matter CFT we get (here $k_i\cdot k_j=0$ so the pure e^{ikX} terms contract to 1):

$$\langle\psi^{\mu}(w_1)e^{ik_1\cdot X(w_1)}\,\psi^{\nu}(w_2)e^{ik_2\cdot X(w_2)}\,(i\dot{X}^{\rho}+2k_3\cdot\psi\psi^{\rho})e^{ik_3\cdot X(w_3)}\rangle$$
$$=2\ell_s^4\delta\!\!\!/^{10}(\Sigma k)\left(\frac{\eta^{\mu\nu}k_1^{\rho}}{w_{12}w_{13}}+\frac{\eta^{\mu\nu}k_2^{\rho}}{w_{12}w_{23}}+\frac{\eta^{\mu\rho}k_3^{\nu}-\eta^{\nu\rho}k_3^{\mu}}{w_{13}w_{23}}\right)$$

So altogether we get an amplitude of (taking $x_1\to0,x_2\to1,x_3\to\infty$)

$$\frac{i}{g_{open}^2\ell_s^4}\times\frac{2i\ell_s^4g_{open}^3}{\sqrt{2}\ell_s}\delta\!\!\!/^{10}(\Sigma k)\left(\frac{\eta^{\mu\nu}k_1^{\rho}x_{23}}{x_{12}}+\frac{\eta^{\mu\nu}k_2^{\rho}x_{13}}{x_{12}}+\eta^{\mu\rho}k_3^{\nu}-\eta^{\nu\rho}k_3^{\mu}\right)([123]-[132])$$
$$=\frac{ig_{open}}{\sqrt{2}\ell_s}\delta\!\!\!/^{10}(\Sigma k)\left(\eta^{\mu\nu}k_{12}^{\rho}+\eta^{\mu\rho}k_{31}^{\nu}+\eta^{\nu\rho}k_{23}^{\mu}\right)([123]-[132])$$

Note unlike the Bosonic string this is *exactly the same* as the ordinary Yang-Mills amplitude, there is no k^3 correction term (what would correspond to at $\text{Tr}F^3$ term in the Lagrangian).

21. This is also in type I. We should put the gaugini in the $-1/2$ picture and the boson in the -1 picture. We have

$$\frac{1}{\ell_s^4g_o^2}\langle:cV^{(-1)}(w_1)::cV^{-1/2}(w_2)::cV^{-1/2}(w_3):\rangle+1\leftrightarrow2,\quad x_1>x_2>x_3$$

The relevant expectation values are

$$\langle c(w_1)c(w_2)c(w_3)\rangle = |w_{12}w_{13}w_{23}|, \quad \langle e^{-\phi(w_1)/2}e^{-\phi(w_2)/2}e^{-\phi(w_3)}\rangle = w_{12}^{-1/4}w_{13}^{-1/2}w_{23}^{-1/2},$$
$$\langle S_\alpha(w_1)S_\beta(w_2)\rangle = C_{\alpha\beta}w_{12}^{-5/4} \qquad \Rightarrow \langle S_\alpha(w_1)S_\beta(w_2)\psi^\mu(w_3)\rangle = \frac{\ell_s^2}{\sqrt{2}}(C\Gamma)^\mu_{\alpha\beta}w_{12}^{-3/4}w_{13}^{-1/2}w_{23}^{-1/2}$$

So altogether this gives

$$\frac{i}{\ell_s^4 g_{open}^2} \times (g_{open}\sqrt{\ell_s})^2 g_{open}\frac{\ell_s^2}{\sqrt{2}}\delta\!\!\!/^{10}(\Sigma k)C\Gamma^\mu_{\alpha\beta} \times ([123]-[132]) = \frac{i g_{open}}{\sqrt{2}\ell_s}\delta\!\!\!/^{10}(\Sigma k)C\Gamma^\mu_{\alpha\beta}([123]-[132])$$

This is k-independent so is an even *simpler* amplitude that the last in some sense.

22. We are now in type II. Gravitons are NS-NS states. We take two of them in the $(-1,-1)$ picture and the remaining one in the $(0,0)$ picture. Again now, the constant demanded from unitarity now gets modified to $\frac{8pii}{g_c^2\ell_s^6}$ We look at

$$\frac{8\pi i}{g_c^2\ell_s^6}\frac{2g_c^3}{\ell_s^2}\Big\langle\big[c\tilde{c}e^{-\phi-\bar{\phi}}\psi^\mu\tilde{\psi}^\sigma e^{ik_1X}\big](z_1)\big[c\tilde{c}e^{-\phi-\bar{\phi}}\psi^\nu\tilde{\psi}^\omega e^{ik_2X}\big](z_2)\big[c\tilde{c}(\partial X^\rho - \frac{i}{2}k\cdot\psi\psi^\rho)(\partial X^\lambda - \frac{i}{2}k\cdot\psi\psi^\lambda)e^{ik_3X}\big](z_3)\Big\rangle$$

Let's just look at the holomorphic part of the matter CFT, and the calculation goes almost exactly as in the last problem

$$\begin{aligned}
&\langle\psi^\mu(z_1)e^{ik_1\cdot X(z_1)}\,\psi^\nu(z_2)e^{ik_2\cdot X(z_2)}\,(i\dot{X}^\rho + \frac{1}{2}k_3\cdot\psi\psi^\rho)e^{ik_3\cdot X(z_3)}\rangle\\
&= \frac{1}{2}\ell_s^4\left(\frac{\eta^{\mu\nu}k_1^\rho}{z_{12}z_{13}} + \frac{\eta^{\mu\nu}k_2^\rho}{z_{12}z_{23}} + \frac{\eta^{\mu\rho}k_3^\nu - \eta^{\nu\rho}k_3^\mu}{z_{13}z_{23}}\right)\\
&\to \frac{\ell_s^4}{4}\underbrace{(\eta^{\mu\nu}k_{12}^\rho + \eta^{\mu\rho}k_{31}^\nu + \eta^{\nu\rho}k_{23}^\mu)}_{=:V^{\mu\nu\rho}}
\end{aligned}$$

So the total amplitude becomes

$$\pi i g_c \delta\!\!\!/^{10}(\Sigma k)V^{\mu\nu\rho}V^{\sigma\omega\lambda}$$

consistent with Polchinski.

23. We can put all our gaugini in the $-1/2$ picture thankfully. Our vertex operators are $g_{open}\sqrt{\ell_s}\lambda^\alpha e^{-\phi/2}S_\alpha e^{ikX}$. The relevant two-point correlator is

$$S_\alpha(z)S_\beta(w) \sim \frac{\ell_s(C\Gamma^\mu)_{\alpha\beta}\psi_\mu}{\sqrt{2}(z-w)}$$

From considerations of the singularity structure, we get that the four-point correlator is:

$$\frac{\ell_s^2(C\Gamma^\mu)_{\alpha\beta}(C\Gamma_\mu)_{\gamma\delta}}{2z_{12}z_{34}z_{23}z_{34}} + \frac{\ell_s^2(C\Gamma^\mu)_{\alpha\gamma}(C\Gamma_\mu)_{\alpha\delta}}{2z_{13}z_{24}z_{32}z_{42}} + \frac{\ell_s^2(C\Gamma^\mu)_{\alpha\delta}(C\Gamma_\mu)_{\beta\gamma}}{2z_{14}z_{23}z_{42}z_{43}}$$

Take $z_1 = 0, z_2 = w, z_3 = 1, z_4 = \infty$. In order for the term going as $1/z$ to cancel so that the integral over the line is well-defined, we need the (physical on-shell condition):

$$\Gamma^\mu_{\alpha\beta}\Gamma^\mu_{\gamma\delta} + \Gamma^\mu_{\alpha\gamma}\Gamma^\mu_{\beta\delta} + \Gamma^\mu_{\alpha\delta}\Gamma^\mu_{\beta\gamma} = 0$$

and defining $-(k_1+k_2)^2 = s$ etc. gives

$$\begin{aligned}
&\frac{i}{\ell_s^4 g_{open}^2} \times (g_{open}\sqrt{\ell_s})^4\delta\!\!\!/^{10}(\Sigma k) \times \frac{\ell_s^2}{2}\int_0^1 x^{-\ell_s^2 s-1}(1-x)^{-\ell_s^2 u-1}(\Gamma^\mu_{\alpha\beta}\Gamma^\mu_{\gamma\delta} + x\Gamma^\mu_{\alpha\gamma}\Gamma^\mu_{\beta\delta})[1234]\\
&= -\frac{i g_{open}^2\ell_s^2}{2}\delta\!\!\!/^{10}(\Sigma k)\,2 \times \left(\frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 u)}{\Gamma(1-\ell_s^2 s - \ell_s^2 u)}(u\Gamma^\mu_{\alpha\beta}\Gamma^\mu_{\gamma\delta} - s\Gamma^\mu_{\alpha\delta}\Gamma^\mu_{\beta\gamma})[1234] + 2\,\text{perms.}\right)
\end{aligned}$$

The minus sign comes from pulling an s or u out of the Γ functions. The factor of 2 comes from summing over both orientations. Altogether we can write this as

$$-8ig_{open}^2\ell_s^2\delta^{10}(\Sigma k)K(u_1,u_2,u_3,u_4)\left(\frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 u)}{\Gamma(1-\ell_s^2 s-\ell_s^2 u)}[1234]+2\,\text{perms.}\right)$$

$$K(u_1,u_2,u_3,u_4)=\frac{1}{8}(u\,\bar{u}_1\Gamma^\mu u_2\bar{u}_3\Gamma_\mu u_4 - s\,\bar{u}_1\Gamma^\mu u_4\bar{u}_3\Gamma_\mu u_2)$$

24. The bosonic action in 11D is:

$$\frac{1}{2\kappa^2}\int d^{11}x\sqrt{-\det\hat{G}}\left[R-\frac{1}{2\cdot 4!}G_4^2+\frac{1}{(144)^2}\epsilon^{M_1\dots M_{11}}G_{M_1\dots M_4}G_{M_4\dots M_8}\hat{C}_{M_9M_{10}M_{11}}\right]$$

where G_4 is the field strength of the 3-form $\hat{C}$. From Appendix F, we have that the dilaton $\Phi=0$ in 11D. So the field σ will just be $\sigma=-2\phi=\frac{1}{2}\log G_{11\,11}$, and A here is as it is in appendix **F**. Directly using the bosonic equation **F.3** gives the terms

$$\frac{1}{2\kappa^2}\int d^8x\sqrt{-g}e^{\sigma}\left[R-\frac{1}{4}e^{2\sigma}F_2^2\right]$$

(Here the $\frac{1}{4}\partial_\mu G_{11\,11}\partial^\mu(G_{11\,11})^{-1}$) will exactly cancel the $4\partial_\mu\phi d^\mu\phi$.

Now let's look at the 3-form potential contribution. Because F is antisymmetric in all four indices, and we only are compactifying along one dimension, only the first two terms of **F.28** can contribute. They give

$$=\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{\sigma}\left[-\frac{1}{2\cdot 4!}F_4^2-\frac{1}{2\cdot 3!}e^{-2\sigma}H_3^2\right]$$

and $(H_3)_{\mu\nu\rho}=\partial_{[\mu}(B_2)_{\nu\rho]}=\partial C_{\mu\nu\,11}$ so that $H_3^2=G^{\mu\sigma}G^{\nu\lambda}G^{\rho\kappa}H_{\mu\nu\rho}H_{\sigma\lambda\kappa}$ and $C_{\mu\nu\rho}=\hat{C}_{\mu\nu\rho}-(\hat{C}_{\nu\rho\,11}A_\mu+2\,\text{perms.})$ consistent with **F.30**.

Finally, let's look at that last $\epsilon^{M_1\dots M11}$ term. At first it looks quite scary. Note we can write this last term as $\frac{1}{6}\mathrm{d}\hat{C}_3\wedge\mathrm{d}\hat{C}_3\wedge\hat{C}_3$. I have 11 indices to pick to be index 11. If I pick any of the indices of the last $\hat{C}$ I get the term

$$\frac{1}{12\kappa^2}\mathrm{d}C_3\wedge\mathrm{d}C_3\wedge B_2$$

If I pick either of the $\mathrm{d}C$ terms, then after an integration by parts I get the same term. So the same term contributes three times **Revisit this logic**. We thus get the requisite action contribution:

$$\frac{1}{4\kappa^2}\int d^{10}xB_2\wedge\mathrm{d}C_3\wedge\mathrm{d}C_3$$

25. Under $A_1\to A_1+\mathrm{d}\epsilon$ and $C_3\to C_3+\epsilon H_3$ we see that obviously R, F_2, B_2, and H_3 will stay the same. Now $dC_3\to dC_3+d\epsilon\wedge H_3$ while $A\wedge H_3\to A\wedge H_3-d\epsilon\wedge H_3$. Thus, F_4 will stay the same.

It remains to look at the variation of $B_2\wedge\mathrm{d}C_3\wedge\mathrm{d}C_3$. This is

$$B_2\wedge\mathrm{d}\epsilon\wedge H_3\wedge\mathrm{d}C_3+B_2\wedge\mathrm{d}C_3\wedge\mathrm{d}\epsilon\wedge H_3.$$

These two terms cancel by antisymmetry of the indices.

26. Defining $C_3'=C_3+A\wedge B_2$ give the above transformation as $A_1\to A_1+\mathrm{d}\epsilon, C_3'\to C_3'+\epsilon H_3+\mathrm{d}\epsilon\wedge B_2$ so that $\mathrm{d}C_3'\to\mathrm{d}C_3'+\mathrm{d}\epsilon\wedge H_3-\mathrm{d}\epsilon\wedge H_3=\mathrm{d}C_3'$.

Now $G_4=\mathrm{d}C_3'-\mathrm{d}A\wedge B_2$ (Kiritsis wrote a small A, which I believe is a typo). Under the same transformation we get that G_4 is invariant as required.

Further, the transformation of $C_3\to C_3+\mathrm{d}\Lambda_2$ implies that $C_3'\to C'+\mathrm{d}\Lambda_2\Rightarrow\mathrm{d}C_3'\to\mathrm{d}C_3'\Rightarrow G_4\to G_4$ as required. So C_3' now transforms trivially under the A transformation.

27. Take $S = S_0 + ie^{-\phi}$. The $\mathrm{SL}_2(\mathbb{R})$ transformation acts on IIB supergravity as:

$$S \to \frac{aS+b}{cS+d}, \quad \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}$$

The latter is also how H_3, F_3 will transform together. Now think of S as the modular parameter τ, $e^{-\phi} = S_2 = \tau_2$ is the imaginary part. Think of H_3, F_3 as periods ω_1, ω_2 The IIB action in the Einstein frame can be written as

$$S_{IIB} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[R - \frac{1}{2} \frac{\partial S \, \partial \bar{S}}{S_2^2} - \frac{1}{2 \cdot 3!} \frac{|G_3|^2}{S_2} - \frac{1}{4 \cdot 5!} F_5^2 \right] + \frac{1}{8i\kappa^2} \int C_4 \wedge \frac{G_3 \wedge \bar{G}_3}{S_2}$$

Now R, C_4, and F_5 do not change. The term $\frac{\partial S \, \partial \bar{S}}{S^2}$ transforms under $\mathrm{SL}(2, \mathbb{R})$ exactly like the invariant measure $\frac{d\tau d\bar{\tau}}{\tau_2^2}$.

Finally, any term consisting of a pair $G_3, \bar{G}_3$ in the numerator (either wedged or wedged with a hodge star) divided by S_2 will also remain modular invariant, as a quick Mathematica check confirms for us:

```
In[2216]:= G = F3 + (S1 + I S2) H3;
           τ2 = S2;
           Gp = Assuming[a d - b c == 1, (a F3 - b H3) + (a (S1 + I S2) + b)/(c (S1 + I S2) + d) (-c F3 + d H3) //
               FullSimplify];
           τ2p = Assuming[a d - b c == 1, Im[(a (S1 + I S2) + b)/(c (S1 + I S2) + d)] // ComplexExpand // FullSimplify];
           (G * Conjugate[G] // ComplexExpand)/τ2 // FullSimplify
           ((Gp * Conjugate[Gp] // ComplexExpand))/τ2p // FullSimplify

Out[2220]= (F3 + H3 S1)^2/S2 + H3^2 S2

Out[2221]= (F3 + H3 S1)^2/S2 + H3^2 S2
```

28. Again for some reason Kiritsis writes a small a, which again I think is a typo. We need to find which gauge transformations need to be modified for C_0, B_2, C_2, C_4. There is a Chern-Simons term only in the definition of $F_5 = dC_4 - C_2 \wedge H_3$ so we see that the $C_0 \to C_0 + c$ (for c a constant, the only closed 0-form) keeps the action invariant.

Taking $B_2 \to B_2 + \mathrm{d}\Lambda_1$ will keep H_3 and therefore F_5 invariant, so this transform is legitimate.

Also, taking $C_4 \to C_4 + \mathrm{d}\Lambda_3$ will keep F_5 invariant as well and will modify the Chern-Simons term in the full action $C_4 \wedge H_4 \wedge F_3$ by closed form, which will give no contribution.

Finally, taking $C_2 \to C_2 + \mathrm{d}\Lambda_1$ will change $F_5 \to F_5 - \mathrm{d}\Lambda_1 \wedge H_3$. This must be compensated by changing $C_4 \to C_4 + \frac{1}{2}\Lambda_1 \wedge H_3 + \frac{1}{2}\mathrm{d}\Lambda_1 \wedge B_2$. Then F_5 will be invariant. Moreover, the Chern Simons term in the action $C_4 \wedge H_3 \wedge F_3$ have a variation

$$\frac{1}{2}\Lambda_1 \wedge H_3 \wedge H_3 \wedge F_3 + \frac{1}{2}\mathrm{d}\Lambda_1 \wedge B_2 \wedge H_3 \wedge F_3$$

After integration by parts this variation will contribute nothing, as required.

29. Clearly $\dim O(32) = 32 \times 31/2 = 496$, which is necessary. For $N = 32$ we also get from **7.9.29** that $\mathrm{Tr}(F^6) = 15\,\mathrm{tr}(F^4)\mathrm{tr}(F^2)$ where tr is the trace of the curvature form in (an associated bundle for) the fundamental representation Also using **7.9.30** we get $\mathrm{Tr}(F^4) = 24\mathrm{tr}(F^4) + 3(\mathrm{tr}(F^2))^2$ and $\mathrm{Tr}(F^2) = 30\mathrm{tr}(F^2)$. Then, both sides of equation **7.9.26** become

$$15\mathrm{tr}(F^4)\mathrm{tr}(F^2) = -\frac{15}{8}\mathrm{tr}(F^2)^3 + \frac{15}{8}\mathrm{tr}(F^2)^3 + 15\mathrm{tr}(F^4)\mathrm{tr}(F^2)$$

and we have agreement!

30. For an individual E_8, we have exactly

$$\mathrm{Tr}(F^6) = \frac{1}{7200}\mathrm{Tr}F^2 = \frac{1}{48}\mathrm{Tr}F^2 \cdot \frac{1}{100}(\mathrm{Tr}F^2)^2 - \frac{1}{14400}(\mathrm{Tr}F^2)^3$$

Now for $E_8 \times E_8$ we have the property that $\mathrm{Tr}(F^n) = \mathrm{Tr}_1(F^n) + \mathrm{Tr}_2(F^n)$ where Tr_i means tracing over the direct summands in the associated bundle that are acted on by the ith E_8. Then we will have the relations

$$\mathrm{Tr}(F^6) = \frac{1}{7200}(\mathrm{Tr}_1(F^2) + \mathrm{Tr}_2(F^2))^3, \quad \mathrm{Tr}(F^4) = \frac{1}{100}(\mathrm{Tr}_1(F^2) + \mathrm{Tr}_2(F^2))^2$$

So replacing $\mathrm{Tr}(F^2)$ with $\mathrm{Tr}_1(F^2) + \mathrm{Tr}_2(F^2)$ in the prior derivation gives us again exact matching and thus anomaly cancelation.

On the other hand $U(1)$ *has* no Casimirs so $\mathrm{Tr}(F^m) = 0$ for all m. In particular this allows us to take $E_8 \times U(1)^{248}$ or $U(1)^{496}$ as a gauge group and remain anomaly-free. **Check with Nick. Reconcile this with**

31. Now let us turn to the $SO(16) \times SO(16)$ theory. To check anomalies, we look at the chiral terms. In this case we have massless content consisting of spin 1/2 Majorana-Weyl fermions transforming in the $(16, 16)$ (positive chirality), and $(1, 128) \oplus (128, 1)$ (negative chirality) representations. We also have *massive* fermion fields in the $(128, 128)$ representation that do *not* contribute to the anomaly. Note that we do *not* have a gravitino, as there is no spacetime SUSY in this theory.

The positive chirality $(16, 16)$ MW fermions have field strength $F_+ = F_1 \otimes 1 + 1 \otimes F_2$ valued in the vector representation.

The negative chirality $(1, 128) \oplus (128, 1)$ MW fermions have field strength $\hat{F}_- = \hat{F}_1 \oplus \hat{F}_2$ valued in the spinor representation.

Our anomaly polynomial is thus

$$\frac{1}{2}(I_{1/2}(R, F_+) - I_{1/2}(R, F_-))$$

Both representations have dimension 256, so the $\frac{n}{64}(\dots)$ term in **7.9.22** cancels (note we did *not* need to use that the dimension of the gauge group was 496 here!). We are left with (using tr for the trace in the fundamental representation and tr_s for the spinor rep'n)

$$\begin{aligned} &-\frac{\mathrm{tr}[F_+^6]}{720} + \frac{\mathrm{tr}[F_+^4]\mathrm{Tr}[R^2]}{24 \cdot 48} - \frac{\mathrm{tr}[F_+^2]}{256}\left(\frac{\mathrm{Tr}[R^4]}{45} + \frac{(\mathrm{Tr}[R^2])^2}{36}\right) \\ &-\left(-\frac{\mathrm{tr}_S[\hat{F}_-^6]}{720} + \frac{\mathrm{tr}_S[\hat{F}_-^4]\mathrm{Tr}[R^2]}{24 \cdot 48} - \frac{\mathrm{tr}_S[\hat{F}_-^2]}{256}\left(\frac{\mathrm{Tr}[R^4]}{45} + \frac{(\mathrm{Tr}[R^2])^2}{36}\right)\right) \end{aligned} \tag{70}$$

From explicitly expanding out $(F_1 \otimes 1 + 1 \otimes F_2)^{2,4,6}$ we get:

$$\begin{aligned} \mathrm{tr}F_+^2 &= 16(\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2) \\ \mathrm{tr}F_+^4 &= 16(\mathrm{tr}F_1^4 + \mathrm{tr}F_2^4) + 6\mathrm{tr}F_1^2\mathrm{tr}F_2^2 \\ \mathrm{tr}F_+^6 &= 16(\mathrm{tr}F_1^6 + \mathrm{tr}F_2^6) + 15\mathrm{tr}F_1^2\mathrm{tr}F_2^4 + 15\mathrm{tr}F_1^4\mathrm{tr}F_2^2 \end{aligned}$$

Together with the results **7.4E, 7.5E** relating tr_S to tr we get

$$\begin{aligned} \mathrm{tr}_S F_-^2 &= 16(\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2) \\ \mathrm{tr}_S F_-^4 &= -8(\mathrm{tr}F_1^4 + \mathrm{tr}F_2^4) + 6(\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2)^2 \\ \mathrm{tr}_S F_-^6 &= 16(\mathrm{tr}F_1^6 + \mathrm{tr}F_2^6) - 15(\mathrm{tr}F_1^2\mathrm{tr}F_1^4 + \mathrm{tr}F_2^2\mathrm{tr}F_2^4) + \frac{15}{4}\left((\mathrm{tr}F_1^2)^3 + (\mathrm{tr}F_2^2)^3\right) \end{aligned}$$

Altogether we get

```
In[3014]:= exp = - (16 (F16 + F26) + 15 (F12 F24 + F22 F14))/720 + (16 (F14 + F24) + 6 F12 F22)/(24 × 48) R2 - (16 (F12 + F22))/256 (R4/45 + R2^2/36) -
    (- ((16 F16 - 15 F12 F14 + 15/4 F12^3) + (16 F26 - 15 F22 F24 + 15/4 F22^3))/720 + (-8 (F14 + F24) + 6 (F12^2 + F22^2))/(24 × 48) R2 - (16 (F12 + F22))/256 (R4/45 + R2^2/36));
    Coefficient[exp, R2] 192/4 // Expand
    (R2 - (F12 + F22)) Coefficient[exp, R2] - exp // Expand

Out[3015]= - F12^2/4 + F14 + (F12 F22)/4 - F22^2/4 + F24

Out[3016]= 0
```

We thus get a Green-Schwarz term

$$\propto \int d^{10}x B\left[\mathrm{Tr}_1(F^4)+\mathrm{Tr}_2(F^4)-\frac{1}{4}(\mathrm{Tr}_1(F^2)^2+\mathrm{Tr}_2(F^2)^2-\mathrm{Tr}(F_1^2)\mathrm{Tr}(F_2^2)\right]$$

We have exhausted the set of supersymmetric chiral anomaly-free theories, so the question remains whether there are any *non*-supersymmetric theories that are chiral and anomaly-free in 10D. We will have only MW fermions and perhaps self-dual 5-form fields contributing. It does not seem possible to cancel the $I_A(R)$ with *just* the $I_{1/2}(R,F)$, so I expect any such 10 D non-SUSY theory will in fact contain *only* MW fermions. They must come in pairs of opposite parities with equal particle number to cancel the gravitational anomaly. **Exhaustively showing this seems really difficult. Xi didn't know the full answer**

I know for a fact there is at least *one* other anomaly free theory in 10D, namely the $USp(32)$ open string Sugimoto theory (c.f. question **7.36**).

32. We are orbifolding by a $\mathbb{Z}_2$. In the sector of the left-moving worldsheet fermions, only $(-1)^{\mathbf{F}}$ acts nontrivially. The twisted blocks are

$$Z_{fermions}\begin{bmatrix}h\\g\end{bmatrix}=\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab+ag+bh+gh}\frac{\theta^4\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}{\eta^4}$$

On the $E_8\times E_8$s the untwisted block is just $(\frac{1}{2}\sum_{ab}\bar\theta^8\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]/\bar\eta^8)^2$. Performing the projection g requires that a,b match for both factors, giving:

$$Z_{E_8^2}\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{4}\sum_{a,b=0}^{1}\frac{(-1)^b\bar\theta^8\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](2\tau)}{\bar\eta^8(2\tau)}=\frac{1}{4}\frac{\bar\theta^8\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]+\bar\theta^8\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\bar\eta^4(\tau)\bar\theta^4\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](\tau)}$$

Taking $\tau\to-1/\tau$ gives

$$Z_{E_8^2}\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{4}\frac{\bar\theta^8\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]+\bar\theta^8\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\bar\eta^4(\tau)\bar\theta^4\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](\tau)}$$

Finally taking $\tau\to\tau+1$ gives

$$Z_{E_8^2}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{4}\frac{\bar\theta^8\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]+\bar\theta^8\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\bar\eta^4(\tau)\bar\theta^4\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](\tau)}$$

The full partition function is thus

$$Z=\frac{1}{2}\sum_{h,g=0}^{1}Z_{E_8^2}\begin{bmatrix}h\\g\end{bmatrix}Z_{fermions}\begin{bmatrix}h\\g\end{bmatrix}$$

We see that this is modular invariant, as individually both $Z_{fermions}$ and $Z_{E_8^2}$ are invariant under $\tau\to-1/\tau$. Their anomalous changes under $\tau\to\tau+1$ from the η^4 powers in the denominator are cancelled in pairs.

The gauge group corresponds to the invariant (diagonal) E_8 sublattice of $E_8\times E_8$ (**Confirm**). At the massless level, we still have the gravity supermultiplet (G,B,Φ), as well as gauge bosons with gauge group E_8 from the untwisted sector. **What about the twisted sector?**

The gravitino has been projected out, so this theory no longer has spacetime supersymmetry. The theory is still chiral, and since the partition function is modular invariant, we are also guaranteed that it is anomaly free. However, it has a tachyon.

Unfinished

33. Let's assume we do not have self-dual 2-form gauge fields that give self-dual 3-form field strengths and we do not consider an I_A contribution. Recall we can write

$$I_{1/2} = \prod_{i=1}^{D/2} \frac{x_i/2}{\sinh(x_i/2)}$$

where x_i are the off-diagonal entries in the 2×2 block decomposition of $R_0 = \mathrm{d}\omega$. All of this is easy to do in Mathematica.

```
PickDegree[poly_, n_] := Module[{tot}, tot = Total@Exponent[#, Variables@#] & /@ List @@ poly;
  Return[Pick[poly, # == n & /@ tot]];]
R2 = -2 (x1^2 + x2^2 + x3^2);
R4 = 2 (x1^4 + x2^4 + x3^4);
deg4 = PickDegree[Series[(x1/2 x2/2 x3/2)/(Sinh[x1/2] Sinh[x2/2] Sinh[x3/2]), {x1, 0, 4}, {x2, 0, 4}, {x3, 0, 4}] // Normal // Expand, 4];
deg2 = 1/2! PickDegree[Series[(x1/2 x2/2 x3/2)/(Sinh[x1/2] Sinh[x2/2] Sinh[x3/2]), {x1, 0, 4}, {x2, 0, 4}, {x3, 0, 4}] // Normal // Expand, 2];
deg0 = 1/4! PickDegree[Series[(x1/2 x2/2 x3/2)/(Sinh[x1/2] Sinh[x2/2] Sinh[x3/2]), {x1, 0, 4}, {x2, 0, 4}, {x3, 0, 4}] // Normal // Expand, 0];
I32 =
  PickDegree[Series[(x1/2 x2/2 x3/2)/(Sinh[x1/2] Sinh[x2/2] Sinh[x3/2]) (-1 + 2 (Cosh[x1] + Cosh[x2] + Cosh[x3])), {x1, 0, 4}, {x2, 0, 4}, {x3, 0, 4}] //
    Normal // Expand, 4];
poly12deg4 = 1/5760 R4 + 1/(576×8) R2^2 // Expand; poly12deg2 = R2/96 // Expand;
poly12deg0 = 1/24 // Expand;
poly32 = 49/576 × 1/2 R4 - 43/(576×8) R2^2 // Expand;
{deg4 - poly12deg4 // Expand, deg2 - poly12deg2 // Expand, deg0 - poly12deg0, I32 - poly32}
Out[830]= {0, 0, 0, 0}
```

For n spin 1/2 fermions and a gravitino we thus get the forms

$$I_{1/2}(R, F) = \frac{n}{576}\left(\frac{\mathrm{Tr}(R^4)}{10} + \frac{(\mathrm{Tr}R^2)^2}{8}\right) - \frac{\mathrm{Tr}F^2}{96}\mathrm{Tr}R^2 + \frac{\mathrm{Tr}F^4}{24}$$
$$I_{3/2}(R) = \frac{49}{576 \times 2}\mathrm{Tr}(R^4) - \frac{43}{576 \times 8}\mathrm{Tr}(R^2)^2$$

The *Anomalies.nb* also has the 10D cancelation if anyone is interested.

34. In the absence of a linear dilaton background, the RR fields simply satisfy the equations of motion $\mathrm{d}\star F_{p+2} = 0$, as well as the Bianchi identities $\mathrm{d}F_{p+2} = 0$. We want to show that, the tree level effective action of type II SUGRA in the string frame will have no coupling at tree level between the RR field strengths and the dilaton.

In a linear dilaton background $\Phi = \frac{Q}{\sqrt{2}\ell_s}X^9$, the supercurrent G will be modified to

$$G = i\frac{\sqrt{2}}{\ell_s^2}\,\psi \cdot \partial X - i\sqrt{2}\,\Phi_{,\mu}\partial\psi^{\mu} \Rightarrow G_0 \propto \frac{1}{\sqrt{2}}\psi_0(p_\mu + i\Phi_{,\mu})$$

I'm not sure about a possible constant factor multiplying the second term in the definition of G, but it is as in Polchsinki 12.1.18. Acting on the RR ground states, ψ_0 gives an additional Γ matrix, but the Φ_μ term will modify the Bianchi and free massless equations as:

$$(\partial_\mu - \Phi_{,\mu}) \wedge F = (\partial_\mu - \Phi_{,\mu}) \wedge \star F = 0 \Rightarrow e^{\Phi}\mathrm{d}e^{-\Phi}F = e^{\Phi}\mathrm{d}\star(e^{-\Phi}F) = 0.$$

This implies that we should view $\hat{F} = e^{-\Phi}F$ as the field strength, and so the RR states correspond to $e^{\Phi}F$ (ie they already incorporate a factor of e^{Φ}). The RR charges are surface integrals of $\hat{F} = \mathrm{d}C$. Thus the dilaton coupling to the RR field strength $\hat{F}^{2m}$ is $e^{2m\Phi}e^{2(k-1)\Phi}F^{2m}$. In particular, at tree level the F^2 term does not couple to the dilaton.

35. The (minimal) supergravity multiplet contains the left-handed 3/2 gravitino as well as a right-handed self-dual 3-form field. The tensor multiplet contains a left-handed anti-self-dual 3-form field and the right-handed 1/2 dilatino. Combining *one* of the N_T tensor multiplets with the gravity multiplet gives an anomaly contribution of:

$$I_{3/2} - I_{1/2}$$

The vector multiplet contains the left-handed gaugino. The hypermultiplet (which BTW Kiritsis has not yet defined this) apparently contains a *right-handed hyperino* (wow fancy).

So far this gives

$$I_{3/2} + (N_V - N_H - 1)I_{1/2}(R)$$

But we have $N_T - 1$ addition tensor multiplets which will then contribute

$$I_{3/2} + (N_V - N_H - N_T)I_{1/2}(R) + (N_T - 1)I_A(R)$$

A quick calculation for an anti-self-dual tensor gives

$$I_{ASD} = -\left(\frac{7\mathrm{Tr}R^4}{1440} - \frac{(\mathrm{Tr}R^2)^2}{144 \times 4}\right)$$

the minus sign out front is from being *anti*-self dual.

As before, in order to have factorization of the anomaly polynomial for the GS mechanism to work, we need the $\mathrm{Tr}R^4$ terms to cancel. This gives our desired constraint

$$\frac{49}{144 \times 8} + \frac{(N_V - N_H - N_T)}{144 \times 40} - \frac{7}{144 \times 10}(N_T - 1) = 0 \Rightarrow N_H - N_V + 29N_T = 273$$

I think Kiritsis has a typo in this equation and it should be $+\mathbf{29}N_T$ rather than -29. This is consistent with **BBS exercise 5.9**.

36. Another 10D nonsupersymmetric string theory without tachyon! This one is open+closed. The $O(16)\times O(16)$ is the only closed non-SUSY string theory in 10D without tachyon. **Is this the only open one?** The relevant reference is arXiv:hep-th/9905159

This is a theory of strings stretching $D9 - D\bar{9}$ branes.

We have $\lambda, \tilde{\lambda}$ are positive chirality spinors belonging to the adjoint of $\mathrm{Sp}(n)$ (equivalent to the symmetric representation $\square\!\square$ and traceless antisymmetric representation $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of $\mathrm{Sp}(m)$ respectively, while $\psi, \bar{\psi}$ are negative chirality spinors belonging to the bi-fundamental representation of $\mathrm{SO}(n) \times \mathrm{SO}(m)$. We take $n = 0, m = 32$.

As in the $\mathrm{SO}(16) \times \mathrm{SO}(16)$ example, the lack of spacetime SUSY means there is no gravitino contribution, and we look only at the massless fermion content:

$$I_\lambda + I_{\bar{\lambda}} - 2I_\psi$$

The gravitational anomaly cancels for free since we have the same number of left and right chirality fermions. $\bar{\lambda}$ does not contribute to the gauge or mixed anomaly, since it transforms trivially under $\mathrm{USp}(32)$.

Finish

Chapter 8: D-Branes

1. First, a simple magnetic monopole for a 1-form gauge field in D spacetime dimensions has a radial magnetic field $B_r = \frac{\tilde{Q}_1}{\Omega_{D-2} r^{D-2}}$ where $\Omega_{D-2} = 2\pi^{d/2}\Gamma(d/2)$ is the volume of a unit $D-2$ sphere. This way, the flux of the solution over any $D-2$ sphere surrounding the (point) monopole will be $\tilde{Q}_1$.

   Upon taking the Hodge star we get the solution is $F = \tilde{Q}_1 \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$. We can write this as $A = \tilde{Q}_1(c - \cos\theta)\mathrm{d}\phi$. Taking $c = 1$ we get A vanishes at $\theta = 0$ (which we need since the ϕ coordinate degnerates there) while taking $c = -1$ we get A vanishes at $\theta = \pi$, which we also need.

