

Chapter 8: D-Branes

1. First, a simple magnetic monopole for a 1-form gauge field in D spacetime dimensions has a radial magnetic field $B_r = \frac{\tilde{Q}_1}{\Omega_{D-2} r^{D-2}}$ where $\Omega_{D-2} = 2\pi^{d/2}\Gamma(d/2)$ is the volume of a unit $D-2$ sphere. This way, the flux of the solution over any $D-2$ sphere surrounding the (point) monopole will be \tilde{Q}_1 .

Upon taking the Hodge star we get the solution is $F = \tilde{Q}_1 \sin\theta d\theta \wedge d\phi$. We can write this as $A = \tilde{Q}_1(c - \cos\theta)d\phi$. Taking $c = 1$ we get A vanishes at $\theta = 0$ (which we need since the ϕ coordinate degenerates there) while taking $c = -1$ we get A vanishes at $\theta = \pi$, which we also need.

We cannot have *both* solutions, and so we realize we are dealing with two A s, corresponding to local sections of a line bundle over S^2 on different hemispheres. Let A^+ be well-defined on all points on S^2 except $\theta = \pi$. Then A^+ is a section of a line bundle on the punctured sphere. The punctured sphere is contractible so any fiber bundle over it is trivial, so A^+ is just a *function* on the punctured sphere $S^2 \setminus \{\theta = \pi\}$. So let's define $A^+ = \tilde{Q}_1(1 - \cos\theta)d\phi$. Similarly, we define A^- to be the nonsingular A on the sphere with $\theta = 0$ removed, namely $A^- = \tilde{Q}_1(-1 - \cos\theta)d\phi$.

On the overlap, $A^+ - A^-$ differ by an integer, which labels the degree of “twisting” of this line bundle over S^2 .

For a p form, our monopole will now be spatially extended in $p-1$ directions. Label these (locally), by $x^1 \dots x^{p-1}$. Time is x^0 . Locally transverse to these coordinates will be $r, \varphi^1, \dots, \varphi^{D-1-p}$, where φ^i parameterize a $D-1-p$ sphere enclosing the monopole. The field strength looks like:

$$F = \tilde{Q}_p \Omega_{D-p-1}$$

where Ω is the canonical $D-p-1$ -sphere area form:

$$\Omega = \sin^{D-p-2}(\varphi_1) \sin^{D-p-3}(\varphi_2) \dots \sin(\varphi_{D-p-2}) d\varphi_1 \wedge \dots \wedge d\varphi_{D-p-1}$$

This can be written (unfortunately unavoidably) in terms of a hypergeometric function:

$$A = {}_2F_1\left(\frac{1}{2}, \frac{D-p-1}{2}, \frac{D-p+1}{2}, \sin^2(\varphi_1)\right) \frac{\sin^{D-p-1}(\varphi_1)}{D-p-1} d\varphi_2 \wedge \dots \wedge d\varphi_{D-p-1}$$

there is no need for an overall constant, as the function above vanishes at both $\varphi_1 = 0$ and π , *however* this is compensated by the hypergeometric function having a branch cut at $\varphi_1 = \pi/2$. Across this cut, it will have a discontinuity set by an integer depending on the convention of the arcsin function, and again we will have $A^+ - A^-$ differing by an integer. The same quantization condition follows.

Again A^+ will be defined on the S^{D-p-1} sphere minus the south-pole (this is homeomorphic to the $D-p-1$ ball, and hence contractible, so again the line bundle trivializes and A^+ is a bona-fide function for any D, p) and A^- is similarly defined on the sphere with the excision of the north pole.

2. Our simply charged point particle with a Wilson line $A_9 = \chi/2\pi R$ turned on will have an action

$$S = \int d\tau \underbrace{\left(\frac{1}{2} \dot{x}^M \dot{x}_M - \frac{m^2}{2} + q A_9 \dot{x}^9 \right)}_{\mathcal{L}}$$

The canonical momentum will be $p_i = \dot{x}^\mu$ for $\mu = 0 \dots 8$ and $p_9 = \dot{x}^9 + \frac{q\chi}{2\pi R}$. Consequently, our hamiltonian is

$$\begin{aligned} H &= p_M \dot{x}^M - \mathcal{L} = \frac{1}{2} p_\mu p^\mu + p_9 \left(p_9 - \frac{q\chi}{2\pi R} \right) - \left[\frac{1}{2} \left(p_9 - \frac{q\chi}{2\pi R} \right)^2 - \frac{m^2}{2} + \left(p_9 - \frac{q\chi}{2\pi R} \right) \frac{q\chi}{2\pi R} \right] \\ &= \frac{1}{2} \left(p_\mu p^\mu + \left(p_9 - \frac{q\chi}{2\pi R} \right)^2 + m^2 \right) \\ &= \frac{1}{2} \left(p_\mu p^\mu + \left(\frac{2\pi n - q\chi}{2\pi R} \right)^2 + m^2 \right) \end{aligned}$$

3. For a string satisfying Dirichlet boundary conditions, the total momentum is not conserved (along the directions associated with the D boundary conditions). This is easily interpretable as momentum transfer to the brane that it is attached to.
4. For an open string of state $|ij\rangle$, A_9 will act as $\frac{\chi_i - \chi_j}{2\pi i}$. Since this is an open string with no winding, we can only have momentum contribution, and so we will get a mass formula

$$m_{ij}^2 = \frac{\hat{N} - \frac{1}{2}}{\ell_s^2} + \left(\frac{n}{R} - \frac{\chi_i - \chi_j}{2\pi R} \right)^2$$

In particular at the lowest (massless) level for a string without momentum we will get the desired spectrum

$$m_{ij}^2 = \left(\frac{\chi_i - \chi_j}{2\pi R} \right)^2$$

5. For completeness, we will do both the gauge boson and scalar scattering. These come from the NS sector, and are given by:

$$V_{-1}^{a,\mu} = g_p \lambda^a \psi^\mu e^{-\phi} c e^{ipX}, \quad V_0^{a,\mu} = \frac{g_p}{\sqrt{2\ell_s}} \lambda^a (i\dot{X} + 2p \cdot \psi \psi^\mu) c e^{ipX}$$

We can explicitly scatter four such gauge bosons - two in the -1 picture and two in the 0 picture.

$$\begin{aligned} & iC_{D^2} \delta^{10}(\Sigma k) \times g_p^4 \langle cV_{-1}(y_1) cV_{-1}(y_2) cV_0(y_3) cV_0(y_4) \rangle + 5 \text{ perms.} \\ &= \frac{i\delta^{p+1}}{g_p^2 \ell_s^4} \times \frac{g_p^4}{2\ell_s^2} \langle [\psi^{\mu_1} e^{ik_1 X}]_{y_1} [\psi^{\mu_2} e^{ik_2 X}]_{y_2} [(i\dot{X}^{\mu_3} + 2k_3 \cdot \psi \psi^{\mu_3}) e^{ik_3 X}]_{y_3} [(i\dot{X}^{\mu_4} + 2k_4 \cdot \psi \psi^{\mu_4}) e^{ik_4 X}]_{y_4} \rangle \\ & \quad \times \langle c(y_1) c(y_2) c(y_3) \rangle \langle e^{-\phi(y_1)} e^{-\phi(y_2)} \rangle \end{aligned}$$

Take $y_1 = 0, y_2 = 1, y_3 = \infty$ and integrate $y_4 = y$ from 0 to 1 (then we'll have 5 more terms coming from permutations). There are five contributions.

- Contracting the $\psi^{\mu_1}(0)\psi^{\mu_2}$ together and allowing the remaining 4 terms at y_3, y_4 to contract either amongst themselves or with various vertex operators.
- Not contracting the first two ψ and Contracting the $i\dot{X}(y_3)$ with any of the vertex operators while contracting the last ψ with the first two
- Not contracting the first two ψ and contracting the $i\dot{X}(y_4)$ with any of the vertex operators while contracting the third ψ with the first two
- Forgetting the $i\dot{X}$, and contracting the ψ at y_1 with the various ψ at y_3 (consequently the ψ at y_2 with the ψ at y_4)
- Swapping 3 with 4 in the above (this gives an overall minus sign by fermionic statistics)

Integrating this will give two types of terms: $\eta^{ab}\eta^{cd}$ and $\eta^{ab}k^c k^d$. Our shorthand replaces the superscript μ_i by just i . Below, I underline the terms that contribute to the first type:

$$\begin{aligned} & \frac{ig_p^2 \delta^{p+1}}{\ell_s^4} \int_0^1 dy \frac{y^{2\ell_s^2 k_1 \cdot k_4} y^{2\ell_s^2 k_2 \cdot k_4}}{2\ell_s^2} \left\{ -\ell_s^2 \eta^{12} \left[\frac{2\ell_s^2 \eta^{34}}{y} + (2\ell_s^2)^2 (-\eta^{34} k_3 \cdot k_4 + k_3^3 k_4^4) + (2\ell_s^2)^2 (k_2^3 + k_4^3) \left(\frac{k_1^4}{y} + \frac{k_2^4}{y-1} \right) \right] \right. \\ & \quad + \ell_s^2 (2\ell_s^2)^2 [(k_2^3 + y k_4^3) (-k_4^1 \eta^{24} + k_4^2 \eta^{14})] \\ & \quad + \ell_s^2 (2\ell_s^2)^2 \left[\left(\frac{k_1^4}{y} + \frac{k_2^4}{y-1} \right) (-k_3^1 \eta^{23} + k_3^2 \eta^{13}) \right] \\ & \quad + \ell_s^2 \frac{(2\ell_s^2)^2}{y-1} [\eta^{13} \eta^{24} k_3 \cdot k_4 + \eta^{34} k_3^1 k_4^2 - \eta^{13} k_4^2 k_3^4 - \eta^{24} k_4^3 k_3^1] \\ & \quad \left. - \ell_s^2 \frac{(2\ell_s^2)^2}{y} [\eta^{14} \eta^{23} k_3 \cdot k_4 + \eta^{34} k_3^2 k_4^1 - \eta^{13} k_4^2 k_3^4 - \eta^{24} k_4^3 k_3^1] \right\} \end{aligned} \quad (1)$$

Using $s = -2\ell_s^2 k_1 \cdot k_2$ etc we see the underlined terms contribute

$$\begin{aligned} & \frac{ig_p^2 \delta^{p+1}}{\ell_s^2} \left[\frac{\Gamma(1-u)\Gamma(1-t)}{\Gamma(2+s)} (-(1+s)\eta^{12}\eta^{34}) + \frac{\Gamma(1-u)\Gamma(-t)}{\Gamma(1+s)} \eta^{14}\eta^{23}s + \frac{\Gamma(-u)\Gamma(1-t)}{\Gamma(1+s)} \eta^{14}\eta^{23}s \right] [1423] \\ &= \frac{ig_p^2 \delta^{p+1}}{\ell_s^2} \frac{\Gamma(-u)\Gamma(-t)}{\Gamma(1+s)} (-t u \eta^{12}\eta^{34} - s u \eta^{13}\eta^{24} - s t \eta^{14}\eta^{23}) [1423] \end{aligned}$$

The non-underlined terms are more involved but end up contributing twelve terms that yield:

$$\frac{ig_p^2 \delta^{p+1}}{\ell_s^2} \left[\frac{\Gamma(-u)\Gamma(1-t)}{\Gamma(1+s)} 2\ell_s^2 \eta^{12} k_2^3 k_1^4 + \dots \right] [1423] = \frac{ig_p^2 \delta^{p+1}}{\ell_s^2} 2\ell_s^2 \frac{\Gamma(-u)\Gamma(-t)}{\Gamma(1+s)} [t \eta^{12} k_2^3 k_1^4 + 11 \text{ perms.}] [1423]$$

Here my Mandelstam variables are dimensionless. The result with dimensionful Mandelstam variables is:

$$ig_p^2 \delta^{p+1} \ell_s^2 \left(\frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 u)}{\Gamma(1+\ell_s^2 t)} K_4(k_i, e_i) ([1234] + [4321]) + 2 \text{ perms.} \right) \quad (2)$$

with $K_4(k_i, e_i) = -tue_1 \cdot e_2 e_3 \cdot e_4 + 2s(e_1 \cdot e_3 e_2 \cdot k_4 e_4 \cdot k_2 + 3 \text{ perms.}) + 2 \text{ perms.}$

Now for the four transverse scalar amplitude, our vertex operators look like:

$$V_{-1}^{a,\mu} = g_p \lambda^a \psi^\mu e^{-\phi} c e^{ipX}, \quad V_0^{a,\mu} = \frac{g_p}{\sqrt{2}\ell_s} \lambda^a (X' + 2p \cdot \psi \psi^\mu) c e^{ipX}$$

here $X' = \partial_\sigma X = 2i\partial X$. Moreover, we have Dirichlet boundary conditions on both the X s and the fermions ψ . The ψ are still in the NS sector since we're looking at the (bosonic) scalar field scattering.

Crucially, the correlators for the ψ field are the same for DD boundary conditions. We had $\langle \dot{X}^\mu(z) \dot{X}^\nu(w) \rangle = -\frac{\ell_s^2}{2} \eta^{\mu\nu} (z-w)^{-2}$ while the correlators for X' pick up a minus sign $\langle X'^i(z) X'^j(w) \rangle = \frac{\ell_s^2}{2} \eta^{ij} (z-w)^{-2}$. This ends up giving the exact same result however, since the vertex operators contain $X'(z)$ while the prior ones contain $i\dot{X}(z)$.

Finally, contracting X'^i with any of the e^{ikX} will give zero, since the open strings only have momenta parallel to the Dp brane while the X'^i is transverse. This gives a simpler amplitude than (2):

$$ig_p^2 \delta^{p+1} \ell_s^2 K'_4 \left(\frac{\Gamma(-\ell_s^2 s)\Gamma(-\ell_s^2 u)}{\Gamma(1+\ell_s^2 t)} ([1234] + [4321]) + 2 \text{ perms.} \right) \quad (3)$$

with $K'_4 = -(tu\delta_{12}\delta_{34} + su\delta_{13}\delta_{24} + st\delta_{14}\delta_{23})$. In the case where there are no CP indices we expand the $\Gamma\Gamma/\Gamma$ functions:

$$ig_p^2 \delta^{p+1} \ell_s^2 K'_4 \times \left(\frac{2}{\ell_s^4 su} + \frac{2}{\ell_s^4 st} + \frac{2}{\ell_s^4 tu} \right) = ig_p^2 \delta^{p+1} \ell_s^2 K'_4 \times \left(\frac{2(s+t+u)}{\ell_s^4 stu} \right) = 0$$

So to leading order in the string length this is zero. This is consistent with the $U(1)$ DBI action, as the scalars do not directly interact with the $U(1)$ gauge field A_μ (in general a real scalar cannot be charged under a $U(1)$ gauge field). That is, at leading order the action is free in the X fields. Taking $\xi^\mu = X^\mu$ for $\mu = 0 \dots p$ and X^i independent functions, we get:

$$\int d^{p+1} \xi \sqrt{\det G_{MN} \partial_\alpha X^M \partial_\beta X^N} = \int d^{p+1} \xi \sqrt{\det \delta_{\alpha\beta} + \delta_{ij} \partial_\alpha X^i \partial_\beta X^j} \rightarrow \int d^{p+1} \xi \frac{\delta^{\alpha\beta} \delta_{ij}}{2} \partial_\alpha X^i \partial_\beta X^j$$

This is just a free theory. Its also quick to see that the 3-point function of the transverse scalars vanishes at tree level in string perturbation theory.

