Magnetic Monopoles, ‘t Hooft Lines, and the Geometric Langlands Correspondence

Alexander B. Atanasov

Advisor: Philsang Yoo

Spring 2018

Abstract

The aim of this thesis is to give the reader a gentle but thorough introduction to the vast web of ideas underlying the realization of the geometric Langlands correspondence in the physics of quantum field theory (QFT). It begins with a pedagogically-motivated introduction to the relevant mathematical concepts in algebraic geometry and topology aimed towards physicists, and goes on to provide an introduction to classical and quantum field theory and gauge theory for a mathematical audience. With the machinery of both sides in place, the more complicated phenomena associated with gauge theory is explored, specifically instantons, topological operators, and electric-magnetic duality. We conclude with a description of the connection between the Langlands correspondence and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) via a striking property known as S-duality. A large part of the goal of this thesis is to fill in gaps otherwise skipped over by the literature related to this topic, so that an advanced undergraduate or graduate student might have a good exposition into this field.

Contents

List of Tables

List of Figures

Chapter 1

Introduction and Overview of the Langlands Program

The aim of this chapter is to give a gentle conceptual and historical overview of both the Langlands program and the development of quantum and conformal field theories. The goal is not so much to develop any mathematical background so much as to illustrate to the reader *why* this great web of ideas is important.

The following two sections are adopted from the lectures and notes of [1]. The third and fourth are motivations adopted from the first lecture of [2] together with various ideas of [1].

1.1 The Langlands Program in Number Theory

Fermat's Last Theorem, once known as the "greatest unsolved problem in mathematics," asserts that there does not exist an integral solution to

$$a^n + b^n = c^n, \qquad n > 2 \tag{1.1}$$

with $abc \neq 0$.

The proof of Fermat's last theorem relied on some of the most intricate mathematics developed at the end of the 20th century. A crucial step towards its completion was put forward by Frey and made rigorous by Ribet and Serre. They showed that if the triple (a, b, c) was a solution to (1.1) for and odd prime $n = p$ (which one might assume without loss of generality), then the so-called Frey curve $y^2 = x(x-a^p)(x+b^p)$ contradicted Taniyama-Shimura-Weil conjecture, now referred to as the Modularity Theorem.

Theorem 1.1.1 (Modularity Theorem for Elliptic Curves)**.** *Every elliptic curve is modular.*

Fermat's last theorem follows from a special case of the modularity conjecture. The modularity conjecture for elliptic curves turns out to follow from a special case of a special case of the *Langlands conjectures*, originally formulated by Robert Langlands in a letter to Andre Weil in 1967 [3]. More precisely, it is a corollary of the Langlands correspondence for $G = \mathrm{GL}_2$ over $\mathbb{Q}$ [1]. This part of the Langlands conjecture remains unproven as of May 2018.

We give a sketch of the statement of the number-theoretic Langlands correspondence, intended towards an audience with some background in *Galois theory* and the language of *adeles*.

Begin by considering the **absolute Galois group** of the rationals:

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$, consisting of all algebraic numbers. This Galois group is tremendously large. It is the profinite group obtained as an inverse limit over all finite Galois extensions of $\mathbb{Q}$. It is an open conjecture whether every finite group appears as a Galois group of some Galois extension.

Conjecture 1.1.2 (Inverse Galois)**.** *Every finite group is contained in* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The number theoretic Langlands correspondence considers the n-dimensional representations of the absolute Galois group (called *Galois representations*) and relates them to certain representations known as *automorphic representations*. To define these latter types of representations, we first make the definition

Definition 1.1.3 (Ring of adeles)**.** The **ring of adeles** of $\mathbb{Q}$ is defined as

$$\mathbb{A}_{\mathbb{Q}} := \mathbb{R} \times \prod_{p\,\mathrm{prime}}^{res} \mathbb{Q}_p,$$

where $\mathbb{Q}_p$ denotes the p-adic completion of the rationals [5] ($\mathbb{R}$ can be viewed as the completion at $p = \infty$) and the above product is *restricted* in the sense that:

$$\prod_{p\,\mathrm{prime}}^{res} \mathbb{Q}_p := \left\{ (x_p) \in \prod_{p\,\mathrm{prime}} \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for all but finitely many } p \right\}.$$

Let $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ denote the set of $n \times n$ matrices with entries in $\mathbb{A}_{\mathbb{Q}}$. Further, because $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$ diagonally, we also have

$$\mathrm{GL}_n(\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$$

which yields a left (and right) action[2]:

$$\mathrm{GL}_n(\mathbb{Q}) \circlearrowright \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}) \circlearrowleft \mathrm{GL}_n(\mathbb{Q}).$$

[1] In fact, the modularity theorem is strictly stronger than necessary. It was enough for Wiles and Taylor to prove that a special family (the so-called semistable ones) of elliptic curves is modular. The case for general elliptic curves has since been proven by Breuil, Conrad, Diamond, and Taylor [4].

[2] In this paper we shall use $G \circlearrowright X$ to denote left action of G on X and $X \circlearrowleft G$ to denote right action.

The left quotient space $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ is well-defined in this case. Since $\mathrm{GL}_n(\mathbb{Q})$ still acts by right action on this space, functions of this space form a (left) representation of $\mathrm{GL}_n(\mathbb{Q})$

$$\mathrm{GL}_n(\mathbb{Q}) \circlearrowleft \mathrm{Fun}\left(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})\right)$$

This can be decomposed into irreducible representations, which are known as the **automorphic representations** of $\mathrm{GL}_n(\mathbb{Q})$. Though not absolutely precise, this is a good first-order description of what an automorphic representation is.

Idea 1.1.4. *The Langlands correspondence associates to each n-dimensional representation of the absolute Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *an automorphic representation of* $\mathrm{GL}_n(\mathbb{Q})$.

More than just an equivalence of sets, though, the Langlands correspondence states that a certain set of *eigenvalue data* must agree on both sides.

From the perspective of the absolute Galois group (henceforth referred to as the *Galois side*), this eigenvalue data is called the **Frobenius eigenvalues** of this representation. For p an unramified prime, the Frobenius automorphism $x \to x^p$ is the generator of the Galois group of any finite extension $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. Given a finite-dimensional representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ as well as a conjugacy class, one can lift the Frobenius automorphism to a conjugacy class. The eigenvalues (well-defined for a given conjugacy class) of these elements are the Frobenius eigenvalues of that representation.

From the perspective of the automorphic representations (henceforth referred to as the *automorphic side*), the eigenvalue data is more difficult to describe. It relies on the construction of linear operators on the space of automorphic representations known as **Hecke Operators**. Though a full description of the Hecke eigendata is beyond the scope of this paper, we can give a rough and "cartoonish" picture of the most basic case of Hecke eigenvalues (c.f. [6, 7] for a deeper exposition). In the GL_2 case, the space of automorphic representations is related to the space of modular forms on the upper half plane corresponding to quotients $\Gamma\backslash\mathbb{H}$ with Γ a special type of discrete subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

When $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, a modular form of weight k can be interpretted as a function f on the set of lattices in $\mathbb{R}^2$ so that $f(a\Lambda) = a^k f(\Lambda)$. The mth Hecke operator is then defined as:

$$T_m f(\Lambda) := m^{k-1} \sum_{[\Lambda':\Lambda]=m} f(\Lambda')$$

These are pairwise-commuting linear operators, and can thus be simultaneously diagonalizable. The modular forms that are eigenvectors for this operator are known as **Hecke eigenforms**, and their eigenvalue data is what we define as the **Hecke eigenvalues** of that representation. The story for more general subgroups Γ gives an analogous construction but the story becomes much more involved beyond the GL_2 case.

With this bare background laid out, we can make at least a parsable statement of the Langlands conjecture.

Conjecture 1.1.5 (Langlands)**.** *To each n-dimensional representation of the absolute Galois group, there is a corresponding automorphic representation of* $\mathrm{GL}_n(\mathbb{Q})$ *so that the Frobenius eigenvalues of the Galois representation agree with the Hecke eigenvalues of the automorphic representation.*

It is worth mentioning that the Langlands conjecture over $G = \mathrm{GL}_1$ is the same as what is known in number theory as *class field theory* [1].

Many questions in number theory can be formulated in terms of questions about the nature of the absolute Galois group. On the other hand, automorphic representations can be studied using analytic methods, which would imply that deep number-theoretic data can be made accessible by studying these analytic objects.

The eigenvalue data plays a particularly important role both in the Langlands correspondence and its geometric analogue. The study of this eigenvalue data will become the study of the *geometric Satake* symmetries acting on both sides of the geometric Langlands equivalence, and this thesis will explore how ideas from physics can give a concrete realization of the eigenvalue data in terms of *operator insertions* in quantum field theory [8].

1.2 Weil's Rosetta Stone and Geometric Langlands

The Langlands correspondence in number theory also has a close analogy for curves defined over finite fields $\mathbb{F}_q$. Indeed, translating the number theoretic statements of the Langlands program has over the past fifty years led to an extremely fruitful set of developments in the field of *Arithmetic Geometry*. These developments have led to the famous proofs of the *Weil Conjectures* and the *Riemann Hypothesis over Finite Fields*. We will not discuss these developments here but refer the reader to [9].

We *will*, however, illustrate this anology to function fields over $\mathbb{F}_q$ to motivate the translation of the Langlands program to a more geometric setting. Consider the 1-dimensional affine space $\mathbb{A}^1(\mathbb{F}_q)$. We have $F := \mathbb{F}_q(t)$ the function field on $\mathbb{A}^1(\mathbb{F}_q)$. This will play the role analogous to the role of $\mathbb{Q}$ before. Before, we could complete $\mathbb{Q}$ at each prime p to get the p-adics. For each point $x \in \mathbb{A}^1(\mathbb{F}_q)$, there is a notion of a *completion* for $\mathbb{F}_q(t)$ at x, and also a notion of a *ring of integers* corresponding to the localization $\mathcal{O}_x$ at x.

To understand these completions, we make the following definitions.

Definition 1.2.1 (Formal Power Series)**.** Let $\mathbf{k}[t]$ be a polynomial ring in one variable over a field $\mathbf{k}$. The **ring of formal power series** around x, $\mathbf{k}[[t-x]]$, is defined as the ring of all (possibly infinite) series of the form

$$\sum_{n=0}^{\infty} a_n q^n,$$

where here there is no restriction that only finitely many a_n are nonzero.

Definition 1.2.2 (Formal Laurent Series)**.** Let $\mathbf{k}[t]$ be a polynomial ring in one variable over a field $\mathbf{k}$. The **ring of formal Laurent series** around x, $\mathbf{k}((t-x))$, is defined as the ring of all (possibly infinite) series of the form

$$\sum_{n=-\infty}^{\infty} a_n q^n,$$

where here there is no restriction that only finitely many $a_n, n \geq 0$ are nonzero but *only finitely many* $a_n, n < 0$ *can be nonzero.*

The field F_x corresponding to the completion of F at x can be viewed as the field of Laurent series around x, denoted $\mathbb{F}_q((t-x))$. $\mathcal{O}_x$ can similarly be viewed in terms of formal power series at x, $\mathbb{F}_q[[t-x]]$. With this machinery in place, we can define the ring of adeles analogously to before.

Definition 1.2.3 (Adele Ring for $\mathbb{F}_q(t)$)**.** The ring of adeles of $\mathbb{F}_q(t)$ is defined as

$$\mathbb{A}_{\mathbb{F}_q(t)} := \prod_{x\in\mathbb{P}^1(\mathbb{F}_q)}^{res} \mathbb{F}_{q_x}((t-x))$$

and the above product is restricted as before in the sense that all but finitely many terms in this product over x lie in $\mathbb{F}_q[[t-x]]$. Here the completion at the point at infinity corresponds to $\mathbb{F}_q((1/t))$.

We naturally have that

$$\mathbb{O}_{\mathbb{F}_q(t)} := \prod_{x\in\mathbb{P}^1(\mathbb{F}_q)} \mathbb{F}_{q_x}[[t-x]]$$

sits inside $\mathbb{A}_{\mathbb{F}_q(z)}$.

All of this can be translated more generally to the function field F for a curve C over $\mathbb{F}_p$. This would correspond to a number field and its ring of integers in the original Langlands conjecture. Here, ramification of various points on the curve becomes an issue and there is more subtlety in defining many of the above concepts.

Already, for a function field of a curve C, the analogue of the Galois group is known to be the **etale fundamental group**, and a Galois representation would be a representation of $\pi_1^{\text{ét}}(C,x) \to \mathrm{GL}_n$ in the unramified case. In analytic language for C a complex curve, the étale fundamental group becomes the usual π_1 and a Galois representation becomes a representation of the fundamental group $\pi_1(C) \to \mathrm{GL}_n$.

In the unramified case, automorphic representations correspond exactly the the $\mathrm{GL}_n(\mathbb{O}_F)$-invariant functions on $\mathrm{GL}_n(\mathbb{F})\backslash\mathrm{GL}_n(\mathbb{A}_F)$. This means that the space of automorphic representations corresponds to:

$$\mathrm{Fun}\left(\mathrm{GL}_n(\mathbb{F})\backslash\mathrm{GL}_n(\mathbb{A}_F)/\mathrm{GL}_n(\mathbb{O}_F)\right).$$

It is the following theorem of Weil that will be crucial in making a connection between with the geometric setting over $\mathbb{C}$.

Number Theory	Curves over $\mathbb{F}_q$	Riemann Surfaces
$\mathbb{Z} \subset \mathbb{Q}$	$\mathbb{F}_q[t] \subset \mathbb{F}_q(t)$	$\mathcal{O}_{\mathbb{C}}^{hol} \subset \mathcal{O}_{\mathbb{C}}^{mer}$
$\operatorname{Spec} \mathbb{Z}$	$\mathbb{A}^1_{\mathbb{F}_q}$	$\mathbb{C}$
$\operatorname{Spec} \mathbb{Z} \cup \{\infty\}$	$\mathbb{P}^1_{\mathbb{F}_q}$ (projective line)	$\mathbb{CP}^1$ (Riemann sphere)
p prime number	$x \in \mathbb{A}^1_{\mathbb{F}_q}$	$x \in \mathbb{C}$
$\mathbb{Z}_p$ (p-adic integers)	$\mathbb{F}_q[[t-x]]$ power series around x	$\mathbb{C}[[z-x]]$ holomorphic on formal disk around x
$\mathbb{Q}_p$ (p-adic numbers)	$\mathbb{F}_q((t-x))$ Laurent series around x	$\mathbb{C}((z-x))$ holomorphic on punctured formal disk around x
$\mathbb{A}_{\mathbb{Q}}$ (adeles)	$\mathbb{A}_{\mathbb{F}_q}$ function field adeles	$\prod^{res}_{x \in \mathbb{C}} \mathbb{C}((z-x))$ restricted product of functions on all punctured disks, with all but finitely many extending to the unpunctured disk
$F/\mathbb{Q}$ (number fields)	$F/\mathbb{F}_q(t)$ or $\mathbb{F}_q(C)/\mathbb{F}_q(\mathbb{P}^1)$	$C \to \mathbb{CP}^1$ (branched covers)
$\operatorname{Gal}(\overline{F}/F)$	$\operatorname{Gal}(\overline{F}/F) = \pi_1^{\text{ét}}(\operatorname{Spec} F, \operatorname{Spec} \bar{F})$	
	$\twoheadrightarrow \operatorname{Gal}(F^{\text{unr}}/F) = \pi_1^{\text{ét}}(C, x)$	$\pi_1(C, x)$

Table 1.1: Weil's *Rosetta stone*

Theorem 1.2.4 (Weil Uniformization)**.** *Take F the function field for a curve C over $\mathbb{F}_q$. There is a canonical bijection as sets between*

$$G(\mathbb{F})\backslash G(\mathbb{A}_F)/G(\mathbb{O}_F)$$

and the set of G-bundles over C. Moreover, this in fact extends to an algebraic correspondence between this space and the stack $\operatorname{Bun}_G(C, \mathbb{F}_q)$.

G-bundles are discussed in Section 3.1.3. So (in the unramified case), the automorphic side is captured by functions on $\operatorname{Bun}_G(C, \mathbb{F}_q)$. This set of functions admits an action by the **spherical Hecke algebra** at every closed point $x \in C$, defined as the space of compactly supported functions on the double coset space:

$$\mathcal{H}_x := \operatorname{Fun}_c(\operatorname{GL}_n(\mathcal{O}_x)\backslash \operatorname{GL}_n(F_x)/\operatorname{GL}_n(\mathcal{O}_x))$$

with multiplication given by an operation known as a **convolution product** of functions. These algebras correspond to the Hecke operators described earlier. The action of these algebras at different x turns out to commute with one another, just like the Hecke operators, and be simultaneously diagonalized to give rise to eigenfunctions. This thesis will aim to explore the corresponding interpretation of this action in the context of topological field theory.

Table 1.1, based off of [1] and [10], captures the analogy described above. This is the *function field analogy*, otherwise known as Weil's *Rosetta stone*.

It is the hope and goal of this correspondence that the extremely difficult number-theoretic Langlands program might become more accessible when phrased in the language of the second or third columns of Table 1.1. A reason to believe this might be so is because the power of modern algebraic geometry, as developed by Grothendieck, Serre, Deligne, and others, becomes a prominent force in driving our understanding of columns two and three.

The analogy between columns one and two is especially strong, and in many cases a statement about the second column can be exactly translated over into a statement about the first.

We are now in a place where we can attempt to discuss and motivate the third column: the geometric Langlands correspondence over $\mathbb{C}$. To do this, we will begin with motivation from a different direction, namely Fourier analysis.

1.3 The Fourier Transform and Pontryagin Duality

In this section, we will attempt to give an alternative motivation for the geometric Langlands program as a generalized non-abelian analogue of the Fourier transform.

First let us begin by working with a locally-compact abelian group G. Recall that these possess a unique (normalized) Haar measure. We make the following definition:

Definition 1.3.1 (Unitary Character)**.** For G locally-compact and abelian, a **unitary character** of G is a group homomorphism $\chi : G \to U(1)$.

Using this definition, we define the following group, which plays a role as a *dual* to G. It is called the **Pontryagin dual**.

Definition 1.3.2. The set of all unitary characters χ together with multiplication $\chi_1 \cdot \chi_2 \in \mathrm{Hom}(G, U(1))$ given by pointwise multiplication of characters, form an abelian group, denoted by $\widehat{G}$.

Example 1.3.3. We have the following examples:

1. Let $G = S^1$, then the space of unitary characters consists precisely of these of the form $e^{inx} : G \to U(1)$. This makes $\widehat{G} = \mathbb{Z}$.

2. Let $G = \mathbb{Z}$, then $\chi(1)$ determines the representation uniquely, and so $\widehat{G} = U(1)$.

3. Let $G = \mathbb{R}$, then $e^{ikx} : \mathbb{R} \to U(1)$ is free to have k vary over $\mathbb{R}$ so $\widehat{G} = \mathbb{R}$.

Notice in all these cases that $\widehat{\widehat{G}} \cong G$. This is in fact true more general, and we have the following theorem:

Theorem 1.3.4 (Pontryagin Duality)**.** *For any locally-compact abelian topological group G, the canonical map*

$$\begin{aligned} G &\to \hat{\hat{G}} \\ g &\mapsto [\chi \mapsto \chi(g)] \end{aligned}$$

is an isomorphism.

Observation 1.3.5. *The space of functions*[3] *on G, $\mathrm{Fun}(G)$ has a basis given by characters.*

Example 1.3.6. We have the following examples:

1. $f : S^1 \to \mathbb{C}$ has $f(\theta) = \sum_n a_n e^{in\theta}$. This is known as the **Fourier series**.
2. $f : \mathbb{Z} \to \mathbb{C}$ has $f(n) = \int_0^{2\pi} F(\theta) e^{in\theta}$. This is known as the **discrete time series**.
3. $f : \mathbb{R} \to \mathbb{R}$ has $f(x) = \int_{-\infty}^{\infty} \widehat{f(k)} e^{ikx}$. This is known as the **Fourier transform**.

Let us now try to generalize the ideas of the Fourier transform to a more direct case. It is useful to view the Fourier transform as letting us see two different sides of the same object. Let that object be the direct product of the group G and $\hat{G}$. The reason this space is worth considering is by noting that there is a unique function on this space, which we can call the **kernel** $K : G \times \hat{G} \to \mathbb{C}$ defined by $K(g, \chi) = \chi(g)$. In the case of $G = \mathbb{R}$, this function is exactly $e^{ikx}, x \in \mathbb{R}, k \in \widehat{\mathbb{R}} = \mathbb{R}$, that is viewed as a function on *both* time and frequency space.

This space comes with two obvious projections.

$$\begin{tikzcd} & G \times \hat{G} \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_{\hat{G}}"] & \\ G & & \hat{G} \end{tikzcd}$$

Any function f on G can be "pulled back" to a function on $G \times \hat{G}$, namely by ignoring the second component $f'(g, \hat{g}) = f(g)$. We will denote this pulled back function by $\pi_G^* f = f \circ \pi_G$.

Further, a suitable distribution on $G \times \hat{G}$ can be "pushed forward" to either G or $\hat{G}$ by integrating it over $\hat{G}$ or G respectively. We will denote these by $(\pi_G)_*$ and $(\pi_{\hat{G}})_*$, again respectively.

