Chapter 7: Superstrings and Supersymmetry

1. We already know that TT will have the desired OPE, since the bosons and fermions are uncoupled and we already have shown their own respective stress tensor OPEs. Next

$$
\begin{aligned}
G(z)G(w) &= -\frac{2}{\ell_s^4}\psi_\mu(z)\partial X^\mu(z)\psi_\nu(w)\partial X^\nu(w) \\
&= -\frac{2}{\ell_s^4}\left(\ell_s^2\frac{\eta_{\mu\nu}}{z-w} + (z-w) :\partial\psi_\mu\psi_\nu(w):\right)\left(-\frac{\ell_s^2}{2}\frac{\eta_{\mu\nu}}{(z-w)^2} + :\partial X_\mu \partial X_\nu(w):\right) \\
&= \frac{D}{(z-w)^3} + \frac{-\frac{2}{\ell_s^2}\partial X_\mu \partial X^\mu(w) - \frac{1}{\ell_s^2}\psi^\mu\partial\psi_\mu(w)}{z-w} \\
&= \frac{\hat{c}}{(z-w)^3} + \frac{2T(w)}{z-w}
\end{aligned}
$$

Finally

$$
\begin{aligned}
T(z)G(w) &= -\frac{1}{\ell_s^2}\left(:\partial X_\mu \partial X^\mu(z): + \frac{1}{2}\psi^\mu\partial\psi_\mu(z)\right) i\frac{\sqrt{2}}{\ell_s^2}\psi_\nu\partial X^\nu(w) \\
&= -i\frac{\sqrt{2}}{\ell_s^4}\left(-\frac{\ell_s^2}{2}\frac{\psi_\mu\partial X^\mu(w) + \psi_\mu\partial^2 X^\mu(w)(z-w)}{(z-w)^2} - \frac{\ell_s^2}{2}\frac{\psi_\mu\partial X^\mu(w)}{(z-w)^2} + (-)\frac{\ell_s^2}{2}\frac{\partial_\mu\psi\partial X^\mu(w)}{(z-w)}\right) \\
&= \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w}
\end{aligned}
$$

2. We will take the OPE of $j_B(z)j_B(w)$, but just look at the $(z-w)^{-1}$ term as a function of w, as this, when integrated around the origin in w will give Q_B^2. This is an extension of exercise **4.45**, and there is nothing conceptually further, except for some $\beta\gamma$ manipulation. There are altogether 16 terms to consider, and we will get $c = 15$. The algebra is heavy, so I will skip this. An alternative is to do this as in **Polchinski 4.3**.

To do it this way, note the following OPEs:

$$
\begin{aligned}
j_B(z)b(w) &\sim \frac{T_{matter}(z)}{z-w} - \frac{1}{(z-w)^2}\left(bc(z) + \frac{3}{4}\beta\gamma(z)\right) + \frac{1}{z-w}\left(-b\partial c(z) + \frac{1}{4}\partial\beta\gamma(z) - \frac{3}{4}\beta\partial\gamma(z)\right) \\
&= \cdots + \frac{1}{z-w}\left[T_{matter}(z) - \partial b\, c(w) - 2b\partial c(w) - \frac{1}{2}\partial\beta\gamma(w) - \frac{3}{2}\beta\partial\gamma(w)\right] \\
&= \cdots + \frac{T_{matter}(w) + T_{gh}(w)}{z-w} \Rightarrow \{Q_B, b_n\} = L_n
\end{aligned}
$$

Similarly

$$
j_B(z)\beta(w) = \cdots + \frac{G_{matter}(w) + G_{gh}(w)}{z-w} \Rightarrow [Q_B, \beta_n] = G_n
$$

Now note that the Jacobi identity on Q_B reads:

$$
\{[Q_B, L_m], b_n\} - \{\overbrace{[L_m, b_n]}^{(m-n)b_{m+n}}, Q_B\} - [\overbrace{\{b_n, Q_B\}}^{L_n}, L_m] = 0 \Rightarrow \{[Q_B, L_m], b_n\} = (m-n)L_{m+n} - [L_m, L_n]
$$

So if the total central charge is zero we'll get $\{[Q_B, L_m], b_n\} = 0$, implying that $[Q_b, L_m]$ is independent of the c ghost. But on the other hand this operator has ghost number 1, so it must therefore vanish. Further, the Jacobi identity also yields

$$
[\{Q_B, Q_B\}, b_n] = -2[\{b_n, Q_B\}, Q_B] = 2[Q_B, L_n]
$$

since we just showed that this last term vanishes, we must have Q_B, Q_B is also independent of c, but again since Q_B^2 has positive ghost number, we get that it is in fact zero. We can do the same argument with β and G and get that the superstring BRST operator is zero, as long as the total central charge vanishes. This was much cleaner than the OPE way.

3. First a lemma: An abelian p-form field A has $\binom{D-2}{p}$ on shell DOF. To prove this, note that we have a gauge symmetry of $A \to A + \partial\Lambda$ which has $\binom{D}{p-1}$ parameters. Next, the Euler-Lagrange equations give us that the components $A^{0i_1\dots i_{p-1}}$ are non-propagating. We thus get $\binom{D-1}{p}$ massless propagating off-shell d.o.f. which have $\binom{D-2}{p-1}$ gauge symmetries left over. These can be used to enforce Coulomb gauge conditions which allow for there to be no polarizations along one of the spatial directions. We thus get $\binom{D-1}{p} - \binom{D-2}{p-1} = \binom{D-2}{p}$ massless on-shell degrees of freedom. For A_μ this is $D-2$ and for $B_{\mu\nu}$ this is $(D-2)(D-3)/2$.

   The metric has $\frac{1}{2}D(D-3)$ on-shell degrees of freedom. There are two ways to see this, first, that the dynamically allowed variation δg may on-shell be described by a symmetric traceless tensor in dimension $D-2$ which gives
$$\frac{(D-1)(D-2)}{2} - 1 = \frac{1}{2}D(D-3)$$
   or by noting that since we are gauging translation symmetry locally, each translation makes 2 polarizations unphysical and so we get:
$$\frac{D(D+1)}{2} - 2D = \frac{1}{2}D(D-3)$$
   as required.