6. I have done the previous problem in full generality, including CP indices. So now let's again look at the s channel. As $s \rightarrow 0$ so that $t = -u$ we get from the $\delta_{12}\delta_{34}$ term a pole in s going as:

$$-ig_p^2 \delta^{p+1} \ell_s^2 t u \times \left(\frac{1}{\ell_s^4 su} ([1234] + [4321]) + \frac{1}{\ell_s^4 st} ([1243] + [3421]) \right) = -i \delta^{p+1} \frac{g_p^2}{\ell_s^2} \frac{t}{s} ([1234] + [4321] - [1243] - [3421])$$

We can rewrite this as:

$$-i\delta^{p+1}\frac{g_p^2}{\ell_s^2}\frac{t}{s}(\text{Tr}(12[34]) - \text{Tr}([34]21)) = -i\delta^{p+1}\frac{g_p^2}{\ell_s^2}\frac{t}{s}\text{Tr}([12][34])$$

The pole at $s = 0$ corresponds to an exchange of a gluon from the $\frac{1}{2}(D_\mu X^I)^2$ term in **8.6.1**.

We also have a further term that does not involve a pole in s . Let's still take 1 and 2 equal. Expanding to this order we find:

$$-i\delta^{p+1}\frac{g_p^2}{\ell_s^2}\left(\text{Tr}([12][34]) + \text{Tr}([13][24]) + \text{Tr}([14][23])\right)$$

This comes from exactly the potential term $\frac{1}{4}[X^I, X^J]^2$ in the effective action **8.6.1**. **Come back to that last term**

7. Momentum conservation will imply $p_{\parallel} = 0$ for the NSNS states. Our vertex operator will take the form $\zeta_{\mu\nu}c\tilde{c}e^{-\phi}e^{-\tilde{\phi}}\psi^\mu\tilde{\psi}^\nu e^{ik_{\perp}X}$. We can use the doubling trick to get $\zeta_{\mu\nu}c(z)c(z^*)e^{-\phi(z)}e^{-\phi(z^*)}\psi^\mu\tilde{\psi}^\nu e^{ik_{\perp}X}$ and we are automatically in the -2 picture.

The states from in the $p+1$ parallel directions give just the correlator $\langle\psi^\mu(i)\psi^\nu(-i)\rangle = -\frac{\eta^{\mu\nu}}{2i}$ (importantly NN fermions in NSNS have 2-point function $-1/(z-\bar{w})$ c.f. **4.16.22**). We also get a δ^{p+1} from momentum conservation.

The states in the transverse (Dirichlet) directions give $\frac{\delta^{ij}}{2i}$ correlator. Defining the diagonal matrix $D^{\mu\nu} = (\eta^{\alpha\beta}, \delta^{ij})$ we get a correlator proportional to

$$-\frac{g_c}{2\ell_s^2g_p^2}\delta^{p+1}(k_{\parallel})D^{\mu\nu} = -\frac{(2\pi\ell_s)^2T_p}{2}V_{p+1}D^{\mu\nu}$$

Check with Victor. Confirm the tension relation. This diagonal tensor $D^{\mu\nu}$ allows for a nonvanishing dilaton and graviton tadpole, but will not couple to the antisymmetric Kalb-Ramond B -field.

8. Our RR fields have picture (r, s) for r, s half-integers. In order to have total picture -2 on the disk, we need to pick this to be the (asymmetric) $(-3/2, -1/2)$ picture. The construction of this operator is complicated. I expect that the $(-1/2, -1/2)$ operator that we are familiar with is basically $e^{-\phi}G_0$ times the $(-3/2, -1/2)$ operator. This means that the $(-3/2, -1/2)$ will be one less power of momentum and one less gamma matrix than the $(-1/2, -1/2)$ operator. Picking out the $p+2$ form part of the $(-1/2, -1/2)$ operator (in Blumenhagen's convention **16.21**) gives

$$\frac{1}{4}\frac{\ell_s}{\sqrt{2}}F^{\alpha\beta}\bar{S}_\alpha(z)\tilde{S}_\beta(\bar{z})e^{-\phi/2-\tilde{\phi}/2} \rightarrow \frac{1}{4}\frac{\ell_s}{\sqrt{2}}\frac{F_{\mu_1\ldots\mu_{p+2}}}{(p+2)!}(\Gamma^{\mu_1\ldots\mu_{p+2}})^{\alpha\beta}\bar{S}_\alpha\tilde{S}_\beta e^{-\phi/2-\tilde{\phi}/2}$$

Here $\bar{S}_\alpha = S_\alpha^\dagger\Gamma^0$ as is standard in a spinor product. BRST will require that F and $\star F$ be closed.

Changing picture means removing one power of momentum and one gamma matrix. This anti-differentiates F , which must be proportional to the potential C since $p_{[\mu_1}C_{\mu_2\ldots\mu_{p+2}]} = F_{\mu_1\ldots\mu_{p+2}}$. It is then reasonable to expect the corresponding $(-3/2, -1/2)$ operator to be proportional to

$$e^{-3\phi/2-\tilde{\phi}/2}\frac{C_{\mu_1\ldots\mu_p}}{(p+1)!}(\Gamma^{\mu_0\ldots\mu_p})^{\alpha\beta}\bar{S}_\alpha\tilde{S}_\beta$$

Note both $e^{-3\phi/3}S_\alpha$ and $e^{-\tilde{\phi}/2}S_\beta$ remain primary operators, having dimensions $3/8 + 5/8$, so this is indeed a reasonable guess. Now, for the D-brane boundary conditions we have

$$(\delta^\perp\tilde{S})_\beta(\bar{z}) = S_\beta(z^*), \quad \delta^\perp = \prod_{i=p+1}^9 \delta^i, \quad \delta^i = \Gamma^i\Gamma_{11}$$

This reflects the S_β spinor along all the Dirichlet directions and keeps it the same across the Neumann directions.

From **5.12.42** of Kiritsis I expect the leading-order of the $S_\alpha \tilde{S}_\beta$ correlator to be $C_{\alpha\beta}/(z - \bar{z})^{10/8}$ and the $e^{-3\phi/2} \tilde{e}^{-\phi/2}$ will contribute $(z - \bar{z})^{-3/4}$ to make this a primary correlator transforming as $C_{\alpha\beta}(z - \bar{z})^{-2}$. For Neumann boundary conditions, $C_{\alpha\beta}$ is the charge conjugation matrix. For the D -brane, it will be $\delta^\perp C$.

Only in the IIB case will $C_{\alpha\beta}$ will be nonzero between \bar{S}_α and \tilde{S}_β since S_α and \tilde{S}_β transform in the same representations. Each β^i changes the chirality. So the amplitude in IIB will vanish if we have an even number of β^i , equivalently $9 - p$ is odd, so we will have only odd dimensional branes in IIB as required and even dimensional branes in IIA as required.

We thus get an amplitude proportional to:

$$\mathcal{A} = i \frac{g_c \delta^{p+1}}{g_p^2 \ell_s^2} \frac{C_{\mu_0 \dots \mu_p}}{(p+1)!} \text{Tr}(\Gamma^{\mu_1 \dots \mu_p} \Gamma^{p+1} \dots \Gamma^9 \Gamma^{11}) = i \frac{g_c \delta^{p+1}}{g_o^2 \ell_s^2} \frac{C_{\mu_0 \dots \mu_p} \epsilon_{(p+1)}^{\mu_0 \dots \mu_p}}{(p+1)!}$$

Comparing with the **8.4.4**, which should factorize as $\mathcal{A}(p_\parallel)^2 G_{9-p}(p_\perp) \delta^{p+1}(p_\parallel)$ we see that the normalization of our on-shell amplitude is in fact:

$$\mathcal{A} = i V_{p+1} \sqrt{2\pi} (2\pi \ell_s)^{3-p} \frac{C_{\mu_0 \dots \mu_p} \epsilon_{(p+1)}^{\mu_0 \dots \mu_p}}{(p+1)!}$$

This is consistent with other results c.f. Di Vecchia, Liccardo *Gauge Theories from D-Branes*, arXiv:0307104 but I think they're not incorporating $1/\alpha_p = 2\kappa_{10}^2$ in the propagator. Taking this factor into account and dividing by it followed by taking a square root gives us an on-shell amplitude of:

$$\mathcal{A} = i V_{p+1} \frac{1}{(2\pi \ell_s)^p \ell_s g_s} \frac{C_{\mu_0 \dots \mu_p} \epsilon_{(p+1)}^{\mu_0 \dots \mu_p}}{(p+1)!} = i V_{p+1} T_p C_{p+1} \wedge \epsilon_{(p+1)}.$$

This is exactly what would come from a minimal coupling term of the form $i T_p \int C_{p+1}$.

9. We take one vertex operator to be in the $(-1, -1)$ picture and gauge fix it to lie at $z = i$, and take the other in the $(0, 0)$ picture, and fix it to range along the line from 0 to i . We wish to calculate the correlator:

$$-\frac{g_c^2}{g_o^2 \ell_s^4} \frac{2}{\ell_s^2} \langle [\psi^\mu \tilde{\psi}^{\bar{\mu}} e^{ik_1 X}]_0 [(i\partial X^\mu + \frac{1}{2} k_2 \cdot \psi \psi^\nu)(i\bar{\partial} X^{\bar{\nu}} + \frac{1}{2} k_2 \cdot \bar{\psi} \bar{\psi}^{\bar{\nu}}) e^{ik_2 X}]_y \rangle$$

This will be very similar to the 4-point gauge boson amplitude in (1). **Finish.**

From this ratio $\Gamma\Gamma/\Gamma$ functions we see that there are open string poles. This corresponds to a closed string splitting in two, with its ends on the D -brane as an intermediate state.

10. In the $D9$ brane case, we have seen that the open string boundary only preserves the sum of $Q + \tilde{Q}$. If we T-dualize in the 9 direction, we act on the right-moving sector by spacetime parity, so that necessarily $\bar{\partial} X^9 \rightarrow -\bar{\partial} X^9, \tilde{\psi}^9 \rightarrow -\tilde{\psi}^9, \tilde{S}_\alpha \rightarrow \delta^9 \tilde{S}^\alpha$ (up to a phase in that last one). Here $\delta^9 = \Gamma^9 \Gamma^{11}$. Our spacetime supersymmetry generator $\tilde{Q}_\alpha = \frac{1}{2\pi i} \int d\bar{z} e^{-\phi/2} S_\alpha$ therefore will be mapped to $\delta^9 \tilde{Q}$. Thus, in the T-dual picture we preserve the supercharge $Q' + \delta^9 \tilde{Q}'$.

Iterating this procedure in other directions we get that in general we preserve $Q + \delta^\perp \tilde{Q}$, with $\delta^\perp = \prod_i \delta^i$, where i runs perpendicular to the brane. Note that T-dualities along different directions do not commute! They commute up to a $(-1)^{\mathbf{F}_R}$, and so the order that we do them matters. In this case the δ^i act by left-action.

11. (As in Polchinski section 13.4) From the previous problem, we see that the first D -brane preserves the supercharges $Q + \delta^\perp \tilde{Q}$ while the second preserves the supercharges $Q + \delta^{\perp'} \tilde{Q} = Q + \delta^\perp (\delta^{\perp-1} \delta^{\perp'} \tilde{Q})$ so the supersymmetries that will be preserved must be of both forms. This is in one-to-one correspondence with spinors invariant under $\delta^{\perp-1} \delta^{\perp'}$. This operator is a reflection in the direction of the ND boundary conditions (the directions orthogonal to the $D_{p'}$ brane in the D_p brane). Since in either IIA or IIB p and p' must differ by an even integer, the number of mixed boundary conditions—call it ν —must be even. Then we can write $\delta^{\perp-1} \delta^{\perp'}$ as a product of rotations by π along each of the $\nu/2$ planes $\delta^{\perp-1} \delta^{\perp'} = e^{i\pi(J_1 + \dots + J_{\nu/2})}$. Each j acts

in a spinor representation, so that $e^{i\pi J_i}$ has eigenvalues $\pm i$. If $\nu/2$ is odd, this makes $\delta^{\perp-1}\delta^{\perp'} = -1$ so this will *not* preserve supersymmetry. We thus need $\nu/2$ even, or $\nu = 0 \bmod 4$.

From this I posit that the static force between two branes vanishes precisely when $\nu = 0 \bmod 4$.

12. Now let's confirm this guess with an amplitude calculation. Take $p' \leq p$. We work in lightcone gauge. We do a trace over an open string with p' NN boundary conditions, $p - p' = \nu$ DN boundary conditions, and $8 - p$ DD boundary conditions.