Now if $\hat{f}$ is a distribution on $\hat{G}$, we get that $\pi_{\hat{G}}^* \hat{f}$ is a distribution on $G \times \hat{G}$. This can be pushed forward to a function on G by integrating over the $\hat{G}$ coordinates, but because $\pi_{\hat{G}}^* \hat{f}$ is constant on the G-coordinate, this function will just be a constant independent of G.

[3] By this, we don't mean $L^2(G)$. $\mathrm{Fun}(G)$ can be taken to mean the space of *tempered distributions* on G, defined as the continuous linear dual of the Schwartz space of functions. See [11].

On the other hand, if we look at:

$$f(g) := (\pi_G)_*([\pi_{\hat{G}}{}^* \hat{f}]K) = \int_{\chi \in \hat{G}} [(\hat{f} \circ \pi_{\hat{G}})(g, \chi)] K(g, \chi)\, d\chi \tag{1.2}$$

we obtain exactly the Fourier transform. For $G = \mathbb{R}$ this gives us:

$$f(x) = \int_{\mathbb{R}} \widehat{f(k)} e^{ikx} dk. \tag{1.3}$$

The reason that the Fourier transform finds so much use in practice is that it serves as an eigendecomposition for the derivative operator. More broadly, on $\mathbb{R}^n$, the eigenfunctions are plane waves $e^{i\vec{k}\cdot\vec{x}}$, which yield eigenvalues both under ∂_x and also under the translation operator more generally $\vec{x} \mapsto \vec{x} + \vec{y}$. Any abelian group acts on itself by translation[4]. Consequently, it acts on the functions living on it, $\mathrm{Fun}(G)$, by translation $f(x) \to f(x - y)$. Note however that the unitary characters satisfy:

$$y \cdot \chi(x) = \chi(x - y) = \chi(-y)\chi(x)$$

so that the characters *diagonalize* the translation operator as an eigenbasis, exactly as e^{ikx} did on the real line.

Fact 1.3.7. *The Fourier transform diagonalizes the action of G on the space of functions $L^2(G) \cong L^2(\hat{G})$.*

We have just treated Fourier analysis successfully for the category of locally-compact abelian groups. A natural next question is:

Question. How could we build upon the ideas Fourier analysis to generalize to non-abelian groups? That is, what could be the non-abelian analogue of the Fourier transform?

Already, one can see that the naive ideas from before will not hold up as well. For one, translation operators no longer commute, and hence cannot be simultaneously diagonalizable with an eigenbasis of unitary characters. As we move to explore the continuous non-abelian setting, the Pontryagin dual group $\hat{G}$ will be replaced by the Langlands dual group ${}^L G$.

It will turn out that to understand the Fourier transform in the non-abelian case, we will have to appeal to *categorification*, which in recent years have proved crucial in many fruitful applications.

1.4 Categorical Harmonic Analysis and Geometric Langlands

As a motivating example of both the algebraic perspective the idea of categorification mentioned in the previous chapter, we will illustrate the **Fourier-Mukai** transform. We will assume basic familiarity with the language of line bundles.

[4] Note that right and left action coincide for an abelian group.

When viewing G as a topological category: a topological space equipped with Haar measure, we consider the space of functions on G $\mathrm{Fun}(G)$. For an algebraic category A, the study of functions on A is often replaced by instead studying *line bundles*, *vector bundles*, or more generally *(quasi-coherent) sheaves* on A.

Let A be an algebraic variety, namely a complex torus of the form $A = \mathbb{C}^g/\Lambda$ such that A is also a projective variety. A is called abelian because it is endowed with the group structure of this torus. We thus have a multiplication operation (along with the two projections):

$$\begin{tikzcd} & A \times A \arrow[dl, "\pi_1"'] \arrow[d, "\mu"] \arrow[dr, "\pi_2"] & \\ A & A & A \end{tikzcd}$$

Just like functions, line bundles can be pulled back along map between varieties. Given a line bundle $\mathcal{L}$ on A, $\mu^*\mathcal{L}, \pi_1^*\mathcal{L}, \pi_2^*\mathcal{L}$ all give line bundles on $A \times A$.

A **geometric character** is a line bundle $\mathcal{L}$ on A such that $\mu^*\mathcal{L} = \pi_1^*\mathcal{L} \otimes \pi_2^*\mathcal{L}$. For geometric characters, there is a canonical isomorphism between $\mathcal{L}_{x+y}$ and $\mathcal{L}_x \otimes \mathcal{L}_y$ given by restricting $\mu^*\mathcal{L}$ to $(x, y) \in A \times A$ and noting that by definition, this must equal $\mathcal{L}_x \otimes \mathcal{L}_y$.

Further, multiplication by an element x gives a map $\mu_x : A \to A$ which is the same as restricting μ to $\{x\} \times A$. Consequently, for a geometric character

$$\mu_x^*\mathcal{L} = \mathcal{L}_x \otimes \mathcal{L}.$$

That is, the group action acts on geometric characters by tensoring each fiber with the 1D vector space of $\mathcal{L}$ at x, $\mathcal{L}_x$. Equivalently, it acts on the line bundle by tensoring it with the trivial line bundle with fiber canonically isomorphic to $\mathcal{L}_x$. Note the similarity between this property of *geometric characters* and the property of ordinary *characters* from before, namely $e^{ik(x+y)} = e^{ikx}e^{iky}$.

It turns out that the set of geometric characters together with the commutative operation $\otimes$ themselves form an abelian variety known as the **dual abelian variety** to A. This is denoted by

$$A^\vee := (\{\text{geometric characters}\}, \otimes)$$

From our birds-eye view of what is going on, it looks like $A^\vee$ is playing an analogous role to $\hat{G}$ of the previous chapter. We have as before the simple diagram

$$\begin{tikzcd} & A \times A^\vee \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"] & \\ A & & A^\vee \end{tikzcd}$$

Just as on $G \times \hat{G}$ there was a universal function K called the kernel from which the Fourier transform was defined, on $A \times A^\vee$ there is a *universal bundle* known as the

number	$\to$	line (vector space in general)
functions on G	$\to$	line bundles on A
vector space of functions/distributions	$\to$	*category* of quasi-coherent sheaves
translations $g : G \to G$	$\to$	translations $\mu_x : A \to A$
$\{e^{ikx}\}_{k\in\hat{G}}$ eigenbasis for translations	$\to$	$\{\mathcal{L}\}_{\mathcal{L}\in A^\vee}$ eigenbasis for translations
eigenvector multiplied by a number	$\to$	eigen-bundle tensored with a line bundle
$e^{ik(x+y)} = e^{ikx}e^{iky}$	$\to$	$\mathcal{L}_{x+y} \cong \mathcal{L}_x \otimes \mathcal{L}_y$
delta function	$\to$	skyscraper sheaf
$\{e^{ikx}\}$ on G is a delta function on $\hat{G}$	$\to$	$\mathcal{L}$ on G is a structure sheaf on $A^\vee$

Table 1.2: The categorification associated to the Fourier-Mukai transform

Poincare line bundle $\mathcal{P}$ so that:

$$\mathcal{P}_{(x,\mathcal{L})} = \mathcal{L}_x.$$

Note that a geometric character $\mathcal{L}$ on A would not correspond to a line bundle on $A^\vee$ but instead to an object with a single fiber at $\mathcal{L} \in A^\vee$ that is zero at all other points. In more precise language, this would be the *structure sheaf* of $\mathcal{L}$ on $A^\vee$. Indeed, the natural objects to consider in place of *functions/distributions on* $G, \hat{G}$ are not line bundles on $A, A^\vee$ but rather objects known as *quasi-coherent sheaves* on these spaces. For a reference about these objects, see [12].

Concept 1.4.1 (Fourier-Mukai Transform)**.** The Fourier-Mukai Transform is a map between the categories of quasi-coherent sheaves:

$$\mathcal{FM} : \mathcal{QC}(A^\vee) \to \mathcal{QC}(A).$$

In terms of the language above, it is given by:

$$\mathcal{F} \mapsto (\pi_1)_*([\pi_2^*\mathcal{F}] \otimes \mathcal{P}).$$

Note the similarity between this and the "classical" or "decategorified" Equation (1.2). In particular the skyscraper sheaf of $\mathcal{L}$ in $A^\vee$, denoted $\mathcal{O}_\mathcal{L}$, is mapped to

$$(\pi_1)_*([\pi_2^*\mathcal{O}_\mathcal{L}] \otimes \mathcal{P}) = \mathcal{L}.$$

The correspondences of this categorification are given in Table 1.2. Note in particular how scalars become vector spaces in this categorification, and how vector spaces become categories.

Everything so far discussed has been about abelian groups, though we have managed to get a much deeper language by using the algebraic picture. This will at least give us some motivation to give a statement of the geometric Langlands conjecture. In the Langlands program, we have G a reductive algebraic group.

Our discussion of the Fourier-Mukai transform would naively lead us to formulate some sort of duality transformation taking us from quasi-coherent sheaves on G to quasi-coherent sheaves on some dual group $\check{G}$. Because the group multiplication is not abelian, the above categorification will not make sense. The correct generalization is more subtle, and the principal geometric objects of study are not G and $\check{G}$.

	Abelian (classical)	Non-abelian (categorified)
Space of "functions"	$\mathrm{Fun}(G) \cong \mathrm{Fun}(\hat{G})$	$\mathcal{D}(\mathrm{Bun}_G) \cong \mathcal{QC}(\mathrm{Flat}_{\check{G}})$
Symmetries acting	$G \circlearrowleft \mathrm{Fun}(G)$	$\mathrm{Sat}_G \circlearrowleft \mathcal{D}(\mathrm{Bun}_G), \mathcal{QC}(\mathrm{Flat}_{\check{G}})$
Eigenbasis	$\{e^{ikx}\}_{t\in\hat{G}}$	Hecke Eigensheaves

Table 1.3: The analogy of the Fourier-Mukai transform as an abelian case of the geometric Langlands correspondence

Taking a hint from the last section, we recall that the Langlands duality for function fields relates certain categories of sheaves on spaces associated to G and $\check{G}$. The automorphic side turned out to correspond to functions on the space of G-bundles, by Weil's uniformization theorem. The Galois side concerned itself with representations of the fundamental group of a curve C into $\pi_1(C)$.

$$\mathcal{D}(\mathrm{Bun_G}(C)) \cong \mathcal{QC}(\mathrm{Flat}_{\check{G}}(C)) \tag{1.4}$$

where Satake symmetries act naturally on both sides. This is supposed to be a nonabelian analogue of the Fourier-Mukai transform, so in particular it should take a skyscraper sheaves on the right (i.e. flat $\check{G}$-connections on C) to a class of D modules known as the "Hecke eigensheaves" on the left.

The original conjecture was formulated by Beilinson and Drinfeld in [13]. This conjecture is true when G is abelian In fact, this conjecture was shown to be false by V. Lafforgue [14]. A refined version of this conjecture is given by Arinkin and Gaitsgory in [15], involving a refinement of the quasi-coherent sheaves on the Galois side to objects known as *ind-coherent* sheaves with a certain support condition.

$$\mathcal{D}(\mathrm{Bun_G}(C)) \cong \mathcal{IC}_N(\mathrm{Flat}_{\check{G}}(C)) \tag{1.5}$$

Though this may seem more complicated, there is reason to believe that these objects can be derived as the right ones to consider on the basis of physical grounds, c.f. [16].

Classical Picture	Geometric Langlands	Topologically twisted $\mathcal{N}=4$ theory
Space of "functions"	$\mathcal{D}(\mathrm{Bun}_G) \cong \mathcal{QC}(\mathrm{Flat}_{\check{G}})$	*Category* of boundary conditions
Symmetries acting	$\mathrm{Sat}_G \circlearrowleft \mathcal{D}(\mathrm{Bun}_G), \mathcal{QC}(\mathrm{Flat}_{\check{G}})$	Insertions of Wilson and ‘t Hooft line defects
Eigenbasis	Hecke Eigensheaves	Electric/Magnetic Eigenbranes

Table 1.4: The connection between the ideas in geometric Langlands and supersymmetric field theory, to be discussed in this thesis.

Although a full discussion of the concepts that appear in Table 1.3 is beyond the scope of this thesis, we can at least give the reader one “final column”, yielding Table 1.4. This column is intended to highlight some key points in the relationship between the concepts of geometric Langlands and physics.

The action of Wilson loops on the Galois side can be very easily understood using the language of holonomy and flat connections, both of which are explained in Section 3.4. On the other hand, the action of the ‘t Hooft operators is much more subtle and involved. To be able to fully appreciate this, we must understand the nature of these so-called “disorder operators” by first understanding the well-known picture of instantons on $\mathbb{R}^4$ and then restricting this to an understanding of monopoles on $\mathbb{R}^3$. Finally, we work in the spirit of Edward Witten’s paper [17] we will make use of our understanding of monopoles and use this to understand the action of line defect operators on boundary conditions in the topological $\mathcal{N} = 4$ theory.

Chapter 2

The Basics of Field Theory

This chapter aims to give a background into the physical ideas needed to understand the remainder of this paper

2.1 Classical Field Theory

Here is a mathematical formulation of classical field theory:

Physical Concept 2.1.1 (Classical Field Theory)**.** A classical field theory $\mathcal{E}$ is a collection of the following data:

- A manifold M known as the **spacetime** of the theory.
- A fiber bundle $E \to M$ (or more generally some set of fiber bundles $E_i \to M$)
- A space $\mathcal{F}$ of sections of $E \to M$ called **fields** on M.
- An action $S[\Phi]$ from the space of fields into $\mathbb{C}$.

Classical field theory studies solutions to the **classical equations of motion**

$$\{\varphi \in \mathcal{F} \mid \ \delta S(\varphi) = 0\}.$$

Example 2.1.2. When $X = \mathbb{R}$, we get a single scalar field ϕ (here Φ is ϕ). An action for this field theory is often given by:

$$S[\phi] = \int_M |\partial_\mu \phi|^2.$$

Example 2.1.3. Classical electromagnetism is defined by $X = T^*M$ with an action given by:

$$S[A] = \int_M F \wedge \star F, \qquad F := \mathrm{d}A.$$

Here $F = dA$ is the *curvature form* or *electromagnetic field-strength tensor*.

More generally, Yang-Mills theory (to be more thoroughly defined and discussed in the next section) takes $X = T^*M \otimes \mathfrak{g}$ and given

$$S[A] = \int_M \mathrm{Tr}\left(F \wedge \star F\right), \qquad F := \mathrm{d}A + A \wedge A.$$

where the trace is taken over the Lie algebra using the Killing form.

2.2 Quantum Field Theory and the Operator-Product Expansion

Though we do not know how to make sense of quantum field theory, the intuitive picture that we have of it is given by the **Feynman Path Integral**. For a given quantum field theory, there is quantity known as the **partition function**, defined as:

$$\mathcal{Z} = \int \mathcal{D}\Phi \, e^{-S[\Phi]}. \tag{2.1}$$

This is an integral taken over the space of all fields. The measure on this space is mathematically ill-defined in general.

Physical Concept 2.2.1 (Classical Observable)**.** A classical observable is a function from the set of field configurations into $\mathbb{C}$.

Physical Concept 2.2.2 (Observable)**.** A **quantum observable** (which we will refer to as just an *observable* in these lectures) is a function from the a field theory into the ground field $\mathbf{k}$. In the Feynman picture, it can be seen as a statistical average of classical observables over all field configurations.

The partition function is an observable, as is the **1-point correlation function** at a point x_1:

$$\langle \Phi(x_1) \rangle := \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \, \Phi(x_1) e^{-S[\Phi]}.$$

In this example, the path integral over all configurations of Φ probes Φ at this single point, giving essentially an expectation value. We can take expectation values of many different operators, e.g. $\phi(x_1), \partial_\mu \phi(x_1), \mathbf{1}, \phi(x_1)\partial_\mu\phi(x_1)$ on X. We denote operators by $\mathcal{O}$. More generally, we define **correlation functions** as

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_g := \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \, \mathcal{O}_1 \dots \mathcal{O}_n e^{-S[\Phi]}.$$

Physical Concept 2.2.3 (TQFT)**.** If the correlation functions of a given quantum field theory are independent of the metric g, then the corresponding theory is called a **topological quantum field theory** (TQFT) in physics.

In fact metric independence implies diffeomorphism invariance.

Example 2.2.4 (Chern Simmons Theory)**.** It turns out the correlation functions of Chern-Simmons theory on a 3-manifold M with Φ being the field $A : M \to T^*M \otimes \mathfrak{g}$ and the action given by

$$S[A] \propto \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

This is clear because the metric has no role in defining the action.

Physical Concept 2.2.5 (Operator Product Expansion)**.** Within the path integral, a product of two local fields can be replaced by a (possibly infinite) sum over individual fields. Namely, given two operators $\mathcal{O}_a, \mathcal{O}_b$ and evaluation points x_1, x_2, there is an open neighborhood U around x_2 such that

$$\mathcal{O}_a(x_1)\mathcal{O}_b(x_2) = \sum_c C_{ab}^c(x_1 - x_2)\mathcal{O}_c(x_2) \tag{2.2}$$

where $\mathcal{O}_c$ are other operators in the quantum field theory, and the C_{ab}^c are analytic functions on $U \backslash \{x_2\}$ (that often become singular as $x_1 \to x_2$).

In the 2D case, this yields the (possibly familiar) Laurent series associated with CFT. The structure constants contain valuable information about the QFT that allow onw to view it *algebraically*. In particular, they satisfy **associativity conditions**. The philosophy of the OPE is as follows:

Idea 2.2.6. *The OPE coefficients, together with the 1-point correlation functions completely determine the n-point correlation functions in a quantum field theory.*

For example, a two-point function is simply given by:

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\rangle = \sum_c C_{ab}^c(x_1 - x_2)\, \langle \mathcal{O}_c(x_2)\rangle \tag{2.3}$$

2.3 Topological Quantum Field Theory

An understanding of topological quantum field theory.

In categorical language, we say:

Definition 2.3.1. A n**-dimensional topological quantum field theory** is a symmetric monoidal functor:

$$\mathcal{Z} : \mathrm{Bord_n} \to \mathrm{Vect}_{\mathbf{k}}$$

Theorem 2.3.2. *The category of 2-dimensional topological quantum field theories is the same as the category of commutative Frobenius algebras.*

In general, besides just considering n-bordisms between $n-1$ manifolds, one might also be inclined to consider the **extended** topological quantum field theory in n-dimensions. These are difficult to define, and would in principle rely on the language of n-categories to give a satisfactory definition.

We can at least summarize

STILL NOT FINISHED

2.4 Supersymmetry

2.4.1 Spin Representations

Given a special orthogonal group in Euclidean or Minkowski space, $\mathrm{SO}(p, q$, the Spin group is defined to be the universal cover of $\mathrm{SO}(p, q)$.

TALK ABOUT S^+ and S^-

Definition 2.4.1. A **Lie superalgebra** is a $\mathbb{Z}_2$-graded Lie algebra with a commutator bracket satisfying:

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

Where $|\cdot|$ is the $\mathbb{Z}_2$ grading.

In our case, we will be extending the familiar *Poincare algebra* of $\mathrm{Lie}\{\mathrm{SO}(3, 1) \ltimes \mathbb{R}^4\}$ by $\mathcal{N}$ "odd" vectors, which transform in the fundamental representation of $\mathrm{SL}(2, \mathbb{C})$, which is a projective *spinor* representation of the Lorentz group. The space of odd vectors is denoted by ΠS.

Definition 2.4.2 (Super-Poincare Algebra)**.** A **super-Poincare algebra**, $\mathfrak{spoin}$, is a Lie superalgebra arising as an extension

$$\Pi S^{\oplus \mathcal{N}} \longrightarrow \mathfrak{spoin} \longrightarrow \mathfrak{poin}$$

of the Poincare algebra $\mathfrak{Poin}$ by the vector space of odd vectors, taken to be in odd degree.

The brackets between the odd vectors $\{Q^A_\alpha, Q^B_\beta\}$ give rise to central elements Z^{AB} in the algebra. These are called *supercharges* and arise as:

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB}.$$

They satisfy

$$Z^{AB} = -Z^{BA}.$$

So that there are a total of $\mathcal{N}(\mathcal{N} - 1)/2$ distinct supercharges in a theory with $\mathcal{N}$ supersymmetry generators.

Definition 2.4.3 (R-symmetry group)**.** The R**-symmetry** group is the group of outer automorphisms of the super-Poincare group which fixes the underlying Poincare group.

For the case of $\mathcal{N} = 4$ the R-symmetry group turns out to be $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$. For a deeper review of the subject, see [18].

Physical Concept 2.4.4 (Sector)**.** Given a supersymmetry operator Q such that $Q^2 = \frac{1}{2}[Q, Q] = 0$, we define the sector of our theory $\mathcal{E}$ associated to Q to the set of Q invariants, and denote this as $(\mathcal{E}, [Q, -])$.

Slightly more precisely, $[Q, -]$ defines a differential operator, and the "observables" become exactly those gauge-invariant quantities annihilated by Q modulo those that are Q-exact.

