Chapter 11: Duality Connections and Nonperturbative Effects

1. Taking the expression for a toroidal heterotic compactification from exercise **9.1**

$$\left[\frac{R}{\sqrt{\tau_2}\eta\bar{\eta}^{17}}\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2} e^{-i\pi\sum_I nY^I(m+n\bar{\tau})Y^IY^I}\frac{1}{2}\sum_{a,b=0}^{1}\prod_{i=1}^{16}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}(Y^I(m+\bar{\tau}n)|\bar{\tau})\right]\times\frac{1}{\tau_2^{7/2}\eta^7\bar{\eta}^7}\frac{1}{2}\sum_{a,b=0}^{1}\frac{\theta^4\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}{\eta^4}$$

Using θ function identites as in the second equation in appendex **E**, we get

$$\Gamma_{1,17}(R,Y) = \frac{R}{\sqrt{\tau_2}}\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\frac{1}{2}\sum_{a,b=0}^{1} e^{i\pi mY^IY^In - i\pi bnY^I}\bar{\theta}\begin{bmatrix}a-2nY^I\\b-2mY^I\end{bmatrix}$$

Now take $Y^I = 0$ for $I = 1\ldots 8$ and $Y^I = 1/2$ for $I = 1\ldots 16$. Then

$$\prod_I e^{i\pi mY^IY^In - i\pi bnY^I} = e^{i\pi m\sum_I (Y^I)^2 - i\pi b\sum_I Y^I} = 1$$

and we can ignore this term. Similarly because we are taking a product over 16 $\bar{\theta}$, no phases will interfere with us replacing $\theta\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]$ with $\theta\left[\begin{smallmatrix}-u\\-v\end{smallmatrix}\right]$ for integer u, v. This gives us the desired first step

$$\Gamma_{1,17}(R,Y) = R\sum_{m,n} e^{-\frac{\pi R^2}{\tau_2}|m+n\tau|^2}\frac{1}{2}\sum_{a,b=0}^{1}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}^8\bar{\theta}\begin{bmatrix}a+n\\b+m\end{bmatrix}^8$$

Now again because we have enough $\theta\left[\begin{smallmatrix}a+n\\b+m\end{smallmatrix}\right]$ that phases do not interfere, we see that we only care about n, m modulo 2 in the fermion term. We know how to divide the partition function of the compact boson into parity odd and even blocks by doing the $\mathbb{Z}^2$ stratification corresponding to the πR translation orbifold of the circle. This gives our desired answer:

$$\frac{1}{2}\sum_{h,g}\Gamma_{1,1}(2R)\begin{bmatrix}h\\g\end{bmatrix}\frac{1}{2}\sum_{a,b}\bar{\theta}\begin{bmatrix}a\\b\end{bmatrix}^8\bar{\theta}\begin{bmatrix}a+h\\b+g\end{bmatrix}^8$$

with

$$\Gamma_{1,1}(2R) = 2R\sum_{m,n}\exp\left[\frac{-\pi R^2}{\tau_2}|2m+g+(2n+h)\tau|^2\right]$$

2. As before, take the ansatz

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}dx^i\cdot dx^i, \qquad A_{012} = \pm e^{C(r)} \Rightarrow G_{r012} = \pm C'(r)\, e^{C(r)}$$

The BPS states in 11D require only the gravitino variation to vanish:

$$\delta\psi_M = \partial_M\epsilon + \frac{1}{4}\omega_M^{PQ}\Gamma_{PQ}\epsilon + \frac{1}{2\cdot 3!\cdot 4!}G_{PQRS}\Gamma^{PQRS}\Gamma_M\epsilon - \frac{8}{2\cdot 3!\cdot 4!}G_{MQRS}\Gamma^{QRS}\epsilon$$

We have worked out ω in **8.43**.

$$\omega_{\hat{\mu}\hat{\nu}} = 0, \quad \omega_{\hat{\mu}\hat{i}} = (-)^{\mu=0}\,\partial_i A\, e^{A-B}dx^\mu, \quad \omega_{\hat{i}\hat{j}} = \partial_j B dx^i - \partial_i B dx^j$$

Let's look first at $M = \mu$ parallel. Since ϵ is Killing we expect no longitudinal variation and we get

$$\begin{aligned}0 &= \cancel{\partial_\mu}\epsilon + \frac{1}{2}A'\, e^{A-B}\Gamma^{\hat{\mu}\hat{r}}\epsilon \pm \frac{1}{2\cdot 3!}C'(r)e^C\cancel{\Gamma^{r012}\Gamma_\mu}\epsilon \mp \frac{1}{3!}C'(r)e^C\Gamma_\mu\Gamma^{r012}\epsilon\\ &= \frac{1}{2}A'\, e^{A-B}\Gamma^{\hat{\mu}\hat{r}}\epsilon \mp \frac{1}{3!}C' e^{C-B-2A}\Gamma^{\hat{\mu}\hat{r}\hat{0}\hat{1}\hat{2}}\epsilon\\ &\Rightarrow 0 = A'\epsilon \mp \frac{1}{3}C' e^{C-3A}\Gamma^{\hat{0}\hat{1}\hat{2}}\epsilon\end{aligned}$$