   We cannot have *both* solutions, and so we realize we are dealing with two As, corresponding to local sections of a line bundle over S^2 on different hemispheres. Let A^+ be well-defined on all points on S^2 except $\theta = \pi$. Then A^+ is a section of a line bundle on the punctured sphere. The punctured sphere is contractible so any fiber bundle over it is trivial, so A^+ is just a *function* on the punctured sphere $S^2 \backslash \{\theta = \pi\}$. So let's define $A^+ = \tilde{Q}_1(1 - \cos\theta)\mathrm{d}\phi$. Similarly, we define A^- to be the nonsingular A on the sphere with $\theta = 0$ removed, namely $A^- = \tilde{Q}_1(-1 - \cos\theta)\mathrm{d}\phi$.

   On the overlap, $A^+ - A^-$ differ by an integer, which labels the degree of "twisting" of this line bundle over S^2.

   For a p form, our monopole will now be spatially extended in $p-1$ directions. Label these (locally), by $x^1 \dots x^{p-1}$. Time is x^0. Locally transverse to these coordinates will be $r, \varphi^1 \dots, \varphi^{D-1-p}$, where φ^i parameterize a $D-1-p$ sphere enclosing the monopole. The field strength looks like:

$$F = \tilde{Q}_p \, \Omega_{D-p-1}$$

   where Ω is the canonical $D-p-1$-sphere area form:

$$\Omega = \sin^{D-p-2}(\varphi_1) \sin^{D-p-3}(\varphi_2) \dots \sin(\varphi_{D-p-2}) \, \mathrm{d}\varphi_1 \wedge \dots \mathrm{d}\varphi_{D-p-1}$$

   This can be written (unfortunately unavoidably) in terms of a hypergeometric function:

$$A = {}_2F_1\left(\frac{1}{2}, \frac{D-p-1}{2}, \frac{D-p+1}{2}, \sin^2(\varphi_1)\right) \frac{\sin^{D-p-1}(\varphi_1)}{D-p-1} \mathrm{d}\varphi_2 \wedge \dots \wedge \mathrm{d}\varphi_{D-p-1}$$

   there is no need for an overall constant, as the function above vanishes at both $\varphi_1 = 0$ and π, *however* this is compensated by the hypergeometric function having a branch cut at $\varphi_1 = \pi/2$. Across this cut, it will have a discontinuity set by an integer depending on the convention of the arcsin function, and again we will have $A^+ - A^-$ differing by an integer. The same quantization condition follows.

   Again A^+ will be defined on the S^{D-p-1} sphere minus the south-pole (this is homeomorphic to the $D-p-1$ ball, and hence contractible, so again the line bundle trivializes and A^+ is a bona-fide function for any D, p) and A^- is similarly defined on the sphere with the excision of the north pole.

2. Our simply charged point particle with a Wilson line $A_9 = \chi/2\pi R$ turned on will have an action

$$S = \int d\tau \underbrace{\left(\frac{1}{2}\dot{x}^M \dot{x}_M - \frac{m^2}{2} + qA_9\dot{x}^9\right)}_{\mathcal{L}}$$

   The canonical momentum will be $p_i = \dot{x}^\mu$ for $\mu = 0 \dots 8$ and $p_9 = \dot{x}^9 + \frac{q\chi}{2\pi R}$. Consequently, our hamiltonian is

$$\begin{aligned} H = p_M\dot{x}^M - \mathcal{L} &= \frac{1}{2}p_\mu p^\mu + p_9(p_9 - \frac{q\chi}{2\pi R}) - \left[\frac{1}{2}(p_9 - \frac{q\chi}{2\pi R})^2 - \frac{m^2}{2} + (p_9 - \frac{q\chi}{2\pi R})\frac{q\chi}{2\pi R}\right] \\ &= \frac{1}{2}\left(p_\mu p^\mu + (p_9 - \frac{q\chi}{2\pi R})^2 + m^2\right) \\ &= \frac{1}{2}\left(p_\mu p^\mu + \left(\frac{2\pi n - q\chi}{2\pi R}\right)^2 + m^2\right) \end{aligned}$$

3. For a string satisfying Dirichlet boundary conditions, the total momentum is not conserved (along the directions associated with the D boundary conditions). This is easily interpretable as momentum transfer to the brane that it is attached to.

4. For an open string of state $|ij\rangle$, A_9 will act as $\frac{\chi_i - \chi_j}{2\pi i}$. Since this is an open string with no winding, we can only have momentum contribution, and so we will get a mass formula

$$m_{ij}^2 = \frac{\hat{N} - \frac{1}{2}}{\ell_s^2} + \left(\frac{n}{R} - \frac{\chi_i - \chi_j}{2\pi R}\right)^2$$

In particular at the lowest (massless) level for a string without momentum we will get the desired spectrum

$$m_{ij}^2 = \left(\frac{\chi_i - \chi_j}{2\pi R}\right)^2$$

5. For completeness, we will do both the gauge boson and scalar scattering. These come from the NS sector, and are given by:

$$V_{-1}^{a,\mu} = g_p \lambda^a \psi^\mu e^{-\phi} c e^{ipX}, \qquad V_0^{a,\mu} = \frac{g_p}{\sqrt{2}\ell_s} \lambda^a (i\dot{X} + 2p \cdot \psi\, \psi^\mu) c e^{ipX}$$

We can explicitly scatter four such gauge bosons - two in the -1 picture and two in the 0 picture.

$$\begin{aligned} &iC_{D^2} \delta\!\!\!/^{10}(\Sigma k) \times g_p^4 \langle cV_{-1}(y_1) cV_{-1}(y_2) cV_0(y_3) cV_0(y_4)\rangle + 5\,\text{perms.} \\ =&\frac{i\delta\!\!\!/^{p+1}}{g_p^2 \ell_s^4} \times \frac{g_p^4}{2\ell_s^2} \langle [\psi^{\mu_1} e^{ik_1 X}]_{y_1}\, [\psi^{\mu_2} e^{ik_2 X}]_{y_2}\, [(i\dot{X}^{\mu_3} + 2k_3 \cdot \psi \psi^{\mu_3}) e^{ik_3 X}]_{y_3}\, [(i\dot{X}^{\mu_4} + 2k_4 \cdot \psi\psi^{\mu_4}) e^{ik_4 3X}]_{y_4} \rangle \\ &\quad \times \langle c(y_1) c(y_2) c(y_3)\rangle \langle e^{-\phi(y_1)} e^{-\phi(y_2)}\rangle \end{aligned}$$

Take $y_1 = 0, y_2 = 1, y_3 = \infty$ and integrate $y_4 = y$ from 0 to 1 (then we'll have 5 more terms coming from permuatations). There are five contributions.

- Contracting the $\psi^{\mu_1}(0)\psi^{\mu_2}$ together and allowing the remaining 4 terms at y_3, y_4 to contract either amongst themselves or with various vertex operators.
- Not contracting the first two ψ and Contracting the $i\dot{X}(y_3)$ with any of the vertex operators while contracting the last ψ with the first two
- Not contracting the first two ψ and contracting the $i\dot{X}(y_4)$ with any of the vertex operators while contracting the third ψ with the first two
- Forgetting the $i\dot{X}$, and contracting the ψ at y_1 with the various ψ at y_3 (consequently the ψ at y_2 with the ψ at y_4)
- Swapping 3 with 4 in the above (this gives an overall minus sign by fermionic statistics)

Integrating this will give two types of terms: $\eta^{ab}\eta^{cd}$ and $\eta^{ab}k^c k^d$. Our shorthand replaces the superscript μ_i by just i. Below, I underline the terms that contribute to the first type:

$$\begin{aligned} \frac{ig_p^2 \delta\!\!\!/^{p+1}}{\ell_s^4} \int_0^1 dy \frac{y^{2\ell_s^2 k_1\cdot k_4} y^{2\ell_s^2 k_2 \cdot k_4}}{2\ell_s^2} \Bigg\{ &-\ell_s^2 \eta^{12} \left[\underline{2\ell_s^2 \eta^{34}} + (2\ell_s^2)^2 (\underline{-\eta^{34} k_3 \cdot k_4} + k_4^3 k_3^4) + (2\ell_s^2)^2 (k_2^3 + k_4^3)\Big(\frac{k_1^4}{y} + \frac{k_2^4}{y-1}\Big)\right] \\ &+ \ell_s^2 (2\ell_s^2)^2 \big[(k_2^3 + y k_4^3)(-k_4^1 \eta^{24} + k_4^2 \eta^{14})\big] \\ &+ \ell_s^2 (2\ell_s^2)^2 \Big[\Big(\frac{k_1^4}{y} + \frac{k_2^4}{y-1}\Big)(-k_3^1 \eta^{23} + k_3^2 \eta^{13})\Big] \\ &+ \ell_s^2 \frac{(2\ell_s^2)^2}{y-1} \big[\underline{\eta^{13}\eta^{24} k_3 \cdot k_4} + \eta^{34} k_3^1 k_4^2 - \eta^{13} k_4^2 k_3^4 - \eta^{24} k_4^3 k_3^1\big] \\ &- \ell_s^2 \frac{(2\ell_s^2)^2}{y} \big[\underline{\eta^{14}\eta^{23} k_3 \cdot k_4} + \eta^{34} k_3^2 k_4^1 - \eta^{13} k_4^2 k_3^4 - \eta^{24} k_4^3 k_3^1\big] \Bigg\} \end{aligned} \tag{71}$$

Using $s = -2\ell_s^2 k_1 \cdot k_2$ etc we see the underlined terms contribute

$$\frac{i g_p^2 \not\delta^{p+1}}{\ell_s^2}\left[\frac{\Gamma(1-u)\Gamma(1-t)}{\Gamma(2+s)}(-(1+s)\eta^{12}\eta^{34}) + \frac{\Gamma(1-u)\Gamma(-t)}{\Gamma(1+s)}\eta^{14}\eta^{23}s + \frac{\Gamma(-u)\Gamma(1-t)}{\Gamma(1+s)}\eta^{14}\eta^{23}s\right][1423]$$
$$= \frac{i g_p^2 \not\delta^{p+1}}{\ell_s^2}\frac{\Gamma(-u)\Gamma(-t)}{\Gamma(1+s)}(-tu\eta^{12}\eta^{34} - su\eta^{13}\eta^{24} - st\eta^{14}\eta^{23})[1423]$$

The non-underlined terms are more involved but end up contributing twelve terms that yield:

$$\frac{i g_p^2 \not\delta^{p+1}}{\ell_s^2}\left[\frac{\Gamma(-u)\Gamma(1-t)}{\Gamma(1+s)}2\ell_s^2\eta^{12}k_2^3k_1^4 + \dots\right][1423] = \frac{i g_p^2 \not\delta^{p+1}}{\ell_s^2}2\ell_s^2\frac{\Gamma(-u)\Gamma(-t)}{\Gamma(1+s)}\left[t\eta^{12}k_2^3k_1^4 + 11\,\text{perms.}\right][1423]$$

Here my Mandelstam variables are dimensionless. The result with dimensionful Mandelstam variables is:

$$i g_p^2 \not\delta^{p+1}\ell_s^2\left(\frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 u)}{\Gamma(1+\ell_s^2 t)}K_4(k_i, e_i)([1234]+[4321]) + 2\,\text{perms.}\right) \tag{72}$$

with $K_4(k_i, e_i) = -tu e_1\cdot e_2 e_3\cdot e_4 + 2s(e_1\cdot e_3\, e_2\cdot k_4 e_4\cdot k_2 + 3\,\text{perms.}) + 2\,\text{perms.}$

Now for the four transverse scalar amplitude, our vertex operators look like:

$$V_{-1}^{a,\mu} = g_p\lambda^a\psi^\mu e^{-\phi}ce^{ipX}, \qquad V_0^{a,\mu} = \frac{g_p}{\sqrt{2}\ell_s}\lambda^a(X' + 2p\cdot\psi\,\psi^\mu)ce^{ipX}$$

here $X' = \partial_\sigma X = 2i\partial X$. Moreover, we have Dirichlet boundary conditions on both the Xs and the fermions ψ. The ψ are still in the NS sector since we're looking at the (bosonic) scalar field scattering.

Crucially, the correlators for the ψ field are the same for DD boundary conditions. We had $\langle \dot{X}^\mu(z)\dot{X}^\nu(w)\rangle = -\frac{\ell_s^2}{2}\eta^{\mu\nu}(z-w)^{-2}$ while the correlators for X' pick up a minus sign $\langle X'^i(z)X'^j(w)\rangle = \frac{\ell_s^2}{2}\eta^{ij}(z-w)^{-2}$. This ends up giving the exact same result however, since the vertex operators contain $X'(z)$ while the prior ones contain $i\dot{X}(z)$.

Finally, contracting X'^i with any of the e^{ikX} will give zero, since the open strings only have momenta parallel to the Dp brane while the X'^i is transverse. This gives a simpler amplitude than (72):

$$i g_p^2 \not\delta^{p+1}\ell_s^2\, K_4'\left(\frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 u)}{\Gamma(1+\ell_s^2 t)}([1234]+[4321]) + 2\,\text{perms.}\right) \tag{73}$$

with $K_4' = -(tu\delta_{12}\delta_{34} + su\delta_{13}\delta_{24} + st\delta_{14}\delta_{23})$. In the case where there are no CP indices we expand the $\Gamma\Gamma/\Gamma$ functions:

$$i g_p^2 \not\delta^{p+1}\ell_s^2 K_4' \times \left(\frac{2}{\ell_s^4 su} + \frac{2}{\ell_s^4 st} + \frac{2}{\ell_s^4 tu}\right) = i g_p^2 \not\delta^{p+1}\ell_s^2 K_4' \times \left(\frac{2(s+t+u)}{\ell_s^4 stu}\right) = 0$$

So to leading order in the string length this is zero. This is consistent with the $U(1)$ DBI action, as the scalars do not directly interact with the $U(1)$ gauge field A_μ (in general a real scalar cannot be charged under a $U(1)$ gauge field). That is, at leading order the action is free in the X fields. Taking $\xi^\mu = X^\mu$ for $\mu = 0\dots p$ and X^i independent functions, we get:

$$\int d^{p+1}\xi\sqrt{\det G_{MN}\partial_\alpha X^M\partial_\beta X^N} = \int d^{p+1}\xi\sqrt{\det \delta_{\alpha\beta} + \delta_{ij}\partial_\alpha X^i\partial_\beta X^j} \to \int d^{p+1}\xi\,\frac{\delta^{\alpha\beta}\delta_{ij}}{2}\partial_\alpha X^i\partial_\beta X^j$$

This is just a free theory. Its also quick to see that the 3-point function of the transverse scalars vanishes at tree level in string perturbation theory.

6. I have done the previous problem in full generality, including CP indices. So now let's again look at the s channel. As $s \to 0$ so that $t = -u$ we get from the $\delta_{12}\delta_{34}$ term a pole in s going as:

$$-i g_p^2 \not\delta^{p+1}\ell_s^2 tu\times\left(\frac{1}{\ell_s^4 su}([1234]+[4321]) + \frac{1}{\ell_s^4 st}([1243]+[3421])\right) = -i\not\delta^{p+1}\frac{g_p^2}{\ell_s^2}\frac{t}{s}([1234]+[4321]-[1243]-[3421])$$

We can rewrite this as:

$$-i\not{\delta}^{p+1}\frac{g_p^2}{\ell_s^2}\frac{t}{s}(\mathrm{Tr}(12[34]) - \mathrm{Tr}([34]21) = -i\not{\delta}^{p+1}\frac{g_p^2}{\ell_s^2}\frac{t}{s}\mathrm{Tr}([12][34])$$

The pole at $s = 0$ corresponds to an exchange of a gluon from the $\frac{1}{2}(D_\mu X^I)^2$ term in **8.6.1**.

We also have a further term that does not involve a pole in s. Let's still take 1 and 2 equal. Expanding to this order we find:

$$-i\not{\delta}^{p+1}\frac{g_p^2}{\ell_s^2}\Big(\mathrm{Tr}([12][34]) + \mathrm{Tr}([13][24]) + \mathrm{Tr}([14][23])\Big)$$

This comes from exactly the potential term $\frac{1}{4}[X^I, X^J]^2$ in the effective action **8.6.1**. **Come back to that last term**

7. Momentum conservation will imply $p_\parallel = 0$ for the NSNS states. Our vertex operator will take the form $\zeta_{\mu\nu}c\tilde{c}e^{-\phi}e^{-\tilde{\phi}}\psi^\mu\tilde{\psi}^\nu e^{ik_\perp X}$. We can use the doubling trick to get $\zeta_{\mu\nu}c(z)c(z^*)e^{-\phi(z)}e^{-\phi(z^*)}\psi^\mu\tilde{\psi}^\nu e^{ik_\perp X}$ and we are automatically in the -2 picture.

   The states from in the $p+1$ parallel directions give just the correlator $\langle\psi^\mu(i)\psi^\nu(-i)\rangle = -\frac{\eta^{\mu\nu}}{2i}$ (importantly NN fermions in NSNS have 2-point function $-1/(z-\bar{w})$ c.f. **4.16.22**). We also get a $\not{\delta}^{p+1}$ from momentum conservation.

   The states in the transverse (Dirichlet) directions give $\frac{\delta^{ij}}{2i}$ correlator. Defining the diagonal matrix $D^{\mu\nu} = (\eta^{\alpha\beta}, \delta^{ij})$ we get a correlator proportional to

$$-\frac{g_c}{2\ell_s^2 g_p^2}\not{\delta}^{p+1}(k_\parallel)D^{\mu\nu} = -\frac{(2\pi\ell_s)^2 T_p}{2}V_{p+1}D^{\mu\nu}$$

   Check with Victor. Confirm the tension relation. This diagonal tensor $D^{\mu\nu}$ allows for a nonvanishing dilaton and graviton tadpole, but will not couple to the antisymmetric Kalb-Ramond B-field.

8. Our RR fields have picture (r, s) for r, s half-integers. In order to have total picture -2 on the disk, we need to pick this to be the (asymmetric) $(-3/2, -1/2)$ picture. The construction of this operator is complicated. I expect that the $(-1/2, -1/2)$ operator that we are familiar with is basically $e^{-\phi}G_0$ times the $(-3/2, -1/2)$ operator. This means that the $(-3/2, -1/2)$ will be one less power of momentum and one less gamma matrix than the $(-1/2, -1/2)$ operator. Picking out the $p+2$ form part of the $(-1/2, -1/2)$ operator (in Blumenhagen's convention **16.21**) gives

$$\frac{1}{4}\frac{\ell_s}{\sqrt{2}}F^{\alpha\beta}\overline{S}_\alpha(z)\tilde{S}_\beta(\bar{z})e^{-\phi/2-\bar{\phi}/2} \to \frac{1}{4}\frac{\ell_s}{\sqrt{2}}\frac{F_{\mu_1\ldots\mu_{p+2}}}{(p+2)!}(\Gamma^{\mu_1\ldots\mu_{p+2}})^{\alpha\beta}\overline{S}_\alpha\tilde{S}_\beta e^{-\phi/2-\bar{\phi}/2}$$

   Here $\overline{S}_\alpha = S_\alpha^\dagger\Gamma^0$ as is standard in a spinor product. BRST will require that F and $\star F$ be closed.

   Changing picture means removing one power of momentum and one gamma matrix. This anti-differentiates F, which must be proportional to the potential C since $p_{[\mu_1}C_{\mu_2\ldots\mu_{p+2}]} = F_{\mu_1\ldots\mu_p+2}$. It is then reasonable to expect the corresponding $(-3/2, -1/2)$ operator to be proportional to

$$e^{-3\phi/2-\bar{\phi}/2}\frac{C_{\mu_1\ldots\mu_p}}{(p+1)!}(\Gamma^{\mu_0\ldots\mu_p})^{\alpha\beta}\overline{S}_\alpha\tilde{S}_\beta$$

   Note both $e^{-3\phi/3}S_\alpha$ and $e^{-\tilde{\phi}/2}S_\beta$ *remain* primary operators, having dimensions $3/8 + 5/8$, so this is indeed a reasonable guess. Now, for the D-brane boundary conditions we have

$$(\delta^\perp\tilde{S})_\beta(\bar{z}) = S_\beta(z^*), \quad \delta^\perp = \prod_{i=p+1}^{9}\delta^i, \quad \delta^i = \Gamma^i\Gamma_{11}$$

   This reflects the S_β spinor along all the Dirichlet directions and keeps it the same across the Neumann directions.

From **5.12.42** of Kiritsis I expect the leading-order of the $S_\alpha \tilde{S}_\beta$ correlator to be $C_{\alpha\beta}/(z-\bar{z})^{10/8}$ and the $e^{-3\phi/2}\tilde{e}^{-\phi/2}$ will contribute $(z-\bar{z})^{-3/4}$ to make this a primary correlator transforming as $C_{\alpha\beta}(z-\bar{z})^{-2}$. For Neumann boundary conditions, $C_{\alpha\beta}$ is the charge conjugation matrix. For the D-brane, it will be $\delta^\perp C$.

Only in the IIB case will $C_{\alpha\beta}$ will be nonzero between $\overline{S}_\alpha$ and $\tilde{S}_\beta$ since S_α and $\tilde{S}_\beta$ transform in the same representations. Each β^i changes the chirality. So the amplitude in IIB will vanish if we have an even number of β^i, equivalently $9-p$ is odd, so we will have only odd dimensional branes in IIB as required and even dimensional branes in IIA as required.

We thus get an amplitude proportional to:

$$\mathcal{A} = i\frac{g_c \not{\delta}^{p+1}}{g_p^2 \ell_s^2}\frac{C_{\mu_0\dots\mu_p}}{(p+1)!}\mathrm{Tr}(\Gamma^{\mu_1\dots\mu_p}\Gamma^{p+1}\dots\Gamma^9\Gamma^{11}) = i\frac{g_c\not{\delta}^{p+1}}{g_o^2\ell_s^2}\frac{C_{\mu_0\dots\mu_p}\epsilon^{\mu_0\dots\mu_p}_{(p+1)}}{(p+1)!}$$

Comparing with the **8.4.4**, which should factorize as $\mathcal{A}(p_\parallel)^2 G_{9-p}(p_\perp)\delta^{p+1}(p_\parallel)$ we see that the normalization of our on-shell amplitude is in fact:

$$\mathcal{A} = iV_{p+1}\sqrt{2\pi}(2\pi\ell_s)^{3-p}\frac{C_{\mu_0\dots\mu_p}\epsilon^{\mu_0\dots\mu_p}_{(p+1)}}{(p+1)!}$$

This is consistent with other results c.f. Di Veccia, Liccardo *Gauge Theories from D-Branes*, arXiv:0307104 but I think they're not incorporating $1/\alpha_p = 2\kappa_{10}^2$ in the propagator. Taking this factor into account and dividing by it followed by taking a square root gives us an on-shell amplitude of:

$$\mathcal{A} = iV_{p+1}\frac{1}{(2\pi\ell_s)^p\ell_s g_s}\frac{C_{\mu_0\dots\mu_p}\epsilon^{\mu_0\dots\mu_p}_{(p+1)}}{(p+1)!} = iV_{p+1}T_p\, C_{p+1}\wedge\epsilon_{(p+1)}.$$

This is exactly what would come from a minimal coupling term of the form $iT_p\int C_{p+1}$.

9. We take one vertex operator to be in the $(-1,-1)$ picture and gauge fix it to lie at $z=i$, and take the other in the $(0,0)$ picture lying at iy, fixing y to range from 0 to 1. In doing the doubling trick, we take $\tilde{X}^\mu = D^\mu_\nu X^\nu, \tilde{\psi}^\mu = D^\mu_\nu\psi^\nu$ and $\tilde{\phi}=\phi, \tilde{c}=c$.

   We wish to calculate the correlator:

$$-\frac{g_c^2}{g_o^2\ell_s^4}\frac{2}{\ell_s^2}\langle[\psi^\mu\tilde{\psi}^{\bar{\mu}}e^{ik_1X}]_0\,[(i\partial X^\mu+\frac{1}{2}k_2\cdot\psi\psi^\nu)(i\bar{\partial}X^{\bar{\nu}}+\frac{1}{2}k_2\cdot\bar{\psi}\bar{\psi}^{\bar{\nu}})e^{ik_2X}]_y\rangle$$

   The simplest way to do this problem is to recognize that after the doubling trick has been applied, we are calculating the a correlator of four fields *at collinear insertions on* the Riemann sphere.

$$\varepsilon^1_{\mu\bar{\mu}}\varepsilon^2_{\nu\bar{n}u}D^{\bar{\mu}}_{\mu'}D^{\bar{\nu}}_{\nu'}\langle V^\mu_{-1}(p_1), V^{\mu'}_{-1}(Dp_1), V^\nu_0(p_2), V^{\nu'}_0(Dp_2)\rangle$$

   We can map these four points to $0,1,y,\infty$ and take the integral to be from $y=0$ to $y=1$, with appropriate jacobian.

   This is nothing more than the 4-gluon amplitude calculated previously, which gave

$$2i\frac{g_o^2}{g_p^2}\not{\delta}^{p+1}\ell_s^2\frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 u)\Gamma(-\ell_s^2 s)}{\Gamma(1+2\ell_s^2 t)}\to K_4(k_i,e_i)$$

   Here we do not have three terms for y in different regions - we only have one. Our momenta are p_1,p_2,Dp_1,Dp_2 giving: The only caveat is that in that case, we had boundary normal ordering. We can view this as $\ell_s^{here} = \ell_s^{there}/2$. This is reflected by halving the momenta of the open strings.

$$\begin{aligned} p_1^{there} &\to p_1^{here}/2, & p_2^{there} &\to p_2^{here}/2 \\ p_3^{there} &\to D\cdot p_1^{here}/2, & p_4^{there} &\to D\cdot p_2^{here}/2 \end{aligned}$$

This gives us (here we do not have three terms for y in different regions - we only have one)

$$2i\frac{g_o^2}{g_p^2}\delta^{p+1}\ell_s^2\frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 u)}{\Gamma(1+2\ell_s^2 t)} \to K_4(k_i,e_i) \to 2i\frac{g_o^2}{g_p^2}\delta^{p+1}\ell_s^2\frac{\Gamma(\ell_s^2 p_1\cdot p_2/2)\Gamma(\ell_s^2 p_1\cdot Dp_1/2)}{\Gamma(1+\ell_s^2 p_1\cdot p_2/2+\ell_s^2 p_1\cdot Dp_1/2)}$$

Following Myers, we can write $t=-2p_1\cdot p_2$ as the momentum transfer to the brane, and $q^2=p_1\cdot Dp_1/2$ as the momentum flowing parallel to the brane. The Gamma ratio then looks like:

$$\frac{\Gamma(-\ell_s^2 t/4)\Gamma(\ell_s^2 q^2)}{\Gamma(1+\ell_s^2 t/4+\ell_s^2 q^2)}$$

If we held t fixed and took the large q limit we would get a series of open string poles. This corresponds to a closed string splitting in two, with its ends on the D-brane as an intermediate state.

Now let's hold q^2 fixed and take the large t limit. This is the limit of large energy transfer - the Regge limit. From the ratio of $\Gamma\Gamma/\Gamma$ we see that there are closed string poles. This can be interpreted as the closed strings interacting with the long-range background fields generated by the presence of the Dp brane.

Comment about pole structure

10. In the D9 brane case, we have seen that the open string boundary only preserves the sum of $Q+\tilde{Q}$. If we T-dualize in the 9 direction, we act on the right-moving sector by spacetime parity, so that necessarily $\bar\partial X^9 \to -\bar\partial X^9, \tilde\psi^9 \to -\tilde\psi^9, \tilde{S}_\alpha \to \delta^9\tilde{S}^\alpha$ (up to a phase in that last one). Here $\delta^9=\Gamma^9\Gamma^{11}$. Our spacetime supersymmetry generator $\tilde{Q}_\alpha=\frac{1}{2\pi i}\int d\bar{z}\,e^{-\phi/2}S_\alpha$ therefore will be mapped to $\delta^9\tilde{Q}$. Thus, in the T-dual picture we preserve the supercharge $Q'+\delta^9\tilde{Q}'$.

    Iterating this procedure in other directions we get that in general we preserve $Q+\delta^\perp\tilde{Q}$, with $\delta^\perp=\prod_i\delta^i$, where i runs perpendicular to the brane. Note that T-dualities along different directions do not commute! They commute up to a $(-1)^{\mathbf{F}_R}$, and so the order that we do them matters. In this case the δ^i act by left-action.

11. (As in Polchinski section 13.4) From the previous problem, we see that the first D-brane preserves the supercharges $Q+\delta^\perp\tilde{Q}$ while the second preserves the supercharges $Q+\delta^{\perp\prime}\tilde{Q}=Q+\delta^\perp({\delta^\perp}^{-1}\delta^{\perp\prime}\tilde{Q})$ so the supersymmetries that will be preserved must be of both forms. This is in one-to-one correspondence with spinors invariant under ${\delta^\perp}^{-1}\delta^{\perp\prime}$. This operator is a reflection in the direction of the ND boundary conditions (the directions orthogonal to the $\mathrm{D}_{p'}$ brane in the D_p brane). Since in either IIA or IIB p and p' must differ by an even integer, the number of mixed boundary conditions–call it ν–must be even. Then we can write ${\delta^\perp}^{-1}\delta^{\perp\prime}$ as a product of rotations by π along each of the $\nu/2$ planes ${\delta^\perp}^{-1}\delta^{\perp\prime}=e^{i\pi(J_1+\cdots+J_{\nu/2})}$. Each j acts in a spinor representation, so that $e^{i\pi J_i}$ has eigenvalues $\pm i$. If $\nu/2$ is odd, this makes ${\delta^\perp}^{-1}\delta^{\perp\prime}=-1$ so this will *not* preserve supersymmetry. We thus need $\nu/2$ even, or $\nu=0$ mod 4.

    From this I posit that the static force between two branes vanishes precisely when $\nu=0$ mod 4.

12. Now let's confirm this guess with an amplitude calculation. Take $p'\leqslant p$. We work in lightcone gauge. We do a trace over an open string with p' NN boundary conditions, $p-p'=\nu$ DN boundary conditions, and $8-p$ DD boundary conditions.

    We begin from the open string point of view in calculating the cylinder amplitude. Chapter 4 has done the NN, DD, and DN boson amplitudes for us. The difficulty lies almost entirely in the fermions. Recall the following:

$$\eta=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

$$\sqrt{\frac{\theta[^0_0]}{\eta}}=q^{-1/48}\prod_{n=0}^{\infty}(1+q^{n+1/2}) \qquad \sqrt{\frac{\theta[^1_0]}{\eta}}=\sqrt{2}\,q^{1/24}\prod_{n=0}^{\infty}(1+q^n) \qquad \sqrt{\frac{\theta[^0_1]}{\eta}}=q^{-1/48}\prod_{n=0}^{\infty}(1-q^{n+1/2})$$

Further from **4.16.2** recall that for modes $b_{n+1/2}$, b_n corresponding to NS and R sectors the NN and DD boundary conditions give:

- NN: $\bar{b}_{n+1/2} = -b_{n+1/2}, \bar{b}_n = b_n$
- DD: $\bar{b}_{n+1/2} = b_{n+1/2}, \bar{b}_n = -b_n$

For DN we have the same result as for DD but now the R sector is half-integrally modded and the NS sector in integrally modded. Now lets compute partition functions. Our final answer will be a sum over spin structures NS+, NS-, R+, R-. Taking $q = e^{-2\pi t}$ we see $\mathrm{Tr}[q^{L_0 - c/24}] =$

- NS+:
  - NN: $q^{-1/48} \prod_n (1 + q^{n+1/2}) = \sqrt{\theta[{}^0_0]/\eta}$
  - DD: $q^{-1/48} \prod_n (1 + q^{n+1/2}) = \sqrt{\theta[{}^0_0]/\eta}$
  - DN: $\sqrt{2} q^{-1/48} q^{1/16} \prod_n (1 + q^n) = \sqrt{\theta[{}^1_0]/\eta}$ ($\sqrt{2}$ when raised to a power counts ground state degeneracy)
- NS-:
  - NN: $q^{-1/48} \prod_n (1 - q^{n+1/2}) = \sqrt{\theta[{}^0_1]/\eta}$
  - DD: $q^{-1/48} \prod_n (1 - q^{n+1/2}) = \sqrt{\theta[{}^0_1]/\eta}$
  - DN: 0
- R+
  - NN: $\sqrt{2} q^{1/24} \prod_n (1 + q^n) = \sqrt{\theta[{}^1_0]/\eta}$
  - DD: $\sqrt{2} q^{1/24} \prod_n (1 + q^n) = \sqrt{\theta[{}^1_0]/\eta}$
  - DN: $q^{-1/48} \prod_n (1 + q^{n+1/2}) = \sqrt{\theta[{}^0_0]/\eta}$
- R-
  - NN: 0
  - DD: 0
  - DN: $q^{-1/48} \prod_n (1 - q^{n+1/2}) = \sqrt{\theta[{}^0_1]/\eta}$

Notice NN vs DD boundary conditions have *no effect* on fermion contribution to partition function. This is because, although the left moving and right-moving modes are identified differently, the mode excitations look exactly the same.

On the other hand for NN and DD the bosons will contribute $1/\eta$ and will contribute $\sqrt{\eta/\theta[{}^0_1]}$ for DN. Thus we have the following contributions to the partition function (here N is the number of NN boundary conditions):

$$NS+ = \frac{V_N}{(2\pi\ell_s)^N} \int \frac{dt}{2t} \frac{e^{-2\pi t \left(\frac{\Delta x}{2\pi \ell_s}\right)^2}}{(\sqrt{2t})^N \eta^{8-\nu} \left(\theta[{}^0_1]/\eta\right)^{\nu/2}} \left(\frac{\theta[{}^0_0]}{\eta}\right)^{(8-\nu)/2} \left(\frac{\theta[{}^1_0]}{\eta}\right)^{\nu/2}$$

$$NS- = -\frac{V_N}{(2\pi\ell_s)^N} \int \frac{dt}{2t} \frac{e^{-2\pi t \left(\frac{\Delta x}{2\pi \ell_s}\right)^2}}{(\sqrt{2t})^N \eta^8} \left(\frac{\theta[{}^0_1]}{\eta}\right)^8 \delta_{\nu=0}$$

$$R+ = -\frac{V_N}{(2\pi\ell_s)^N} \int \frac{dt}{2t} \frac{e^{-2\pi t \left(\frac{\Delta x}{2\pi \ell_s}\right)^2}}{(\sqrt{2t})^N \eta^{8-\nu} \left(\theta[{}^0_1]/\eta\right)^{\nu/2}} \left(\frac{\theta[{}^1_0]}{\eta}\right)^{(8-\nu)/2} \left(\frac{\theta[{}^0_0]}{\eta}\right)^{\nu/2}$$

$$R- = \frac{V_N}{(2\pi\ell_s)^N} \int \frac{dt}{2t} \frac{e^{-2\pi t \left(\frac{\Delta x}{2\pi \ell_s}\right)^2}}{(\sqrt{2t})^N} \delta_{\nu=8}$$

All theta and eta functions are evaluated at it. The circumference of the cylinder is $2\pi t$. The relative signs in front of the different contributions come from a combination of defining the NS vacuum to have negative fermion and modular invariance (equivalently spacetime spin-statistics). Note when $\nu = 4$ we only get contributions from NS+ and R+, which exactly cancel. Similarly when $\nu = 4$ or 8, by the abtruse identity of Jacobi we will get cancelation again.

We can interpret our result as a one-loop free energy. Differentiating this w.r.t. Δx would then give us our force. For $\nu = 0, 4, 8$ we do not get a force, consistent with the D-brane configuration preserving supersymmetry.

For the sake of completeness, and to clear my own confusion once and for all, I will also do this from the POV of the boundary state formalism (not developed in Kiritsis). For a good reference see the last chapter of Blumenhagen's text on conformal field theory.

For a single free boson, after the flip $(\sigma, \tau)_{open} \to (\tau, \sigma)_{closed}$ the boundary states $|N\rangle, |D\rangle$ must satisfy

$$(\alpha_n + \tilde{\alpha}_{-n})\,|N\rangle = 0, \qquad (\alpha_n - \tilde{\alpha}_{-n})\,|D_x\rangle = 0,$$

This gives boundary states:

$$|N\rangle = \frac{1}{(2\pi\ell_s\sqrt{2})^{1/2}} \prod_n e^{-\frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}}\,|0,0;0\rangle = \sum_{\vec{m}=\{m_i\}} |\vec{m}, \Theta\vec{m}; 0\rangle$$
$$|D_x\rangle = (2\pi\ell_s/\sqrt{2})^{1/2} \int \frac{dk}{2\pi} e^{ipx} \prod_n e^{-\frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}}\,|0,0;k\rangle$$

The overall normalization came from comparing with cylinder amplitudes. Θ here is CPT reversal. Similarly for a fermion

$$(\psi_n + \tilde{\psi}_{-n})\,|N\rangle = 0, \qquad (\psi_n - \tilde{\psi}_{-n})\,|D_x\rangle = 0,$$

So with GSO projection we get:

$$|N, \mathrm{NSNS}\rangle = P_L P_R \prod_r e^{\psi_{-r}\tilde{\psi}_{-r}}\,|0\rangle, \qquad |N, \mathrm{RR}\rangle = P_L P_R \prod_n e^{\psi_{-n}\tilde{\psi}_{-n}}\,|0\rangle$$
$$|D, \mathrm{NSNS}\rangle = P_L P_R \prod_r e^{-\psi_{-r}\tilde{\psi}_{-r}}\,|0\rangle, \qquad |D, \mathrm{RR}\rangle = P_L P_R \prod_n e^{-\psi_{-n}\tilde{\psi}_{-n}}\,|0\rangle$$

Here r runs over half-integers in the NSNS sector and n runs over integers in the RR sector. $P_L = \frac{1}{2}(1 + (-1)^F), P_R = \frac{1}{2}(1 + (-1)^{\tilde{F}})$ are our GSO projections, defined to project out the tachyon in the NS sector and project out one of the spinors in the R sector.

