Chapter 13: Black Holes and Entropy in String Theory

1. We begin with

$$ds^2 = -F(r)C(r)dt^2 + \frac{dr^2}{C(r)} + H(r)r^2 d\Omega_2^2$$

and $C(r)$ vanishes at the horizon $r = r_0$ while all other functions are positive for $r \geqslant r_0$ and everything asymptotes to 1 as $r \to \infty$. Now, let's do a wick rotation $t \to i\tau$ with τ Euclidean time. We get

$$F(r)C(r)dt^2 + \frac{dr^2}{C(r)} + H(r)r^2 d\Omega_2^2$$

Now at $r = r_0 + \epsilon$ we see that the geometry takes the form

$$F(r_0)C'(r_0)(r - r_0)dt^2 + \frac{dr^2}{C'(r_0)(r - r_0)} + r_0^2 d\Omega_2^2$$

The last term is simply the expected metric on a 2-sphere of fixed radius r_0. The other two terms give a metric

$$ds^2 = F(r_0)C'(r_0)\epsilon dt^2 + \frac{d\epsilon^2}{C'(r_0)\epsilon}$$

Take $u = \frac{2\sqrt{\epsilon}}{\sqrt{C'(r_0)}}$ then $du = \frac{d\epsilon}{\sqrt{C'(r_0)\epsilon}}$ giving us

$$ds^2 = F(r_0)\frac{C'(r_0)^2}{4}u^2 dt^2 + du^2$$

This describe a conical deficit geometry in polar coordinates. In order to obtain a smooth geometry, we need the requirement that

$$\tau + 2\pi \times \frac{2}{C'(r_0)\sqrt{F(r_0)}} = \tau$$

Giving an inverse temperature of

$$\frac{1}{T} = \beta = \frac{4\pi}{C'(r_0)\sqrt{F(r_0)}}$$

This formula generalizes directly to higher-dimensional black holes.

2. In what follows, recall the area formula for a general KN Black hole of mass charge and spin (M, Q, J) is:

$$A = 4\pi(r_+^2 + a^2), \quad r_+ = M + \sqrt{M^2 - a^2 - Q^2}, \quad a = J/M$$

In particular an extremal Kerr black hole has area $8\pi M^2$.

   (a) The areas of the individual Schwarzschild black holes are

$$4\pi(2M_i)^2 = 16\pi M_i^2$$

each. The area of their composite must then be $\geqslant 16\pi(M_1^2 + M_2^2)$. Because they start as almost stationary, the total angular momentum in the center of mass frame is zero, so the final black hole will be (essentially) Schwarzschild. So we get

$$M_f^2 \geqslant M_1^2 + M_2^2$$

If the initial masses were equal, we'd get $M_f \geqslant \sqrt{2}M$ so that $E = 2M - \sqrt{2}M$ and $E/(M_1 + M_2) = 1 - 1/\sqrt{2}$. Let's write WLOG $M_2 = \gamma M_1$ with $\gamma \leqslant 1$ then

$$M_f^2 \geqslant (1 + \gamma^2)M_1^2 \Rightarrow E = (1 + \gamma)M_1 - \sqrt{1 + \gamma^2}M_1$$

$$\Rightarrow \frac{E}{M_1 + M_2} = \frac{(1 + \gamma)M_1 - \sqrt{1 + \gamma^2}M_1}{(1 + \gamma)M_1} = 1 - \frac{\sqrt{1 + \gamma^2}}{(1 + \gamma)} \geqslant 1 \leqslant 1 - \frac{1}{\sqrt{2}}$$

as required.

(b) For two extremal RN black holes we have $r_+ = M$ so each has area $4\pi M^2$. They will collide to form a neutral (perhaps rotating) black hole. The area law gives us

$$4\pi((M_f + \sqrt{M_f^2 - a^2})^2 + a^2) \geqslant 2 \times 4\pi M^2.$$

This bound is sharpest if we take the final state to be extremal Kerr $a = M$, giving

$$8\pi M_f^2 \geqslant 8\pi M^2 \Rightarrow M_f \geqslant M$$

We get

$$E \leqslant 2M - M \Rightarrow \frac{E}{2M} \leqslant \frac{1}{2}.$$

(c) Such a decay would look like

$$2M_f^2 \geqslant M^2 \Rightarrow \sqrt{2}M_f \geqslant M \Rightarrow M - 2M_f \leqslant (\sqrt{2} - 2)M_F < 0.$$

This is a contradiction.

3. We have that $n = \frac{1}{\sqrt{g_{rr}}}\partial_r = \sqrt{f(r)}\partial_r$ so that

$$K_{\mu\nu} = \frac{1}{2}\frac{1}{\sqrt{g_{rr}}}\partial_r G_{\mu\nu} = \frac{\sqrt{f(r)}}{2}\text{diag}\left(f'(r), -\frac{f'(r)}{f(r)}, 2r, 2r\sin^2\theta\right)$$

Contracting with the 3×3 boundary inverse metric $h^{\mu\nu} = \text{diag}(f(r)^{-1}, r^{-2}, r^{-2}\sin^{-2}\theta)$ which has *no r component* gives

$$K = \frac{\sqrt{f}}{2}\left(\frac{f'}{f} + \frac{4}{r}\right) = \sqrt{f}\frac{rf' + 4f}{2rf}\bigg|_{r=r_0}$$

where r_0 is large and formally infinite. We can then evaluate

$$\frac{1}{8\pi G}\int_{\partial M}\sqrt{h}K = \frac{4\pi r^2 \beta \sqrt{f}}{8\pi G}K\bigg|_{r=r_0} = \beta\frac{r}{4G}(rf' + 4f)\bigg|_{r=r_0}$$

Directly evaluating this for $f(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}$ gives

$$\frac{\beta}{2G}\left(\frac{Q^2}{r_0^2} + 2r_0 - 3GM\right)$$

This is the gravitational boundary term contribution to the classical action. The gravitational bulk term is zero since the Ricci scalar vanishes for the RN solution. The electromagnetic contribution is

$$\frac{1}{16\pi G}\int_M \sqrt{g}F_{\mu\nu}F^{\mu\nu} = \frac{1}{8\pi G}\int_0^\beta d\tau \int d\Omega_2 \int_{r_+}^{r_0} r^2 dr\frac{Q^2}{r^4} = \beta\frac{4\pi Q^2}{8\pi G}\left(\frac{1}{r_+} - \frac{1}{r_0}\right) = \frac{\beta}{2G}Q^2\left(\frac{1}{r_+} - \frac{1}{r_0}\right)$$

All together, as $r_0 \to \infty$ we get action:

$$S_{RN} = -\frac{\beta}{2G}\left(2r_0 - 3GM + \frac{Q^2}{r_+}\right)$$

Note that there is one divergent term, namely the one linear in r_0 in the boundary action, but this is insensitive to the properties of the RN black hole and is also present in flat space. It is then sensible to define a regularized (renormalized) action by subtracting this term off. In doing this subtraction, there is an ambiguity of how we should define the inverse temperature of the reference flat space subtraction. The appropriately redshifted temperature **Justify** is $\beta\sqrt{f}$, giving reference action:

$$S_{flat} = -\frac{\beta}{G}r_0\sqrt{f(r_0)} = -\frac{\beta}{2G}(2r_0 - 2GM + O(1/r_0))$$

The renormalized Euclidean action is thus

$$I_{RN} = S_{RN} - S_{flat} = \frac{\beta}{2}(M - \frac{Q^2}{Gr_+}) = \frac{\beta}{2}(M - \mu Q) = \beta\mathcal{F}$$

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4. The specific heat C is given by the coefficient in

$$dM = MCdT$$

For Schwarzschild, $T = (8\pi GM)^{-1}$ so this is

$$dM = -MC\frac{dM}{8\pi GM^2} \Rightarrow C = -8\pi GM$$

Which is negative. This should not be so surprising, given that by increasing the energy (ie mass) of the Schwarzschild black hole we make a larger one which thus have *lower* temperature. It is worth noting that, including units, this is proportional to $\frac{1}{\hbar}$.

