

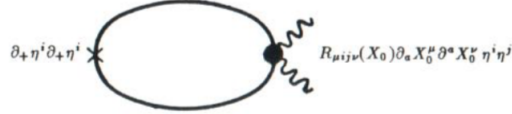
## Chapter 6: Strings in Background Fields

Note this chapter is specific to *closed oriented strings*. As such, we will not consider the effects of the boundary.

0. This is not a required problem but it certainly should be <sup>1</sup>. Let's *calculate the  $\beta$ -functions of the nonlinear sigma model*. Here, I will borrow diagrams from the very nice set of TASI lecture notes of Callan and Thorlacius

First, it is worth using a normal coordinate system for the  $X^\mu$  (one in which all of the  $\Gamma$  symbols vanish and all higher symmetrized  $\Gamma$  symbols also vanish). We want to look at radiative corrections to  $\langle T_{++} \rangle$ , since they have integrals that are easier to handle than those for  $\langle T_{+-} \rangle$ . From conservation this will give us the trace anomaly for  $\langle T_{+-} \rangle$ . We will first look at how  $G$ ,  $B$  affect the trace on a flat worldsheet.

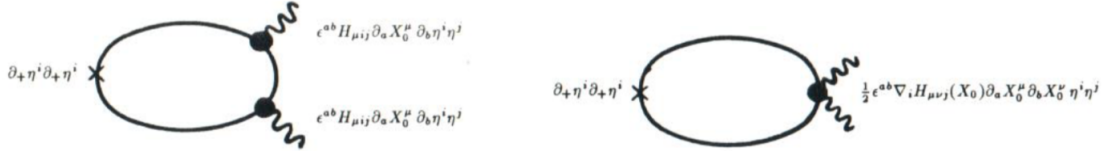
For the graviton contribution to  $\beta^G$ , we have only one diagram



This contributes an anomalous trace of

$$\langle T_{+-} \rangle = \frac{1}{4} R_{\mu\nu} \partial_a X_0^\mu \partial^a X_0^\nu$$

For the  $B$  contribution to  $\beta^B$ , we have two such diagrams:



These contribute anomalous traces of:

$$-\frac{1}{16} H_{\mu\rho\sigma} H_{\nu\rho\sigma} \partial_a X_0^\mu \partial^a X_0^\nu, \quad \frac{1}{8} \nabla^\lambda H_{\mu\nu\lambda} \epsilon^{ab} \partial_a X_0^\mu \partial_b X_0^\nu$$

respectively.

The dilaton contribution *also* affects the trace on the flat world sheet (even though it does not couple at  $R = 0$ ), by affecting the stress energy tensor as it is defined by varying the action w.r.t. the metric. Kiritsis has worked this out before and shown that the dilaton contributes  $(\partial_a \partial_b - g_{ab} \square) \Phi$  to the stress energy tensor, from which we get a dilaton contribution of  $\square_\xi \Phi(X(\xi))$  to the trace. Using covariant expressions for the D'alambertian we arrive at a contribution

$$\nabla_\mu \nabla_\nu \Phi(X_0) \partial_a X_0^\mu \partial^a X_0^\nu - \frac{1}{2} \nabla^\lambda \Phi(X_0) H_{\mu\nu\lambda}(X_0) \partial_a X_0^\mu \partial_b X_0^\nu \epsilon^{ab}$$

Combining all of this together, we see that we will get the  $\beta$ -functions:

$$\beta^G = R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}^{\rho\sigma} + 4 \nabla_\mu \nabla_\nu \Phi, \quad \beta^B = -\frac{1}{2} \nabla^\lambda H_{\lambda\mu\nu} - 2 \nabla^\lambda \Phi H_{\lambda\mu\nu}.$$

As pointed out, these are not quite that RG beta functions (for example compare  $\beta^B$  to the correct form in Kiritsis), but around the fixed point, they capture the correct first order behavior. In particular their vanishing will mean that we have no Weyl anomaly.

<sup>1</sup>After seeing the details of this calculation, I can understand why it was omitted.