For the boson, it is quick to see that ($\ell = 1/t$)

$$\langle N|\, e^{-\pi\ell(L_0 + \tilde{L}_0 - c/12)}\,|N\rangle = \frac{V}{2\pi\ell_s\sqrt{2}\eta(i\ell)} = \frac{V}{(2\pi\ell_s)\sqrt{2t}\eta(it)}$$
$$\langle D|\, e^{-\pi\ell(L_0 + \tilde{L}_0 - c/12)}\,|D\rangle = \frac{2\pi\ell_s}{\sqrt{2t}\eta(it)} \int \frac{dk}{2\pi} e^{ik\Delta x} e^{-\pi\ell_s^2 p^2/2t} = \frac{e^{-2\pi t\left(\frac{\Delta x}{2\pi\ell_s}\right)^2}}{\eta(it)}$$
$$\langle D|\, e^{-\pi\ell(L_0 + \tilde{L}_0 - c/12)}\,|N\rangle = \frac{1}{\sqrt{2}} \frac{1}{\prod_n (1 + q^{2n})} = \sqrt{\frac{\eta(i\ell)}{\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](i\ell)}} = \sqrt{\frac{\eta(it)}{\theta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](it)}}$$

These are exactly what we've already gotten many times before from our trace over the open string bosonic states. The states $|N\rangle, |D\rangle$ must be a sum of both the RR and NSNS sector fermion states. We do not know the relative coefficients.

Let's look at the NSNS contributions. For the NN boundary conditions, the NSNS sector with projection consists of two terms:

$$\langle N, \mathrm{NSNS}_{unproj}|\, e^{-\pi\ell(L_0 + \tilde{L}_0 - c/12)}\,|N, \mathrm{NSNS}_{unproj}\rangle = \left(\frac{\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](i\ell)}{\eta(i\ell)}\right)^{\#NN/2} = \left(\frac{\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](it)}{\eta(it)}\right)^{\#NN/2}$$
$$\langle N, \mathrm{NSNS}_{unproj}|\, (-1)^{F_L = F_R} e^{-\pi\ell(L_0 + \tilde{L}_0 - c/12)}\,|N, \mathrm{NSNS}_{unproj}\rangle = \left(\frac{\theta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](i\ell)}{\eta(i\ell)}\right)^{\#NN/2} = \left(\frac{\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](it)}{\eta(it)}\right)^{\#NN/2}$$

Replacing N with D would give the *exact same* factor in both cases **WHY?** (explain: bc we need to match on both sides and so both minuses cancel in the exponent). For DN boundary conditions the NSNS sector give the two terms:

$$\langle D, \text{NSNS}_{unproj}| \, e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} \, |N, \text{NSNS}_{unproj}\rangle = \left(\frac{\theta[^0_1](i\ell)}{\eta(i\ell)}\right)^{\nu/2} = \left(\frac{\theta[^1_0](it)}{\eta(it)}\right)^{\nu/2}$$

$$\langle D, \text{NSNS}_{unproj}| \, (-1)^{F_L=F_R} e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} \, |N, \text{NSNS}_{unproj}\rangle = \left(\frac{\theta[^0_0](i\ell)}{\eta(i\ell)}\right)^{\nu/2} = \left(\frac{\theta[^0_0](it)}{\eta(it)}\right)^{\nu/2}$$

Now let's look at the RR sector. For NN boundary conditions, it contributes:

$$\langle N, \text{RR}_{unproj}| \, e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} \, |N, \text{RR}_{unproj}\rangle = \left(\frac{\theta[^1_0](i\ell)}{\eta(i\ell)}\right)^{\#NN/2} = \left(\frac{\theta[^0_1](it)}{\eta(it)}\right)^{\#NN/2}$$

$$\langle N, \text{RR}_{unproj}| \, (-1)^{F_L=F_R} e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} \, |N, \text{RR}_{unproj}\rangle = 0$$

By the argument before, we get the same for DD boundary conditions. Finally, with DN boundary conditions we get

$$\langle D, \text{RR}_{unproj}| \, e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} \, |N, \text{RR}_{unproj}\rangle = 0$$

$$\langle D, \text{RR}_{unproj}| \, (-1)^{F_L=F_R} e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} \, |N, \text{RR}_{unproj}\rangle = \left(\frac{\theta[^1_0](i\ell)}{\eta(i\ell)}\right)^{\nu/2} = \left(\frac{\theta[^0_1](it)}{\eta(it)}\right)^{\nu/2}$$

Together this is exactly consistent with what we get from tracing over the open string. We can work back to get relative normalizations.

This shows that the massless RR and NSNS fields mediate the force. Moreover the NSNS fields without and with projection correspond respectively to the unprojected NS and R open string states while the RR fields without and with projection correspond to the *projected* NS and R open string states.

13. First recall that for a constant vector potential $A_9 = \frac{\chi_9}{2\pi R}$ corresponds to a T-dual picture of a D-brane at position $-\chi\tilde{R} = -2\pi\ell_s^2 A_9$. Now consider a magnetic flux F_{12} we can write a (nonconstant now) vector potential that gives this flux as $A_2 = F_{12}X^1$. We T-dualize along X^2 and get $X^2 = -2\pi\ell_s^2 F_{12}X^1$. Then $\tan\theta = -2\pi\ell_s^2 F_{12}$.

Although we were working with D1 and D2 branes, we could have done the exact same calculation for F_{01} on a D1 brane and recovered a D0 brane tilted in the $X^0 - X^1$ plane (ie boosted). Such a D0 brane has the usual point-particle action:

$$S_{D0} = -T_0 \int dX^0 \sqrt{1 + (\partial_0 X'^1)^2}$$

Because the D0 brane and the D1 brane describe the same physics, this action should be identical to the D1 action. Note that $\partial_0 X'^1$ is infinitesimally exactly $\tan\theta$ calculated above. We get the action

$$S_{D1} = -T_1 \int dX^1 dX^2 \sqrt{1 + (2\pi\ell_s^2 F_{12})^2}$$

Of course, because the branes couple to strings, the only gauge invariant combination under transformations of the Kalb-Ramond B field is $\mathcal{F} = B + 2\pi\ell_s^2 F$. We thus get:

$$S_{D1} = -T_1 \int dX^1 dX^2 \sqrt{-\det(G + \mathcal{F})}$$

We can tilt this brane and T-dualize to pick up EM field strengths in arbitrary dimension up to 9.

14. Let's T-dualize. This describes two D4 branes that are tilted only along the x_1-x_5 plane, and are otherwise parallel in the x_2, x_3, x_4 directions. T-dualizing x_2, x_3, x_4 makes these into D1 branes tilted in the $x_1 - x_5$ plane. See the solution (74) to the next problem. Now setting $\nu_{2,3,4} = 0$ will give poles from the theta function in the denominator. This is to be expected, from the NN boundary conditions that always come with a volume divergence factor in that direction. We regulate this divergence by replacing:

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(i\nu t, it)^{-1} \to i\frac{L}{\eta(it)^{-3}2\pi\ell_s\sqrt{2t}}$$

Thus we get a potential:

$$-i\frac{L^3}{(2\pi\ell_s)^4}\int_0^\infty \frac{dt}{t}\frac{e^{-2\pi t\left(\frac{\Delta x}{2\pi\ell_s}\right)^2}}{(2t)^{4/2}\eta(it)^9}\frac{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](i\nu t/2, it)^4}{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](i\nu t, it)}$$

Let's take the distance to be large. The small t contributions are then most important. The θ-function ratio contributes a factor of $e^{-3\pi/4t}t^{-3/2}$ while the η^{-9} contributes $e^{3\pi/4t}t^{9/2}$ **Finish the details here**

We get a potential that decays as $-\frac{L^3\ell_s}{(\Delta x)^5}\frac{\sin^4(\theta/2)}{\sin(\theta)}$ giving another attractive force going as $1/(\Delta x)^6$.

15. Following Polchinski, we define variables $Z^i = X^i + iX^{i+4}, i = 1, \ldots, 4$. Let the $\sigma = 0$ endpoint be on the untilted string. Then at $\sigma = 0$ we have $\partial_1 \Re Z^a = \Im Z^a = 0$ and at $\sigma = \pi$ on the tilted string we have $\partial_1\Re(e^{i\theta_a}Z^a) = \Im(e^{i\theta_a}Z^a)$.

We see that the field that satisfies this is given by $Z^a(w,\bar w) = \mathcal{Z}^a(w) + \bar{\mathcal{Z}}^a(-\bar w)$ for $w = \sigma^1 + i\sigma^2$. Using the doubling trick we have $\mathcal{Z}^a(w+2\pi) = e^{2i\theta_a}\mathcal{Z}^a(w)$ (and similarly for the conjugate). This gives a mode expansion with $\nu_a = \theta_a/\pi$

$$\mathcal{Z}^a(w) = i\frac{\ell_s}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\nu_a}\frac{\alpha_r^a}{r}e^{irw}.$$

The $a^\dagger$ modes are then linearly independent. Taking $q = e^{-2\pi t}$ as usual for open string partition functions, and restricting $0 < \phi_a < \pi$ we get:

$$\frac{q^{\frac{1}{24}-\frac{1}{2}(\nu_a-\frac{1}{2})^2}}{\prod_{m=0}^\infty(1-q^{m+\nu_a})(1-q^{m+1-\nu_a})} = -i\frac{q^{-\nu_a^2/2}\eta(it)}{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](i\nu_a t, it)}$$

Where we have used

$$\theta\begin{bmatrix}1\\1\end{bmatrix}(i\nu t, it) = iq^{\frac{1}{8}}(q^{\nu/2}-q^{-\nu/2})\prod_{m=1}^\infty(1-q^m)(1-q^{m+\nu})(1-q^{m-\nu}) = i\eta(it)q^{\frac{1}{8}-\frac{1}{24}+\frac{\nu}{2}}\prod_{m=0}^\infty(1-q^{m+\nu})(1-q^{m+1-\nu})$$

So the angles act like chemical potentials to make the theta functions nonzero. Now its time for the fermions (oh boy!). In each NS and R sector (projected and unprojected) the boundary conditions shift by ν_a. We thus get e.g. for NS unprojected:

$$Z\begin{bmatrix}0\\0\end{bmatrix} = q^{-\frac{1}{24}+\nu_a^2/2}\prod_{m=1}^\infty(1-q^{m+1/2+\nu_a})(1-q^{m+1/2-\nu_a}) = q^{\nu_a^2/2}\frac{\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](i\nu_a t, it)}{\eta(it)}$$

In total, then we will see that the fermion part gives us

$$\frac{\prod_a q^{\nu_a^2/2}}{2\eta(it)^4}\left[\prod_{a=1}^4\theta\begin{bmatrix}0\\0\end{bmatrix}(i\nu_a t, it) - \prod_{a=1}^4\theta\begin{bmatrix}1\\0\end{bmatrix}(i\nu_a t, it) - \prod_{a=1}^4\theta\begin{bmatrix}0\\1\end{bmatrix}(i\nu_a t, it) - \prod_{a=1}^4\theta\begin{bmatrix}1\\1\end{bmatrix}(i\nu_a t, it)\right] = \prod_{a=1}^4\frac{e^{\nu_a^2/2}\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](i\nu_a' t, it)}{\eta(it)}$$

This last equality follows from the full abstruse identity of Jacobi, where $\phi_1' = \frac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4)$, $\phi_2' = \frac{1}{2}(\phi_1+\phi_2-\phi_3-\phi_4)$, $\phi_3' = \frac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4)$, $\phi_4' = \frac{1}{2}(\phi_1-\phi_2-\phi_3+\phi_4)$ and the ν_a' are defined identically. Inserting the DD conditions along the 9 direction that denotes separation, we get the full potential as a function of the separation Δx

$$V = -2\times\int_0^\infty\frac{dt}{2t}\frac{e^{-2\pi t\left(\frac{\Delta x}{2\pi\ell_s}\right)^2}}{2\pi\ell_s\sqrt{2t}}\prod_{a=1}^4\frac{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](i\nu_a' t, it)}{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](i\nu_a t, it)} \tag{74}$$

The initial overall factor of two comes from two orientations of the open string. At long enough distances, the exponential factor forces small t to contribute primarily. The complicated ratio of θ-functions becomes a ratio of sines. Then we get

$$\prod_{a=1}^{4} \frac{\sin(\pi\nu_a')}{\sin(\pi\nu_a)} \int_0^\infty \frac{dt}{2\pi\ell_s\sqrt{2t}t} e^{-\frac{(\Delta x)^2}{2\pi\ell_s^2}t}$$

Taking the integral and analytically continuing, we get a potential that looks like $-\frac{|\Delta x|}{2\pi\ell_s^2}\prod_{a=1}^4 \frac{\sin(\pi\nu_a')}{\sin(\pi\nu_a)}$, giving an attractive, constant force, of $-\frac{1}{2\pi\ell_s^2}\prod_{a=1}^4 \frac{\sin(\pi\nu_a')}{\sin(\pi\nu_a)}$.

16. Let the first brane at $\sigma = 0$ have no electric field and put an electric field F_{01} on the second brane at $\sigma = \pi$. The endpoints of the string are charged. We have the following action (take e, the charge of the string endpoint to be 1)

$$-\frac{1}{4\pi\ell_s^2}\int d\sigma d\tau[(\partial_\sigma X)^2 + (\partial_\tau X)^2] + \int_{\sigma=\pi} d\tau A_\mu \partial_\tau X^\mu$$

Upon variation, we get a boundary term:

$$\begin{aligned}
&-\frac{1}{2\pi\ell_s^2}\int d\tau \partial_\sigma X^\mu \delta X_\mu + \int d\tau \delta(A_\nu \partial_\tau X^\nu)\\
&= -\frac{1}{2\pi\ell_s^2}\int d\tau \partial_\sigma X^\mu \delta X_\mu + \int d\tau \partial_\mu A_\nu \delta X^\mu \partial_\tau X^\nu - \partial_\tau A_\nu\\
&= -\frac{1}{2\pi\ell_s^2}\int d\tau \partial_\sigma X_\mu \delta X^\mu + \int d\tau F_{\mu\nu}\delta X^\mu \partial_\tau X^\nu\\
&\Rightarrow \partial_\sigma X_\mu - 2\pi\ell_s^2 F_{\mu\nu}\partial_\tau X^\nu = 0
\end{aligned}$$

This gives mixed boundary conditions on the X^0 and X^1 which can be written as

$$\begin{pmatrix}\partial_\sigma X^0\\ \partial_\sigma X^1\end{pmatrix} = 2\pi\ell_s^2 E\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\partial_\tau X^0\\ \partial_\tau X^1\end{pmatrix}$$

with $E = F_{10}$. Note that we have been careful in raising the μ index. I will define $Z^\pm = X^0 \pm X^1$ and have $\partial_\sigma Z^+ = \partial_\sigma Z^- = 0$ at $\sigma = 0$ and $(\partial_\sigma - 2\pi\ell_s^2 E\partial_\tau)Z^+ = (\partial_\sigma + 2\pi\ell_s^2 E\partial_\tau)Z^- = 0$ at $\sigma = \pi$. The modes thus satisfy Neumann-Mixed boundary conditions. Following a modification of exercise **2.14** and solving these boundary conditions we get that the modes must be labeled by $\nu = -iu/\pi + \mathbb{Z}$. Here $u = \text{atanh}v$ is the *rapidity*. Now let's compute a cylinder diagram. Let's assume for now that we are scattering D1 branes (the problem does not explicitly give p, p'). It will look like the wick-rotation of the integral considered in the previous two questions. We have an amplitude

$$-iV_p \times 2 \times \int_0^\infty \frac{dt}{2t}\frac{e^{-t\frac{(\Delta x)^2}{2\pi\ell_s^2}}}{(2\pi\ell_s)^p(2t)^{p/2}}\frac{\theta\big[{}^1_1\big](ut/2\pi, it)^4}{\eta(it)^9\theta\big[{}^1_1\big](ut/\pi, it)}$$

Note however that since the first argument of the θ functions is real, we have poles at $t = \pi n/u$, $\nu = u/\pi$ for n an integer. Upon deforming the integration contour, we can use the identity $\frac{1}{x-i\epsilon} = \pi\delta(x) + \mathrm{P}(x)$ to pick up poles at $t = n/\nu$, at *odd* integers n (so that the four-order zero in the numerator doesn't cancel them), giving:

$$\pi V_p \sum_{n\in\mathbb{Z}^{odd}} \frac{e^{-n\frac{(\Delta x)^2}{2\pi\ell_s^2\nu}}}{(2\pi\ell_s)^p(2n/\nu)^{(p+2)/2}}\frac{\theta\big[{}^1_1\big](\frac{n}{2}, it)^4}{2\eta(in/\nu)^{12}} \tag{75}$$

The imaginary part of the amplitude can be interpreted (after T-duality) as resonances (ie bound states) of the D-branes.

17. From the last problem we can write: In terms of ∂_+, ∂_-, we have:

$$\begin{pmatrix}\partial_+ X^0\\ \partial_+ X^1\end{pmatrix} = \begin{pmatrix}\frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} & \frac{2\mathcal{E}}{1-\mathcal{E}^2}\\ \frac{2\mathcal{E}}{1-\mathcal{E}^2} & \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2}\end{pmatrix}\begin{pmatrix}\partial_- X^0\\ \partial_- X^1\end{pmatrix} \tag{76}$$

Here $\mathcal{E} = 2\pi\ell_s^2 E$. Taking $\mathcal{E}$ close to zero recovers NN boundary conditions on X^0, X^1. (It's worth noting that that the speed of light here will translate to a maximum electric field $|E| < T$ on the brane. This provides one motivation for the necessity of a nonlinear electrodynamics, namely DBI). taking $E = 0$ give N boundary conditions on X^0, X^1. T-dualizing X^1 gives D boundary conditions on $\tilde{X}^1$. Now, boosting the brane along X^0 and

$$\begin{pmatrix}\partial_+ X^0\\ \partial_+\tilde{X}^1\end{pmatrix} = \begin{pmatrix}\gamma & v\gamma\\ v\gamma & \gamma\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\gamma & -v\gamma\\ -v\gamma & \gamma\end{pmatrix}\begin{pmatrix}\partial_- X^0\\ \partial_-\tilde{X}^1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\partial_+ X^0\\ \partial_+ X^1\end{pmatrix} = \begin{pmatrix}\gamma & v\gamma\\ v\gamma & \gamma\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\gamma & -v\gamma\\ -v\gamma & \gamma\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\partial_- X^0\\ \partial_-\tilde{X}^1\end{pmatrix} = \begin{pmatrix}\frac{1+v^2}{1-v^2} & \frac{2v}{1-v^2}\\ \frac{2v}{1-v^2} & \frac{1+v^2}{1-v^2}\end{pmatrix}\begin{pmatrix}\partial_- X^0\\ \partial_-\tilde{X}^1\end{pmatrix}$$

This exactly what we had before, with $v = 2\pi\ell_s E$.

This argument is quite simple from the abstract picture: taking $A_1 = EX^0$, T-dualizing in the direction of X^1 gives a D-brane lying at $X_1 = 2\pi\ell_s^2 EX^0$ giving a velocity $2\pi\ell_s^2 E$. This can also be obtained by analytic continuation of question **8.13**.

18. Using the fact that this scattering problem is exactly T-dual to the electric field problem mentioned before, we return to (75) and consider $b = \Delta x$ to be small. In this case the large t regime dominates (corresponding to a loop of light open strings). First we perform a modular transformation to get

$$\mathcal{A} = \frac{V_p}{(8\pi^2\ell_s^2)^{p/2}}\int_0^\infty \frac{dt}{t} t^{(6-p)/2} e^{-\frac{tb^2}{2\pi\ell_s^2}} \frac{\theta\big[{}^1_1\big](i\nu/2, i/t)^4}{\eta(i/t)^9\theta\big[{}^1_1\big](i\nu, i/t)}$$

Now we follow Polchinski and rewrite $\mathcal{A}$ in terms of an integral over the particle's path $r(\tau)^2 = b^2 + v^2\tau^2$, $\mathcal{A} = -i\int d\tau V(r(\tau), v)$. Then we get V from reversing a Gaussian integral to be

$$V(r, v) = i\frac{2V_p}{(8\pi^2\ell_s^2)^{(p+1)/2}}\int_0^\infty dt t^{(5-p)/2} e^{-\frac{tr^2}{2\pi\ell_s^2}} \frac{\tanh\pi\nu\,\theta\big[{}^1_1\big](i\nu/2, i/t)^4}{\eta(i/t)^9\theta\big[{}^1_1\big](i\nu, i/t)}$$

The large-t limit is now direct:

$$V(r, v) = -\frac{2V_{p+1}}{(8\pi^2\ell_s^2)^{(p+1)/2}}\int_0^\infty \frac{dt}{t^{(1+p)/2}} e^{-\frac{tr^2}{2\pi\ell_s^2}} \frac{\tanh u\,\sin^4(ut/2)}{\sin(ut)}$$

Using steepest descent at zeroth order, the t that dominates is of order $2\pi\ell_s^2/r^2$ so that $ut \approx 2\pi\ell_s^2 v/r^2$ is the leading contribution. **Justify why ut is $O(1)$.** This then gives that $r \sim \ell_s\sqrt{v}$. If we go at very small velocities we can probe below the string scale.

On the other hand, a slower velocity means that the time it takes to probe this distance is longer $\delta t = r/v = \ell_s/\sqrt{v}$. This implies that $\delta x\delta t \geqslant \ell_s^2$. This looks like a type of *noncommutative geometry* with α' playing the role of Planck's constant now.

Combining this with the usual position-momentum uncertainty relation

$$1 \leqslant \delta x\, m\delta v = g\ell_s\delta x\delta v \Rightarrow \Delta x = \frac{g\ell_s}{\delta v},$$

we can minimize simultaneously $\ell_s\sqrt{v}, g\ell_s/v$ by having $v \sim g^{2/3}$ giving $\delta x = \ell_s g^{1/3}$. At weak coupling this is smaller than the string scale.

19. The action is a factor of two off from Polchinski's. The momenta are $p_i = \frac{2}{g_s\ell_s}(\dot{X}^i + [A_t, X^i])$. Then we get:

$$H = \int dt \mathrm{Tr}\Big[\frac{g_s\ell_s}{4}p_i p^i + \frac{1}{2g_s\ell_s(2\pi\ell_s^2)^2}[X^i, X^j]^2\Big]$$

Defining $g^{1/3}\ell_s Y^i = X^i, p_i = p_{Y_i}/g_s^{1/3}\ell_s$, this sets the length scale, which coincides with what we got in the previous question by less rigorous arguments. Now we have Y^i is dimensionless and get a hamiltonian

$$H = \frac{g_s^{1/3}}{\ell_s}\int dt \mathrm{Tr}\Big[\frac{1}{4}p_{Y_i}p_{Y_i} + \frac{1}{2(2\pi)^2}[Y^i, Y^j]^2\Big]$$

So the only dimensionful content of this hamiltonian comes from g, ℓ_s appearing in the overall normalization. This gives an energy scale of $g_s^{1/3}/\ell_s$. For strong coupling $g_s > 1$ this probes deeper than the string scale.

20. This is pretty direct. Since the metric $G_{\mu\nu}$ does not depend on X^i for $i = p+1\dots 9$, we have that each X^i is killing, in particular the metric takes a block-diagonal form where only the first $(p+1)^2$ $G_{\mu\nu}$ entries have nontrivial coordinate dependence and the remaining metric is just the identity matrix δ_{ij} along the X_i directions (we didn't even have to do this since 8.5.1 has $\eta_{\mu\nu}$ the flat metric. Is my logic here even right?).

    Take the ansatz $A \to (A_\mu, \Phi_i)$. We thus get $F_{\mu\nu}$ in the first $(p+1)^2$ entries and $\partial_\mu \Phi^i$ in the off-block-diagonal piece. We can rewrite this as a determinant of just the $(p+1)$ piece **Justify this step**

$$\sqrt{-\det(G_{\mu\nu} + 2\pi F_{\mu\nu} + \partial_\mu \Phi^i \partial_\nu \Phi^i)}$$

21. The bosonic part of this is immediate. Write the fields A_i in the dimensionally reduced dimensions as X^I and we immediately get $\mathrm{Tr}F_{10}^2 \to \mathrm{Tr}[F_{d+1}^2 + 2[D_\mu, X^I]^2 + \sum_{I,J}[X^I, X^J]^2]$. The fermionic part will reduce to:

$$(\mathrm{Tr}\bar{\chi}\Gamma^\mu D_\mu \chi)_{10D} \to \mathrm{Tr}[\bar{\chi}\Gamma^\mu D_\mu \chi + \bar{\lambda}_a \Gamma^i [X_i, \lambda^b]]$$

    Where now the χ_i are fermions that break the **16** representation of SO$(9,1)$ into a representation of $SO(d-1,1) \times SO(10-d)$. For $d = 3$ we get $\mathcal{N} = 4$ SYM and this is $(2,4) + (\bar{2}, \bar{4})$, corresponding to four Weyl spinors.

22. At the minimum of the potential, all X^I lie in a cartan and mutually commute. The A_{ij} correspond to open strings moving between the D-branes at positions X_I. The ground states of these open strings have a mass squared of $(X_I - X_J)^2/2\pi\ell_s^2$, so indeed the mass is linear in the separation. **Confirm. Understand Lie-Theoretic perspective**.

23. The worldvolume coupling to the RR 2-form looks like $iT_2 \int C_2$. For the brane tilted in the x^1, x^2 plane we can write this explicitly as:

$$i \int dx^0 dx^1 (C_{01} + C_{02} \tan(\theta))$$

    Now T-dualizing in the x^2 direction changes the C_{01} form to the RR 3-form C_3, giving the standard $i \int C_3$ term. On the other hand, the second term gets reduced to $-2\pi\ell_s^2 \int dx^0 dx^1 dx^2 C_0 F_{12}$, where I have used exercise **8.13** to write $\tan\theta = -2\pi\ell_s^2 F_{12}$. So we get a leading coupling to the three-form and a sub-leading coupling (in ℓ_s^2) to the one-form. This is a hint of the *Meyers effect*.

24. For the D2 brane the CP odd terms are C_3, $C_1 \wedge \mathcal{F}$ and $-C_1 \wedge \frac{(2\pi\ell_s)^4}{48}[p_1(\mathcal{T}) - p_1(\mathcal{N})]$. In the frame described by exercise **7.26**, C_3 transforms trivially under A transformations, and under its own gauge transformations it only adds an exact term which does not modify the CS action.

    $\mathcal{F}$ transforms trivially under A transformations and under B transformations only modifies the action by a closed term again.

    Equation **I.14** is not in any standard frame. The Dilaton plays no role here. The

    I feel I am missing something.

25. Gauge transformations of the axion C_0 are just shifts $C_0 \to C_0 + a$. C_0 couples to F_2 through the Chern-Simons term:

$$\int C_0 \wedge \mathrm{Tr} e^{\mathcal{F}} \wedge \mathcal{G}$$

    Because of the Bianchi identity, $\mathrm{d}\mathcal{F} = 0$, and the same holds for any trace of any polynomial of F. Similarly $\mathcal{G}$ is also a closed form. Therefore shifting C_0 gives an integrand term $\mathrm{Tr}e^{\mathcal{F}} \wedge \mathcal{G}$ which is closed.

26. For trivial flat-space background $\eta_{\mu\nu}$, we have $g_{ab} = \partial_a X_\mu \partial_b X^\mu$. Take $M_{ab} = \partial_a X_\mu \partial_b X^\mu + 2\pi\ell_s^2 F_{ab}$ and $M = \det M_{ab}$. Taking the DBI variation w.r.t. X_μ^a and A_a respectively gives:

$$\frac{T}{2}\partial^a\left(\sqrt{-M}M_{ab}^{-1}\partial_b X^\mu\right) = 0$$
$$\frac{2\pi\ell_s^2 T}{2}\partial^a\left(\sqrt{-M}M_{ab}^{-1}\right) = 0$$

Its rather nasty to evaluate that inverse matrix. On the other hand, taking X^9 to be the only nontrivial function of the ξ, and depending only on the radial distance r from a central point, and setting all $A_i = 0$ with A_0 a function of r alone, we get $M = 1 + \delta_{a=r,b=r}(\partial_r X^9)^2 + 2\pi\ell_s^2(\delta_{a=r,b=0} - \delta_{a=0,b=r})E$. We take $E = \partial_r X^9$. The determinant is then $(\nabla X^9)^2(1 + 2\pi\ell_s^2)$.

Note that if the second equation of motion holds, the first equation of motion implies that we would want for $\partial^r\partial_r X^9 = 0$, namely that X^9 is a harmonic function of r. On a p brane this is $X^9 = \frac{C_p}{r^{p-2}}$. In this case, the determinant, as well as M^{-1} will vanish when covariantly differentiated by ∂^r, giving us our desired second equation of motion.

This solution is known as a BI-on (BI for Born-Infeld). It represents an infinitely long open string ending on our p-brane.

27. Let's have G, B, Φ, C_2 trivial. We get

$$S = \frac{1}{2\pi\ell_s^2 g}\int d^2\xi\sqrt{1 - (2\pi\ell_s^2 F_{01})^2} + \frac{1}{2\pi\ell_s^2}\int d^2\xi\, C_0(2\pi\ell_s^2)F_{01}$$

Pick the gauge $A_0 = 0$. Our variable is then A_1. From this, we get a canonical momentum conjugate to A_1 equal to:

$$-\frac{1}{2\pi\ell_s^2 g}\frac{(2\pi\ell_s^2)^2 F_{01}}{\sqrt{1 - (2\pi\ell_s^2 F_{01})^2}} + C_0 F_{01}$$

Consider putting the D1 brane in a circle. Now since C_0 acts as a θ term, consider putting an integer m for C_0. The momentum is quantized, and in particular there is a gap between the zero momentum ground state and the next state up. We get:

$$\frac{2\pi\ell_s^2 F_{01}}{\sqrt{1 - (2\pi\ell_s^2 F_{01})^2}} = gm \Rightarrow 2\pi\ell_s^2 F_{01} = \frac{gm}{\sqrt{1 + m^2 g^2}}$$

We have a Hamiltonian

$$\mathcal{H} = \frac{1}{2\pi\ell_s^2 g}\frac{1}{\sqrt{1 - (2\pi\ell_s^2 F_{01})^2}}$$

And from the quantization condition on the electric field we obtain from the Hamiltonian a set of quantized tensions

$$T = \frac{1}{2\pi\ell_s^2 g}\sqrt{1 + m^2 g^2}$$

28. The D3-D$_{-1}$ system has #ND$=4$ and so preserves 1/4 supersymmetry (1/2 the SUSY of the D3 brane itself). Similarly, the instanton configurations satisfying $\star F_2 = \pm F_2$. The supersymmetric variation of the gaugino is $\delta\lambda \propto F_{\mu\nu}\Gamma^{\mu\nu}$. The $\Gamma^{\mu\nu}$ are generators of SO(4) $=$ SU(2) $\times$ SU(2), and the (A)SD conditions on F_2 will ensure that only half the generators (the first or second SU(2)) will appear in the variation. Thus instanton configurations are also 1/2 BPS on the worldvolume.

To confirm that these instantons really *are* D$_{-1}$ branes, note that the CS term contains $\frac{1}{2}(2\pi\ell_s^2)^2 T_3 \int C_0 F_2 \wedge F_2$. For a nontrivial instanton configuration we get $\int F_2 \wedge F_2 = 8\pi^2$. Thus the instanton coupling to C_0 is $(2\pi\ell_s)^4 T_3 = T_{-1}$, exactly the charge of the D$_{-1}$ brane. **Does this exclude the possibility of objects with the same charge and BPS properties as D_{-1} branes, but that don't have interpretations as endpoints of open strings?**

I expect the moduli space to have dimension $4n$, corresponding to the space (technically Hilbert scheme) of n points on $\mathbb{R}^4$.

29. This configuration is invariant under x^1 translations as well as under time x^0. The exact same BPS properties discussed in the previous question apply here. The state is half-BPS on the worldvolume both from the POV of string theory and from the POV of the low energy SYM theory having half the gaugino variations vanish.

The same instanton action argument in the previous question gives us that $\frac{1}{2}(2\pi\ell_s^2)^2 T_4 \int C_1 F_2 \wedge F_2$ yields a coupling $(2\pi\ell_s)^4 T_4 = T_0$ to the C_1 form.

For N D5 branes the low-energy effective theory is SU(N) SYM, and we obtain the moduli space of $SU(N)$ instantons. The dimension now becomes $4Nk$ **justify** . I expect that the moduli spaces of D1-D5 bound states are identical to the moduli space in the previous problem.

30. First, the pair of 5-branes in the 12345 and 12367 dimensions are parallel in the 123 directions and 90∘ rotated in two directions. This gives 2 sets of ND and 2 sets of DN boundary conditions, on the strings which gives us $\nu = 4$. In this case, following Polchinski the spinor $\beta = {\beta^{\perp}}^{-1}\beta^{\perp\prime}$ has an equal number of eigenvalues -1 and 1. So half of the original 16 spinors preserved by the first D-brane will be preserved by the combination of both.

Now take a third D-brane in the 12389 direction, perpendicular to both the first two. The same argument shows that we brane another half of the supersymmetry, giving 4 supersymmetries left in this configuration. In other words it is $\frac{1}{8}$ BPS. **Confirm**

31. Adding a D1 string gives 4 ND boundary conditions with each of the other D-branes. This breaks the supersymmetry in half again, preserving 2 supersymmetries now. It is $\frac{1}{16}$ BPS.

If I were to add it along direction 4 it would have 5 ND boundary conditions with the second two branes, which preserves no SUSY, so the latter configuration has nothing preserved.

32. Note this is O(2) and not SO(2), so instead of getting one D-brane at $\theta\tilde{R}$ and the other at $-\theta\tilde{R}$, we get one "half" D8 brane at $x^9 = 0$ (the location of one orientifold plane) and its image at $\pi\tilde{R}$ (the location of the other).

33. We work with the compact real form USp($2N$) = Sp($2N$) ∩ U($2N$). In this case any symplectic matrix can again be diagonalized to be of the form $(e^{i\theta_1}, e^{-i\theta_1}, \dots)$. Again we interpret this as D-branes on both sides $\pm\theta_i\tilde{R}$ of the orientifold plane. The generic gauge group is $U(1)^{2N}$. If m branes lie at either orientifold plane we get an enhancement Sp($2m$). When all N branes and their images lie on one of the orientifold planes, we recover the full symmetry.

34. Due to the negative tension, an excitation on it has even lower energy, corresponding to a negative norma state which is forbidden in a unitary theory by positivity. **What more can I say?**

35. There is a mistake in Kiritsis' equation **G.8**. We should have $A = -H^{-1}(\rho)$ not $-H(\rho)$. The way to see that is by noting that $-H^{-1}(\rho) = -\frac{\rho}{\rho+Q} = -\frac{r-Q}{r} = 1 + \frac{Q}{r}$. The constant 1 is gauge and hence irrelevant, while the second term is the proper electric potential that will give rise to a $F_{tr} = \frac{Q}{r^2}$.

Also, this problem asks us to work in $\mathcal{N} = 2, D = 4$ SUGRA, so the appropriate equations should be that the variation of *each* dilatino by a Killing spinor is zero. In this SUGRA, there are two Majorana gravitinos $\psi_{\mu,A}, A = 1,2$ with four components each, for a total of 8 SUSYs. Consequently, the variations involve two Majorana spinor parameters $\epsilon_A, A = 1,2$. We will use lower indices to indicate chiral and upper indices to indicate anti-chiral fermions. The gravitino variation is then (c.f. Freedman *Supergravity* Section 22.4)

$$\delta\psi_{\mu,A} = \nabla_\mu\epsilon_A - \frac{1}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\varepsilon_{AB}\epsilon^B \tag{77}$$

Here we have $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\Gamma^{ab}$ with ω spin connection [3]

Because of the chirality we can replace F with F^- in the above equation. Now, if we use spatial coordinates $\vec{x}, |x| = \rho$, the metric takes the form

$$ds^2 = -H^{-2}dt + H^2 d\vec{x}^2$$

[3]This corresponds to setting $\kappa = \sqrt{2}$ in Friedman's *Supergravity* **22.69**.