5. First off, at $a = 0$ Kerr-Newman reproduces the RN black hole, which we already know is a solution of the Einstein-Maxwell system.

   Further, it is quick to check using Mathematica that at $Q = 0$ the Kerr metric is itself Ricci-Flat: $R_{\mu\nu} = 0$ so is indeed a solution of the vacuum Einstein equations (away from $r = 0$).

```
In[591]:= (*Kerr*)
xx = {t, r, θ, ϕ};
Δ = r² + a² - 2 G M r;
Σ = r² + a² Cos[θ]²;
g = {{-(Δ - a² Sin[θ]²)/Σ, 0, 0, -(a Sin[θ]² (r² + a² - Δ))/Σ}, {0, Σ/Δ, 0, 0}, {0, 0, Σ, 0}, {-(a Sin[θ]² (r² + a² - Δ))/Σ, 0, 0, ((r² + a²)² - Δ a² Sin[θ]²)/Σ Sin[θ]²}};
ginv = InverseMetric[g];
Riem = RiemannTensor[g, xx];
Ricc = RicciTensor[g, xx];
Ricc // MatrixForm
R = RicciScalar[g, xx]
```

Out[598]//MatrixForm=
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[599]= 0

When $Q \neq 0$ we get a nonzero Ricci tensor (the Ricci scalar still vanishes since classical electrodynamics is conformal).

```
In[605]:= (*Kerr-Newman*)
xx = {t, r, θ, ϕ};
Δ = r² + a² + Q² - 2 G M r;
Σ = r² + a² Cos[θ]²;
g = {{-(Δ - a² Sin[θ]²)/Σ, 0, 0, -(a Sin[θ]² (r² + a² - Δ))/Σ}, {0, Σ/Δ, 0, 0}, {0, 0, Σ, 0}, {-(a Sin[θ]² (r² + a² - Δ))/Σ, 0, 0, ((r² + a²)² - Δ a² Sin[θ]²)/Σ Sin[θ]²}};
ginv = InverseMetric[g];
Riem = RiemannTensor[g, xx];
Ricc = RicciTensor[g, xx];
Ricc // MatrixForm
R = RicciScalar[g, xx]
```

Out[612]//MatrixForm=
$$\begin{pmatrix} \frac{4Q^2 \left(3a^2+2\left(Q^2-2GMr+r^2\right)-a^2\cos[2\theta]\right)}{\left(a^2+2r^2+a^2\cos[2\theta]\right)^3} & 0 & 0 & -\frac{8aQ^2 \left(2a^2+Q^2+2r(-GM+r)\right)\sin[\theta]^2}{\left(a^2+2r^2+a^2\cos[2\theta]\right)^3} \\ 0 & -\frac{2Q^2}{\left(a^2+Q^2+r(-2GM+r)\right)\left(a^2+2r^2+a^2\cos[2\theta]\right)} & 0 & 0 \\ 0 & 0 & \frac{2Q^2}{a^2+2r^2+a^2\cos[2\theta]} & 0 \\ -\frac{8aQ^2 \left(2a^2+Q^2+2r(-GM+r)\right)\sin[\theta]^2}{\left(a^2+2r^2+a^2\cos[2\theta]\right)^3} & 0 & 0 & -\frac{4Q^2 \left(-3a^4-2r^4-a^2\left(Q^2-2GMr+5r^2\right)+a^2\left(a^2+Q^2-2GMr+r^2\right)\cos[2\theta]\right)\sin[\theta]^2}{\left(a^2+2r^2+a^2\cos[2\theta]\right)^3} \end{pmatrix}$$

Out[613]= 0

The Ricci tensor must correspond to an electromagnetic stress-energy tensor. It comes from an electric potential of the form $A_\mu = (\frac{rQ}{\Sigma}, 0, 0, -\frac{arQ\sin^2\theta}{\Sigma})$

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For r very large we get an electric field going as $qr^2/\Sigma^2 \sim q/r^2$ corresponding to the electric field for a charge q, and we also get a magnetic field dying off as $a\cos\theta/r^3$ corresponding to the field from a spinning charged source. Said another way, we see that $\frac{1}{4\pi}\int \star F = q$ and $\frac{1}{4\pi}\int F = 0$ asymptotically, so we have just an electric charge q.

We can verify mass and angular momentum using the killing vectors ∂_t and ∂_ϕ respectively using the formulas in **Wald 12.3.8-9**

$$-\frac{1}{8\pi G}\int \epsilon_{abcd}\nabla^c(\partial_t)^d = M \qquad \frac{1}{16\pi G}\int \epsilon_{abcd}\nabla^c(\partial_\phi)^d = aM$$

For the KN black hole metric, the only singularities can come from $\Sigma = 0$ or $\Delta = 0$. Σ is only zero for $a > 0$ when $r = 0, \theta = \pi/2$. This corresponds to the curvature singularity of the black hole (in fact despite deceptive coordinate choice, this takes the form of a ring $S_1 \times \mathbb{R}$ as is revealed in Kerr-Schild coordinates). The horizons come from g_{rr} becoming singular, namely $\Delta = 0$ which occurs at

$$r^2 - 2GMr + a^2 + Q^2 = 0 \Rightarrow r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}.$$

These give the outer and inner horizons.

The horizon area is given by

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}}\Big|_{r=r_+} = 2\pi \int_0^\pi d\theta \sin\theta \sqrt{(r_+^2 + a^2)^2 - \Delta a^2 \sin^2\theta}$$

But $\Delta = 0$ at the horizon so this trivializes to

$$4\pi(r_+^2 + a^2) = 4\pi((m + \sqrt{m^2 - a^2 - Q^2})^2 + a^2)$$

The entropy of the black hole is then

$$S = \frac{A}{4} = \pi(r_+^2 + a^2)$$

Taking care to write things in terms of J and not a now, by holding J, Q fixed, let's vary M and get

The Hawking temperature is thus

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}$$

Now let's fix S and Q. We get

In[707]:= $\left(\frac{-\text{D}[\text{Pi}\,((\text{M}+\text{Sqrt}[\text{M}^2-\text{J}^2/\text{M}^2-\text{Q}^2])^2+(\text{J}/\text{M})^2),\,\text{J}]}{\text{D}[\text{Pi}\,((\text{M}+\text{Sqrt}[\text{M}^2-\text{J}^2/\text{M}^2-\text{Q}^2])^2+(\text{J}/\text{M})^2),\,\text{M}]} \,\,//\,\text{Simplify} \right) == \frac{\text{J}/\text{M}}{\text{rp}^2+(\text{J}/\text{M})^2} \,\,//\,\text{Simplify}$