Now we need to account for the effects of a curved worldsheet geometry. We can account for this by looking at a  $\langle T_{+-} T_{+-} \rangle$  correlator:

$$\frac{\delta}{\delta\phi(\xi)} \langle T_{+-}(0) \rangle_{e^\phi \delta_{ab}} = -\frac{1}{4\pi} \langle T_{+-}(\xi) T_{+-}(0) \rangle_{\delta_{ab}} \quad (1)$$

Again we can get this by first looking at  $\langle T_{++} T_{++} \rangle$  and appealing to conservation. The Weyl anomaly comes from this diagram:

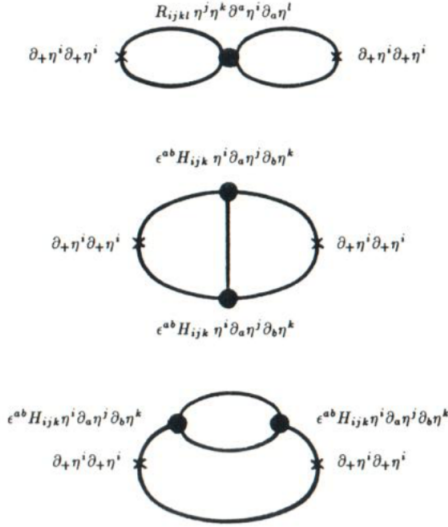


This gives  $\langle T_{+-} T_{+-} \rangle = \frac{\pi D}{12} \square \delta^{(2)}(\xi)$ . Here we have a factor of  $D$  coming from each degree of freedom. This can be used to integrate equation (1) to yield:

$$\langle T_{+-} \rangle = -\frac{D}{48} \square \phi = \frac{D}{24} \sqrt{\gamma} R$$

Note that the ghosts (which are otherwise decoupled) will here contribute their factor of  $-26$ .

We also now need to consider *two-loop* contributions of  $G, B$  to the  $TT$  correlator. The following diagrams contribute:



The calculations here are very involved, but will precisely give us

$$\frac{\alpha}{8} \left( -R + \frac{H^2}{12} \right)$$

Finally, the dilaton both modifies the energy-momentum tensor, giving rise to a tree-level propagator contribution to the two-point function:



This contributes  $\langle T_{+-}^{dil} T_{+-}^{dil} \rangle = \pi \alpha' (\nabla \Phi)^2 \square \delta^{(2)}(\xi)$  which will integrate to give a factor of  $\frac{\alpha'}{2} (\nabla \Phi)^2 \sqrt{\gamma} R$ .

Also, the dilaton gives a loop-contribution to the unmodified energy-momentum tensor:



Which contributes the term  $\langle T_{+-}^{dil} T_{+-}^{dil} \rangle = -\pi\alpha' \square \Phi \square \delta^{(2)}(\xi)$  which will integrate to give a factor of  $-\frac{\alpha'}{2} \square \Phi \sqrt{\gamma} R$ .

Altogether this gives:

$$\beta^\Phi = D - 26 + \frac{3}{2}\alpha' \left[ 4(\nabla\Phi)^2 - 4\square\Phi - R + \frac{1}{12}H^2 \right].$$

as required.

1. Each  $\beta$ -function of a coupling constant  $G, B, \Phi$  as given in **6.1.5**, **6.1.6**, **6.1.7** is  $\frac{\delta}{\delta\phi}$  of that coupling constant, since our scaling  $\mu = e^\phi \Rightarrow \log \mu = \phi$ . Since

$$T_a^a = \frac{\beta^\Phi}{12} R^{(2)} + \frac{1}{2\ell_s^2} (\beta_{\mu\nu}^G g^{\alpha\beta} + \beta_{\mu\nu}^B \varepsilon^{\alpha\beta}) \partial_\alpha X \partial_\beta X$$

The change in effective action under an infinitesimal Weyl transformation  $\delta g^{\alpha\beta} = -g^{\alpha\beta} \delta\phi$  is

$$\delta \log Z = -\delta S = \frac{1}{4\pi} \int d^2\xi \sqrt{g} T_a^a \delta\phi = \frac{1}{4\pi} \int d^2\xi \left[ \frac{\beta^\Phi}{12} \sqrt{g} R^{(2)} + \frac{1}{2\ell_s^2} (\beta_{\mu\nu}^G \sqrt{g} g^{ab} + \beta_{\mu\nu}^B \varepsilon^{ab}) \partial_a X \partial_b X \right] \delta\phi$$

We can integrate this to get the change after a finite conformal transformation:

$$\frac{1}{4\pi} \int d^2\xi \left[ \sqrt{g} \beta^\Phi \left( R^{(2)} \phi - \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi \right) + \frac{\phi}{2\ell_s^2} (\beta_{\mu\nu}^G + \beta_{\mu\nu}^B \varepsilon^{ab}) \partial_a X \partial_b X \right]$$

this vanishes, of course, when all beta functions are zero. When  $\beta^G, \beta^B$  are zero we can show (exercise 3) that  $\beta^\Phi$  is a constant, and we recover the Liouville action from before.