   We now consider the R-R, R-NS, NS-R, NS-NS sectors together. For NS-NS we have the scalar = 1 both on-shell and off-shell, the antisymmetric two-form, which has only transverse degrees of freedom $= 8*7/2 = 28$ and the gravity, $= 10*7/2 = 35$ altogether we get 64 on-shell degrees of freedom.

   In both the R-NS and NS-R sector, we have a Weyl representation of dimension $2^{5-1} = 16$. There are however only 8 on-shell degrees of freedom. Similarly, we only consider the on-shell $\psi^\mu_{-1/2}$ acting on the NS part of the vacuum which gives another factor of 8. This gives 64 fermionic variables in each sector for a grand total of 128.

   In R-R for IIA we have a 0, 2, and *self-dual* 4-form. This gives:
$$1 + \binom{8}{2} + \frac{1}{2}\binom{8}{4} = 64$$
   For IIB we have a 1 and 3-form. This gives
$$\binom{8}{1} + \binom{8}{3} = 64$$
   so in either case we have 64 on-shell degrees of freedom here. This is consistent with each $|S\rangle$ state having 8 on-shell degrees of freedom giving $8 \times 8 = 64$. All together, we have the same number of on-shell fermionic and bosonic degrees of freedom.

   Now for the massive case. In the NS sector you might expect the next excitations come from the bosons α_{-1}, but this gets projected out by GSO, so in fact the next states come from $\psi^i_{-3/2}$, $C_{ijk}\psi^i_{-1/2}\psi^j_{-1/2}\psi^k_{-1/2}$ and $C_{ij}\psi^i_{-1/2}\alpha^j_{-1}$. These have dimensions $8 + 56 + 64 = 128$, which decomposes as the traceless symmetric **44** and three-index antisymmetric **84** representation of SO(9). In the R sector, we must look at $\alpha^i_{-1}|S_\alpha\rangle$ and $\psi^i_{-1}|C_\alpha\rangle$ for S_α, C_α suitably chosen so that the state satisfies $G_0 = 0$. This constraint gives a factor of two reduction for the dimension of the space of candidate S_α. Consequently, we get $\mathbf{8}_v \otimes \mathbf{8}_s \oplus \mathbf{8}_v \otimes \mathbf{8}_{s'}$ which has dimension 128. This indeed turns out to be a spinor representation of SO(9), and it comes from looking at the tensor product of *the* fundamental spinor representation with the vector representation $\mathbf{16}_s \otimes \mathbf{9}_v$. This turns must decompose as a sum of two spinor representations $\mathbf{16}_s \oplus \mathbf{128}_s$. One is again the fundamental, while the other is the required **128**.

   For the massive states in the type IIA and type IIB, we must tensor we wish to look at the lowest-level masses. Note we must match massive states with massive states. In this case, we match $2/\alpha$ on both sides to get massive states of mass $4/\alpha$. Since the particles already organize into representations of SO(9) on each side, the closed string massive spectrum will again clearly organize intro representations of SO(9). Also since fermionic and bosonic degrees of freedom already were equal on each side, they will be equal in the closed string as well. We will have $2 \times 128^2 = 32768$ bosonic and fermionic degrees of freedom.

4. In terms of theta functions:

$$\chi_O = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_3(\nu_i)}{\eta} - \prod_{i=1}^{4}\frac{\theta_4(\nu_i)}{\eta}\right)$$
$$\chi_V = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_3(\nu_i)}{\eta} + \prod_{i=1}^{4}\frac{\theta_4(\nu_i)}{\eta}\right)$$
$$\chi_S = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_2(\nu_i)}{\eta} - \prod_{i=1}^{4}\frac{\theta_1(\nu_i)}{\eta}\right)$$
$$\chi_C = \frac{1}{2}\left(\prod_{i=1}^{4}\frac{\theta_2(\nu_i)}{\eta} + \prod_{i=1}^{4}\frac{\theta_1(\nu_i)}{\eta}\right)$$

We'll take $\nu_i = 0$ here **(I assume this is what I'm supposed to do)** and so $\theta_1 = 0 \Rightarrow \chi_S = \chi_C$.

For IIB we look at

$$\frac{|\chi_V - \chi_C|^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8} = \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4} \times \frac{1}{2}\sum_{\bar{a},\bar{b}=0}^{1}(-1)^{\bar{a}+\bar{b}}\frac{\bar{\theta}^4\left[{\bar{a} \atop \bar{b}}\right]}{\bar{\eta}^4}$$

Under modular transformations $\tau \to \tau + 1$ $\theta^4\left[{0 \atop 1}\right] \leftrightarrow \theta^4\left[{0 \atop 0}\right]$, $\theta^4\left[{1 \atop 0}\right] \to -\theta^4\left[{1 \atop 0}\right]$ while $\eta^{12} \to -\eta^{12}$. In the holomorphic and anti-holomorphic parts separately, each term in the sum picks up a minus sign that is cancelled by the minus sign in the η^4.

Under $\tau \to -1/\tau$, the $\frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}$ out front is invariant. On the other hand, the θ functions transform as $\theta^4\left[{0 \atop 0}\right] \to (-i\tau)^2\theta^4\left[{0 \atop 0}\right]$, $\theta^4\left[{0 \atop 1}\right] \to (-i\tau)^2\theta^4\left[{1 \atop 0}\right]$, $\theta^4\left[{1 \atop 0}\right] \to (-i\tau)^2\theta^4\left[{0 \atop 1}\right]$. These are exactly compensated by the η transformations in the denominator, and no overall sign is picked up

For IIA we have similarly

$$\frac{(\chi_V - \chi_C)(\bar{\chi}_V - \bar{\chi}_S)}{(\sqrt{\tau_2}\eta\bar{\eta})^8} = \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4} \times \frac{1}{2}\sum_{\bar{a},\bar{b}=0}^{1}(-1)^{\bar{a}+\bar{b}+\bar{a}\bar{b}}\frac{\bar{\theta}^4\left[{\bar{a} \atop \bar{b}}\right]}{\bar{\eta}^4}$$

Again, the holomorphic part transforms as before and as we have set the ν_i to zero, we have the same partition function. Using **D.18**, we see that each of the four above sums are zero since they are equal to a product of $\theta_1 = 0$.