We begin from the open string point of view in calculating the cylinder amplitude. Chapter 4 has done the NN, DD, and DN boson amplitudes for us. The difficulty lies almost entirely in the fermions. Recall the following:

$$\eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

$$\sqrt{\frac{\theta[0]}{\eta}} = q^{-1/48} \prod_{n=0}^{\infty} (1 + q^{n+1/2}) \quad \sqrt{\frac{\theta[1]}{\eta}} = \sqrt{2} q^{1/24} \prod_{n=0}^{\infty} (1 + q^n) \quad \sqrt{\frac{\theta[1]}{\eta}} = q^{-1/48} \prod_{n=0}^{\infty} (1 - q^{n+1/2})$$

Further from **4.16.2** recall that for modes $b_{n+1/2}$, b_n corresponding to NS and R sectors the NN and DD boundary conditions give:

- NN: $\bar{b}_{n+1/2} = -b_{n+1/2}$, $\bar{b}_n = b_n$
- DD: $\bar{b}_{n+1/2} = b_{n+1/2}$, $\bar{b}_n = -b_n$

For DN we have the same result as for DD but now the R sector is half-integrally modded and the NS sector is integrally modded. Now let's compute partition functions. Our final answer will be a sum over spin structures NS+, NS-, R+, R-. Taking $q = e^{-2\pi t}$ we see $\text{Tr}[q^{L_0 - c/24}] =$

- NS+:
 - NN: $q^{-1/48} \prod_n (1 + q^{n+1/2}) = \sqrt{\theta[0]/\eta}$
 - DD: $q^{-1/48} \prod_n (1 + q^{n+1/2}) = \sqrt{\theta[0]/\eta}$
 - DN: $\sqrt{2} q^{-1/48} q^{1/16} \prod_n (1 + q^n) = \sqrt{\theta[1]/\eta}$ ($\sqrt{2}$ when raised to a power counts ground state degeneracy)
- NS-:
 - NN: $q^{-1/48} \prod_n (1 - q^{n+1/2}) = \sqrt{\theta[1]/\eta}$
 - DD: $q^{-1/48} \prod_n (1 - q^{n+1/2}) = \sqrt{\theta[1]/\eta}$
 - DN: 0
- R+
 - NN: $\sqrt{2} q^{1/24} \prod_n (1 + q^n) = \sqrt{\theta[0]/\eta}$
 - DD: $\sqrt{2} q^{1/24} \prod_n (1 + q^n) = \sqrt{\theta[0]/\eta}$
 - DN: $q^{-1/48} \prod_n (1 + q^{n+1/2}) = \sqrt{\theta[1]/\eta}$
- R-
 - NN: 0
 - DD: 0
 - DN: $q^{-1/48} \prod_n (1 - q^{n+1/2}) = \sqrt{\theta[1]/\eta}$

Notice NN vs DD boundary conditions have *no effect* on fermion contribution to partition function. This is because, although the left moving and right-moving modes are identified differently, the mode excitations look exactly the same.

On the other hand for NN and DD the bosons will contribute $1/\eta$ and will contribute $\sqrt{\eta/\theta[1]^{[0]}}$ for DN. Thus we have the following contributions to the partition function (here N is the number of NN boundary conditions):

$$\begin{aligned}
NS+ &= \frac{V_N}{(2\pi\ell_s)^N} \int \frac{dt}{2t} \frac{e^{-2\pi t \left(\frac{\Delta x}{2\pi\ell_s}\right)^2}}{(\sqrt{2t})^N \eta^{8-\nu} (\theta[1]^{[0]}/\eta)^{\nu/2}} \left(\frac{\theta[0]^{[1]}}{\eta}\right)^{(8-\nu)/2} \left(\frac{\theta[1]^{[0]}}{\eta}\right)^{\nu/2} \\
NS- &= -\frac{V_N}{(2\pi\ell_s)^N} \int \frac{dt}{2t} \frac{e^{-2\pi t \left(\frac{\Delta x}{2\pi\ell_s}\right)^2}}{(\sqrt{2t})^N \eta^8} \left(\frac{\theta[1]^{[0]}}{\eta}\right)^8 \delta_{\nu=0} \\
R+ &= -\frac{V_N}{(2\pi\ell_s)^N} \int \frac{dt}{2t} \frac{e^{-2\pi t \left(\frac{\Delta x}{2\pi\ell_s}\right)^2}}{(\sqrt{2t})^N \eta^{8-\nu} (\theta[1]^{[0]}/\eta)^{\nu/2}} \left(\frac{\theta[1]^{[1]}}{\eta}\right)^{(8-\nu)/2} \left(\frac{\theta[0]^{[1]}}{\eta}\right)^{\nu/2} \\
R- &= \frac{V_N}{(2\pi\ell_s)^N} \int \frac{dt}{2t} \frac{e^{-2\pi t \left(\frac{\Delta x}{2\pi\ell_s}\right)^2}}{(\sqrt{2t})^N} \delta_{\nu=8}
\end{aligned}$$

All theta and eta functions are evaluated at it . The circumference of the cylinder is $2\pi t$. The relative signs in front of the different contributions come from a combination of defining the NS vacuum to have negative fermion and modular invariance (equivalently spacetime spin-statistics). Note when $\nu = 4$ we only get contributions from NS+ and R+, which exactly cancel. Similarly when $\nu = 4$ or 8, by the abtruse identity of Jacobi we will get cancelation again.

We can interpret our result as a one-loop free energy. Differentiating this w.r.t. Δx would then give us our force. For $\nu = 0, 4, 8$ we do not get a force, consistent with the D-brane configuration preserving supersymmetry.

For the sake of completeness, and to clear my own confusion once and for all, I will also do this from the POV of the boundary state formalism (not developed in Kiritsis). For a good reference see the last chapter of Blumenhagen's text on conformal field theory.

For a single free boson, after the flip $(\sigma, \tau)_{open} \rightarrow (\tau, \sigma)_{closed}$ the boundary states $|N\rangle, |D\rangle$ must satisfy

$$(\alpha_n + \tilde{\alpha}_{-n}) |N\rangle = 0, \quad (\alpha_n - \tilde{\alpha}_{-n}) |D_x\rangle = 0,$$

This gives boundary states:

$$\begin{aligned}
|N\rangle &= \frac{1}{(2\pi\ell_s\sqrt{2})^{1/2}} \prod_n e^{-\frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}} |0, 0; 0\rangle = \sum_{\vec{m}=\{m_i\}} |\vec{m}, \Theta\vec{m}; 0\rangle \\
|D_x\rangle &= (2\pi\ell_s/\sqrt{2})^{1/2} \int \frac{dk}{2\pi} e^{ipx} \prod_n e^{-\frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}} |0, 0; k\rangle
\end{aligned}$$

The overall normalization came from comparing with cylinder amplitudes. Θ here is CPT reversal. Similarly for a fermion

$$(\psi_n + \tilde{\psi}_{-n}) |N\rangle = 0, \quad (\psi_n - \tilde{\psi}_{-n}) |D_x\rangle = 0,$$

So with GSO projection we get:

$$\begin{aligned}
|N, \text{NSNS}\rangle &= P_L P_R \prod_r e^{\psi_{-r}\tilde{\psi}_{-r}} |0\rangle, & |N, \text{RR}\rangle &= P_L P_R \prod_n e^{\psi_{-n}\tilde{\psi}_{-n}} |0\rangle \\
|D, \text{NSNS}\rangle &= P_L P_R \prod_r e^{-\psi_{-r}\tilde{\psi}_{-r}} |0\rangle, & |D, \text{RR}\rangle &= P_L P_R \prod_n e^{-\psi_{-n}\tilde{\psi}_{-n}} |0\rangle
\end{aligned}$$

Here r runs over half-integers in the NSNS sector and n runs over integers in the RR sector. $P_L = \frac{1}{2}(1 + (-1)^F)$, $P_R = \frac{1}{2}(1 + (-1)^{\tilde{F}})$ are our GSO projections, defined to project out the tachyon in the NS sector and project out one of the spinors in the R sector.

For the boson, it is quick to see that ($\ell = 1/t$)

$$\begin{aligned}\langle N | e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | N \rangle &= \frac{V}{2\pi\ell_s\sqrt{2}\eta(i\ell)} = \frac{V}{(2\pi\ell_s)\sqrt{2t}\eta(it)} \\ \langle D | e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | D \rangle &= \frac{2\pi\ell_s}{\sqrt{2t}\eta(it)} \int \frac{dk}{2\pi} e^{ik\Delta x} e^{-\pi\ell_s^2 p^2/2t} = \frac{e^{-2\pi t\left(\frac{\Delta x}{2\pi\ell_s}\right)^2}}{\eta(it)} \\ \langle D | e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | N \rangle &= \frac{1}{\sqrt{2}} \frac{1}{\prod_n (1+q^{2n})} = \sqrt{\frac{\eta(i\ell)}{\theta_{[0]}^1(i\ell)}} = \sqrt{\frac{\eta(it)}{\theta_{[1]}^0(it)}}$$

These are exactly what we've already gotten many times before from our trace over the open string bosonic states. The states $|N\rangle, |D\rangle$ must be a sum of both the RR and NSNS sector fermion states. We do not know the relative coefficients.

Let's look at the NSNS contributions. For the NN boundary conditions, the NSNS sector with projection consists of two terms:

$$\begin{aligned}\langle N, \text{NSNS}_{unproj} | e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | N, \text{NSNS}_{unproj} \rangle &= \left(\frac{\theta_{[0]}^0(i\ell)}{\eta(i\ell)} \right)^{\#NN/2} = \left(\frac{\theta_{[0]}^0(it)}{\eta(it)} \right)^{\#NN/2} \\ \langle N, \text{NSNS}_{unproj} | (-1)^{F_L=F_R} e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | N, \text{NSNS}_{unproj} \rangle &= \left(\frac{\theta_{[1]}^0(i\ell)}{\eta(i\ell)} \right)^{\#NN/2} = \left(\frac{\theta_{[0]}^1(it)}{\eta(it)} \right)^{\#NN/2}\end{aligned}$$

Replacing N with D would give the *exact same* factor in both cases **WHY?** (explain: bc we need to match on both sides and so both minuses cancel in the exponent). For DN boundary conditions the NSNS sector give the two terms:

$$\begin{aligned}\langle D, \text{NSNS}_{unproj} | e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | N, \text{NSNS}_{unproj} \rangle &= \left(\frac{\theta_{[1]}^0(i\ell)}{\eta(i\ell)} \right)^{\nu/2} = \left(\frac{\theta_{[0]}^1(it)}{\eta(it)} \right)^{\nu/2} \\ \langle D, \text{NSNS}_{unproj} | (-1)^{F_L=F_R} e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | N, \text{NSNS}_{unproj} \rangle &= \left(\frac{\theta_{[0]}^0(i\ell)}{\eta(i\ell)} \right)^{\nu/2} = \left(\frac{\theta_{[0]}^0(it)}{\eta(it)} \right)^{\nu/2}\end{aligned}$$

Now let's look at the RR sector. For NN boundary conditions, it contributes:

$$\begin{aligned}\langle N, \text{RR}_{unproj} | e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | N, \text{RR}_{unproj} \rangle &= \left(\frac{\theta_{[0]}^1(i\ell)}{\eta(i\ell)} \right)^{\#NN/2} = \left(\frac{\theta_{[1]}^0(it)}{\eta(it)} \right)^{\#NN/2} \\ \langle N, \text{RR}_{unproj} | (-1)^{F_L=F_R} e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | N, \text{RR}_{unproj} \rangle &= 0\end{aligned}$$

By the argument before, we get the same for DD boundary conditions. Finally, with DN boundary conditions we get

$$\begin{aligned}\langle D, \text{RR}_{unproj} | e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | N, \text{RR}_{unproj} \rangle &= 0 \\ \langle D, \text{RR}_{unproj} | (-1)^{F_L=F_R} e^{-\pi\ell(L_0+\tilde{L}_0-c/12)} | N, \text{RR}_{unproj} \rangle &= \left(\frac{\theta_{[0]}^1(i\ell)}{\eta(i\ell)} \right)^{\nu/2} = \left(\frac{\theta_{[1]}^0(it)}{\eta(it)} \right)^{\nu/2}\end{aligned}$$

Together this is exactly consistent with what we get from tracing over the open string. We can work back to get relative normalizations.

This shows that the massless RR and NSNS fields mediate the force. Moreover the NSNS fields without and with projection correspond respectively to the unprojected NS and R open string states while the RR fields without and with projection correspond to the *projected* NS and R open string states.

13. First recall that for a constant vector potential $A_9 = \frac{\chi_9}{2\pi R}$ corresponds to a T -dual picture of a D -brane at position $-\chi\tilde{R} = -2\pi\ell_s^2 A_9$. Now consider a magnetic flux F_{12} we can write a (nonconstant now) vector potential that gives this flux as $A_2 = F_{12}X^1$. We T -dualize along X^2 and get $X^2 = -2\pi\ell_s^2 F_{12}X^1$. Then $\tan\theta = -2\pi\ell_s^2 F_{12}$.

Although we were working with D1 and D2 branes, we could have done the exact same calculation for F_{01} on a D1 brane and recovered a D0 brane tilted in the $X^0 - X^1$ plane (ie boosted). Such a D0 brane has the usual point-particle action:

$$S_{D0} = -T_0 \int dX^0 \sqrt{1 + (\partial_0 X'^1)^2}$$

Because the D0 brane and the D1 brane describe the same physics, this action should be identical to the D1 action. Note that $\partial_0 X'^1$ is infinitesimally exactly $\tan\theta$ calculated above. We get the action

$$S_{D1} = -T_1 \int dX^1 dX^2 \sqrt{1 + (2\pi\ell_s^2 F_{12})^2}$$

Of course, because the branes couple to strings, the only gauge invariant combination under transformations of the Kalb-Ramond B field is $\mathcal{F} = B + 2\pi\ell_s^2 F$. We thus get:

$$S_{D1} = -T_1 \int dX^1 dX^2 \sqrt{-\det(G + \mathcal{F})}$$

We can tilt this brane and T -dualize to pick up EM field strengths in arbitrary dimension up to 9.

14. Let's T -dualize. This describes two D4 branes that are tilted only along the x_1 - x_5 plane, and are otherwise parallel in the x_2, x_3, x_4 directions. T -dualizing x_2, x_3, x_4 makes these into D1 branes tilted in the $x_1 - x_5$ plane. See the solution (4) to the next problem. Now setting $\nu_{2,3,4} = 0$ will give poles from the theta function in the denominator. This is to be expected, from the NN boundary conditions that always come with a volume divergence factor in that direction. We regulate this divergence by replacing:

$$\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (i\nu t, it)^{-1} \rightarrow i \frac{L}{\eta(it)^{-3} 2\pi\ell_s \sqrt{2t}}$$

Thus we get a potential:

$$-i \frac{L^3}{(2\pi\ell_s)^4} \int_0^\infty \frac{dt}{t} \frac{e^{-2\pi t \left(\frac{\Delta x}{2\pi\ell_s}\right)^2}}{(2t)^{4/2} \eta(it)^9} \frac{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (i\nu t/2, it)^4}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (i\nu t, it)}$$

Let's take the distance to be large. The small t contributions are then most important. The θ -function ratio contributes a factor of $e^{-3\pi/4t} t^{-3/2}$ while the η^{-9} contributes $e^{3\pi/4t} t^{9/2}$ **Finish the details here**

We get a potential that decays as $-\frac{L^3 \ell_s}{(\Delta x)^5} \frac{\sin^4(\theta/2)}{\sin(\theta)}$ giving another attractive force going as $1/(\Delta x)^6$.