Out[707]= True

Which gives us that

$$\Omega = \left(\frac{\partial M}{\partial J}\right)_{Q,S} = -\left(\frac{dS}{dJ}\right)_{Q,M}\left(\frac{dS}{dM}\right)_{Q,J}^{-1} = \frac{a}{r_+^2 + a^2}$$

Finally let's hold S, J fixed and do the same procedure, giving

In[709]:= $\left(\frac{-\text{D}[\text{Pi}\,((\text{M}+\text{Sqrt}[\text{M}^2-\text{J}^2/\text{M}^2-\text{Q}^2])^2+(\text{J}/\text{M})^2),\,\text{Q}]}{\text{D}[\text{Pi}\,((\text{M}+\text{Sqrt}[\text{M}^2-\text{J}^2/\text{M}^2-\text{Q}^2])^2+(\text{J}/\text{M})^2),\,\text{M}]} \,\,//\,\text{Simplify} \right) == \frac{\text{Q}\,\text{rp}}{\text{rp}^2+(\text{J}/\text{M})^2} \,\,//\,\text{Simplify}$

Out[709]= True

$$\mu = \left(\frac{\partial M}{\partial Q}\right)_{J,S} = -\left(\frac{dS}{dQ}\right)_{J,M}\left(\frac{dS}{dM}\right)_{Q,J}^{-1} = \frac{Qr_+}{r_+^2 + a^2}$$

The full form of the first law is then

$$dM = TdS + \Omega dJ + \mu dQ$$

We obtain an extremal black hole when $M = a^2 + Q^2$, as this is the minimum value of M where r_+ is a well-defined radius. At this value, $r_+ = r_-$.

Thermodynamic stability comes from minimizing the Gibbs free energy:

$$G = M - TS - \Omega J - \mu Q$$

Note that for flat space, $G = 0$, so if $G > 0$ for any of these black holes, thermal fluctuations will eventually drive their decay to flat space.

Plugging in what we have gives

In[726]:= $\text{T} = \frac{1}{2\,\text{Pi}} \frac{\sqrt{-\frac{\text{J}^2}{\text{M}^2}+\text{M}^2-\text{Q}^2}}{\text{rp}^2+\left(\frac{\text{J}}{\text{M}}\right)^2}; \,\, \Omega = \frac{\text{J}/\text{M}}{\text{rp}^2+(\text{J}/\text{M})^2}; \,\, \mu = \frac{\text{Q}\,\text{rp}}{\text{rp}^2+(\text{J}/\text{M})^2};$

$\text{S} = \text{Pi}\,(\text{rp}^2+(\text{J}/\text{M})^2);$

$\text{M} - \text{T}\,\text{S} - \mu\,\text{Q} - \Omega\,\text{J} \,\,//\,\text{FullSimplify}$

Out[728]= $\dfrac{\text{J}^2\,(4\,\text{M}^2-2\,\text{Q}^2)+\text{M}\,\text{Q}^4\,\sqrt{-\frac{\text{J}^2}{\text{M}^2}+\text{M}^2-\text{Q}^2}}{2\,\text{M}\,(4\,\text{J}^2+\text{Q}^4)}$

Notice that if $J > 0$ then this will *always* be greater than zero, by virtue of the fact that $M > Q$ always. If we take $J = 0$, we get that this is still thermodynamically unstable unless $Q = M$ and the black hole is extremally charged.

6. The Hawking evaporation rate gets modified as

$$\Gamma_H = \frac{\sigma_{abs}(\omega)}{\exp(\beta(\hbar\omega - \vec{s}\cdot\vec{\Omega} - q\Phi)) \mp 1} \frac{d^3k}{(2\pi)^3}$$

where $\vec{s}\cdot\vec{\Omega}$ is the angular momentum product (orientation of $\vec{s}$ relative to $\vec{\Omega}$ matters). The $\mp$ is for bosons and fermions respectively.

Return to understand how this generalizes to systems more broadly

7. This is direct - we take M theory on T^6. Wrap Q_1 M2 branes along $x^9 - x^{10}$, Q_2 M2 branes along $x^7 - x^8$, and Q_3 M2 branes along $x^5 - x^6$.

   Take the M-theory S^1 to be along x^{10}. Now, taking this to be microscopic first, we get Q^1 strings wrapping x^9 together with Q_2, Q_3 D2 branes in IIA.

   T-dualize along x^5, x^6 to get Q_3 D0 branes, Q_2 D4 branes wrapping x^{5-8}, and Q^2 D2 branes wrapping x^5, x^6. The F1 around x^9 is unaffected.

   Finally, T-dualize along x^9, taking IIA to IIB and giving Q_3 D1 branes and Q_2 D5 branes, while replacing the F1 (ie B-flux) with KK momentum of the system along the x^9. This is exactly the D1-D5 system.

8. The D5 and D5 branes are both BPS. We know that, upon toroidal compactification,

   In $D = 10$ have the D1 stretch $x_0 = t, x_5 = \gamma$ and the D5 stretch $x_0, \ldots, x_5$, where we write $\gamma^a, a = 1 \ldots 4$ to be the new D5 directions. These will form the direction of the T^4.

   Upon compactifying on $T^4 \times S^1$, the logic we used to for the 10D solution will still carry over to 5D. We will still write the extremal metric in terms of functions $H_{1,5}$ that must be harmonic w.r.t. the flat metric of the 4D transverse space.

   Then the D1 brane solution gives

   $$ds_{D1}^2 = \frac{-dt^2 + d\gamma^2}{\sqrt{H_1}} + \sqrt{H_1}d\gamma^a \cdot d\gamma^a + \sqrt{H_1}\, dx^i \cdot dx^i, \quad H_1 = 1 + \frac{r_1^6}{r^6}, \quad e^{-2\Phi} = H_1^{-1}, \quad F_{05i} = \partial_i(H_1^{-1})$$

   While the D5 brane gives

   $$ds_{D5}^2 = \frac{-dt^2 + d\gamma^2}{\sqrt{H_5}} + \frac{d\gamma^a \cdot d\gamma^a}{\sqrt{H_5}} + \sqrt{H_5}\, dx^i \cdot dx^i, \quad H_5 = 1 + \frac{r_5^2}{r^2}, \quad e^{-2\Phi} = H_5, \quad F_{ijk} = -\epsilon_{ijk}\partial_r H$$

   I think this problem has a typo and Kiritsis means D_1, D_5 not N_1, N_5. Further, Kiritsis (likely borrowing from Maldacena's thesis) writes $F_{05i} = -\frac{1}{2}\partial_i(H^{-1} - 1)$. This factor of $1/2$ is different from what I'm used to seeing both in Kirtsis and Blumenhagen. This stems from a different choice of normalization for the Kalb-Ramond and RR forms in Maldacena's thesis. I am unsure why this different normalization exists, but at any rate I will *ignore* the factor of $1/2$. Finally, the overall sign in F disagrees also with Kiritsis and Blumenhagen, and I think the total D-brane charge in Maldacena counts anti-D-branes in our scheme.