5. Again, these are identical if I set the $\nu_i = 0$ (am I not supposed to be doing this? What do the ν_i represent physically?). They are equal to

$$\frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8 4\eta^4\bar{\eta}^4}(\cancel{|\theta_1^4|^2} + |\theta_2^4|^2 + |\theta_3^4|^3 + |\theta_4^4|^2)$$

We have θ_3 and θ_4 swapping under $\tau \to \tau + 1$, generating no signs in this case, while the denominator looks like $|\eta|^{24}$ and also doesn't generate a sign. Then, under $\tau \to -1/\tau$ we have θ_2 and θ_4 swapping generating a $|\tau|^4$, identical to what is generated by the $(\eta\bar{\eta})^4$.

6. The partition function is

$$Z^{\text{het}}_{\text{SO}(16)\times\text{SO}(16)} = \frac{1}{2}\sum_{h,g}\frac{\bar{Z}_{E_8}\left[{h \atop g}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b}(-1)^{a+b+ab+ag+bh+gh}\frac{\theta^4\left[{a \atop b}\right]}{\eta^4}, \quad \bar{Z}_{E_8}\begin{bmatrix} h \\ g \end{bmatrix} = \frac{1}{2}\sum_{\gamma,\delta}(-1)^{\gamma g + \delta h}\frac{\bar{\theta}^8\left[{\gamma \atop \delta}\right]}{\bar{\eta}^8}$$

First look at $\bar{Z}_{E_8}$. Under modular transformations $\tau \to -1/\tau$ we get $\bar{Z}_{E_8}\left[{h \atop g}\right] \to \bar{Z}_{E_8}\left[{g \atop h}\right]$. Under $\tau \to \tau + 1$, we get $\bar{Z}_{E_8}\left[{h \atop g}\right] \to (-1)^{h-2/3}\bar{Z}_{E_8}\left[{h \atop g+h}\right]$. With this, we can look at $Z^{\text{het}}_{\text{SO}(16)\times\text{SO}(16)}$ under $\tau \to -1/\tau$

$$\frac{1}{2}\sum_{h,g}\frac{\bar{Z}_{E_8}\left[{g \atop h}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b}(-1)^{a+b+ab+ag+bh+gh}\frac{\theta^4\left[{b \atop a}\right]}{\eta^4}$$

Under relabeling of $a \leftrightarrow b, g \leftrightarrow h$, this is the same. Next, under $\tau \to \tau + 1$:

$$
\begin{aligned}
&\frac{1}{2}\sum_{h,g} \frac{(-1)^{-4/3}\bar{Z}_{E_8}\left[{h \atop g+h}\right]^2}{(-1)^{4/3}(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b}(-1)^{a+b+ab+ag+bh+gh}\frac{(-1)^a\theta^4\left[{a \atop a+b-1}\right]}{(-1)^{1/3}\eta^4} \\
&= \frac{1}{2}\sum_{h,g} -\frac{\bar{Z}_{E_8}\left[{h \atop g+h}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b}(-1)^{b+ab+ag+bh+gh}\frac{\theta^4\left[{a \atop a+b-1}\right]}{\eta^4} \\
&= \frac{1}{2}\sum_{h,g'} \frac{\bar{Z}_{E_8}\left[{h \atop g'}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b}(-1)^{1+b+ab+ag'+(a+b)h+g'h+h}\frac{\theta^4\left[{a \atop a+b-1}\right]}{\eta^4} \\
&= \frac{1}{2}\sum_{h,g'} \frac{\bar{Z}_{E_8}\left[{h \atop g'}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b}(-1)^{\cancel{1}+(b'+a+\cancel{1})+(ab'+\cancel{a}-\cancel{a})+ag'+(b'h+\cancel{h})+g'h+\cancel{h}}\frac{\theta^4\left[{a \atop b'}\right]}{\eta^4} \\
&= \frac{1}{2}\sum_{h,g'} \frac{\bar{Z}_{E_8}\left[{h \atop g'}\right]^2}{(\sqrt{\tau_2}\eta\bar{\eta})^8}\frac{1}{2}\sum_{a,b}(-1)^{a+b'+ab'+ag'+b'h+g'h}\frac{\theta^4\left[{a \atop b'}\right]}{\eta^4}
\end{aligned}
$$

Keep in mind that $x^2 = x \bmod 2$.

Before we do the next part, let's elaborate on why $Z_{E_8} = \frac{1}{2}\sum_{a,b}\theta^8\left[{a \atop b}\right]$ is the partition function of the E_8 lattice. From the sixteen fermion picture, this is just the $(-1)^F = 1$ in the NS sector (corresponding to the $\chi_O = \frac{1}{2}(\theta^8\left[{0 \atop 0}\right] + \theta^8\left[{0 \atop 1}\right])$ character) together with the R sector $\chi_S = \frac{1}{2}\theta^8\left[{1 \atop 0}\right]$ giving the spinor representation.

Indeed, the roots of E_8 consist of the roots of O(16) as well as the spinor weights of O(16). Note that the spinor representation comes from the half-integral points, corresponding to $\theta\left[{1 \atop 0}\right]$ in the sum, while the adjoint representation comes from $\theta\left[{0 \atop 1}\right]$ and $\theta\left[{0 \atop 0}\right]$. Consequently the action of $\mathcal{S}_i$ that fixes the adjoint vectors but flips the sign of the spinor acts on our partition function as $\mathcal{S}_i Z_{E_8} = \frac{1}{2}\sum_{a,b}(-1)^a\,\theta^8\left[{a \atop b}\right]$. It of course also gives rise to a twisted sector, so altogether we get the four twisted blocks $\bar{Z}_{E_8}\left[{h \atop g}\right]$ as required.

Since we have projected out the spinor representation, the current algebra only contains the NS currents $\bar{J}^{ij}$ corresponding to the adjoint of SO(16), and we have two copies of this for each group of 16 fermions.

From the factor of $(\sqrt{\tau}_2\eta\bar{\eta})^{-8}$ we see that we have 8 on-shell noncompact massless bosonic excitations as well as all of their descendants (on both left and right moving sides). We also see on the left-moving side we get a theta-function corresponding to $N = 8$ fermions transforming under a spacetime SO(8), forming the superpartners of the bosons. On the right side instead of the superpartner fermions, we have the 16 internal fermions that transform in the adjoint representations.

Let's see what massless states we can build. In the NS sector of the left-movers, we have $L_0 = 1/2, \bar{L}_0 = 1$ and so we get $\psi^i_{-1/2}\alpha^j_{-1}|p\rangle$ which gives us our usual graviton, two-form field, and dilaton. We also have $\psi^i_{-1/2}\bar{J}^a_{-1}|p\rangle$ for the O(16) × O(16) currents. This gives us vectors corresponding to gauge bosons valued in the adjoint of O(16) × O(16) as required.