15. Following Polchinski, we define variables $Z^i = X^i + iX^{i+4}, i = 1, \dots, 4$. Let the $\sigma = 0$ endpoint be on the untilted string. Then at $\sigma = 0$ we have $\partial_1 \Re Z^a = \Im Z^a = 0$ and at $\sigma = \pi$ on the tilted string we have $\partial_1 \Re(e^{i\theta_a} Z^a) = \Im(e^{i\theta_a} Z^a)$.

We see that the field that satisfies this is given by $Z^a(w, \bar{w}) = \mathcal{Z}^a(w) + \bar{\mathcal{Z}}^a(-\bar{w})$ for $w = \sigma^1 + i\sigma^2$. Using the doubling trick we have $\mathcal{Z}^a(w + 2\pi) = e^{2i\theta_a} \mathcal{Z}^a(w)$ (and similarly for the conjugate). This gives a mode expansion with $\nu_a = \theta_a/\pi$

$$\mathcal{Z}^a(w) = i \frac{\ell_s}{\sqrt{2}} \sum_{r \in \mathbb{Z} + \nu_a} \frac{\alpha_r^a}{r} e^{irw}.$$

The a^\dagger modes are then linearly independent. Taking $q = e^{-2\pi t}$ as usual for open string partition functions, and restricting $0 < \phi_a < \pi$ we get:

$$\frac{q^{\frac{1}{24} - \frac{1}{2}(\nu_a - \frac{1}{2})^2}}{\prod_{m=0}^\infty (1 - q^{m+\nu_a})(1 - q^{m+1-\nu_a})} = -i \frac{q^{-\nu_a^2/2} \eta(it)}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (i\nu_a t, it)}$$

Where we have used

$$\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (i\nu t, it) = iq^{\frac{1}{8}}(q^{\nu/2} - q^{-\nu/2}) \prod_{m=1}^{\infty} (1 - q^m)(1 - q^{m+\nu})(1 - q^{m-\nu}) = i\eta(it)q^{\frac{1}{8} - \frac{1}{24} + \frac{\nu}{2}} \prod_{m=0}^{\infty} (1 - q^{m+\nu})(1 - q^{m+1-\nu})$$

So the angles act like chemical potentials to make the theta functions nonzero. Now its time for the fermions (oh boy!). In each NS and R sector (projected and unprojected) the boundary conditions shift by ν_a . We thus get e.g. for NS unprojected:

$$Z \begin{bmatrix} 0 \\ 0 \end{bmatrix} = q^{-\frac{1}{24} + \nu_a^2/2} \prod_{m=1}^{\infty} (1 - q^{m+1/2+\nu_a})(1 - q^{m+1/2-\nu_a}) = q^{\nu_a^2/2} \frac{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (i\nu_a t, it)}{\eta(it)}$$

In total, then we will see that the fermion part gives us

$$\frac{\prod_a q^{\nu_a^2/2}}{2\eta(it)^4} \left[\prod_{a=1}^4 \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (i\nu_a t, it) - \prod_{a=1}^4 \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (i\nu_a t, it) - \prod_{a=1}^4 \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (i\nu_a t, it) - \prod_{a=1}^4 \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (i\nu_a t, it) \right] = \prod_{a=1}^4 \frac{e^{\nu_a^2/2} \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (i\nu'_a t, it)}{\eta(it)}$$

This last equality follows from the full abstruse identity of Jacobi, where $\phi'_1 = \frac{1}{2}(\phi_1 + \phi_2 + \phi_3 + \phi_4)$, $\phi'_2 = \frac{1}{2}(\phi_1 + \phi_2 - \phi_3 - \phi_4)$, $\phi'_3 = \frac{1}{2}(\phi_1 - \phi_2 + \phi_3 - \phi_4)$, $\phi'_4 = \frac{1}{2}(\phi_1 - \phi_2 - \phi_3 + \phi_4)$ and the ν'_a are defined identically. Inserting the DD conditions along the 9 direction that denotes separation, we get the full potential as a function of the separation Δx

$$V = -2 \times \int_0^{\infty} \frac{dt}{2t} \frac{e^{-2\pi t \left(\frac{\Delta x}{2\pi \ell_s} \right)^2}}{2\pi \ell_s \sqrt{2t}} \prod_{a=1}^4 \frac{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (i\nu'_a t, it)}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (i\nu_a t, it)} \quad (4)$$

The initial overall factor of two comes from two orientations of the open string. At long enough distances, the exponential factor forces small t to contribute primarily. The complicated ratio of θ -functions becomes a ratio of sines. Then we get

$$\prod_{a=1}^4 \frac{\sin(\pi \nu'_a)}{\sin(\pi \nu_a)} \int_0^{\infty} \frac{dt}{2\pi \ell_s \sqrt{2t}} e^{-\frac{(\Delta x)^2}{2\pi \ell_s^2} t}$$

Taking the integral and analytically continuing, we get a potential that looks like $-\frac{|\Delta x|}{2\pi \ell_s^2} \prod_{a=1}^4 \frac{\sin(\pi \nu'_a)}{\sin(\pi \nu_a)}$, giving an attractive, constant force, of $-\frac{1}{2\pi \ell_s^2} \prod_{a=1}^4 \frac{\sin(\pi \nu'_a)}{\sin(\pi \nu_a)}$.

16. Let the first brane at $\sigma = 0$ have no electric field and put an electric field F_{01} on the second brane at $\sigma = \pi$. The endpoints of the string are charged. We have the following action (take e , the charge of the string endpoint to be 1)

$$-\frac{1}{4\pi \ell_s^2} \int d\sigma d\tau [(\partial_{\sigma} X)^2 + (\partial_{\tau} X)^2] + \int_{\sigma=\pi} d\tau A_{\mu} \partial_{\tau} X^{\mu}$$

Upon variation, we get a boundary term:

$$\begin{aligned} & -\frac{1}{2\pi \ell_s^2} \int d\tau \partial_{\sigma} X^{\mu} \delta X_{\mu} + \int d\tau \delta(A_{\nu} \partial_{\tau} X^{\nu}) \\ & = -\frac{1}{2\pi \ell_s^2} \int d\tau \partial_{\sigma} X^{\mu} \delta X_{\mu} + \int d\tau \partial_{\mu} A_{\nu} \delta X^{\mu} \partial_{\tau} X^{\nu} - \partial_{\tau} A_{\nu} \\ & = -\frac{1}{2\pi \ell_s^2} \int d\tau \partial_{\sigma} X_{\mu} \delta X^{\mu} + \int d\tau F_{\mu\nu} \delta X^{\mu} \partial_{\tau} X^{\nu} \\ & \Rightarrow \partial_{\sigma} X_{\mu} - 2\pi \ell_s^2 F_{\mu\nu} \partial_{\tau} X^{\nu} = 0 \end{aligned}$$

This gives mixed boundary conditions on the X^0 and X^1 which can be written as

$$\begin{pmatrix} \partial_{\sigma} X^0 \\ \partial_{\sigma} X^1 \end{pmatrix} = 2\pi \ell_s^2 E \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_{\tau} X^0 \\ \partial_{\tau} X^1 \end{pmatrix}$$

with $E = F_{10}$. Note that we have been careful in raising the μ index. I will define $Z^\pm = X^0 \pm X^1$ and have $\partial_\sigma Z^+ = \partial_\sigma Z^- = 0$ at $\sigma = 0$ and $(\partial_\sigma - 2\pi\ell_s^2 E \partial_\tau)Z^+ = (\partial_\sigma + 2\pi\ell_s^2 E \partial_\tau)Z^- = 0$ at $\sigma = \pi$. The modes thus satisfy Neumann-Mixed boundary conditions. Following a modification of exercise **2.14** and solving these boundary conditions we get that the modes must be labeled by $\nu = -iu/\pi + \mathbb{Z}$. Here $u = \text{atanh} v$ is the *rapidity*. Now let's compute a cylinder diagram. Let's assume for now that we are scattering D1 branes (the problem does not explicitly give p, p'). It will look like the wick-rotation of the integral considered in the previous two questions. We have an amplitude

$$-iV_p \times 2 \times \int_0^\infty \frac{dt}{2t} \frac{e^{-t \frac{(\Delta x)^2}{2\pi\ell_s^2}}}{(2\pi\ell_s)^p (2t)^{p/2}} \frac{\theta\left[\frac{1}{1}\right](ut/2\pi, it)^4}{\eta(it)^9 \theta\left[\frac{1}{1}\right](ut/\pi, it)}$$

Note however that since the first argument of the θ functions is real, we have poles at $t = \pi n/u$, $\nu = u/\pi$ for n an integer. Upon deforming the integration contour, we can use the identity $\frac{1}{x-i\epsilon} = \pi\delta(x) + P(x)$ to pick up poles at $t = n/\nu$, at *odd* integers n (so that the four-order zero in the numerator doesn't cancel them), giving:

$$\pi V_p \sum_{n \in \mathbb{Z}_{\text{odd}}} \frac{e^{-n \frac{(\Delta x)^2}{2\pi\ell_s^2 \nu}}}{(2\pi\ell_s)^p (2n/\nu)^{(p+2)/2}} \frac{\theta\left[\frac{1}{1}\right](\frac{n}{2}, it)^4}{2\eta(in/\nu)^{12}} \quad (5)$$

The imaginary part of the amplitude can be interpreted (after T -duality) as resonances (ie bound states) of the D-branes.

17. From the last problem we can write: In terms of ∂_+, ∂_- , we have:

$$\begin{pmatrix} \partial_+ X^0 \\ \partial_+ X^1 \end{pmatrix} = \begin{pmatrix} \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} & \frac{2\mathcal{E}}{1-\mathcal{E}^2} \\ \frac{2\mathcal{E}}{1-\mathcal{E}^2} & \frac{1+\mathcal{E}^2}{1-\mathcal{E}^2} \end{pmatrix} \begin{pmatrix} \partial_- X^0 \\ \partial_- X^1 \end{pmatrix} \quad (6)$$

Here $\mathcal{E} = 2\pi\ell_s^2 E$. Taking \mathcal{E} close to zero recovers NN boundary conditions on X^0, X^1 . (It's worth noting that that the speed of light here will translate to a maximum electric field $|E| < T$ on the brane. This provides one motivation for the necessity of a nonlinear electrodynamics, namely DBI). taking $E = 0$ give N boundary conditions on X^0, X^1 . T-dualizing X^1 gives D boundary conditions on \tilde{X}^1 . Now, boosting the brane along X^0 and

$$\begin{aligned} \begin{pmatrix} \partial_+ X^0 \\ \partial_+ \tilde{X}^1 \end{pmatrix} &= \begin{pmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} \partial_- X^0 \\ \partial_- \tilde{X}^1 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \partial_+ X^0 \\ \partial_+ X^1 \end{pmatrix} &= \begin{pmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial_- X^0 \\ \partial_- \tilde{X}^1 \end{pmatrix} = \begin{pmatrix} \frac{1+v^2}{1-v^2} & \frac{2v}{1-v^2} \\ \frac{2v}{1-v^2} & \frac{1+v^2}{1-v^2} \end{pmatrix} \begin{pmatrix} \partial_- X^0 \\ \partial_- \tilde{X}^1 \end{pmatrix} \end{aligned}$$

This exactly what we had before, with $v = 2\pi\ell_s E$.

This argument is quite simple from the abstract picture: taking $A_1 = EX^0$, T -dualizing in the direction of X^1 gives a D-brane lying at $X_1 = 2\pi\ell_s^2 EX^0$ giving a velocity $2\pi\ell_s^2 E$. This can also be obtained by analytic continuation of question **8.13**.

18. Using the fact that this scattering problem is exactly T-dual to the electric field problem mentioned before, we return to (5) and consider $b = \Delta x$ to be small. In this case the large t regime dominates (corresponding to a loop of light open strings). First we perform a modular transformation to get

$$\mathcal{A} = \frac{V_p}{(8\pi^2\ell_s^2)^{p/2}} \int_0^\infty \frac{dt}{t} t^{(6-p)/2} e^{-\frac{tb^2}{2\pi\ell_s^2}} \frac{\theta\left[\frac{1}{1}\right](i\nu/2, i/t)^4}{\eta(i/t)^9 \theta\left[\frac{1}{1}\right](i\nu, i/t)}$$

Now we follow Polchinski and rewrite \mathcal{A} in terms of an integral over the particle's path $r(\tau)^2 = b^2 + v^2\tau^2$, $\mathcal{A} = -i \int d\tau V(r(\tau), v)$. Then we get V from reversing a Gaussian integral to be

$$V(r, v) = i \frac{2V_p}{(8\pi^2\ell_s^2)^{(p+1)/2}} \int_0^\infty dt t^{(5-p)/2} e^{-\frac{tr^2}{2\pi\ell_s^2}} \frac{\tanh \pi\nu \theta\left[\frac{1}{1}\right](i\nu/2, i/t)^4}{\eta(i/t)^9 \theta\left[\frac{1}{1}\right](i\nu, i/t)}$$

The large- t limit is now direct:

$$V(r, v) = -\frac{2V_{p+1}}{(8\pi^2\ell_s^2)^{(p+1)/2}} \int_0^\infty \frac{dt}{t^{(1+p)/2}} e^{-\frac{tr^2}{2\pi\ell_s^2}} \frac{\tanh u \sin^4(ut/2)}{\sin(ut)}$$

Using steepest descent at zeroth order, the t that dominates is of order $2\pi\ell_s^2/r^2$ so that $ut \approx 2\pi\ell_s^2 v/r^2$ is the leading contribution. **Justify why ut is $O(1)$.** This then gives that $r \sim \ell_s\sqrt{v}$. If we go at very small velocities we can probe below the string scale.

On the other hand, a slower velocity means that the time it takes to probe this distance is longer $\delta t = r/v = \ell_s/\sqrt{v}$. This implies that $\delta x \delta t \geq \ell_s^2$. This looks like a type of *noncommutative geometry* with α' playing the role of Planck's constant now.

Combining this with the usual position-momentum uncertainty relation

$$1 \leq \delta x m \delta v = g\ell_s \delta x \delta v \Rightarrow \Delta x = \frac{g\ell_s}{\delta v},$$

we can minimize simultaneously $\ell_s\sqrt{v}, g\ell_s/v$ by having $v \sim g^{2/3}$ giving $\delta x = \ell_s g^{1/3}$. At weak coupling this is smaller than the string scale.