If we would like these two terms to be proportional, then we should take $C = 3A$, and we get the following condition for ϵ

$$(1 \mp \Gamma^{\hat{0}\hat{1}\hat{2}})\epsilon = 0$$

So half the dimension of the space of spinors satisfies this at any given point. We thus get

For $M = i$ transverse, we recall Γ_{ij} generates rotations, so assuming rotational invariance in the transverse space, we'll cancel this. We get

$$\begin{aligned}
&\partial_r \epsilon + \cancel{\frac{1}{4}\omega_r^{jk}}\Gamma_{jk}\epsilon + \cancel{\frac{1}{2 \cdot 3!} G_{r012}\Gamma^{r012}}\Gamma_r \epsilon \mp \frac{1}{3!} G_{r012}\Gamma^{012}\epsilon = 0 \\
\Rightarrow &\partial_r \epsilon \mp \frac{1}{3!} G_{r012}\Gamma^{012}\epsilon = 0 \\
\Rightarrow &\partial_r \epsilon \mp \frac{e^{-3A}}{3!} C' e^{C} \Gamma^{\hat{0}\hat{1}\hat{2}}\epsilon
\end{aligned}$$

Solving this gives us that

$$\epsilon(r) = e^{C(r)/6}\epsilon_0$$

for ϵ_0 some constant spinor. We still do not have a relationship between C and B. This can be obtained by not assuming rotational invariance but rather imposing cancelation of the second and third terms above as follows:

$$\begin{aligned}
&\frac{1}{2}\partial_j B\, \Gamma^{\hat{i}\hat{j}}\epsilon \pm \frac{1}{2 \cdot 3!}\partial_j C\, e^{C}\, \Gamma^{j012}\Gamma_i \epsilon \\
&= \frac{1}{2}\partial_j B\, \Gamma^{\hat{i}\hat{j}}\epsilon \pm \frac{1}{2 \cdot 3!}\partial_j C\, e^{C-3A}\, \Gamma^{\hat{i}\hat{j}\hat{0}\hat{1}\hat{2}}\epsilon \\
&\Rightarrow \partial_j B + \frac{1}{3!}\partial_j C = 0
\end{aligned}$$

where we have used the condition on ϵ already obtained. Thus $C = 3A = -6B$. Finally Let's look at G's equation of motion:

$$\mathrm{d}G = 0, \qquad \frac{1}{3!}\mathrm{d}\star G + \frac{3}{(144)^2}\epsilon^{MNOPQRST}G_{MNOP}G_{QRST} = 0$$

By assumption, the term quadratic in G vanishes. What remains gives us:

$$0 = \partial_r(e^{3A+8B}e^{-6A-2B}C'(r)e^{C}) = \partial_r(e^{-3A+6B+C}C') = \partial_r(C'e^{-C}) \Rightarrow \partial_r^2 e^{-C} = 0$$

So we have that $e^{-C} = H(r)$ as required, where

$$H(r) = 1 + \frac{L^6}{r^6}$$

I'm happy with this. I could use Mathematica to show that the other EOM:

$$R_{MN} - \frac{1}{2}g_{MN}R = \kappa^2 T_{MN}, \quad \kappa^2 T_{MN} = \frac{1}{2 \cdot 4!}\left(4G_{MPQR}G_N^{PQR} - \frac{1}{2}g_{MN}G^2\right)$$

is satisfied - but this is barely different from what I've done several times before for the D-branes and fundamental string solutions in chapter 8.

As before, this generalizes straightforwardly to multi-membrane configurations.

The charge of the M2 brane with $H = 1 + \frac{32\pi^2 N \ell_s^6}{r^6}$ is given by integrating $\frac{\star G}{2\kappa_{11}^2}$ on a seven-sphere at infinity. Here $2\kappa_{11}^2 = (2\pi)^8 \ell_{11}^9$ Asymptotically we will get the field strength going as

$$\frac{32 \times 6\pi^2 N \ell_{11}^6}{r^6}$$

Altogether, using $\Omega_7 = \frac{\pi^4}{3}$ this gives a total charge of

$$\frac{\pi^4}{3}\frac{32 \times 6\pi^2 N \ell_{11}^6}{(2\pi)^8 \ell_{11}^9} = \frac{N}{(2\pi)^2 \ell_{11}^2}$$

This is exactly consistent with **11.4.10-13**, with $\mu = N = 1$ corresponding to a single M2 brane.

Calculating the Ricci scalar curvature gives a divergence going as $1/r^2$ as we approach $r = 0$.