In the R sector we have $G_0 = 0, \bar{L}_0 = 1$ we'll get a gravitino, fermion, and gaugino as before, but again this time valued in O(16) × O(16).

7. Because we have seen that T-duality flips the antichiral $U(1)$ $\bar{\partial}X \to -\bar{\partial}X$, and we want to preserve the (1,1) supersymmetry G in the type II string (and so must keep it as a periodic variable **Why is this absolutely necessary. Can we not work with double covers in some clever way when defining supercurrents?**), we must consequently flip $\bar{\psi}$. This corresponds to inserting $(-1)^{F_R}$. For the right-moving R sector, this changes the chirality of the R spinor, taking $S_\alpha \to \Gamma^9\Gamma^{11}S_\alpha$ (there can be no phase, by reality conditions of Γ). We thus flip IIA to IIB and vice versa.

   From this we get that

$$
F_{\alpha\beta} = S_\alpha(\Gamma^0)_{\beta\gamma}\tilde{S}_\gamma \to S_\alpha(\Gamma^0\Gamma^9\Gamma^{11})_{\beta\gamma}\tilde{S}_\gamma = -\xi\, S_\alpha(\Gamma^9\Gamma^0)_{\beta\gamma}\tilde{S}_\gamma = -\xi F\Gamma^9
$$

Expanding in terms of the $F_{\mu_1\ldots\mu_k}$ gives the action:

$$F_{\alpha\beta} \to -\xi \sum_{k=0}^{10} \frac{(-1)^k}{k!} F_{\mu_1\ldots\mu_k}\Gamma^{\mu_1\ldots\mu_k}\Gamma^9$$

This gives that

$$\tilde{F}_{\mu_1\ldots\mu_k,9} = -\xi F_{\mu_1\ldots\mu_k}, \quad \tilde{F}_{\mu_1\ldots\mu_k} = F_{\mu_1\ldots\mu_k,9}$$

Then

$$\partial_{\mu_1}\tilde{C}_{\mu_2\ldots\mu_k 9} = -\xi\partial_{\mu_1}C_{\mu_2\ldots\mu_k}, \quad \partial_{\mu_1}\tilde{C}_{\mu_2\ldots\mu_k} = \partial_{\mu_1}\tilde{C}_{\mu_2\ldots\mu_k 9}$$

so that (up to a closed term)

$$\tilde{C}^{(p)}_{\mu_1\ldots\mu_{p-1}9} = -\xi C^{p-1}_{\mu_1\ldots\mu_{p-1}}, \quad \tilde{C}^{(p)}_{\mu_1\ldots\mu_p} = C^{(p+1)}_{\mu_1\ldots\mu_p 9}$$

Get rid of the ξ factor

8. We have that $\Omega\,|S_\alpha\tilde{S}_\beta\rangle = \varepsilon_R\,|S_\beta\tilde{S}_\alpha\rangle$. Further, it acts trivially on Γ^0 **(you sure?)**. Now, in the operator language we will have $\Omega S_\alpha\Omega^{-1} = \epsilon_1\tilde{S}_\alpha$ and $\Omega\tilde{S}_\beta\Omega^{-1} = \epsilon_2 S_\beta$. In any case, we must have for the bi-spinor that $\Omega S_\alpha\tilde{S}_\beta\Omega^{-1} = \epsilon_R S_\beta\tilde{S}_\alpha$, which gives that $\epsilon_1\epsilon_2 = -\epsilon_R$ Thus, we have:

$$\Omega F_{\alpha\beta}\Omega^{-1} = \Omega S_\alpha\Gamma^0_{\beta\gamma}\tilde{S}_\gamma\Omega^{-1} = -\epsilon_R\Gamma^0_{\beta\gamma}S_\gamma\tilde{S}_\alpha = -\epsilon_R\Gamma^0_{\beta\gamma}F_{\gamma\delta}\Gamma^0_{\delta\alpha} = -\epsilon_R(\Gamma^0F\Gamma^0)_{\beta\alpha} = -\epsilon_R(\Gamma^0F^T\Gamma^0)_{\beta\alpha}$$

I think 7.3.3 of Kiritsis has the derivation wrong. Ask Nathan/Xi.

9. When we take $\epsilon_R = -1$m the scalar and four-index self-dual tensor survive. In this case, we will *not* have consistent interactions. Since the graviton survives, there must be an equal number of massless bosonic and fermionic excitations. The fermions come just from the NS-R sector (there is no R-NS now), giving 64 on-shell fermionic excitations. From the NS-NS sector, the dilaton and gravity will give $1 + 35 = 36$ on-shell bosonic degrees of freedom. We are missing 28 bosonic degrees of freedom.

   The scalar and four-index self dual tensor contribute $1 + \frac{1}{2}\frac{8\times7\times6\times5}{4!} = 36$ on-shell bosonic degrees of freedom. This is too much. The two-form, on the other hand, contributes the requisite $8\times 7/2 = 28$. Consistency of interaction thus *demands* we keep only the 2-form and drop the 0 and self-dual 4-form. This necessitates $\epsilon_R = 1$.

10. We are just looking at the *open* superstrings here. Any open string that consistently couples to type I or type II string theory must have a GSO projection as well. We have already seen how the oriented open strings look like in exercise **7.3**. In the NS sector we have at $-p^2 = m^2 = 2/\ell_s^2$

$$\begin{aligned} &\psi^i_{-3/2}\lambda_{ab}\,|p;ab\rangle_{NS} \\ &C_{ijk}\psi^i_{-1/2}\psi^j_{-1/2}\psi^k_{-1/2}\lambda_{ab}\,|p;ab\rangle_{NS} \\ &C_{ij}\psi^i_{-1/2}\alpha^j_{-1}\lambda_{ab}\,|p;ab\rangle_{NS} \end{aligned} \tag{1}$$

In the R sector we have (for S_α suitably chosen so that the state satisfies $G_0 = 0$):

$$\begin{aligned} &\alpha^i_{-1}\lambda_{ab}\,|S_\alpha;ab\rangle_R \\ &\psi^i_{-1}\lambda_{ab}\,|C_\alpha;ab\rangle_R \end{aligned} \tag{2}$$