Take $e^{2A} = H^{-2}$, then we have the frame fields $e^{\hat{0}} = e^A dt, e^{\hat{i}} = e^{-A}dx^i$. We will use hats to denote frame indices a, b. Our spin connection is then:

$$\omega_{\hat{0}\hat{i}} = -e^{2A}\partial_i A\, dt, \quad \omega_{\hat{i}\hat{j}} = -\partial_j A dx^i + \partial_i A dx^j$$

First let's look at the $\mu = 0$ constraint of equation (77)

$$\partial_t \epsilon_A + \frac{1}{4}\omega_{0\,ab}\Gamma^{ab}\epsilon_A - \frac{1}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_0 \varepsilon_{AB}\epsilon^B$$

Now because the solution is static, ∂_t is killing and we expect that $\partial_t \epsilon_A = 0$. Further, the only contribution to $\omega_{0\,ab}$ is $\omega_{0\,\hat{0}\hat{i}}$ since only this has a dt (NB the double sum gives a factor of 2). Similarly for the second term, since there is only an electric field, we only care about $\nu, \rho \in \{0, i\}$ (NB the double gives a factor of 2). Finally, we have only an electric field $F_{0i} = -\partial_i A_t$ (A_t is the vector potential, not to be confused with A). This yields:

$$\begin{aligned}
&-\frac{1}{2}e^{2A}\,\partial_i A\,\gamma^{\hat{0}}\gamma^{\hat{i}}\epsilon_A - \frac{1}{2}(-\partial_i A_t)\gamma^0\gamma^i\gamma_0\varepsilon_{AB}\epsilon^B = 0\\
&\Rightarrow -\frac{1}{2}e^A\,\partial_i e^A\,\gamma^{\hat{i}}\gamma^{\hat{0}}\epsilon_A + \frac{1}{2}(-\partial_i A_t)\gamma^i\varepsilon_{AB}\epsilon^B = 0\\
&\Rightarrow e^A\,\partial_i e^A\,\gamma^{\hat{i}}\gamma^{\hat{0}}\epsilon_A - \partial_i A_t\, e^A\gamma^{\hat{i}}\varepsilon_{AB}\epsilon^B = 0\\
&\Rightarrow \partial_i e^A\,\gamma^{\hat{0}}\epsilon_A = \partial_i A_0\,\varepsilon_{AB}\epsilon^B
\end{aligned}$$

Here, the hatted γ-matrices are the familiar ones from flat space. We thus need (up to an irrelevant gauge constant) $A_0 = \pm H^{-1}$ and

$$\epsilon_A = \mp\gamma^{\hat{0}}\varepsilon_{AB}\epsilon^B \tag{78}$$

Since we require $-H^{-1}$ to match the electromagnetic potential A, and so that it is asymptotically unity, we have $H = (1 - \frac{Q}{r})^{-1} = \frac{\rho+Q}{\rho} = 1 + \frac{Q}{\rho}$. This verifies the extremal RN solution.

We have not yet derived the spatial dependence of ϵ. Taking $\mu = i$ we get

$$\partial_i \epsilon_A + \frac{1}{4}\omega_{i\,ab}\Gamma^{ab}\epsilon_A - \frac{1}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_i\varepsilon_{AB}\epsilon^B$$

Now we must use that $\omega_{i\,\hat{j}\hat{k}} = -\partial_n A(\delta_{ij}\delta^n_k - \delta_{ik}\delta^n_j)$. Using the $\mu = 0$ constraint we get:

$$\begin{aligned}
&\partial_i\epsilon_A - \frac{1}{2}\partial_k A\,\gamma_{i\,\hat{k}}\epsilon_A - \frac{1}{2}F_{0i}\gamma^0\cancel{\gamma^i\gamma_i}\varepsilon_{AB}\epsilon^B = 0\\
&\partial_i\epsilon_A + \frac{1}{2}\partial_k A\,e^{-A}\,\gamma^{i\,\hat{k}}\epsilon_A \mp \frac{1}{2}\partial_i e^A H\gamma^{\hat{0}}\varepsilon_{AB}\epsilon^B = 0\\
&\partial_i\epsilon_A + \frac{1}{2}\partial_k A\,\gamma^{\hat{i}\,\hat{k}}\epsilon_A - \frac{1}{2}\partial_i e^A e^{-A}\epsilon_A = 0
\end{aligned}$$

Now the $\gamma^{\hat{i}\hat{k}}$ is nothing more than a *generator of rotations* acting on ϵ_A. Since we are assuming spherical symmetry, $\gamma^{\hat{i}\hat{k}}\epsilon_A = 0$ and we are left with the differential equation:

$$\partial_i\epsilon_A = \frac{1}{2}\partial_i A\epsilon_A \Rightarrow \epsilon_A = e^{1/2A}\epsilon_0$$

where ϵ_0 is a constant spinor satisfying (78).

Because the constraint $\epsilon_A = \mp\gamma^{\hat{0}}\varepsilon_{AB}\epsilon^B$ applies to half the space of spinors at any given point, we have that the extremal RN solution is half-BPS.

36. Again take coordinates x_i so that

$$ds^2 = -\frac{dt^2}{H^2(\rho)} + H^2(\rho)(dx_i^2)$$

Upon the choice $\epsilon_A = -\mp\gamma^{\hat{0}}\varepsilon_{AB}\epsilon^B$, we had the relation $\partial_i H^{-1} = \partial_i A_0 = F_{i0}$. The field equation $\partial \star F = 0$

$$\partial_i\sqrt{-g}g^{00}g^{ii}F_{i0} = \partial_i H^2\partial_i H^{-1} = \partial_i^2 H(x_i)$$

Thus we have that H is a harmonic function of the *flat* Laplacian in transverse space.

We see that H from the previous problem takes the simple form $1+\frac{Q}{|x|}$, which is obviously harmonic in 3+1 dimensions.

A more general solution allowing for for multiple charged extremal black holes amounts to nothing more than replacing H with $1+\sum_i \frac{Q_i}{|x-x_i|}$, which remains harmonic, and thus preserves half supersymmetry. This looks like a bunch of extremal black holes whose pairwise electric repulsions cancel their gravitational attractions.

37. Again we are working in $\mathcal{N}=2$ SUGRA. The metric takes the form

$$ds^2 = Q^2\left[\frac{-dt^2+d\rho^2}{\rho^2}+d\Omega_2^2\right]$$

This corresponds to 2D AdS times a sphere of *constant radius*. Both spaces have radius Q. The spinor equation is as before, now with $T_{\mu\nu}=-\frac{1}{L}g_{\mu\nu}$. Consider just AdS_2. Define the operator

$$\hat{D}_\mu \epsilon_A = \nabla_\mu \epsilon_A - \frac{1}{2Q}\gamma_\mu \varepsilon_{AB}\epsilon^B$$

Consider now

$$[\hat{D}_\mu, \hat{D}_\nu]\epsilon_A = (\frac{1}{4}R_{\mu\nu ab}\gamma^{ab}+\frac{1}{2Q^2})\epsilon$$

And since AdS_2 is a maximally symmetric space $R_{\mu\nu ab}=-(e_{a\mu}e_{b\nu}-e_{a\nu}e_{b\mu})/L^2$ and so the commutator vanishes identically. This is the integrability condition we need. At each point, the spinor bundle is dimension $\mathcal{N}\times 2^{[D/2]}$ - for $\mathcal{N}=2$ AdS2 this is 4. We see that any spinor can be transported by the connection $\hat{D}_\mu$ to define a spinor field on all of AdS_2, and thus we get that the space is maximally supersymmetric.

The exact same arguments (with $Q\to -Q$) apply for the positively curved 2-sphere of the same radius.

The product of two maximally supersymmetric spaces is maximally supersymmetric **Confirm**. We get 8 Killing spinors. We now see that the Bertotti-Robertson universe preserves full supersymmetry, and thus the extremal RN black hole plays the role of a *half-BPS soliton* that interpolates between two fully supersymmetric backgrounds (flat space and $\mathrm{AdS}_2\times S^2$).

38. The variation of $\frac{1}{2(p+2)!}F_{p+2}^2 = F_{p+2}\wedge \star F_{p+2}$ directly gives $\star F_{p+2}=0$.

Varying the dilaton gives

$$\begin{aligned} 0 = -2e^{-2\Phi}[R+4(\nabla\Phi)^2] - \nabla(e^{-2\Phi}8\nabla\Phi) &= -2e^{-2\Phi}[R+4(\nabla\Phi)^2] + 16e^{-2\Phi}(\nabla\Phi)^2 - 8e^{-2\Phi}\Box\Phi \\ &\Rightarrow R = 4(\nabla\Phi)^2 - 4\Box\Phi \end{aligned}$$

Finally, writing the metric explicitly in the action

$$\sqrt{-g}e^{-2\Phi}[g^{\mu\nu}R_{\mu\nu}+4g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi] - \frac{1}{2(p+2)!}\sqrt{-g}F_{p+2}^2$$

Let's look how each term changes when we vary $\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}$.

- $\sqrt{-g}e^{-2\Phi}R$

$$\begin{aligned} &\to (R_{\mu\nu}+g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)e^{-2\Phi} - \frac{1}{2}g_{\mu\nu}e^{-2\Phi}R \\ &= e^{-2\Phi}\Big(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}(-2\Box\Phi+4(\partial\Phi)^2) - (-2\nabla_\mu\nabla_\nu\Phi + 4\partial_\mu\Phi\partial_\nu\Phi)\Big) \end{aligned}$$

- $\sqrt{-g}e^{-2\Phi}4g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \to 4e^{-2\Phi}\partial_\mu\Phi\partial_\nu\Phi - 2e^{-2\Phi}(\partial\Phi)^2$
- $-\frac{1}{2(p+2)!}\sqrt{-g}g^{\mu_1\nu_1}\dots g^{\mu_{p+2}\nu_{p+2}}F_{\mu_1\dots\mu_{p+2}}F_{\nu_1\dots\nu_{p+2}} \to -\frac{1}{2(p+1)!}F^2_{\mu\nu}+\frac{1}{4(p+2)!}g_{\mu\nu}F^2$. Here $F^2_{\mu\nu}=F_{\mu\dots}F^{\nu\dots}$

Combining these all together and using the dilaton equations of motion gives

$$e^{-2\Phi}R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi-\frac{1}{2(p+1)!}F^2_{\mu\nu}+\frac{1}{4(p+2)!}g_{\mu\nu}F^2=0\Rightarrow R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi=\frac{e^{2\Phi}}{2(p+1)!}\Big(F^2_{\mu\nu}-\frac{g_{\mu\nu}}{2p+2}F^2\Big)$$

exactly as required.

39. This problem was very time-consuming to do out explicitly. The only resource that was of any help for cross-checking was Ortín's "*Gravity and Strings*".

First consider the possibility of $\Phi=0$, $F=0$. In this case we have no stress tensor and are left with static vacuum Einstein equations, spherically symmetric in $10-p$ dimensions and translationally invariant in p dimensions. In that case we know that our solution is nothing more than the Schwarzschild solution in $10-p$ dimensions times a transverse p-dimensional space:

$$ds^2=-f(r)dt^2+dx_i^2+\frac{1}{f(r)}dr^2+r^2d\Omega^2_{8-p}$$

Reproducing the arguments from black holes in 4D, we see that

$$\frac{d}{dr}(r^{8-p}f(r)\frac{d}{dr}\log(f))=\frac{d}{dr}(r^{8-p}f'(r))=0$$

ie $f(r)$ must be harmonic in the transverse dimension. After rescaling coordinates to have appropriate asymptotic behavior, we get:

$$f(r)=1-\frac{r_0^{7-p}}{r^{7-p}}$$

for some constant r_0 related to the ADM mass of the solution. So we see that $H(r)=1$ when the dilaton and $p+2$-form field strength are turned off. The curvature is $R=0$. Now let's turn on Φ. We expect small Φ will correspond to small H.

To do this, let's look at the simplest case, $p=0$. Take the Ansatz (which you can convince yourself to be completely general given the symmetry of the problem)

$$ds^2=-\lambda(r)dt^2+\frac{dr^2}{\mu(r)}+R(r)^2d\Omega^2_{d-2}$$

We will later set $d=10, \lambda=\mu=f(r)/\sqrt{H}$. Let's explicitly calculate the Christoffel symbols. There are three categories: Γs involving just r,t Γs involving mixed r,Ω, and Γs involving just the Ω variables. I will use a,b,c to index the angular variables ψ_a, whose metric is $R^2d\Omega^2_{ab}=q_a\delta_{ab}$, and I use $'$ to denote ordinary partial differentiation w.r.t. r.

$$\Gamma^r_{tt}=\tfrac{1}{2}\mu\lambda' \qquad \Gamma^t_{tr}=\tfrac{1}{2}\lambda^{-1}\lambda' \qquad \Gamma^r_{rr}=-\tfrac{1}{2}\mu^{-1}\mu'$$
$$\Gamma^a_{rb}=\delta^a_b\tfrac{R'}{R} \qquad \Gamma^r_{ab}=\delta_{ab}\mu(R^2)'\tfrac{q_a}{R^2}$$
$$\Gamma^a_{bc}=\theta_{b>c}\delta^a_b\cot\psi_b+\delta_{c>b}\theta_{ac}\cot\psi_c-\theta_{b>a}\delta_{bc}\cot\psi_a\tfrac{q_b}{q_a}$$

That last Christoffel symbol looks particularly nasty. Thankfully, by using the fact that the sphere is a symmetric space, we will not need to use it explicitly.

Now let's directly compute the Ricci tensor.

$$R_{\mu\nu}=\partial_\rho\Gamma^\rho_{\mu\nu}-\partial_\mu\Gamma^\rho_{\rho\nu}+\Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\mu\nu}-\Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu}$$

In what follows, it is useful to recall the identity $\Gamma^\mu_{\mu\nu}=\partial_\nu\log\sqrt{-g}$. The nonzero terms will be R_{tt},R_{rr},R_{ab}. Respectively they are:

$$\begin{aligned}R_{tt}&=\partial_r\Gamma^r_{tt}-\cancel{\partial_t\Gamma^\rho_{\rho t}}+\Gamma^\rho_{\rho r}\Gamma^r_{tt}-\Gamma^\sigma_{t\rho}\Gamma^\rho_{\sigma t}\\&=\frac{1}{2}(\mu\lambda')'+\partial_r\log\sqrt{g}\,(\mu\lambda')-2\frac{1}{2}\frac{\lambda'}{\lambda}\frac{1}{2}\mu\lambda'\\&=\frac{1}{2}\frac{\lambda}{\sqrt{g}}\partial_r\Big(\sqrt{-g}\mu\frac{\lambda'}{\lambda}\Big)=\frac{1}{2}\lambda\nabla^2\log\lambda\end{aligned}$$

It's important for this next one to note that q_a/R^2 is independent of r. It's equally important to appreciate that the final combination of Γ symbols is the only thing that would appear in the absence of r dependence in R. In this case, because the $d-2$ sphere is a symmetric space, we'd have $R_{ab} = \frac{d-3}{R^2} g_{ab}$. Indeed, this is exactly what the final term gives. We thus get

$$
\begin{aligned}
R_{ab} &= \partial_r \Gamma^r_{ab} - \cancel{\partial_a \Gamma^\rho_{\rho b}} + \Gamma^\rho_{\rho r}\Gamma^r_{ab} - \Gamma^\sigma_{a\rho}\Gamma^\rho_{\sigma b} \\
&= -\frac{1}{2}\delta_{ab}\partial_r\left(\mu (R^2)' \frac{q_a}{R^2}\right) + \partial_r \log\sqrt{-g}\,\delta_{ab}\mu(R^2)'\frac{q_a}{R^2} + \frac{d-3}{R^2} q_a \delta_{ab} \\
&= g_{ab}(-\nabla^2 \log R + \frac{d-3}{R^2})
\end{aligned}
$$

The next one is a bit different. Less cancelation. The last term will sum over $(\sigma, \rho) = (r, r), (t, t), (a, a)$. Also note $\sqrt{-g} = \sqrt{\lambda/\mu} R^{d-2}$.

$$
\begin{aligned}
R_{rr} &= \partial_r \Gamma^r_{rr} - \partial_r \Gamma^\rho_{\rho r} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{rr} - \Gamma^\sigma_{r\rho}\Gamma^\rho_{\sigma r} \\
&= \cancel{-\frac{1}{2}(\mu^{-1}\mu')'} - \partial_r^2 \log(\sqrt{\lambda/\mu}R^{d-2}) - \frac{1}{2}\partial_r \log(\sqrt{\lambda/\mu}R^{d-2})\,(\mu^{-1}\mu') - \frac{1}{4}\frac{(\lambda')^2}{\lambda^2} - \frac{1}{4}\mu^2(\mu')^2 - (d-2)\frac{(R')^2}{R^2} \\
&= -\frac{1}{2}\partial_r^2 \log(\lambda) - \frac{d-2}{R}R'' - \frac{1}{2}(d-2)\frac{R'}{R}\partial_r \log\sqrt{\lambda/\mu} \\
&= -\frac{1}{2}\mu^{-1}\nabla^2 \log\lambda - \frac{d-2}{R}\sqrt{\frac{\lambda}{\mu}}\left(R'\sqrt{\frac{\mu}{\lambda}}\right)'
\end{aligned}
$$

Altogether we get a Ricci scalar:

$$
R = -\nabla^2 \log(\lambda R^{d-2}) + \frac{(d-2)(d-3)}{R^2} - \frac{d-2}{R}\sqrt{\lambda\mu}\left(R'\sqrt{\frac{\mu}{\lambda}}\right)'
$$

Now let's take $\lambda = \mu = f(r)/\sqrt{H}$ and $R = H^{1/4}$, $\sqrt{-g} = H^{(d-2)/4}$.

$$
-\nabla^2 \log(f(r) r^{d-2} H^{\frac{d-4}{4}}) + \frac{(d-2)(d-3)}{r^2 H^{1/2}} - (d-2)\frac{f(r)}{rH^{3/4}}(rH^{1/4})''
$$

The Laplacian takes the form $\frac{\partial_r [H^{(d-4)/4} r^8 f(r) \partial_r]}{r^{d-2} H^{(d-2)/2}}$ which simplifies the above to:

$$
-\frac{\partial_r^2 (f(r) r^{d-2} H^{(d-4)/4})}{r^{d-2} H^{(d-2)/2}} + \frac{(d-2)(d-3)}{r^{d-2} H^{1/2}}
$$

The dilaton equation of motion is

$$
R = 4(\nabla\Phi)^2 - 4\nabla^2\Phi
$$

Since a nonzero Φ is what gives an H away from 1, we might hypothesize a relationship $\log H \propto \Phi$, meaning we should replace Φ with $\log(H^\alpha)$ in the dilaton equation. Let's also take $f(r)$ to be *not different* from the Schwarzschild solution: $f(r) = 1 - \frac{r_0^{d-3}}{r^{d-3}}$. So far we will not be so bold as to assume *anything* about H. We also at this point need to specialize to $d = 10$, otherwise no nice simplification occurs. Straightforward algebra then gives:

```
In[2382]:= f[r_] := 1 - r0^(d-3)/r^(d-3);
sqrtg = r^(d-2) H[r]^((d-2)/4);
grr = Sqrt[H[r]]/f[r];
(-1/(r^(d-2) H[r]^((d-2)/4)) ∂{r,2} (f[r] r^(d-2) H[r]^((d-4)/4)) + ((d-2) (d-3))/(r^2 Sqrt[H[r]]) - (d-2) f[r]/(r H[r]^(3/4)) ∂{r,2} (r H[r]^(1/4)) /. d → 10) -
  (4 grr^-1 (∂r Log[H[r]^α])^2 - 4/sqrtg ∂r (sqrtg grr^-1 ∂r (Log[H[r]^α])) /. d → 10) // FullSimplify

Out[2385]= (r (r^7 - r0^7) (3 + 8 (1 - 2 α) α) H'[r]^2 + 2 H[r] (2 (r0^7 (7 - 4 α) + 4 r^7 (-7 + 8 α)) H'[r] + r (r^7 - r0^7) (-7 + 8 α) H''[r]))/(4 r^8 H[r]^(5/2))
```

To get rid of the term quadratic in $H'(r)$ we need $3 + 8(1-2\alpha)\alpha = 0 \Rightarrow \alpha = 3/4$.

After that, these will only be equal if

$$8H' - rH'' = 0 \Rightarrow H = 1 - \frac{L^7}{r^7}$$

The above solution is the most general given that $H \to 1$ as $r \to \infty$.

Now let us generalize this to higher dimensions. We add p x_i in the parallel dimensions of the solution.

$$ds^2 = -\lambda(r)dt^2 + \nu(r)d\vec{x}_i^2 + \frac{dr^2}{\mu(r)} + R(r)^2 d\Omega_{d-2}^2$$

This gives two new Christoffels. Here is a complete list

$$\begin{gathered}
\Gamma^r_{tt} = \tfrac{1}{2}\mu\lambda' \qquad \Gamma^t_{tr} = \tfrac{1}{2}\lambda^{-1}\lambda' \qquad \Gamma^r_{rr} = -\tfrac{1}{2}\mu^{-1}\mu' \\
\Gamma^a_{rb} = \delta^a_b \tfrac{R'}{R} \qquad \Gamma^r_{ab} = \delta_{ab}\mu(R^2)' \tfrac{q_a}{R^2} \\
\Gamma^a_{bc} = \theta_{b>c}\delta^a_b \cot\psi_b + \delta_{c>b}\theta_{ac}\cot\psi_c - \theta_{b>a}\delta_{bc}\cot\psi_a \tfrac{q_b}{q_a} \\
\Gamma^r_{ij} = -\tfrac{1}{2}\delta_{ij}\mu\nu' \qquad \Gamma^i_{rj} = \tfrac{1}{2}\delta^i_j \nu^{-1}\nu'
\end{gathered}$$

Our nonzero Ricci components are now $R_{tt}, R_{rr}, R_{ab}, R_{ij}$. The primary way that the new dimensions will contribute is by modifying $\sqrt{-g}$ We get:

$$R_{tt} = R_{tt}^{(10-p)} - \frac{1}{4}p\mu\lambda'(\log\nu)'$$

$$R_{ab} = R_{ab}^{(10-p)} - \frac{1}{2}pg_{ab}\mu(\log\nu)'(\log R)'$$

$$R_{rr} = R_{rr}^{(10-p)} + \frac{1}{2}p(\mu\nu)^{-1/2}((\mu\nu)^{1/2}(\log\nu)')'$$

$$R_{ij} = \frac{1}{2}\delta_{ij}\nu\nabla^2\nu$$

This gives a Ricci curvature of:

$$R = R^{(10-p)} + \frac{1}{2}p(-\nabla^2_{10-p}\log\nu - \nu^{-1}\nabla^2\nu + \frac{1}{2}\mu((\log\nu)')^2)$$

Making the necessary replacements we get

$$\begin{aligned}
&-\frac{\partial_r^2(f(r)r^{d-2}H^{(d-4)/4})}{r^{d-2}H^{(d-2)/2}} + \frac{(d-2)(d-3)}{r^{d-2}H^{1/2}} \\
&+\frac{1}{2}p\left(-\frac{\partial_r[fr^{8-p}H^{(6-p)/4}\partial_r \log H^{-1/2}]}{r^{8-p}H^{(8-p)/4}} - \frac{\sqrt{H}}{r^{8-p}H^{(8-2p)/4}}\partial_r[fr^{8-p}H^{(6-2p)/4}\partial_r H^{-1/2}] + \frac{1}{2}\frac{f}{H^{1/2}}(\partial_r \log H^{-1/2})^2 \right)
\end{aligned}$$

It makes sense to take the ansatz $f(r) = 1 - \frac{r_0^{7-p}}{r^{7-p}}$ and $H = 1 + \frac{L^{7-p}}{r^{7-p}}$. Further, the relationship between H and Φ can be guessed from reasoning in the $p = 0$ case to go as $\Phi \propto H^{(3-p)/4}$, or alternatively we can establish this from first principles by algebra

```
In[2732]:= H[r_] := 1 + L^(7-p)/r^(7-p);
f[r_] := 1 - r0^(7-p)/r^(7-p);
grr = Sqrt[H[r]]/f[r];
sqrtg = r^(8-p) H[r]^((8-2p)/4);
FullSimplify[-1/(r^(8-p) H[r]^((8-p)/4)) ∂{r,2} (f[r] r^(8-p) H[r]^((6-p)/4)) + ((8-p) (7-p))/(r^2 Sqrt[H[r]]) - (8-p) f[r]/(r H[r]^(3/4)) ∂{r,2} (r H[r]^(1/4)) +
  1/2 p (1/(2 r^(8-p) H[r]^((8-p)/4)) ∂r (f[r] r^(8-p) H[r]^((6-p)/4) ∂r (Log[H[r]])) - H[r]^(1/2)/(r^(8-p) H[r]^((8-2p)/4)) ∂r (f[r] r^(8-p) H[r]^((6-2p)/4) ∂r (H[r]^(-1/2))) +
   1/2 f[r]/Sqrt[H[r]] (∂r (Log[H[r]^(-1/2)]))^2), p < 7]
FullSimplify[(4 grr^-1 (∂r Log[H[r]^α])^2 - 4/sqrtg ∂r (sqrtg grr^-1 ∂r (Log[H[r]^α])))]
```

Out[2736]=
$$-\frac{L^7\,(-7+p)^2\,(-3+p)\,r^{-9+2p}\,r0^{-p}\left(4\,L^p\,r^7\,r0^7+L^7\left(-(-3+p)\,r^p\,r0^7+(1+p)\,r^7\,r0^p\right)\right)}{4\sqrt{1+L^{7-p}\,r^{-7+p}}\,\left(L^p\,r^7+L^7\,r^p\right)^2}$$

Out[2737]=
$$\frac{2\,L^7\,(-7+p)^2\,r^{-9+2p}\,r0^{-p}\,\alpha\left(2\,L^p\,r^7\,r0^7+L^7\left(-r^p\,r0^7\,(-3+p+2\alpha)+r^7\,r0^p\,(-1+p+2\alpha)\right)\right)}{\sqrt{1+L^{7-p}\,r^{-7+p}}\,\left(L^p\,r^7+L^7\,r^p\right)^2}$$

This immediately gives that the dilaton term will equal the scalar curvature only when $\alpha = (3-p)/4$.

We have thus proved the form of f, H, Φ. Let's finally look at the RR field.

For now let us ignore the issues with self-duality at $p = 3$. Take the the $p+1$ form has flux in the radial but not angular directions in transverse space. The only nonzero component of F_{p+2} is given by $F_{r0\ldots p}$. The equation of motion gives:

$$\partial_r \left(\sqrt{-g} g^{\mu_0 \nu_0} g^{\cdots} F_{\mu_0 \ldots}\right) = 0$$

Now we already have $\sqrt{g} = r^{8-p} H^{(4-p)/2}$, while we will have raising for each index $0 \ldots p$ as well as r, giving a factor of $f(r) H^{(p+1)/2} H^{-1/2}/f(r) = H^{p/2}$. Altogether the differential equation becomes:

$$\partial_r r^{8-p} H^2 H^r$$

Immediately we must have $F = \frac{\kappa/r^{8-p}}{H^2}$. This means that F is proportional to $H'(r)/H(r)^2$.

I don't know how to easily get this constant of proportionality without knowing the decay properties of $R_{\mu\nu}$ as $r \to \infty$ **Return to this**. I know it must scale roughly as a positive power of L. It turns out to be:

$$F_{r0\ldots p} = -\sqrt{1 - \frac{r_0^{7-p}}{L^{7-p}}} \frac{H_p'(r)}{H_p^2(r)}$$

Can I do this all by somehow "boosting" Schwarzschild?

40. From the expression **8.8.9** of the electric field in terms of H we have (assuming $p < 7$)

$$E_r = (7-p) L^{(7-p)/2} \sqrt{L^{7-p} + r_0^{7-p}} \frac{r^{6+p}}{(r^7 + L r^p)^2} \to (7-p) L^{(7-p)/2} \sqrt{L^{7-p} + r_0^{7-p}} r^{p-8}$$

Integrating this over the $8-p$ sphere will give

$$T_p N = \frac{\Omega_{8-p}(7-p)}{2\kappa_{10}^2} L^{(7-p)/2} \sqrt{L^{7-p} + r_0^{7-p}}.$$

41. Using the standard ADM formula (cf, eg Carrol)

$$M = \frac{1}{2\kappa_{10}^2} \int_{S^{8-p}} g^{\mu\nu}(g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha}) n^\alpha dS$$

Then for a metric that looks like $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu} = \frac{c_{\mu\nu}}{r^{7-p}}$.

$$\begin{aligned}
M = T_{00} V_p &= \frac{\Omega_{8-p}}{2\kappa_{10}^2}((7-p)c_{00} - \eta_{00}\eta^{ab}c_{ab}) \\
&= \frac{\Omega_{8-p} V_p}{2\kappa_{10}^2}((7-p)(r_0^{7-p} + \frac{1}{2}L^{7-p}) + r_0^{7-p} + \frac{1}{2}L^{(7-p)/4}) \\
&= \frac{\Omega_{8-p} V_p}{2\kappa_{10}^2}((8-p)r_0^{7-p} + (7-p)L^{7-p})
\end{aligned}$$

Revisit- something seems off

42. Note that

$$f_-(\rho) = 1 - \frac{L^{7-p}}{r^{7-p} + L^{7-p}} = \frac{1}{1 + L^{7-p}r^{7-p}} = \frac{1}{H(r)}$$

Similarly

$$f_+(\rho) = 1 - \frac{r_0^{7-p} + L^{7-p}}{r^{7-p} + L^{7-p}} = \frac{r^{7-p} + r_0^{7-p}}{r^{7-p} + L^{7-p}} = \frac{f(r)}{H(r)}$$

This confirms that the dt and $d\vec{x}$ terms are indeed consistent, and that the $f_-^{-1/2}(\rho)$ in front of the transverse part is our desired $\sqrt{H}$. Next note that:

$$f_-(\rho)^{\frac{1}{7-p}} = \left[\frac{\rho^{7-p} - L^{7-p}}{\rho^{7-p}}\right]^{1/(7-p)} = \frac{r}{\rho} \Rightarrow \rho^2 f_-(\rho)^{1-\frac{5-p}{7-p}} = \rho^2\frac{r^2}{\rho^2} = r^2$$

So the angular part is consistent. Lastly, $f_+/f_- = f(r)$ which is the required coefficient for dr^2. It remains to cancel the jacobian:

$$dr = \frac{\rho^{6-p}}{r^{6-p}}d\rho = f_-^{-\frac{6-p}{7-p}}d\rho \Rightarrow dr^2 = f_-^{-\frac{12-2p}{7-p}}d\rho^2 = f_-^{-1-\frac{5-p}{7-p}}d\rho^2$$

So

$$\frac{\sqrt{H(r)}}{f(r)}dr^2 = f_-^{-1/2}\frac{f_-}{f_+}f_-^{-1-\frac{5-p}{7-p}}d\rho^2 = f_-^{-\frac{1}{2}-\frac{5-p}{7-p}}\frac{d\rho^2}{f_+(\rho)}$$

43. Before doing any supersymmetric manipulations, we should know the spin connection.

Take the extremal p-brane metric to be of the form

$$ds^2 = e^{2A(r)}dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)}dx^i dx^j \delta_{ij}$$

In this case we have $A = -B = \frac{1}{4}\log H(r)$. Take the frame fields

$$e^{\hat{\mu}} = e^A dx^\mu, \quad e^{\hat{i}} = e^B dx^i$$

Then

$$\begin{aligned}
\mathrm{d}e^{\hat{\mu}} &= \partial_r A e^A \mathrm{d}r \wedge \mathrm{d}x^\mu = \sum_i \partial_i A e^A \mathrm{d}x^i \wedge dx^\mu = e^{\hat{i}} \wedge \omega^{\hat{\mu}}{}_{\hat{i}} \Rightarrow \omega_{\hat{\mu}\hat{\nu}} = 0,\ \omega_{\hat{\mu}\hat{i}} = (-)^{\mu=0}\partial_i A\, e^{A-B}\mathrm{d}x^\mu \\
\mathrm{d}e^{\hat{i}} &= \partial_r B e^B \mathrm{d}r \wedge \mathrm{d}x^i = \sum_j \partial_i B e^B \mathrm{d}x^j \wedge dx^i = e^{\hat{j}} \wedge \omega^{\hat{i}}{}_{\hat{j}} \Rightarrow \omega_{\hat{i}\hat{j}} = \partial_j B \mathrm{d}x^i - \partial_i B \mathrm{d}x^j
\end{aligned}$$

Using our extremal form of the solution, I can further write

$$e^\Phi = g_s^2 H^{(3-p)/4} = g_s^2 e^{(p-3)A}, \quad F_{r01\ldots p} = \mp\frac{H'}{H^2} = \pm 4A' e^{4A}$$

The $\pm$ corresponds to brane/anti-brane solutions.

In 10D $\mathcal{N}=2$ SUGRA coupled to matter, represent the Killing spinor as $\epsilon = \binom{\epsilon^1}{\epsilon^2}$. We have the gravitino and dilatino variations:

$$\begin{aligned} 0 = \delta\psi_{\mu,A} &= (\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab})\epsilon + \frac{e^\Phi}{8}\not{F}\Gamma_\mu\mathcal{P}_{p+2}\epsilon \\ 0 = \delta\lambda &= \not{\partial}\Phi\epsilon + \frac{e^\Phi}{4}(-1)^p(3-p)\not{F}\mathcal{P}_{p+2}\epsilon \end{aligned}$$

Here we are took what was written the democratic formulation of Kiritsis Appendix **I.4**, setting all fields equal to zero except for the dilaton and relevant RR $p+2$ field strength. The extra factor of two in the last terms on both lines comes from counting $\hat{F}_n \cdot \hat{F}_n$ and $\hat{F}_{10-n} \cdot \hat{F}_{10-n}$ on equal footing.

As written there, for IIA we have $\mathcal{P}_n = (\Gamma_{11})^{n/2}\sigma^1$ and for IIB we have $\mathcal{P}_n = \sigma^1$ for $\frac{1+n}{2}$ even and $i\sigma^2$ for $\frac{1+n}{2}$ odd.

Let's first look at the dilatino variation. We get [4]

$$\begin{aligned} &(p-3)A'\,\Gamma^r\epsilon \pm e^{A(3-p)}(-1)^p(p-3)A'\,e^{4A}\Gamma^{r0\dots p}\mathcal{P}_{p+2}\epsilon = 0 \\ &\Rightarrow (1 \pm (-1)^p e^{A(1+p)}\Gamma^{01\dots p}\mathcal{P}_{p+2})\epsilon = 0 \\ &\Rightarrow (1 \pm (-1)^p\Gamma^{\hat{0}\hat{1}\dots\hat{p}}\mathcal{P}_{p+2})\epsilon = 0 \end{aligned}$$

Here the Γ matrices with hatted (vielbein) indices are the familiar 10D Dirac matrices, as in Freedman and Van Proyen *Supergravity.* We need our constant of proportionality $\kappa = 2$ in order for the above combination of matrices to have a nontrivial null space. Note we could inversely have taken this as a way to take the profile of $\Phi = e^{(p-3)A}$ and get the profile of F to be $\pm 4A'e^{4A}$.

Locally, then, this is a linear algebraic constraint on the space of spinors at a given point, which half of the spinors will satisfy.

Now let's look at *longitudinal* the gravitino variation. Similar to the case of the RN black hole, we expect $\partial_\mu\epsilon = 0$ since ∂_μ in the longitudinal direction is Killing.

$$\begin{aligned} &\partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\epsilon \mp \frac{e^{(p-3)A}}{8}4A'e^{4A}\Gamma^{r0\dots p}\Gamma_\mu\mathcal{P}_{p+2}\epsilon = 0 \\ \Rightarrow &-\frac{1}{2}e^{A-B}A'\,\Gamma_{\hat{r}\hat{\mu}}\epsilon \mp \frac{1}{2}e^{(p+1)A}A'\Gamma^{r0\dots p}\Gamma_\mu\mathcal{P}_{p+2}\epsilon = 0 \\ \Rightarrow &-\frac{1}{2}A'\,\Gamma_{\hat{r}\hat{\mu}}\epsilon \mp \frac{1}{2}A'\Gamma^{\hat{r}\hat{0}\dots\hat{p}}\Gamma_{\hat{\mu}}\mathcal{P}_{p+2}\epsilon = 0 \\ \Rightarrow &\Gamma_{\hat{\mu}}\epsilon \pm \Gamma^{\hat{0}\dots\hat{p}}\Gamma_{\hat{\mu}}\mathcal{P}_{p+2}\epsilon = 0 \\ \Rightarrow &(1 \pm (-1)^p\Gamma^{\hat{0}\dots\hat{p}}\mathcal{P}_{p+2})\epsilon = 0 \end{aligned}$$

This is exactly the same constraint as the one that the dilatino gave us. This also directly confirms our assumption: $\partial_\mu\epsilon = 0$ longitudinally, since we can subtract the dilatino variation from the above gravitino one.