Chapter 3

Gauge Theory

Gauge theory will play a central role in understanding the geometric Langlands correspondence physically. The role of the group G in the Langlands correspondence is played by the gauge group in the physical theory.

3.1 Fiber Bundles

3.1.1 Definitions and Examples

We will be working on a manifold M (not necessarily Riemannian). In the first definition, we can assume M is just a topological space.

Definition 3.1.1 (Fiber Bundle)**.** We define a **fiber bundle** on a topological space

- A topological space E called the **total space**
- A topological space M called the **base space**
- A topological space F called the **fiber**
- A **projection map** $\pi : E \to M$ that is surjective so that $\pi^{-1}(p) := E_p$ is homeomorphic to F. This is the fiber over p.
- For each $x \in E$ there is an open neighborhood $U \subseteq M$ of $p = \pi(x)$ so that there is a homeomorphism ψ from $U \times F$ to $\pi^{-1}(U)$ in such a way that projection p_1 onto the first factor of $U \times F$ gives π

$$\begin{tikzcd} U \times F \arrow[rr, "\psi"] \arrow[dr, "p_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

Fiber bundles generalize the notion of cartesian products of two spaces M and F by allowing for the same local product structure but much more interesting global "twisted structure". For $p \in M$, the space $E_p := \pi^{-1}(p)$ is called the **fiber over p of E** and is homeomorphic to the fiber F.

In physics, especially when calculations are to be performed, manifolds are often described in terms of a set of coordinate charts U_α that are homeomorphic to $\mathbb{R}^n$ with $n = \dim M$ and $\alpha \in I$ is an index in some indexing set, not necessarily finite[1]. A covering of M in terms of coordinate charts

$$M = \bigcup_{\alpha \in I} U_\alpha.$$

together with homeomorphisms $\psi_\alpha : U_\alpha \to \mathbb{R}^n$ is called an **atlas** for M. In order to make sense of M in terms of an atlas, we define **transition maps** between different U_α that intersect.

Definition 3.1.2 (Transition Map)**.** Given an atlas $\{U_\alpha\}_{\alpha \in I}$, the transition maps $\tau_{\alpha \to \beta} : U_\alpha \to U_\beta$ defined by $\varphi_\beta \circ \varphi_\alpha^{-1}$

By using transition maps, we can transport data locally defined on U_α to other parts of M by "moving it across" other U_β. This data often comes from the fiber bundles over M.

In physics, this perspective is particularly important, as it gives us an ability to "glue together" locally trivial bundles on the U_α to construct a globally nontrivial fiber bundle. For the fiber bundles of interest to us, there will be a group G of automorphisms that acts on the fibers when comparing the data across different U_α. We will later refer to E as an **associated bundle** to G. We define this more precisely:

Definition 3.1.3 (Coordinate Bundle)**.** A **coordinate bundle** consists of

- A fiber bundle, defined as before

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

- A group G, called the **structure group** of E acting effectively on each fiber[2].

[1]But in the case of M compact, I can always be made finite.

[2]A G-action is effective if only the identity element acts trivially i.e. $\forall g \in G \, \exists f \in F \mid gf \neq x$. The reason for this is that if G did not act effectively, then elements that act trivially would give a normal subgroup N. Upon passing to the quotient we would get an effective action of G/N on F.

- A set of open coverings $\{U_\alpha\}_{\alpha\in I}$ of M with diffeomorphisms $\phi_\alpha : U_\alpha \times F \to \pi^{-1}(U_i)$ called **local trivializations** so that the following diagram commutes.

$$\begin{tikzcd} U_\alpha \times F \arrow[rr, "\psi_\alpha"] \arrow[dr, "p_1"'] & & \pi^{-1}(U_\alpha) \arrow[dl, "\pi"] \\ & U_\alpha & \end{tikzcd}$$

- For each $p \in U_\alpha \cap U_\beta$, $\psi_\beta^{-1}\psi_\alpha$ act continuously on the fiber $\pi^{-1}(p)$, coinciding the action of an element of G.

In gauge theory, G is taken to be a **Lie group** called the **structure group** of E.

Definition 3.1.4. A Lie group is a group that is also a differentiable manifold so that the group operations of multiplication and inversion are compatible with the differentiable structure.

A basic working knowledge of Lie theory is assumed, however we will go over relevant aspects of Lie groups in the following sections of this chapter.

Note. In the above, we described φ_α, $\tau_{\alpha\to\beta}$, and $\psi_\beta^{-1}\psi_\alpha$ as *homeomorphisms*, which are indeed morphisms in the category of topological spaces. If we wish to work in other categories, such as C^r-differentiable, smooth, analytic, or complex manifolds, then the transition functions would have to be C^r-differentiable, smooth, convergent Taylor series, or holomorphic respectively. If we were working in the category of algebraic varieties, the corresponding maps we consider would have to be *regular*.

At the fiber over each point, since we can identify $\psi_{\beta,p}^{-1} \circ \psi_{\alpha,p}$ with an element in G, we write $g_{\alpha,\beta} : U_{\alpha\beta} \to G$ to denote the G action fiberwise on the overlap of the two bundles over U_α, U_β. This translates data from one coordinate patch into the other.

Proposition 3.1.5. *$g_{\alpha\beta}$ satisfies*

- *(identity)* $g_{\alpha\alpha} = 1$
- *(inversion)* $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$
- *(cocycle condition) On* $U_\alpha \cap U_\beta \cap U_\gamma$ $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$

The equivalence class of a set of coordinate bundles on M gives the corresponding fiber bundle over M.

Fiber bundles whose fibers are vector spaces are called **vector bundles**. The **rank** of a vector bundle is the dimension of the vector space fiber. A rank n vector bundle over a field $\mathbf{k}$ will have its structure group $G \subseteq \mathrm{GL}_n(\mathbf{k})$. Examples are the tangent/cotangent bundles of a manifold, and any tensor/symmetric/exterior powers thereof. We will see that we can view vector fields, p-forms, and many other interesting and physically-relevant objects as **sections** of fiber bundles, to be described in the later sections.

3.1.2 Morphisms and Extensions

The morphisms in the category of fiber bundles are called **bundle maps**:

Definition 3.1.6 (Bundle Map)**.** For two fiber bundles $\pi : E \to M, \pi' : E' \to M'$ a bundle map is a smooth map $\overline{f} : E \to E'$ that naturally induces a smooth map on the base spaces so that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\overline{f}} & E' \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{f} & M. \end{array}$$

From this we obtain the way which we will identify two bundles as identical.

Definition 3.1.7 (Equivalence of fiber bundles)**.** Two bundles are equivalent if there is a bundle map so that both $\overline{f}$ and f are diffeomorphisms.

If we have a fiber bundle $\pi : E \to M$ and $\varphi : N \to M$ for another manifold N, then we can pull back E to form a bundle over N

$$\varphi^* E = \{(y, [f, p]) \in N \times E \mid \varphi(y) = p\}. \tag{3.1}$$

We have projection on the second factor of $\varphi^* E$ as a map $g : \varphi^* E \to E$. This is the **pullback bundle** $\varphi^* E$.

Definition 3.1.8 (Pullback Bundle)**.** For a map $\varphi : N \to M$ and E a fiber bundle over M so that $\pi : E \to M$, we define the pullback bundle $\varphi^* E$ so that the following diagram commutes:

$$\begin{array}{ccc} \varphi^* E & \xrightarrow{g} & E \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\varphi} & M. \end{array}$$

Let us consider an example which will appear later in the context of studying a monopole placed at the origin of $\mathbb{R}^3$.

Example 3.1.9. Consider a vector bundle over $\mathbb{R}^3 \backslash \{0\}$. The restriction of this vector bundle to the sphere S^2 gives rise to a vector bundle on S^2 which is the same as the pullback bundle induced from $\iota : S^2 \to \mathbb{R}^3$

We can take products of fiber bundles as topological spaces in the obvious way to obtain a fiber bundle over $M \times M'$,

$$E \times E' \xrightarrow{\pi \times \pi'} M \times M'.$$

In the special case where $M = M'$ we can also define

Definition 3.1.10 (Whitney Sum of Vector Bundles)**.** For E, E' vector bundles over M with structure groups G, G' respectively, we can define their sum as $E \oplus E'$ to be pullback bundle $E \times E'$ along the diagonal map $\Delta : M \to M \times M$.

More explicitly, this is a fiber bundle over M with $F \oplus F'$ fibered over every point. The structure group of $E \oplus E'$ is the product $G \times G'$ of the structure groups of the original bundles and it acts diagonally on their sum.

$$G^{E\oplus E'} = \left\{ \begin{pmatrix} g^E & 0 \\ 0 & g^{E'} \end{pmatrix} : g^E \in G, g^{E'} \in G' \right\} \tag{3.2}$$

and the transition functions act diagonally in the same way.

Similarly, we can define arbitrary direct sums of bundles recursively:

$$E_1 \oplus \cdots \oplus E_r. \tag{3.3}$$

For some intuition about when fiber bundles are *nontrivial*, consider the following theorem which we state without proof but refer to [19] chapter 3. Stated simply: taking the pullback of a bundle along a map is topologically invariant under homotopy of the map.

Theorem 3.1.11. *Let $\pi : E \to M$ be a fiber bundle over M and consider maps f, g from $N \to M$ so that f, g are homotopic, then the pullback bundles are equivalent: $f^*E \cong g^*E$ over N.*

An important fact is the following corollary:

Corollary 3.1.12. *If M is contractible, every fiber bundle $\pi : E \to M$ is topologically trivial*[3]*.*

Proof. Let $f : pt \to M$ and $g : M \to pt$ be such that $f \circ g \sim id|_M$ and $g \circ f \sim id|_{pt}$. Then because pullback respects homotopy equivalence, we will have that $E \sim (f \circ g)^*E \sim f^*(g^*E)$ but g^*E is the (necessarily trivial) bundle on a point, so this will pull back along f to the trivial bundle along f. □

3.1.3 Principal Bundles

We have seen that in general, the structure group of a fiber bundle acts effectively on the fibers. More strictly, when G acts freely[4] and transitively[5] *from the right*[6] on the fiber, we can identify F with G. In this case, we get a **principal G-bundle**. This will be an object of central interest in what follows.

[3] In the language of classifying spaces, M being trivial implies there is only one homotopy class of map $M \to BG$, so that consequently the only fiber bundle over M is the trivial one.

[4] A free G-action on F is one where $\forall f \in F, gf = f \Rightarrow g = 1$ i.e. each element has only the identity fixing it. This is a more restrictive form of effective action.

[5] A transitive G-action on F is one with a single G-orbit, i.e. any element can be taken to any other.

[6] The reason for defining this to be a *right* action is so that it can commute with transition maps, which are taken to act from the *left* [20].

Observation 3.1.13. *The fibers of a principal G-bundle are* homeomorphic *to G*

Proof. Let $P \to M$ be a principal G-bundle and pick a point $p \in M$. Take a point $f \in \pi^{-1}(p)$. We can construct a homeomorphism $\varphi : G \to \pi^{-1}(p)$ by sending $g \mapsto pg$. To prove that this is invertible, note that the action is transitive, so φ is certainly surjective. Further, if $pg = pg'$ then $p = pgg'^{-1}$ so necessarily $g = g'$, and we have injectivity as a map between topological spaces. □

Though *topologically* each fiber F of a principal G-bundle looks like G, unlike G, F need not have a canonical choice of identity element, and consequently does not generically have canonical groups structure. Indeed, if it did then the bundle would necessarily have to be the trivial $M \times G$. Such a space, that looks like G after "forgetting" which point is the identity is called a **G-torsor**.

We give a an example for motivation:

Example 3.1.14 (Frame Bundle)**.** The fiber bundle of all **frames**, namely choices of bases in an n-dimensional space is a principal GL_n bundle.

The frame bundle is generally nontrivial.

Example 3.1.15. As another example, taking G to be a discrete group and $\tilde{X} \to X$ be the universal cover of a topological space X, we get that $\tilde{X}$ is a principal G-bundle on X with $G = \pi_1(X)$.

Since G acts transitively on the fiber, there is only one G orbit and we can form the quotient P/G in a well-defined way. We then have that P/G is homeomorphic to M.

Corollary 3.1.16. *For a principal bundle $P(M,G)$ we get* $\dim P = \dim M + \dim G$

If M, F are two manifolds and G has an action $G \times F \to F$, then for an open cover $\{U_\alpha\}$ of M with a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying the conditions of Proposition 3.1.5 we can construct a fiber bundle by first building the set

$$X = \bigcup_\alpha U_\alpha \times F \tag{3.4}$$

and quotienting out by the relation

$$(x, f) \in U_\alpha \times F \sim (x', f') \in U_\beta \times F \iff x = x', f = g_{\alpha\beta}(x) f' \tag{3.5}$$

Then $E = X/\sim$ is a fiber bundle over M. We can locally denote elements of E by $[x, f]$ so that

$$\pi(x, f) = x, \;\; \psi_\alpha(x, f) = [x, f]. \tag{3.6}$$

Proposition 3.1.17. *For a fiber bundle $\pi : E \to M$ with overlap functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ between charts, we can form a principal bundle $P(M,G)$ so that*

$$P = X/\sim, \;\; X = \bigcup_\alpha U_\alpha \times G \tag{3.7}$$

In certain contexts that we will encounter later, the $g_{\alpha\beta}$ are referred to as **clutching functions**.

Example 3.1.18. Take $M = \mathbb{CP}^1$ the Riemann sphere and consider constructing a G-bundle over it. The Riemann sphere can be decomposed as a union of two copies of $\mathbb{C}$ with overlap exactly on the cylinder $\mathbb{C}^\times$. On each copy of $\mathbb{C}$ the G-bundle is trivializable since $\mathbb{C}$ is contractible. A clutching function would be a map $\rho : \mathbb{C}^\times \to G$, and this gives rise to a principal G-bundle on M.

This discussion leads naturally to the next subsection.

3.1.4 Associated Bundles

Take a principal bundle $P(M, G)$ and let F be a space with associated automorphism $\mathrm{Aut}(F)$ so that $\rho : G \to \mathrm{Aut}(F)$ is a *faithful* representation. Then $g \cdot f$ is a well-defined faithful left G-action.

Definition 3.1.19. Given a principal bundle $\pi : P \to M$ and group action $\rho : G \to \mathrm{Aut}(F)$, the **associated bundle** is given by taking the product space $P \times F$ and forming the quotient space:

$$(P \times F)/G$$

given by identifying:

$$(xg, f) \sim (x, \rho(g)f).$$

This is the **fiber product** $P \times_{G,\rho} F$. The projection map:

$$\pi' : (P \times F)/G \to M$$

given by sending (x, f) to $\pi(x)$ is well-defined since $\pi(xg) = \pi(x)$.

Note that the (equivalence classes of) a coordinate bundles in section 3.1.1 gives an associated bundle.

Two associated bundles that we'll care about are $P(M, G) \times_{\mathrm{Ad}} G$ and $P(M, G) \times_{\mathrm{ad}} \mathfrak{g}$. Every fiber bundle with some structure group G arises as an associated bundle to some principal G-bundle.

Importantly, the study of equivalence classes of G-bundles can be equivalently cast as a study of certain associated bundles.

3.1.5 Sections and Lifts

As mentioned before, any specific smooth vector field on a manifold M can be viewed as a smooth map from M to the the tangent bundle of M: TM. This motivates the notion of a **section** of a fiber bundle that associates to each base point $p \in M$ an element f in the fiber E_p. Explicitly:

Definition 3.1.20 (Section of a Fiber Bundle)**.** A **global section** of the fiber bundle $\pi : E \to M$ is a map $s : M \to E$ so that $\pi \circ s = \mathrm{id}$.

When we have, $s : U \subseteq M \to E$, we call s a **local section**. The set of global sections is denoted by $\Gamma(M, E)$. In different contexts, this may mean sections that are continuous, smooth, holomorphic, regular, etc. For smooth sections, this space is often denoted $\Gamma^\infty(M, E)$.

Example 3.1.21. The set of all smooth r-forms on M is $\Gamma^\infty(M, \Lambda^r(T^*M))$ on which the structure group G of T^*M acts on each component.

Proposition 3.1.22. *For a principal bundle $P(M, G)$, any local trivialization $\psi : U \times G \to \pi^{-1}(U)$ defines a local section by $s : p \mapsto \psi(p, e)$ and conversely any local section defines a trivialization by $\psi(p, g) = s(p)g$*

3.2 Lie Theory

Although standard knowledge on the definition of a Lie Group/Algebra is assumed, let's try to motivate the ideas within this field in a more geometric way than is often done.

Consider a manifold M, and consider $\mathrm{Vect}(M)$, the space of all smooth vector fields on M. For a map $\varphi : M \to N$ we have a notion of **pushforward** $\varphi_* : \mathrm{Vect}(M) \to \mathrm{Vect}(N)$ on vector fields given by their actions on functions as

$$[\varphi_*(v)](f) = v(\varphi^*(f)) \tag{3.8}$$

A smooth vector field X on M gives rise to **flows** that are solutions to the differential equation of motion

$$\frac{d}{dt} f(\gamma(t)) = Xf. \tag{3.9}$$

One could argue, more strongly, that in fact the *entire field* of ordinary differential equations has an interpretation as equations of motion along flows of vector fields. Such a viewpoint has brought forward the lucrative insights of symplectic geometry.

The motion along this flow is expressed as the exponential:

$$f(\gamma(t)) = e^{tX} f(p), \; p = \gamma(0) \tag{3.10}$$

Now consider two vector fields X, Y on M. Let Y flow along X so we move along X giving:

$$e^{tX} Y = Y(\gamma(t)) \in T_{\gamma(t)} M \tag{3.11}$$

Note that the reverse flow e^{-tX} maps $T_{\gamma(t)}M \to T_{\gamma(0)}M = T_pM$, so acts by pushforward on $e^{tX}Y$ equivalent to:

$$e^{tX} Y e^{-tX} \in T_p \tag{3.12}$$

We can compare this to Y and take the local change by dividing through by t as $t \to 0$, giving the Lie derivative

$$\mathcal{L}_X Y := \frac{e^{tX} Y e^{-tX} - Y}{t} \tag{3.13}$$

It is easy to check that this is in fact antisymmetric and gives rise to a bilinear form on $\mathrm{Vect}(M)$

$$[X, Y] := L_X Y \tag{3.14}$$

A vector space endowed with such a bilinear form and satisfying the Jacobi identity is a **Lie algebra**.

Most important is when M itself has group structure, so is a Lie group, which we will denote by G. Then the vector fields on G of course also form a Lie algebra, just by virtue of the manifold structure of G.

We state the following proposition without proof

Proposition 3.2.1. *Let $\varphi : G_1 \to G_2$ be a homomorphism of Lie groups, then $\varphi_* : \mathrm{Vect}(G_1) \to \mathrm{Vect}(G_2)$ is a homomorphism of Lie algebras.*

For a Lie group, group elements induce automorphisms on the manifold by left multiplication, denoted L_g and by right multiplication R_g:

$$\begin{aligned} R_g : G \to G,\ g : h \mapsto gh \\ L_g : G \to G,\ g : h \mapsto hg \end{aligned} \tag{3.15}$$

We have that each group element induces (by pushforward) a map between tangent spaces

$$\begin{aligned} (L_g)_* : T_h G \to T_{gh} G \\ (R_g)_* : T_h G \to T_{hg} G \end{aligned} \tag{3.16}$$

A vector field X is left-invariant if $(L_g)_* X(h) = X(gh)$. By the proposition, we get that $(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y]$ so these left-invariant vector fields in fact form a Lie algebra for the group. Physically, this is the set of vector fields corresponding to the isometries of G.

In local coordinates, the commutator can be written as:

$$\begin{aligned} X =& X^\mu \partial_\mu,\ Y = Y^\mu \partial_\mu \\ [X, Y] =& (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu)\partial_\mu. \end{aligned} \tag{3.17}$$

Left-invariant vectors flow in a way that is consistent with the group action:

$$(L_g)_* X(e) = X(g). \tag{3.18}$$

The set of all left-invariant vector fields can be uniquely extracted from their value at the identity by this rule, and in fact for any vector $x \in T_e G$, there is a corresponding left-invariant vector field $X(g) = (L_g)_* x$. Therefore the tangent space to the identity gives rise to a Lie algebra which we will call *the* Lie algebra of G and denote by $\mathfrak{g}$. The Lie algebra of G is finite dimensional when G is and its dimension is equal to the dimension of G.

Now because we define the Lie algebra as the "tangent space to the identity", it is worth asking "how does the Lie algebra appear at a generic point, g, on the group?". The idea is to bring that vector back to the identity using G and see what it looks like.

This is accomplished by using the **Maurer-Cartan form** Θ, which is a $\mathfrak{g}$-valed 1-form on G so that

$$\Theta(g) = (L_{g^{-1}})_*. \tag{3.19}$$

Note that this maps from $\mathrm{Vect}(G) \to \mathfrak{g}$. It takes a vector v at point g and traces it back to the natural vector at the identity that would have gotten pushed forward to v under g.