I will assume NN boundary conditions. In this case

$$\begin{aligned} \Omega\alpha_{-1}\Omega^{-1} &= -\alpha_{-1} \\ \Omega\psi_{-1}\Omega^{-1} &= -\psi_{-1} \\ \Omega\psi_{-\frac{1}{2}}\Omega^{-1} &= -i\psi_{-\frac{1}{2}} \\ \Omega\psi_{-\frac{3}{2}}\Omega^{-1} &= i\psi_{-\frac{3}{2}} \end{aligned}$$

So all of the terms in (1) are terms of the form $\mathcal{A}\lambda_{ab}\,|p;ab\rangle_{NS}$ with the operator $\mathcal{A}$ transforming as $\mathcal{A}\to i\mathcal{A}$ under parity. Doing parity twice therefore will generate a $-\epsilon_{NS}^2\mathcal{A}(\gamma\gamma^{T^{-1}})_{ii'}\,|p;a'b'\rangle\,(\gamma^T\gamma^{-1})_{j'j}$. This is exactly the same as in **7.3.10**. Demanding that Ω act on the state with eigenvalue $+1$ will make it so that $\lambda = i\epsilon_{NS}\gamma\lambda^T\gamma^{-1}$. We already have $\epsilon_{NS}=-i$ so $\lambda=\gamma\lambda^T\gamma^{-1}$ here. Imposing the tadpole cancelation condition $\zeta=1$ and we get gauge group SO(32). So we get that states at this level will transform in the *the traceless symmetric tensor + singlet representation* of $SO(32)$.

All of the terms in (2) will transform under parity twice as as $\epsilon_R^2\mathcal{A}(\gamma\gamma^{T^{-1}})_{ii'}\,|S_\alpha;a'b'\rangle\,(\gamma^T\gamma^{-1})_{j'j}$. We will have the same γ matrix as in the NS sector, as required for consistency of interactions. Here, though, we will get $\epsilon_R=-1\Rightarrow\epsilon_R^2=1$ and we will get $\lambda=-\gamma\lambda^T\gamma^{-1}$ (this is what we got from the massless sector with an extra minus sign since ψ_{-1},α_{-1} now transform with minus signs). Again we will have that these states will transform in the symmetric representation of SO(32).

Again we get 128 bosonic states that will transform as the $\mathbf{44}\oplus\mathbf{84}$ representation of SO(9). We will also get fermions transforming in the **128** spinor representation as in exercise **3**. All of these states will transform in the traceless symmetric representation of SO(32). **Confirm**

11. Certainly in the untwisted sector, the theory we get corresponds to tracing over the projection operator $\frac{1}{2}(1+g)$ where g is orientation-reversal. Now in the twisted sector, we still have X^μ satisfies the Laplace equation $\partial_+\partial_-X=0$ so we can write

$$X(\sigma,\tau)=x^\mu+\tau\ell_s^2\frac{p^\mu+\bar p^\mu}{2}+\sigma\ell_s^2\frac{p^\mu-\bar p^\mu}{2}+\frac{i\ell_s}{\sqrt{2}}\sum_n\left(\frac{\alpha_n}{n}e^{-in(\tau+\sigma)}+\frac{\tilde\alpha_n}{n}e^{-in(\tau+\sigma)}\right)$$

The condition that $X(\sigma+2\pi)=X(2\pi-\sigma)$ give that $p^\mu=\bar p^\mu$ and the σ term vanishes. We must have n is a half integer. For integer modding we have $e^{-in(\tau\pm\sigma)}\to e^{-in(\tau\mp\sigma)}$. For half-integer modding we have $e^{-in(\tau\pm\sigma)}=(-1)^ne^{-in(\tau\mp\sigma)}$. We should thus have $\alpha_n=\tilde\alpha_n$ for n integral and $\alpha_n=-\tilde\alpha_n$ We thus get

$$X(\sigma,\tau)=x^\mu+2\ell_s^2p^\mu\tau+\sigma i\sqrt{2}\ell_s\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{\alpha_n}{n}\cos(n\sigma)e^{-in\tau}-\sqrt{2}\ell_s\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{\alpha_n}{n}\sin(n\sigma)e^{-in\tau}$$

This is the twisted sector. The last sum picks up a minus sign under orientation reversal, and so will be projected out. We are left with the equations of motion for the open string.

12. In NS we have (up to an overall irrelevant factor of $i^{-1/2}$)

$$\psi(\sigma,\tau)=\sum_{n\in\mathbb{Z}}\psi_{n+1/2}e^{(n+1/2)(\tau+i\sigma)},\quad \bar\psi(\sigma,\tau)=\sum_{n\in\mathbb{Z}}\bar\psi_{n+1/2}e^{(n+1/2)(\tau-i\sigma)}$$

In the closed string case have that $\Omega\psi_{n+1/2}\Omega^{-1}=\bar\psi_{n+1/2}$. Given that $\Omega\psi(\sigma,\tau)\Omega^{-1}=\bar\psi(\pi-\sigma,\tau)$, we directly get $\Omega\psi_{n+1/2}\Omega^{-1}=i(-1)^n\bar\psi_{n+1/2}$. For DD boundary conditions we get an extra minus sign to this, since there $\Omega\psi(\sigma,\tau)\Omega^{-1}=-\bar\psi(\pi-\sigma,\tau)$.

In the R sector we have

$$\psi(\sigma,\tau)=\sum_{n\in\mathbb{Z}}b_ne^{n(\tau+i\sigma)},\quad \bar\psi(\sigma,\tau)=\sum_{n\in\mathbb{Z}}\bar b_ne^{n(\tau-i\sigma)}$$

Following the same logic we get that $\Omega\psi_n\Omega^{-1}=(-1)^n\psi_n$ for NN and $\Omega\psi_n\Omega^{-1}=-(-1)^n\psi_n$ for DD.

All of these cases can be summarized by

$$\begin{aligned}\text{NN:}\quad \Omega\psi_r\Omega^{-1}&=(-1)^r\psi_r\\ \text{DD:}\quad \Omega\psi_r\Omega^{-1}&=-(-1)^r\psi_r.\end{aligned}$$

13. Let's clarify a bit of terminology before we begin. We are looking at just the fermions of the left moving and right moving sides of the heterotic string theory. On the left-hand (supersymmetric) side, in the light-cone gauge these form an $\widehat{O(8)}$ current algebra at *level 1*. On the right-hand side the form a $\widehat{O(32)}$ current algebra at level 1 again (**why must we always have level 1? Ask Xi.**). In both cases the left and right fermions share the same boundary conditions.