19. The action is a factor of two off from Polchinski's. The momenta are $p_i = \frac{2}{g_s\ell_s}(\dot{X}^i + [A_t, X^i])$. Then we get:

$$H = \int dt \text{Tr} \left[\frac{g_s\ell_s}{4} p_i p^i + \frac{1}{2g_s\ell_s(2\pi\ell_s^2)^2} [X^i, X^j]^2 \right]$$

Defining $g^{1/3}\ell_s Y^i = X^i, p_i = p_{Y_i}/g_s^{1/3}\ell_s$, this sets the length scale, which coincides with what we got in the previous question by less rigorous arguments. Now we have Y^i is dimensionless and get a hamiltonian

$$H = \frac{g_s^{1/3}}{\ell_s} \int dt \text{Tr} \left[\frac{1}{4} p_{Y_i} p_{Y_i} + \frac{1}{2(2\pi)^2} [Y^i, Y^j]^2 \right]$$

So the only dimensionful content of this hamiltonian comes from g, ℓ_s appearing in the overall normalization. This gives an energy scale of $g_s^{1/3}/\ell_s$. For strong coupling $g_s > 1$ this probes deeper than the string scale.

20. This is pretty direct. Since the metric $G_{\mu\nu}$ does not depend on X^i for $i = p+1 \dots 9$, we have that each X^i is killing, in particular the metric takes a block-diagonal form where only the first $(p+1)^2$ $G_{\mu\nu}$ entries have nontrivial coordinate dependence and the remaining metric is just the identity matrix δ_{ij} along the X_i directions (we didn't even have to do this since 8.5.1 has $\eta_{\mu\nu}$ the flat metric. Is my logic here even right?).

Take the ansatz $A \rightarrow (A_\mu, \Phi_i)$. We thus get $F_{\mu\nu}$ in the first $(p+1)^2$ entries and $\partial_\mu \Phi^i$ in the off-block-diagonal piece. We can rewrite this as a determinant of just the $(p+1)$ piece **Justify this step**

$$\sqrt{-\det(G_{\mu\nu} + 2\pi F_{\mu\nu} + \partial_\mu \Phi^i \partial_\nu \Phi^i)}$$

21. The bosonic part of this is immediate. Write the fields A_i in the dimensionally reduced dimensions as X^I and we immediately get $\text{Tr} F_{10}^2 \rightarrow \text{Tr} [F_{d+1}^2 + 2[D_\mu, X^I]^2 + \sum_{I,J} [X^I, X^J]^2]$. The fermionic part will reduce to:

$$(\text{Tr} \bar{\chi} \Gamma^\mu D_\mu \chi)_{10D} \rightarrow \text{Tr} [\bar{\chi} \Gamma^\mu D_\mu \chi + \bar{\lambda}_a \Gamma^i [X_i, \lambda^b]]$$

Where now the χ_i are fermions that break the **16** representation of $SO(9, 1)$ into a representation of $SO(d-1, 1) \times SO(10-d)$. For $d=3$ we get $\mathcal{N}=4$ SYM and this is $(2, 4) + (\bar{2}, \bar{4})$, corresponding to four Weyl spinors.

22. At the minimum of the potential, all X^I lie in a cartan and mutually commute. The A_{ij} correspond to open strings moving between the D-branes at positions X_I . The ground states of these open strings have a mass squared of $(X_I - X_J)^2/2\pi\ell_s^2$, so indeed the mass is linear in the separation. **Confirm. Understand Lie-Theoretic perspective.**

23. The worldvolume coupling to the RR 2-form looks like $iT_2 \int C_2$. For the brane tilted in the x^1, x^2 plane we can write this explicitly as:

$$i \int dx^0 dx^1 (C_{01} + C_{02} \tan(\theta))$$

Now T-dualizing in the x^2 direction changes the C_{01} form to the RR 3-form C_3 , giving the standard $i \int C_3$ term. On the other hand, the second term gets reduced to $-2\pi\ell_s^2 \int dx^0 dx^1 dx^2 C_0 F_{12}$, where I have used exercise **8.13** to write $\tan \theta = -2\pi\ell_s^2 F_{12}$. So we get a leading coupling to the three-form and a sub-leading coupling (in ℓ_s^2) to the one-form. This is a hint of the *Meyers effect*.

24. For the D2 brane the CP odd terms are C_3 , $C_1 \wedge \mathcal{F}$ and $-C_1 \wedge \frac{(2\pi\ell_s)^4}{48} [p_1(\mathcal{T}) - p_1(\mathcal{N})]$. In the frame described by exercise **7.26**, C_3 transforms trivially under A transformations, and under its own gauge transformations it only adds an exact term which does not modify the CS action.

\mathcal{F} transforms trivially under A transformations and under B transformations only modifies the action by a closed term again.

Equation **I.14** is not in any standard frame. The Dilaton plays no role here. The

I feel I am missing something.

25. Gauge transformations of the axion C_0 are just shifts $C_0 \rightarrow C_0 + a$. C_0 couples to F_2 through the Chern-Simons term:

$$\int C_0 \wedge \text{Tr} e^{\mathcal{F}} \wedge \mathcal{G}$$

Because of the Bianchi identity, $d\mathcal{F} = 0$, and the same holds for any trace of any polynomial of F . Similarly \mathcal{G} is also a closed form. Therefore shifting C_0 gives an integrand term $\text{Tr} e^{\mathcal{F}} \wedge \mathcal{G}$ which is closed.

26. For trivial flat-space background $\eta_{\mu\nu}$, we have $g_{ab} = \partial_a X_\mu \partial_b X^\mu$. Take $M_{ab} = \partial_a X_\mu \partial_b X^\mu + 2\pi\ell_s^2 F_{ab}$ and $M = \det M_{ab}$. Taking the DBI variation w.r.t. X_μ^a and A_a respectively gives:

$$\begin{aligned} \frac{T}{2} \partial^a \left(\sqrt{-M} M_{ab}^{-1} \partial_b X^\mu \right) &= 0 \\ \frac{2\pi\ell_s^2 T}{2} \partial^a \left(\sqrt{-M} M_{ab}^{-1} \right) &= 0 \end{aligned}$$

Its rather nasty to evaluate that inverse matrix. On the other hand, taking X^9 to be the only nontrivial function of the ξ , and depending only on the radial distance r from a central point, and setting all $A_i = 0$ with A_0 a function of r alone, we get $M = 1 + \delta_{a=r, b=r} (\partial_r X^9)^2 + 2\pi\ell_s^2 (\delta_{a=r, b=0} - \delta_{a=0, b=r}) E$. We take $E = \partial_r X^9$. The determinant is then $(\nabla X^9)^2 (1 + 2\pi\ell_s^2)$.

Note that if the second equation of motion holds, the first equation of motion implies that we would want for $\partial^r \partial_r X^9 = 0$, namely that X^9 is a harmonic function of r . On a p brane this is $X^9 = \frac{C_p}{r^{p-2}}$. In this case, the determinant, as well as M^{-1} will vanish when covariantly differentiated by ∂^r , giving us our desired second equation of motion.

This solution is known as a BI-on (BI for Born-Infeld). It represents an infinitely long open string ending on our p -brane.

27. Let's have G, B, Φ, C_2 trivial. We get

$$S = \frac{1}{2\pi\ell_s^2 g} \int d^2 \xi \sqrt{1 - (2\pi\ell_s^2 F_{01})^2} + \frac{1}{2\pi\ell_s^2} \int d^2 \xi C_0 (2\pi\ell_s^2) F_{01}$$

Pick the gauge $A_0 = 0$. Our variable is then A_1 . From this, we get a canonical momentum conjugate to A_1 equal to:

$$-\frac{1}{2\pi\ell_s^2 g} \frac{(2\pi\ell_s^2)^2 F_{01}}{\sqrt{1 - (2\pi\ell_s^2 F_{01})^2}} + C_0 F_{01}$$

Consider putting the D1 brane in a circle. Now since C_0 acts as a θ term, consider putting an integer m for C_0 . The momentum is quantized, and in particular there is a gap between the zero momentum ground state and the next state up. We get:

$$\frac{2\pi\ell_s^2 F_{01}}{\sqrt{1 - (2\pi\ell_s^2 F_{01})^2}} = gm \Rightarrow 2\pi\ell_s^2 F_{01} = \frac{gm}{\sqrt{1 + m^2 g^2}}$$

We have a Hamiltonian

$$\mathcal{H} = \frac{1}{2\pi\ell_s^2 g} \frac{1}{\sqrt{1 - (2\pi\ell_s^2 F_{01})^2}}$$

And from the quantization condition on the electric field we obtain from the Hamiltonian a set of quantized tensions

$$T = \frac{1}{2\pi\ell_s^2 g} \sqrt{1 + m^2 g^2}$$

28. The D3-D₋₁ system has #ND= 4 and so preserves 1/4 supersymmetry (1/2 the SUSY of the D3 brane itself). Similarly, the instanton configurations satisfying $\star F_2 = \pm F_2$. The supersymmetric variation of the gaugino is $\delta\lambda \propto F_{\mu\nu} \Gamma^{\mu\nu}$. The $\Gamma^{\mu\nu}$ are generators of $SO(4) = SU(2) \times SU(2)$, and the (A)SD conditions on F_2 will ensure that only half the generators (the first or second $SU(2)$) will appear in the variation. Thus instanton configurations are also 1/2 BPS on the worldvolume.

To confirm that these instantons really *are* D₋₁ branes, note that the CS term contains $\frac{1}{2}(2\pi\ell_s^2)^2 T_3 \int C_0 F_2 \wedge F_2$. For a nontrivial instanton configuration we get $\int F_2 \wedge F_2 = 8\pi^2$. Thus the instanton coupling to C_0 is $(2\pi\ell_s)^4 T_3 = T_{-1}$, exactly the charge of the D₋₁ brane. **Does this exclude the possibility of objects with the same charge and BPS properties as D₋₁ branes, but that don't have interpretations as endpoints of open strings?**

I expect the moduli space to have dimension $4n$, corresponding to the space (technically Hilbert scheme) of n points on \mathbb{R}^4 .

29. This configuration is invariant under x^1 translations as well as under time x^0 . The exact same BPS properties discussed in the previous question apply here. The state is half-BPS on the worldvolume both from the POV of string theory and from the POV of the low energy SYM theory having half the gaugino variations vanish.

The same instanton action argument in the previous question gives us that $\frac{1}{2}(2\pi\ell_s^2)^2 T_4 \int C_1 F_2 \wedge F_2$ yields a coupling $(2\pi\ell_s)^4 T_4 = T_0$ to the C_1 form.

For N D5 branes the low-energy effective theory is $SU(N)$ SYM, and we obtain the moduli space of $SU(N)$ instantons. The dimension now becomes $4Nk$ **justify**. I expect that the moduli spaces of D1-D5 bound states are identical to the moduli space in the previous problem.

30. First, the pair of 5-branes in the 12345 and 12367 dimensions are parallel in the 123 directions and 90° rotated in two directions. This gives 2 sets of ND and 2 sets of DN boundary conditions, on the strings which gives us $\nu = 4$. In this case, following Polchinski the spinor $\beta = \beta^{\perp-1} \beta^{\perp'}$ has an equal number of eigenvalues -1 and 1 . So half of the original 16 spinors preserved by the first D-brane will be preserved by the combination of both.

Now take a third D -brane in the 12389 direction, perpendicular to both the first two. The same argument shows that we brane another half of the supersymmetry, giving 4 supersymmetries left in this configuration. In other words it is $\frac{1}{8}$ BPS. **Confirm**

31. Adding a D1 string gives 4 ND boundary conditions with each of the other D-branes. This breaks the supersymmetry in half again, preserving 2 supersymmetries now. It is $\frac{1}{16}$ BPS.

If I were to add it along direction 4 it would have 5 ND boundary conditions with the second two branes, which preserves no SUSY, so the latter configuration has nothing preserved.

32. Note this is $O(2)$ and not $SO(2)$, so instead of getting one D -brane at $\theta\tilde{R}$ and the other at $-\theta\tilde{R}$, we get one “half” D8 brane at $x^9 = 0$ (the location of one orientifold plane) and its image at $\pi\tilde{R}$ (the location of the other).

33. We work with the compact real form $\mathrm{USp}(2N) = \mathrm{Sp}(2N) \cap \mathrm{U}(2N)$. In this case any symplectic matrix can again be diagonalized to be of the form $(e^{i\theta_1}, e^{-i\theta_1}, \dots)$. Again we interpret this as D-branes on both sides $\pm\theta_i \tilde{R}$ of the orientifold plane. The generic gauge group is $U(1)^{2N}$. If m branes lie at either orientifold plane we get an enhancement $\mathrm{Sp}(2m)$. When all N branes and their images lie on one of the orientifold planes, we recover the full symmetry.

34. Due to the negative tension, an excitation on it has even lower energy, corresponding to a negative norma state which is forbidden in a unitary theory by positivity. **What more can I say?**

35. There is a mistake in Kiritsis' equation **G.8**. We should have $A = -H^{-1}(\rho)$ not $-H(\rho)$. The way to see that is by noting that $-H^{-1}(\rho) = -\frac{\rho}{\rho+Q} = -\frac{r-Q}{r} = 1 + \frac{Q}{r}$. The constant 1 is gauge and hence irrelevant, while the second term is the proper electric potential that will give rise to a $F_{tr} = \frac{Q}{r^2}$.

Also, this problem asks us to work in $\mathcal{N} = 2, D = 4$ SUGRA, so the appropriate equations should be that the variation of *each* dilatino by a Killing spinor is zero. In this SUGRA, there are two Majorana gravitinos $\psi_{\mu,A}$, $A = 1, 2$ with four components each, for a total of 8 SUSYs. Consequently, the variations involve two Majorana spinor parameters ϵ_A , $A = 1, 2$. We will use lower indices to indicate chiral and upper indices to indicate anti-chiral fermions. The gravitino variation is then (c.f. Freedman *Supergravity* Section 22.4)

$$\delta\psi_{\mu,A} = \nabla_{\mu}\epsilon_A - \frac{1}{4}F_{\nu\rho}\gamma^{\nu\rho}\gamma_{\mu}\epsilon_{AB}\epsilon^B \quad (7)$$

Because of the chirality we can replace F with F^- in the above equation. Now, if we use spatial coordinates \vec{x} , $|x| = \rho$, the metric takes the form

$$ds^2 = -H^{-2}dt + H^2d\vec{x}^2$$

Take the frame fields $e^{\hat{0}} = H^{-1}dt$, $e^{\hat{i}} = Hdx^i$. We will use hats to denote frame indices a, b . Our spin connection is then:

$$\omega_{0i} = -H^{-2}\partial_i \log H dt, \quad \omega_{ij} = \partial_j \log H dx^i - \partial_i \log H dx^j$$

Now because the solution is static, ∂_t is killing and we expect that $\partial_t \epsilon_A = 0$. Looking at $\mu = 0$, equation (7) becomes. We only care about $\omega^{\hat{0}\hat{i}}$ since only this has a lower $\mu = 0$ index corresponding to dt . Similarly for the second term we only care about $\nu, \rho = 0, i$ because of there only being an electric field. We note $\gamma^{\hat{0}\hat{0}}\gamma_0 = H^{-1}\gamma^{\hat{i}}$

$$\begin{aligned} -\frac{1}{2}H^{-2}\partial_i \log H \gamma^{\hat{i}}\gamma^{\hat{0}}\epsilon_A + \frac{1}{2}H^{-1}\partial_i A_0 \gamma^{\hat{i}}\epsilon_{AB}\epsilon^B &= 0 \\ \Rightarrow \partial_i H^{-1}\gamma^{\hat{i}}\gamma^{\hat{0}}\epsilon_A &= \partial_i A_0 \gamma^{\hat{i}}\epsilon_{AB}\epsilon^B \end{aligned}$$

We thus need (up to an irrelevant gauge constant) $A_0 = \pm H^{-1}$ and $\epsilon_A = \mp \gamma^{\hat{0}}\epsilon_{AB}\epsilon^B$. Since we require $-H^{-1}$ to match the potential A , and so that it is asymptotically unity, we have $H = (1 - \frac{Q}{r})^{-1} = \frac{\rho+Q}{\rho} = 1 + \frac{Q}{\rho}$. This verifies (and in fact derives) the extremal RN solution.