   When superimposing a D1 and D5 solution, the dilaton and field strength contributions add while the metric contributions get multiplied. One way to see this is, because the solution remains BPS, we only need to solve the first-order BPS equations.

   For a p-Brane, as we have seen, the Killing spinors have spatial profile $\epsilon(r) = H^{1/8}\epsilon_0$ regardless of p. The linear equations for spinors coincide with the D-brane equations $\epsilon_L = \pm\Gamma^0 \ldots \Gamma^p\epsilon_R$. We know that for the 1-5 system these can be simultaneously solved, giving a $1/4$ BPS state.

   I could do this in more detail... but I've computed enough Killing spinors by this point.

   The combined 10D solution thus gives:

   $$\frac{-dt^2 + d\gamma^2}{\sqrt{H_1 H_5}} + \sqrt{\frac{H_1}{H_5}}d\gamma^a \cdot d\gamma^a + \sqrt{H_1 H_5}dx^i \cdot dx^i, \quad F_{r05} = \partial_r H^{-1}, \quad F_{ijk} = -H'(r), \quad , e^{-2\Phi} = \frac{H_5}{H_1}$$

   The next step is compactification. Upon wrapping D5 and D1 around a T^5, dimensional reduction freezes out γ, γ^a dependence of the metric and fields. The T^5 is parallel to the D5, so the D5 solution will look identical to how it looked before. The D1 also wraps a cycle of the T^5. Compactifying the other 4 directions will look like a periodic arrangement of $D1$ branes, which effectively serves to remove γ^a dependence from the $D1$ contribution to the solution. **Think about this. Is it really true that the metric warps the same regardless of where on T^5 I am? More likely that they are taking T^5 small and neglecting it, or we're thinking about a uniform distribution of D1s on T^5.**

   Finally, the D1 solution can be given momentum.

9. Ignoring the $-1/2$ discussed before, we can verify the charge from direct integration along a S^3. First, the electric charge

$$\frac{1}{2\kappa_{10}^2} \int_{S^3 \times T^4} \star F = \frac{1}{2\kappa_{10}^2} \int_{S^3 \times T^4} \frac{2r_1^2}{r^3} = -\frac{4\pi^2(2\pi\ell_s)^4 V}{(2\pi)^7\ell_s^8 g_s^2} r_1^2 = \frac{Q_1}{2\pi\ell_s^2 g} = Q_1 T_1$$

for $r_1^2 = \ell_s^2 g_s/V$, as required.

For the magnetic charge, $F_{\theta\phi\psi} = \epsilon_{\theta\phi\psi r}\partial_r H_5 = -H_5'$ so we get

$$\frac{1}{2\kappa_{10}^2} \int_{S^3} F = \frac{1}{2\kappa_{10}^2} \int_{S^3} d\Omega_3 \frac{2r_5^2}{r^3} = \frac{4\pi^2}{(2\pi)^7\ell_s^8 g_s^2} r_5^2 = \frac{Q_5}{(2\pi)^5\ell_s^6 g_s} = Q_5 T_5$$

for $r_5^2 = \ell_s^2 g_s$, as required.

We can also derive c_p from the KK solution **Do this**.

In the non-extremal case, this generalizes quite directly.

$$\frac{Q_1}{2\pi\ell_s^2 g_s} = \frac{1}{2\kappa_{10}^2} \int_{S^3 \times T^4} \star F = \frac{(2\pi\ell_s)^4 V g_s}{(2\pi)^7\ell_s^8 g_s^2} \int_{S^3} \coth a_1 \frac{2r_1^2}{r^3} = \frac{V r_0^2 \sinh^2 a_1 \coth a_1}{2\pi g_s^2 \ell_s^4} = \frac{r_0^2 \sinh 2a_1}{4\pi g_s \ell_s^2 c_1} \Rightarrow Q_1 = \frac{r_0^2 \sinh 2a_1}{2c_1}$$

Similarly:

$$\frac{Q_5}{(2\pi)^5\ell_s^6 g_s} = \frac{1}{2\kappa_{10}^2} \int_{S^3} \star F = \frac{1}{(2\pi)^7\ell_s^8 g_s^2} \int_{S^3} \coth a_5 \frac{2r_5^2}{r^3} = \frac{r_0^2 \sinh^2 a_5 \coth a_5}{(2\pi)^5 g_s^2 \ell_s^8} = \frac{r_0^2 \sinh 2a_5}{2(2\pi)^5\ell_s^6 g_s c_5} \Rightarrow Q_5 = \frac{r_0^2 \sinh 2a_5}{2c_5}$$

For the KK momentum, I assume it can be read off from the $dtd\gamma$ term **justify** (Maldacena writes this too, in his thesis below 2.34), which goes as $r_0^2 \sinh a_p \cosh a_p = \frac{r_0^2 \sinh 2a_p}{2} = c_p Q_p$, giving KK momentum

$$Q_p = \frac{r_0^2 \sinh 2a_p}{2c_p}$$

10. First, by analogy to 5D we expect an extremal metric of the form

$$-\lambda^{-1/2}dt^2 + \lambda^{1/2}(dr^2 + r^2 d\Omega_2^2), \quad \lambda = \prod_{i=1}^4 \left(1 + \frac{r_i}{r}\right)$$

This will have a nonzero area $4\pi\sqrt{r_1 r_2 r_3 r_4}$ only when all the $r_i \neq 0$. On the other hand the total mass is $M = \sum_{i=1}^4 M_i$ with $M_i = r_i/4G$. The question is what the charges correspond to at the level of a brane construction.

Towards this end, let's take IIA and compactify on T^6. We consider a D6 brane wrapping $x^1 \ldots x^6$ together with a D2 wrapping x^1, x^6. $6 - 2 = 4$ is good, makes the state 1/4 BPS. We can also add KK momentum along the 1 direction.

The crucial principle (Maldacena 2.5) is that if a scalar diverges at the horizon, the d-dimensional character of the solution is lost. For a single p-brane $p \neq 3$ the dilaton goes either to ∞ or 0. In the case of the $D1$-$D5$ system, we needed branes symmetric about $p = 3$ and differing by 4 in order to give a BPS state with the dilaton tending to a constant $\frac{1}{4}\log H_1/H_5 \rightarrow \frac{1}{2}\log r_1/r_5$.

We see now that this does not work with a D6 and D2. For a p-brane $e^{-2\Phi} = H^{(p-3)/2}$ giving that D6-D2 gas a dilaton going as $e^{-2\Phi} = H_6^{3/2}H_2^{-1/2}$. There's no (even dimensional) D-brane we could add in type 1 that would save us, and adding fundamental strings would only give $e^{-2\phi} = H_f$, which would not help.