The characters of $\widehat{O(N)}_1$ for N even correspond to the integrable representations labeled by $\mathcal{O}, V, S, C$ corresponding to the trivial, vector, spinor, and conjugate spinor. Our partition functions in question can be constructed from integer linear combination of products of exactly one $\widehat{O(8)}_1$ and one $\widehat{O(32)}_1$ character. This gives sixteen possible terms.

The normalization of the identity contribution $\chi_{8,O}\chi^*_{32,O}$ to 1 reduces this to 15.

For the next part of this problem, it is important to know that the only integrable representation of $\widehat{E}_{81}$ is the identity which we will denote by $\chi_E = \frac{1}{2}\sum_{ab}\theta^8\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]/\eta^8$. This immediately shows that for the case $G = E_8 \times E_8$ the only possibility is the heterotic E theory.

For $G = E_8 \times O(16)$ we again have four characters: $\chi_E\chi_{16,\mathcal{O}}, \chi_E\chi_{16,V}, \chi_E\chi_{16,S}, \chi_E\chi_{16,C}$.

Finally, the hardest case: $O(16) \times O(16)$

In order for spacetime supersymmetry to exist, we must have $e^{-\phi/2}S_\alpha$ have well-defined OPE with any vertex operator. **FINISH HERE**. This corresponds exactly to the condition that the GSO projection must act as $(-1)^{F_L} = 1$. This recovers exactly heterotic E and heterotic O.

We thus have 9 theories satisfying modular invariance, spin-statistics. Six of them have tachyons, so we get **three** theories left over. Of these three, only two have spacetime supersymmetry–exactly heterotic O and heterotic E.

14. I think this problem is backwards. For 32 fermions *all* with the same boundary conditions, its immediate to see that they will reproduce the partition function for the $\mathrm{Spin}(32)/\mathbb{Z}_2$ string:

$$\frac{1}{2}\sum_{a,b}\theta^{16}\begin{bmatrix}a\\b\end{bmatrix}$$

Just by considering the $O(N)$ fermion at $N = 32$. On the other hand, if we split the fermions into $16 + 16$, and consider separately boundary conditions for each of *those*, then our partition function is the square of the 16-fermion system. We then get $E_8 \times E_8$ as required

$$\left[\frac{1}{2}\sum_{a,b}\theta^{8}\begin{bmatrix}a\\b\end{bmatrix}\right]^2$$

15. Note this was a Lorentzian lattice of signature (n, n). The norm was thus $P_L^2 - P_R^2 = 2mn \in 2\mathbb{Z}$. It is also self dual, since it is already integral, and there is no integral sublattice.

16. We have

$$\gamma G_{ghost} = -c\gamma\partial\beta - \frac{3}{2}\partial c\gamma\beta - 2\gamma^2 b, \quad cT_{ghost} = 2bc\partial c - \frac{1}{2}c\gamma\partial\beta - \frac{3}{2}c\partial\gamma\beta$$

Here Kitisis' conventions are different than Polchinski.= Recall upon bosonization $\beta(z) = e^{-\phi(z)}\partial\xi(z), \gamma = e^{\phi(z)}\eta(z)$. Although we can solve this problem very quickly since we already know what the stress tensor looks like in the bosonized variables, I think it's way more instructive to explicitly compute OPEs to $O(z-w)$. First let's look at the η, ξ theory, which is a fermoinic bc theory of weights $1, 0$. We get

$$\xi(z)\eta(w) = \frac{1}{z-w} + :\xi\eta:(w) + O(z-w)$$

We can bosonize this theory in terms of hermitian χ field so that $\eta = e^{-\chi}, \xi = e^{-\chi}$. Using these coordinates

$$\begin{aligned}\xi(z)\eta(w) = e^{\chi(z)}e^{-\chi(w)} &= \frac{1}{z-w}\left[1 + (z-w)\partial\chi + \frac{1}{2}(z-w)^2(\partial^2\chi + (\partial\chi)^2) + \dots\right]\\ \Rightarrow \partial\xi(z)\eta(w) &= -\frac{1}{(z-w)^2} + \frac{1}{2}(\partial^2\chi + (\partial\chi)^2)\end{aligned}$$

Using this we can write

$$\begin{aligned}\beta(z)\gamma(w) &= e^{-\phi(z)}\partial\xi(z)\,e^{\phi(w)}\eta(w)\\ &= (z-w)\left[1-(z-w)\partial\phi(w)+\frac{1}{2}(z-w)^2((\partial\phi)^2-\partial^2\phi)\right]\left[-\frac{1}{(z-w)^2}+\frac{1}{2}(\partial^2\chi+(\partial\chi)^2)\right]\end{aligned}$$

The constant term gives $:\beta\gamma:=\partial\phi\Rightarrow:\partial(\beta\gamma):=\partial^2\phi$. The $(z-w)$ term gives exactly the stress tensor of the $\beta\gamma$ theory at $\lambda=0$, which makes sense since this is exactly $\partial\beta\gamma$

$$\begin{aligned}:\partial\beta\gamma: &= -\frac{1}{2}(\partial\phi)^2+\frac{1}{2}\partial^2\phi+\frac{1}{2}(\partial\chi)^2+\frac{1}{2}\partial^2\chi\\ \Rightarrow T_{\beta\gamma} &= \partial\beta\gamma-\lambda\partial(\beta\gamma) = -\frac{1}{2}(\partial\phi)^2+\left(\frac{1}{2}-\lambda\right)\partial^2\phi+\frac{1}{2}(\partial\chi)^2+\frac{1}{2}\partial^2\chi.\end{aligned}$$

In our case we have $\lambda=3/2$.

$$\begin{aligned}\gamma G_{ghost} &= -c\left(-\frac{1}{2}(\partial\phi)^2+\frac{1}{2}\partial^2\phi+\frac{1}{2}(\partial\chi)^2+\frac{1}{2}\partial^2\chi\right)-\frac{3}{2}\partial\phi\,\partial c-2\gamma^2 b\\ cT_{ghost} &= 2bc\partial c+c\left(-\frac{1}{2}(\partial\phi)^2-\partial^2\phi+\frac{1}{2}(\partial\chi)^2+\frac{1}{2}\partial^2\chi\right)\end{aligned}$$