Proposition 3.2.2 (Properties of exp)**.** *For G a compact and connected Lie group, with Lie algebra $\mathfrak{g}$, we have a map $\exp : \mathfrak{g} \to G$.*

1. $[X, Y] = 0 \Leftrightarrow e^X e^Y = e^Y e^X$
2. *The map $t \to \exp(tX)$ is a homomorphism from $\mathbb{R}$ to G.*
3. *If G is connected then $\exp$ generates G as a group, meaning all elements can be written as some product $\exp(X_1) \dots \exp(X_n)$ for $X_i \in \mathfrak{g}$*
4. *If G is connected and compact then $\exp$ is surjective. It is almost never injective.*

Example 3.2.3. The Lie algebra associated to the Lie group $\mathrm{U}(n)$ of unitary matrices is $\mathfrak{u}(n)$ of antihermitian matrices. This is the same as the Lie algebra for the group $\mathrm{SU}(n)$

Definition 3.2.4 (Adjoint Action on G)**.** For each g we can consider the homomorphism $\mathrm{Ad}_g : h \mapsto ghg^{-1}$ or $\mathrm{Ad}_g = L_g \circ R_{g^{-1}}$. This defines a representation

$$\mathrm{Ad} : g \to \mathrm{Diff}(G) \tag{3.20}$$

Definition 3.2.5 (Adjoint Representation of $\mathfrak{g}$)**.** The pushforward of this action gives rise to the **adjoint representation** of the Lie group $\mathfrak{g}$ by

$$(\mathrm{Ad}_g)_* = (L_g \circ R_{g^{-1}})_* \tag{3.21}$$

From the product rule, this acts as $[g, -]$ at the identity. We denote this as

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\, \mathfrak{g} \tag{3.22}$$

The Jacobi identity ensures that ad is a homomorphism. If the center of G is zero then ad is faithful and we have an embedding into $\mathrm{GL}(n)$. This is nice because it also shows that after a central extension, every Lie algebra can be represented into $\mathrm{GL}(n)$, a weaker form of Ado's theorem.

Moreover the adjoint representation gives rise to a natural metric on $\mathfrak{g}$ called the **Killing Form** given by

$$\kappa(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) \tag{3.23}$$

Proposition 3.2.6. *For $\mathfrak{g}$ a semisimple Lie algebra, the above gives rise to a non-degenerate bilinear form.*

For a proof see [21].

3.3 The Group of Gauge Transformations

We use the ideas form the section on gauge transformation in [22] to build the following definition

Definition 3.3.1 (Gauge Transformation)**.** Let P be a principle bundle over M with structure group G. A diffeomorphism $\Phi : P \to P$ is a **gauge transformation** if it satisfies the following two properties

- Φ preserves fibers so that the following diagram commutes

$$\begin{tikzcd} P \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

- Φ commutes with the right G action on P.

Diffeomorphisms satisfying these conditions form a group referred to as the **group of gauge transformations**[7].

3.4 Connections on Principal Bundles

There are several different and equivalent ways to characterize the notion of a **connection** on a principal G-bundle.

3.4.1 The Ehresman Connection

Take a G-principal bundle $\pi : P \to M$. Just like $\xi \in \mathfrak{g}$ gives rise to a vector field X_ξ on G, it also canonically gives rise to a vector field $\sigma(\xi)$ on P.

Definition 3.4.1 (**Fundamental Vector Field** of ξ)**.** Let $\xi \in \mathfrak{g}$ and consider $\exp(t\xi) \in G$ so that for $p \in P(M, G)$ we get $c_p(t) = R_{\exp(t\xi)}p$ which depends smoothly on p. Note $c_p'(0) \in T_pP(M, G)$ at each point.

$$\sigma : \mathfrak{g} \to \mathrm{Vect}(P(G, M)),\ [\sigma(\xi)](p) \mapsto \left[\frac{d}{dt}pe^{t\xi}\right]_{t=0} \tag{3.24}$$

The **vertical subspace** V_pP at a point p of a fiber bundle is the tangent space at p restricted to the fiber over x, i.e. $T_p(\pi^{-1}(x))|_{F_x}$. Equivalently, this is $\ker \pi_*$. Note

$$\pi_* \circ \sigma(x) = \frac{d}{dt}(\pi \circ c_p(t))|_{t=0} = \frac{d}{dt}(p) = 0 \tag{3.25}$$

so $\sigma(x) \in V_pP$. Since E is a manifold of dimension $\dim M + \dim G$, $\pi_* : T_pE \to T_{\pi(p)}M$ has a kernel of dimension $\dim G = \dim \mathfrak{g}$ In fact:

[7]Some authors may refer to this as the *gauge group*. For us, the gauge group will be the G we started with while this (much larger) group will be denoted $\mathcal{G}$.

Proposition 3.4.2. *σ_p is a Lie algebra isomorphism between $\mathfrak{g}$ and V_pP*

Proof. Since G acts freely on principal bundles, σ is injective, so in fact it must be an isomorphism. □

Lemma 3.4.3 (Properties of σ)**.** *We get that σ satisfies:*

1. $[R_g]_* \sigma(x) = \sigma(ad_{g^{-1}} x)$
2. $[g_i]_* \sigma(x) = g_i(p)x$

Proof. 1. We have

$$\begin{aligned} [R_g]_* \left[\sigma(x)\right](p) &= \frac{d}{dt}(R_g p e^{tx}) \\ &= \frac{d}{dt} pg \mathrm{Ad}_{g^{-1}} e^{tx} \\ &= \frac{d}{dt} pg \exp[t(\mathrm{ad}_{g^{-1}} x)] \\ &= [\sigma(\mathrm{ad}_{g^{-1}} x)](pg) \end{aligned} \tag{3.26}$$

2. And

$$\begin{aligned} [g_i]_* \left[\sigma(x)\right](p) &= \frac{d}{dt} g_i p e^{tx} \\ &= g_i(p)x \end{aligned} \tag{3.27}$$

□

Now σ respects the Lie algebra structure and forms a homomorphism from $\mathfrak{g}$ to $\mathrm{Vect}(P(M,G))$ so that in fact

Corollary 3.4.4. *$(R_g)_* V_p = V_{pg}$: pushforward acts equivariantly on vertical subspaces.*

Proof. Let $X(p) \in V_p$ pick $A \in \mathfrak{g}$ so that the corresponding fundamental vector field is $\sigma(A)(p) = X(p)$. Then we just look at

$$(R_g)_* \sigma(A)(p) = \sigma(\mathrm{ad}_{g^{-1}} A)(pg) \tag{3.28}$$

which is vertical. It's easy to go back from pg to g as well by picking $A \in \mathfrak{g}$ so that $X(pg) = \mathrm{ad}_{g^{-1}} A$. □

Now note:

$$0 \longrightarrow V_pP \longrightarrow T_pP \xrightarrow{\pi_*} T_{\pi(p)}M \longrightarrow 0$$

An injection of $T_{\pi(p)}P$ into P to make the above sequence split is called a **horizontal subspace** H_pP.

Definition 3.4.5 (Horizontal Subspace)**.** A horizontal subspace is a subspace H_pP of T_pP such that

$$T_pP = V_pP \oplus H_pP. \tag{3.29}$$

We'll abbreviate this by H_p and the vertical subspace by V_p when our principal bundle is unambiguous.

Crucially, there is *no canonical choice of* H_p, reflecting the physical fact there is no "god-given" way to compare local gauges between different points. For a gauge g at x, a vector on $T_x M$ should lift to a vector on $T_{[x,g]}P$ given by lifting to a horizontal subspace. A choice of horizontal gives rise to the following:

Definition 3.4.6. An **Ehresmann connection** is a choice of horizontal subspace at each point $p \in P(M, G)$ so that

1. Any smooth vector field X splits as a sum of two smooth vector fields: a **vertical field** X_V and a **horizontal field** X_H so that at each point $p \in P(M, G)$ we have $X_V \in V_p$, $X_H \in H_p$. That is, the choice of H_p varies smoothly.

2. G acts equivariantly on H_{pg}:

$$H_{pg} = (R_g)_* H_p \tag{3.30}$$

We will denote the collection of our choice of H_pP by HP and similarly define VP to be the (always canonical) collection of vertical subspaces. We say any vector field can be split into a vector field $X^H \in HP$ and $X^V \in VP$.

Naturally, for any choice of HP, we have a corresponding projection operator π_H on vector fields $\pi_H : \mathrm{Vect}(P(M, G)) \to HP$ and similarly $\pi_V = id - \pi_H$, both with corresponding equivariance conditions.

3.4.2 Differential Forms on Principal Bundles

Proposition 3.4.7. *We have the following correspondence:*

$$\begin{matrix} \text{Ehresman} & & \text{Horizontal/Vertical} & & \mathfrak{g}\text{-valued} \\ \text{Connections } HP & \longleftrightarrow & \text{Projection Operators } H/V & \longleftrightarrow & \text{1-forms } \alpha \end{matrix}$$

Each of the above are smooth on E, and have appropriate equivariance conditions:

- $R_g H_p = H_{pg}$*: Horizontal subspaces are G-equivariant*
- $[R_g]_* H = H[R_g]$*: Horizontal projection commutes with G action of "changing gauge"*
- $\alpha(pg) = R_g^* \alpha = g^{-1}\alpha(p)g$*: The 1-form is G-covariant*

Definition 3.4.8. Given two $\mathfrak{g}$-valued differential forms α, β of ranks p and q respectively their wedge product is defined as

$$(\alpha \wedge \beta)(v_1, \ldots, v_{p+q}) = \frac{1}{(p+q)!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \left[\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)}), \beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})\right].$$

The following lemmas, which we take from [23], are important in translating from a picture of k-forms on P and k-forms on M.

Lemma 3.4.9. *Let α be a k form on a G-principal bundle $P \to M$. α will descend to a unique k-form $\overline{\alpha}$ on M if the following are satisfied:*

- $\alpha(v_1, \ldots, v_k) = 0$ *if v_i is vertical for any i,*
- $R_g^* \alpha = \alpha$, *i.e.* $\alpha(R_g v_1, \ldots, R_g v_k) = \alpha(v_1, \ldots, v_k)$.

In this case, we will have $\alpha = \pi^* \overline{\alpha}$.

Proof. Let $\{\overline{v}_i\}_{i=1}^k$ be set of k vectors in T_pM and $\{v_i\}_{i=1}^k$ be a set of k vectors in T_xM for any $x \in \pi^{-1}(p)$ so that $\pi_* v_i = \overline{v}_i$. We define

$$\overline{\alpha}(\overline{v}_1, \ldots, \overline{v}_k) := \alpha(v_1, \ldots, v_k)$$

This is well-defined regardless of the choice of $\{v_i\}$ for given $\{\overline{v}_i\}$ since by hypothesis α is zero on the kernel of π_*. It is also independent of the choice of $x \in \pi^{-1}(p)$ by the hypothesis of α's invariance under right G action. $\square$

Lemma 3.4.10. *If $\alpha \in \Omega^1(P, \mathfrak{g})$ descends to a form $\overline{\alpha}$ on M, then we have:*

$$\mathrm{d}_\omega \alpha = \mathrm{d}\alpha \tag{3.31}$$

Proof. This follows from the following manipulation:

$$\begin{aligned}
(\mathrm{d}_\omega \alpha)(v_1, \ldots, v_k) &= (\mathrm{d}\alpha)(hv_1, \ldots, hv_n) \\
&= (\mathrm{d}\pi^*\overline{\alpha})(hv_1, \ldots, hv_n) \\
&= (\pi^*\mathrm{d}\overline{\alpha})(hv_1, \ldots, hv_n) \\
&= (\mathrm{d}\overline{\alpha})(\pi_* hv_1, \ldots, \pi_* hv_n) \\
&= (\mathrm{d}\overline{\alpha})(\pi_* v_1, \ldots, \pi_* v_n) \\
&= (\pi^*\mathrm{d}\overline{\alpha})(v_1, \ldots, v_n) \\
&= (\mathrm{d}\alpha)(v_1, \ldots, v_n).
\end{aligned}$$

$\square$

3.4.3 Holonomy

A particularly important aspect of this thesis will be the action of Wilson loops when inserted into gauge theories. Wilson loops are defined in terms of something known as the **Holonomy**.

The **concatenation** of two paths γ_1, γ_2 such that $\gamma_1(1) = \gamma_2(0)$ is the (piecewise smooth) curve given by

$$\gamma'(t) := \begin{cases} \gamma_(2t) \text{ if } t \leq 1/2 \\ \gamma_(2t-1) \text{ if } 1/2 \leq t \leq 1. \end{cases}$$

Proposition 3.4.11. *Given a principal G-bundle $\pi : P \to M$, consider a smooth path $\gamma : [0,1] \to M$. Given a point $p \in \pi^{-1}(\gamma(0))$, there is a unique lift $\tilde{\gamma}$ so that $\pi(\tilde{\gamma}) = \gamma$ and $\tilde{\gamma}'(t) \in H_{\tilde{\gamma}'(t)}P$. This is called the* ***horizontal lift*** *of γ*

Proof. The result follows by noting that the condition that the lift be horizontal is a first order differential equation with unique specified initial conditions. By smoothness, there exists a unique solution. □

This can be generalized to piecewise smooth curves similarly.

Definition 3.4.12. The **holonomy group** for the connection ω at point $p \in P$, denoted $\mathrm{Hol}_p(\omega)$, is the subgroup of G consisting of elements that are holonomies around some loop $\gamma \subseteq M$.

The **restricted holonomy group** $\mathrm{Hol}^0_p(\omega)$ is analogous, but considers only curves that are *contractible*.

Note that both of these are indeed subgroups, with multiplication of elements corresponding to the concatenation of the associated loops.

3.5 Chern-Weil Theory

In physics, relevant quantities such as the action, the instanton number, and the gauge field Lagrangian are expressed in terms of polynomials of the field strength F. Mathematically, Chern-Weil theory is concerned with the study of polynomials of the curvature form Ω on the associated principal G-bundle. These can be related to the cohomology classes of M.

3.5.1 Symmetric Invariant Polynomials on $\mathfrak{g}$

Consider $\mathfrak{g}$ as an affine algebraic variety ($\cong \mathbb{C}^{\dim \mathfrak{g}}$), and consider the ring of functions $\mathbb{C}[\mathfrak{g}]$. Since $G \circlearrowleft \mathfrak{g}$ by Ad_G-action, we naturally have a G-action on this space of polynomials

$$\mathbb{C}[\mathfrak{g}] \circlearrowleft G.$$

Taking $f(x) \to f(\mathrm{Ad}_g x)$. Polynomials that are fixed by this action are called **invariant polynomials** on $\mathfrak{g}$, and are denoted by $\mathbb{C}[\mathfrak{g}]^G$.

Example 3.5.1. Take $\mathfrak{g} = \mathfrak{gl}_n$. The following are invariant polynomials on $\mathfrak{g}$:

- $\mathrm{Tr}\, x^n$ for any $n \in \mathbb{Z}^+$,
- $\det(x - \lambda \cdot 1)$ for any $\lambda \in \mathbb{C}$.

Invariance under $x \to gxg^{-1}$ follows from the cyclic properties of the trace in the first case and the fact that the determinant map is a homomorphism in the second case.

Definition 3.5.2. A polynomial f on $\mathbb{C}[\mathfrak{g}]$ is called **homogenous** of degree k if $f(ax) = a^k f(x)$ for $x \in \mathfrak{g}, a \in \mathbb{C}$.

Observation 3.5.3. *A homogenous degree k polynomial corresponds to an element of* $\mathrm{Sym}^k(\mathfrak{g}^*)$*: a k-linear symmetric functional $f : \prod_{i=1}^k \mathfrak{g} \to \mathbb{C}$.*

We ask what it would mean to apply f to the $\mathfrak{g}$-valued 2-form Ω. By using Definition 3.4.8 to construct a k-fold wedge products of 2-forms, we get a $2k$ form:

$$f(\Omega)(v_1, \ldots, v_{2k}) = \frac{1}{(2k)!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, f\left(\Omega(v_{\sigma(1)}, \ldots, v_{\sigma(2)}), \ldots, \Omega(v_{\sigma(2k-1)}, \ldots, v_{\sigma(2k)})\right).$$

We now note that $f(\Omega)$ satisfies the requirements of Lemmas 3.4.9 and 3.4.10 so that

$$\mathrm{d}f(\Omega) = \mathrm{d}_\omega f(\Omega).$$

Since d_ω acts as a graded derivation

$$\mathrm{d}_\omega(\alpha \wedge \beta) = (\mathrm{d}_\omega \alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (\mathrm{d}_\omega \beta),$$

and since $\mathrm{d}_\omega \Omega = 0$ we get that $f(\Omega)$ is closed. Further, since $f(\Omega)$ descends to a $2k$-form $\overline{f(\Omega)}$, we get a closed $2k$ form on M, so that

$$[\overline{f(\Omega)}] \in H^{2k}(M). \tag{3.32}$$

We formulate the following proposition:

Theorem 3.5.4 (Chern-Weil)**.** *Let f be an invariant homogenous polynomial of degree k on $\mathfrak{g}$ and Ω be the curvature 2-form associated to some connection ω on a principle bundle P. Then $\overline{f(\Omega)}$ is a representative of a cocycle class in $H^{2k}(M)$ independent of the choice of connection.*

Proof. (Adopted from [24]) We have proved everything other than connection independence. For this, let ω_0, ω_1 be two different connection 1-forms on P. We can perform a homotopy and use the fact that cohomology is homotopy invariant. Consider P as a principal G-bundle on $M \times [0,1]$ and let $\omega' := tp^*\omega_0 + (1-t)p^*\omega_1$ be the 1-form given by pulling back the appropriate combination of ω_0 and ω_1. Then using $\iota_t : M \to M \times [0,1]$ sending $p \to (p,t)$, $f(\Omega')$ can be pulled back from $M \times [0,1]$ to a $2k$-form on M. Since ι_0 and ι_1 are homotopic:

$$\iota_0^* \overline{f(\Omega')} \sim \iota_1^* \overline{f(\Omega')}$$

must lie in the same cohomology class. This are easily seen to be equal to $\overline{f(\Omega_0)}$ and $\overline{f(\Omega_1)}$, respectively. ∎

We have the following corollary.

Corollary 3.5.5. *For a manifold M, $\overline{f(\Omega_1)}$ is locally exact on each coordinate patch. The form K so that $\mathrm{d}K = \overline{f(\Omega_1)}$ on a given U_α is the **Chern-Simmons** form.*

3.5.2 Chern Classes

Let P be a principal G bundle and E be an associated complex vector bundle on which G acts nontrivially. For G semisimple, this can be taken to be the adjoint bundle, but also for G a classical, linear algebraic group, we can take the bundle to be associated to the defining representation. Let n denote the rank of E.

In either case, the curvature form $F \in \Omega^2(M, \mathfrak{g})$ corresponding to a connection on E gives rise to the following polynomial in F that is easily seen to be symmetric-invariant:

$$c(F) := \det(1 - \frac{tF}{2\pi i}) \tag{3.33}$$

This polynomial is not homogenous, but rather splits into a sum of homogenous polynomials in even degree:

$$c(F) = 1 + tc_1(F) + t^2 c_2(F) + \cdots = \sum_{k=1}^{n} t^k c_k(F) \tag{3.34}$$

where $c_k \in \Omega^{2k}(M)$. Clearly $c_k(F) = 0$ if $2k > \dim M$.

By using simple matrix identities such as $\exp \operatorname{Tr} A = \det \exp A$ one can arrive at a more explicit form of the first few of these polynomials

$$c(F) = 1 + i\frac{\operatorname{Tr}(F)}{2\pi}t + \frac{\operatorname{Tr}(F \wedge F) - \operatorname{Tr}(F) \wedge \operatorname{Tr}(F)}{8\pi^2}t^2 + \cdots + \frac{i \det F}{2\pi}t^n$$

By theorem by Theorem 3.5.4, the cohomology classes $[c_i(F)]$ are independent of the connection used to define F. Consequently, we can define

Definition 3.5.6 (Chern class)**.** The **Chern classes** for the bundle E are the cohomology classes in $H^*(M)$ associated with each $c_i(F)$. We write

$$c_i(E) := [c_i(F)].$$

The **Chern numbers** for the bundle E are given by

$$c_i(E) := \int_M c_i(F)$$

and are again independent of the connection.

Proposition 3.5.7. *Let $E = \bigoplus_{j=1}^m E_j$. Then*

$$c(E) = c(E_1) \cdot \cdots \cdot c(E_m)$$

Proof. Because the gauge group acts as block matrices, the field strength tensor can be decomposed into blocks acting separately on each E_j so that the determinant factors:

$$\det\left(I - \frac{tF}{2\pi i}\right) = \det\left(I - \frac{tF_1}{2\pi i}\right) \wedge \cdots \wedge \det\left(I - \frac{tF_m}{2\pi i}\right).$$

Thus, the associated Chern class is a cup product of the cohomology classes corresponding to each differential form in this wedge. $\square$

Chapter 4

Instantons and the ADHM Construction

Instantons are objects of significant interest to both physicists and mathematicians. For physicists, they represent *classical solutions to the equations of motion.* In the context of field theory, and more specifically *Yang-Mills Field Theory*, instantons correspond to nontrivial field configurations on a given spacetime manifold.

Donaldson used the interesting mathematical properties of Yang-Mills instantons on $\mathbb{R}^4$ to prove novel and extremely surprising statements about the nontrivial smooth structures that can be associated to $\mathbb{R}^4$ uniquely among all Euclidean spaces[25].

