Chapter 16: String Theory and Matrix Models

1. The Nambu-Goto action is
$$-T_2 \int d^3\xi[\sqrt{\det \hat{g}} + \hat{C}_{\alpha\beta\gamma}\epsilon^\alpha\beta\gamma], \quad \hat{g}_{\alpha\beta} = G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \quad C_{\alpha\beta\gamma} = C_{\mu\nu\rho}\partial_\alpha X^\mu \partial_\beta X^n u \partial_\gamma X^\rho$$
Let's set $C_{\alpha\beta\gamma} = 0$. The EOM for the scalar field is quickly seen to be $\Box X = 0$, where $\Box$ is the Laplacian from the induced metric.

In the Polyakov action, the equations of motion for γ are the vanishing the energy-momentum tensor, giving:
$$\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\gamma_{\alpha\beta}(\gamma^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu - 1)$$
This is harder to solve than the $p = 1$ case, as we can't just take the square root of the determinant of both sides. Taking the ansatz that $\gamma_{\alpha\beta} = \lambda\partial_\alpha X^\mu \partial_\beta X_\mu$ we get:
$$\lambda\gamma_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}(3\lambda - 1)$$
And we get a solution with $\lambda = 1$. Note there is no Weyl rescaling here. Similarly, the X field must satisify
$$\frac{1}{\sqrt{-h}}\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu) = 0$$
which agrees with $\Box X = 0$ upon the identification of the induced and auxiliary metrics.

Note importantly that the *p-brane action for $p \neq 1$ requires a cosmological constant term.*

2. Take the gauge $\gamma_{00} = -\det \hat{g}_{ij}$ with $\gamma_{0i} = 0$ so that $\sqrt{-\gamma} = \det g_{ij}$. The action then becomes:
$$-\frac{T_2}{2}\sqrt{\gamma}\left(\gamma^{00}\dot{X}\cdot\dot{X} - 1\right) = \frac{T_2}{2}(\dot{X}\cdot\dot{X} + \det \hat{g}_{ij})$$
Here there is a small typo in Kiritsis. Rewriting
$$\det \hat{g}_{ij} = \partial_1 X^\mu \partial_1 X^\nu \partial_2 X_\mu \partial_2 X_\nu - \partial_1 X^\mu \partial_2 X^\nu \partial_1 X_\mu \partial_2 X_\nu = -\frac{1}{2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\}$$
We thus get total action:
$$\frac{T_2}{2}\int d^3\xi \left(\dot{X}^\mu \dot{X}_\mu - \frac{1}{2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\}\right)$$
Giving equations of motion
$$\ddot{X}^\mu = \{\{X^\mu, X^\nu\}, X_\nu\}$$
Taking now lightcone gauge $X^+(\tau, \sigma_1, \sigma_2) = \tau$

The transverse momenta are:
$$p^i = \frac{\delta L}{\delta(\partial_\tau X^i)} = T_2\dot{X}^i = \frac{p^+}{V}\dot{X}^i$$
The Hamiltonian is thus
$$p_-\dot{X}^- - \mathcal{L} + \int d^2\xi\, p_i\dot{X}^i$$

3. This rescaling is very straightforward once one has the hamiltonian 16.1.14. One rescales $X^i \to \left(\frac{N}{V_2}\right)^{1/4} X^i$ and $t \to \left(\frac{N}{4V_2}\right)^{-1/4}$ yielding:
$$\frac{T_2}{2}\int d^2\sigma \dot{X}^i\dot{X}^i \to \frac{T_2}{4}\frac{N}{V_2}\int d^2\sigma\frac{\dot{X}^i\dot{X}^i}{2} = \frac{T_2}{4}\mathrm{Tr}[\frac{\dot{X}^i\dot{X}^i}{2}]$$
and
$$\frac{T_2}{4}\int d^2\sigma\{X^i, X^j\}\{X_i, X_j\} \to -\frac{T_2}{4}\frac{N}{V_2}\int d^2\sigma\frac{1}{4}[X^i, X^j][X_i, X_j] = \frac{T_2}{4}\mathrm{Tr}\left(-\frac{1}{4}[X^i, X^j][X_i, X_j]\right)$$

4. For a string, imagine a rectangular spike of cross-section ϵ and length L. Its total energy is $2L + \epsilon$, where ϵ does not multiply L now. Therefore, taking L large will give a large energy deviation, regardless of how small we take ϵ. Thus, the string is stable against decaying into these small spikes.

5. Its immediate that the $C_{\mu\nu\rho}$ term multiplies a Nambu bracket, by antisymmetry. Now by permutation invariance we can write:

$$\frac{1}{6}\{X^\mu, X^\nu, X^\rho\}\{X_\mu, X_\nu, X_\rho\} = \partial_1 X^\mu \partial_2 X^\nu \partial_3 X^\rho \epsilon_{\alpha\beta\gamma}(\partial_\alpha X_\mu \partial_\beta X_\nu \partial_\gamma X_\rho)$$

   Its not hard to see that this reproduces the formula for a 3x3 determinant, as we have an antisymmetric object involving one element from every row and column multiplied together, all with unital coefficients.

   The bracket is not associative **show**

6.