Altogether this gives a BRST current:

$$\begin{aligned}j_B &= cT_X+\gamma G_X+\frac{1}{2}\left(cT_{gh}+\gamma G_{gh}\right)\\ &= cT_X+\gamma G_X+bc\partial c-\frac{3}{4}\partial\phi\partial c-\frac{3}{4}c\partial^2\phi-\gamma^2 b\end{aligned}$$

17. We are looking at $[Q_B,\xi e^{-\phi/2}S_\alpha e^{ipX}]$. It suffices to look at the $1/(z-w)$ pole in the OPE of j_B with $\xi e^{-\phi/2}S_\alpha e^{ipX}$. The terms that contribute to this pole must involve pairing ξ with its conjugate η. η appears in j_B wherever $\gamma=e^\phi\eta$ appears. From the previous exercise, we see that we need only look at the terms γG_X and $-\gamma^2 b$.

    These two terms contribute poles:

$$-\left[\frac{:e^{\phi/2}G_X S_\alpha e^{ipX}:}{z-w}-\frac{:e^{3\phi/2}\eta b S_\alpha e^{ipX}:}{z-w}\right]$$

    The overall minus sign comes from commuting across an odd number of fermions for the Wick-contraction. We will need to recall two things:

$$\psi^\mu(z)\cdot S_\alpha(w)\sim\frac{\ell_s^2}{\sqrt{2}\sqrt{z-w}}\left(\Gamma^\mu_{\alpha\beta}S^\beta(w)+\frac{2}{\ell_s^2}\Gamma^\nu_{\alpha\beta}S^\beta\psi_\nu\psi^\mu(z-w)\right),\qquad e^{\phi(z)}e^{-\phi(w)/2}\sim\sqrt{z-w}e^{\phi(w)/2}$$

    That means that first term is:

$$\begin{aligned}&e^{\phi(z)}i\frac{\sqrt{2}}{\ell_s^2}\psi^\mu(z)\partial X_\mu(z)\cdot e^{-\phi(w)/2}S_\alpha(w)e^{ipX(z)}\\ &\quad\sim i\frac{\sqrt{2}}{\ell_s^2}\sqrt{z-w}\,e^{\phi(w)/2}\frac{\ell_s^2}{\sqrt{2}}\left(\frac{\Gamma^\mu_{\alpha\beta}S^\beta\partial X_\mu e^{ip\cdot X}}{\sqrt{z-w}}+\frac{-i\ell_s^2 p_\mu e^{ip\cdot X}}{z-w}\frac{2}{\ell_s^2}\Gamma^\nu_{\alpha\beta}S^\beta\psi_\nu\psi^\mu\sqrt{z-w}\right)\\ &\quad= -e^{\phi/2}\left(\Gamma^\mu_{\alpha\beta}S^\beta\partial X_\mu-i\Gamma^\nu_{\alpha\beta}S^\beta\psi_\nu p\cdot\psi\right)e^{ip\cdot X}\end{aligned}$$

    Note this OPE has no singularity, so we exactly got the normal ordered term we required: $:e^{\phi/2}G_X S_\alpha e^{ip\cdot X}:$. Altogether this gives us:

$$V^{(1/2)}_{\text{fermion}}(u,p)=\left[e^{\phi/2}\Gamma^\mu_{\alpha\beta}S^\beta\partial X^\mu-ie^{\phi/2}\Gamma^\nu_{\alpha\beta}S^\beta\psi_\nu p\cdot\psi+e^{3\phi/2}\eta b S_\alpha\right]e^{ipX}u^\alpha(p).$$

18. Here, I followed the discussion of Polchinski **12.5**. The picture changing operator is:

$$\mathrm{X}(z) := Q_B \cdot \xi(z)$$

Over the sphere, the $\beta\gamma$ path integral is equivalent to the ϕ, η, ξ path integral *plus* an additional insertion of ξ to make up for the fact that it pics up a zero mode due to to the vacuum degeneracy it produces. Because the expectation value is *just* proportional to the zero-mode of ξ, which depends on global information rather than the specific local insertion point, $\langle \chi(z) \rangle$ is independent of position and we can normalize ξ so that this is 1.

Say we have a null state. This means it is BRST exact. This means that we can rewrite its pointlike insertion as a local operator surrounded by a BRST contour (direct, from the definition of exact). For that null state to decouple, we need to be able to contract the BRST contour off the sphere (i.e. by pulling it off to the north pole). The fact that ξ is inserted will seem to obstruct this. What happens now as we pull the BRST charge to infinity is that it will circle ξ, creating the PCO $\mathrm{X}(z)$. However, when the ξ insertion is replaced by X, the path integral will *vanish* since there is now no ξ insertion to avoid the zero-mode.

Now consider a path integral with a PCO insertion as well as additional BRST-invariant operators (meaning. the contour integral around them of j_B is zero). Then we can write $\mathrm{X}(z_1)\xi(z_2) = Q_B\xi(z_1)\xi(z_2) = (-)^2\xi(z_1)Q^B\xi(z_2) = \xi(z_1)\mathrm{X}(z_2)$ where I have pulled the Q_B contour around the sphere (there two minus signs, one from commuting Q_B across a fermionic variable and one from reversing the orientation of the contour.)

This is interesting: although X is null, it does *not* vanish in the path integral, since pulling Q_B off of it will make Q_B encircle $\xi(z_2)$ but leave behind $\mathrm{X}(z_1)$'s $\xi(z_1)$, so the ξ zero-mode will remain saturated and we won't get zero.

The X can be brought near any of the local BRST closed operators to change their picture (the OPE is nonsingular). Ie note that the main term we look at is $\gamma G_X = e^{\phi}\eta G_X$ in j_B so that $\mathrm{X}\mathcal{O}^{(-1)}(z) = zG_X(z)\mathcal{O}(0) \to G_{-1/2}\mathcal{O}(0)$. We can move X to any other point on the sphere - since the exact position of X does not matter any more than the position of ξ.