A useful picture comes from quantum mechanics, of a particle in a double-well potential. Having a particle localized at the bottom of either well gives rise to a classical solution. Perturbative corrections around this minimum due to the quantum theory may give rise to harmonic-oscillator-type structure within the well, but is completely unable to account for the possibility of *quantum tunneling* across the barrier into the second well of the potential. To account for this, we must understand the space of classical solutions in addition to performing perturbation theory.

Mathematically, one way that this can manifest itself is in the fact that $e^{-1/x}$ has every higher derivative vanish as $x \to 0^+$. It is the same phenomenon that allows for the existence of *bump functions* in real analysis and also for *asymptotic expansions* in various areas of physics and engineering.

For the purposes of this thesis, instantons will not themselves play a central role, but their close relatives in three dimensions will: magnetic monopoles. In order to understand the construction of monopoles, however, it will be important to first understand the famous self-duality equation and ADHM construction of instantons.

4.1 Instantons in Classical Yang-Mills Field Theory

4.1.1 The Equations of Motion

Yang-Mills gauge theory is a gauge theory with gauge group $G = \mathrm{SU}(n)$. In four dimensions, the objects of study are G-bundles and associated G-bundles on Euclidean 4-space $M = \mathbb{R}^4$. $\mathbb{R}^4$ has a Riemannian metric, so we have a Hodge-star operator giving an isomorphism:

$$\star : \Omega^k \to \Omega^{n-k}.$$

From the prior section, gauge theory on $\mathbb{R}^4$ involves a connection 1-form A transforming in the $\mathrm{ad}\,\mathfrak{g}$ representation. From this, we obtain the field-strength F, again transforming in the adjoint action, by applying the covariant exterior derivative:

$$F = \mathrm{d}_A A = \mathrm{d}A + [A, A] \tag{4.1}$$

Both F and $\star F$ are $\mathfrak{g}$-valued 2-forms. On the other hand $F \wedge \star F$ is a $\mathfrak{g}$-valued 4-form. Taking the trace of this over the Lie algebra gives a 4-form that can be integrated over M, $\mathrm{Tr}\, F \wedge \star F$. This is equivalently denoted by $||F||^2$ since $\mathrm{Tr}\,(F \wedge \star F)$ corresponds exactly to the inner product norm on $\mathfrak{g}$-valued 2-forms induced by the killing form.

Proposition 4.1.1. $\mathrm{Tr}\,(F \wedge \star F)$ *is gauge independent and globally defined.*

Proof. Since F transforms in the adjoint representation, the cyclic property of the trace gives:

$$\mathrm{Tr}\,(F \wedge \star F) \to \mathrm{Tr}\,(gFg^{-1} \wedge g \star Fg^{-1}) = \mathrm{Tr}\,(F \wedge \star F).$$

□

It is important to recall that the field strength corresponds to a curvature 2-form on some principal $\mathrm{SU}(n)$-bundle, P. Given such a field strength 2-form on M, it can be pulled back to any bundle E associated to P.

In Yang-Mills theory, the action is given by:

$$S[A] := \frac{1}{8\pi} \int_M \mathrm{Tr}\,(F \wedge \star F) \tag{4.2}$$

We aim to find A so that $S_E[A]$ is a minimum. To do this, we use standard calculus of variations. Consider a small perturbation $A + t\alpha$.

$$\begin{aligned}\mathcal{F}[A + t\alpha] &= \mathrm{d}(A + t\alpha) + A \wedge A + t[A, \alpha] + O(t^2) \\ &= \mathcal{F}[A] + t(d\alpha + A \wedge \alpha) \\ &= \mathcal{F}[A] + \mathrm{d}_A \alpha\end{aligned}$$

so that to order t:

$$||\mathcal{F}[A+t\alpha]||^2 = ||\mathcal{F}[A+t\alpha]||^2 + 2t(\mathcal{F}[A], \mathrm{d}_A\alpha)$$
$$\Rightarrow (\mathcal{F}[A], \mathrm{d}_A\alpha) = 0 \ \forall\alpha.$$

The adjoint of the covariant derivative is the codifferential $\star\mathrm{d}_A\star$, so that we can equivalently write this as:

$$\forall\alpha \ (\star\mathrm{d}_A \star \mathcal{F}[A], \alpha) = 0 \Rightarrow \mathrm{d}_A \star F = 0.$$

Except for the case of an abelian gauge theory, these will in general give second-order nonlinear differential equations in the connection that are difficult to solve for explicitly. Though we will not be able to easily talk about general field configurations, we *will* be able to talk about field configurations that are minima for the action on the principal $\mathrm{SU}(n)$ bundle P that the theory is defined on. To do this, we must first understand a connection between a certain integral of the field strength and the topology of P.

4.1.2 The Instanton Number

Though the action is defined by $\int_M \mathrm{Tr}\,(F \wedge \star F)$, we define

Definition 4.1.2 (Instanton Number)**.** The **instanton number** k for a given field configuration is given by

$$k := \int_M \mathrm{Tr}\,(F \wedge F). \tag{4.3}$$

Recall from the definition of Chern classes in 3.5.6 that the Chern numbers are independent of the choice of connection. Recall further that the first few Chern numbers were given by:

$$c_1(E) := \frac{i}{2\pi}\int_M \mathrm{Tr}\,(F) \qquad c_2(E) := \frac{1}{8\pi^2}\int_M [\mathrm{Tr}\,(F \wedge F) - \mathrm{Tr}\,(F) \wedge \mathrm{Tr}\,(F)]$$

Note that since $\mathfrak{su}(n)$ consists of only traceless matrices, c_1 vanishes, and thus for any associated bundle $\mathfrak{su}(n)$-bundle E we have:

$$c_1(E) = 0 \qquad c_2(E) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}\,(F \wedge F) = k.$$

Thus in our case, the instanton number is simply the second Chern class, and in particular is a *topological invariant of the bundle E, independent of the connection.*

4.1.3 The ASD Equations

We are now in a place where we can understand the equations defining the local minima of the action. Note first by basic properties of $\star$ that

$$\star\star : \Omega^2(M, \mathfrak{g}) \to \Omega^2(M, \mathfrak{g}) \tag{4.4}$$

is equal to 1 for $M = \mathbb{R}^4$. This means that this operator has two eigenspaces corresponding to $+1$ and -1, giving a decomposition

$$\Omega^2(M, \mathfrak{g}) = \Omega^2(M, \mathfrak{g})^+ \oplus \Omega^2(M, \mathfrak{g})^-. \tag{4.5}$$

So in general F can be expressed as a sum $F = F_+ + F_-$ of 2-forms in these two spaces. Moreover since these two spaces are orthogonal by the Hermiticity of $\star$, $(F_+, F_-) = 0$. On one hand, then:

$$\begin{aligned} S[A] &= \int_M \mathrm{Tr}\,(F \wedge \star F) \\ &= \int_M \mathrm{Tr}\,((F_+ + F_-) \wedge \star(F_+ + F_-)) \\ &= \int_M \mathrm{Tr}\,(F_+ \wedge \star F_+) + \int_M \mathrm{Tr}\,(F_+ \wedge \star F_+) \end{aligned}$$

Note that the action integral is the integral of $||F||^2$ is is necessarily positive. Now consider the following manipulation:

$$\begin{aligned} 8\pi^2 k &= \int_M \mathrm{Tr}\,(F \wedge F) \\ &= \int_M \mathrm{Tr}\,((F_+ + F_-) \wedge (F_+ + F_-)) \\ &= \int_M \mathrm{Tr}\,(F_+ \wedge F_+) + \int_M \mathrm{Tr}\,(F_- \wedge F_-) \\ &= \int_M \mathrm{Tr}\,(F_+ \wedge F_+) + \int_M \mathrm{Tr}\,(F_- \wedge F_-) \\ &= \int_M \mathrm{Tr}\,(F_+ \wedge \star F_+) - \int_M \mathrm{Tr}\,(F_- \wedge \star F_-) \\ &= \int_M ||F_+||^2 - \int_M ||F_-||^2. \end{aligned}$$

Using the triangle inequality we get:

$$S[A] \geq |8\pi^2 k|. \tag{4.6}$$

It is also easy to see that equality will be satisfied iff $F = F_+$ or $F = F_-$.

We thus have the **anti-self-dual equations** for instantons:

$$\star F = -F, \tag{4.7}$$

or component-wise:

$$\begin{aligned} F_{12} + F_{34} &= 0 \\ F_{14} + F_{23} &= 0 \\ F_{14} + F_{32} &= 0. \end{aligned} \tag{4.8}$$

We see that the instanton number depends on the principal bundle, and that the instanton number of the trivial bundle is zero.

Note. $\mathfrak{su}(n)$-instantons do not exist in Minkowski space $\mathbb{R}^{3,1}$, since $\star^2 = -1$ would have eigenvalues $\pm i$ and $F = \pm iF$ would contradict that F is a real object as an $\mathfrak{su}(n)$-valued 2-form.

4.1.4 Classifying Principal Bundles over S^4

In our above analysis, and the construction of instantons that is to follow, we make several assumptions about F and A.

- For the above integrals to have made sense, we must require that $F(\vec{x})$ decays "sufficiently quickly" as $|\vec{x}| \to \infty$.
- Consequently we must also have A "tend to a constant". In the language of gauge theory, A must become "pure gauge" $g\mathrm{d}g^{-1}$ as $|\vec{x}| \to \infty$.
- We thus restrict the gauge group to consist of only **framed** gauge transformations, defined below.

Definition 4.1.3. A framed gauge transformation on $\mathbb{R}^4$ is one that tends to a constant group element as $|\vec{x}| \to \infty$.

We first change the setting from $\mathbb{R}^4$ to S^4. Because of the decay of the fields, this will still give a well-defined field strength and vector potential on S^4. The following argument is directly from [22].

We will now understand the instanton number in terms of a *clutching function* defined on S^3 connecting the two hemispheres of S^4. First, note that on an open disk, the form $\mathrm{Tr}\,(F \wedge F)$ (by virtue of being locally exact) can be written as

$$\mathrm{dTr}\left[F \wedge A - \frac{1}{3}A^3\right] = \mathrm{Tr}\,(F \wedge F)$$

where $A^3 = A \wedge A \wedge A$. Now take D_N and D_S be two disks overlapping on S^3. The G-bundle must have an overlap function $\rho : S^3 \to G$.

Now the integral becomes:

$$\begin{aligned}
8\pi k &= \int_{S^4} \mathrm{Tr}\,(F \wedge F) \\
&= \int_{D_S} \mathrm{Tr}\,(F_S \wedge F_S) + \int_{D_N} \mathrm{Tr}\,(F_N \wedge F) \\
&= \int_{\partial D_S} \mathrm{Tr}\left[F_S \wedge A_S - \frac{1}{3}A_S^3\right] + \int_{\partial D_N} \mathrm{Tr}\left[F_N \wedge A_N - \frac{1}{3}A_N^3\right] \\
&= \int_{S^3} \left(\mathrm{Tr}\left[F_S \wedge A_S - \frac{1}{3}A_S^3\right] + \mathrm{Tr}\left[F_N \wedge A_N - \frac{1}{3}A_N^3\right]\right).
\end{aligned}$$

After some manipulations, changing A_N, F_N to A_S, F_S by transforming according to ρ, this all reduces to:

$$k = -\frac{1}{24\pi^2}\int_{S^3} \mathrm{Tr}\left((\rho \mathrm{d}\rho)^3\right)$$

and this can now be expressed as the pullback of ρ acting on the Mauer-Cartan form on some SU(2)-homotopic subgroup of G by Bott's theorem. Hence,

$$k = -\frac{1}{24\pi^2}\int_{S^3} \rho^*\mathrm{Tr}\left(\Theta^3\right) = \frac{\deg \rho}{24}\int_{\mathrm{SU}(2)} \mathrm{Tr}\left(\Theta^3\right).$$

On SU(2), the triple wedge of the Mauer-Cartan form gives a volume form whose integral is exactly $24\pi^2$.

Proposition 4.1.4. *The homotopy classes of maps $S^3 \to \mathrm{SU}(2)$ are classified by integers.*

Proof. This follows from noting that $\mathrm{SU}(2) \cong S^3$ and $\pi_3(S^3) = \mathbb{Z}$. □

Consequently, we have our result.

Proposition 4.1.5. *The instanton number k must be an integer equal to the negative of the degree of the clutching function ρ defining the principal G-bundle on S^4.*

With the stage set, will now discuss the method for constructing *all* instantons on $\mathbb{R}^4$. This is the **ADHM construction** of Atiyah, Hitchin, Drinfeld, and Mannin [26].

4.2 Construction of Instantons

In the ADHM construction, we make use of an identification $\mathbb{R}^4 \cong \mathbb{C}^2$.

We will show how this construction will give a bundle E over S^4 with topological charge $-k$. The proof that this exhaustively gives *all* instantons can be found in [27].

4.2.1 Holomorphic and Hermitian Vector Bundles

FINISH THIS

4.2.2 The Data

Let x_1, x_2, x_3, x_4 parameterize a $\mathbb{R}^4$, and write this as $\mathbb{C}^2$ using $z_1 = x_2 + ix_1, z_2 = x_4 + ix_3$. In terms of the complex coordinates, we get

$$\begin{aligned} D_1 &= \frac{1}{2}(\mathrm{d}_{A2} - i\mathrm{d}_{A1}) \\ D_2 &= \frac{1}{2}(\mathrm{d}_{A4} - i\mathrm{d}_{A3}) \end{aligned} \tag{4.9}$$

We can express anti-self duality of $\mathcal{F}_{\mu\nu}$ in terms of these D_μ through two equations:

$$\begin{aligned} [D_1, D_2] &= 0 \\ [D_1, D_1^\dagger] + [D_2, D_2^\dagger] &= 0 \end{aligned} \tag{4.10}$$

Definition 4.2.1 (ADHM System)**.** Let U be a 4-dimensional space with complex structure. An **ADHM System** on $\mathbb{C}^2$ is a set of linear data:

1. Vector spaces V, W over $\mathbb{C}$ of dimensions k, n respectively.
2. Complex $k \times k$ matrices B_1, B_2, a $k \times n$ matrix I, and an $n \times k$ matrix J.

We can see this diagrammatically by the following quiver:

$$\begin{tikzcd} W \arrow[r, bend left, "I"] & V \arrow[l, bend left, "J"] \arrow[loop above, "B_1"] \arrow[loop below, "B_2"] \end{tikzcd}$$

A set of ADHM Data is an ADHM system if it satisfies the following constraints:

1. The ADHM equations:

$$\begin{aligned} [B_1, B_2] + IJ &= 0 \\ [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J &= 0 \end{aligned} \tag{4.11}$$

2. For $(x, y) \in \mathbb{C}^2$ with $x = (z_1, z_2), y = (w_1, w_2)$, the map:

$$\alpha_{x,y} = \begin{pmatrix} w_2 J - w_1 I^\dagger \\ -w_2 B_1 - w_1 B_2^\dagger - z_1 \\ w_2 B_2 - w_1 B_1^\dagger + z_2 \end{pmatrix} \tag{4.12}$$

   is injective from V to $W \oplus (V \otimes \mathbb{C}^2)$ while

$$\beta_{x,y} = \begin{pmatrix} w_2 I + w_1 J^\dagger & w_2 B_2 - w_1 B_1^\dagger + z_2 & w_2 B_1 + w_1 B_2^\dagger + z_1 \end{pmatrix}$$

   is surjective from $W \oplus (V \otimes \mathbb{C}^2)$ to V.

It is an easy check to see

Observation 4.2.2. *If B_1, B_2, I, J satisfy the above conditions, then for $g \in (k), h \in$ SU(n),*

$$(gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1})$$

also satisfies the ADHM equations.

We can recast the ADHM equations into a more succinct form.

Proposition 4.2.3. *The ADHM equations are satisfied iff*

$$0 \longrightarrow V \xrightarrow{\alpha_{x,y}} W \oplus (V \otimes \mathbb{C}^2) \xrightarrow{\beta_{x,y}} V \longrightarrow 0$$

is a complex, namely $\beta \circ \alpha = 0$.

Proof. We need both $\beta\alpha = 0$ as well as surjectivity of β and injectivity of α. The equation $\beta\alpha = 0$ reduces to a quadratic polynomial in the w_1, w_2 with the two ASD equations emerging as coefficients. □

Theorem 4.2.4 (ADHM construction)**.** *There is a one-to-one correspondence between equivalence classes of solutions to the ADHM system and gauge equivalence classes of anti-self-dual* $\mathrm{SU}(n)$*-connections* $\mathcal{A}$ *with instanton number* k*.*

A full proof of this theorem is beyond the scope of this thesis. Nonetheless, we show how such a set of data gives rise to a 2-dimensional $\mathrm{SU}(n)$-associated bundle E over S^4.

Succinctly: the only nontrivial cohomology group of this complex is $\ker \beta_{x,y}/\mathrm{im}\, \alpha_{x,y}$. This gives a vector bundle over $\mathbb{C}^2 \times \mathbb{C}^2$ which can be identified with $\mathbb{H}^2$. An equivariance condition on the data under quaternionic action will let this descend to a vector bundle on $\mathbb{HP}^1 \cong S^4$. This 2D complex vector bundle will be associated to some appropriate principal bundle and have instanton number k.

In quaternionic language, the ADHM equations become easier to work with. To each $x = (q_1, q_2) \in \mathbb{C}^2$, we can associate a quaternionic operator acting on $\mathbb{C}^2$ as:

$$(q_1, q_2) \mapsto z = \begin{pmatrix} \bar{q}_2 & -q_1 \\ \bar{q}_2 & q_2 \end{pmatrix}. \tag{4.13}$$

For $(q_1, q_2) \neq 0$ this is a rank two linear operator.

We can we write the ADHM equations by defining an operator:

$$\Delta_{x,y} := \begin{pmatrix} \beta^\dagger_{x,y} & \alpha_{x,y} \end{pmatrix}. \tag{4.14}$$

Then it is easy to see that (with $x = (z_1, z_2)$ and $y = (w_1, w_2)$)

$$\Delta_{x,y} = aw + bz \tag{4.15}$$

where w, z are the quaternionic matrices corresponding to the complex pairs $(w_1, w_2), (z_1, z_2)$ and

$$a = \begin{pmatrix} I^\dagger & J \\ B_2^\dagger & -B_1 \\ B_1^\dagger & B_2 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 \\ I_k & 0 \\ 0 & I_k \end{pmatrix} \tag{4.16}$$

are the by $n+2n$ by $2k$ matrices, with I_k here denoting the identity. We similarly have [1]

$$\Delta^\dagger_{x,y} = \begin{pmatrix} \beta_{x,y} \\ \alpha^\dagger_{x,y} \end{pmatrix} = (aw+bz)^\dagger. \tag{4.17}$$

Importantly, the kernel of this operator is $\ker\beta_{x,y} \cap \ker\alpha^\dagger_{x,y}$ which can be rewritten as $\ker\beta_{x,y} \cap \mathrm{im}(\alpha)^\perp_{x,y}$. By the definition of orthogonal complement together with $\beta_{x,y} \circ \alpha_{x,y} = 0 \Rightarrow \mathrm{im}\,\alpha_{x,y} \subseteq \ker\beta_{x,y}$, this intersection is seen to be isomorphic to $\ker\beta_{x,y}/\mathrm{im}\,\alpha_{x,y}$.

We see that x, y can be interpretted as two quaternions on $\mathbb{H}^2$. We have an action of the quaternionic operators on this space by $(x,y) \to (xq, yq)$. The space $\ker\Delta^\dagger_{x,y} \to (x,y)$ gives rise to a rank two vector bundle $\tilde{E}$ on $\mathbb{H}^2$. Observe of the following equivariance condition:

$$\Delta^\dagger_{xq,yq} = (awq+bzq)^\dagger = q^\dagger\Delta^\dagger. \tag{4.18}$$

For $q \neq 0$, $q^\dagger$ maintains full rank, so the kernel of $\Delta^\dagger_{xq,yq}$ is the same as the kernel of $\Delta^\dagger_{x,y}$. This means that $\tilde{E}$ descends to a vector bundle on $\mathbb{HP}^1 \cong S^4$.

[1] The notation here is suggestive. $\Delta^\dagger$ is a Dirac operator, and solutions to the ADHM equations are $\Psi(x,y)$ so that $\Delta^\dagger\Psi = 0$.

Chapter 5

Magnetic Monopoles and the Equations of Bogomolny and Nahm

With the machinery of gauge theory and instantons developed, the goal of this chapter is to give the reader a gentle introduction to the notable discoveries in the study of monopoles on $\mathbb{R}^3$.

In section 1 we give two derivations of the Bogomolny equations. The first approach derives the equations directly from the anti-self-duality (ASD) conditions for instanton solutions in $\mathbb{R}^4$ by treating the fourth component of the connection 1-form, A_4, as a scalar field ϕ and ignoring translations ∂_4 along the x_4 direction. The second approach works directly with the action to derive not only the Bogomolny equations but also an integrality condition on the asymptotics of ϕ that allow $\mathfrak{su}(2)$ monopole solutions, much like instantons, to be characterized by a single number k: the magnetic charge[1].