But there is another extended object with the correct dilaton dependence as $e^{-2\Phi} = H^{-1}$. This is the NS5 brane! But will adding it break supersymmetry completely? On the contrary, the SUSY constraints from the D6 and D2 and KK momentum are:

$$\epsilon_L = \Gamma^{0123456}\epsilon_R, \quad \epsilon_L = \Gamma^{016}\epsilon_R, \quad \epsilon_L = \Gamma^{01}\epsilon_L, \quad \epsilon_R = -\Gamma^{01}\epsilon_R$$

The NS5 brane wrapping $1 \ldots 5$ would give $\epsilon_L = \Gamma^{012345}\epsilon_L, \epsilon_R = -\Gamma^{012345}\epsilon_R$. This can be rewritten as

$$\epsilon_{L,R} = \pm\Gamma^6\Gamma^{0123456}\epsilon_{R,L} = \Gamma^6\epsilon_{L,R}$$

But $\epsilon_L = \pm\Gamma^6\epsilon_L$ already follows from the prior supersymmetry constraints, so adding NS5 breaks nothing!

7

11. Directly applying the formula derived in problem **1** with $F = f^{-1/3}, C = f^{-1/3}h$ gives

$$2\pi \frac{2}{\sqrt{F(r_0)}C'(r_0)} = 2\pi r_0 \cosh(a_1)\cosh(a_5)\cosh(a_p)$$

12. This is direct by writing the differentials in terms of variables r_0, a_1, a_5, a_p:

```
In[279]:= Q1 = (r0^2 Sinh[2 a1])/(2 c1); Q5 = (r0^2 Sinh[2 a5])/(2 c5); Qp = r0^2 (Sinh[2 ap])/(2 cp);

S = 2 Pi (r0^3)/(Sqrt[c1 c5 cp]) Cosh[a1] Cosh[a5] Cosh[ap];

M = (r0^2)/(2 Sqrt[c1 c5 cp]) (Cosh[2 a1] + Cosh[2 a5] + Cosh[2 ap]);

T = 1/(2 Pi r0 Cosh[a1] Cosh[a5] Cosh[ap]);

μ1 = c1/(Sqrt[c1 c5 cp]) Tanh[a1]; μ5 = c5/(Sqrt[c1 c5 cp]) Tanh[a5]; μp = cp/(Sqrt[c1 c5 cp]) Tanh[ap];

dQ1 = D[Q1, a1] da1 + D[Q1, r0] dr0 // Simplify;
dQ5 = D[Q5, a5] da5 + D[Q5, r0] dr0 // Simplify;
dQp = D[Qp, ap] dap + D[Qp, r0] dr0 // Simplify;
dS = D[S, r0] dr0 + D[S, a1] da1 + D[S, a5] da5 + D[S, ap] dap // Simplify;
dM = D[M, r0] dr0 + D[M, a1] da1 + D[M, a5] da5 + D[M, ap] dap // Simplify;

dM - (T dS + μ1 dQ1 + μ5 dQ5 + μp dQp) // Simplify

Out[289]= 0
```

The variations do not involve arbitrary changes in the $N_{\pm i}$. This is not obvious from the form of the first law **as far as I can tell**, but $N_{\pm i}$ do need to be discrete in the brane interpretation.

13. I have done this problem for Andy's class on quantum black holes. I will copy the full answer below: **BTZ as a Quotient of AdS$_3$** The objective of this problem is to describe the precise way in which the BTZ black hole arises as a quotient of AdS_3. Take the embedding space to be $\mathbb{R}^{2,2}$, with metric:

$$\eta = \text{diag}(-1, -1, 1, 1)$$

Denote the coordinates of the embedding space by $x^\mu = (x^0, x^1, x^2, x^3)$. AdS_3 is given by $x_\mu x^\mu = -\ell^2$. The Killing vectors generating isometries are given by $J_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu$. The most general Killing vector is then $\omega^{\mu\nu} J_{\mu\nu}$.

Define the identification subgroup of AdS_3 by

$$P \sim e^{t\xi} P, t \in 2\pi n, \quad n \in \mathbb{Z}$$

For this identification to make physical sense, it should not give rise to closed timelike curves. Unfortunately, in some regions, the ξ used in this construction do give rise to CTCs. Luckily, however, they are bounded by a region where $\xi \cdot \xi = 0$. The part of the spacetime where $\xi \cdot \xi = 0$ is then interpreted as a singularity in the causal structure, and the region where $\xi \cdot \xi < 0$ is cut out of the spacetime. Let

$$\xi = \frac{r_+}{l} J_{12} - \frac{r_-}{l} J_{03}$$

(a) **Write down $\omega_{\mu\nu}$ for this Killing vector, and the corresponding Casimir invariants for the SO(2, 2) isometry group, which are given by**

$$I_1 = \omega_{\mu\nu}\omega^{\mu\nu}, \quad I_2 = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\omega_{\mu\nu}\omega_{\rho\sigma}$$

We have

$$\omega_{12} = -\omega_{21} = \frac{r_+}{l}, \quad \omega_{03} = -\omega_{30} = -\frac{r_-}{l}$$

Giving Casimirs:

$$I_1 = -2\frac{r_+^2 + r_-^2}{l^2}, \quad I_2 = 4\frac{r_+ r_-}{l^2}$$

(b) **Find the allowed region $\xi \cdot \xi > 0$ in terms of x^1, x^2.**

We can write:

$$\xi = \frac{r_+}{l}(x_2\partial_1 - x_1\partial_2) - \frac{r_-}{l}(x_3\partial_0 - x_0\partial_3) = \begin{pmatrix} -\frac{r_-}{l}x_3 \\ \frac{r_+}{l}x_2 \\ -\frac{r_+}{l}x_1 \\ \frac{r_-}{l}x_0 \end{pmatrix} = \begin{pmatrix} -\frac{r_-}{l}x^3 \\ \frac{r_+}{l}x^2 \\ \frac{r_+}{l}x^1 \\ -\frac{r_-}{l}x^0 \end{pmatrix}$$

This vector has norm:

$$\xi^2 = \frac{1}{l^2}[r_+^2(x_1^2 - x_2^2) - r_-^2(x_3^2 - x_0^2)] = \frac{1}{\ell^2}(r_+^2 - r_-^2)(x_1^2 - x_2^2) + r_-^2$$

This is $\geqslant 0$ when (assuming $r_+ > r_-$)

$$x_1^2 - x_2^2 \geqslant -\frac{r_-^2 l^2}{r_+^2 - r_-^2}$$

(c) **Find the regions for $\xi \cdot \xi \in \{(0, r_-^2), (r_-^2, r_+^2), (r_+^2, \infty)\}$ and identify whether the boundaries between them are timelike, spacelikeor null:**

Now let's look at

$$r_+^2 \leqslant \xi^2 \Rightarrow (r_+^2 - r_-^2)(x_1^2 - x_2^2) \geqslant (r_-^2 + r_+^2)l^2 \Rightarrow l^2 \leqslant x_1^2 - x_2^2$$

Next

$$r_-^2 \leqslant \xi^2 \leqslant r_+^2 \Rightarrow 0 \leqslant x_1^2 - x_2^2 \leqslant l^2$$

Finally

$$0 \leqslant \xi^2 \leqslant r_-^2 \Rightarrow -\frac{r_-^2 l^2}{r_+^2 - r_-^2} \leqslant x_1^2 - x_2^2 \leqslant 0$$

All boundaries between these regions are null. The boundary $x_1^2 - x_2^2 = l^2$ implies $x_0^2 - x_3^2 = 0$ and so is a null surface (cone). Similarly, the boundary $x_1^2 - x_2^2 = 0$ implies $x_+ = \pm x_-$ which is again null surface.