2. First write  $G$  explicitly in the action:

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\det G} e^{-2\Phi} \left[ R + 4G^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi - \frac{1}{12} G^{\alpha\delta} G^{\beta\epsilon} G^{\gamma\zeta} H_{\alpha\beta\gamma} H_{\delta\epsilon\zeta} + 2 \frac{26-D}{3\ell_s^2} \right]$$

The classical equations of motion from varying the action with respect to  $G$  give

$$\begin{aligned} 0 &= \overbrace{R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - 4\nabla_\mu \Phi \nabla_\nu \Phi - 2G_{\mu\nu} \square \Phi + 4G_{\mu\nu} (\nabla\Phi)^2}^{R \text{ variation}} + \overbrace{4\nabla_\mu \Phi \nabla_\nu \Phi}^{(\nabla\Phi)^2 \text{ variation}} \\ &\quad - \underbrace{\frac{1}{4} H_{\mu\rho\sigma} H_\nu^{\rho\sigma}}_{H^2 \text{ variation}} - \underbrace{\frac{1}{2} G_{\mu\nu} \left( R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 + \frac{2}{3\ell_s^2} (26-D) \right)}_{\sqrt{-\det G} \text{ variation}} \\ &= \underbrace{R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\rho\sigma} H_\nu^{\rho\sigma}}_{:=\beta_{\mu\nu}^G} - \frac{1}{2} G_{\mu\nu} \left( R - 4(\nabla\Phi)^2 + 4\square\Phi - \frac{1}{12} H^2 + 2 \frac{26-D}{3\ell_s^2} \right) \end{aligned} \quad (2)$$

With respect to  $B$  we get:

$$-\frac{1}{12} e^{-2\Phi} (2(\delta_{B^{\mu\nu}} (\partial_\alpha B_{\beta\gamma} + 2 \text{ perms.})) H^{\alpha\beta\gamma}) \xrightarrow{IBP} \frac{2 \times 3}{12} e^{-2\Phi} (\nabla^\alpha H_{\alpha\mu\nu}) \xrightarrow{IBP} \underbrace{-\frac{1}{4} \nabla^\alpha (e^{-2\Phi} H_{\alpha\mu\nu})}_{:=\beta_{\mu\nu}^B} = 0$$

Finally, with respect to  $\Phi$  we get:

$$0 = -2 \left( R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 + 2 \frac{26-D}{3\ell_s^2} \right) - 8\square\Phi - 16(\nabla\Phi)^2 = -2 \underbrace{\left( R - 4(\nabla\Phi)^2 + 4\square\Phi - \frac{1}{12} H^2 + 2 \frac{26-D}{3\ell_s^2} \right)}_{:= -\frac{2}{3} \beta^\Phi}$$

The term in parentheses is the same as the term in parentheses the bottom line of (2). This agrees with **Polchinski 3.7.21** (with appropriate conventions adopted)

$$\delta S = -\frac{1}{2\kappa^2} \int d^D x \sqrt{-\det G} e^{-2\Phi} \left[ \delta G^{\mu\nu} \left( \beta_{\mu\nu}^G - \frac{1}{2} G_{\mu\nu} \frac{2}{3} \beta^\Phi \right) + \delta B^{\mu\nu} \beta_{\mu\nu}^B + 2\delta\Phi \frac{2}{3} \beta^\Phi \right]$$

3. Let's look at  $\frac{2}{3\ell_s^2}\nabla\beta^\Phi$ . We get:

$$8\nabla_\nu\Phi\nabla_\mu\nabla^\nu\Phi - 4\Box\nabla_\mu\Phi - \nabla_\mu R + \frac{1}{6}(\nabla_\mu H_{\alpha\beta\gamma})H^{\alpha\beta\gamma}$$

The contracted Bianchi identity  $\nabla_\mu R = 2\nabla^\nu R_{\mu\nu}$  together with the vanishing of  $\beta_{\mu\nu}^G$  gives:

$$\nabla_\mu R = 2\nabla^\nu R_{\mu\nu} = \frac{1}{2}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) - 4\Box\nabla_\mu\Phi$$

which in turn gives

$$8\nabla_\nu\Phi\nabla_\mu\nabla^\nu\Phi - \frac{1}{2}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) + \frac{1}{6}(\nabla_\mu H_{\alpha\beta\gamma})H^{\alpha\beta\gamma}$$