Because the constraint $\epsilon_A = \mp \gamma^{\hat{0}}\epsilon_{AB}\epsilon^B$ applies to half the space of spinors at any given point, we have that the extremal RN solution is half-BPS.

36. Again take coordinates x_i so that

$$ds^2 = -\frac{dt^2}{H^2(\rho)} + H^2(\rho)(dx_i^2)$$

Upon the choice $\epsilon_A = -\mp \gamma^{\hat{0}}\epsilon_{AB}\epsilon^B$, we had the relation $\partial_i H^{-1} = \partial_i A_0 = F_{i0}$. The field equation $\partial \star F = 0$

$$\partial_i \sqrt{-g} g^{00} g^{ii} F_{i0} = \partial_i H^2 \partial_i H^{-1} = \partial_i^2 H(x_i)$$

Thus we have that H is a harmonic function of the *flat* Laplacian in transverse space.

We see that H from the previous problem takes the simple form $1 + \frac{Q}{|x|}$, which is obviously harmonic in 3+1 dimensions.

A more general solution allowing for multiple charged extremal black holes amounts to nothing more than replacing H with $1 + \sum_i \frac{Q_i}{|x-x_i|}$, which remains harmonic, and thus preserves half supersymmetry. This looks like a bunch of extremal black holes whose pairwise electric repulsions cancel their gravitational attractions.

37. Again we are working in $\mathcal{N} = 2$ SUGRA. The metric takes the form

$$ds^2 = Q^2 \left[\frac{-dt^2 + d\rho^2}{\rho^2} + d\Omega_2^2 \right]$$

This corresponds to 2D AdS times a sphere of *constant radius*. Both spaces have radius Q . The spinor equation is as before, now with $T_{\mu\nu} = -\frac{1}{L}g_{\mu\nu}$. Consider just AdS₂. Define the operator

$$\hat{D}_\mu \epsilon_A = \nabla_\mu \epsilon_A - \frac{1}{2Q} \gamma_\mu \varepsilon_{AB} \epsilon^B$$

Consider now

$$[\hat{D}_\mu, \hat{D}_\nu] \epsilon_A = \left(\frac{1}{4} R_{\mu\nu ab} \gamma^{ab} + \frac{1}{2Q^2} \right) \epsilon_A$$

And since AdS₂ is a maximally symmetric space $R_{\mu\nu ab} = -(e_{a\mu}e_{b\nu} - e_{a\nu}e_{b\mu})/L^2$ and so the commutator vanishes identically. This is the integrability condition we need. At each point, the spinor bundle is dimension $\mathcal{N} \times 2^{[D/2]}$ - for $\mathcal{N} = 2$ AdS₂ this is 4. We see that any spinor can be transported by the connection \hat{D}_μ to define a spinor field on all of AdS₂, and thus we get that the space is maximally supersymmetric.

The exact same arguments (with $Q \rightarrow -Q$) apply for the positively curved 2-sphere of the same radius.

The product of two maximally supersymmetric spaces is maximally supersymmetric **Confirm**. We get 8 Killing spinors. We now see that the Bertotti-Robertson universe preserves full supersymmetry, and thus the extremal RN black hole plays the role of a *half-BPS soliton* that interpolates between two fully supersymmetric backgrounds (flat space and AdS₂ \times S^2).

38. The variation of $\frac{1}{2(p+2)!} F_{p+2}^2 = F_{p+2} \wedge \star F_{p+2}$ directly gives $\star F_{p+2} = 0$.

Varying the dilaton gives

$$\begin{aligned} 0 &= -2e^{-2\Phi} [R + 4(\nabla\Phi)^2] - \nabla(e^{-2\Phi} 8\nabla\Phi) = -2e^{-2\Phi} [R + 4(\nabla\Phi)^2] + 16e^{-2\Phi} (\nabla\Phi)^2 - 8e^{-2\Phi} \square\Phi \\ &\Rightarrow R = 4(\nabla\Phi)^2 - 4\square\Phi \end{aligned}$$

Finally, writing the metric explicitly in the action

$$\sqrt{-g}e^{-2\Phi} [g^{\mu\nu} R_{\mu\nu} + 4g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi] - \frac{1}{2(p+2)!} \sqrt{-g} F_{p+2}^2$$

Let's look how each term changes when we vary $\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}}$.

- $\sqrt{-g}e^{-2\Phi} R \rightarrow (R_{\mu\nu} + g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) e^{-2\Phi} - \frac{1}{2} g_{\mu\nu} e^{-2\Phi} R$
 $= e^{-2\Phi} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} (-2\square\Phi + 4(\partial\Phi)^2) - (-2\nabla_\mu \nabla_\nu \Phi + 4\partial_\mu \Phi \partial_\nu \Phi) \right)$
- $\sqrt{-g}e^{-2\Phi} 4g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \rightarrow 4e^{-2\Phi} \partial_\mu \Phi \partial_\nu \Phi - 2e^{-2\Phi} (\partial\Phi)^2$
- $-\frac{1}{2(p+2)!} \sqrt{-g} g^{\mu_1 \nu_1} \dots g^{\mu_{p+2} \nu_{p+2}} F_{\mu_1 \dots \mu_{p+2}} F_{\nu_1 \dots \nu_{p+2}} \rightarrow -\frac{1}{2(p+1)!} F_{\mu\nu}^2 + \frac{1}{4(p+2)!} g_{\mu\nu} F^2$. Here $F_{\mu\nu}^2 = F_{\mu\dots} F^{\nu\dots}$

Combining these all together and using the dilaton equations of motion gives

$$e^{-2\Phi} R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{2(p+1)!} F_{\mu\nu}^2 + \frac{1}{4(p+2)!} g_{\mu\nu} F^2 = 0 \Rightarrow R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi = \frac{e^{2\Phi}}{2(p+1)!} \left(F_{\mu\nu}^2 - \frac{g_{\mu\nu}}{2p+2} F^2 \right)$$

exactly as required.

39. This problem was very time-consuming to do out explicitly. The only resource that was of any help for cross-checking was Ortín's "*Gravity and Strings*".

First consider the possibility of $\Phi = 0$, $F = 0$. In this case we have no stress tensor and are left with static vacuum Einstein equations, spherically symmetric in $10 - p$ dimensions and translationally invariant in p dimensions. In that case we know that our solution is nothing more than the Schwarzschild solution in $10 - p$ dimensions times a transverse p -dimensional space:

$$ds^2 = -f(r)dt^2 + dx_i^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{8-p}^2$$

Reproducing the arguments from black holes in 4D, we see that

$$\frac{d}{dr}(r^{8-p}f(r))\frac{d}{dr}\log(f) = \frac{d}{dr}(r^{8-p}f'(r)) = 0$$

ie $f(r)$ must be harmonic in the transverse dimension. After rescaling coordinates to have appropriate asymptotic behavior, we get:

$$f(r) = 1 - \frac{r_0^{7-p}}{r^{7-p}}$$

for some constant r_0 related to the ADM mass of the solution. So we see that $H(r) = 1$ when the dilaton and $p + 2$ -form field strength are turned off. The curvature is $R = 0$. Now let's turn on Φ . We expect small Φ will correspond to small H .

To do this, let's look at the simplest case, $p = 0$. Take the Ansatz (which you can convince yourself to be completely general given the symmetry of the problem)

$$ds^2 = -\lambda(r)dt^2 + \frac{dr^2}{\mu(r)} + R(r)^2 d\Omega_{d-2}^2$$

We will later set $d = 10$, $\lambda = \mu = f(r)/\sqrt{H}$. Let's explicitly calculate the Christoffel symbols. There are three categories: Γ s involving just r, t Γ s involving mixed r, Ω , and Γ s involving just the Ω variables. I will use a, b, c to index the angular variables ψ_a , whose metric is $R^2 d\Omega_{ab}^2 = q_a \delta_{ab}$, and I use $'$ to denote ordinary partial differentiation w.r.t. r .

$$\begin{aligned} \Gamma_{tt}^r &= \frac{1}{2}\mu\lambda' & \Gamma_{tr}^t &= \frac{1}{2}\lambda^{-1}\lambda' & \Gamma_{rr}^r &= -\frac{1}{2}\mu^{-1}\mu' \\ \Gamma_{rb}^a &= \delta_b^a \frac{R'}{R} & \Gamma_{ab}^r &= \delta_{ab}\mu(R^2)' \frac{q_a}{R^2} \\ \Gamma_{bc}^a &= \theta_{b>c}\delta_b^a \cot\psi_b + \delta_{c>b}\theta_{ac} \cot\psi_c - \theta_{b>a}\delta_{bc} \cot\psi_a \frac{q_b}{q_a} \end{aligned}$$

That last Christoffel symbol looks particularly nasty. Thankfully, by using the fact that the sphere is a symmetric space, we will not need to use it explicitly.

Now let's directly compute the Ricci tensor.

$$R_{\mu\nu} = \partial_\rho \Gamma_{\mu\nu}^\rho - \partial_\mu \Gamma_{\rho\nu}^\rho + \Gamma_{\rho\sigma}^\rho \Gamma_{\mu\nu}^\sigma - \Gamma_{\mu\rho}^\sigma \Gamma_{\sigma\nu}^\rho$$

In what follows, it is useful to recall the identity $\Gamma_{\mu\nu}^\mu = \partial_\nu \log \sqrt{-g}$. The nonzero terms will be R_{tt}, R_{rr}, R_{ab} . Respectively they are:

$$\begin{aligned} R_{tt} &= \partial_r \Gamma_{tt}^r - \cancel{\partial_t \Gamma_{\rho t}^\rho} + \Gamma_{\rho r}^\rho \Gamma_{tt}^r - \Gamma_{t\rho}^\sigma \Gamma_{\sigma t}^\rho \\ &= \frac{1}{2}(\mu\lambda')' + \partial_r \log \sqrt{g} (\mu\lambda') - 2\frac{1}{2}\frac{\lambda'}{\lambda}\frac{1}{2}\mu\lambda' \\ &= \frac{1}{2}\frac{\lambda}{\sqrt{g}}\partial_r\left(\sqrt{-g}\mu\frac{\lambda'}{\lambda}\right) = \frac{1}{2}\lambda\nabla^2 \log \lambda \end{aligned}$$

It's important for this next one to note that q_a/R^2 is independent of r . It's equally important to appreciate that the final combination of Γ symbols is the only thing that would appear in the absence of r dependence

in R . In this case, because the $d - 2$ sphere is a symmetric space, we'd have $R_{ab} = \frac{d-3}{R^2} g_{ab}$. Indeed, this is exactly what the final term gives. We thus get

$$\begin{aligned} R_{ab} &= \partial_r \Gamma_{ab}^r - \cancel{\partial_a \Gamma_{\rho b}^\rho} + \Gamma_{\rho r}^\rho \Gamma_{ab}^r - \Gamma_{a\rho}^\sigma \Gamma_{\sigma b}^\rho \\ &= -\frac{1}{2} \delta_{ab} \partial_r \left(\mu (R^2)' \frac{q_a}{R^2} \right) + \partial_r \log \sqrt{-g} \delta_{ab} \mu (R^2)' \frac{q_a}{R^2} + \frac{d-3}{R^2} q_a \delta_{ab} \\ &= g_{ab} \left(-\nabla^2 \log R + \frac{d-3}{R^2} \right) \end{aligned}$$

The next one is a bit different. Less cancelation. The last term will sum over $(\sigma, \rho) = (r, r), (t, t), (a, a)$. Also note $\sqrt{-g} = \sqrt{\lambda/\mu} R^{d-2}$.

$$\begin{aligned} R_{rr} &= \partial_r \Gamma_{rr}^r - \partial_r \Gamma_{\rho r}^\rho + \Gamma_{\rho\sigma}^\rho \Gamma_{rr}^\sigma - \Gamma_{r\rho}^\sigma \Gamma_{\sigma r}^\rho \\ &= \cancel{-\frac{1}{2} (\mu^{-1} \mu')'} - \partial_r^2 \log(\sqrt{\lambda/\mu} R^{d-2}) - \frac{1}{2} \partial_r \log(\sqrt{\lambda/\mu} R^{d-2}) (\mu^{-1} \mu') - \frac{1}{4} \frac{(\lambda')^2}{\lambda^2} - \frac{1}{4} \mu^2 (\mu')^2 - (d-2) \frac{(R')^2}{R^2} \\ &= -\frac{1}{2} \partial_r^2 \log(\lambda) - \frac{d-2}{R} R'' - \frac{1}{2} (d-2) \frac{R'}{R} \partial_r \log \sqrt{\lambda/\mu} \\ &= -\frac{1}{2} \mu^{-1} \nabla^2 \log \lambda - \frac{d-2}{R} \sqrt{\frac{\lambda}{\mu}} \left(R' \sqrt{\frac{\mu}{\lambda}} \right)' \end{aligned}$$

Altogether we get a Ricci scalar:

$$R = -\nabla^2 \log(\lambda R^{d-2}) + \frac{(d-2)(d-3)}{R^2} - \frac{d-2}{R} \sqrt{\lambda\mu} \left(R' \sqrt{\frac{\mu}{\lambda}} \right)'$$

Now let's take $\lambda = \mu = f(r)/\sqrt{H}$ and $R = H^{1/4}$, $\sqrt{-g} = H^{(d-2)/4}$.

$$-\nabla^2 \log(f(r) r^{d-2} H^{\frac{d-4}{4}}) + \frac{(d-2)(d-3)}{r^2 H^{1/2}} - (d-2) \frac{f(r)}{r H^{3/4}} (r H^{1/4})''$$