In[120]:= `R = RicciScalar[g, xx]`

Out[120]= $$-\frac{6144\, \mathtt{ll}^{12}\, \mathtt{NN}^2 \pi^4}{\left(1+\frac{32\, \mathtt{ll}^6\, \mathtt{NN}\, \pi^2}{r^6}\right)^{1/3} \left(32\, \mathtt{ll}^6\, \mathtt{NN}\, \pi^2 r + r^7\right)^2}$$

In[122]:= $$\mathtt{Series}\left[-\frac{6144\, \mathtt{ll}^{12}\, \mathtt{NN}^2 \pi^4}{\left(1+\frac{32\, \mathtt{ll}^6\, \mathtt{NN}\, \pi^2}{r^6}\right)^{1/3} \left(32\, \mathtt{ll}^6\, \mathtt{NN}\, \pi^2 r + r^7\right)^2},\ \{r, 0, 0\}\right]$$

Out[122]= $$-\frac{3}{(2\pi)^{2/3} \left(\frac{\mathtt{ll}^6\, \mathtt{NN}}{r^6}\right)^{1/3} r^2} + O[r]^1$$

Finally, we can take the near-horizon limit and get

$$\begin{aligned} ds^2 &= \frac{r^4}{L^4}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dx^i \cdot dx^i \\ &= \frac{r^4}{L^4}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_7^2 \end{aligned}$$

Take now $r = L/\sqrt{z}$ to get the first term to look like $1/z^2$ while not affecting the second term much:

$$\frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + 4L^2 dz^2) + L^2 d\Omega_7^2$$

We can rescale z, x^μ and see that this geometry is $\mathrm{AdS}_4 \times S^7$

3. The M5 brane is now magnetically charged under C_3. Now the equations of motion $\mathrm{d} \star \mathrm{d}C = 0$ are trivially satisfied but the Bianchi identity is nontrivial, giving

$$\partial_r^2 H = 0 \Rightarrow H = 1 + \frac{L^3}{r^3}$$

The metric form can be fixed by analyzing the gravitino variation similar to before. Longitudinally:

$$\begin{aligned} 0 &= \frac{1}{2}A' e^{A-B}\Gamma^{\hat{\mu}\hat{r}} + \frac{1}{2\cdot 3!}C' e^{C+A-4B}\Gamma^{\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4\hat{\mu}} \\ &\Rightarrow A'\epsilon + \frac{1}{3!}C' e^{C-3B}\Gamma^{\hat{r}\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4}\epsilon \end{aligned}$$

We see that we must take $C = 3B$ and $A = -C/6$, and we get the half-BPS condition:

$$(1 - \Gamma^{\hat{7}\hat{8}\hat{9}\hat{10}\hat{11}})\epsilon = 0$$

The transverse components will give the profile for ϵ.

$$\partial_r \epsilon + \frac{1}{2\cdot 3!}C' e^{C-3B}\Gamma^{\hat{\theta}_1\hat{\theta}_2\hat{\theta}_3\hat{\theta}_4\hat{r}}\epsilon$$

and this gives a profile

$$\epsilon = e^{-C/12}\epsilon_0$$

The membrane charge is given by integrating G on a 4-sphere whose area is given by $8\pi^2/3$, so we get

$$\frac{8\pi^2}{3}\frac{3\pi N \ell_{11}^3}{(2\pi^8)\ell_{11}^9} = \frac{N}{(2\pi\ell_{11})^5 \ell_{11}}$$

We again get a quadratic divergence of the Ricci curvature scalar

In[138]:= `R = RicciScalar[g, xx]`

Out[138]= $$\frac{3\, \mathtt{ll}^6\, \mathtt{NN}^2 \pi^2}{2\left(1+\frac{\mathtt{ll}^3\, \mathtt{NN}\, \pi}{r^3}\right)^{2/3} \left(\mathtt{ll}^3\, \mathtt{NN}\, \pi r + r^4\right)^2}$$

In[139]:= `Series[R, {r, 0, 0}]`

Out[139]= $$\frac{3}{2\pi^{2/3}\left(\frac{\mathtt{ll}^3\, \mathtt{NN}}{r^3}\right)^{2/3} r^2} + O[r]^1$$

Taking the near-horizon limit we arrive at

$$ds^2 = \frac{r}{L}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dx^i \cdot dx^i = \frac{r}{L}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_4^2$$

Now take $r = L/z^2$ yielding

$$\frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + 4L^2 dr^2) + L^2 d\Omega_4^2$$

so again after rescaling the same was as before we get $\mathrm{AdS}_7 \times S^4$.

As before, a solution can consist of an arbitrary number of $M5$ branes at different places, in which case we get

$$H(r) = 1 + \sum_i \frac{L_i}{|r - r_i|^3}$$

This remains half-bps.

4. First look at the field strengths. The general M5 brane solution For a uniform distribution of M5 charges, we know that in the transverse (3D) space the potential must now decay as

$$H = 1 + \int dx^{11} \frac{L}{|\vec{r} - x^{11}\hat{e}_{11}|^2} = 1 + \frac{2L}{r_{10D}^2}$$

where L depends on the density of the distribution **work out explicitly**. Then the 3-form field strength in 10D will just be

$$(dB)_{abc} = \epsilon_{abce}\partial_e H$$

Given this source in 10D, we have already worked out Einstein's equations in **Chapter 8**. Another way to see this is that we remain half-BPS.

The dilaton comes from $G_{11,11} = H^{2/3}$