The fact that  $H$  is exact gives us  $dH = 0$  so  $\partial_{[\alpha}H_{\beta\gamma\delta]} = 0$ . The symmetry properties of  $H$  imply that summing over the four cyclic permutations of this gives zero. Contracting with the metric then implies a contracted Bianchi-type identity for  $H$ , namely that  $\nabla^\alpha H_{\alpha\beta\gamma} = 0$ .

Using  $\beta^B = 0$  together with the Bianchi identity, we have  $0 = \nabla^\rho H_{\mu\nu\rho} = 2\nabla^\rho\Phi H_{\mu\nu\rho}$ . So we have that  $H$  is divergence-free, and  $\nabla^\rho\Phi$  dotted with any component of  $H$  is zero. This lets us rewrite:

$$\begin{aligned} -\frac{1}{2}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) &= -\frac{1}{2}H^{\nu\rho\sigma}\nabla_\nu H_{\mu\rho\sigma} \\ \frac{1}{6}\nabla_\mu(H_{\alpha\beta\gamma})H^{\alpha\beta\gamma} &= -\frac{1}{6}H^{\alpha\beta\gamma}(\nabla_\alpha H_{\beta\gamma\mu} - \nabla_\beta H_{\gamma\alpha\mu} + \nabla_\gamma H_{\alpha\beta\mu}) = -\frac{1}{6}H^{\nu\rho\sigma}\nabla_\nu H_{\mu\rho\sigma} \\ \Rightarrow \frac{1}{12\ell_s^2}\nabla_\mu\beta^\Phi &= \nabla_\nu\Phi\nabla_\mu\nabla^\nu\Phi - \frac{1}{12}\nabla^\nu(H_{\mu\rho\sigma}H_\nu^{\rho\sigma}) = -\frac{1}{2}\nabla^\nu\Phi R_{\mu\nu} - \frac{1}{12}\nabla^\nu H_{\mu\nu} \end{aligned}$$

**One last step. I am missing something.**

This gives that  $\nabla_\mu\beta^\Phi = 0$  as required. So  $\beta^\Phi = c$  is a constant.

4. We get a linear dilaton giving rise to a Liouville action with  $Q = 0$ . This is our familiar free massless boson in  $2D$  with  $1D$  target space. So we get a string propagating in a single dimension.
5. Note that the only relevant parameters are  $\ell_s$ , with units of length, and whatever length scales there are on the manifold, all of which depend on its volume (since its compact) as  $V^{1/D}$ . In particular  $c = \beta^\Phi$  depends on  $\ell_s$  as

$$c = D + O(\ell_s^2/V^{2/D}).$$

**I think this is correct, though it is different from Kiristis' equation.**

6. Note that a nonzero total flux of  $H$  over any closed 3-manifold is incompatible with  $H = dB$  for a single-valued  $B$ . We can write:

$$e^{\frac{i}{2\pi\ell_s^2}\int_M B} = e^{\frac{i}{2\pi\ell_s^2}\int_N H}$$

where  $M$  is the 2D manifold corresponding to the embedding of the world-sheet into the target space and  $N$  is any manifold whose boundary is  $M$ . We need this to be independent of  $N$ , so for any three-cycle  $M_3$  we need:

$$\frac{1}{2\pi\ell_s^2}\int_{M_3} H \in 2\pi\mathbb{Z} \Rightarrow \frac{1}{4\pi^2\ell_s^2}\int_{M_3} H \in \mathbb{Z}$$

7. (a) We have

$$H = 2R^2 \sin^2 \psi \sin \theta d\psi \wedge d\theta \wedge d\phi \Rightarrow \int_{S^3} H = \frac{(2\pi R)^2}{4\pi^2 \ell_s^2} = \frac{R^2}{\ell_s^2} \in \mathbb{Z}$$