In section 2, we then study the (moduli) space of directed lines on $\mathbb{R}^3$ and make the identification between this space and the (holomorphic) tangent bundle of the Riemann sphere $T\mathbb{CP}^1$. From here, we motivate Hitchin's use of a 1-dimensional scattering equation along a line $(D_t - i\phi)s = 0$ to characterize monopole solutions to the Bogomolny equations as giving rise to a holomorphic vector bundle $\tilde{E}$ over $T\mathbb{CP}^1$ corresponding to the solution space of the scattering equation for a given line. An asymptotic analysis of the solutions to this equation naturally leads to both Hitchin's spectral curve Γ and Donaldson's rational map theorem.

In section 3, we motivate the Nahm transform by analogy to the ADHM construction for instantons from the prior chapter. The story is a little bit more complicated here, since rather than a reduction to linear data, we have a reduction to a Sobolev space of functions on the line segment $(0, 2)$. The Nahm equations are the related to the spectral curve Γ. We finally show how a solution of Nahm's equation gives rise to a monopole solution (A, ϕ) on $\mathbb{R}^3$.

---

[1]For general $\mathfrak{su}(n)$ instantons, $n-1$ numbers are required, associated to the Cartan subalgebra of $\mathfrak{g}$. We restrict to the $\mathfrak{su}(2)$ case, as most authors do, although the generalization of many of these statements to other real Lie groups is not difficult. For the purposes of the Langlands program $\mathfrak{su}(2)$ will play a special role [28].

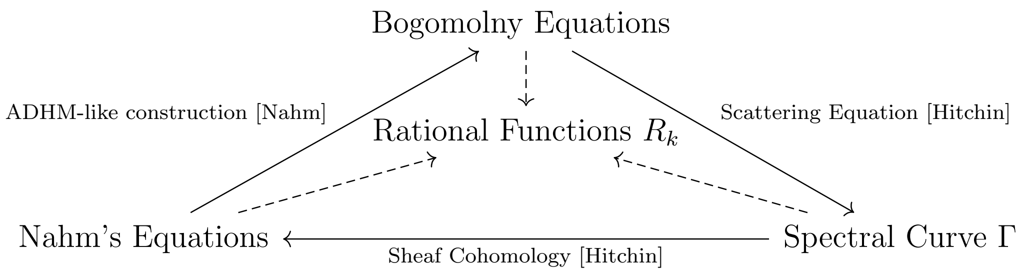

Figure 5.1: The triangle of ideas in the construction of monopoles.

The main ideas relating to understanding the Bogomolny equations can be simply diagrammed in the triangle of Figure 5.1.

Historically, the Bogomolny equations were first introduced by Bogomolny [29] together with Prasad and Sommerfield [30] in their studies of spherically-symmetric single-monopole solutions to nonabelian gauge theories. Explicitly, the $\mathfrak{su}(2)$ single-monopole solution takes the form

$$A = \left(\frac{1}{\sinh|x|} - \frac{1}{|x|}\right)\epsilon_{ijk}\frac{x_j}{|x|}\sigma_k dx^i$$
$$\phi = \left(\frac{1}{\tanh|x|} - \frac{1}{|x|}\right)\frac{x_i}{|x|}\sigma_i$$

where σ_i are the generators of $\mathfrak{su}(2)$ and we are using Eisntein summation convention.

In [31], Hitchin considered the complex structure of geodesics (i.e. directed lines) in $\mathbb{R}^3$ and used this together with the previous scattering ideas in the Atiyah-Ward $\mathcal{A}_k$ ansatz [32] to develop his approch using the spectral curve (righthand arrow in Figure 5.1). In a separate approach, Nahm [33] made use of the ADHM ansatz to formulate the solutions to the Bogomolny equations for $\mathfrak{su}(2)$ in terms of solutions to a coupled system of differential equations, now known as the Nahm equations:

$$\frac{dT_j}{ds}(s) = \epsilon_{ijk}[T_j(s), T_k(s)]$$

where T_i for $i \in \{1,2,3\}$ are $k \times k$-matrix valued functions of s on the interval $(0,2)$, subject to certain conditions. This is the lefthand arrow of Figure 5.1.

The equivalence of these two approaches, corresponding to the bottom arrow in Figure 5.1 was demonstrated by Hitchin in [34]. Hitchin considered the spectral curve of a monopole and constructed a set of Nahm data associated to it, from which one could obtain Nahm's equations. This construction involved methods from sheaf cohomology for the construction of a necessary set of bundles $\mathcal{L}^s$ over $T\mathbb{CP}^1$. This general circle of ideas for $\mathrm{SU}(n)$ monopoles was completed in [35].

Remarkably, these three various descriptions of monopoles can all be related using relatively straightforward constructions to a fourth object: the space of rational functions of a complex variable z with denominator of degree k. This is the rational map constructed by Donaldson [36].

In general, the role of the Nahm transform in understanding the moduli space instanton-like solutions in $\mathbb{R}^4/\Lambda$ for Λ a subgroup of translations in $\mathbb{R}^4$ is as follows:

$$\text{Yang-Mills(-Higgs) on } \mathbb{R}^4/\Lambda \underset{\text{Nahm Transform}}{\Longleftrightarrow} \text{Nahm Equations on } (\mathbb{R}^4)^*/\Lambda^*$$

5.1 Monopoles on $\mathbb{R}^3$

We give here an exposition to magnetic monopoles, following the book of Atiyah and Hitchin [37].

5.1.1 From the Reduction of the ASD Equations

Taking the source-free Yang-Mills equations on $\mathbb{R}^4$, consider solutions that are translation invariant under one coordinate, say x_4. There are two ways forward: either by immediately considering the ASD connections together with translation invariance or by building up the action and seeing how the 3D analogue of the ASD connections emerges.

Observation 5.1.1 (ASD Connection)**.** *The ASD conditions for instantons on $\mathbb{R}^4$ can be explicitly written as*

$$F_{14} = -F_{32}, \quad F_{24} = -F_{13}, \quad F_{34} = -F_{21} \tag{5.1}$$

For F translation invariant w.r.t. x_4, we get

$$\partial_2 A_3 - \partial_3 A_2 + [A_2, A_3] = \partial_1 A_4 + [A_1, A_4] \tag{5.2}$$

and the two other permutations. Taking $A_4 = \phi$ gives that all three of these equations can be written as

$$\star F = \mathrm{d}_A \phi. \tag{5.3}$$

*These are the **Bogomolny equations**. Any solution to this gives us a translation-invariant instanton in $\mathbb{R}^4$. Note that these do not satisfy the decay conditions necessary for the instantons of the ADHM construction.*

5.1.2 From the Yang-Mills-Higgs Action on $\mathbb{R}^3$

To derive an effective action for the $\mathbb{R}^3$ field theory from translation invariance in $\mathbb{R}^4$ we first write:

$$A_{4D} = A_1\, dx^1 + A_2\, dx^2 + A_3\, dx^3 + \phi dx^4.$$

Under the translation assumption, the spatial symmetry group of 4D Euclidean transformations $\mathrm{ISO}(4) = \mathbb{R}^4 \rtimes SO(4)$ reduces down to the 3D group $\mathrm{ISO}(3) = \mathbb{R}^3 \rtimes SO(3)$. With this reduced symmetry, the x^4 component of A (namely ϕ) remains invariant under SO(3) transformations and does not mix with the other three components. Thus, we have a reduction of A from lying in $\Omega^1(\mathbb{R}^4)$, as a fundamental representation of $\mathrm{SO}(4,\mathbb{R})$ fiberwise to lying in an inhomogeneous direct sum $\Omega^1(\mathbb{R}^3)\oplus\Omega^0(\mathbb{R}^3)$ of the fundamental $\mathrm{SO}(3,\mathbb{R})$ representation of SO(3) with the trivial one.

Note that both A and ϕ are still valued in $\mathfrak{g}$ and transform in the adjoint representation. The covariant derivative becomes $(\mathrm{d}_A)_{3D} = \mathrm{d}_{3D} + A$, since $\phi\, dx^4 = 0$ on any vector in $\mathbb{R}^3$. Now note that the 4D curvature form becomes

$$(\mathrm{d}_A)_{3D}(A_{3D}+\phi) = F_{3D} + (\mathrm{d}_A)_{3D}\phi. \tag{5.4}$$

From now on we write F for F_{3D} and d_A for $(\mathrm{d}_A)_{3D}$. The associated action is then

$$S = \frac{1}{8\pi}\int \mathrm{Tr}\left[F\wedge\star F + (\mathrm{d}_A\phi)\wedge\star(\mathrm{d}_A\phi)\right] = \frac{1}{8\pi}\int\left[(F,F) + (\mathrm{d}_A\phi,\mathrm{d}_A\phi)\right]. \tag{5.5}$$

where $(\Omega,\Omega) := \mathrm{Tr}[\Omega\wedge\star\Omega]$ denotes the inner product on p-forms induced by the metric on $\mathbb{R}^3$. From now on, we restrict to the case $\mathfrak{g}=\mathfrak{su}(2)$, though many of the more general results for $\mathfrak{su}(n)$ follow analogously.

Letting B_R be ball of radius R centered at the origin in $\mathbb{R}^3$, we recover the action as the limit of the integral:

$$\lim_{R\to\infty}\frac{1}{8\pi}\int_{B_R}\left[(F-\star\mathrm{d}_A\phi, F-\star\mathrm{d}_A\phi) + 2\,(\star\mathrm{d}_A\phi, F)\right]$$

Before tackling this last term, make the following observations:

Observation 5.1.2. *For the above action to be well-defined, we require $|F(\vec{x})| = O(|x|^{-2})$ and $|\mathrm{d}\phi(\vec{x})| = O(|x|^{-2})$. This implies that the killing norm of ϕ, $|\phi|$, tends to a constant value as $|x|\to\infty$.*

Observation 5.1.3. *If $(A(\vec{x}),\phi(\vec{x}))$ is solution to the equations of motion, then $(cA(\vec{x}/c), c\phi(\vec{x}/c))$ is also a solution.*

For this reason, without loss of generality we may assume $|\phi(\vec{x})|\to 1$ as $|x|\to\infty$. For R large, this makes $\phi|_{S_R} : S^2_R \to S^2$ map from the sphere of radius R in $\mathbb{R}^3$ to the unit sphere S^2 in $\mathfrak{su}(2)$.

Let's make one more observation before tackling the second term

$$\begin{aligned}
\mathrm{d}(\phi,\star F) &= \mathrm{d}\mathrm{Tr}[\phi F]\\
&= \mathrm{Tr}[\mathrm{d}\phi\wedge F - \phi\,\mathrm{d}F]\\
&= \mathrm{Tr}[\mathrm{d}_A\phi\wedge F - \phi A\wedge F + \phi\, A\wedge F]\\
&= (\mathrm{d}_A\phi,\star F)\\
&= (\star\mathrm{d}_A\phi, F)
\end{aligned}\tag{5.6}$$

This implies that the second term can be written as a boundary term:

$$\int_{B_R}(\star\mathrm{d}_A\phi, F) = \int_{S^2_R}\mathrm{Tr}[F\phi]$$

Note ϕ acting on a bundle E transforming in the fundamental representation of $\mathfrak{su}(2)$ has two eigenspaces of opposite imaginary eigenvalues, and by assumption that $|\phi| \to 1$, these eigenvalues cannot both be zero. Thus, they cannot cross and this gives us two well-defined line bundles L_+, L_- over S^2_R corresponding to the positive and the negative eigenvalues.

Proposition 5.1.4. *$E = L_+ \oplus L_-$ has vanishing first Chern class $c_1(E) = 0$.*

Proof. This follows from the fact that $\mathfrak{su}(2)$ is traceless □

Corollary 5.1.5. *The first Chern class of L_+ is $c_1(L_+) = +k$ and L_- is $c_1(L_-) = -k$ for an integer k* [2]*.*

Proof. After picking an orientation so that the first Chern class of L_+ is positive, the corollary immediately follows upon observing that the Chern classes of complex line bundles over the sphere are always integral, and the first Chern class of a direct sum is the sum of the individual first Chern classes. □

Proposition 5.1.6. $\lim_{R\to\infty} \int_{S^2_R} (F, \phi) = \pm 4\pi k.$

Proof. By definition, the first Chern class of a vector bundle E is $\frac{i}{2\pi}\int_{S^R} \mathrm{Tr}(\Omega)$ for Ω the curvature two-form associated to E. Now note that on the eigenbundles of ϕ, we have that since $|\phi| \to 1$, it acts as $\pm i$ (σ_3 up to gauge) so that we must have (from before)

$$\lim_{R\to\infty} i\int_{S^2_R} \mathrm{Tr}(F_{L_+}) - i\int_{S^2_R} \mathrm{Tr}(F_{L_+}) = \pm(2\pi k c_1(L_+) + 2\pi k c_1(L_-)) = \pm 4\pi k. \quad (5.7)$$

□

As we take $R \to \infty$, this proposition gives us an action of

$$S = \frac{1}{8\pi}\int_{B_R} ||F - \star \mathrm{d}_A\phi||^2 \pm k. \quad (5.8)$$

In this case, the absolute minimum is achieved when (A, ϕ) satisfy the following:

Proposition 5.1.7 (Bogomolny Equations). *The monopole solutions for Yang-Mills theory on $\mathbb{R}^3$ satisfy*

$$\star\, F(\vec{x}) = \mathrm{d}_A\phi(\vec{x}) \quad (5.9)$$

subject to the constraints (after rescaling of axes and fields) that:

1. $|\phi(\vec{x})| \to 1 - \frac{k}{2r}$ *as* $|x| = r \to \infty$,

[2] It should be noted that (besides the non-monopole case of $k = 0$), this makes the bundle E nontrivial. This means that E cannot just be the restriction of a (necessarily trivial) vector bundle over $\mathbb{R}^3$. To understand this: the non-triviality of E can be seen to come from singularities induced on the vector bundle by the insertion of monopole. In the $k = 1$ BPS case, this corresponds to E being a nontrivial bundle on $\mathbb{R}^3\backslash\{0\}$

2. $\partial|\phi(\vec{x})|/\partial\Omega = O(r^{-2})$, *where* Ω *denotes any angular variable in polar coordinates,*

3. $|\mathrm{d}_A\phi(\vec{x})| = O(r^{-2})$.

The norm $|\phi|$ *is the standard killing norm on* $\mathfrak{g} = \mathfrak{su}(2)$. *These equations can also describe* $\mathfrak{su}(n)$ *monopoles, with adapted decay conditions.*

Note under $\phi \to -\phi$ we get that the Bogomolny equations with $k \leq 0$ become the anti-Bogomolny equations and $F = -\star \mathrm{d}_A\phi$ and $k \geq 0$. Further, spatial inversion together with $A \to -A$ can flip these to the Bogomolny equations with $k \geq 0$. Therefore, it is enough look at solutions to the Bogomolny equations for $k \geq 0$.

Definition 5.1.8 (Magnetic Charge)**.** The positive integer k is called the **monopole number** or **magnetic charge** of the monopole solution.

Though our analysis has been for $\mathfrak{su}(2)$, the $\mathfrak{u}(1)$ case has the same equations characterizing a monopole solution.

Observation 5.1.9. *Note when* $\mathfrak{g} = \mathfrak{u}(1)$, *and using the notation* $B_k = \epsilon_{ijk}F_{ij}$ *the Bogomolny equation becomes* $B = \nabla\phi$, *giving the first known magnetic monopole, the* ***Dirac Monopole****:*

$$\phi = \frac{k}{2r}.$$

Note. We aim to study the solutions of the Bogomolny equations modulo the action of the gauge group $\mathcal{G}$. However, not all gauge transformations preserve the decay conditions on $\mathrm{d}_A\phi$ and $|\partial\phi/\partial\Omega|$. Consequently, we study the Bogomolny equations modulo the restricted gauge group $\tilde{\mathcal{G}}$ of transformations that tend to a constant element g as $|x| \to \infty$.

5.2 Hitchin's Scattering Equation, Donaldson's Rational Map, and the Spectral Curve

5.2.1 The moduli spaces N_k and M_k

We make the following notational definition

Definition 5.2.1. Let N_k be the space of gauge-equivalent $\mathfrak{su}(2)$ monopoles of magnetic charge k.

This is our main object of study in what follows.

This section involves studying the solutions of "scattering-type" equations along directed lines in $\mathbb{R}^3$. Consequently, the covariant derivative operator when restricted to a line, say along a line parallel to the x_1 axis, becomes:

$$\mathrm{d}_A \to \frac{d}{dx_1} + A_1 \tag{5.10}$$

In this case, we can make a gauge transformation

$$A \to gAg^{-1} + g^{-1}dg$$

so as to make $A_1 = 0$. This simplifies the covariant derivative along lines parallel to the x_1 axis to become just $\mathrm{d}_A \to \frac{d}{dx_1}$.

A copy of $U(1)$ still remains to act on A_2 and A_3. Thus, as $x_1 \to \infty$, because the decay conditions on ϕ, we have that any gauge transformation tends to a constant element in this $U(1)$ subgroup. In this context, define:

Definition 5.2.2 (Framing)**.** Define a **framed gauge transformation** [34, 38] to be one that tends to the identity as $x_1 \to \infty$.

If we only identify solutions modulo *framed* gauge, then the asymptotic $U(1)$ element as $x_1 \to \infty$ will differentiate between solutions that are otherwise equivalent modulo the full gauge group. We thus make a definition

Definition 5.2.3. Define M_k to the the space of solutions to the Bogomolny equations modulo framed gauge. This is fibered over N_k with fiber S_1

$$S_1 \hookrightarrow M_k \twoheadrightarrow N_k$$

Proof. We have seen that upon choosing $A_1 = 0$, gauge transformations can still have an asymptotic value in a $U(1) \cong S^1$ subgroup. Thus, quotienting out by only *framed* gauge transformations to get M_k leaves a piece of S^1 information that N_k does not have. We will call this S^1 element the *phase* of a given monopole solution. □

Note. M_k depends on a choice of oriented x_1-axis in $\mathbb{R}^3$. A more coordinate-free way of defining this extension M_k of N_k is given in [37]. It relies on a simple observation from the previous section that asymptotically the restriction of E over S^2_R is a direct sum of k-twisted bundles: $E_k = L_{-k} \oplus L_k$. The automorphism group in SU(2) fixing this direct sum is exactly the $U(1)$ diagonal action:

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Thus, up to this $U(1)$ automorphism determining phase, every k-monopole solution is asymptotically equivalent to a fixed E_k. Informally: restricting the gauge transformation group so as to retain this automorphism information gives us $M_k = N_k \times S^1$.

5.2.2 Hitchin's Scattering Transform

In [31] Hitchin made use of a scattering method to show the following equivalence:

Theorem 5.2.4 (Hitchin)**.** *Given a solution (A, ϕ) to the Bogomolny equations satisfying the criteria of 5.1.7, then let ℓ be a directed line in $\mathbb{R}^3$ pointing along a direction $\hat{n}$ with distance parameterized by t and consider the following **scattering equation** along ℓ*

$$(D_{\hat{n}} - i\phi)\psi = 0. \tag{5.11}$$

Here $D_{\hat{n}}$ is a restriction of the covariant derivative d_A to act along ℓ, ϕ is the scalar field restricted to ℓ, and ψ is a section of the restriction of the vector bundle E associated to the fundamental representation $\mathbb{C}^2$ to the line ℓ.

The solutions to this equation form a complex two-dimensional space $\tilde{E}_\ell$ of sections. If A, ϕ satisfy the Bogomolny equations, then $\tilde{E}_\ell$ is a holomorphic vector bundle over the space of directed lines in $\mathbb{R}^3$.

There are several propositions that need to be developed before this theorem can be made sense of. Firstly,

Proposition 5.2.5. *The space of directed lines in $\mathbb{R}^3$ forms a complex variety isomorphic to the tangent bundle to the Riemann sphere $T\mathbb{CP}^1$ with a real structure σ.*

Proof. Once a normal direction $\hat{n}$ is chosen, a directed line ℓ in $\mathbb{R}^3$ is uniquely determined by a vector $\vec{v} \perp \hat{n}$. Thus our space is

$$\{(n, v) : |n| = 1, u \cdot v = 0\} \tag{5.12}$$

Clearly $\hat{n}$ sits on a sphere S^2 and $(\hat{n}, v)$ form TS^2. It is sufficient to find a complex structure to make this into the complex variety $T\mathbb{CP}^1$. We will form a complex structure on $\mathbb{CP}^1$ and then this lifts to one on the tangent bundle. The complex structure J acting on a point (n, v) is given by taking $J(v) = \hat{n} \times v$. This corresponds exactly to the complex structure on the holomorphic tangent bundle of the Riemann sphere.