The Laplacian takes the form $\frac{\partial_r [H^{(d-4)/4} r^8 f(r) \partial_r]}{r^{d-2} H^{(d-2)/2}}$ which simplifies the above to:

$$-\frac{\partial_r^2 (f(r) r^{d-2} H^{(d-4)/4})}{r^{d-2} H^{(d-2)/2}} + \frac{(d-2)(d-3)}{r^{d-2} H^{1/2}}$$

The dilaton equation of motion is

$$R = 4(\nabla\Phi)^2 - 4\nabla^2\Phi$$

Since a nonzero Φ is what gives an H away from 1, we might hypothesize a relationship $\log H \propto \Phi$, meaning we should replace Φ with $\log(H^\alpha)$ in the dilaton equation. Let's also take $f(r)$ to be *not different* from the Schwarzschild solution: $f(r) = 1 - \frac{r_0^{d-3}}{r^{d-3}}$. So far we will not be so bold as to assume *anything* about H . We also at this point need to specialize to $d = 10$, otherwise no nice simplification occurs. Straightforward algebra then gives:

$$\begin{aligned} \text{In[2382]:= } f[r_] &:= 1 - \frac{r_0^{d-3}}{r^{d-3}}; \\ \text{sqrtg} &= r^{d-2} H[r]^{(d-2)/4}; \\ \text{grr} &= \frac{\text{Sqrt}[H[r]]}{f[r]}; \\ &\left(-\frac{1}{r^{d-2} H[r]^{(d-2)/4}} \partial_{\{r,2\}} \left(f[r] r^{d-2} H[r]^{(d-4)/4} \right) + \frac{(d-2)(d-3)}{r^2 \text{Sqrt}[H[r]]} - (d-2) \frac{f[r]}{r H[r]^{3/4}} \partial_{\{r,2\}} \left(r H[r]^{1/4} \right) /. d \rightarrow 10 \right) - \\ &\quad \left(4 \text{grr}^{-1} \left(\partial_r \text{Log}[H[r]^\alpha] \right)^2 - \frac{4}{\text{sqrtg}} \partial_r \left(\text{sqrtg} \text{grr}^{-1} \partial_r \left(\text{Log}[H[r]^\alpha] \right) \right) \right) /. d \rightarrow 10 // \text{FullSimplify} \\ \text{Out[2385]= } &\frac{r \left(r^7 - r_0^7 \right) \left(3 + 8 (1 - 2 \alpha) H[r] \right) H'[r]^2 + 2 H[r] \left(2 \left(r_0^7 (7 - 4 \alpha) + 4 r^7 (-7 + 8 \alpha) \right) H'[r] + r \left(r^7 - r_0^7 \right) (-7 + 8 \alpha) H''[r] \right)}{4 r^8 H[r]^{5/2}} \end{aligned}$$

To get rid of the term quadratic in $H'(r)$ we need $3 + 8(1 - 2\alpha)\alpha = 0 \Rightarrow \alpha = 3/4$.

After that, these will only be equal if

$$8H' - rH'' = 0 \Rightarrow H = 1 - \frac{L^7}{r^7}$$

The above solution is the most general given that $H \rightarrow 1$ as $r \rightarrow \infty$.

Now let us generalize this to higher dimensions. We add p x_i in the parallel dimensions of the solution.

$$ds^2 = -\lambda(r)dt^2 + \nu(r)d\vec{x}_i^2 + \frac{dr^2}{\mu(r)} + R(r)^2 d\Omega_{d-2}^2$$

This gives two new Christoffels. Here is a complete list

$$\begin{aligned} \Gamma_{tt}^r &= \frac{1}{2}\mu\lambda' & \Gamma_{tr}^t &= \frac{1}{2}\lambda^{-1}\lambda' & \Gamma_{rr}^r &= -\frac{1}{2}\mu^{-1}\mu' \\ \Gamma_{rb}^a &= \delta_b^a \frac{R'}{R} & \Gamma_{ab}^r &= \delta_{ab}\mu(R^2)' \frac{q_a}{R^2} \\ \Gamma_{bc}^a &= \theta_{b>c}\delta_b^a \cot\psi_b + \delta_{c>b}\theta_{ac} \cot\psi_c - \theta_{b>a}\delta_{bc} \cot\psi_a \frac{q_b}{q_a} \\ \Gamma_{ij}^r &= -\frac{1}{2}\delta_{ij}\mu\nu' & \Gamma_{rj}^i &= \frac{1}{2}\delta_j^i \nu^{-1}\nu' \end{aligned}$$

Our nonzero Ricci components are now $R_{tt}, R_{rr}, R_{ab}, R_{ij}$. The primary way that the new dimensions will contribute is by modifying $\sqrt{-g}$ We get:

$$\begin{aligned} R_{tt} &= R_{tt}^{(10-p)} - \frac{1}{4}p\mu\lambda'(\log\nu)' \\ R_{ab} &= R_{ab}^{(10-p)} - \frac{1}{2}pg_{ab}\mu(\log\nu)'(\log R)' \\ R_{rr} &= R_{rr}^{(10-p)} + \frac{1}{2}p(\mu\nu)^{-1/2}((\mu\nu)^{1/2}(\log\nu)')' \\ R_{ij} &= \frac{1}{2}\delta_{ij}\nu\nabla^2\nu \end{aligned}$$

This gives a Ricci curvature of:

$$R = R^{(10-p)} + \frac{1}{2}p(-\nabla_{10-p}^2 \log\nu - \nu^{-1}\nabla^2\nu + \frac{1}{2}\mu((\log\nu)')^2)$$

Making the necessary replacements we get

$$\begin{aligned} & -\frac{\partial_r^2(f(r)r^{d-2}H^{(d-4)/4})}{r^{d-2}H^{(d-2)/2}} + \frac{(d-2)(d-3)}{r^{d-2}H^{1/2}} \\ & + \frac{1}{2}p\left(-\frac{\partial_r[f r^{8-p}H^{(6-p)/4}\partial_r \log H^{-1/2}]}{r^{8-p}H^{(8-p)/4}} - \frac{\sqrt{H}}{r^{8-p}H^{(8-2p)/4}}\partial_r[f r^{8-p}H^{(6-2p)/4}\partial_r H^{-1/2}] + \frac{1}{2}\frac{f}{H^{1/2}}(\partial_r \log H^{-1/2})^2\right) \end{aligned}$$

It makes sense to take the ansatz $f(r) = 1 - \frac{r_0^{7-p}}{r^{7-p}}$ and $H = 1 + \frac{L^{7-p}}{r^{7-p}}$. Further, the relationship between H and Φ can be guessed from reasoning in the $p = 0$ case to go as $\Phi \propto H^{(3-p)/4}$, or alternatively we can establish this from first principles by algebra

$$\begin{aligned}
\text{In[2732]} &:= \mathbf{H[r_]} := 1 + \frac{L^{7-p}}{r^{7-p}}; \\
\mathbf{f[r_]} &:= 1 - \frac{r\theta^{7-p}}{r^{7-p}}; \\
\mathbf{grr} &= \frac{\text{Sqrt}[H[r]]}{f[r]}; \\
\mathbf{srtg} &= r^{8-p} H[r]^{(8-2p)/4}; \\
\mathbf{FullSimplify} &\left[-\frac{1}{r^{8-p} H[r]^{(8-p)/4}} \partial_{(r,2)} \left(f[r] r^{8-p} H[r]^{(6-p)/4} \right) + \frac{(8-p)(7-p)}{r^2 \text{Sqrt}[H[r]]} - (8-p) \frac{f[r]}{r H[r]^{3/4}} \partial_{(r,2)} \left(r H[r]^{1/4} \right) + \right. \\
&\quad \frac{1}{2} p \left(\frac{1}{2 r^{8-p} H[r]^{(8-p)/4}} \partial_r \left(f[r] r^{8-p} H[r]^{(6-p)/4} \partial_r (\text{Log}[H[r]]) \right) - \frac{H[r]^{1/2}}{r^{8-p} H[r]^{(8-2p)/4}} \partial_r \left(f[r] r^{8-p} H[r]^{(6-2p)/4} \partial_r (H[r]^{-1/2}) \right) + \right. \\
&\quad \left. \left. \frac{1}{2} \frac{f[r]}{\text{Sqrt}[H[r]]} \left(\partial_r (\text{Log}[H[r]^{-1/2}]) \right)^2 \right), p < 7 \right] \\
\mathbf{FullSimplify} &\left[\left(4 grr^{-1} \left(\partial_r \text{Log}[H[r]^\alpha] \right)^2 - \frac{4}{\text{srtg}} \partial_r \left(\text{srtg} grr^{-1} \partial_r (\text{Log}[H[r]^\alpha]) \right) \right) \right] \\
\text{Out[2736]} &= -\frac{L^7 (-7+p)^2 (-3+p) r^{-9+2p} r\theta^{-p} \left(4 L^p r^7 r\theta^7 + L^7 (-(-3+p) r^p r\theta^7 + (1+p) r^7 r\theta^p) \right)}{4 \sqrt{1 + L^{7-p} r^{-7+p}} (L^p r^7 + L^7 r^p)^2} \\
\text{Out[2737]} &= \frac{2 L^7 (-7+p)^2 r^{-9+2p} r\theta^{-p} \alpha \left(2 L^p r^7 r\theta^7 + L^7 (-r^p r\theta^7 (-3+p+2\alpha) + r^7 r\theta^p (-1+p+2\alpha)) \right)}{\sqrt{1 + L^{7-p} r^{-7+p}} (L^p r^7 + L^7 r^p)^2}
\end{aligned}$$

This immediately gives that the dilaton term will equal the scalar curvature only when $\alpha = (3-p)/4$.

We have thus proved the form of f, H, Φ . Let's finally look at the RR field.

For now let us ignore the issues with self-duality at $p = 3$. Take the the $p+1$ form has flux in the radial but not angular directions in transverse space. The only nonzero component of F_{p+2} is given by $F_{r0\dots p}$. The equation of motion gives:

$$\partial_r (\sqrt{-g} g^{\mu_0 \nu_0} g^{\dots} F_{\mu_0 \dots}) = 0$$

Now we already have $\sqrt{g} = r^{8-p} H^{(4-p)/2}$, while we will have raising for each index $0 \dots p$ as well as r , giving a factor of $f(r) H^{(p+1)/2} H^{-1/2} / f(r) = H^{p/2}$. Altogether the differential equation becomes:

$$\partial_r r^{8-p} H^2 H^r$$

Immediately we must have $F = \frac{\kappa/r^{8-p}}{H^2}$. This means that F is proportional to $H'(r)/H(r)^2$.

I don't know how to easily get this constant of proportionality without knowing the decay properties of $R_{\mu\nu}$ as $r \rightarrow \infty$ **Return to this**. I know it must scale roughly as a positive power of L . It turns out to be:

$$F_{r0\dots p} = -\sqrt{1 - \frac{r_0^{7-p}}{L^{7-p}} \frac{H'_p(r)}{H_p^2(r)}}$$

Can I do this all by somehow “boosting” Schwarzschild?

40. From the expression **8.8.9** of the electric field in terms of H we have (assuming $p < 7$)

$$E_r = (7-p) L^{(7-p)/2} \sqrt{L^{7-p} + r_0^{7-p}} \frac{r^{6+p}}{(r^7 + L r^p)^2} \rightarrow (7-p) L^{(7-p)/2} \sqrt{L^{7-p} + r_0^{7-p}} r^{p-8}$$

Integrating this over the $8-p$ sphere will give

$$T_p N = \frac{\Omega_{8-p}(7-p)}{2\kappa_{10}^2} L^{(7-p)/2} \sqrt{L^{7-p} + r_0^{7-p}}.$$

41. Using the standard ADM formula (cf, eg Carrol)

$$M = \frac{1}{2\kappa_{10}^2} \int_{S^{8-p}} g^{\mu\nu} (g_{\mu\alpha, \nu} - g_{\mu\nu, \alpha}) n^\alpha dS$$

Then for a metric that looks like $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu} = \frac{c_{\mu\nu}}{r^{7-p}}$.

$$\begin{aligned} M = T_{00}V_p &= \frac{\Omega_{8-p}}{2\kappa_{10}^2}((7-p)c_{00} - \eta_{00}\eta^{ab}c_{ab}) \\ &= \frac{\Omega_{8-p}V_p}{2\kappa_{10}^2}((7-p)(r_0^{7-p} + \frac{1}{2}L^{7-p}) + r_0^{7-p} + \frac{1}{2}L^{(7-p)/4}) \\ &= \frac{\Omega_{8-p}V_p}{2\kappa_{10}^2}((8-p)r_0^{7-p} + (7-p)L^{7-p}) \end{aligned}$$

Revisit- something seems off

42. Note that

$$f_-(\rho) = 1 - \frac{L^{7-p}}{r^{7-p} + L^{7-p}} = \frac{1}{1 + L^{7-p}r^{7-p}} = \frac{1}{H(r)}$$

Similarly

$$f_+(\rho) = 1 - \frac{r_0^{7-p} + L^{7-p}}{r^{7-p} + L^{7-p}} = \frac{r^{7-p} + r_0^{7-p}}{r^{7-p} + L^{7-p}} = \frac{f(r)}{H(r)}$$

This confirms that the dt and $d\vec{x}$ terms are indeed consistent, and that the $f_-^{-1/2}(\rho)$ in front of the transverse part is our desired \sqrt{H} . Next note that:

$$f_-(\rho)^{\frac{1}{7-p}} = \left[\frac{\rho^{7-p} - L^{7-p}}{\rho^{7-p}} \right]^{1/(7-p)} = \frac{r}{\rho} \Rightarrow \rho^2 f_-(\rho)^{1-\frac{5-p}{7-p}} = \rho^2 \frac{r^2}{\rho^2} = r^2$$

So the angular part is consistent. Lastly, $f_+/f_- = f(r)$ which is the required coefficient for dr^2 . It remains to cancel the jacobian:

$$dr = \frac{\rho^{6-p}}{r^{6-p}} d\rho = f_-^{-\frac{6-p}{7-p}} d\rho \Rightarrow dr^2 = f_-^{-\frac{12-2p}{7-p}} d\rho^2 = f_-^{-1-\frac{5-p}{7-p}} d\rho^2$$

So

$$\frac{\sqrt{H(r)}}{f(r)} dr^2 = f_-^{-1/2} \frac{f_-}{f_+} f_-^{-1-\frac{5-p}{7-p}} d\rho^2 = f_-^{-\frac{1}{2}-\frac{5-p}{7-p}} \frac{d\rho^2}{f_+(\rho)}$$

43. Before doing any supersymmetric manipulations, we should know the spin connection.

Take the extremal p -brane metric to be of the form

$$ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)} dx^i dx^j \delta_{ij}$$

We will shortly set $A = -B = \frac{1}{2} \log H(r)$. Take the frame fields

$$e^{\hat{\mu}} = e^A dx^\mu, \quad e^{\hat{i}} = e^B dx^i$$

Then

$$\begin{aligned} de^{\hat{\mu}} &= \partial_r A e^A dr \wedge dx^\mu = \sum_i \partial_i A e^A dx^i \wedge dx^\mu = e^{\hat{i}} \wedge \omega_{\hat{i}}^{\hat{\mu}} \Rightarrow \omega_{\hat{\mu}\hat{\nu}} = 0, \quad \omega_{\hat{\mu}\hat{i}} = \partial_i A e^{A-B} dx^\mu \\ de^{\hat{i}} &= \partial_r B e^B dr \wedge dx^i = \sum_j \partial_j B e^B dx^j \wedge dx^i = e^{\hat{j}} \wedge \omega_{\hat{j}}^{\hat{i}} \Rightarrow \omega_{\hat{i}\hat{j}} = \partial_j B dx^i - \partial_i B dx^j \end{aligned}$$

Using our extremal form of the solution, I can further write

$$e^\Phi = g_s^2 H^{(3-p)/4} = e^{(3-p)A}, \quad F_{r01\dots p} = \mp \frac{H'}{H^2} = \mp \kappa \frac{A'}{e^{4A}}$$

Here I have appropriately absorbed g_s^2 into the definition of A . I expect $F_{p+2} = \mp H'/H^2$ to be proportional (with an appropriate choice of κ that we will find) to what we have above.