Meanwhile in the *transverse* directions we no longer expect $\partial_i\epsilon = 0$. It is important to note that the components of the spin connection $\omega_i{}^{ab}\Gamma_{ab}$ will only be nonvanishing for $a, b = \{j, k\}$ being transverse coordinates, in which case Γ_{jk} is proportional to the infinitesimal rotation generator. By assumption of spherical symmetry (just as in the RN case of problem **35**), this must vanish $\Gamma_{jk}\epsilon = 0$.

$$\begin{aligned} 0 &= \partial_r\epsilon + \cancel{\frac{1}{4}\omega_r^{ab}\Gamma_{ab}}\epsilon \pm \frac{e^{A(p-3)}}{8}4A'e^{4A}\Gamma^{r0\dots p}\Gamma_r\mathcal{P}_{p+2}\epsilon \\ &= \partial_r\epsilon \pm \frac{1}{2}A'\,\Gamma^{\hat{r}\hat{0}\dots\hat{p}}\Gamma_r\mathcal{P}_{p+2}\epsilon \\ &= \partial_r\epsilon \pm (-1)^{p+1}\frac{1}{2}A'\,\Gamma^{\hat{0}\dots\hat{p}}\mathcal{P}_{p+2}\epsilon \\ &= \partial_r\epsilon - \frac{1}{2}A'\epsilon \Rightarrow \epsilon = e^{A/2}\epsilon_0 \end{aligned}$$

[4] I have set $g_s = 1$ for all of this. I don't understand how any of this could work without being modified for arbitrary g_s.

where ϵ_0 is a constant spinor satisfying the linear algebraic constraints previously given.

We thus have that indeed our configuration is half-BPS.

44. We write again the spin connection found in the last problem:

$$e^{\hat{\mu}} = e^A dx^\mu, \quad e^{\hat{i}} = e^B dx^i$$

Hatted indices always denote the vielbein indices.

Then

$$\mathrm{d}e^{\hat{\mu}} = \partial_r A e^A \mathrm{d}r \wedge \mathrm{d}x^\mu = \sum_i \partial_i A e^A \mathrm{d}x^i \wedge dx^\mu = e^{\hat{i}} \wedge \omega^{\hat{\mu}}_{\ \hat{i}} \Rightarrow \omega_{\hat{\mu}\hat{\nu}} = 0,\ \omega_{\hat{\mu}\hat{i}} = (-)^{\mu=0} \partial_i A\, e^{A-B} \mathrm{d}x^\mu$$

$$\mathrm{d}e^{\hat{i}} = \partial_r B e^B \mathrm{d}r \wedge \mathrm{d}x^i = \sum_j \partial_i B e^B \mathrm{d}x^j \wedge dx^i = e^{\hat{j}} \wedge \omega^{\hat{i}}_{\ \hat{j}} \Rightarrow \omega_{\hat{i}\hat{j}} = \partial_j B \mathrm{d}x^i - \partial_i B \mathrm{d}x^j$$

From this, we can get the Riemann curvature using $\mathbf{R}_{\hat{\alpha}\hat{\beta}} = \mathrm{d}\boldsymbol{\omega}_{\alpha\beta} + \boldsymbol{\omega}_{\alpha\gamma} \wedge \boldsymbol{\omega}^{\gamma}_{\hat{\beta}}$. First $\mathbf{R}_{\hat{\mu}\hat{\nu}}$ is the easiest:

$$\mathbf{R}_{\hat{\mu}\hat{\nu}} = \cancel{\mathrm{d}\omega_{\hat{\mu}\hat{\nu}}} + \omega_{\hat{\mu}\hat{i}} \wedge \omega^{\hat{i}}_{\ \hat{\nu}} = e^{2(A-B)} (\partial A)^2 dx^\mu \wedge dx^\nu = \mathbf{R}_\mu^{\ \nu}$$

Note that last expression is unhatted.

Next is $\mathbf{R}_{\hat{\mu}\hat{i}}$

$$\begin{aligned}
\mathbf{R}_{\hat{\mu}\hat{i}} &= \mathrm{d}\omega_{\hat{\mu}\hat{i}} & &+\omega_{\hat{\mu}\hat{j}} \wedge \omega^{\hat{j}}_{\ \hat{\nu}} \\
&= [\partial_j \partial_i A e^{A-B} + \partial_i A(\partial_j A - \partial_j B)] dx^j \wedge dx^\mu & &-\partial_i A e^{A-B} \partial_j B dx^\mu \wedge dx^i + \partial_j A e^{A-B} \partial_i B dx^\mu \wedge dx^j \\
&\quad = e^{A-B} [(\partial_i \partial_j A + \partial_i A \partial_j A - \partial_i B \partial_j A - \partial_j B \partial_i A)] dx^j \wedge dx^\mu - e^{A-B} \partial_j A \partial_j B dx^\mu \wedge dx^i
\end{aligned}$$

We can get $R_\mu^{;i}$ (note unhatted) by multiplying this by e^{A-B} and $R_i^{;\mu}$ by multiplying this by $-e^{B-A}$.

Finally $\mathbf{R}_{\hat{i}\hat{j}}$:

$$\begin{aligned}
\mathbf{R}_{\hat{i}\hat{j}} &= \mathrm{d}\omega_{\hat{i}\hat{j}} \qquad + \cancel{\omega_{\hat{i}\hat{\mu}} \wedge \omega^{\hat{\mu}}_{\ \hat{j}}} \qquad +\omega_{\hat{i}\hat{k}} \wedge \omega^{\hat{k}}_{\ ;\hat{j}} \\
&= \partial_k \partial_j B dx^k \wedge dx^i - \partial_k \partial_i B dx^k \wedge dx^j \quad +\partial_k B \partial_j B dx^i \wedge dx^k - \partial_k B \partial_i B dx^j \wedge dx^k - (\partial B)^2 dx^i \wedge dx^j \\
&\quad = -(\partial B)^2 dx^i \wedge dx^j + (\partial_k \partial_i B - \partial_k B \partial_i B) dx^j \wedge dx^k - (\partial_k \partial_j B - \partial_k B \partial_j B) dx^i \wedge dx^k = \mathbf{R}_i^{;j}
\end{aligned}$$

To evaluate $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ amounts to summing the squares of all the entries in the curvature two form when expressed in only vielbein indices. We can do this in Mathematica:

I can't get c_+, c_- exactly right. The best attempt is in *"exact p brane solutions.nb"*. The general L and r dependence in both cases matches though, and I'm not getting any $p-3$ factors, so I can believe this result.

To get the Ricci tensor, we must to the appropriate contractions. Importantly, if a longitudinal index must be summed over this gives an extra factor of $p+1$ while if a transverse index must be summed over this gives an extra factor of $9-p$.

$$R_{\mu\nu} = R_{\mu\rho\nu}^{\ \ \ \rho} + R_{\mu i \nu}^{\ \ \ i} = -\eta_{\mu\nu} e^{2(A-B)} \Big((p+1)(A')^2 + A'' + \frac{8-p}{r} A' + A'B'(9-p-2) \Big)$$

Here $A' = \partial_r A$ is differentiation with respect to the radial coordinate. The Ricci tensor has no components mixing transverse and longitudinal directions:

$$R_{\mu i} = \cancel{R_{\mu\rho i}^{\ \ \ \rho}} + \cancel{R_{\mu j i}^{\ \ \ j}} = 0$$

Finally the annoying one, for which I looked at Stelle's *Lectures on p-Branes 9701088* :

$$\begin{aligned}
R_{ij} = R_{i\mu j}^{\ \ \ \mu} + R_{ikj}^{\ \ \ k} =& -\delta_{ij} \left(B'' + (p+1)A'B' + (7-p)(B')^2 + \tfrac{2(7-p)+1}{r} B' + \tfrac{d}{r} A' \right) \\
&+ \tfrac{x^i x^j}{r^2} \left((7-p)B'' - \tfrac{7-p}{r} B' + (p+1)A'' - \tfrac{p+1}{r} A' - 2(p+1)A'B' + (p+1)(A')^2 - (7-p)(B')^2 \right)
\end{aligned}$$

In this last part I rewrote $\partial_i = \frac{x^i}{r} \partial_r$.

We can evaluate this directly in Mathematica. For $R_{\mu\nu} R^{\mu\nu}$ and R we get:

```
In[391]:= H[r_] := 1 + L^(7-p)/r^(7-p);
A[r_] := -1/4 Log[H[r]]
B[r_] := 1/4 Log[H[r]]
Rμν = -Exp[2 (A[r] - B[r])] (A''[r] + (p + 1) A'[r]^2 + (7 - p) A'[r] × B'[r] + (8 - p)/r A'[r]) // FullSimplify;
Rmn = -(B''[r] + (p + 1) A'[r] × B'[r] + (7 - p) (B'[r])^2 + (2 (7 - p) + 1)/r B'[r] + (p + 1)/r A'[r]) -
   1/(9 - p) ((7 - p) B''[r] + (p + 1) A''[r] - 2 (p + 1) A'[r] × B'[r] + (p + 1) (A'[r])^2 - (7 - p) (B'[r])^2 -
      (7 - p)/r B'[r] - (p + 1)/r A'[r]) // FullSimplify;

In[396]:= main = -(B''[r] + (p + 1) A'[r] × B'[r] + (7 - p) (B'[r])^2 + (2 (7 - p) + 1)/r B'[r] + (p + 1)/r A'[r]);
offdiag =
  -((7 - p) B''[r] + (p + 1) A''[r] - 2 (p + 1) A'[r] × B'[r] + (p + 1) (A'[r])^2 - (7 - p) (B'[r])^2 -
     (7 - p)/r B'[r] - (p + 1)/r A'[r]) // FullSimplify;
RicciSquared = (p + 1) (Exp[-2 A[r]])^2 Rμν^2 + (Exp[-2 B[r]])^2 ((9 - p) main^2 + 2 main * offdiag + offdiag^2) //
   FullSimplify
R = (p + 1) Exp[-2 A[r]] Rμν + Exp[-2 B[r]] ((9 - p) main + offdiag) // FullSimplify

Out[398]= 1/(32 (L^p r^7 + L^7 r^p)^5) L^(14+p) (-7 + p)^2 r^(3+2p) (8 L^(2p) (-9 + p) (-8 + p) (-3 + p)^2 r^14 +
    L^14 (1 + p) (137 + p (-1 + (-9 + p) p)) r^(2p) - 8 L^(7+p) (-8 + p) (-5 + p) (-3 + p) (1 + p) r^(7+p))

Out[399]= -(L^14 (-7 + p)^2 (-3 + p) (1 + p) r^(-2+2p))/(4 Sqrt[1 + L^(7-p) r^(-7+p)] (L^p r^7 + L^7 r^p)^2)
```

The last line is in agreement with the expression for R in Kiritsis **8.8.31**

45. Exercise 7.7 shows that, upon T-dualizing along the x^9 direction we get

$$\tilde{C}^{(p+1)}_{\mu_1\ldots\mu_p 9} = C^{(p)}_{\mu_1\ldots\mu_p}, \quad \tilde{C}^{(p)}_{\mu_1\ldots\mu_p} = \tilde{C}^{(p+1)}_{\mu_1\ldots\mu_p 9}$$

In transverse space, our $(p+1)$-form C has components only along the longitudinal directions. Upon T-dualizing, we pick up the 9 index in the C form, and thus get that our brane has a $(p+2)$ form charge. We thus expect this to be a $p+1$ brane wrapping that additional x^9 direction. I'm unsure if this wants us to explicitly give the form of that solution, since doing it in a compact space seems a bit harder.

46. Let's assume $p < 7$. When $\lambda \gg 1$ the perturbative stringy description is no longer valid. For an extremal p-brane, we know from problem 40 that:

$$L^{7-p} = \frac{2\kappa_{10}^2 T_p N}{(7-p)\Omega_{8-p}} \Rightarrow \left(\frac{L}{2\pi\ell_s}\right)^{7-p} = \frac{g_s N}{7-p}\frac{\Gamma(\frac{9-p}{2})}{2\pi^{\frac{9-p}{2}}}$$

So $\lambda = 2\pi g_s N \gg 1$ gives that $L \gg \ell_s$, meaning that the throat size is macroscopic. We can thus probe it without having to see distances smaller than the string scale.

When $p > 3$ we see from our calculation of R in problem **44** that R blows up as $\frac{L^{2(7-p)}}{r^{(p-3)/2}}$ as $r \to 0$. This will become order ℓ_s^{-2} at

$$r \approx \left(\frac{\ell_s^2}{L^{(7-p)/2}}\right)^{2/(p-3)}$$

When $p < 3$ the formula for R indeed is seen to go to zero. On the other hand the string coupling grows as

$$e^\Phi = g_s H^{(3-p)/4} = g_s\left(1 + \frac{L^{7-p}}{r^{7-p}}\right)^{(3-p)/4}$$

So if g_s is the string coupling "at infinity" which we can take to initially be small, then it will become appreciable at

$$r = L(-1 + g_s^{-4/(3-p)})^{1/(p-7)}$$

so for g_s sufficiently small, the quantity in parentheses will be quite large and be raised to a negative power, so that this is a small fraction of the throat size.

47. This is only a slight variant of exercises **8.38-9**, and in fact is a bit easier. Our action is

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{2\cdot 3!}(dB)^2\right]$$

Let's first vary Φ. We get

$$\begin{aligned}
0 &= -2e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{2\cdot 3!}(dB)^2\right] - \nabla(e^{-\Phi}8\nabla\Phi) \\
&= -2e^{-2\Phi}R - 8e^{-2\Phi}(\nabla\Phi)^2 + \frac{e^{-2\Phi}}{3!}(dB)^2 - 8e^{-2\Phi}\Box\Phi + 16(\nabla\Phi)^2 e^{-2\Phi} \\
\Rightarrow & R = 4(\nabla\Phi)^2 - 4\Box\Phi + \frac{(dB)^2}{2\cdot 3!}
\end{aligned}$$

Next for the B-field we will just have

$$\mathrm{d}\star e^{-2\Phi}dB = 0$$

Finally, varying g is the hardest, but we've done most of the work already in the other problem:

- $\sqrt{-g}e^{-2\Phi}R$

$$\begin{aligned}
&\to (R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)e^{-2\Phi} - \frac{1}{2}g_{\mu\nu}e^{-2\Phi}R \\
&= e^{-2\Phi}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}(-2\Box\Phi + 4(\partial\Phi)^2) - (-2\nabla_\mu\nabla_\nu\Phi + 4\partial_\mu\Phi\partial_\nu\Phi)\Big)
\end{aligned}$$

- $\sqrt{-g}e^{-2\Phi}4g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \to 4e^{-2\Phi}\partial_\mu\Phi\partial_\nu\Phi - 2e^{-2\Phi}(\partial\Phi)^2$
- $-\frac{e^{-2\Phi}}{2(p+2)!}\sqrt{-g}g^{\mu_1\nu_1}\ldots g^{\mu_{p+2}\nu_{p+2}}F_{\mu_1\ldots\mu_{p+2}}F_{\nu_1\ldots\nu_{p+2}} \to -\frac{e^{-2\Phi}}{2(p+1)!}F_{\mu\nu}^2 + \frac{e^{-2\Phi}}{4(p+2)!}g_{\mu\nu}F^2$. Here $F_{\mu\nu}^2 = F_{\mu\ldots}F^{\nu\ldots}$

Combining these all together, dividing through by $e^{-2\Phi}$ and using the dilaton equations of motion gives

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi = \frac{1}{2\cdot 2!}H_{\mu\nu}^2$$

Here $H_{\mu\nu} = H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}$ as we've had before (i.e. in chapter 6).

Ok next let's take the ansatz as in Kiritsis:

$$ds^2 = -f(R)dt^2 + dx_i^2 + H(r)\left(\frac{dr^2}{f(r)} + r^2 d\Omega_3^2\right).$$

With dilaton

$$e^{2\Phi} = g_s^2 H(r)$$

The field strength written is wrong (as you can see by noting that as $r \to \infty$ the magnetic flux integral goes to zero). We can find the correct expression by noting that $\mathrm{d}\star\mathrm{d}B = 0$ trivially since $\star\mathrm{d}B$ has a $\mathrm{d}r$ component. The only nontrivial equation is the Bianchi identity, giving (by spherical symmetry)

$$dB = 0 \Rightarrow B = c\omega$$

for $\omega = d\psi \wedge \sin\psi d\theta \wedge \sin\psi\sin\theta d\phi$ the unit volume form on the sphere. Let's see what this constant c should be from the dilaton equations. We get

```
In[1703]:= xx = {t, r, ψ, θ, ϕ};
           H[r_] := 1 + L^2/r^2
           f[r_] := 1 - r0^2/r^2
           g = {{-f[r], 0, 0, 0, 0}, {0, H[r]/f[r], 0, 0, 0}, {0, 0, H[r] r^2, 0, 0},
              {0, 0, 0, H[r] r^2 Sin[ψ]^2, 0}, {0, 0, 0, 0, H[r] r^2 Sin[ψ]^2 Sin[θ]^2}};
           ginv = InverseMetric[g];
           R = RicciScalar[g, xx];
           Φ[r_] := 1/2 Log[H[r]]
           dΦ2 = ginv[[2, 2]] D[Φ[r], r]^2 // FullSimplify;
           d2Φ = 1/(H[r]^2 r^3) D[H[r]^2 r^3 ginv[[2, 2]] D[Φ[r], r], r] // FullSimplify;
           dB = c Sin[ψ]^2 Sin[θ] // FullSimplify;
           R - (4 dΦ2 - 4 d2Φ + 1/2 ginv[[3, 3]] ginv[[4, 4]] ginv[[5, 5]] dB^2) // FullSimplify

Out[1713]= -(c^2 - 4 L^2 (L^2 + r0^2))/(2 (L^2 + r^2)^3)
```

So when $c = -2L\sqrt{1 + r_0^2/L^2}$ we get our dilaton.

By Hodge-dualizing, this also gives credibility for the $\sqrt{1 - r_0^2/L^2}$ constant in the p-brane solution, which would have required the more complicated $R_{\mu\nu}$ equation.

Finally the least trivial equation of motion is also straightforward:

```
In[1778]:= RicciTensor[g, xx] +
              2 Table[D[D[Φ[r], xx[[i]]], xx[[j]]] - Sum[Γ[[k, i, j]] D[Φ[r], xx[[k]]], {k, 1, 5}],
                {i, 1, 5}, {j, 1, 5}] -
              1/2 {{0, 0, 0, 0, 0},
                   {0, 0, 0, 0, 0},
                   {0, 0, ginv[[4, 4]] ginv[[5, 5]], 0, 0},
                   {0, 0, 0, ginv[[3, 3]] ginv[[5, 5]], 0},
                   {0, 0, 0, 0, ginv[[3, 3]] ginv[[4, 4]]}} dB^2 //
             FullSimplify // MatrixForm

Out[1778]//MatrixForm=
    0 0 0 0 0
    0 0 0 0 0
    0 0 0 0 0
    0 0 0 0 0
    0 0 0 0 0
```

48. Let's review the extremal near horizon limit first. There, when $r \ll L$ we can just write

$$ds^2 = -dt^2 + d\vec{x}\cdot d\vec{x} + L^2\frac{dr^2}{r^2} + L^2 d\Omega_3^2$$

Defining $\gamma = \sqrt{N}\ell_s \log\frac{r}{g_s\ell_s\sqrt{N}}$ gives $d\gamma^2 = L^2/r^2$ giving

$$ds^2 = -dt^2 + d\vec{x}\cdot d\vec{x} + d\gamma^2 + N\ell_s^2 d\Omega_3^2$$

This looks like flat space times a constant-radius sphere with a linear dilaton background going as $\Phi = \gamma/\sqrt{N}\ell_s$

Next let's look at the near-extremal case. We take $r = r_0\cosh\sigma$ so that $f(r) = 1 - r_0^2/r^2 = \tanh^2\sigma$. Meanwhile

$$\frac{H(r)}{f(r)}dr^2 = \left(1 + \frac{L^2}{r_0^2\cosh\sigma}\right)\frac{r_0^2\sinh^2\sigma d\sigma^2}{\tanh^2\sigma} = L^2 + r_0^2\cosh\sigma^2 = H(r)r^2$$

So we get a metric

$$-\tanh^2\sigma dt^2 + d\vec{x}\cdot d\vec{x} + (N\ell_s^2 + r_0^2\cosh^2\sigma)(d\sigma^2 + d\Omega_3^2)$$

At large N, rescaling t this looks like

$$-\tanh^2\sigma N\ell_s^2 dt^2 + d\vec{x}\cdot d\vec{x} + N\ell_s^2(d\sigma^2 + d\Omega_3^2),$$

which looks like a 2D black hole solution in σ, t space, after rescaling

49. Let's write the spin connection. Take $e^{2A} = H(r)$ so that $\phi - \phi_0 = A$. Our frame fields look like:

$$e^{\hat{\mu}} = dx^\mu, \quad e^{A(r)}dx^i$$

for μ parallel and i transverse. It looks like $\omega_{\mu\nu} = \omega_{\mu i} = 0$ while

$$\omega_{\hat{i}\hat{j}} = -\partial_j A dx^i + \partial_i A dx^j$$

similar to what we had before.

We again write the gravitino and dilatino variation in 10D type II SUGRA, neglecting this time the RR forms but incorporating the N 2-form contribution:

$$\begin{aligned} 0 = \delta\psi_{\mu,A} &= (\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab})\epsilon + \frac{1}{4}\not{H}_\mu\mathcal{P}\epsilon \\ 0 = \delta\lambda &= \not{\partial}\Phi\epsilon + \frac{1}{2}\not{H}\mathcal{P}\epsilon \end{aligned}$$

Here $\mathcal{P} = \Gamma^{11}\otimes 1_2$ in type IIA and $-1_{32}\otimes\sigma^3$ in type IIB.

The dilatino variation gives

$$\begin{aligned} &\partial_r\phi\Gamma^r\epsilon \pm \frac{1}{2}(-2L^2)\sin^2\psi\sin\theta\Gamma^{\psi\theta\phi}\mathcal{P}\epsilon \\ &= \frac{H'}{2H}\Gamma^r\epsilon \mp \frac{L^2}{H^{3/2}r^3}\Gamma^{\hat{\psi}\hat{\theta}\hat{\phi}}\mathcal{P}\epsilon \\ &= \frac{H'}{2H^{3/2}}\Gamma^{\hat{r}}\epsilon \mp \frac{L^2}{H^{3/2}r^3}\Gamma^{\hat{\psi}\hat{\theta}\hat{\phi}}\mathcal{P}\epsilon \\ &\Rightarrow -L^2(1 \pm \Gamma^{\hat{r}\hat{\psi}\hat{\theta}\hat{\phi}}\mathcal{P})\epsilon = 0 \end{aligned}$$

This is an algebraic constraint that is satisfied by half the space of spinors at any given point. This makes the solution half-BPS, so long as the profile of ε can be chosen so that the gravitino vanishes.

The $\delta\psi_\mu$ variation longitudinal to the solution is trivial. The transverse variation is

$$(\partial_i + \frac{1}{4}\omega_{ijk}\Gamma^{jk})\epsilon_i + \frac{1}{4}H_{ijk}\Gamma^{jk}\mathcal{P}\epsilon$$

Crucially, though, Γ^{jk} is the generator of rotations. By rotational symmetry we thus reduce this to $\partial_i\epsilon = 0$, implying that $\epsilon(r) = \epsilon_0$ is a constant spinor.

It is also worth noting that transverse to the NS5 brane is precisely the extremal BH solution in 5D, which preserves half SUSY by the same arguments as before. Parallel to it is flat space (which preserves all SUSY). The product spacetime therefore preserves half.

50. We have the same equations as when we were solving the for the NS5 brane. This time, the $\mathrm{d}e^{-2\Phi}\star dB$ constraint is nontrivial, and we must have a field strength. Because the field is electrically charged under the field, I expect

$$B \sim H^{-1}(r)$$

For $H = 1 + \frac{L^6}{r^6}$ the relevant harmonic form in transverse space. I don't have much justification for this other than the fact that - in every problem I've seen this seems to hold true. Now let's take the ansatz that the metric and dilaton look like

$$ds^2 = H^\alpha(-dt^2 + dx_1^2) + H^\beta d_\perp x^2, \qquad e^\Phi = H^\gamma$$

Then $\sqrt{-g} = H^{\alpha+4\beta}$ and we get

$$e^{-2\Phi} \star dB = H^{-\alpha+3\beta-2\gamma-2} r^7 H'(r)$$

We want $\mathrm{d}e^{-2\Phi} \star \mathrm{d}B = 0$ so we must have

$$-\alpha + 3\beta - 2\gamma - 2 = 0$$

The simplest guess would be $\alpha = -1, \gamma = -1/2$. This turns out to work. First look at the dilaton EOM:

```
In[2158]:= xx = {t, x1, r, θ1, θ2, θ3, θ4, θ5, θ6, θ7};
g = {{-H[r]^-1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, H[r]^-1, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
   {0, 0, 0, r^2, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, r^2 Sin[θ1]^2, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, r^2 Sin[θ1]^2 Sin[θ2]^2, 0, 0, 0, 0},
   {0, 0, 0, 0, 0, 0, r^2 Sin[θ1]^2 Sin[θ2]^2 Sin[θ3]^2, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, r^2 Sin[θ1]^2 Sin[θ2]^2 Sin[θ3]^2 Sin[θ4]^2, 0, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, r^2 Sin[θ1]^2 Sin[θ2]^2 Sin[θ3]^2 Sin[θ4]^2 Sin[θ5]^2, 0},
   {0, 0, 0, 0, 0, 0, 0, 0, 0, r^2 Sin[θ1]^2 Sin[θ2]^2 Sin[θ3]^2 Sin[θ4]^2 Sin[θ5]^2 Sin[θ6]^2}};
Γ = ChristoffelSymbol[g, xx];
R = RicciScalar[g, xx]

Out[2161]= -(126 L^12)/(r^2 (L^6 + r^6)^2)
```

```
In[2129]:= Φ[r_] := -1/2 Log[H[r]]
dΦ2 = D[Φ[r], r]^2 // FullSimplify;
d2Φ = 1/(H[r]^-1 r^7) D[H[r]^-1 r^7 D[Φ[r], r], r] // FullSimplify;
dB = -H'[r]/H[r]^2 // FullSimplify;
R - (4 dΦ2 - 4 d2Φ + 1/2 ginv[[1, 1]] ginv[[2, 2]] dB^2) // FullSimplify

Out[2133]= 0
```

Next, look at the metric's EOM

```
In[2148]:= (RicciTensor[g, xx] +
    2 Table[D[D[Φ[r], xx[[i]]], xx[[j]]] - Sum[Γ[[k, i, j]] D[Φ[r], xx[[k]]], {k, 1, 10}], {i, 1, 10}, {j, 1, 10}]) // FullSimplify //
  MatrixForm
```