(d) **For region I use the coordinate transform given by $x^0 = \sqrt{B(r)}\sinh \tilde{t}, x^1 = \sqrt{A(r)}\cosh \tilde{\phi}, x^2 = \sqrt{A(r)}\sinh \tilde{\phi}, x^3 = \sqrt{A(r)}\cosh \tilde{t}$ where $A, B = l^2\frac{r^2 - r_\mp^2}{r_+^2 - r_-^2}$ and $\tilde{t}, \tilde{\phi} = \frac{1}{l}(\pm\frac{tr_\pm}{l} \mp r_\mp\phi)$ to write the metric in a form**

$$-N_\perp^2 dt^2 + N_\perp^{-2}dr^2 + r^2(N_\phi dt + d\phi)^2$$

Note we get $x_1^2 - x_2^2 = A(r)$, $x_3^2 - x_4^2 = B(r)$, and $B(r) = A(r) + l^2$ so that $B - A = l^2$ as required in AdS. At $r = r_+$, $A = -\frac{(r_+^2 - r_-^2)l^2}{r_+^2 - r_-^2}$ exactly saturating the boundary of region 1. Simple differential manipulations

In[1698]:= A = $\frac{\text{l}^2 (\text{r}^2 - \text{rm}^2)}{\text{rp}^2 - \text{rm}^2}$; B = $\frac{\text{l}^2 (\text{r}^2 - \text{rp}^2)}{\text{rp}^2 - \text{rm}^2}$;

x0 = Sqrt[B] Sinh[t];   x3 = Sqrt[B] Cosh[t];   x1 = Sqrt[A] Cosh[φ];   x2 = Sqrt[A] Sinh[φ];
dx0 = D[x0, r] dr + D[x0, t] dt; dx3 = D[x3, r] dr + D[x3, t] dt; dx1 = D[x1, r] dr + D[x1, φ] dφ; dx2 = D[x2, r] dr + D[x2, φ] dφ;
ds = -dx0² + dx3² - dx1² + dx2² // FullSimplify;

ds2 = ds /. dφ → - $\frac{\text{rm}}{\text{l}^2}$ dt2 + $\frac{\text{rp}}{\text{l}}$ dφ2 /. dt → $\frac{\text{rp}}{\text{l}^2}$ dt2 - $\frac{\text{rm}}{\text{l}}$ dφ2 // FullSimplify

Nperp = Sqrt$\left[\frac{1}{\text{Coefficient}[\text{ds2, dr}^2]}\right]$

Nφ = Sqrt$\left[\frac{1}{\text{r}^2} (\text{Coefficient}[\text{ds2, dt2}^2] + \text{Nperp}^2)\right]$ // Simplify

Out[1702]= dφ2² r² - $\frac{2 \text{ dt2 dφ2 rm rp}}{\text{l}}$ + $\frac{\text{dr}^2 \text{ l}^2 \text{ r}^2}{(\text{r}^2 - \text{rm}^2)(\text{r}^2 - \text{rp}^2)}$ + $\frac{\text{dt2}^2 (-\text{r}^2 + \text{rm}^2 + \text{rp}^2)}{\text{l}^2}$

Out[1703]= $\sqrt{\frac{(\text{r}^2 - \text{rm}^2)(\text{r}^2 - \text{rp}^2)}{\text{l}^2 \text{ r}^2}}$

Out[1704]= $\sqrt{\frac{\text{rm}^2 \text{ rp}^2}{\text{l}^2 \text{ r}^4}}$

gives us

$$N_\perp = \frac{\sqrt{(r^2 - r_-^2)(r^2 - r_+^2)}}{lr}, \quad N_\phi = \frac{r_+ r_-}{lr^2}$$

For region II, taking $r_- < r < r_+$ makes B negative, so we will keep x_1, x_2 as before and instead define

$$x^0 = -(-B(r))^{1/2} \cosh \tilde{t}, \quad x^3 = -(-B(r))^{1/2} \sinh \tilde{t}$$

Here we have flipped the sign of B together with exchanging sinh and cosh (so as to remain in coordinates satisfying the AdS constraint). **This is exactly as in Kiritsis 13.7.8, referred to as the "standard BTZ form"**

We keep $\tilde{t}, \tilde{\phi}$ the same. This gives the same value for $N_\perp$ and N_ϕ. Lastly for region III, A also becomes negative, and we redefine x^1, x^2 while keeping x^0, x^3 from region II:

$$x^1 = (-A(r))^{1/2} \sinh \tilde{\phi}, \quad x^3 = (-A(r))^{1/2} \cosh \tilde{\phi}$$

The here r ranges from 0 to ∞ while t, ϕ are unrestricted and range from $-\infty$ to ∞

(e) **Compute the Killing vector ξ in the (t, r, ϕ) coordinates and perform the identification. You should recognize the metric found in the previous part as the BTZ geometry. Identify M, J in terms of $r_\pm$ and write the casimir invariants from part a) in terms of M, J**

By computing the Jacobian $\frac{\partial(x^0, x^1, x^2, x^3)}{\partial(t, r, \phi)}$ and judiciously guessing what vector I should push forward (way easier than trying to compute inverse Jacobians to pull back ξ), I see that $\xi = \partial_\phi$ in our new basis:

```
In[1679]:= ξ = {- rm/l x3, rp/l x2, rp/l x1, - rm/l x0} // Simplify;

J = {{D[x0, t], D[x0, r], D[x0, ϕ]},
     {D[x1, t], D[x1, r], D[x1, ϕ]},
     {D[x2, t], D[x2, r], D[x2, ϕ]},
     {D[x3, t], D[x3, r], D[x3, ϕ]}};

J2 = {{rp/l², 0, -rm/l}, {0, 1, 0}, {-rm/l², 0, rp/l}};

ξϕ = {0, 0, 1};
J.J2.ξϕ == ξ // Reduce

Out[1683]= True
```

Indeed, it is easy to see that ∂_ϕ is killing from directly applying the Killing equation, and now we see it comes directly from a combination of the manifest $J_{\mu\nu}$ symmetries in the embedding space. We can thus identify ϕ as a periodic variable and retain the space as a solution to Einstein's equations. Looking at the $N_\perp^2$ and N_ϕ^2 contributions to g_{00} we get:

$$-g_{00} = -\frac{r_-^2 + r_+^2}{l^2} + \frac{r^2}{l^2}$$

This is a black hole in AdS with mass $M = \frac{r_-^2 + r_+^2}{l^2}$. Similarly, from the $dt\, d\phi$ component, we see that $2r^2 N_\phi$ corresponds exactly to the angular momentum. We thus get