The real structure σ comes from reversing the orientation of a line $(\hat{n}, v) \to (-\hat{n}, v)$. It is easy to see $\sigma^2 = 0$, and since it reverses orientation in $\mathbb{R}^3$ is takes $J \to -J$. □

Example 5.2.6. To make this picture clearer for the reader, let's note that given a point (x_1, x_2, x_3), each direction $\hat{n}$ has a unique line $(\hat{n}, v)$ passing through this point. Thus, a point $\vec{x} \in \mathbb{R}^3$ determines a section $s : \mathbb{CP}^1 \to T\mathbb{CP}^1$. Explicitly, picking a local coordinate ζ on $\mathbb{CP}^1$ we get:

$$s(\zeta) = ((x_1 + ix_2) - 2x_3\zeta - (x_1 - ix_2)\zeta^2)\frac{d}{d\zeta}. \tag{5.13}$$

The fact that the coefficient is a degree 2 polynomial in ζ is a consequence of the tangent bundle being a bundle of degree 2 over $\mathbb{CP}^1$. Note further that this corresponds to describing $\mathbb{R}^3$ as the space of real holomorphic vector fields on the Riemann sphere, namely $\mathfrak{so}(3, \mathbb{R})$.

Next, let us try to study this scattering equation. It will be useful to restrict, without loss of generality, to lines parallel to the x_1 axis.

Proposition 5.2.7. *The solutions to the scattering equation on a line form a two dimensional space.*

Proof. In the gauge $A_1 = 0$ developed before, this is an easy consequence of the fact that E is rank two and so upon decomposing E into eigenspaces of ϕ, $L_+ \oplus L_-$, the scattering equation decouples into two linear differential equations:

$$\left[\frac{d}{dx} - i\lambda_j(x_1)\right] s_j = 0, \qquad j = 1, 2. \tag{5.14}$$

Because these equations are both linear and first-order, they each have a one-dimensional space of solutions. □

We can now understand the vector bundle that Hitchin constructed on $T\mathbb{CP}^1$.

Observation 5.2.8. *Let $\tilde{E} \to T\mathbb{CP}^1$ denote the two-dimensional space of solutions to the scattering equation associated to a given line in $\mathbb{R}^3$. This forms a vector bundle.*

We are now ready to prove Hitchin's theorem.

Proposition 5.2.9 (Construction of a Holomorphic Vector Bundle)**.** *If (A, ϕ) satisfy the Bogomolny equations, then $\tilde{E}$ is holomorphic.*

Proof. Hitchin appeals to a theorem of Nirenberg [39]: that it is sufficient to construct an operator

$$\bar{\partial} : \Gamma(T\mathbb{CP}^1, \tilde{E}) \to \Gamma(T\mathbb{CP}^1, \Omega^{(0,1)}(\tilde{E})).$$

The existence of $\bar{\partial}$ on $\tilde{E}$ would give $\tilde{E}$ a holomorphic structure for which $\bar{\partial}$ plays the role of the anti-holomorphic differential. Let s be a section of $\tilde{E}$ for a given directed line ℓ in $\mathbb{R}^3$. Let t be the coordinate alone this line an x, y be orthogonal coordinates in the plane perpendicular to ℓ. In this case, define:

$$\bar{\partial} s = [D_x + iD_y]\, s(dx - idy). \tag{5.15}$$

Where D_x, D_y are shorthand for the x and y components of the covariant derivative d_A.

It is easy to show that this operator satisfies the Leibniz rule together with $(\bar{\partial})^2 = 0$, but we must show that it is *well-defined* as an operator from $\Gamma(T\mathbb{CP}^1, \tilde{E}) \to \Gamma(T\mathbb{CP}^1, \Omega^{(0,1)}(\tilde{E}))$. Namely, we must show that it fixes $\tilde{E}$, meaning that:

$$\left(\frac{d}{dt} - i\phi\right)(D_x + iD_y) = 0. \tag{5.16}$$

But this can be written as the requirement that the following commutator vanishes:

$$\begin{aligned} 0 &= \left[\frac{d}{dt} - i\phi, D_x + iD_y\right] = F_{12} + iF_{13} - D_y\phi + iD_x\phi \\ &\Rightarrow F_{12} = D_y\phi \quad F_{31} = D_x\phi. \end{aligned} \tag{5.17}$$

These are exactly the Bogomolny equations, as desired. We have thus shown that Hitchin's construction works. □

5.2.3 The Spectral Curve

Given the above discussion, it is worth trying to understand what the solutions of this scattering equation mean. We know from before that the null space of the scattering operator consists of two linearly independent solutions, s_0 and s_1. Let us look at their asymptotics. Again, let ℓ be a line parallel to the x_1 axis with $A_1 = 0$. Then

Proposition 5.2.10. *As $t \to \infty$, the two solutions to Hitchin's scattering equation are combinations of the following two solutions:*

$$s_0(t) = t^{k/2} e^{-t}\, e_0, \qquad s_1(t) = t^{-k/2} e^{t}\, e_1 \tag{5.18}$$

where e_0 and e_1 are constant vectors in E in the asymptotic gauge.

Proof. Since $A_1 = 0$, the scattering equation becomes

$$\frac{d}{dt} - i\phi = 0. \tag{5.19}$$

Using asymptotics on ϕ from the prior section, we get

$$\frac{d}{dt} - i\left(1 - \frac{k}{2t}\right)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + O(1/t^2) = 0. \tag{5.20}$$

This yields two differential equations:

$$\frac{d}{dt} + \left(1 - \frac{k}{2t}\right) + O(1/t^2) = 0, \qquad \frac{d}{dt} - \left(1 - \frac{k}{2t}\right) + O(1/t^2) = 0, \tag{5.21}$$

which in turn yield two solutions as $t \to \infty$:

$$s_0(t) \to t^{k/2} e^{-t}\, e_0, \qquad s_1(t) \to t^{-k/2} e^{t}\, e_1. \tag{5.22}$$

□

Note that (by t-reversal symmetry) we must have the same type of solutions as $t \to -\infty$. Namely, there is a basis where one solution blows up as $t \to -\infty$ and the other decays to zero. The solution that decays to zero, s', must necessarily be some linear combination of the $t \to \infty$ solutions s_0 and s_1. We thus have:

$$s' = as_0 + bs_1. \tag{5.23}$$

In the special case that $b = 0$, we get that s' decays not only as $t \to -\infty$ but also as $t \to \infty$. Physically, this is called a **bound state**.

Physical Concept 5.2.11 (Bound state)**.** A bound state $\psi(\vec{x})$ is a state of a physical system that decays "sufficiently quickly" (i.e. as $e^{-|x|}$) as $|x| \to \infty$). It captures the notion of a localized particle.

Since the linear combination for s' is a relationship between sections of a holomorphic line bundle, the ratio a/b is a well-defined meromorphic function on $T\mathbb{CP}^1$. Fixing $\hat{n}$, the poles of this function generically give k points on $T_{\hat{n}}\mathbb{CP}^1$. Letting $\hat{n}$ vary gives Hitchin's **spectral curve** Γ on $T\mathbb{CP}^1$. Note this is a k-fold cover of $\mathbb{CP}^1$, and an application of the Riemann-Hurwitz formula would yield that Γ in fact has genus $k-1$. We will illustrate more on why this curve deserves its name using the Nahm transform in section 4.

Hitchin gives the following theorem, which we will state without proof:

Theorem 5.2.12 (Hitchin)**.** *If two monopole solutions $(A, \phi), (A', \phi')$ have spectral curves Γ', Γ, then (A, ϕ) is a gauge transform of (A', ϕ').*

Note that here there is no assumption on framing. The spectral curve itself does not carry information about the phase of the monopole solution. On the other hand, the section s' associated to a given line for a monopole solution gives rise to a distinguished line bundle $\mathcal{L}$ over Γ, alongside the standard restriction of the vector bundle $\widetilde{E}$ to Γ.

Note that Γ is holomorphic and *real* in the sense that it is preserved by the real structure σ on $T\mathbb{CP}^1$.

The proof that a spectral curve satisfying the conditions imposed on Γ will give rise to a monopole solution is done by going through the Nahm equations. As mentioned before, Hitchin [34] showed using ideas from sheaf cohomology that a spectral curve on $T\mathbb{CP}^1$ naturally gives rise to a set of Nahm data from which the Nahm equations can be constructed. In this way, the construction of monopoles goes in the direction of Figure 5.1.

5.2.4 The Rational Map

Let $x_1 = t$ and $z = x_2 + ix_3$. Let ℓ be a line parallel to the x_1 axis. Note it is determined by its intersection z with the x_2, x_3 plane. a and b are as before: the linear combination of $s' = as_0 + bs_1$, the solution decaying as $t \to -\infty$.

It is a powerful result of Donaldson [36] that tells us: for a fixed direction x_1 we not only obtain a meromorphic function of the lines ℓ parallel to x_1, namely $S(z) = a(z)/b(z)$, but that in fact *any* meromorphic function on $\mathbb{CP}^1$ with denominator degree k has an interpretation as a k-monopole solution. This rational function depends on the point of M_k specifying the monopole. In this sense it is *almost* gauge invariant, except for the S^1 phase associated to it. The poles of this rational function correspond to when the solution has $s' = s_0$ from before, namely a bound state.

We state Donaldson's result:

Theorem 5.2.13 (Donaldson)**.** *For any $m \in M_k$, the scattering function S_m is a rational function of degree k with $S_m(\infty) = 0$. Denote this space of rational functions by R_k. The identification of $m \to S_m$ gives a scattering map diffeomorphism $M_k \to R_k$.*

Example 5.2.14. For $k = 1$ we have R_k takes functions of the form $\frac{\alpha}{z-\beta}$, which turns out to correspond to a monopole at $(\log 1/\sqrt{|\alpha|}, \mathrm{Re}(\beta), \mathrm{Im}(\beta))$. The argument of α describes the $U(1)$ phase at $t \to \infty$. This means M_1 has complex structure $\mathbb{C} \times \mathbb{C}^{\times}$.

Example 5.2.15. For higher k, in the generic case a rational function in R_k will split as a sum of simple poles

$$\sum_i \frac{\alpha_i}{z - \beta_i}.$$

This has the interpretation of monopoles having centers at positions $(\log 1/\sqrt{|\alpha_i|}, \mathrm{Re}(\beta_i), \mathrm{Im}(\beta_i))$ and phases described by the arguments of the α_i.

5.3 The Nahm Equations

5.3.1 Motivation

By adopting the monad construction of ADHM, Nahm succeeded in adapting their formalism to solving the 3D Bogomolny equation. The idea of Nahm (and indeed, the idea behind the Nahm transform more broadly) was to recognize monopoles on $\mathbb{R}^3$ as solutions to the anti-self-duality equations in $\mathbb{R}^4$ that were invariant under translation along one direction, and then appropriately modify ADHM to account for the different decay conditions and symmetries of the configuration.

We present a review of the ADHM construction from the prior section. In what follows, a **quaternionic vector space of dimension** k is taken to mean k copies of $\mathbb{C}^2$, $\mathbb{C}^{2k}$, where each copy has quaternionic structure.

Review. The ADHM construction for $\mathfrak{su}(2)$ starts with W a real vector space of dimension k and V a quaternionic vector space of dimension $k+1$ with inner product respecting the quaternionic structure. Then, for a given $x \in \mathbb{R}^4$ it forms the operator:

$$\Delta(x) : W \to V. \tag{5.24}$$

The operator $\Delta(x)$ is written as $Cx + D$ where C, D are constant matrices and $x \in \mathbb{H}$ is viewed a quaternionic variable once a correspondence is made $\mathbb{R}^4 \cong \mathbb{H}$.

If Δ is of maximal rank, then the adjoint $\Delta^*(x) : V \to W$ has a one-dimensional quaternionic subspace E_x that, as x varies, can be described as a bundle over $\mathbb{H} \cong \mathbb{R}^4$. The orthogonal projection to E_x (viewed as a horizontal subspace) in V defines the (Ehresman) connection on the vector bundle $E \to \mathbb{R}^4$. [34]

Here, we will use the zero-indexed (x_0, x_1, x_2, x_3) to label the coordinates so that the imaginary quaternionic structure of the latter three becomes more clear. Nahm's approach [33] was to seek vector spaces W, V fulfilling the same function, and look for the following conditions:

1. $\Delta(x)^*\Delta(x)$ is real and invertible (as before).

2. $\ker \Delta(x)^*\Delta(x)$ has quaternionic dimension 1 (as before).

3. $\Delta(x + x_0) = U(x_0)^{-1}\Delta(x)U(x_0)$.

This last point is equivalent to the translation invariance of the connection in x_0, up to gauge transformation.

Because of this new condition, unlike the case of ADHM, V and W turn out to be infinite dimensional. Consequently, Δ, Δ^* become differential (Dirac) operators.

5.3.2 Construction

To construct V, first consider the space of all complex-valued L^2 integrable functions on the interval $(0, 2)$. Denote this space by H^0 (this notation coming from the fact that this is the zeroth Sobolev space). This space has a real structure coming not only from $f(s) \to \bar{f}(s)$ but also from $f(s) \to \bar{f}(2-s)$. Define $V = H^0 \otimes \mathbb{C}^k \otimes \mathbb{H}$, where $\mathbb{C}^k$ is taken to have a real structure.

Similarly, we define W by considering the space of functions whose derivatives are L^2 integrable. This will be denoted by H^1 (again with motivation deriving from a corresponding Sobolev space concept). Define

$$W = \{H^1 \otimes \mathbb{C}^k : f(0) = f(1) = 0\}.$$

Now define $\Delta : W \to V$ by

$$\Delta(x)f = i\frac{df}{ds} + x_0 f + \sum_{i=1}^{3}(x_i e_i + iT_i(s)e_i)f, \tag{5.25}$$

where e_i denote multiplication by the quaternions i, j, k respectively and $T_i(s)$ are $k \times k$ matrices. It is clear that this operator is the form $Cx + D$ with $C = 1$ and $D = i\frac{d}{ds} + i\sum T_j e_j$.

Using the language of [34] we make the following proposition

Proposition 5.3.1. *The following hold:*

1. *The requirement that Δ is quaternionic implies $T_i(s) = T_i(2-s)^*$.*

2. *The requirement that Δ is real implies $T_i(s)$ are anti-hermitian and also that $[T_i, T_j] = \epsilon_{ijk}\frac{dT_k}{dt}$.*

3. *The requirement that Δ is invariant under x_0 translation is automatically satisfied*

4. *The requirement that Δ^* has kernel of quaternionic dimension 1 comes from requiring that the residues of T_i at $s = 0, 2$ form a representation of* SU(2)

Proof. The first two are relatively straightforward to see. The new condition follows immediately from

$$\begin{aligned} e^{ix_0(s-1)}[\Delta(x)]e^{-ix_0(s-1)}f &= e^{ix_0(s-1)}\left[i\frac{d}{ds}+\dots\right](e^{-ix_0(s-1)}f) \\ &= \Delta(x)f + x_0 f \\ &= \Delta(x+x_0)f. \end{aligned} \tag{5.26}$$

The last item states that since the residues of a $k \times k$ matrix valued functions are themselves $k \times k$ matrices, that in fact the commutation relations of these residue matrices at $s = 0$ and 2 form k-dimensional representations of SU(2). This requires a bit of work, and can be found in [34]. □

We thus have the following data:

$T_1(s), T_2(s), T_3(s)$ $k \times k$ matrix-valued functions for $s \in (0, 2)$ satisfying

$$\frac{dT_i}{ds} + \epsilon_{ijk}[T_j, T_k] = 0. \tag{5.27}$$

together with the requirements

1. $T_i(s)^* = -T_i(s)$
2. $T_i(2-s) = -T_i(s)$
3. T_i has simple poles at 0 and 2 and is otherwise analytic
4. At each pole, the residues T_1, T_2, T_3 define an irreducible representation of $\mathfrak{su}(2)$.

These are **Nahm's equations**.

For a given solution of Nahm's equations, the associated Dirac operator $\Delta^*(x)$, depending on a chosen $\vec{x}$, can be shown to again yield a 1-dimensional quaternionic (2-dimensional complex) kernel E_x. Here, though, it does not specify a connection on $\mathbb{R}^4$ but instead gives rise to A and ϕ through the following way construction:

Construction 5.3.2 (3D Monopole from Nahm's Equations)**.** Pick an orthonormal basis of $E_x = \ker \Delta^*(x) \cong C^2$. Call this v_1, v_2. We view E_x as a fiber at x corresponding to a $\mathbb{C}^2$ bundle, and construct ϕ and A by their actions on a given v_a at x.

$$\begin{aligned} \phi(\vec{x})(v_a) &= i\frac{v_1}{||v_1||_{L^2}}\int_0^2 (v_1, (1-s)v_a)ds + i\frac{v_2}{||v_2||_{L^2}}\int_0^2 (v_2, (1-s)v_a)ds, \\ A(\vec{x})(v_a) &= \frac{v_1}{||v_1||_{L^2}}\int_0^2 (v_1, \partial_i v_a)ds + \frac{v_2}{||v_2||_{L^2}}\int_0^2 (v_2, \partial_i v_a)ds. \end{aligned} \tag{5.28}$$

5.3.3 The Spectral Curve in Nahm's Equations

For any complex number ζ we can make a definition:

$$\begin{aligned} A(\zeta) &= (T_1 + iT_2) + 2T_3\zeta - (T_1 - iT_2)\zeta^2, \\ A_+ &= iT_3 - (iT_1 + T_2)\zeta. \end{aligned} \tag{5.29}$$

Nahm's equations can then be recast as:

$$\frac{dA}{ds} = [A_+, A]. \tag{5.30}$$

This is the **Lax Form** of Nahm's equations. This can be solved by considering the curve $\mathbf{S}$ in $\mathbb{C}^2$ with coordinates (η, ζ) defined by

$$\det\left(\eta - A(\zeta)\right).$$

Proposition 5.3.3. *The above equation is independent of s.*

Proof. Let v be an eigenvector of A and let it evolve as $\frac{dv}{ds} = A_+ v$. Then

$$\frac{d(Av)}{ds} = [A_+, A]v + AA_+v = A_+Av = \lambda A_+ v, \tag{5.31}$$

so this gives

$$\frac{d}{ds}(A - \lambda v) = 0. \tag{5.32}$$

Since $A - \lambda v = 0$ at $s = 0$, it is always zero. Thus, this curve of eigenvalues is independent of s. □

It is in fact a remarkable result that:

Proposition 5.3.4. *The curve $\mathbf{S}$ constructed above is the same as the spectral curve Γ constructed previously.*

Hitchin showed this by associating to a given spectral curve Γ a set of Nahm data in [34].

5.4 The Nahm Transform and Periodic Monopoles

The Nahm transform is a nonlinear generalization of the Fourier transform, related to the Fourier-Mukai transform. It allows for the construction of instantons on $\mathbb{R}^4/\Lambda$. Some examples are below:

1. $\Lambda = 0$: ADHM Construction of Instantons on $\mathbb{R}^4$,
2. $\Lambda = \mathbb{R}$: The monopole construction that this paper has described,
3. $\Lambda = \mathbb{R} \times \mathbb{Z}$: Periodic monopoles on $\mathbb{R}^3$ (calorons, c.f. [40]),
4. $\Lambda = (\mathbb{R} \times \mathbb{Z})^2$: Hitchin system on a torus.

Chapter 6

The Physical Picture

The aim of this chapter is to first develop for the reader a picture of $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory together with its topological twists. With this, we bring together the ideas of the previous chapters and study the actions of line defects on the categories of boundary conditions of two topological twists of $\mathcal{N} = 4$ SYM.

6.1 Setting the Stage

6.1.1 Reduction from Ten Dimensions

One of the simplest ways to arrive at 4D $\mathcal{N} = 4$ SYM is to begin with gauge theory in 10 dimensions with gauge group G [8]. In the 10D theory, we have two fields, A and λ. A is a connection on a G-bundle E while λ transforms as what physicists would call a "positive chirality spinor with values in the adjoint representation", so $\lambda \in \Gamma(M, S^+ \otimes E)$. We have $F = \mathrm{d}_A A$.

"Bosonic" will be taken to mean terms consisting of only the connection A and its derivatives. "Ferminionic" will be taken to mean terms involving a the spinor λ. For more information and motivation see [41].

The generator of supersymmetry is a spinor in S^+. This theory has 16 supercharges Q_a. The *supersymmetric variation* of the fields are given by:

$$\sum_a [\epsilon^a Q_a, A_I] = i\bar{\epsilon}\Gamma_I \lambda$$
$$\sum_a [\epsilon^a Q_a, \lambda] = \Gamma^{IJ} F_{IJ} \epsilon$$

The action here is:

$$S = \frac{1}{e^2} \int \mathrm{Tr}\, \left(F_{10D} \wedge \star F_{10D} - i\overline{\lambda}\Gamma \mathrm{d}_A \lambda \right) \tag{6.1}$$

The reduction to 4 dimensions is done in a similar manner to how we proceeded in Chapter 5. Namely, we assume all fields are independent of the last six direction.

This gives us a connection $A = A_\mu d^\mu$ in 4D together with six scalar fields ϕ_i. The curvature F_{10} decomposes into three terms. The first is the curvature in 4D, F, the second consists of covariant derivatives of the ϕ_i, $\mathrm{d}_A\phi_i$, and the last consists of commutators $[\phi_i, \phi_j]$. All together, the bosonic part of the action can be written in physics convention as:

$$\frac{1}{e^2}\int_M \mathrm{Tr}\left(F \wedge \star F + \sum_i \mathrm{d}_A\phi \wedge \star(\mathrm{d}_A\phi) + \sum_{i<j}[\phi_i, \phi_j]^2 \mathrm{Vol}_M\right) \tag{6.2}$$

The fermionic part can be similarly decomposed into Weyl spinors λ^a to give the full action. On $M = \mathbb{R}^4$ the theory has 16 supersymmetries $\overline{Q}_{a,\alpha}$ and $Q^a_{\cdot\alpha}$ transforming as spinors and dual spinors in the R-symmetry group Spin(6).