In $\mathcal{N} = 2$ SUGRA coupled to matter, represent the Killing spinor as $\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$. We have the gravitino and dilatino variations:

$$\begin{aligned} 0 = \delta\psi_{\mu,A} &= (\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab})\epsilon - \frac{e^\Phi}{16}\not{F}\Gamma_\mu\mathcal{P}_{p+2}\epsilon \\ 0 = \delta\lambda &= \not{\Phi}\epsilon + \frac{e^\Phi}{8}(-1)^p(3-p)\not{F}\mathcal{P}_{p+2}\epsilon \end{aligned}$$

Here we are following the democratic formulation of Kiritsis Appendix **I.4**, setting all fields equal to zero except for the dilaton and relevant RR $p+2$ field strength. As written there, for IIA we have $\mathcal{P}_n = (\Gamma_1 1)^{n/2}\sigma^1$ and for IIB we have $\mathcal{P}_n = \sigma^1$ for $\frac{1+n}{2}$ even and $i\sigma^2$ for $\frac{1+n}{2}$ odd.

Let's first look at the dilatino variation. We get

$$\begin{aligned} (3-p)A'\Gamma^r\epsilon \mp e^{A(3-p)}(-1)^p(3-p)\frac{A'}{e^{4A}}\Gamma^{r0\dots p}\mathcal{P}_{p+2}\epsilon &= 0 \\ \Rightarrow (1 \mp (-1)^p e^{-A(1+p)}\Gamma^{01\dots p}\mathcal{P}_{p+2})\epsilon &= 0 \\ \Rightarrow (1 \mp (-1)^p\Gamma^{\hat{0}\hat{1}\dots\hat{p}}\mathcal{P}_{p+2})\epsilon &= 0 \end{aligned}$$

Here the Γ matrices with hatted (vielbein) indices are the familiar 10D Dirac matrices, as in Freedman and Van Proyen *Supergravity*¹. Locally, this is a linear algebraic constraint on the space of spinors at a given point, which half of the spinors will satisfy.

Now let's look at *longitudinal* the gravitino variation.

$$\begin{aligned} \partial_i\epsilon + \frac{1}{4}\omega_i^{ab}\Gamma_{ab}\epsilon \mp \frac{e^{A(3-p)}}{16}\frac{4A'}{e^{4A}}\Gamma^{r0\dots p}\Gamma_i\mathcal{P}_{p+2} \\ = \partial_i\epsilon + \frac{1}{2}\partial^k B \Gamma_{ki}\epsilon \pm (-1)^p\frac{e^{A(3-p)}}{4}\frac{A'}{e^{4A}}\Gamma^{0\dots p}\mathcal{P}_{p+2} \end{aligned}$$

44. We write again the spin connection found in the last problem:

$$e^{\hat{\mu}} = e^A dx^\mu, \quad e^{\hat{i}} = e^B dx^i$$

Hatted indices always denote the vielbein indices.

Then

$$\begin{aligned} de^{\hat{\mu}} &= \partial_r A e^A dr \wedge dx^\mu = \sum_i \partial_i A e^A dx^i \wedge dx^\mu = e^{\hat{i}} \wedge \omega_{\hat{i}}^{\hat{\mu}} \Rightarrow \omega_{\hat{\mu}\hat{\nu}} = 0, \quad \omega_{\hat{\mu}\hat{i}} = \partial_i A e^{A-B} dx^\mu \\ de^{\hat{i}} &= \partial_r B e^B dr \wedge dx^i = \sum_j \partial_j B e^B dx^j \wedge dx^i = e^{\hat{j}} \wedge \omega_{\hat{j}}^{\hat{i}} \Rightarrow \omega_{\hat{i}\hat{j}} = \partial_j B dx^i - \partial_i B dx^j \end{aligned}$$

From this, we can get the Riemann curvature using $\mathbf{R}_{\hat{\alpha}\hat{\beta}} = d\omega_{\hat{\alpha}\hat{\beta}} + \omega_{\hat{\alpha}\hat{\gamma}} \wedge \omega_{\hat{\beta}}^{\hat{\gamma}}$. First $\mathbf{R}_{\hat{\mu}\hat{\nu}}$ is the easiest:

$$\mathbf{R}_{\hat{\mu}\hat{\nu}} = \cancel{d\omega_{\hat{\mu}\hat{\nu}}} + \omega_{\hat{\mu}\hat{i}} \wedge \omega_{\hat{\nu}}^{\hat{i}} = e^{2(A-B)}(\partial A)^2 dx^\mu \wedge dx^\nu = \mathbf{R}_\mu{}^\nu$$

Note that last expression is unhatted.

Next is $\mathbf{R}_{\hat{\mu}\hat{i}}$

$$\begin{aligned} \mathbf{R}_{\hat{\mu}\hat{i}} &= d\omega_{\hat{\mu}\hat{i}} + \omega_{\hat{\mu}\hat{j}} \wedge \omega_{\hat{i}}^{\hat{j}} \\ &= [\partial_j \partial_i A e^{A-B} + \partial_i A (\partial_j A - \partial_j B)] dx^j \wedge dx^\mu - \partial_i A e^{A-B} \partial_j B dx^\mu \wedge dx^i + \partial_j A e^{A-B} \partial_i B dx^\mu \wedge dx^j \\ &= e^{A-B}[(\partial_i \partial_j A + \partial_i A \partial_j A - \partial_i B \partial_j A - \partial_j B \partial_i A)] dx^j \wedge dx^\mu - e^{A-B} \partial_j A \partial_j B dx^\mu \wedge dx^i \end{aligned}$$

We can get $R_\mu{}^i$ (note unhatted) by multiplying this by e^{A-B} and $R_i{}^\mu$ by multiplying this by $-e^{B-A}$.

¹Here I'm off by two in the second factor unless I use Blumenhagen's conventions (not democratic)

Finally $\mathbf{R}_{\hat{i}\hat{j}}$:

$$\begin{aligned}\mathbf{R}_{\hat{i}\hat{j}} &= d\omega_{\hat{i}\hat{j}} + \cancel{\omega_{\hat{i}\hat{\mu}} \wedge \omega_{\hat{j}}^{\hat{\mu}}} + \omega_{\hat{i}\hat{k}} \wedge \omega_{\hat{j}}^{\hat{k}} \\ &= \partial_k \partial_j B dx^k \wedge dx^i - \partial_k \partial_i B dx^k \wedge dx^j + \partial_k B \partial_j B dx^i \wedge dx^k - \partial_k B \partial_i B dx^j \wedge dx^k - (\partial B)^2 dx^i \wedge dx^j \\ &= -(\partial B)^2 dx^i \wedge dx^j + (\partial_k \partial_i B - \partial_k B \partial_i B) dx^j \wedge dx^k - (\partial_k \partial_j B - \partial_k B \partial_j B) dx^i \wedge dx^k = \mathbf{R}_i^j\end{aligned}$$

Find Kreschman

To get the Ricci tensor, we must to the appropriate contractions:

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho + R_{\mu i\nu}{}^i = -\eta_{\mu\nu} e^{2(A-B)} \left((p+1)(A')^2 + A'' + \frac{8-p}{r} A' + A' B' (9-p-2) \right)$$

Here $A' = \partial_r A$ is differentiation with respect to the radial coordinate. The Ricci tensor has no components mixing transverse and longitudinal directions:

$$R_{\mu i} = \cancel{R_{\mu\rho i}{}^\rho} + \cancel{R_{\mu j i}{}^j} = 0$$

Finally the annoying one:

$$R_{ij} = R_{i\mu j}{}^\mu + R_{ikj}{}^k$$

FINISH

We can evaluate this directly in Mathematica. For $R_{\mu\nu} R^{\mu\nu}$ and R we get:

```
In[391]:= H[r_] := 1 +  $\frac{L^{7-p}}{r^{7-p}}$ ;
A[r_] := - $\frac{1}{4}$  Log[H[r]]
B[r_] :=  $\frac{1}{4}$  Log[H[r]]
R $\mu\nu$  = -Exp[2 (A[r] - B[r])] (A'[r] + (p+1) A'[r]^2 + (7-p) A'[r] B'[r] +  $\frac{8-p}{r}$  A'[r]) // FullSimplify;
Rmn = - (B'[r] + (p+1) A'[r] B'[r] + (7-p) (B'[r])^2 +  $\frac{2(7-p)+1}{r}$  B'[r] +  $\frac{p+1}{r}$  A'[r]) -
 $\frac{1}{9-p}$  ((7-p) B'[r] + (p+1) A'[r] - 2 (p+1) A'[r] B'[r] + (p+1) (A'[r])^2 - (7-p) (B'[r])^2 -
 $\frac{7-p}{r}$  B'[r] -  $\frac{p+1}{r}$  A'[r]) // FullSimplify;
In[396]:= main = - (B'[r] + (p+1) A'[r] B'[r] + (7-p) (B'[r])^2 +  $\frac{2(7-p)+1}{r}$  B'[r] +  $\frac{p+1}{r}$  A'[r]);
offdiag =
- ((7-p) B'[r] + (p+1) A'[r] - 2 (p+1) A'[r] B'[r] + (p+1) (A'[r])^2 - (7-p) (B'[r])^2 -
 $\frac{7-p}{r}$  B'[r] -  $\frac{p+1}{r}$  A'[r]) // FullSimplify;
RicciSquared = (p+1) (Exp[-2 A[r]] R $\mu\nu$ ^2 + (Exp[-2 B[r]])^2 ((9-p) main^2 + 2 main*offdiag + offdiag^2) //
FullSimplify
R = (p+1) Exp[-2 A[r]] R $\mu\nu$  + Exp[-2 B[r]] ((9-p) main + offdiag) // FullSimplify
Out[398]=  $\frac{1}{32 (L^p r^7 + L^7 r^p)^5} L^{14+p} (-7+p)^2 r^{3+2p} (8 L^{2p} (-9+p) (-8+p) (-3+p)^2 r^{14} +$ 
 $L^{14} (1+p) (137+p (-1+(-9+p)p)) r^{2p} - 8 L^{7+p} (-8+p) (-5+p) (-3+p) (1+p) r^{7+p})$ 
Out[399]= -  $\frac{L^{14} (-7+p)^2 (-3+p) (1+p) r^{-2+2p}}{4 \sqrt{1 + L^{7-p} r^{-7+p} (L^p r^7 + L^7 r^p)^2}}$ 
```

The last line is in agreement with the expression for R in Kiritsis **8.8.31**

45. Exercise 7.7 shows that, upon T -dualizing along the x^9 direction we get

$$\tilde{C}_{\mu_1 \dots \mu_p 9}^{(p+1)} = C_{\mu_1 \dots \mu_p}^{(p)}, \quad \tilde{C}_{\mu_1 \dots \mu_p}^{(p)} = \tilde{C}_{\mu_1 \dots \mu_p 9}^{(p+1)}$$

In transverse space, our $(p+1)$ -form C has components only along the longitudinal directions. Upon T -dualizing, we pick up the 9 index in the C form, and thus get that our brane has a $(p+2)$ form charge. We thus expect this to be a $p+1$ brane wrapping that additional x^9 direction. I'm unsure if this wants us to explicitly give the form of that solution, since doing it in a compact space seems a bit harder.

46. Let's assume $p < 7$. When $\lambda \gg 1$ the perturbative stringy description is no longer valid. For an extremal p-brane, we know from problem 40 that:

$$L^{7-p} = \frac{2\kappa_{10}^2 T_p N}{(7-p)\Omega_{8-p}} \Rightarrow \left(\frac{L}{2\pi\ell_s} \right)^{7-p} = \frac{g_s N}{7-p} \frac{\Gamma(\frac{9-p}{2})}{2\pi^{\frac{9-p}{2}}}$$

So $\lambda = 2\pi g_s N \gg 1$ gives that $L \gg \ell_s$, meaning that the throat size is macroscopic. We can thus probe it without having to see distances smaller than the string scale.

When $p > 3$ we see from our calculation of R in problem 44 that R blows up as $\frac{L^{2(7-p)}}{r^{(p-3)/2}}$ as $r \rightarrow 0$. This will become order ℓ_s^{-2} at

$$r \approx \left(\frac{\ell_s^2}{L^{(7-p)/2}} \right)^{2/(p-3)}$$

When $p < 3$ the formula for R indeed is seen to go to zero. On the other hand the string coupling grows as

$$e^\Phi = g_s H^{(3-p)/4} = g_s \left(1 + \frac{L^{7-p}}{r^{7-p}} \right)^{(3-p)/4}$$

So if g_s is the string coupling “at infinity” which we can take to initially be small, then it will become appreciable at

$$r = L(-1 + g_s^{-4/(3-p)})^{1/(p-7)}$$

so for g_s sufficiently small, the quantity in parentheses will be quite large and be raised to a negative power, so that this is a small fraction of the throat size.

- 47.
- 48.
49. Transverse to the NS5 brane is precisely the extremal BH solution in 5D, which preserves half SUSY by the same arguments as before. Parallel to it is flat space (which preserves all SUSY). The product therefore preserves half.
- 50.
- 51.