(b) The dilaton is  $\Phi = 0$ . Using Mathematica, the Ricci tensor is:

$$R_{\mu\nu} = \text{diag}(2, 2 \sin^2 \psi, 2 \sin^2 \psi \sin^2 \theta)$$

Which gives a Ricci scalar of  $6/R^2$ . From the previous part,  $H_{123} = 2R^2 \sin^2 \psi \sin \theta$ . From the metric being diagonal, we get that  $H_{\mu\nu}^2 := H_{\mu\rho\sigma} H_\nu^{\rho\sigma}$  is diagonal. We have

$$H_{\mu\nu}^2 = \text{diag}(8, 8 \sin^2 \psi, 8 \sin^2 \psi \sin^2 \theta) \Rightarrow \beta^G = R_{\mu\nu} - \frac{1}{4} H_{\mu\nu}^2 = 0$$

as desired. Next,  $\beta_{\mu\nu}^B = -\frac{1}{2} \nabla^\alpha (H_{\mu\nu\alpha})$ . To take a contravariant divergence we divide by the volume element and differentiate, but the volume element is  $\sin^2 \psi \sin \theta$  which will give  $H/\sqrt{g}$  is a constant, so  $\beta_{\mu\nu}$  will vanish.

Lastly,  $H^2 = (2R^2)^2/R^6 = 2/R^6$  so that  $-R + \frac{1}{12} H^2 = -\frac{4}{R^2}$ . Ignoring ghosts, this gives a central charge of:

$$D - 6 \frac{\ell_s^2}{R^2} + O(\ell_s^4) = D - \frac{6}{k} + O(\ell_s^4)$$

as desired.

(c) Without using coordinates, the isometry of  $S^3$  is  $G = \text{SO}(4) = [\text{SU}(2) \times \text{SU}(2)]/\mathbb{Z}_2$ . To see that equivalence, think of  $S^3$  as the unit quaternions, and take  $\text{SU}(2) \times \text{SU}(2)$  act as unit quaternions on the left and right. We get a right  $G$ -action by:  $x \rightarrow a^{-1}xb$ . Note the kernel is the set of  $(a, b) \in G$   $ax = xb$  for all  $x$ . In particular, for  $x = 1$  we get  $a = b$  so the kernel lies in the diagonal subgroup. To act trivially on all quaternions,  $a$  must be in the center, and for the unit quaternions this is exactly  $\pm 1$ . So this is an injection  $\varphi : [\text{SU}(2) \times \text{SU}(2)]/\mathbb{Z}_2 \rightarrow \text{SO}(4)$ . Since  $\text{SO}(4)$  is compact and connected, it is generated by the image of exponentiating  $\mathfrak{so}(4)$ , and so surjectivity of  $\varphi$  at the level of the Lie algebras (which is true by dimension-counting) implies surjectivity and hence equivalence at the level of Lie groups.

So we see that  $\mathfrak{so}(4)$  acting on  $S^3$  is just a simultaneous left and right copt of  $\mathfrak{su}(2)$  acting on  $\text{SU}(2)$ . Thus, we view this as the CFT of a nonlinear sigma model with target space  $G = \text{SU}(2)$  and the left, right copies of the  $\mathfrak{su}(2)$  action correspond to currents  $J = g^{-1}\partial g$  and  $\bar{J} = \bar{\partial}g g^{-1}$

We indeed get the central charge  $c = \frac{3k}{k+2}$  which has the large  $k$  expansion  $3 - 6/k + O(1/k^2)$ . Since  $k$  in a non-negative integer in WZW models, except for the case  $k = 0$  corresponding to the trivial CFT, we must have  $k \geq 1$ , where we get  $R \geq \ell_s$ .

8. Here the metric has three degrees of freedom and  $B_{\mu\nu}, \Phi$  both have only one degree of freedom (which can be spatially varying).  $H$ , being a 3-index antisymmetric tensor, must vanish in 1+1D, and so we will always have  $\beta^B = 0$ . The other two constraints become:

$$0 = \beta_{\mu\nu}^G = \frac{1}{2} R g_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \Phi, \quad 0 = \beta^\Phi = -24 + \frac{3}{2} \ell_s^2 [4(\nabla\Phi)^2 - 4\Box\Phi - R]$$

Translational isometry implies that  $R, g$  depend on only the time variable  $t$ . The  $x$  variable can therefore parameterize either  $S^1$  or  $\mathbb{R}$  endowed with constant metric.