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad J = \frac{2r_+ r_-}{l}$$

Note for global AdS when $r_- \to 0, r_+ \to -l^2$, we get mass $-l^2$. **Kiritsis adjusts the definition of mass by $+1$ making it 0 in global AdS, so that it counts only the mass of the black hole.** This gives

$$I_1 = -2M, \quad I_2 = 2J/l$$

14. The KK reduction is not too bad. Following appendix F directly gives

$$\int d^5x \sqrt{\det g}\left(e^{-2\phi}\left[R + 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta} - \frac{1}{4}G_{\alpha\beta}F^\alpha_{\mu\nu}F^{\mu\nu\,\beta}\right]\right.$$
$$\left. - \frac{1}{4}G^{\alpha\beta}G^{\gamma\delta}H_{\mu\alpha\gamma}H^\mu_{\gamma\delta} - \frac{1}{4}G^{\alpha\beta}H_{\mu\nu\alpha}H^{\mu\nu}_\beta - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right)$$

with $\phi = \Phi - \frac{1}{4}\log\det G_{\alpha\beta}$. Here H comes from the *two form* not from the B field. We can rewrite this as

$$\int d^5x \sqrt{\det g}\left(e^{-2\phi}\left[R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}G^{\alpha\beta}G^{\gamma\delta}\left(\partial G_{\alpha\gamma}\partial G_{\beta\delta} - \partial C_{\alpha\gamma}\partial C_{\beta\delta}\right) - \frac{1}{4}G_{\alpha\beta}F^\alpha_{\mu\nu}F^{\mu\nu\,\beta}\right]\right.$$
$$\left. - \frac{1}{4}\sqrt{G}G^{\alpha\beta}H_{\mu\nu\alpha}H^{\mu\nu}_\beta - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right)$$

Now we must take this to the Einstein frame. We perform a Weyl rescaling $g \to e^{4\phi/3}g$. This rescales fields (and changes the kinetic ϕ term) to give us the requisite action

$$S_5 = \frac{1}{2\kappa_5^2}\int\sqrt{-g}\left[R - \frac{4}{3}(\partial\phi)^2 - \frac{1}{4}G^{\alpha\beta}G^{\gamma\delta}(\partial G_{\alpha\gamma}\partial G_{\beta\delta} + e^{2\phi}\partial C_{\alpha\gamma}\partial C_{\beta\delta})\right.$$
$$\left. - \frac{e^{-4\phi/3}}{4}G_{\alpha\beta}F^\alpha_{\mu\nu}F^{\mu\nu\,\beta} - \frac{e^{2\phi/3}}{4}\sqrt{G}G^{\alpha\beta}H_{\mu\nu\alpha}H^{\mu\nu}_\beta - \frac{e^{4\phi/3}}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right]$$

equations of motion

show solution

15. Write $G_{\alpha\beta} = \sqrt{\frac{H_1}{H_2}}(\delta_{\alpha\beta} + h_{\alpha\beta})$

16. This problem definitely has some errors in the typesetting of equation 13.7E.

By redefining $\tilde\phi = \phi - \frac{1}{2}\nu_5, \lambda = -\frac{1}{2}\phi + \frac{3}{4}\nu_5$, we get

$$-(\partial\tilde\phi)^2 - \frac{4}{3}(\partial\lambda)^2 = -\frac{4}{3}(\partial\phi)^2 - (\partial\nu_5)^2$$

The ν_5 kinetic term exactly reproduces the expected $-\frac{1}{4}\partial_\mu G_{\alpha\beta}\partial^\mu G^{\alpha\beta}$

17. This is direct:

```
In[483]:= R1 = Sqrt[(Pi ω^3 / 2) (A/2 + B/π (-2/(ω^2 r^2) Log[ω r]))];

R2 = A2 (1 - r0^2/r^2)^(-I (a+b)/2) Hypergeometric2F1[-I a, -I b, 1 - I a - I b, (1 - r0^2/r^2)];

cond1 = Assuming[r0 > 0, Series[R2, {r0, 0, 1}]]
cond2 = Assuming[r0 > 0, Series[D[R2, r], {r0, 0, 1}]]

Solve[Sqrt[(Pi ω^3 / 2) (A/2 + B/π (-2/(ω^2 r^2) Log[ω r]))] == cond1 && -2 B/(π r^3 ω^2) + 4 B Log[r ω]/(π r^3 ω^2) == cond2, {A, B}]

Out[485]= A2 Gamma[1 - i a - i b]/(Gamma[1 - i a] Gamma[1 - i b]) + O[r0]^2

Out[486]= O[r0]^2

Out[487]= {{A → 2 A2 Sqrt[2/π] Gamma[1 - i a - i b]/(Sqrt[ω^3] Gamma[1 - i a] Gamma[1 - i b]), B → 0}}
```

Importantly, the derivative of the inner R profile goes as $O(r_0^2)$, which makes it subleading in determining B. Thus, B is $O(r_0^2)$. On the other hand, this derivative's r_0^2 dependence is important for determining the incoming flus $\Im(h r^3 R^*\partial_r R)$. For completeness I will add this part of the notebook as well:

In[531]:= `Assuming[r0 > 0, cond1 Series[r³ D[R2, r], {r0, 0, 2}] /. r → r0 // FullSimplify]`

Out[531]= $\dfrac{\text{A2}^2 \, \text{Gamma}[-i\,(i+a+b)]^2 \,(-i\,(a+b)+2\,a\,b\,(\text{HarmonicNumber}[-i\,a]+\text{HarmonicNumber}[-i\,b]))\,\text{r0}^2}{\text{Gamma}[1-i\,a]^2\,\text{Gamma}[1-i\,b]^2} + O[\text{r0}]^3$

18. Redefining K to be unitless, e have a relationship

$$e^{ix\cos\theta} = K\frac{e^{-i\omega x}}{x^{3/2}}Y_{000} + \text{higher moments}$$

Let's integrate both over S^3, giving

$$\int_0^\pi d\theta \int_0^\pi d\phi \int_0^{2\pi} d\psi \sin^2\theta \sin\phi\, e^{ix\cos\theta} = 2\pi^2 \frac{J_1(x)}{x} = K\frac{e^{i\omega x}}{x^{3/2}}\sqrt{2\pi^2}$$

The asymptotic form of J_1 is

$$J_1(x) \sim \sqrt{\frac{2}{\pi x}}\cos(z - 3\pi/4)$$

So then, up to a phase

$$2\pi\sqrt{2\pi} = K\sqrt{2\pi^2} \Rightarrow K = \sqrt{4\pi}$$

as required.

19. Even in higher spacetime dimension $d+1$, nothing stops me from writing the metric as

$$ds^2 = -f^{-1+1/(d-1)}h\,dt^2 + f^{1/(d-1)}\left[\frac{dr^2}{h} + r^2 d\Omega_{d-1}^2\right] \Rightarrow \sqrt{-g} = f^{1/(d-1)}r^3, \quad h = 1 - \frac{r_0^{d-1}}{r^{d-1}}$$

I will ignore the details of f for now, since it depends a lot on the dimension and charges. h is simply what reproduces the Schwarzschild solution for a totally uncharged black hole. What does matter is that the horizon is at $r = r_0$, giving a horizon area $A = \Omega_{d-1}r_0^{(d-1)}f(r_0)^{1/2}$, meaning that to leading order as $r \to r_0$ we have

$$f(r) \approx \frac{R_H^{2(d-1)}}{r_0^{2(d-1)}}, \quad R_H^{d-1} = \frac{A}{\Omega_{d-1}}$$

This is all we need for the near-horizon data.