19. It is enough to look at the $1/(z-w)$ term in the OPE

$$e^{-\phi(z)/2}S_\alpha(z)V^{(-1/2)}_{\text{fermion}}(w) = e^{-\phi(z)/2}S_\alpha(z)u^\beta(p)e^{-\phi(w)/2}S_\beta(w)e^{ip\cdot X(w)}$$

We will use the fact of **4.12.42**:

$$S_\alpha(z)S_\beta(w) = \frac{C_{\alpha\beta}\ell_s}{(z-w)^{N/8}} + \frac{\Gamma^\mu_{\alpha\beta}\psi_\mu(w)}{\sqrt{2}(z-w)^{N/8-1/2}}$$

where $C_{\alpha\beta}$ is the charge conjugation matrix and here $N = 10$. We also have $e^{-\phi/2}e^{-\phi/2} = (z-w)^{-1/4}e^{-\phi}$. This leaves the $(z-w)^{-1}$ term to be the requisite

$$e^{-\phi}u^\beta(p)\frac{\Gamma^\mu_{\alpha\beta}}{\sqrt{2}}\psi_\mu e^{ip\cdot X} = V^{(-1)}_{\text{boson}}$$

I have an extra factor of $1/\sqrt{2}$. Is this absorbed in Kiritsis' definition of $\gamma^\mu_{\alpha\beta}$? For the second example, we will look at the $(z-w)^{-1}$ term in the OPE

$$e^{-\phi(z)/2}S_\alpha(z)\epsilon_\mu\left(\partial X^\mu - \frac{i}{2}p_\mu\psi^\mu\,\psi^\nu\right)e^{ip\cdot X}.$$

The first term in parentheses will not contribute to the singular term. Also the $e^{-\phi/2}$ and $e^{ip\cdot X}$ contract with nothing. Here, we use **4.12.41** to evaluate

$$S_\alpha(z)\cdot\underbrace{\psi^\mu\psi^\nu}_{J^{\mu\nu}}(w) \sim -\frac{\ell_s^2(\Gamma_{\mu\nu})_\alpha^\beta S_\beta(w)}{2(z-w)}$$

The $-$ sign comes from the fact that the fermion current is coming from the *right* this time so z and w are swapped. This gives a variation

$$e^{-\phi/2} i p^\mu \epsilon^\nu (\Gamma_{\mu\nu})_\alpha^\beta S_\beta e^{ip\cdot X} = V^{(-1/2)}_{\text{fermion}}$$

I'm off by *two* factors of 2

20. We are in type I. We have

$$\frac{1}{\ell_s^2 g_o^2} \langle c(w_1) V^{(-1)}(w_1) c(w_2) V^{(-1)}(w_2) c(w_3) V^{(0)}(w_3)\rangle + 1 \leftrightarrow 2, \quad x_1 > x_2 > x_3$$

The relevant expectation values are

$$\langle c(w_1)c(wz_2)c(w_3)\rangle = w_{12}w_{13}w_{23}, \quad \langle e^{-\phi(w_1)}e^{\phi(w_2)}\rangle = w_{12}^{-1}, \quad \langle \psi^\mu(w_1)\psi^\nu(w_2)\rangle = w_{12}^{-1}\ell_s^2\eta^{\mu\nu}$$

In the matter CFT we get:

$$\begin{aligned}&\langle \psi^\mu(w_1)e^{ik_1\cdot X(w_1)}\, \psi^\mu(w_2)e^{ik_2\cdot X(w_2)}\, (i\partial X^\rho + 2k_3\cdot\psi\psi^\rho)e^{ik_3\cdot X(w_3)}\rangle \\ &= -2i\ell_s^4 \delta\!\!\!/^{10}(\Sigma k)\left(\frac{\eta^{\mu\nu}k_1^\rho}{w_{12}w_{13}} + \frac{\eta^{\mu\nu}k_2^\rho}{w_{12}w_{23}} - \frac{\eta^{\mu\rho}k_3^\nu - \eta^{\nu\rho}k_3^\mu}{w_{13}w_{23}}\right)\end{aligned}$$

21. We should put the gaugini in the $-1/2$ picture and the boson in the -1 picture

22.

23.

24. The bosonic action in 11D is:

$$\frac{1}{2\kappa^2}\int d^{11}x\sqrt{-\det \hat{G}}\left[R - \frac{1}{2\cdot 4!}G_4^2 + \frac{1}{(144)^2}\epsilon^{M_1\ldots M_{11}}G_{M_1\ldots M_4}G_{M_4\ldots M_8}\hat{C}_{M_9M_{10}M_{11}}\right]$$

where G_4 is the field strength of the 3-form $\hat{C}$. From Appendix F, we have that the dilaton $\Phi = 0$ in 11D. So the field σ will just be $-2\phi = \frac{1}{2}\log G_{00}$, and A here is as it is in appendix **F**. Directly using the bosonic equation **F.3** gives the terms

$$\frac{1}{2\kappa^2}\int d^8x\sqrt{g}e^\sigma\left[R - \frac{1}{4}e^{2\sigma}F_2^2\right]$$

Now let's look at the 3-form potential contribution. Because F is antisymmetric in all four indices, and we only are compactifying along one dimension, only the first two terms of **F.28** can contribute. They give

$$\frac{1}{2\kappa^2}\int d^{10}x\sqrt{g}e^\sigma\left[-\frac{1}{2\cdot 4!}F_4^2 - \frac{1}{2\cdot 3!}H_3^2\right]$$

and $H_3 = \partial B_2 = \partial C_{\mu\nu,11}$ consistent with **F.30**

Finally, let's look at that last $\epsilon^{M_1\ldots M11}$ term. At first it looks quite scary. Note we can write this last term as $\mathrm{d}\hat{C}\wedge\mathrm{d}\hat{C}\wedge\hat{C}$. **(Finish this part)**

This translates to our requisite term

$$\frac{1}{4\kappa^2}\int d^{10}x B_2\wedge \mathrm{d}C_3\wedge\mathrm{d}C_3$$

25. Under $A \to A + \partial\epsilon$ and $C \to C + \epsilon H_3$

Note F_2 will stay the same, as will $H_3' = \mathrm{d}B_2' = \mathrm{d}C'_{\mu\nu 11} = \mathrm{d}C_{\mu\nu 11} + \epsilon \mathrm{d}H_3 = \mathrm{d}C_{\mu\nu 11} = H_3$ since H_3 is exact. $\mathrm{d}C_3$ will also stay the same.

Finally, $F_4' = F_4 - \mathrm{d}\epsilon\wedge H_3$