Now taking the trace of the first equation implies  $R(t) = -2\Box\Phi(x, t)$ . Then the second equation will give:

$$\frac{16}{\ell_s^2} = 4(\nabla\Phi(x, t))^2 - 2(\Box\Phi)(t)$$

The only way for this to work is for  $R = \Box\Phi = 0$  so that  $\nabla\Phi$  can be a constant. We then have  $\Phi = \alpha x + \beta t$  so that  $\alpha^2 + \beta^2 = 4/\ell_s^2$ , and  $g$  is Ricci flat everywhere (so we can pick it to be constant). In the case of either  $\alpha, \beta = 0$ , we can also safely take  $x, t$  respectively to be periodic without having  $\Phi$  be multi-valued.

9. We still have  $\beta^B = 0$ , but  $\beta^G = R_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi$  while  $\beta^\Phi = D - 26 + \frac{3}{2}\ell_s^2(4(\nabla\Phi)^2 - 4\Box\Phi - R)$

This can be recast in terms of a new 4D *Ricci flat* metric  $ds^2 = F(\phi)d\phi^2 + \phi R^2 d\Omega_3^2$ .

Using Mathematica again to take the trace of this gives  $R_{ij}$  for  $i = j \geq 1$  proportional to  $R^2\phi F'(\phi) + 8\phi F(\phi)^2 - R^2 F(\phi)$ . Solving this differential equation for  $F$  gives

$$F(\phi) = \frac{R^2\phi}{4\phi^2 + R^2 c_1}$$

Setting  $c_1 = 0$ ,  $F(\phi) = R^2/4\phi$  will also make  $R_{00}$  vanish. Then we can take the dilaton to be zero  $\Phi(\phi) = 0$ .

10. As stated in the problem, upon gauging the adapted compact  $U(1) : \theta \rightarrow \theta + \epsilon$ , which has radius  $2\pi$ , we modify our derivative operator to act as  $\partial_\alpha \theta \rightarrow \partial_\alpha \theta + A_\alpha$ , where  $A_\alpha$  gives our connection on the  $U(1)$  principal bundle associated with gauging the Killing symmetry. The action gets modified:

$$S \supseteq \frac{R^2}{4\pi\ell_s} \int |\partial\theta|^2 \rightarrow \frac{R^2}{4\pi\ell_s^2} \int |\partial\theta + A|^2$$

This is a new theory, but we can *return to the old one* by enforcing that  $A$  be pure gauge as follows: introduce an auxiliary field  $\phi$  and add to  $S$  the term

$$\frac{i}{2\pi} \int \phi \epsilon^{\alpha\beta} \partial_\alpha A_\beta = -\frac{i}{2\pi} \int d\phi \wedge A.$$

Integrating out  $\phi$  gives exactly a  $\delta$ -function enforcing  $\epsilon^{\alpha\beta} \partial_\alpha A_\beta = 0$ . This gives that  $A$  is closed, but it need not be exact if our manifold has nontrivial topology. Going around any cycle,  $\int A$  can pick up a factor of  $2\pi n$ .

For a closed, genus  $g$  Riemann surface, there are  $2g$  cycles labeled by  $a_i, b_i$ ,  $1 \leq i \leq g$  coming from viewing it as a  $2g$ -gon. we have *Riemann's bilinear identity*, namely for two closed 1-forms  $\omega_1, \omega_2$ ,

$$\int_\Sigma \omega_1 \wedge \omega_2 = \sum_{i=1}^g \left( \int_{a_i} \omega_1 \int_{b_i} \omega_2 - \int_{a_i} \omega_2 \int_{b_i} \omega_1 \right) \quad (3)$$

Now take  $\omega_1 = A$ ,  $\omega_2 = d\phi$ . Now (3) gives us that  $\frac{1}{2\pi} \int d\phi \wedge A$  will not be zero in general, but in the path integral, it suffices to have it be an integral multiple of  $2\pi$ , since then the nontrivial holonomies will have no contribution to the action. We have that  $A$  can have winding  $2\pi\mathbb{Z}$ , so the only solution is to have  $\phi$  have winding  $2\pi\mathbb{Z}$ . This will exactly leave over a factor of  $2\pi\mathbb{Z}$ . So we return to our original action by introducing the field  $\phi$  of period  $2\pi$ . (NB if I had kept the fields dimensionful, then  $\phi$  would have period  $2\pi/R$  when  $\theta$  has period  $2\pi R$ )

In this new, equivalent action, we can gauge-fix  $\theta = 0$  (do I need ghosts? No because this is abelian  $U(1)$ ) and integrate out  $A$ . We get:

$$\frac{\ell_s^4/R^2}{4\pi\ell_s^2} \int d^2\xi (\partial\phi)^2$$

so we have obtained the same action but now on a circle of radius  $\ell_s^2/R$  instead of  $R$ .