Physical Concept 6.1.1. $\mathcal{N} = 4$ Super-Yang Mills theory is the field unique theory of maximal supersymmmetry in four dimensions.

6.1.2 Montonen-Olive Duality

Concept 6.1.2 (Montonen-Olive Duality)**.** In 4D $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group G and complex coupling constant τ, any correlator of observables

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_{\tau,G} := \int \mathcal{D}\{Fields\}\, \mathcal{O}_1 \dots \mathcal{O}_n\, e^{-S}$$

can be rewritten in terms of Yang-Mills theory with inverse coupling constant $-1/\tau$ on the Langlands dual group $\check{G}$ as a correlator of dual operators $\tilde{\mathcal{O}}_1 \dots \tilde{\mathcal{O}}_n$

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_{\tau,G} = \left\langle \tilde{\mathcal{O}}_1 \dots \mathcal{O}_n \right\rangle_{-1/\tau,\check{G}}.$$

In particular $\mathcal{N} = 4$ super Yang-Mills theory has a $\mathbb{CP}^1$ family of topological twists. Two of these will be relevant here, known as the $\hat{A}$-model and the $\hat{B}$-model[1]. This twisting introduces an asymmetry between G and $\check{G}$.

6.1.3 Topological Twisting

First recall from Chapter 2 Section 2.4 the following definition:

Definition 6.1.3 (Subsector)**.** Given a supersymmetry operator Q such that $Q^2 = \frac{1}{2}[Q, Q] = 0$, we define the subsector of our theory $\mathcal{E}$ by the set of Q invariants, and denote this as $(\mathcal{E}, [Q, -])$.

Slightly more precisely, $[Q, -]$ defines a differential operator, and the "observables" become exactly those gauge-invariant quantities annihilated by Q modulo those that are Q-exact.

[1] This notation comes from the fact that, upon compactification down to two dimensions, these models become the A and B topological sigma models discussed before

Definition 6.1.4 (Topological Twist)**.** Given a supersymmetric (SUSY) field theory $\mathcal{E}$, a topological twist is a procedure for extracting a sector of $\mathcal{E}$ that depends only on the topology of the spacetime manifold. The resulting field theory is **topological** in the definition of Section 2.3

In general this involves a homomorphism from the universal cover of the structure group of the spacetime tangent space TM to the R-symmetry group. For our four-dimensional $\mathcal{N} = 4$ case this is

$$\rho : \mathrm{Spin}(4) \to \mathrm{Spin}(6)$$

This redefines how the fields transform under the cover of the Lorentz group, Spin(4). We have an equivalence-class of obvious embeddings.

$$\mathrm{Spin}(4) \hookrightarrow \mathrm{Spin}(6)$$

given by:

$$\begin{pmatrix} * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and this is what Geometric Langlands will be concerned with.

After twisting by ρ, the group Spin(4) acts differently on the supersymmetry operators. In particular one of the left-handed and one of the right-handed supersymmetries become scalars under Spin(4). We thus get scalars Q_l, Q_r, and any linear combination of these gives rise to a different "sector" of invariants. Clearly overall scaling does not matter, so we have $\mathbb{P}^1(\mathbb{C})$ of subsectors to chose from.

Proposition 6.1.5. *Any such subsector defines a theory that is independent of the Riemannian metric (i.e. diffeomorphism invariant). The path integral localizes to Q-invariant configurations.*

6.2 Wilson Lines

In general, the connection 1-form, A, gives a way to transport data along any given vector bundle E associated to a representation R of G. This allows us to compare the values of fields operators at different points by integrating along E using our connection. Recalling the definition of holonomy from Section 3.4.3 The result is:

$$W_R(\gamma) = \exp\left(\int_\gamma A\right) \tag{6.3}$$

This classical operator is called a **Wilson line**. Wilson lines transform (under a general transformation $g \in \mathcal{G}$), as:

$$W_R(\gamma) = g(\gamma(1))W_R(\gamma)g(\gamma(0))^{-1} \tag{6.4}$$

in the special case of γ closed, we see this is gauge-invariant. In this case, it called a **Wilson loop**. It can be viewed as yielding an element of the group G in the representation R. In this case, the trace of this element gives an invariant scalar quantity (known in physics as a c-number), and so for γ closed we further add a trace.

Definition 6.2.1 (Wilson Loop)**.** Given a field theory with gauge group G and a finite-dimensional representation R of G together with a closed loop γ, we define the Wilson loop operator:

$$\mathcal{W}_R(\gamma) := \operatorname{Tr} R(\operatorname{Hol}(A, \gamma)). \tag{6.5}$$

The algebra of Wilson loops is simple. For $\gamma \to \gamma'$ the operator product expansion gives us that

$$\lim_{\gamma \to \gamma'} W_R(\gamma)W_{R'}(\gamma') = \sum_\alpha n_\alpha W_{R_\alpha}(L'). \tag{6.6}$$

In our picture, let M be a 4-manifold and let $L \subset M$ be an oriented 1-manifold embedded in M. On the $\hat{B}$-twist, we can consider taking the holonomy of the connection $\mathcal{A}$ along L, when L is closed, giving us a Wilson loop.

Proposition 6.2.2. *The $\hat{B}$ model condition on the flatness of $\mathcal{A}$ implies that the holonomy of the Wilson loop only depends on the homotopy class of L*

If M has boundaries, we can let L be an open 1-manifold connecting two ends of M. Then, the Wilson operator will give us matrix elements between the initial and final states of the theory. Because Wilson operators geometrize $\operatorname{Rep}(\check{G})$, the space of physical states living on the boundary of M is exactly $\check{R}$ for some $\check{R} \in \operatorname{Rep}(\check{G})$. A Wilson loop connecting boundary components gives us a matrix element between initial and final vectors in $\check{R}$.

In the G theory: the $\hat{A}$-twist, A and ϕ instead obey a different equation:

$$F - \phi \wedge \phi = \star D_A \phi. \tag{6.7}$$

This equation is analogous to the equation of motion for the 2D A models. We will see how the Bogomolny equations for magnetic monopoles arise as a special restriction of this equation in the next section.

From the above discussion, we should ask

Question. What is the dual operator to a Wilson line?

From the physics viewpoint, ‘t Hooft showed in the 1980s that MO duality will exchange a Wilson line (a type of “order operator”) on one side with something known as a ‘t Hooft line (a type of “disorder operator”) on the other side.

We can intuitively understand the insertions of ‘t Hooft lines in the path integral as imposing divergence conditions on the curvature form F so that in local coordinates $x^1 \dots x^3$ perpendicular to the line we have

$$F(\vec{x}) \sim \star_3 d\left(\frac{\mu}{2r}\right) \tag{6.8}$$

where μ is an element of the lie algebra $\mathfrak{g}$. It turns out that for us to be able to find a gauge field A whose curvature F satisfies this condition, we must have that μ is a Lie algebra homomorphism $\mathbb{R} \to \mathfrak{g}$ obtained as the pushforward of a Lie group homomorphism $U(1) \to G$.

Another way to say this is (after using gauge freedom to conjugate μ to a particular Cartan subalgebra) that μ must lie in the coweight lattice Λ_{cw}. In fact the ‘t Hooft operator remains the same after the action of the Weyl group $\mathcal{W}$ on μ so we have that ‘t Hooft operators are classified by the space:

$$\Lambda_{cw}(G)/\mathcal{W}.$$

But this is also the same as

$$\Lambda_w(\check{G})/\mathcal{W}.$$

We know that this is the space of representations of the Langlands dual group.

Proposition 6.2.3. *‘t Hooft operators in gauge group G are classified by irreducible representations of $\check{G}$.*

The operator product expansion of Wilson lines captures the monoidal category structure of $\mathrm{Rep}(\check{G})$. By duality, this category must also be capturing the OPE of ‘t Hooft lines. Can we say anything about the OPE of ‘t Hooft lines in terms of G?

6.3 Operator Product Expansion of ‘t Hooft Lines

6.3.1 Reduction to 3D

Because the operator product expansion is a local process, we can assume our base manifold looks like anything. It turns out to be fruitful to take $X = I \times C \times \mathbb{R}$. Here, I is the unit interval $(0, 1)$, C is a Riemann surface (which we can take to be $\mathbb{CP}^1$ WLOG) and $\mathbb{R}$ is regarded as the “time” direction and adopt a Hamiltonian point of view on $W = I \times C$.

The boundary conditions on I matter here, and it turns out that in the $\hat{A}$ model we should consider *Dirichlet* boundary conditions on one end and *Neumann* boundary conditions on the other. In the language of gauge theory, Dirichlet boundary conditions demand the bundle to be trivial on that boundary, while Neumann boundary conditions allow for it to be arbitrary.

Now ‘t Hooft lines look like points on the 3-manifold $W = I \times C$. We can locally take $\phi = \phi_4 dx^4$ so that on W, ϕ behaves as a scalar. Then, on W, Equation (6.8) reduces exactly to the Bogomolny equations for monopoles:

$$F = \star_3 D_A \phi.$$

Let's write a local coordinate $z \in \mathbb{C}$ parameterizing C and $\sigma \in \mathbb{R}$ parameterizing I. Gauging away $A_\sigma = 0$, these equations reduce to the following:

$$\partial_\sigma A_{\bar{z}} = -i D_{\bar{z}} \phi.$$

This condition can be interpreted as stating that the isomorphism class of the holomorphic G-bundle corresponding to the connection $A_{\bar{z}}$ is independent of y. This is because the right hand side corresponds to changing A by a gauge transformation generated by $-i\phi$. Thus, gauge transforming $A \to A + i\phi$ gives us a holomorphic connection on the new G-bundle, putting it in the same holomorphic class.

The only place where this is violated is at the values σ where the Bogomolny equations become singular. This is where we have the insertion of a 't Hooft operator.

It is worth noting that this construction follows very closely the inverse scattering approach of Hitchin[31][37]. In that case, the curve C corresponded to the (non-compact) Riemann surface $\mathbb{C}$ parameterizing the $x_1 - x_2$ plane, and lines along the x_3 direction take the place of our s variable along the unit interval I.

6.3.2 The Affine Grassmannian

The Langlands dual is defined to have the property that any highest weight representation $\hat{\rho} : \hat{G} \to U(1)$ is dual to a morphism $\rho : U(1) \to G$ which can be viewed as a *clutching function* for a G bundle on the Riemann sphere $\mathbb{CP}^1$. Complexifying this gives $\rho : G \to \mathbb{C}^\times \cong \mathbb{CP}^1 \backslash \{p, q\}$, AKA gluing a trivial bundle over $\mathbb{CP}^1 \backslash \{p\}$ to a trivial bundle over $\mathbb{CP}^1 \backslash \{q\}$. This is exactly what we call a Hecke modification of type ρ. Every holomorphic G-bundle over $\mathbb{CP}^1$ arises in this way. We can recognize this space of Hecke modifications as the affine Grassmannian $Gr_G = G((z))/G[[z]]$.

6.3.3 The Space of Physical States

MOTIVATE THE NEXT PARAGRAPH

It turns out that for $\mathcal{N} = 4$ supersymmetric Yang Mills, the space of physical states is the (intersection) cohomology of the space of solutions to the Bogomolny equations with prescribed singularities labeled by $\check{R}_i, p_i$[2]. We denote this space by $\mathcal{Z}(\check{R}_1, p_1, \ldots, \check{R}_k, p_k)$. Because the underlying field theory is topological, and because the space of n-tuples on W is simply connected (so no monodromy can occur), we have that $\mathcal{Z}$ does not depend on the explicit positions of any of the p_i. Thus we can write $\mathcal{H}(\check{R}_1, \ldots \check{R}_k) = H^*(\mathcal{Z})$ and define this as the *space of physical states* for this given set of line defect insertions.

[2]In general, there are so-called "instanton corrections" to this space of states, but they are absent in this situation for reasons relating to supersymmetry.

Further, $\mathcal{Z}(\check{R}_1, \ldots \check{R}_k)$ turns out to topologically be a simple product $\prod_{i=1}^{k} \mathcal{Z}(\check{R}_i)$ where $\mathcal{Z}(\check{R}_i)$ is the same as the compactified space $\mathcal{N}(\check{R}_i)$ of Hecke modifications of type $\check{R}_i$, then by using the fact that *the product of cohomologies is the cohomology of the product* we obtain:

$$\mathcal{H}(\check{R}_1, \ldots \check{R}_k) = \bigotimes_{i=1}^{k} \mathcal{H}(\check{R}_i) \tag{6.9}$$

This suggests that there is an isomorphism of $\check{R}_i$ and $\mathcal{H}(\check{R}_i)$ as vector spaces. Indeed, it can be shown that such an isomorphism is the only way for these categories of (finite dimensional) vector spaces to have the same monoidal structure.

6.4 The Action of Wilson Loops on Boundary Conditions

If we assume that $M = \Sigma \times C$ for C a compact Riemann surface and Σ a (not necessarily compact) surface with boundary, we can study loop insertions more naturally. The following is a simplified picture of the general case:

Definition 6.4.1 (Hitchin's Moduli Space)**.** $\mathcal{M}_H(G, C)$ is the space of solutions to the Hitchin equations on a curve C.

If we consider C to be "small" relative to Σ, for each point in Σ, the additional data for the field configurations on the space C must give us a point in this moduli space. That is, we get a nonlinear sigma model on $\Sigma \to \mathcal{M}(G, C)$.

Let the curve defining a (Wilson or 't Hooft) operator be $\gamma = \gamma_0 \times p$ in $\Sigma \times C$ with p a point on C and γ_0 a curve on Σ. Let $\partial\Sigma_0$ be a connected component of $\partial\Sigma$. A boundary condition for the field theory on Σ_0 is called a **brane**.

Let γ_0 approach this boundary. On the $\hat{B}$ side, the insertion of a Wilson loop acts as an associative endofunctor for the category of boundary conditions on the topological sigma model on Σ with target $\mathcal{M}_H(G, C)$. This target space, with choice of complex structure J, can be identified with $\mathcal{M}_{flat}(G, C)$.

MAKE YOUR OWN GRAPHIC HERE

This functor will depend on the point $p \in C$ corresponding to the Wilson line. Consider the product $\mathcal{M}_{flat}(G, C) \times C$. There is a universal G-bundle $\mathcal{E}$ over this space, given by taking a point in $\mathcal{M}_{flat}$ and restricting the corresponding bundle to a point in C.

Given any coherent sheaf on $\mathcal{M}_{flat}$, we can tensor this with $R(\mathcal{E})$. This is the action of the Wilson loop insertion on the space.

Consider the structure sheaf $\mathcal{O}_x$ of a point $x \in \mathcal{M}_{flat}(\check{G}, C)$. For any representation $\check{R}$, the Wilson loop maps $\mathcal{O}_x$ to $\mathcal{O}_x \otimes \check{R}$. Thus $\mathcal{O}_x$ is an eigenobject for the functor $W_{\check{R}}(p)$, which acts on it by tensoring it with the vector space $\check{R}(\mathcal{E}_p)_x$. In fact, letting p vary we see that it is an eigenobject for all $W_{\check{R}}(p)$. Another way of saying this is that the eigenvalue is the flat $\check{G}$-bundle $\check{R}(\mathcal{E})_x$ on C.

More directly, this flat bundle is obtained by taking the flat principle bundle on C corresponding to x and forming the associated bundle via $\check{R}$.

The action of the ‘t Hooft operators is more difficult to see. They will end up acting by Hecke transformations on the space of boundary conditions. By Monotonen-Olive duality, it turns out that the brane corresponding to a fiber of the Hitchin fibration in $\mathcal{M}_H(G, C)$ is a common eigenobject for all operators.

Bibliography

[1] Philsang Yoo. Seminar on the Langlands Program, 2018.

[2] Philsang Yoo. Seminar on Geometric Langlands and Derived Algebraic Geometry, 2018.

[3] Robert P Langlands. Letter to André Weil. *unpublished, January*, 1967.

[4] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over Q: wild 3-adic exercises. *Journal of the American Mathematical Society*, 14(4):843–940, 2001.

[5] George Bachman. *Introduction to p-adic numbers and valuation theory.* Academic Press. Inc, 1964.

[6] Toshitsune Miyake. On automorphic forms on GL2 and Hecke operators. *Annals of Mathematics*, pages 174–189, 1971.

[7] Stephen S Kudla. From modular forms to automorphic representations. In *An Introduction to the Langlands Program*, pages 133–151. Springer, 2004.

[8] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *arXiv preprint hep-th/0604151*, 2006.

[9] Brian Osserman. The Weil conjectures, 2008.

[10] nLab authors. function field analogy. `http://ncatlab.org/nlab/show/function%20field%20analogy`, April 2018. Revision 23.

[11] James G Arthur. Harmonic Analysis of Tempered Distributions on Semisimple Lie Groups. *Representation theory and harmonic analysis on semisimple Lie groups*, (31):13, 1989.

[12] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 1977.

[13] Alexander Beilinson and Vladimir Drinfeld. Quantization of Hitchins integrable system and Hecke eigensheaves, 1991.

[14] Vincent Lafforgue. Quelques calculs reliésa la correspondance de Langlands géométrique pour P1 (version provisoire). *Available at authors website: http://www. math. jussieu. fr/˜ vlafforg*, 2009.

[15] Dima Arinkin and Dennis Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Mathematica*, 21(1):1–199, 2015.

[16] Chris Elliott and Philsang Yoo. A Physical Origin for Singular Support Conditions in Geometric Langlands. *arXiv preprint arXiv:1707.01292*, 2017.

[17] Edward Witten. Geometric Langlands duality and the equations of Nahm and Bogomolny. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 140(4):857–895, 2010.

[18] Fernando Quevedo, Sven Krippendorf, and Oliver Schlotterer. Cambridge lectures on supersymmetry and extra dimensions. Technical report, 2010.

[19] Raoul Bott and Loring W Tu. *Differential forms in algebraic topology*, volume 82. Springer Science & Business Media, 1982.

[20] Qfwfq (https://mathoverflow.net/users/4721/qfwfq). Why does the group act on the right on the principal bundle? MathOverflow. URL: https://mathoverflow.net/q/50587 (version: 2010-12-30).

[21] James E Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.

[22] AJ Lindenhovius. Instantons and the ADHM Construction. *URL: http://dare. uva. nl/cgi/arno/show. cgi*, 2011.

[23] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry*, volume 1. Interscience publishers New York, 1963.

[24] Akhil Mathew. Notes on the Kodaira vanishing theorem. 2012.

[25] Simon K Donaldson. Instantons and geometric invariant theory. *Communications in Mathematical Physics*, 93(4):453–460, 1984.

[26] MF Atiyah, NJ Hitchin, VG Drinfeld, and Yu I Manin. Construction of instantons 1978 Phys. *Lett. A*, 65:185.

[27] Simon K Donaldson. Geometry of Four-manifolds: Icm 86; International Congress of Mathematicians, Berkeley, California, USA, August 1986. American Math. Soc., 1988.

[28] Edward Witten. Geometric Langlands duality and the equations of Nahm and Bogomolny. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 140(4):857–895, 2010.

[29] EB Bogomolny. The stability of classical solutions. *Yad. Fiz.*, 24:861–870, 1976.

[30] MK Prasad and Charles M Sommerfield. Exact classical solution for the't Hooft monopole and the Julia-Zee dyon. *Physical Review Letters*, 35(12):760, 1975.

[31] Nigel Hitchin. Monopoles and geodesics. *Comm. Math. Phys.*, 83(4):579–602, 1982.

[32] Michael F Atiyah and Richard S Ward. Instantons and algebraic geometry. *Communications in Mathematical Physics*, 55(2):117–124, 1977.

[33] Werner Nahm. The construction of all self-dual multimonopoles by the ADHM method. Technical report, International Centre for Theoretical Physics, 1982.

[34] N. J. Hitchin. On the construction of monopoles. *Comm. Math. Phys.*, 89(2):145–190, 1983.

[35] Jacques Hurtubise and Michael K. Murray. On the construction of monopoles for the classical groups. *Comm. Math. Phys.*, 122(1):35–89, 1989.

[36] Simon K Donaldson. Nahm's equations and the classification of monopoles. *Communications in Mathematical Physics*, 96(3):387–407, 1984.

[37] Michael Francis Atiyah and Nigel Hitchin. *The Geometry and Dynamics of Magnetic Monopoles*. Princeton University Press, 1988.

[38] Hiraku Nakajima. Monopoles And Nahm's Equations. In *Einstein Metrics And Yang-Mills Connections: Proceedings of the 27th Taniguchi International Symposium, T. Mabuchi and*, 1993.

[39] A. Newlander and L. Nirenberg. Complex Analytic Coordinates in Almost Complex Manifolds. *Annals of Mathematics*, 65(3):391–404, 1957.

[40] Tom MW Nye. The geometry of calorons. *arXiv preprint hep-th/0311215*, 2003.

[41] Matthew D Schwartz. *Quantum field theory and the standard model*. Cambridge University Press, 2014.