What matters is that the throat size that is determined by f is much larger than the extremality parameter r_0. For a minimal scalar $\Box\phi = 0$ in $d+1$ dimensions we get

$$\left[\frac{h}{r^{d-1}}\partial_r\left(hr^{d-1}\partial_r\right) + \omega^2 f\right]R_\omega(r) = 0$$

Redefining $\psi = r^{\frac{d-1}{2}}R$ and introducing the tortoise coordinate $dr_* = dr/h$ so that $\partial_{r_*} = h\partial_r$ we get

$$\left[-\frac{d^2}{dr_*^2} + \underbrace{\frac{(d-1)(d-3)}{4r^2}\left(1 - \frac{r_0^{d-2}}{r^{d-2}}\right)\left(1 + \frac{d-1}{d-3}\frac{r_0^{d-2}}{r^{d-2}}\right) - \omega^2 f(r)}_{V(r_*)}\right]\psi(r) = 0$$

This reproduces **13.8.4** when $d = 4$. We don't expect this to be solvable, and so we will work at it by matching. Again, $r_0, r_p \ll r_m \ll r_1, r_5$.

For r large, $r = r_*$ and we can divide through by ω giving $\rho = r\omega$. The equation then reduces to

$$\left(\frac{d^2}{d\rho^2} + 1 - \frac{(d-1)(d-3)}{4\rho^2}\right)\psi$$

This has a solution in terms of Bessel functions:

$$\psi = \sqrt{\frac{\pi\rho}{2}}[AJ_{-1+d/2}(\rho)+BY_{-1+d/2}(\rho)] \Rightarrow R \approx \frac{1}{r^{(d-1)/2}}[e^{i\omega r}(Ae^{i(1-d)\frac{\pi}{4}}-Be^{i(3-d)\frac{\pi}{4}})+e^{-i\omega r}(Ae^{-i(1-d)\frac{\pi}{4}}-Be^{-i(3-d)\frac{\pi}{4}})]$$

Taking $\rho = r\omega \ll 1$ at $r = r_m$ for matching, the Bessel functions will instead become at leading order:

$$R = \frac{1}{r^{(d-1)/2}}\sqrt{\frac{\pi\omega r}{2}}\left(A\frac{2^{1-d/2}}{\Gamma(d/2)}(r\omega)^{-1+d/2} + B\frac{2^{(d-2)/2}\omega^{2-d}}{\Gamma(2-d/2)r^{(d-1)/2}}\right)$$

For r near the horizon, at *zeroth order* keeping only the incoming (positive frequency wave), I see that the equations become:

$$\left[\frac{d^2}{dr_*^2} + \frac{\omega^2 R_H^{2(d-1)}}{r_0^{2(d-1)}}\right]\psi(r) \Rightarrow \psi = \frac{1}{r^{(d-1)/2}}e^{-i\omega r_*\frac{R_H^{d-1}}{r_0^{d-1}}} \approx \frac{1}{r^{(d-1)/2}}(1 - i\omega R_H^{d-1}/(d-2)r^{(d-2)})$$

It is not obvious that zeroth order is good enough to do the matching between the near-field and far-field at the intermediate $r = r_m$. One can, however argue, that any corrections of this zeroth order equation for ψ will receive corrections that are subleading in r_0, which will translate over to the matching conditions. We will therefore continue with our zeroth order solution.

Finish

20.

21. For $N = 0$ $(T^4)^N/S_N$ is just a point which has trivial cohomology ring with dimension 0. For $N = 1$ we recover T^4 which has 4×4 cocycles generated as an alternating algebra by the elements $dx^i, i = 1\ldots4$, giving dimension 16. For $N = 2$ we get T^8/S_2 identifying points of two separate T_4s. Each individual T_4 has all of its cycles remaining intact, giving $2 \times 2^4 = 32$ cycles untouched. The remaining $2^8 - 2^4$ cycles are half-killed, giving $2 \times 2^4 + (2^8 - 2 \times 2^4)/2 = 144$. Finally, let's do $N = 3$, the first case where S_N becomes nonabelian.

Finish

22. A derivation of Cardy's Formula:

Take the CFT to have continuous spectrum, which we can write in terms of a δ-function based $\rho(\Delta)$ as

$$Z(\tau) = \int_0^\infty d\Delta\rho(\Delta)e^{2\pi i\tau\Delta}$$

We can invert this using a Bromwich integral:

$$\rho(\Delta) = \int_C d\tau Z(\tau)e^{-2\pi i\tau\Delta}$$

where C is the contour running parallel but slightly above the real axis, enclosing the upper half plane. In the q-disk this would run close to the boundary of the disk.

Now, we would like an expression for $Z(\tau)$ as $\Im\tau \to 0$, namely the high-temperature limit. We know that $Z(\tau \to \infty) = \dim\mathcal{H}_0$, the space of ground states, which we take to consist of only a unique $|0\rangle$, so we take this to be 1. Further, we know that

$$q^{-c/24}Z(\tau) = e^{-2\pi i\tau c/24}Z(\tau) = Z(-1/\tau)e^{\frac{2\pi ic}{24\tau}}$$

is modular invariant. This implies that

$$Z(\tau \to 0) \approx Z(\infty)e^{2\pi i\frac{c}{24\tau}} = e^{2\pi i\frac{c}{24\tau}}$$

We now can approximate the integral:

$$\rho(\Delta) \approx \int_{-\infty}^\infty d\tau e^{-2\pi i(\tau\Delta - \frac{c}{24\tau})}$$

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This gives a stationary value at

$$\Delta + \frac{c}{24\tau^2} = 0 \Rightarrow \tau = i\sqrt{\frac{c}{24\Delta}}$$

Plugging this back and interpreting ρ as just an expected number of states at a given level $\Omega(N)$ gives

$$\Omega(N) = \rho(\Delta) \approx e^{2\pi\sqrt{\frac{c\Delta}{6}}} \Rightarrow S = \log\Omega(N) \approx 2\pi\sqrt{\frac{c}{6}}\sqrt{N}$$

I did this just for the left-movers, but taking left and right movers together gives the desired result:

$$S = \log\Omega(N_L, N_R) \approx 2\pi\sqrt{\frac{c}{6}}(\sqrt{N_L} + \sqrt{N_R})$$

23. As we've seen before, a single free boson the partition function is $\text{Tr}(e^{-\beta L_0}) = \eta^{-1}$ while for a two free fermions, bosonization give the identical result. For a single free fermion then, it is $\eta^{-1/2}$. For n_f copies of this system we simply exponentiate to obtain

$$\left(\prod_{n=1}^{\infty} \frac{1}{1 - e^{-\beta n/R}}\right)^{\frac{3}{2}n_f}$$

This can be directly written using $q = e^{-\beta/R}$ ie $\tau = i\beta/2\pi R$

$$\left(\frac{q^{1/24}}{\eta(\tau)}\right)^{\frac{3}{2}n_f}$$

We need the high temperature limit, for which $q^{1/24}$ is subleading and can be ignored. Now the η function satisfies $\eta(\tau) = \sqrt{i/\tau}\,\eta(-1/\tau)$

$$\eta(\beta n/R \to 0) = \sqrt{\frac{2\pi R}{\beta n}}\, e^{-\frac{(2\pi)^2 R}{24\beta}}$$

The square-root term is also subleding and we obtain to leading order

$$\log Z = \frac{3}{2}n_f \frac{(2\pi)^2 R}{24\beta}$$

which is exactly what we want, once we remove units from R, $R \to \ell_s R$.