In doing this path integral we get a determinant factor of  $\sqrt{4\pi^2\ell_s^2/R^2} = 2\pi\ell_s/R$  for each mode. Using zeta function regularization this is equal to  $\sqrt{R/2\pi\ell_s}$  which we can understand as adding a  $-\frac{1}{2}\log(R/2\pi\ell_s)$  term to the action that will couple to the curvature  $R$  (**Show why**), this shifting the dilaton as required.

11. Let's use the conventions of the next problem to do this one. Here, we have a *single* compact coordinate  $\theta$ . From formula **F.3** specialized to this case, we get that the metric and Dilaton terms become

$$\int d^D x \sqrt{-\det \hat{G}_{\mu\nu}} e^{-2\Phi} \left[ \hat{R} + 4(\partial_\mu \Phi)^2 \right] = \int d^{D-1} x \sqrt{-\det G_{ij}} e^{-2\phi} \left[ R + 4(\partial_\mu \Phi)^2 + \frac{1}{4}(\partial_\mu G_{00})^2 - \frac{1}{4}G_{00}F_{\mu\nu}F^{\mu\nu} \right]$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  and  $R$  corresponds to  $G_{ij}$  while  $\hat{R}$  corresponds to the original  $\hat{G}_{\mu\nu}$ . From **F.6-F.9**, the antisymmetric tensor changes as:

$$-\frac{1}{12} \int d^D \sqrt{-\det \hat{G}} e^{-2\Phi} \hat{H}_{ijk} \hat{H}^{ijk} = - \int d^{D-1} x \sqrt{-\det G} \left[ \frac{1}{12} H_{ijk} H^{ijk} + \frac{1}{4} \hat{H}_{ij0} \hat{H}^{ij0} \right]$$

where  $H_{ij0} = \hat{H}_{ij0}$  and  $H_{ijk} = \hat{H}_{ijk} - (A_i H_{0jk} + 3 \text{ perms.})$ .

Now under T-duality  $G_{00} \rightarrow G_{00}^{-1}$  so that  $(\partial_\mu G_{00})^2 \rightarrow G_{00}^{-2} (\partial_\mu G_{00})^2$ . **FINISH**

12. This one is quick. We have

$$ds^2 = G_{00} d\theta^2 + 2G_{00} A_i dx^i dx^0 + G_{ij} dx^i dx^j, \quad B = B_j d\theta \wedge dx^j + (b_{ij} + A_i B_j) dx^i \wedge dx^j$$

Certainly we have  $\tilde{G}_{00} = 1/G_{00}$ ,  $\tilde{B}_i = G_{00} A_i / G_{00}$ . Then  $\tilde{A}_i = B_i$  is consistent both for the  $i, 0$  components of the line element and the  $dx^i \wedge dx^j$  components of the  $B$ -field as long as we keep  $\tilde{b}_{ij} = b_{ij}$  and  $\tilde{g}_{ij} = g_{ij}$ . Finally, the dilaton must be shifted by  $\Phi = \Phi - \frac{1}{2} \log G_{00}$ .

13. The  $N$  commuting isometries correspond to a fibration by  $N$ -dimensional tori over each point in the base space. As we have seen before (for strings valued in a  $N$ -dimensional torus target space), we have that modes are described by two momenta  $p_L, p_R$  that Lie on an integral lattice. Naively, we can rotate  $p_L, p_R$  by any  $\text{GL}(N)$  transformation, but the integrality condition restricts us to  $\text{GL}(N, \mathbb{Z})$ . Now  $\text{GL}(N)$  acts separately on the left and the right momenta, but we are allowed to exchange between these two by applying  $T$ -duality, which still preserves our Lorentzian norm, so the  $T$ -duality group gets enhanced to  $O(N, N, \mathbb{Z})$ .

14. This is clear, since orientation reversal acts trivially on  $g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$  while it acts with a minus sign on  $\epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$ . The corresponding vertex operators are:

$$: \partial X^\mu \bar{\partial} X_\mu e^{ikX} :, \quad : G_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu :, \quad R : e^{ikX} :$$

If we assume the tachyon  $: e^{ikX} :$  is negative under parity then so are the dilaton and graviton.

This is incompatible with **6.1.10**, as then parity will flip the sign of the dilaton in the exponential, substantially changing the action of the theory.