In[2153]:=

$$\begin{pmatrix}
\frac{18 L^{12} r^4}{(L^6+r^6)^3} - \frac{1}{2}\,\text{ginv}[\![2,2]\!]\,\text{ginv}[\![3,3]\!]\,\text{dB}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{18 L^{12} r^4}{(L^6+r^6)^3} - \frac{1}{2}\,\text{ginv}[\![1,1]\!]\,\text{ginv}[\![3,3]\!]\,\text{dB}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{18 L^{12}}{r^2 (L^6+r^6)^2} - \frac{1}{2}\,\text{ginv}[\![1,1]\!]\,\text{ginv}[\![2,2]\!]\,\text{dB}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} //$$

```
FullSimplify

Out[2153]= {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

Perhaps the easier thing to do was look for a BPS solution. In either case we are done. My (reasonable) guess for the non-extremal version of this would be to keep the dilaton and NS field the same and modify the metric to be

$$H^{-1}(r)(-f(r)dt^2 + dx_1^2) + \frac{dr^2}{f(r)} + r^2 d\Omega_7^2$$

where $f(r) = 1 - \frac{r_0^6}{r^6}$. I have not checked this, but it seems right based on experience at this point. s

51. Orbifolding by this symmetry indeed leaves invariant a 5 plane given by $x_0 \dots x_5$ where $x_6 \dots x_9$ were the coordinates that were reflected under the orbifold. The twisted sector must thus localize on that invariant 5-plane.

I will come back to this problem - it should not be too difficult once I know what I'm doing. In some way this is relating the NS5 brane wordvolume theory to the theory of type I strings

The IIB NS5 brane has $(1,1)$ SUSY and contains a vector multiplet

The IIA NS5 brane has $(2,0)$ SUSY and contains a *tensor* multiplet with *six* scalars.

Chapter 9: Compactification and Supersymmetry Breaking

In collaboration with Alek Bedroya

1. We compactify the heterotic string along just one dimension, making it a compact circle of radius R with all 16 Wilson lines turned on.

   Each noncompact boson contributes
$$\frac{1}{\sqrt{\tau_2}\eta\bar{\eta}}$$
   The fermions on the supersymmetric side contribute
$$\sum_{a,b=0}^{1}(-1)^{a+b+ab}\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}^4}{\eta^4}$$
   The (p,p) compact bosons and 16 complex right-moving fermions that can be written as the pair $\psi^I(\bar{z}), \bar{\psi}^I(\bar{z})$ have the action as in **E.1** (setting $\ell_s = 1$)
$$\frac{1}{4\pi}\int d^2\sigma\sqrt{\det g}g^{ab}G_{\alpha\beta}\partial_a X^\alpha\partial_b X^\beta + \frac{1}{4\pi}\int d^2\sigma\epsilon^{ab}B_{\alpha\beta}\partial_a X^\alpha\partial_b X^\beta + \frac{1}{4\pi}\int d^2\sigma\sqrt{-\det g}\sum_I \psi^I[\bar{\nabla} + Y^I_\alpha\bar{\partial}X^\alpha]\bar{\psi}^I$$
   Here α, β are the toral coordinates for the compact spacetime and Y^I_α is the Wilson line along torus cycle α. To evaluate the path integral, as we did in the purely bosonic case, we have a factor of
$$\frac{\sqrt{\det G}}{\tau_2^{p/2}(\eta\bar{\eta})^p}$$
   coming from evaluating the determinant $(\det \nabla^2)^{-1/2}$ of the bosons. This multiplies a sum over instanton contributions labelled by m^α, n^α taking values in a (p,p)-signature lattice with classical action
$$\sum_{m^\alpha, n^\alpha} e^{-\frac{\pi}{\tau}(G+B)_{\alpha\beta}(m+\tau n)^\alpha(m+\bar{\tau}n)^\beta} \times \text{fermions}.$$
   The fermion contribution depends via the Wilson lines on the configuration of the X^α. In each such instanton sector, the fermion path integral with a constant background Wilson line is equivalent to a free fermion with twisted boundary conditions. For simplicity, let's compactify just on S^1, and denote $\theta^I = Y^I n, \phi^I = -Y^I m$. We get boundary conditions:
$$\psi^I(\sigma+1, \sigma_2) = -(-1)^a e^{2\pi i\theta^I}$$
$$\psi^I(\sigma, \sigma_2+1) = -(-1)^b e^{-2\pi i\phi^I}$$
   where $a, b = 0, 1$ denotes anti-periodic/periodic boundary conditions respectively. We know that (in the absence of Wilson lines) the determinant of ∂ acting on complex fermions is:
$$\det_{a,b}\partial = \frac{\theta\begin{bmatrix}a\\b\end{bmatrix}}{\eta}$$
   Let us now investigate the twisted boundary conditions. For simplicity its enough to take $a = b = 0$ (all antiperiodic). We have two different ways to write the partition function. As a product over modes, we have $\psi_m, \bar{\psi}_m$ modes, with respective weights $m - \frac{1}{2} - \theta, m - \frac{1}{2} + \theta$ **Check against Polch 16.1.16** and respective fermion numbers ± 1 *relative to the ground state.* The fermion number of the ground state has no canonical value (as far as I can see). On the other hand, the ground state energy is given by the standard mneumonic to be $-\frac{1}{24} + \frac{1}{2}\theta^2$. This gives:
$$\mathrm{Tr}_\theta[e^{2i\pi\phi F}q^H] = q^{\frac{\theta^2}{2}-\frac{1}{24}}\prod_{m=1}^{\infty}(1+q^{m-1/2+\theta}e^{2\pi i\phi})(1+q^{m-1/2-\theta}e^{-2\pi i\phi}) = q^{\theta^2/2}\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(\phi+\theta\tau|\tau)}{\eta}$$

For other boundary conditions, we can apply the same logic to get

$$q^{\theta^2/2}\frac{\theta\left[{a \atop b}\right](\phi+\theta\tau|\tau)}{\eta}$$

The overall phase is still a mystery. Writing $\theta\left[{a \atop b}\right]\left[{\theta \atop \phi}\right]$ as a new theta function, we can fix the phase by requiring modular invariance

$$\begin{aligned}
&\theta\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix}(\tau+1)=\theta\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}\theta\\\phi+\theta\end{bmatrix}(\tau) \qquad &&\theta\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix}(\tau+1)=\theta\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}\theta\\\phi+\theta\end{bmatrix}(\tau)\\
&\theta\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix}(\tau+1)=e^{i\pi/4}\theta\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}\theta\\\phi+\theta\end{bmatrix}(\tau) \qquad &&\theta\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix}(\tau+1)=e^{i\pi/4}\theta\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}\theta\\\phi+\theta\end{bmatrix}(\tau)
\end{aligned} \tag{79}$$

Even from the first of these conditions, we see that we need a term going as $e^{i\theta\phi}$ out front. After adding this in, all other transformations will hold automatically. The $\tau\to -1/\tau$ transformation will thus hold automatically. **Interpret this as an anomaly? Yes, Narain, Witten do this in Section 3 of their paper. It seems careful anomaly analysis is not enough and one must indeed impose modular invariance by hand.**

Altogether then the 16 complex antiholomorphic fermions contribute in each instanton sector:

$$e^{-i\pi\sum_I\theta^I(\phi^I+\bar\tau\theta^I)}\frac{1}{2}\sum_{a,b=0}^{1}\prod_{i=1}^{16}\frac{\bar\theta\left[{a \atop b}\right](\phi+\bar\tau\theta|\bar\tau)}{\bar\eta}$$

Giving a total partition function as in the second (unnumbered) equation of **Appendix E**:

$$\left[\frac{R}{\sqrt{\tau_2}\eta\bar\eta^{17}}\sum_{m,n}e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\underline{e^{-i\pi\sum_I nY^I(m+n\bar\tau)Y^IY^I}\frac{1}{2}\sum_{a,b=0}^{1}\prod_{i=1}^{16}\bar\theta\begin{bmatrix}a\\b\end{bmatrix}(Y^I(m+\bar\tau n)|\bar\tau)}\right]\times\frac{1}{\tau_2^{7/2}\eta^7\bar\eta^7}\frac{1}{2}\sum_{a,b=0}^{1}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4}$$

From the properties of the theta functions in Equation (79), the underlined fermionic sum has the exact same transformation properties as a sum of θ^{16} terms and thus makes the full partition function modular invariant.

Each theta function can be written in sum form as:

$$\theta\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}\theta\\\phi\end{bmatrix}=e^{\pi i\theta\phi}q^{\theta^2/2}\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}(n-\frac{a}{2})^2}e^{2\pi i(n-\frac{a}{2})(\phi+\tau\theta-\frac{b}{2})}=\sum_{n\in\mathbb{Z}}q^{\frac{1}{2}(n+\theta-\frac{a}{2})^2}e^{2\pi i\phi(n+\frac{1}{2}\theta-\frac{a}{2})-\pi ib(n-\frac{a}{2})}$$

Then we get the following expression for the underlined fermionic term:

$$\begin{aligned}
&\frac{1}{2}\sum_{a,b=0}^{1}\prod_{I=1}^{16}\sum_{k\in\mathbb{Z}}\bar q^{\frac{1}{2}(k+nY^I-\frac{a}{2})^2}e^{-2\pi imY^I(k+\frac{1}{2}nY^I-\frac{a}{2})+\pi ib(k-\frac{a}{2})}\\
&=\frac{1}{2}\sum_{a,b=0}^{1}\sum_{q^I\in\mathbb{Z}^{16}}\bar q^{\frac{1}{2}(q^I+nY^I-\frac{a}{2})^2}e^{-2\pi imY^I(q^I+nY^I-\frac{a}{2})+\pi ib(k-\frac{a}{2})}\\
&=\frac{1}{2}\sum_{q^I\in\mathbb{Z}^{16}}\left[\bar q^{\frac{1}{2}(q^I+nY^I)^2}e^{-2\pi imY^I(q^I+\frac{1}{2}nY^I)}(1+(-1)^{\sum_I q^I})+\bar q^{\frac{1}{2}(q^I+nY^I-\frac{1}{2})^2}e^{-2\pi imY^I(q^I+\frac{1}{2}nY^I-\frac{1}{2})}(1+(-1)^{\sum_I(q^I-\frac{1}{2})})\right]\\
&=\sum_{q^I\in\Lambda^{16}}q^{(q^I+nY^I)^2}e^{-2\pi imY^I(q^I+\frac{1}{2}nY^I)}
\end{aligned}$$

We note that the second-to last line is indeed the sum over the roots of $O(32)$ augmented with one of the spinor weight lattices. Altogether the compact dimensions contribute:

$$\frac{R}{\sqrt{\tau_2}\eta\bar\eta^{17}}\sum_{m\in\mathbb{Z},n\in\mathbb{Z},q^I\in\Lambda^{16}}\exp\left[\frac{\pi R^2}{\tau_2}(m+n\tau)(m+n\bar\tau)+\pi i\tau(q^I+nY^I)^2-2\pi imY^I(k+\frac{1}{2}Y^I)\right]$$

To put this whole thing into Hamiltonian form, we proceed as in the bosonic case and perform a Poisson summation over m. The terms that contribute are:

$$
\begin{aligned}
e^{-\frac{\pi R^2}{\tau_2}n^2\tau_1^2-n^2\pi R^2\tau_2}\sum_m e^{-\frac{\pi R^2}{\tau_2}m^2-2\pi i m Y^I(q^I+\frac{1}{2}nY^I)-i\frac{nR^2\tau_1}{\tau_2}} \\
&= e^{-\frac{\pi R^2}{\tau_2}n^2\tau_1^2-n^2\pi R^2\tau_2}\frac{\sqrt{\tau_2}}{R}\sum_m e^{-\frac{\pi\tau_2}{R^2}(m+Y^I(q^I+\frac{1}{2}nY^I)-in\frac{R^2\tau_1}{\tau_2})^2} \\
&= e^{-\frac{\pi R^2}{\tau_2}n^2\tau_1^2-n^2\pi R^2\tau_2}\frac{\sqrt{\tau_2}}{R}\sum_m e^{-\frac{\pi\tau_2}{R^2}(m+Y^I(q^I+\frac{1}{2}nY^I))^2+\pi R^2\frac{\tau_1^2}{\tau_2}n^2+2\pi i(m+q^I+\frac{1}{2}nY^I)n\tau_1} \\
&= e^{-n^2\pi R^2\tau_2}\frac{\sqrt{\tau_2}}{R}\sum_m e^{-\frac{\pi\tau_2}{R^2}(m+Y^I(q^I+\frac{1}{2}nY^I))^2+2\pi i(m+q^I+\frac{1}{2}nY^I)n\tau_1}
\end{aligned}
$$

Together with the other terms this gives us

$$
\begin{aligned}
&\frac{1}{\eta\bar{\eta}^{17}}\sum_{n,m,q^I} q^{\frac{1}{2}(q^I+nY^I)^2}e^{-n^2\pi R^2\tau_2}e^{-\frac{\pi\tau_2}{R^2}(m+Y^I(q^I+\frac{1}{2}nY^I))^2+2\pi i(m+q^I+\frac{1}{2}nY^I)n\tau_1} \\
&=\frac{1}{\eta\bar{\eta}^{17}}\sum_{n,m,q^I} q^{\frac{1}{2}(q^I+nY^I)^2}q^{\frac{1}{2}(\frac{1}{R}(m-Y^I(q^I+\frac{1}{2}nY^I)+nR)^2}\bar{q}^{\frac{1}{2}(\frac{1}{R}(m-Y^I(q^I+\frac{1}{2}nY^I)-nR)^2}
\end{aligned}
$$

where I've flipped $m\to -m$ at the end there. We get momenta

$$
\begin{aligned}
k_L &= \frac{1}{R}(m-q^IY^I-\frac{1}{2}nY^IY^I)+nR=\frac{m}{R}+n(R-\frac{1}{2}Y^IY^I)-q^IY^I \\
k_R &= \frac{1}{R}(m-q^IY^I-\frac{1}{2}nY^IY^I)-nR=\frac{m}{R}-n(R+\frac{1}{2}Y^IY^I)-q^IY^I \\
k_R^I &= q^I+nY^I
\end{aligned}
$$

consistent with Polchinski with $m\leftarrow n_m, n\leftarrow w^n, Y^I\leftarrow RA^I$ and $\alpha'=0$ **(might be off by a factor of 2 for k_R^I rel. to Polchinski but I think I'm consistent with Ginsparg)**. We only care about the $SO(1,1,\mathbb{Z})$ T-duality group coming from the compact x^9. This does not act on the Y^I as far as I can see **CHECK**

The SO$(16,\mathbb{Z})$ on the other hand acts on the Y^I as in the standard vector representation.

2. I am going to re-do the computations of appendix F Hatted indices denote the 10D terms. Greek indices from the start of the alphabet denote compact 10-D-dimensional indices while greek indices from the middle of the alphabet denote noncompact D-dimensional indices.

   The 10D action is
$$\int d^{10}x\sqrt{-\hat{G}_{10}}\,e^{-2\hat{\Phi}}[\hat{R}+4(\nabla\hat{\Phi})^2-\frac{1}{12}\hat{H}^2-\frac{1}{4}\mathrm{Tr}\hat{F}^2]+O(\ell_s^2)$$
   with $\hat{F}^I_{\mu\nu}=\partial_\mu\hat{A}^I_\nu-\partial_\nu\hat{A}^I_\mu$ and $\hat{H}_{\mu\nu\rho}=\partial_\mu\hat{B}_{\nu\rho}-\frac{1}{2}\sum_I\hat{A}^I_\mu\hat{F}^I_{\nu\rho}+2\,\text{perms.}$. Here I is the internal 16-dimensional index for the heterotic string.

   We take the 10-bein (r,a denote D and $10-D$ 10-bein indices, hatted indices $\hat{r},\hat{\mu}$ should not be confused for 10-bein indices!!)
$$e^{\hat{r}}_{\hat{\mu}}=\begin{pmatrix} e^r_\mu & A^\beta_\mu E^a_\beta \\ 0 & E^a_\alpha\end{pmatrix}\qquad e^{\hat{\mu}}_{\hat{r}}=\begin{pmatrix} e^\mu_r & -e^\nu_r A^\alpha_\nu \\ 0 & E^\alpha_a\end{pmatrix}$$
   This gives us the metric:
$$G_{\hat{\mu},\hat{\nu}}=\begin{pmatrix} G_{\mu\nu}-A^\alpha_\mu G_{\alpha\beta}A^\beta_\nu & G_{\alpha\beta}A^\beta_\mu \\ G_{\alpha\beta}A^\beta_\nu & G_{\alpha\beta}\end{pmatrix}$$
   As we've done before in chapter 7, we then define
$$\phi=\Phi-\frac{1}{4}\log\det G_{\alpha\beta},\qquad F^A_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

With this, the compactification of $R + 4(\nabla\phi)^2$ is clear:

$$\int d^D \sqrt{g} e^{-2\phi}[R + 4\partial_\mu \phi \partial^\mu \phi + \frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta} - \frac{1}{4}G_{\alpha\beta}F^{A\ \alpha}_{\mu\nu}F^{A\ \beta}_{\mu\nu}]$$

The first and second terms are clear. The third term makes up for the redefinition of Φ in terms of ϕ while the last term is the standard KK mechanism generating a gauge field strength from the compact dimensions.

Next, let's look $\hat{H}$. Because we have no sources for the H field, $\hat{H}$ is on the compact cycles. We can define the D-dimensional fields using the 10-bein as:

$$H_{\mu\alpha\beta} = e^r_\mu e^{\hat{\mu}}_r \hat{H}_{\hat{\mu}\alpha\beta} = \hat{H}_{\mu\alpha\beta} \tag{80}$$

$$H_{\mu\nu\alpha} = e^r_\mu e^s_\nu e^{\hat{\mu}}_r e^{\hat{\nu}}_s H_{\hat{\mu}\hat{\nu}\alpha} = \hat{H}_{\mu\nu\alpha} - A^\beta_\mu \hat{H}_{\nu\alpha\beta} + A^\beta_\nu \hat{H}_{\mu\alpha\beta} \tag{81}$$

$$H_{\mu\nu\rho} = e^r_\mu e^s_\nu e^t_\rho e^{\hat{\mu}}_r e^{\hat{\nu}}_s e^{\hat{\rho}}_t \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} + [-A^\alpha_\mu \hat{H}_{\alpha\nu\rho} + A^\alpha_\mu A^\beta_\nu \hat{H}_{\alpha\beta\rho} + 2\,\text{perms.}] \tag{82}$$

The point of defining these coordinates in terms of the 10-bein coordinate is that now, we can just directly separate the $\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}}$ sum into terms without worrying about the metric, and yield directly:

$$\int d^D \sqrt{-g} e^{-2\phi}[-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{3}{12}H_{\mu\nu\alpha}H^{\mu\nu\alpha} - \frac{3}{12}H_{\mu\alpha\beta}H^{\mu\alpha\beta}]$$

The method is the same for the F tensor. We define new Wilson lines and field strengths:

$$Y^I_\alpha = A^I_\alpha, \qquad A^I_\mu = e^r_\mu e^{\hat{\mu}}_r \hat{A}^I_{\hat{\mu}} = \hat{A}^I_\mu - Y^I_\alpha A^\alpha_\mu$$

I can define F in the standard $F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$, $\tilde{F}^I_{\mu\alpha} = \partial_\mu Y^I_\alpha$. This gives me $\hat{F}^I_{\mu\nu} = F^I_{\mu\nu} + \partial_\mu(Y^I_\alpha A^\alpha_\nu) - \partial_\nu(Y^I_\alpha A^\alpha_\nu)$. By redefining

$$\tilde{F}^I_{\mu\nu} = F^I_{\mu\nu} + Y^I_\alpha F^{A,\alpha}_{\mu\nu}$$

we can equate this with $\hat{F}^I_{\mu\nu}$. For the compact coordinates its more simple and I take $\tilde{F}_{\mu\alpha} = \partial_\mu Y^I_\alpha$. Again $\tilde{F}_{\alpha\beta}$ vanishes since we cannot have internal sources. This yields directly

$$\int d^D x \sqrt{-g} e^{-2\phi}[-\frac{1}{4}\sum_I^{16} \tilde{F}^I_{\mu\nu}\tilde{F}^{I,\mu\nu} - \frac{2}{4}\tilde{F}^I_{\mu\alpha}\tilde{F}^{I,\mu\alpha}]$$

Its not good enough for us to write everything in terms of an abstract H 3-form. We want to relate H to B and Y. From our relationship in 10D we can directly write:

$$H_{\mu\alpha\beta} = \partial_\mu B_{\alpha\beta} + \frac{1}{2}\sum_I (Y^I_\alpha \partial_\mu Y^I_\beta - Y^I_\beta \partial_\mu Y^I_\alpha)$$

Taking $C_{\alpha\beta} = \hat{B}_{\alpha\beta} - \frac{1}{2}\sum_I Y^I_\alpha Y^I_\beta$ we get

$$H_{\mu\alpha\beta} = \partial_\mu C_{\alpha\beta} + \sum_I Y^I_\alpha \partial_\mu Y^I_\beta$$

Next

$$H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} - \partial_\nu B_{\mu\alpha} + \frac{1}{2}\sum_I (\hat{A}^I_\nu \partial_\mu Y^I_\alpha - \hat{A}^I_\mu \partial_\nu Y^I_\alpha - Y^I_\alpha F^I_{\mu\nu})$$

We define the B field using not just the vielbein but also the gauge connection:

$$B_{\mu\alpha} := \hat{B}_{\mu\alpha} + B_{\alpha\beta}A^\beta_\mu + \frac{1}{2}\sum_I Y^I_\alpha A^I_\mu, \qquad F^B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

Then using (81) we get

$$H_{\mu\nu\alpha} = F^B_{\alpha\mu\nu} - C_{\alpha\beta}F^{A\ \beta}_{\mu\nu} - \sum_I Y^I_\alpha F^I_{\mu\nu}$$

Finally, using both vielbein and connection

$$B_{\mu\nu} = \hat{B}_{\mu\nu} + \frac{1}{2}[A^\alpha_\mu B_{\nu\alpha} + \sum_I A^I_\mu A^\alpha_\nu Y^I_\alpha - (\nu \leftrightarrow \mu)] - A^\alpha_\mu A^\beta_\nu B_{\alpha\beta}$$

And this gives us

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2} L_{ij} A^i_\mu F^j_{\nu\rho} + 2\,\text{perms.}$$

where L_{ij} is the $(10-D, 26-D)$-invariant metric and we have combined $A^\alpha_\mu, B_{\alpha\mu}, A^I_\mu$ into a length $36-2D$ vector.

Now the full action is:

$$\int d^D \sqrt{g} e^{-2\phi}[R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}$$
$$- \frac{1}{4}G^{\alpha\beta}H_{\mu\nu\alpha}H^{\mu\nu\beta} - \frac{1}{4}G_{\alpha\beta}F^{A\ \alpha}_{\mu\nu}F^{A\mu\nu\beta} - \frac{1}{4}\tilde{F}^I_{\mu\nu}\tilde{F}^{I,\mu\nu}$$
$$- \frac{1}{4}H_{\mu\alpha\beta}H^{\mu\alpha\beta} + \frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta} - \frac{1}{2}\tilde{F}^I_{\mu\alpha}\tilde{F}^{I,\mu\alpha}]$$

Using our expressions for $H_{\mu\nu\alpha}$ and $\tilde{F}^A_{\mu\nu}$, the middle line can be combined into

$$-\frac{1}{4}\begin{pmatrix} G + C^TG^{-1}C + Y^TY & -C^TG^{-1} & C^TG^{-1}Y^T + Y^T \\ -G^{-1}C & G^{-1} & -G^{-1}Y^T \\ YG^{-1}C + Y & -YG^{-1} & 1 + YG^{-1}Y^T \end{pmatrix}_{ij} F^i_{\mu\nu}F^{\mu\nu\,j}$$

here $F^i = (F^{A^\alpha}, F^B{}_\alpha, F^I)$. Call the matrix M^{-1} and notice that $LML = M^{-1}$, and indeed we get M transforms in the adjoint of $\mathrm{SO}(26-D, 10-D)$.

Similar arguments would give that the last line becomes $\frac{1}{8}\mathrm{Tr}\partial_\mu M\partial^\mu M^{-1}$ (Too much algebra).

From this, its immediate that any $\mathrm{SO}(10-D, 26-D)$ transformation on the scalar matrix (adjoint rep) and array of vector bosons (vector rep) will preserve both of these last two terms. It will also preserve H since it depends on the invariant $B_{\nu\rho}$ and SO-invariant combination $L_{ij}A^i_\mu F^j_{\nu\rho}$.

3. The action for IIA in the string frame is

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-\hat{G}}\left[e^{-2\hat{\Phi}}[\hat{R} + 4(\nabla\hat{\Phi})^2 - \frac{1}{12}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}}] - \frac{1}{4}F_2^2 - \frac{1}{2\cdot 4!}F_4^2\right] + \frac{1}{4\kappa^2}\int B_2 \wedge \mathrm{d}C_3 \wedge \mathrm{d}C_3$$

Doing the same reduction as before, the $\hat{R} + 4(\nabla\hat{\Phi})^2 - \frac{1}{12}H^2$ term becomes:

$$\int d^4\sqrt{-g}e^{-2\phi}\Big[R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F^{A\ \alpha}_{\mu\nu}F^{A\mu\nu}_{\ \ \alpha} + \frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta} - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}H_{\mu\alpha\beta}H^{\mu\alpha\beta} - \frac{1}{4}G^{\alpha\beta}H_{\mu\nu\alpha}H^{\mu\nu\alpha}\Big]$$
$$= \int d^4\sqrt{-g}e^{-2\phi}\Big[R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}M^{-1}_{ij}F^i_{\mu\nu}F^{\mu\nu\,j} + \frac{1}{8}\mathrm{Tr}[\partial_\mu M\partial^\mu M^{-1}]\Big]$$

Here we used H as in the last problem and the matrix M consisting of the 21 $G_{\alpha\beta}$ and 15 $B_{\alpha\beta}$. The F^i are the field strengths of the $6+6$ $U(1)$ vectors coming from G and B compactification.

$$H_{\mu\nu\rho} = \partial_\mu B_{\mu\rho} - \frac{1}{2}L_{ij}A^i_\mu F^j_{\nu\rho} + 2\,\text{perms.} \qquad M^{-1} = \begin{pmatrix} G + B^TG^{-1}B & -B^TG^{-1} \\ -G^{-1}BG^{-1} & G \end{pmatrix}$$

The $H_{\mu\nu\rho}$ can be dualized to provide a *sixteenth* scalar coming from the B field. By analogy to **9.1.13**, in the string frame I would expect to write:

$$e^{-2\phi}H_{\mu\nu\rho} = E_{\mu\nu\rho\sigma}\nabla^\sigma a$$

The $B_{\mu\nu}$ equations $\nabla^\mu(e^{-2\phi}H_{\mu\nu\rho})$ are now automatically satisfied. The axion EOMs come from the Bianchi identity:

$$E^{\mu\nu\rho\sigma}\partial_\mu H_{\nu\rho\sigma} = -\frac{1}{2}L_{ij}E^{\mu\nu\rho\sigma}F^i_{\rho\sigma}F^j_{\mu\nu} = -L_{ij}\tilde{F}^i_{\mu\nu}F^{j\,\mu\nu}, \qquad \tilde{F}^i_{\mu\nu} = \frac{1}{2}E^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

Here we have defined the dual 2-form as required. This can now be recast as the equation of motion for the axion (contracting the Es gives a 4):

$$\nabla^\mu(e^{2\phi}\nabla_\mu a) = -\frac{1}{4}L_{ij}F^i_{\mu\nu}\tilde{F}^{j\,\mu\nu}$$

With this, we can dualize the action in terms of the axion to yield:

$$\int d^4\sqrt{-g}e^{-2\phi}\Big[R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{4\phi}(\partial a)^2 + \frac{1}{4}e^{2\phi}aL_{ij}F^i_{\mu\nu}\tilde{F}^{j\,\mu\nu} - \frac{1}{4}M^{-1}_{ij}F^i_{\mu\nu}F^{\mu\nu\,j} + \frac{1}{8}\mathrm{Tr}[\partial_\mu M\partial^\mu M^{-1}]\Big]$$

We could also do this in the Einstein frame and get *exactly* the same action as in **9.1.15** with the M matrix as we have it (no sum over heterotic internals).

The only thing left is the RR fields. We follow Kiritis' treatment of the 4-form field strength. We use the 10-bein to get:

$$\begin{aligned}
C_{\alpha\beta\gamma} &= \hat{C}_{\alpha\beta\gamma}\\
C_{\mu\alpha\beta} &= \hat{C}_{\mu\alpha\beta} - C_{\alpha\beta\gamma}A^\gamma_\mu\\
C_{\mu\nu\alpha} &= \hat{C}_{\mu\nu\alpha} + \hat{C}_{\mu\alpha\beta}A^\beta_\nu - \hat{C}_{\nu\alpha\beta}A^\beta_\mu + C_{\alpha\beta\gamma}A^\beta_\mu A^\alpha_\nu\\
C_{\mu\nu\rho} &= \hat{C}_{\mu\nu\rho} - (A^\alpha_\mu\hat{C}_{\nu\rho\alpha} + A^\alpha_\mu A^\beta_\nu C_{\alpha\beta\rho} + 2\,\text{perms.}) - C_{\alpha\beta\gamma}A^\alpha_\mu A^\beta_\nu A^\gamma_\rho
\end{aligned}$$

Let's now define the field strengths. Now we must have $F_{\alpha\beta\gamma\delta} = 0$ since the internal dimensions do not contain sources for the field. What remains is

$$\begin{aligned}
F_{\mu\alpha\beta\gamma} &= \partial_\mu C_{\alpha\beta\gamma}\\
F_{\mu\nu\alpha\beta} &= \partial_\mu C_{\nu\alpha\beta} - \partial_\nu C_{\mu\alpha\beta} + C_{\alpha\beta\gamma}F^\gamma_{\mu\nu}\\
F_{\mu\nu\rho\alpha} &= \partial_\mu C_{\nu\rho\alpha} + C_{\mu\alpha\beta}F^\beta_{\nu\rho} + 2\,\text{perms.}\\
F_{\mu\nu\rho\sigma} &= (\partial_\mu C_{\alpha\beta\gamma} + 3\,\text{perms.}) + (C_{\sigma\rho\alpha}F^\alpha_{\mu\nu} + 5\,\text{perms.})
\end{aligned}$$

Then this gives the contribution (here all two-lower one-upper index $F^\alpha_{\mu\nu}$ are taken to mean F^A):

$$S^{(4)}_{RR} = -\frac{1}{2\cdot 4!}\int d^4\sqrt{-g}\sqrt{\det G_{\alpha\beta}}[F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} + 4F_{\mu\nu\rho\alpha}F^{\mu\nu\rho\alpha} + 6F_{\mu\nu\alpha\beta}F^{\mu\nu\alpha\beta} + 4F_{\mu\alpha\beta\gamma}F^{\mu\alpha\beta\gamma}]$$

It is important to realize that in 4-D the 4-form field strength coming from the 3-form has *no* dynamical degrees of freedom. It plays the role of a cosmological constant **Check w/ Alek**.

The two-spacetime-index term can be directly dualized. It corresponds to $6\times 5/3 = 15$ vectors. The three-spacetime-index term can be dualized to become the kinetic term for 6 scalar axions a_α with no interaction term.

The $F_{\mu\alpha\beta\gamma}$ correspond to kinetic terms of the $6\times 5\times 4/3! = 20$ scalars $C^{(4)}_{\alpha\beta\gamma}$.

Let's do a similar thing for the 2-form field strength. There, we get $C_\alpha = \hat{C}_\alpha, C_\mu = \hat{C}_\mu - C_\alpha A^\alpha_\mu$. The corresponding field strength is $F_{\alpha\beta} = 0, F_{\mu\alpha} = \partial_\mu C_\alpha$ and $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu + C_\alpha F^\alpha_{\mu\nu}$. We then get contribution

$$S^{(2)}_{RR} = -\frac{1}{4}\int d^4\sqrt{-g}\sqrt{\det G_{\alpha\beta}}[F_{\mu\nu}F^{\mu\nu} + 2F_{\mu\alpha}F^{\mu\alpha}]$$

Again $F_{\mu\nu}$ can be written in terms of dual fields $\tilde{F}^{(2)}_{\mu\nu} = E_{\mu\nu\rho\sigma}F^{(2)\,\rho\sigma}$. This is one gauge fields and six further scalars.

Return and think about the effect of the CS terms. I bet they make the RR field equations non-free.

4. First note that using the OPE

$$\Sigma^I(z)\bar{\Sigma}^J(w) = \frac{\delta^{IJ}}{(z-w)^{3/4}} + (z-w)^{1/4} J^{IJ}(w)$$

the $\langle J^{II}\Sigma^J\bar{\Sigma}^J\rangle$ correlator can be evaluated as

$$\langle J^{II}(z_1)\Sigma^J(z_2)\bar{\Sigma}^J(z_3)\rangle = (\delta^{IJ} - \tfrac{1}{4})\frac{z_{23}^{1/4}}{z_{12}z_{13}}$$

Taking $z_1 \to z_2$ we see a singularity going as $\frac{(\delta^{IJ}-\frac{1}{4})}{z_{12}} z_{23}^{-3/4}$. Meanwhile taking the $J\Sigma$ OPE gives

$$q\frac{\langle\Sigma(z_2)\bar{\Sigma}(z_3)\rangle}{z_{12}} = \frac{q}{z_{12}} z_{23}^{-3/4}$$

So we see that under J^I the charge of Σ^J is $3/4$ if $I = J$ and $-1/4$ otherwise. We have 4 J^{II}, and notice that the total charge under all four of each Σ^I is always zero. Consider the following combination of charges, which provides a basis for the Σ^I charge space

$$\begin{aligned}
\tilde{J}^1 &= J^{11} + J^{22} - J^{33} - J^{44}\\
\tilde{J}^2 &= J^{11} - J^{22} + J^{33} - J^{44}\\
\tilde{J}^3 &= J^{11} - J^{22} - J^{33} + J^{44}
\end{aligned}$$

Under each of $\tilde{J}^i$ we have the following charges

$$\begin{array}{llll}
\Sigma^1 \to (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & \Sigma^2 \to (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), & \Sigma^3 \to (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}), & \Sigma^4 \to (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\\
\bar{\Sigma}^1 \to (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), & \bar{\Sigma}^2 \to (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), & \bar{\Sigma}^3 \to (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), & \bar{\Sigma}^4 \to (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})
\end{array}$$

These are exactly all combinations, and we can define the three bosonic fields ϕ_i with $T = \sum_i \frac{1}{2}(\partial\phi_i)^2$ so that

$$\Sigma^1 = \exp\left[i(\tfrac{1}{2}\phi_1 + \tfrac{1}{2}\phi_2 + \tfrac{1}{2}\phi_3)\right], \quad \Sigma^2 = \exp\left[i(\tfrac{1}{2}\phi_1 - \tfrac{1}{2}\phi_2 - \tfrac{1}{2}\phi_3)\right], \quad \text{etc.}$$

Each of these $\Sigma^I, \bar{\Sigma}^I$ has dimension $3/8$ as required.

Let's look at the supercurrent G^{int}. It can be written in terms of an eigenbasis of the commuting $\tilde{J}^i$. In particular look at $\tilde{J}^1$.

$$G^{int} = \sum_q e^{iq\phi_1} T^{(q)}$$

Now consider the OPEs $G^{int}\cdot\Sigma^1$ and $G^{int}\cdot\bar{\Sigma}^1$. As observed in the chapter, both of these have only the singular term going as $(z-w)^{-1/2}$. Together both of these require that q in G can only be ± 1. We can repeat this argument for $\tilde{J}^2, \tilde{J}^3$ to see that G^{int} must be a sum of 6 terms:

$$e^{iq_1\phi_1} Z_1 + e^{-iq_1\phi_1}\bar{Z}_1 + e^{iq_2\phi_2} Z_2 + e^{-iq_2\phi_2}\bar{Z}_2 + e^{iq_3\phi_3} Z_3 + e^{-iq_3\phi_3}\bar{Z}_3$$

Each $Z_i, \bar{Z}_i$ must be dimension one operators, so they are themselves bosonic fields $i\partial X^i_\pm$. We thus have that $G^{int} = \sum_{i=1,\pm}^3 \psi_i^\pm \partial X^i_\pm$. This is exactly the supercurrent for six free boson-fermion systems and will give (under anticommutator) the stress tensor of a six free boson-fermion systems. This is exactly a toroidal CFT.

5. The relevant partition function is not difficult to compute, as we can follow 9.4's example but not do the twist on the internal $(0,16)$ part. Firstly the fermions on the left-moving (SUSY) side have orbifold blocks under the shifts as before:

$$Z_\psi\begin{bmatrix}h\\g\end{bmatrix} = \frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\frac{\theta^2\begin{bmatrix}a\\b\end{bmatrix}\theta\begin{bmatrix}a+h\\b+g\end{bmatrix}\theta\begin{bmatrix}a-h\\b-g\end{bmatrix}}{\eta^4}$$

Similarly we've already constructed the bosonic blocks before. They are given by **4.12.10** as:

$$Z_{4,4}\begin{bmatrix}0\\0\end{bmatrix} = \frac{\Gamma_{4,4}}{\eta^4\bar{\eta}^4}, \quad Z_{4,4}\begin{bmatrix}h\\g\end{bmatrix} = 2^4 \frac{\eta^2\bar{\eta}^2}{\theta^2\left[{1-h \atop 1-g}\right]\bar{\theta}^2\left[{1-h \atop 1-g}\right]}$$

Then the $(2,2)$ part is untouched, yielding $\frac{\Gamma_{2,2}}{\eta^2\bar{\eta}^2}$ as is the $(0,16)$ part. We get the partition function

$$Z^{het} = \underbrace{\frac{\Gamma_{2,2}}{\eta^2\bar{\eta}^2}}_{\mathbf{1}} \times \underbrace{\frac{1}{2}\sum_{h,g=0}^{1} \frac{Z_{4,4}\left[{h \atop g}\right]}{\tau_2\eta^2\bar{\eta}^2}}_{\mathbf{2}} \times \underbrace{\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\frac{\theta^2\left[{a \atop b}\right]\theta\left[{a+h \atop b+g}\right]\theta\left[{a-h \atop b-g}\right]}{\eta^4}}_{\mathbf{3}} \times \underbrace{\frac{\left(\frac{1}{2}\sum_{a,b=0}^{1}\bar{\theta}\left[{a \atop b}\right]^8\right)^2}{\bar{\eta}^{16}}}_{\mathbf{4}}$$

Let's see how each term transforms under $\tau \to -1/\tau$. **1** stays invariant. **2** have $Z_{4,4}\left[{h \atop g}\right] \to Z_{4,4}\left[{g \atop h}\right]$ with $\tau_2\eta^2\bar{\eta}^2$ invariant. **3** is the only nontrivial one. We will do it explicitly in the next step. **4** will remain invariant.

Under $\tau \to \tau+1$, we must be careful, as $\theta\left[{1 \atop 0}\right]$ picks up an $e^{i\pi/4}$ while $\theta\left[{-1 \atop 0}\right]$ picks up $e^{-3i\pi/4}$. The other two nonzero theta functions simply do $\theta\left[{a \atop b}\right] \to \theta\left[{a \atop a+b-1}\right]$

1, **2**, remain invariant, with **2** making us change variables $g', h' = g, h+g-1$. The η functions in the denominators of **3** and **4** leave over an $1/\bar{\eta}^{12}$ which contributes a $-$ sign.

Let's look at 3. First when $h=0, g=0$ we have $(-1)^{a+b+ab}\theta^4\left[{a \atop b}\right]$ and $\tau+\tau+1$ will send this to $-$ itself as required to cancel the $\bar{\eta}^{12}$ $-$ sign.

The other terms looks like (after canceling $\theta\left[{1 \atop 1}\right]$)

$$h=0, h=0 : \theta\begin{bmatrix}0\\0\end{bmatrix}^4 - \theta\begin{bmatrix}1\\0\end{bmatrix}^4 - \theta\begin{bmatrix}0\\1\end{bmatrix}^4 - \theta\begin{bmatrix}1\\1\end{bmatrix}^4 = 0$$

$$h=1, g=0 : \theta\begin{bmatrix}0\\0\end{bmatrix}^2\theta\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}-1\\0\end{bmatrix} - \theta\begin{bmatrix}1\\0\end{bmatrix}^2\theta\begin{bmatrix}2\\0\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix} - \theta\cancel{\begin{bmatrix}0\\1\end{bmatrix}^2\theta\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}-1\\1\end{bmatrix}} - \theta\cancel{\begin{bmatrix}1\\1\end{bmatrix}^2\theta\begin{bmatrix}2\\1\end{bmatrix}\theta\begin{bmatrix}0\\1\end{bmatrix}} = \theta\begin{bmatrix}0\\0\end{bmatrix}^2\theta\begin{bmatrix}1\\0\end{bmatrix}^2 - \theta\begin{bmatrix}1\\0\end{bmatrix}^2\theta\begin{bmatrix}0\\0\end{bmatrix}^2 = 0$$

$$h=0, g=1 : \theta\begin{bmatrix}0\\0\end{bmatrix}^2\theta\begin{bmatrix}0\\1\end{bmatrix}\theta\begin{bmatrix}0\\-1\end{bmatrix} - \theta\cancel{\begin{bmatrix}1\\0\end{bmatrix}^2\theta\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}1\\-1\end{bmatrix}} - \theta\begin{bmatrix}0\\1\end{bmatrix}^2\theta\begin{bmatrix}0\\2\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix} - \theta\cancel{\begin{bmatrix}1\\1\end{bmatrix}^2\theta\begin{bmatrix}1\\2\end{bmatrix}\theta\begin{bmatrix}1\\0\end{bmatrix}} = \theta\begin{bmatrix}0\\0\end{bmatrix}^2\theta\begin{bmatrix}0\\1\end{bmatrix}\theta\begin{bmatrix}0\\1\end{bmatrix} - \theta\begin{bmatrix}0\\1\end{bmatrix}^2\theta\begin{bmatrix}0\\0\end{bmatrix}^2 = 0$$

$$h=1, g=1 : \theta\cancel{\begin{bmatrix}0\\0\end{bmatrix}^2\theta\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}-1\\-1\end{bmatrix}} - \theta\begin{bmatrix}1\\0\end{bmatrix}^2\theta\begin{bmatrix}2\\1\end{bmatrix}\theta\begin{bmatrix}0\\-1\end{bmatrix} - \theta\begin{bmatrix}0\\1\end{bmatrix}^2\theta\begin{bmatrix}1\\2\end{bmatrix}\theta\begin{bmatrix}-1\\0\end{bmatrix} - \theta\cancel{\begin{bmatrix}1\\1\end{bmatrix}^2\theta\begin{bmatrix}2\\2\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix}} = -\theta\begin{bmatrix}1\\0\end{bmatrix}^2\theta\begin{bmatrix}0\\1\end{bmatrix}^2 + \theta\begin{bmatrix}0\\1\end{bmatrix}^2\theta\begin{bmatrix}1\\0\end{bmatrix}^2 = 0$$

Ok, so in fact this partition function is zero. This should not be surprising, since naively we are just breaking supersymmetry in half, and so we should still expect fermions and bosons to run in loops such that the vacuum energy vanishes. Naively, then we would again say "zero is modular invariant" and be done with it- but not so fast. There are still phases we can pick up, say from $\tau \to \tau+1$ that would not be visible given the vanishing of the partition function, but would nonetheless spoil modular invariance.

One way around this is to turn on the chemical potential ν_i in the theta functions to prevent vanishing. Effectively, then, we ignore the Jacobi identity and don't just set $\theta\left[{1 \atop 1}\right] = 0$. Then, let's look at how each term transforms under $\tau \to \tau+1$. Again, the terms not involving $\theta\left[{1 \atop 1}\right]$ will cancel independently of $\nu_i = 0$ or not, and after simplifying things ,we have

$$(0,0) : \theta\begin{bmatrix}0\\0\end{bmatrix}^4 - \theta\begin{bmatrix}1\\0\end{bmatrix}^4 - \theta\begin{bmatrix}0\\1\end{bmatrix}^4 - \theta\begin{bmatrix}1\\0\end{bmatrix}^4 \to \theta\begin{bmatrix}0\\1\end{bmatrix}^4 + \theta\begin{bmatrix}1\\0\end{bmatrix}^4 - \theta\begin{bmatrix}0\\0\end{bmatrix}^4 + \theta\begin{bmatrix}1\\1\end{bmatrix}^4 \Leftarrow -(0,0)$$

$$(1,0) : -2\theta\begin{bmatrix}0\\1\end{bmatrix}^2\theta\begin{bmatrix}1\\1\end{bmatrix}^2 \to -2i\,\theta\begin{bmatrix}0\\0\end{bmatrix}^2\theta\begin{bmatrix}1\\1\end{bmatrix}^2 \Leftarrow i \times (1,1)$$

$$(0,1) : 2\theta\begin{bmatrix}1\\0\end{bmatrix}^2\theta\begin{bmatrix}1\\1\end{bmatrix}^2 \to -2\theta\begin{bmatrix}1\\0\end{bmatrix}^2\theta\begin{bmatrix}1\\1\end{bmatrix}^2 \Leftarrow -(0,1)$$

$$(1,1) : -2\theta\begin{bmatrix}0\\0\end{bmatrix}^2\theta\begin{bmatrix}1\\1\end{bmatrix}^2 \to -2i\theta\begin{bmatrix}0\\1\end{bmatrix}^2\theta\begin{bmatrix}1\\1\end{bmatrix}^2 \Leftarrow i \times (1,0)$$

So we see $(0,1)$ (ie the projected part of the untwisted sector) goes to its negative as required. On the other hand, the twisted sector has $(1,0)$ and $(1,1)$ swap, but with a factor of i instead of -1. This is not good enough for modular invariance.

Under $\tau \to -1/\tau$ the sectors appropriately get sent to one another except for the twisted projected sector which picks up a factor of -1 from the $\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]^2$, so this too is not modular invariant.

It is worth adding that Polchinski remarks in 16.1 that for abelian orbifolds (of the type T^n/H with H and abelian group), the only obstruction to modular invariance is $\tau \to \tau + 1$

Indeed, we see that this twist violates **16.1.28** of Polchinski, where we hae $r_2 = 0, r_3 = r_4 = 1$ and so $\sum_{i=2}^4 r_i - \sum_{k=1}^{16} s_k 2 \neq 0 \bmod 2N$ when $N = 2$.

6. Now the partition function is given by

$$Z^{het}_{N=2} = \underbrace{\frac{\Gamma_{2,2}}{\eta^2\bar\eta^2}}_{\mathbf{1}} \times \underbrace{\frac{1}{2}\sum_{h,g=0}^{1} \frac{Z_{4,4}\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]}{\tau_2\eta^2\bar\eta^2}}_{\mathbf{2}} \times \underbrace{\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\frac{\theta^2\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a+h\\b+g\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a-h\\b-g\end{smallmatrix}\right]}{\eta^4}}_{\mathbf{3}} \times \underbrace{\frac{1}{2}\sum_{\gamma,\delta=0}^{1}\frac{\bar\theta^6\left[\begin{smallmatrix}\gamma\\\delta\end{smallmatrix}\right]\bar\theta\left[\begin{smallmatrix}\gamma+h\\\delta+g\end{smallmatrix}\right]\bar\theta\left[\begin{smallmatrix}\gamma-h\\\delta-g\end{smallmatrix}\right]}{\bar\eta^8}}_{\mathbf{4}} \times \underbrace{\frac{\frac{1}{2}\sum_{a,b=0}^{1}\bar\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]^8}{\bar\eta^8}}_{\mathbf{5}}$$

Things will still remain invariant under $\tau \to -1/\tau$ for the reasons given above, now applied to both **3** and **4**. The only important subtlety is now in the $(1,1)$ sector the E_8 $\bar\theta^6\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ will contribute a -1 sign, as necessary to cancel the twisted projected left-moving fermion sector.

Next, under $\tau \to \tau + 1$, the exact same arguments apply to **3** and **4**, namely the untwisted sector of the left-handed fermions picks up -1 phase as required to cancel with the $\bar\eta$. The twisted sectors look like:

$$(0,0) : \bar\theta\begin{bmatrix}0\\0\end{bmatrix}^8 + \bar\theta\begin{bmatrix}1\\0\end{bmatrix}^8 + \bar\theta\begin{bmatrix}0\\1\end{bmatrix}^8 + \bar\theta\begin{bmatrix}1\\0\end{bmatrix}^8 \to \bar\theta\begin{bmatrix}0\\1\end{bmatrix}^4 + \bar\theta\begin{bmatrix}1\\0\end{bmatrix}^4 + \bar\theta\begin{bmatrix}0\\0\end{bmatrix}^4 + \bar\theta\begin{bmatrix}1\\1\end{bmatrix}^4 \Leftarrow (0,0)$$

$$(1,0) : \bar\theta\begin{bmatrix}0\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\0\end{bmatrix}^2 + \bar\theta\begin{bmatrix}1\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\0\end{bmatrix}^2 + \bar\theta\begin{bmatrix}0\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\1\end{bmatrix}^2 + \bar\theta\begin{bmatrix}1\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\1\end{bmatrix}^2 \to -i\bar\theta\begin{bmatrix}0\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\0\end{bmatrix}^2 - i\bar\theta\begin{bmatrix}1\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\1\end{bmatrix}^2 - i\bar\theta\begin{bmatrix}0\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\1\end{bmatrix}^2 - i\bar\theta\begin{bmatrix}1\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\0\end{bmatrix}^2 \Leftarrow i \times (1,1)$$

$$(0,1) : \bar\theta\begin{bmatrix}0\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\1\end{bmatrix}^2 - \bar\theta\begin{bmatrix}1\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\1\end{bmatrix}^2 + \bar\theta\begin{bmatrix}0\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\0\end{bmatrix}^2 - \bar\theta\begin{bmatrix}1\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\0\end{bmatrix}^2 \to \bar\theta\begin{bmatrix}0\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\0\end{bmatrix}^2 - \bar\theta\begin{bmatrix}1\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\1\end{bmatrix}^2 + \bar\theta\begin{bmatrix}0\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\1\end{bmatrix}^2 - \bar\theta\begin{bmatrix}1\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\0\end{bmatrix}^2 \Leftarrow (0,1)$$

$$(1,1) : -\bar\theta\begin{bmatrix}0\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\1\end{bmatrix}^2 - \bar\theta\begin{bmatrix}1\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\1\end{bmatrix}^2 - \bar\theta\begin{bmatrix}0\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\0\end{bmatrix}^2 - \bar\theta\begin{bmatrix}1\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\0\end{bmatrix}^2 \to i\bar\theta\begin{bmatrix}0\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\1\end{bmatrix}^2 + i\bar\theta\begin{bmatrix}1\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\1\end{bmatrix}^2 + i\bar\theta\begin{bmatrix}0\\0\end{bmatrix}^6\bar\theta\begin{bmatrix}1\\1\end{bmatrix}^2 + i\bar\theta\begin{bmatrix}1\\1\end{bmatrix}^6\bar\theta\begin{bmatrix}0\\0\end{bmatrix}^2 \Leftarrow i \times (0,1)$$

So we get that the untwisted sector remains the same, while each of the two twisted sector components change by a factor of i. This combines with what we know about the left-moving fermions to make *every* combined contribution change with a $-$ phase which exactly cancels the η-functions. The result is modular invariant.

To verify the spectrum, as remarked in the text when we act by orbifold on the $E_8 \times E_8$ we break down $[120] \oplus [128]$ of O(16). We get: $[120] \to [3,1,1] \oplus [1,3,1] \oplus [1,1,66] \oplus [2,1,12] \oplus [1,2,12]$ and $128 \to [2,1,32] \oplus [1,\bar{2},32]$ in $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{O}(12)$.

The $\mathbb{Z}_2$ action takes the spinors of the two SU(2) subgroups to minus themselves, keeping the conjugate spinors in variant. Projecting by this keeps $[3,1,1] \oplus [1,3,1] \oplus [1,1,66], [1,\bar{2},32]$. This organizes into $[3,1] \oplus [1,133] \oplus [2,56] \in \mathrm{SU}(2) \times E_7$. Here 56 is the fundamental representation and 133 is the adjoint representation of E_7.

Now let's organize our coordinates into $\mu = 2,3$ indicating the spatial coordinates in lightcone gauge, and pair the remaining 6 coordinates into $Z^i = \frac{1}{\sqrt{2}}(X^{2i} \pm iX^{2i+1})$, $i = \{2,3,4\}$. Let's organize the different sector contributions based on how they transform under the $\mathbb{Z}_2$ twist:

- Untwisted Sector
  - Left-handed side:
    - NS - The zero-point energy is $-1/2$ and we thus have massless states coming from single fermion excitations.

$$+ : \psi^{\mu}_{-1/2}, \psi^{4,5}_{-1/2}$$
$$- : \psi^{6,7,8,9}_{-1/2}$$

* R - The zero-point energy is 0 from equal number of bosons and fermions and our massless excitation comes from the ground state. Under the rotation $e^{2\pi i(s_2\phi_2 - s_3\phi_3)}$ the ground states organize as follows:

$$\begin{aligned} +: &\quad \left|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle, \left|-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\rangle, \left|\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right\rangle, \left|-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \\ -: &\quad \left|\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \left|\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\right\rangle \left|-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \left|-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\right\rangle \end{aligned}$$

  Note we only have an even number of + signs in any of the ground states by GSO projection. *These won't matter for the massless bosonic spectrum.*

- Right-handed side
  The zero-point energy is -1, so we either have a bosonic excitation:

$$\begin{aligned} +: &\quad \tilde{\alpha}^{\mu}_{-1}, \alpha^{4,5}_{-1} \\ -: &\quad \tilde{\alpha}^{6,7,8,9}_{-1} \end{aligned}$$

  Or a weight 1 excitation from the current algebra:

$$\begin{aligned} +: &\quad |a^+\rangle \in [3,1,1] \oplus [1,133,1] \oplus [1,1,128] \\ -: &\quad |a^-\rangle \in [2,56,1] \end{aligned}$$

So, the untwisted bosonic massless states must be the $\mathbb{Z}_2$-invariant combinations of left (NS) and right movers. We get

- $\psi^{\mu}_{-1/2}\tilde{\alpha}^{\nu}_{-1}$: $G_{\mu\nu}, B_{\mu\nu}, \Phi$.
- $\psi^{\mu}_{-1/2}|a^+\rangle$ - vector boson in the adjoint of SU(2) $\times E_7 \times E_8$. This combines together with $\psi^{\mu}_{-1/2}\tilde{\alpha}^{4,5}_{-1}$ and $\psi^{4,5}_{-1/2}\tilde{\alpha}^{\mu}_{-1}$ to produce an extra $U(1)^4$.
- $\psi^{4,5}_{-1/2}|a\rangle \cup \psi^{4,5}_{-1/2}\tilde{\alpha}^{4,5}_{1}$ - complex scalar transforming in the adjoint of $U(1)^4 \times$ SU(2) $\times E_7 \times E_8$
- $\psi^{6,7,8,9}_{-1/2}\tilde{\alpha}^{6,7,8,9}_{-1}$ - 16 neutral real scalars.
- $\psi^{6,7,8,9}_{-1/2}|a^-\rangle$ 4 real scalars transforming in the $[2,56,1]$ representation of SU(2) $\times E_7 \times E_8$

Here Kiritsis does not mention the presence of the dilaton with the other 16 real scalars. I assume this is an accidental omission.

- Twisted Sector

For the transformation g, we have 4 points on each T^2 that are equivalence classes with the transformed point gx. This means that we have 4×4 equivalence classes that we must include in the spectrum for the twisted sector. This will be the same as looking at the spectrum for 1 class of twist and taking it 16-fold.

Equivalently, because fixed points correspond to the equivalence classes in this case, note that our transformation has fixed points given by $(0, \frac{1}{2}, \frac{\tau_2}{2}, \frac{1}{2} + \frac{\tau_2}{2}) \times (0, \frac{1}{2}, \frac{\tau_3}{2}, \frac{1}{2} + \frac{\tau_3}{2})$ on the respective T^2s. The products give 16 fixed points. So we will have 16 copies of the spectrum at the fixed point $(0,0)$ on our T^4 **Appreciate this. Are you sure its not 32?**

- Left side The bosonic oscillators will be shifted by 1/2
  The fermionic oscillators will also be shifted by 1/2.
  * NS - The zero-point energy is now $-\frac{1}{4} + \frac{1}{4} = 0$ and so we get only one ground state - the vacuum.
  * R - The zero-point energy remains zero. The zero modes that give the vacuum are now obtained from $\psi^{2,3,4,5}$. We thus get 2 ground states after GSO projection, which will end up giving us the two requisite gravitinos
- Right side:
  This is the hardest part. We use complex fermion language for the current algebra. We separate it into two parts $\lambda^{\pm,1\ldots 8}, \lambda^{\pm,9\ldots 16}$. We get massless states from the (R, NS) and (NS, NS) states.

* (NS,NS) Here the ground state energy is $-1/2$. We thus get the following states contributing:

$$\alpha_{-1/2}^{6,7,8,9}, \quad \lambda_{-1/2}^{\pm 3\ldots 8}$$

  The first one will get GSO projected out (as will anything with an even number of fermions). The second one will transform as the [12] of SO(12). In line with this, we can also construct three other copies of [12] (or $[\overline{12}]$):

$$\lambda_{-1/2}^{\pm 3\ldots 8}\lambda_0^{\pm 1}\lambda_0^{\pm 2}$$

  ISNT THIS 5?
  The other state we can build that does *not* get GSO projected out is:

$$\alpha_{-1/2}^{6,7,8,9}\lambda_0^{\pm,1,2}$$

  This gives 4×2 copies of the [2] of SU(2).
* (R, NS) Here the ground state energy is 0. We have zero modes coming from the 12 fermions $\lambda^{\pm,3\ldots 8}$ giving 2^6 ground states giving the 32 and $\overline{32}$ spinors of SO(12), one of which will get projected out by GSO.
  α_0 alone will get GSO projected out, so does not contribute to the spectrum.

Together the two copies of [32] + [12] + [12] of SO(12) combine together to form the two copies of the [56] of E_7 and we get 8 copies of the 2 of SU(2).
Altogether our gauge multiplets lie in $2 \times [1, 56, 1]$ and $8 \times [2, 1, 1]$.

Thus we get the twisted bosonic states coming from $|0\rangle_{NS}|a\rangle$ giving us 32 scalars in the $[1, 56, 1]$ and 128 scalars in the $[2, 1, 1]$.
The zero-point energy calculations are here:

```
In[276]:= bose1 = 1/2 (1/24 - 1/2 (θ - 1/2)^2) /. θ → 0;
          bose2 = 1/2 (1/24 - 1/2 (θ - 1/2)^2) /. θ → 1/2;
          fermi1 = -1/2 (1/24 - 1/2 (θ - 1/2)^2) /. θ → 1/2;
          fermi2 = -1/2 (1/24 - 1/2 (θ - 1/2)^2) /. θ → 0;
          4 bose1 + 4 bose2 + 4 fermi1 + 4 fermi2
Out[280]= 0

          (*NS, NS*)4 bose1 + 4 bose2 + 4 fermi2 + 12 fermi1 + 16 fermi1
          (*R, NS*) 4 bose1 + 4 bose2 + 4 fermi1 + 12 fermi2 + 16 fermi1
          (*NS R*)4 bose1 + 4 bose2 + 4 fermi2 + 12 fermi1 + 16 fermi2
          (*R R*) 4 bose1 + 4 bose2 + 4 fermi1 + 12 fermi2 + 16 fermi2
Out[272]= -1/2
Out[273]= 0
Out[274]= 1/2
Out[275]= 1
```

7. Under $\tau \to \tau + 1$ its quick to see that compactifying on any $(d, d + 16)$ Lorentzian lattice and orbifolding by a $\mathbb{Z}_n$ shift symmetry of ϵ/N will give a transformation

$$\tau \to \tau + 1 : Z^N \begin{bmatrix} h \\ g \end{bmatrix} = e^{4\pi i/3} e^{\frac{i\pi h^2 \epsilon^2}{N^2}} Z^N \begin{bmatrix} h \\ h + g \end{bmatrix}$$

   where the first exponential factor comes from the $\bar{\eta}^{-16}$ and the second factor comes from shifting $p_L^2 - p_R^2$ which is otherwise even by $\epsilon h/N$ which gives $\frac{h^2}{N^2}(\epsilon_L^2 - \epsilon_R^2) = h^2\epsilon^2/N^2$.
   The $\tau \to -1/\tau$ phase

$$\tau \to -1/\tau : Z^N \begin{bmatrix} h \\ h \end{bmatrix} \to e^{-\frac{2\pi i h g \epsilon^2}{N}} Z^N \begin{bmatrix} g \\ -h \end{bmatrix}$$

can similarly be proven from straightforward Poisson resummation.

This problem specializes to $N = 2$.

For $\epsilon^2/2 = 1 \bmod 4$ the twisted sector picks up a phase under $\tau \to \tau + 1$ and one can see that this phase is $+i$, just as in the last problem. This is what was necessary to combine with the left-moving fermions to give a modular invariance. Note this happens *only* when $\epsilon^2/2 = 1 \bmod 4$.

Under $\tau \to -1/\tau$ the twisted sector's projected part picks up a factor of -1, exactly what we need to cancel the -1 on the left-moving side.

8. The partition function for our general heterotic $\mathcal{N} = 2$ compactification takes the form:

$$Z^{het}_{N=2} = \frac{1}{2}\sum_{h,g=0}^{1} \frac{\Gamma_{2,18}\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]\Gamma_{4,4}\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]}{\tau_2\eta^8\bar{\eta}^{24}} \frac{1}{2}\sum_{a,b=0}^{1} \frac{\theta^2\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a+h\\b+g\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a-h\\b-g\end{smallmatrix}\right]}{\eta^4}$$

We seek to compute $\tau_2 B_2$ where $B_2 = \mathrm{Tr}[(-1)^{2\lambda}\lambda^2]$ over our string's Hilbert space. To do this, consider the following *helicity generating* partition function:

$$\mathcal{Z}(\nu,\bar{\nu}) = \mathrm{Tr}[q^{L_0}\bar{q}^{\bar{L}_0}e^{2\pi i\nu\lambda_L - 2\pi i\bar{\nu}\lambda_R}] = \frac{1}{2}\sum_{h,g=0}^{1} \frac{\Gamma_{2,18}\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]\Gamma_{4,4}\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]}{\tau_2\eta^8\bar{\eta}^{24}}\xi(\nu)\bar{\xi}(\bar{\nu})\frac{1}{2}\sum_{a,b=0}^{1} \frac{\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a+h\\b+g\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a-h\\b-g\end{smallmatrix}\right]}{\eta^4} \tag{83}$$

Here

$$\xi(\nu) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n e^{2\pi i n\nu})(1-q^n e^{-2\pi i n\nu})} = \frac{\sin \pi\nu}{\pi}\frac{\theta_1'(\nu)}{\theta_1(\nu)}$$

plays the role of exchanging the traces over the bosons in the non-compact spatial (3,4) directions with traces that involve the helicity.

I apply formula **D.21** in Kiritsis to simplify the theta functions to:

$$\frac{1}{2}\sum_{a,b=0}^{1} \frac{\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a+h\\b+g\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}a-h\\b-g\end{smallmatrix}\right]}{\eta^4} = \frac{\theta^2\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](\frac{\nu}{2})\theta\left[\begin{smallmatrix}1-h\\1-g\end{smallmatrix}\right](\frac{\nu}{2})\theta\left[\begin{smallmatrix}1+h\\1+g\end{smallmatrix}\right](\frac{\nu}{2})}{\eta^4} \tag{84}$$

This vanishes at least as fast as ν^2

We must now take (83) this and apply

$$\left(\frac{1}{2\pi i}\partial_\nu - \frac{1}{2\pi i}\bar{\partial}_{\bar{\nu}}\right)^2 \mathcal{Z}(\nu,\bar{\nu}).$$

Because our generating function (83) vanishes as ν^2 thanks to (84), we only need to look at ∂_ν^2.

To obtain a nonzero result we thus need to act with ∂_ν^2. On these terms for each h, g. First note that for $h = g = 0$ (84) vanishes as ν^4 so will not contribute. For $(h, g) \neq (0, 0)$, the terms vanish as ν^2 due to the $\theta^2\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, exactly cancellable by taking two derivatives on that term. Thus, we need only worry about the zeroth order behavior of everything else: $\xi \sim 1$ and $\theta\left[\begin{smallmatrix}1-h\\1-g\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1+h\\1+g\end{smallmatrix}\right](\nu/2) \sim \theta\left[\begin{smallmatrix}1-h\\1-g\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1+h\\1+g\end{smallmatrix}\right](0)$. We are left with

$$\begin{aligned}
&-\frac{\pi^2}{4}\frac{1}{(2\pi)^2}\frac{1}{2}\sum_{h,g\neq(0,0)} \frac{\Gamma_{2,18}\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]}{\tau_2\eta^8\bar{\eta}^{20}}\frac{16\eta^2\bar{\eta}^2}{\theta^2\left[\begin{smallmatrix}1-h\\1-g\end{smallmatrix}\right]\bar{\theta}^2\left[\begin{smallmatrix}1-h\\1-g\end{smallmatrix}\right]}\theta\begin{bmatrix}1-h\\1-g\end{bmatrix}\theta\begin{bmatrix}1+h\\1+g\end{bmatrix}\left(\partial_\nu\theta\begin{bmatrix}1\\1\end{bmatrix}_{\nu=0}\right)^2\\
&= -\frac{4\eta^6}{2\tau_2\eta^6\bar{\eta}^{18}}\left[\frac{\Gamma_{2,18}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}{\bar{\theta}^2\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} + \frac{\Gamma_{2,18}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\bar{\theta}^2\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} - \frac{\Gamma_{2,18}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\bar{\theta}^2\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right]
\end{aligned}$$

Where we have used $\theta\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] = -\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, as well as $\theta_1'|_{\nu=0} = 2\eta^3$ We now use the identity

$$\bar{\theta}_2\bar{\theta}_3\bar{\theta}_4 = 2\bar{\eta}^3$$

and recover

$$\tau_2 B_2 = -\frac{\Gamma_{2,18}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\bar{\theta}_2\bar{\theta}_3}{\bar{\eta}^{24}} - \frac{\Gamma_{2,18}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\bar{\theta}_2\bar{\theta}_3}{\bar{\eta}^{24}} + \frac{\Gamma_{2,18}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\bar{\theta}_3\bar{\theta}_4}{\bar{\eta}^{24}}.$$

9. The gravitini can only come from the untwisted left-moving R sector (spinor spacetime index) tensored with an $\tilde{\alpha}^{2,3}_{-1}$ on the right (vector spacetime index). The zero-point energy of the left-moving R sector is 0 from equal numbers of bosons and fermions. Because our group acts on the (bosonized) fermions the same way it acts on the bosons, we get that $\mathbb{Z}_2^2$ gives the three nontrivial elements given by rotations $e^{2\pi i(s_1\phi_1 - s_2\phi_2)}, e^{2\pi i(s_1\phi_1 - s_3\phi_2)}, e^{2\pi i(s_2\phi_1 - s_3\phi_2)}$, with ϕ_0 corresponding to the spacetime fermions not appearing. We see that the only spinors which are invariant under these three transformations take the form

$$|\pm\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\rangle, |\pm\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

   And we must have an even number of signs by GSO projection, so we in fact get two supersymmetries preserved: $\tilde{\alpha}^{2,3}_{-1}|\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\rangle, \tilde{\alpha}^{2,3}_{-1}|-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\rangle$, providing the $\pm 3/2$ states only *one* gravitino.

10.

11. As before, the twist acts the same way on the bosons and (left moving) fermions. Already at this level, we see that the only invariant states $|s_1, s_2, s_3, s_4\rangle$ must satisfy $s_2 = s_3 = s_4$ so we will have the (GSO projected) possibilities:

$$|\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\rangle, |-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

   providing again the $\pm 3/2$ states of a *single gravitino.*

   To avoid anomaly from ground state energy mismatch, we need the condition of Polchinski **16.1.28**

$$\sum_{i=2}^{4} r_i^2 - \sum_{I=1}^{16} s_I^2 = 0 \text{ mod } 2N$$

   Here $N = 6$. Note that our $r_i = (1, 1, -2)$ already sums to 6, so we must have the same for our s_i that determines the Γ_{16} action.

   I am confused why Kiritsis is saying there is only one such action of $\mathbb{Z}_3$ on Γ_{16}. As long as $\sum s_i^2 = 0$ mod 6 we should get a consistent theory, as shown in **Table 16.1** of Polchinski.

   The simplest such twist (aside from the trivial one that leaves the $E_8 \times E_8$ untouched) would be to act on the first 3 complex fermions of forming the first E_8 group in the same way as we act on the complexified bosons and left-moving fermions, namely by

$$\tilde{\lambda}^{\pm,1,2,3} \to e^{\pm 2\pi i \beta_{1,2,3}} \tilde{\lambda}^{\pm,1,2,3}, \quad \beta_1 = \beta_2 = \tfrac{1}{3}, \beta_3 = -\tfrac{2}{3}$$

   while the remaining $\lambda^{\pm,4\ldots16}$ are left untouched. Let's now get the massless spectrum under $\mathbb{Z}_3 = \{1, r, r^2\}$

   - Untwisted
     - Left-moving The bosons are labeled by
   - Twisted by r
   - Twisted by r^2

1 Chapter 10: Loop Corrections to String Effective Couplings

1.

Chapter 11: Duality Connections and Nonperturbative Effects

1. Taking the expression for a toroidal heterotic compactification from exercise **9.1**

$$\left[\frac{R}{\sqrt{\tau_2}\eta\bar{\eta}^{17}}\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2} e^{-i\pi\sum_I nY^I(m+n\bar{\tau})Y^IY^I}\frac{1}{2}\sum_{a,b=0}^{1}\prod_{i=1}^{16}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}(Y^I(m+\bar{\tau}n)|\bar{\tau})\right]\times\frac{1}{\tau_2^{7/2}\eta^7\bar{\eta}^7}\frac{1}{2}\sum_{a,b=0}^{1}\frac{\theta^4\begin{bmatrix}a\\b\end{bmatrix}}{\eta^4}$$

Using θ function identites as in the second equation in appendex **E**, we get

$$\Gamma_{1,17}(R,Y)=\frac{R}{\sqrt{\tau_2}}\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\frac{1}{2}\sum_{a,b=0}^{1} e^{i\pi mY^IY^In-i\pi bnY^I}\bar{\theta}\begin{bmatrix}a-2nY^I\\b-2mY^I\end{bmatrix}$$

Now take $Y^I=0$ for $I=1\ldots 8$ and $Y^I=1/2$ for $I=1\ldots 16$. Then

$$\prod_I e^{i\pi mY^IY^In-i\pi bnY^I}=e^{i\pi m\sum_I(Y^I)^2-i\pi b\sum_I Y^I}=1$$

and we can ignore this term. Similarly because we are taking a product over 16 $\bar{\theta}$, no phases will interfere with us replacing $\theta\begin{bmatrix}u\\v\end{bmatrix}$ with $\theta\begin{bmatrix}-u\\-v\end{bmatrix}$ for integer u,v. This gives us the desired first step

$$\Gamma_{1,17}(R,Y)=R\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\frac{1}{2}\sum_{a,b=0}^{1}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}^8\bar{\theta}\begin{bmatrix}a+n\\b+m\end{bmatrix}^8$$

Now again because we have enough $\theta\begin{bmatrix}a+n\\b+m\end{bmatrix}$ that phases do not interfere, we see that we only care about n,m modulo 2 in the fermion term. We know how to divide the partition function of the compact boson into parity odd and even blocks by doing the $\mathbb{Z}^2$ stratification corresponding to the πR translation orbifold of the circle. This gives our desired answer:

$$\frac{1}{2}\sum_{h,g}\Gamma_{1,1}(2R)\begin{bmatrix}h\\g\end{bmatrix}\frac{1}{2}\sum_{a,b}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}^8\bar{\theta}\begin{bmatrix}a+h\\b+g\end{bmatrix}^8$$

with

$$\Gamma_{1,1}(2R)=2R\sum_{m,n}\exp\left[\frac{-\pi R^2}{\tau_2}|2m+g+(2n+h)\tau|^2\right]$$

2. As before, take the ansatz

$$ds^2=e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu+e^{2B(r)}dx^i\cdot dx^i,\qquad A_{012}=\pm e^{C(r)}\Rightarrow G_{r012}=\pm C'(r)\,e^{C(r)}$$

The BPS states in 11D require only the gravitino variation to vanish:

$$\delta\psi_M=\partial_M\epsilon+\frac{1}{4}\omega_M^{PQ}\Gamma_{PQ}\epsilon+\frac{1}{2\cdot 3!\cdot 4!}G_{PQRS}\Gamma^{PQRS}\Gamma_M\epsilon-\frac{8}{2\cdot 3!\cdot 4!}G_{MQRS}\Gamma^{QRS}\epsilon$$

We have worked out ω in **8.43**.

$$\omega_{\hat{\mu}\hat{\nu}}=0,\quad \omega_{\hat{\mu}\hat{i}}=(-)^{\mu=0}\,\partial_iA\,e^{A-B}dx^\mu,\quad \omega_{\hat{i}\hat{j}}=\partial_jBdx^i-\partial_iBdx^j$$

Let's look first at $M=\mu$ parallel. Since ϵ is Killing we expect no longitudinal variation and we get

$$\begin{aligned}0&=\cancel{\partial_\mu\epsilon}+\frac{1}{2}A'\,e^{A-B}\Gamma^{\hat{\mu}\hat{r}}\epsilon\pm\frac{1}{2\cdot 3!}C'(r)e^C\cancel{\Gamma^{r012}\Gamma_\mu}\epsilon\mp\frac{1}{3!}C'(r)e^C\Gamma_\mu\Gamma^{r012}\epsilon\\&=\frac{1}{2}A'\,e^{A-B}\Gamma^{\hat{\mu}\hat{r}}\epsilon\mp\frac{1}{3!}C'e^{C-B-2A}\Gamma^{\hat{\mu}\hat{r}\hat{0}\hat{1}\hat{2}}\epsilon\\&\Rightarrow 0=A'\epsilon\mp\frac{1}{3}C'e^{C-3A}\Gamma^{\hat{0}\hat{1}\hat{2}}\epsilon\end{aligned}$$

If we would like these two terms to be proportional, then we should take $C = 3A$, and we get the following condition for ϵ

$$(1 \mp \Gamma^{\hat{0}\hat{1}\hat{2}})\epsilon = 0$$

So half the dimension of the space of spinors satisfies this at any given point. We thus get

For $M = i$ transverse, we recall Γ_{ij} generates rotations, so assuming rotational invariance in the transverse space, we'll cancel this. We get

$$\begin{aligned}
&\partial_r \epsilon + \cancel{\frac{1}{4}\omega_r^{jk}\Gamma_{jk}\epsilon} + \cancel{\frac{1}{2\cdot 3!}G_{r012}\Gamma^{r012}}\Gamma_r\epsilon \mp \frac{1}{3!}G_{r012}\Gamma^{012}\epsilon = 0\\
\Rightarrow &\partial_r \epsilon \mp \frac{1}{3!}G_{r012}\Gamma^{012}\epsilon = 0\\
\Rightarrow &\partial_r \epsilon \mp \frac{e^{-3A}}{3!}C' e^C \Gamma^{\hat{0}\hat{1}\hat{2}}\epsilon
\end{aligned}$$

Solving this gives us that

$$\epsilon(r) = e^{C(r)/6}\epsilon_0$$

for ϵ_0 some constant spinor. We still do not have a relationship between C and B. This can be obtained by not assuming rotational invariance but rather imposing cancelation of the second and third terms above as follows:

$$\begin{aligned}
&\frac{1}{2}\partial_j B\,\Gamma^{\hat{i}\hat{j}}\epsilon \pm \frac{1}{2\cdot 3!}\partial_j C\, e^C\,\Gamma^{j012}\Gamma_i\epsilon\\
&= \frac{1}{2}\partial_j B\,\Gamma^{\hat{i}\hat{j}}\epsilon \pm \frac{1}{2\cdot 3!}\partial_j C\, e^{C-3A}\,\Gamma^{\hat{i}\hat{j}\hat{0}\hat{1}\hat{2}}\epsilon\\
&\Rightarrow \partial_j B + \frac{1}{3!}\partial_j C = 0
\end{aligned}$$

where we have used the condition on ϵ already obtained. Thus $C = 3A = -6B$. Finally Let's look at G's equation of motion:

$$\mathrm{d}G = 0, \qquad \frac{1}{3!}\mathrm{d}\star G + \frac{3}{(144)^2}\epsilon^{MNOPQRST}G_{MNOP}G_{QRST} = 0$$

By assumption, the term quadratic in G vanishes. What remains gives us:

$$0 = \partial_r(e^{3A+8B}e^{-6A-2B}C'(r)e^C) = \partial_r(e^{-3A+6B+C}C') = \partial_r(C'e^{-C}) \Rightarrow \partial_r^2 e^{-C} = 0$$

So we have that $e^{-C} = H(r)$ as required, where

$$H(r) = 1 + \frac{L^6}{r^6}$$

I'm happy with this. I could use Mathematica to show that the other EOM:

$$R_{MN} - \frac{1}{2}g_{MN}R = \kappa^2 T_{MN}, \quad \kappa^2 T_{MN} = \frac{1}{2\cdot 4!}\left(4G_{MPQR}G_N^{PQR} - \frac{1}{2}g_{MN}G^2\right)$$

is satisfied - but this is barely different from what I've done several times before for the D-branes and fundamental string solutions in chapter 8.

As before, this generalizes straightforwardly to multi-membrane configurations.

The charge of the M2 brane with $H = 1 + \frac{32\pi^2 N\ell_s^6}{r^6}$ is given by integrating $\frac{\star G}{2\kappa_{11}^2}$ on a seven-sphere at infinity. Here $2\kappa_{11}^2 = (2\pi)^8\ell_{11}^9$ Asymptotically we will get the field strength going as

$$\frac{32\times 6\pi^2 N\ell_{11}^6}{r^6}$$

Altogether, using $\Omega_7 = \frac{\pi^4}{3}$ this gives a total charge of

$$\frac{\pi^4}{3}\frac{32\times 6\pi^2 N\ell_{11}^6}{(2\pi)^8\ell_{11}^9} = \frac{N}{(2\pi)^2\ell_{11}^2}$$

This is exactly consistent with **11.4.10-13**, with $\mu = N = 1$ corresponding to a single M2 brane.

Calculating the Ricci scalar curvature in fact gives a *constant* as $r \to 0$ so we do *not* encounter a divergence. This signifies that this is just a coordinate singularity and we can extend past.

In[120]:= `R = RicciScalar[g, xx]`

Out[120]= $-\frac{6144\,\mathrm{ll}^{12}\,\mathrm{NN}^2\,\pi^4}{\left(1+\frac{32\,\mathrm{ll}^6\,\mathrm{NN}\,\pi^2}{r^6}\right)^{1/3}\left(32\,\mathrm{ll}^6\,\mathrm{NN}\,\pi^2\,r+r^7\right)^2}$

In[122]:= `Series[` $-\frac{6144\,\mathrm{ll}^{12}\,\mathrm{NN}^2\,\pi^4}{\left(1+\frac{32\,\mathrm{ll}^6\,\mathrm{NN}\,\pi^2}{r^6}\right)^{1/3}\left(32\,\mathrm{ll}^6\,\mathrm{NN}\,\pi^2\,r+r^7\right)^2}$ `, {r, 0, 0}]`

Out[122]= $-\frac{3}{(2\pi)^{2/3}\left(\frac{\mathrm{ll}^6\,\mathrm{NN}}{r^6}\right)^{1/3} r^2} + O[r]^1$

Finally, we can take the near-horizon limit and get

$$\begin{aligned} ds^2 &= \frac{r^4}{L^4}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dx^i \cdot dx^i \\ &= \frac{r^4}{L^4}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_7^2 \end{aligned}$$

Take now $r = L/\sqrt{z}$ to get the first term to look like $1/z^2$ while not affecting the second term much:

$$\frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + 4L^2 dz^2) + L^2 d\Omega_7^2$$

We can rescale z, x^μ and see that this geometry is $\mathrm{AdS}_4 \times S^7$

3. The M5 brane is now magnetically charged under C_3. Now the equations of motion $\mathrm{d} \star \mathrm{d}C = 0$ are trivially satisfied but the Bianchi identity is nontrivial, giving

$$\partial_r^2 H = 0 \Rightarrow H = 1 + \frac{L^3}{r^3}$$

The metric form can be fixed by analyzing the gravitino variation similar to before. Longitudinally:

$$\begin{aligned} 0 &= \frac{1}{2}A' e^{A-B}\Gamma^{\hat{\mu}\hat{r}} + \frac{1}{2\cdot 3!}C' e^{C+A-4B}\Gamma^{\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4\hat{\mu}} \\ &\Rightarrow A'\epsilon + \frac{1}{3!}C' e^{C-3B}\Gamma^{\hat{r}\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4}\epsilon \end{aligned}$$

We see that we must take $C = 3B$ and $A = -C/6$, and we get the half-BPS condition:

$$(1 - \Gamma^{\hat{7}\hat{8}\hat{9}\hat{1}\hat{0}\hat{1}\hat{1}})\epsilon = 0$$

The transverse components will give the profile for ϵ.

$$\partial_r \epsilon + \frac{1}{2\cdot 3!}C' e^{C-3B}\Gamma^{\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4\hat{r}}\epsilon$$

and this gives a profile

$$\epsilon = e^{-C/12}\epsilon_0$$

The membrane charge is given by integrating G on a 4-sphere whose area is given by $8\pi^2/3$, so we get

$$\frac{8\pi^2}{3}\frac{3\pi N \ell_{11}^3}{(2\pi^8)\ell_{11}^9} = \frac{N}{(2\pi\ell_{11})^5\ell_{11}}$$

Again we get that the Ricci scalar tends to a constant as $r \to 0$, giving regularity at the horizon. Again, this signifies that this is just a coordinate singularity and we can extend past.

```
In[138]:= R = RicciScalar[g, xx]

Out[138]= 3 ll^6 NN^2 π^2 / (2 (1 + ll^3 NN π / r^3)^(2/3) (ll^3 NN π r + r^4)^2)

In[139]:= Series[R, {r, 0, 0}]

Out[139]= 3 / (2 π^(2/3) (ll^3 NN / r^3)^(2/3) r^2) + O[r]^1
```

Taking the near-horizon limit we arrive at

$$ds^2 = \frac{r}{L}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dx^i \cdot dx^i = \frac{r}{L}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_4^2$$

Now take $r = L/z^2$ yielding

$$\frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + 4L^2 dr^2) + L^2 d\Omega_4^2$$

so again after rescaling the same was as before we get $\mathrm{AdS}_7 \times S^4$.

As before, a solution can consist of an arbitrary number of $M5$ branes at different places, in which case we get

$$H(r) = 1 + \sum_i \frac{L_i}{|r - r_i|^3}$$

This remains half-BPS.

4. First look at the field strengths. The general M5 brane solution For a uniform distribution of M5 charges, we know that in the transverse (3D) space the potential must now decay as

$$H = 1 + \int dx^{11} \frac{L}{|\vec{r} - x^{11}\hat{e}_{11}|^2} = 1 + \frac{2L}{r_{10D}^2}$$

where L depends on the density of the distribution. Then the 3-form field strength in 10D will just be

$$(dB)_{abc} = \epsilon_{abce}\partial_e H$$

Given this source in 10D, we have already worked out Einstein's equations in **Chapter 8**. Another way to see this is that we remain half-BPS after adding even an infinite number of parallel branes.

We have that $e^{4\Phi/3} = G_{11,11}$ so that $e^\Phi = H^{1/2}$ consistent with the NS5 solution.

Using the perscription of dimensional reduction in appendix **I.2**, we take $e^\sigma = e^{2\Phi/3} = H^{1/3}$. Using $g_{\mu\nu} = e^{-\sigma} g^S_{\mu\nu}$, we see that multiplying by $H^{1/3}$ takes us to the *string frame* NS5 metric solution.

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + H(r) dx^i \cdot dx^i$$

This is exactly the NS5 metric in string frame.

We can further take $g^S_{\mu\nu} = e^{\Phi/2} g^E_{\mu\nu}$ and multiply the string frame by $e^{-\Phi/2} = H^{-1/4}$ to get us to the Einstein frame.

5. Recall the BPS D6 brane in 10D is described by

$$H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2} d\vec{x} \cdot d\vec{x}, \quad H = 1 + \frac{L}{|x|}, L = g_s \ell_s N/2, \qquad F = L\, \mathrm{d}\Omega_2, \quad e^\Phi = g_s^2 H^{-3/4}$$

This means that $e^{-2\Phi/3} = H^{1/2}$. Multiplying ds^2_{string} by this factor, we the 10D part of 11D metric

$$\eta_{ab} d\gamma^a d\gamma^b + V d\vec{x} \cdot d\vec{x}$$

Here we've picked notation consistent with the problem so that $\gamma^{0\ldots6} = x^{0\ldots6}$, $H(r) = V(r)$, and x^i is the same.

Note also that

$$\frac{1}{2\kappa_{10}^2}\int_{S^2} F = \frac{L4\pi}{(2\pi)^7\ell_s^8 g_s^2} = nT_p \Rightarrow 2L = \ell_s n g_s$$

This should be supplemented by $e^{4\Phi/3}(d\gamma + A_\mu \cdot d\vec{x})^2 = V^{-1}(d\gamma + A_\mu \cdot d\vec{x})^2$ where A_μ is the 10D gauge field generated by the monopole solution.

Now A cannot be globally defined because of the monopole. Given $L = 2N$, it takes the same form as A_μ does in 3D about a monopole of charge $n = N/\ell_s$.

We could have taken a more "active" approach, demonstrating that this metric ansatz does indeed solve Einstein's equations, and shown that for the field strength to satisfy the Bianchi identity in this geometry it needed to indeed be a harmonic function of the transverse coordinates taken with flat metric.

6. The DBI action for a two-brane *in flat space with vanishing B-field and constant dilaton* is given in euclidean signature as

$$-T_2\int d^3x\sqrt{\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + 2\pi\ell_s^2 F_{ab})} + i\int C^{(3)} \wedge \mathrm{Tr}[e^{\mathcal{F}}] \wedge \mathcal{G},$$

where the second integral consists of Chern-Simons terms that we will ignore in this argument. We can work with the field variable F rather than A by imposing the Bianchi identity "by hand", namely writing the (non-CS) part of the action as

$$-T_2\int d^3x\left[\sqrt{\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + 2\pi\ell_s^2 F_{ab})} + \frac{i}{2}\lambda\epsilon^{abc}\partial_a F_{bc}\right]$$

This last term can just as well be integrated by parts to give $\epsilon^{abc}\partial_a\lambda F_{bc}$.

We now introduce an auxiliary V variable to rewrite the action as

$$\begin{aligned}&-T_2\int d^3x\left[\frac{1}{2}V\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + 2\pi\ell_s^2 F_{ab}) + \frac{1}{2}\frac{1}{V} + \frac{i}{2}\epsilon^{abc}\partial_a\lambda F_{bc}\right]\\ &= -T_2\int d^3x\left[\frac{1}{2}V(1 + \frac{1}{2}(2\pi\ell_s^2)^2 F_{ab}^2 + \dots) + \frac{1}{2}\frac{1}{V} + \frac{i}{2}\epsilon^{abc}\partial_a\lambda F_{bc}\right]\end{aligned}$$

here $\dots$ involves terms depending on the $\partial_a X^\mu$. The equations of motion for F then give

$$F_{ab} = -i\frac{\epsilon^{abc}\partial_a\lambda}{(2\pi\ell_s^2)^2 V}$$

Substituting this back in gives

$$-T_2\int d^3x\left[\frac{1}{2}V(1 + (-\frac{1}{2} + 1)(2\pi\ell_s^2)^{-2}(\partial\lambda)^2 + \dots) + \frac{1}{2}\frac{1}{V}\right]$$

Integrating out V gives us the square root action again, but now with F replaced by $\partial\lambda$, a new coordinate

$$-T_2\int d^3x\sqrt{\det(\delta_{ab} + \partial_a X^\mu \partial_b X^\nu + (2\pi\ell_s^2)^{-2}\partial_a\lambda\partial_b\lambda)}$$

Taking $X = \lambda/2\pi\ell_s^2$ gives our desired result

I have only shown classical equivalence. How to I prove this is quantum-mechanically true as well?

7. We are looking at the transformation $\tau \to -1/\tau$. We see taht

$$C_0 + ie^{-\Phi} \to \frac{-1}{C_0 + ie^{-\Phi}} = \frac{-C_0 + ie^{-\Phi}}{C_0^2 + e^{-2\Phi}}$$

So we see $C_0 \to -\frac{C_0}{C_0^2 + e^{-2\phi}}$ and $e^{-\Phi} \to \frac{e^{-\Phi}}{C_0^2 + e^{-2\Phi}}$. On the other hand, C_0 will not affect the C_2, B_2 transformations. Nor will it affect C_4, which remains invariant

In the Einstein frame the metric is invariant. That means that $e^{\Phi/2} g_{string}$ is invariant, which means g_{string} transforms as $e^{-\Phi/2}$ times the Einstein frame metric. Consequently, in the string frame $g'_{string} = e^{-\Phi} g_{string}$ (I think Kiritsis is wrong here, and Polchinski agrees with this)

Am I missing anything with that last one?

8. There's effectively nothing to derive. Translating the the Einstein frame means multiplying all lengths by $e^{-\Phi/4}$. At fixed dilaton this is $g_s^{-1/4}$. Given ℓ_s^2 in the denominator will then contribute a factor $\sqrt{g_s}$ overall, that's exactly what was done here.

9. We have that C_4 is invariant. That means that objects charged under C_4 remain charged under C_4, with the same charge. These are precisely the D3/anti-D3 branes. Now recall the DBI action has coupling constant

$$g_{YM}^2 = \frac{1}{(2\pi \ell_s^2)^2 T_3} = 2\pi g_s$$

note that this is dimensionless, as it should be for a gauge theory in 4D. At low energies, the closed strings decouple we can reliably trust the DBI action, considering the D-brane gauge theory on its own. In the absence of axion, the SL$(2, \mathbb{Z})$ of IIB takes $g_s \to 1/g_s$. This corresponds to

$$g_{YM}^2 \to \frac{4\pi^2}{g_{YM}^2}$$

So this is the Weak-Strong Montonen-Olive duality of $\mathcal{N} = 4$ SYM.

The only subtlety is that one must take care to include the Chern-Simons terms in the DBI action in order to get the full duality, specifically

$$\int C_0 \mathrm{Tr}[F \wedge F].$$

At fixed $C_0 = \theta/2\pi$ this produces the instanton number. The duality $C_0 \to C_0 + 1$ is a bona-fide duality of the $\mathcal{N} = 4$ theory, a consequence of the fact that instanton charge is quantized.

Is there anything else that I can say that constitutes any form of "showing" that this fact is true?

The only thing is I think I'm assuming that the D3 brane is the only object charged under C_3 at leading order in ℓ_s. Can I safely assume this?

10. We should go to the Einstein frame, ie multiply the F1 solution by $H^{1/4}$. The F1 solution is then:

$$ds_E^2 = H^{-3/4}(-dt^2 + (dx^1)^2) + H^{1/4} d\vec{x} \cdot d\vec{x}, \qquad H = 1 + \frac{L^6}{r^6}$$

Here $L^6 = \frac{2\kappa_{10}^2 T_p}{6\Omega_7} = 32\ell_s^6 \pi^2$ Note this is the same metric as the D1 solution, and indeed the metric will stay the same for all (p, q) strings.

The C_0 field has been set to zero. For F1 the dilaton and B-field have the profile

$$e^{\Phi} = g_s H^{-1/2}, \quad B_{01} = H^{-1}(r)$$

and indeed the dilaton has the inverse of this for the D1 while B and C exchange. Indeed, consider the SL$_2(\mathbb{Z})$ action

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here, we have $ad - bc = 1$, implying c, d are relatively prime. Further $\mathcal{S} = C_0 + ie^{-\phi}$ and C_2, B_2 transform as

$$\mathcal{S} \to \frac{a\mathcal{S} + b}{c\mathcal{S} + d}, \qquad \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to (\Lambda^T)^{-1} \cdot \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}$$

There is only one subtlety in the problem, which gives us the unique Λ we should take, which I got from reading arXiv:hep-th/9508143. We need to fix the dilaton's asymptotic value as $r \to \infty$ so as to define the vacuum of our string theory. First, consider $\phi, C_0 = 0$ asymptotically, ie $\mathcal{S} \to i$. We then stay within the $\mathrm{SO}(2) \subset \mathrm{SL}_2(\mathbb{R})$ that fixes $\mathcal{S} = i$. We want to take $(1,0)$ to the string p, q. This is now uniquely determined:

$$\Lambda = \frac{1}{\sqrt{q_1^2 + q_2^2}} \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$$

Applying this to B_2, C_2 gives

$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} = \frac{H^{-1}}{\sqrt{p^2+q^2}} \begin{pmatrix} p \\ q \end{pmatrix}$$

Upon doing this, the B_2, C_2 fluxes will have coefficients that get modified from just p, q by a factor of $\frac{1}{\sqrt{p^2+q^2}}$, so will no longer be integers satisfying the quantization condition. We can fix this by modifying $T \to T_{p,q} = \sqrt{p^2+q^2}T$. Since this only serves to modify L, which was an arbitrary parameter of the classical solution, we still remain in a solution.

This means: $H_{p,q} = 1 + \frac{L_{p,q}^6}{r^6}$, $L_{p,q}^6 = \frac{2\kappa^2 T_{p,q}}{6\Omega_7} = \sqrt{q_1^2+q_2^2}\,\frac{2\kappa^2 T_{1,0}}{6\Omega_7} = \sqrt{q_1^2+q_2^2}L^6$.

Our solution is now:

$$ds_E^2 = H_{p,q}^{-3/4}(-dt^2 + (dx^1)^2) + H_{p,q}^{1/4} d\vec{x} \cdot d\vec{x} \qquad \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} = \frac{H_{p,q}^{-1}}{\sqrt{p^2+q^2}} \begin{pmatrix} p \\ q \end{pmatrix} \qquad \mathcal{S} = \frac{ipH_{p,q}^{-1/2} - q}{iqH_{p,q}^{-1/2} + p}$$

Now, let us generalize this for different asymptotic values of the dilaton and axion. After applying Λ, compose with:

$$\Lambda' = \begin{pmatrix} e^{-\phi_0/2} & \chi_0 e^{\phi_0/2} \\ 0 & e^{\phi_0/2} \end{pmatrix}$$

$\mathcal{S}$ originally asymptotes to i, and now it asymptotes to

$$\frac{e^{-\phi_0/2} i + \chi_0 e^{\phi_0/2}}{0 + e^{\phi_0/2}} = \chi + ie^{-\phi_0}$$

exactly as we want. To get this to play well with the field strengths, we should make our first Λ take the form

$$(\Lambda^T)^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Again, applying this will break our quantization condition. The asymptotic value of the charges of B_2, C_2 is given by $(p,q)/\Delta_{p,q}^{1/2}$ where $\Delta_{p,q}$ is the invariant

$$\Delta_{p,q} = \begin{pmatrix} p & q \end{pmatrix} \mathcal{S}_2^{-2} \begin{pmatrix} |\mathcal{S}|^2 & \mathcal{S}_1 \\ \mathcal{S}_1 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = e^{\phi_0}(p - q\chi_0)^2 + e^{-\phi_0} q^2$$

Worth working out a bit more explicitly?

This gives in full generality:

$$T_{p,q} = \sqrt{e^{\phi_0}(p - q\chi_0)^2 + e^{-\phi_0} q^2}\; T_{F1}$$

Because (aside from redefining L) the metric is unchanged, the singularity structure of (p,q) strings is no different from $(1,0)$ or $(0,1)$ strings. Neither of these has a regular horizon.

11. The 11-D SUGRA Lagrangian is

$$L_{D=11} = \frac{1}{2\kappa_{11}^2}\left[R - \frac{1}{2}|G_4|^2 + G_4 \wedge G_4 \wedge \hat{C}_3\right]$$

Let's take M-theory to 9 dimensions. The R term becomes:

$$\frac{1}{2\kappa_{11}^2}e^{-2\phi}\Big[R+4\partial^\mu\phi\partial_\mu\phi+\frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta}-\frac{1}{4}G_{\alpha\beta}F^{A\ \alpha}_{\mu\nu}F^{A\,\mu\nu\beta}\Big]$$

with $\phi=-\frac{1}{4}\det G_{\alpha\beta}, F^A_{\mu\nu}=\partial_\mu A^\alpha_\nu-\partial_\nu A^\mu\alpha$. The kinetic 3-form potential yields

$$\frac{1}{2\kappa_{11}^2}e^{-2\phi}\Big[-\frac{1}{2}|F_4|^2-4\times\frac{1}{2}|F_3|^2-6\times\frac{1}{2}|F_2|^2\Big]$$

The IIB SUGRA Lagrangian is

$$e^{-2\Phi}\left[R+4(\nabla\Phi)^2-\frac{1}{2}|H_3|^2\right]-\frac{1}{2}|F_1|^2-\frac{1}{2}|F_3|^2-\frac{1}{4}|F_5|^2$$

supplemented by $\star F_5=F_5$. Taking this to 9 dimensions, the $R+4(\nabla\Phi)^2$ term becomes

$$\frac{1}{2\kappa_{10}^2}e^{-2(\Phi)+\sigma}\Big[R+4(\nabla\Phi)^2+(\partial_\mu\sigma)^2-\frac{1}{4}e^{-2\sigma}F^A_{\mu\nu}F^{A\,\mu\nu}-\frac{1}{2}|H_3|^2-\frac{1}{2}e^{-4\rho}|H_2|^2\Big]$$

with $G_{10,10}=e^{2\sigma}$. The RR forms give

$$e^{\sigma}\Big[-\frac{1}{2}(\partial_\mu C_0)^2-\frac{1}{2}|F_3|^2-\frac{1}{2}e^{-2\sigma}|F_2|^2-\underbrace{\frac{1}{4}|F_5|^2}_{\text{dualize}}-\frac{1}{4}e^{-2\sigma}|F_4|^2\Big]$$

Here F_2 comes from F_3 and we can dualize the 9D F_5 to an F_4.

To get to the 9D Einstein frame

12.

13. Again, we know the $(1,0)$ and $(0,1)$ 5-brane, namely the D5 and NS5.

14.

Chapter 12: Compactifications with Fluxes

1.

Chapter 13: Black Holes and Entropy in String Theory

1. We begin with
$$ds^2 = -F(r)C(r)dt^2 + \frac{dr^2}{C(r)} + H(r)r^2 d\Omega_2^2$$
and $C(r)$ vanishes at the horizon $r = r_0$ while all other functions are positive for $r \geqslant r_0$ and everything asymptotes to 1 as $r \to \infty$. Now, let's do a wick rotation $t \to i\tau$ with τ Euclidean time. We get
$$F(r)C(r)dt^2 + \frac{dr^2}{C(r)} + H(r)r^2 d\Omega_2^2$$
Now at $r = r_0 + \epsilon$ we see that the geometry takes the form
$$F(r_0)C'(r_0)(r - r_0)dt^2 + \frac{dr^2}{C'(r_0)(r - r_0)} + r_0^2 d\Omega_2^2$$
The last term is simply the expected metric on a 2-sphere of fixed radius r_0. The other two terms give a metric
$$ds^2 = F(r_0)C'(r_0)\epsilon dt^2 + \frac{d\epsilon^2}{C'(r_0)\epsilon}$$
Take $u = \frac{2\sqrt{\epsilon}}{\sqrt{C'(r_0)}}$ then $du = \frac{d\epsilon}{\sqrt{C'(r_0)\epsilon}}$ giving us
$$ds^2 = F(r_0)\frac{C'(r_0)^2}{4}u^2 dt^2 + du^2$$
This describe a conical deficit geometry in polar coordinates. In order to obtain a smooth geometry, we need the requirement that
$$\tau + 2\pi \times \frac{2}{C'(r_0)\sqrt{F(r_0)}} = \tau$$
Giving an inverse temperature of
$$\frac{1}{T} = \beta = \frac{4\pi}{C'(r_0)\sqrt{F(r_0)}}$$
This formula generalizes directly to higher-dimensional black holes. **Confirm.**

2. In what follows, recall the area formula for a general KN Black hole of mass charge and spin (M, Q, J) is:
$$A = 4\pi(r_+^2 + a^2), \quad r_+ = M + \sqrt{M^2 - a^2 - Q^2}, \quad a = J/M$$
In particular an extremal Kerr black hole has area $8\pi M^2$.

   (a) The areas of the individual Schwarzschild black holes are
   $$4\pi(2M_i)^2 = 16\pi M_i^2$$
   each. The area of their composite must then be $\geqslant 16\pi(M_1^2 + M_2^2)$. Because they start as almost stationary, the total angular momentum in the center of mass frame is zero, so the final black hole will be (essentially) Schwarzschild. So we get
   $$M_f^2 \geqslant M_1^2 + M_2^2$$
   If the initial masses were equal, we'd get $M_f \geqslant \sqrt{2}M$ so that $E = 2M - \sqrt{2}M$ and $E/(M_1 + M_2) = 1 - 1/\sqrt{2}$. Let's write WLOG $M_2 = \gamma M_1$ with $\gamma \leqslant 1$ then
   $$\begin{aligned} M_f^2 \geqslant (1 + \gamma^2)M_1^2 &\Rightarrow E = (1 + \gamma)M_1 - \sqrt{1 + \gamma^2}M_1 \\ &\Rightarrow \frac{E}{M_1 + M_2} = \frac{(1 + \gamma)M_1 - \sqrt{1 + \gamma^2}M_1}{(1 + \gamma)M_1} = 1 - \frac{\sqrt{1 + \gamma^2}}{(1 + \gamma)} \geqslant 1 \leqslant 1 - \frac{1}{\sqrt{2}} \end{aligned}$$
   as required.

(b) For two extremal RN black holes we have $r_+ = M$ so each has area $4\pi M^2$. They will collide to form a neutral (perhaps rotating) black hole. The area law gives us

$$4\pi((M_f + \sqrt{M_f^2 - a^2})^2 + a^2) \geqslant 2 \times 4\pi M^2.$$

This bound is sharpest if we take the final state to be extremal Kerr $a = M$, giving

$$8\pi M_f^2 \geqslant 8\pi M^2 \Rightarrow M_f \geqslant M$$

We get

$$E \leqslant 2M - M \Rightarrow \frac{E}{2M} \leqslant \frac{1}{2}.$$

(c) Such a decay would look like

$$2M_f^2 \geqslant M^2 \Rightarrow \sqrt{2}M_f \geqslant M \Rightarrow M - 2M_f \leqslant (\sqrt{2} - 2)M_F < 0.$$

This is a contradiction.

3. We have that $n = \frac{1}{\sqrt{g_{rr}}}\partial_r = \sqrt{f(r)}\partial_r$ so that

$$K_{\mu\nu} = \frac{1}{2}\frac{1}{\sqrt{g_{rr}}}\partial_r G_{\mu\nu} = \frac{\sqrt{f(r)}}{2}\text{diag}\Big(f'(r), -\frac{f'(r)}{f(r)}, 2r, 2r\sin^2\theta\Big)$$

Contracting with the 3×3 boundary inverse metric $h^{\mu\nu} = \text{diag}(f(r)^{-1}, r^{-2}, r^{-2}\sin^{-2}\theta)$ which has *no r component* gives

$$K = \frac{\sqrt{f}}{2}\left(\frac{f'}{f} + \frac{4}{r}\right) = \sqrt{f}\,\frac{rf' + 4f}{2rf}\Big|_{r=r_0}$$

where r_0 is large and formally infinite. We can then evaluate

$$\frac{1}{8\pi G}\int_{\partial M}\sqrt{h}K = \frac{4\pi r^2\beta\sqrt{f}}{8\pi G}K\Big|_{r=r_0} = \beta\frac{r}{4G}(rf' + 4f)\Big|_{r=r_0}$$

Directly evaluating this for $f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}$ gives

$$\frac{\beta}{2G}\left(\frac{Q^2}{r_0^2} + 2r_0 - 3GM\right)$$

This is the gravitational boundary term contribution to the classical action. The gravitational bulk term is zero since the Ricci scalar vanishes for the RN solution. The electromagnetic contribution is

$$\frac{1}{16\pi G}\int_M \sqrt{g}F_{\mu\nu}F^{\mu\nu} = \frac{1}{8\pi G}\int_0^\beta d\tau\int d\Omega_2\int_{r_+}^{r_0} r^2 dr\frac{Q^2}{r^4} = \beta\frac{4\pi Q^2}{8\pi G}\left(\frac{1}{r_+} - \frac{1}{r_0}\right) = \frac{\beta}{2G}Q^2\left(\frac{1}{r_+} - \frac{1}{r_0}\right)$$

All together, as $r_0 \to \infty$ we get action:

$$S_{RN} = -\frac{\beta}{2G}\left(2r_0 - 3GM + \frac{Q^2}{r_+}\right)$$

Note that there is one divergent term, namely the one linear in r_0 in the boundary action, but this is insensitive to the properties of the RN black hole and is also present in flat space. It is then sensible to define a regularized (renormalized) action by subtracting this term off. In doing this subtraction, there is an ambiguity of how we should define the inverse temperature of the reference flat space subtraction. The appropriately redshifted temperature **Justify** is $\beta\sqrt{f}$, giving reference action:

$$S_{flat} = -\frac{\beta}{G}r_0\sqrt{f(r_0)} = -\frac{\beta}{2G}(2r_0 - 2GM + O(1/r_0))$$

The renormalized Euclidean action is thus

$$I_{RN} = S_{RN} - S_{flat} = \frac{\beta}{2}(M - \frac{Q^2}{Gr_+}) = \frac{\beta}{2}(M - \mu Q) = \beta\mathcal{F}$$

4. The specific heat C is given by the coefficient in

$$dM = MCdT$$

For Schwarzschild, $T = (8\pi GM)^{-1}$ so this is

$$dM = -MC\frac{dM}{8\pi GM^2} \Rightarrow C = -8\pi GM$$

Which is negative. This should not be so surprising, given that by increasing the energy (ie mass) of the Schwarzschild black hole we make a larger one which thus have *lower* temperature. It is worth noting that, including units, this is proportional to $\frac{1}{\hbar}$.

5. First off, at $a = 0$ Kerr-Newman reproduces the RN black hole, which we already know is a solution of the Einstein-Maxwell system.

Further, it is quick to check using Mathematica that at $Q = 0$ the Kerr metric is itself Ricci-Flat: $R_{\mu\nu} = 0$ so is indeed a solution of the vacuum Einstein equations (away from $r = 0$).

```
In[591]:= (*Kerr*)
xx = {t, r, θ, ϕ};
Δ = r^2 + a^2 - 2 G M r;
Σ = r^2 + a^2 Cos[θ]^2;
g = {{-(Δ - a^2 Sin[θ]^2)/Σ, 0, 0, -(a Sin[θ]^2 (r^2 + a^2 - Δ))/Σ}, {0, Σ/Δ, 0, 0}, {0, 0, Σ, 0}, {-(a Sin[θ]^2 (r^2 + a^2 - Δ))/Σ, 0, 0, ((r^2 + a^2)^2 - Δ a^2 Sin[θ]^2)/Σ Sin[θ]^2}};
ginv = InverseMetric[g];
Riem = RiemannTensor[g, xx];
Ricc = RicciTensor[g, xx];
Ricc // MatrixForm
R = RicciScalar[g, xx]
```

Out[598]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[599]= 0

When $Q \neq 0$ we get a nonzero Ricci tensor (the Ricci scalar still vanishes since classical electrodynamics is conformal).

```
In[605]:= (*Kerr-Newman*)
xx = {t, r, θ, ϕ};
Δ = r^2 + a^2 + Q^2 - 2 G M r;
Σ = r^2 + a^2 Cos[θ]^2;
g = {{-(Δ - a^2 Sin[θ]^2)/Σ, 0, 0, -(a Sin[θ]^2 (r^2 + a^2 - Δ))/Σ}, {0, Σ/Δ, 0, 0}, {0, 0, Σ, 0}, {-(a Sin[θ]^2 (r^2 + a^2 - Δ))/Σ, 0, 0, ((r^2 + a^2)^2 - Δ a^2 Sin[θ]^2)/Σ Sin[θ]^2}};
ginv = InverseMetric[g];
Riem = RiemannTensor[g, xx];
Ricc = RicciTensor[g, xx];
Ricc // MatrixForm
R = RicciScalar[g, xx]
```

Out[612]//MatrixForm=

$$\begin{pmatrix} \frac{4 Q^2 \left(3 a^2+2 \left(Q^2-2 G M r+r^2\right)-a^2 \cos[2\theta]\right)}{\left(a^2+2 r^2+a^2 \cos[2\theta]\right)^3} & 0 & 0 & -\frac{8 a Q^2 \left(2 a^2+Q^2+2 r (-G M+r)\right) \sin[\theta]^2}{\left(a^2+2 r^2+a^2 \cos[2\theta]\right)^3} \\ 0 & -\frac{2 Q^2}{\left(a^2+Q^2+r (-2 G M+r)\right) \left(a^2+2 r^2+a^2 \cos[2\theta]\right)} & 0 & 0 \\ 0 & 0 & \frac{2 Q^2}{a^2+2 r^2+a^2 \cos[2\theta]} & 0 \\ -\frac{8 a Q^2 \left(2 a^2+Q^2+2 r (-G M+r)\right) \sin[\theta]^2}{\left(a^2+2 r^2+a^2 \cos[2\theta]\right)^3} & 0 & 0 & -\frac{4 Q^2 \left(-3 a^4-2 r^4-a^2 \left(Q^2-2 G M r+5 r^2\right)+a^2 \left(a^2+Q^2-2 G M r+r^2\right) \cos[2\theta]\right) \sin[\theta]^2}{\left(a^2+2 r^2+a^2 \cos[2\theta]\right)^3} \end{pmatrix}$$

Out[613]= 0

The Ricci tensor must correspond to an electromagnetic stress-energy tensor. It comes from an electric potential of the form $A_\mu = (\frac{rQ}{\Sigma}, 0, 0, -\frac{arQ\sin^2\theta}{\Sigma})$

```
In[619]:= A = {r Q/Σ, 0, 0, -(a r Q Sin[θ]^2)/Σ};
F = Table[D[A[[i]], xx[[j]]] - D[A[[j]], xx[[i]]], {i, 1, 4}, {j, 1, 4}] // FullSimplify;
F2 = Sum[ginv[[i, k]] ginv[[j, l]] F[[i, j]] F[[k, l]], {i, 1, 4}, {j, 1, 4}, {k, 1, 4}, {l, 1, 4}] // FullSimplify;
T = 2 (Table[Sum[ginv[[k, l]] F[[i, k]] F[[j, l]], {k, 1, 4}, {l, 1, 4}], {i, 1, 4}, {j, 1, 4}] - F2/4 g) // FullSimplify;
T - Ricc // Simplify // MatrixForm
```

Out[623]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For r very large we get an electric field going as $qr^2/\Sigma^2 \sim q/r^2$ corresponding to the electric field for a charge q, and we also get a magnetic field dying off as $a\cos\theta/r^3$ corresponding to the field from a spinning charged source. Said another way, we see that $\frac{1}{4\pi}\int \star F = q$ and $\frac{1}{4\pi}\int F = 0$ asymptotically, so we have just an electric charge q.

We can verify mass and angular momentum using the killing vectors ∂_t and ∂_ϕ respectively using the formulas in **Wald 12.3.8-9**

$$-\frac{1}{8\pi G}\int \epsilon_{abcd}\nabla^c(\partial_t)^d = M \frac{1}{16\pi G}\int \epsilon_{abcd}\nabla^c(\partial_\phi)^d \quad = aM$$

```
In[784]:= Γ = ChristoffelSymbol[g, xx];
-1/(8 π G) 4 π Limit[-2 r^2 Γ[[1, 2, 1]], r → Infinity]
1/(16 Pi G) 2 Pi Integrate[(-2 Limit[r^2 Γ[[1, 2, 4]], r → Infinity]) Sin[θ], {θ, 0, Pi}]

Out[785]= M

Out[786]= a M
```

For the KN black hole metric, the only singularities can come from $\Sigma = 0$ or $\Delta = 0$. Σ is only zero for $a > 0$ when $r = 0, \theta = \pi/2$. This corresponds to the curvature singularity of the black hole (in fact despite deceptive coordinate choice, this takes the form of a ring $S_1 \times \mathbb{R}$ as is revealed in Kerr-Schild coordinates). The horizons come from g_{rr} becoming singular, namely $\Delta = 0$ which occurs at

$$r^2 - 2GMr + a^2 + Q^2 = 0 \Rightarrow r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}.$$

These give the outer and inner horizons.

The horizon area is given by

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}}\Big|_{r=r_+} = 2\pi \int_0^\pi d\theta \sin\theta \sqrt{(r_+^2 + a^2)^2 - \Delta a^2 \sin^2\theta}$$

But $\Delta = 0$ at the horizon so this trivializes to

$$4\pi(r_+^2 + a^2) = 4\pi((m + \sqrt{m^2 - a^2 - Q^2})^2 + a^2)$$

The entropy of the black hole is then

$$S = \frac{A}{4} = \pi(r_+^2 + a^2)$$

Taking care to write things in terms of J and not a now, by holding J, Q fixed, let's vary M and get

```
In[678]:= rp = M + Sqrt[M^2 - (J/M)^2 - Q^2];
D[Pi (rp^2 + (J/M)^2), M]^-1 // Expand
```

Out[679]=

$$\frac{1}{\pi\left(-\frac{2 J^2}{M^3} + 2\left(1 + \frac{\frac{2 J^2}{M^3} + 2 M}{2\sqrt{-\frac{J^2}{M^2} + M^2 - Q^2}}\right)\left(M + \sqrt{-\frac{J^2}{M^2} + M^2 - Q^2}\right)\right)}$$

In[683]:=

$$\frac{\sqrt{-\frac{J^2}{M^2} + M^2 - Q^2}}{\pi\left(-\frac{2 J^2}{M^3}\sqrt{-\frac{J^2}{M^2} + M^2 - Q^2} + 2\left(\sqrt{-\frac{J^2}{M^2} + M^2 - Q^2} + \frac{J^2}{M^3} + M\right)\left(M + \sqrt{-\frac{J^2}{M^2} + M^2 - Q^2}\right)\right)} == \frac{1}{2 \mathrm{Pi}} \frac{\sqrt{-\frac{J^2}{M^2} + M^2 - Q^2}}{rp^2 + \left(\frac{J}{M}\right)^2} \quad \text{// Simplify}$$

Out[683]= True

The Hawking temperature is thus

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}$$

Now let's fix S and Q. We get

In[707]:= $\left(\frac{\texttt{-D[Pi ((M + Sqrt[M}^2 \texttt{ - J}^2 \texttt{/M}^2 \texttt{ - Q}^2\texttt{]))}^2 \texttt{ + (J/M)}^2\texttt{), J]}}{\texttt{D[Pi ((M + Sqrt[M}^2 \texttt{ - J}^2 \texttt{/M}^2 \texttt{ - Q}^2\texttt{]))}^2 \texttt{ + (J/M)}^2\texttt{), M]}} \texttt{ // Simplify} \right) \texttt{ == } \frac{\texttt{J/M}}{\texttt{rp}^2 \texttt{ + (J/M)}^2} \texttt{ // Simplify}$

Out[707]= True

Which gives us that

$$\Omega = \left(\frac{\partial M}{\partial J}\right)_{Q,S} = -\left(\frac{dS}{dJ}\right)_{Q,M} \left(\frac{dS}{dM}\right)_{Q,J}^{-1} = \frac{a}{r_+^2 + a^2}$$

Finally let's hold S, J fixed and do the same procedure, giving

In[709]:= $\left(\frac{\texttt{-D[Pi ((M + Sqrt[M}^2 \texttt{ - J}^2 \texttt{/M}^2 \texttt{ - Q}^2\texttt{]))}^2 \texttt{ + (J/M)}^2\texttt{), Q]}}{\texttt{D[Pi ((M + Sqrt[M}^2 \texttt{ - J}^2 \texttt{/M}^2 \texttt{ - Q}^2\texttt{]))}^2 \texttt{ + (J/M)}^2\texttt{), M]}} \texttt{ // Simplify} \right) \texttt{ == } \frac{\texttt{Q rp}}{\texttt{rp}^2 \texttt{ + (J/M)}^2} \texttt{ // Simplify}$

Out[709]= True

$$\mu = \left(\frac{\partial M}{\partial Q}\right)_{J,S} = -\left(\frac{dS}{dQ}\right)_{J,M} \left(\frac{dS}{dM}\right)_{Q,J}^{-1} = \frac{Q r_+}{r_+^2 + a^2}$$

The full form of the first law is then

$$dM = TdS + \Omega dJ + \mu dQ$$

We obtain an extremal black hole when $M = a^2 + Q^2$, as this is the minimum value of M where r_+ is a well-defined radius. At this value, $r_+ = r_-$.

Thermodynamic stability comes from minimizing the Gibbs free energy:

$$G = M - TS - \Omega J - \mu Q$$

Note that for flat space, $G = 0$, so if $G > 0$ for any of these black holes, thermal fluctuations will eventually drive their decay to flat space.

Plugging in what we have gives

In[726]:= $\texttt{T = } \frac{1}{\texttt{2 Pi}} \frac{\sqrt{-\frac{J^2}{M^2} + M^2 - Q^2}}{\texttt{rp}^2 + \left(\frac{J}{M}\right)^2}\texttt{; } \Omega \texttt{ = } \frac{\texttt{J/M}}{\texttt{rp}^2 \texttt{ + (J/M)}^2}\texttt{; } \mu \texttt{ = } \frac{\texttt{Q rp}}{\texttt{rp}^2 \texttt{ + (J/M)}^2}\texttt{;}$

$\texttt{S = Pi (rp}^2 \texttt{ + (J/M)}^2\texttt{);}$

$\texttt{M - T S - } \mu \texttt{ Q - } \Omega \texttt{ J // FullSimplify}$

Out[728]= $\frac{J^2 \left(4 M^2 - 2 Q^2\right) + M Q^4 \sqrt{-\frac{J^2}{M^2} + M^2 - Q^2}}{2 M \left(4 J^2 + Q^4\right)}$

Notice that if $J > 0$ then this will *always* be greater than zero, by virtue of the fact that $M > Q$ always. If we take $J = 0$, we get that this is still thermodynamically unstable unless $Q = M$ and the black hole is extremally charged.

6. The Hawking evaporation rate gets modified as

$$\Gamma_H = \frac{\sigma_{abs}(\omega)}{\exp(\beta(\hbar\omega - \vec{s} \cdot \vec{\Omega} - q\Phi)) \mp 1} \frac{d^3k}{(2\pi)^3}$$

where $\vec{s} \cdot \vec{\Omega}$ is the angular momentum product (orientation of $\vec{s}$ relative to $\vec{\Omega}$ matters). The $\mp$ is for bosons and fermions respectively.

Return to understand how this generalizes to systems more broadly

7.

8. The D5 and D5 branes are both BPS. We know that, upon toroidal compactification,

   In $D = 10$ have the D1 stretch $x_0 = t, x_5 = \gamma$ and the D5 stretch $x_0, \dots, x_5$, where we write $\gamma^a, a = 1 \dots 4$ to be the new D5 directions. These will form the direction of the T^4.

   Upon compactifying on $T^4 \times S^1$, the logic we used to for the 10D solution will still carry over to 5D. We will still write the extremal metric in terms of functions $H_{1,5}$ that must be harmonic w.r.t. the flat metric of the 4D transverse space.

   Then the D1 brane solution gives

$$ds^2_{D1} = \frac{-dt^2 + d\gamma^2}{\sqrt{H_1}} + \sqrt{H_1} d\gamma^a \cdot d\gamma^a + \sqrt{H_1}\, dx^i \cdot dx^i, \quad H_1 = 1 + \frac{r_1^6}{r^6}, \quad e^{-2\Phi} = H_1^{-1}, \quad F_{05i} = -\partial_i (H_1^{-1})$$

   While the D5 brane gives

$$ds^2_{D5} = \frac{-dt^2 + d\gamma^2}{\sqrt{H_5}} + \frac{d\gamma^a \cdot d\gamma^a}{\sqrt{H_5}} + \sqrt{H_5}\, dx^i \cdot dx^i, \quad H_5 = 1 + \frac{r_5^2}{r^2}, \quad e^{-2\Phi} = H_5^2$$

   We are also assuming here that at $r \to \infty$ we asymptote to the $g_s = 1$ vacuum solution of string theory.

   The dilaton and field strength contributions add while the metric contributions get multiplied **Justify**.

   The combined solution thus gives:

$$\frac{-dt^2 + d\gamma^2}{\sqrt{H_1 H_5}}$$

   Finish