Magnetic Monopoles, ‘t Hooft Lines, and the Geometric Langlands Correspondence

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Abstract

The aim of this thesis is to give the reader a gentle but thorough introduction to the vast web of ideas underlying the realization of the geometric Langlands correspondence in the physics of quantum field theory (QFT). It begins with a pedagogically-motivated introduction to the relevant mathematical concepts in algebraic geometry and topology aimed towards physicists, and goes on to provide an introduction to classical and quantum field theory and gauge theory for a mathematical audience. With the machinery of both sides in place, the more complicated phenomena associated with gauge theory is explored, specifically instantons, topological operators, and electric-magnetic duality. We conclude with a description of the connection between the Langlands correspondence and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) via a striking property known as S-duality. A large part of the goal of this thesis is to fill in gaps otherwise skipped over by the literature related to this topic, so that an advanced undergraduate or graduate student might have a good exposition into this field.

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Chapter 1

Introduction to the Langlands Program

The aim of this chapter is to give a both a conceptual and historical overview of both the Langlands program and the development of quantum and conformal field theory. The goal is not so much to develop any mathematical background so much as to illustrate to the reader *why* this great web of ideas is important.

1.1 The Langlands Program in Number Theory

Fermat's Last Theorem, once known as the "greatest unsolved problem in mathematics", conjectures that there does not exist any integer solution to the following equation:

$$a^n + b^n = c^n, \qquad n > 2$$

that is nontrivial. A solution is nontrivial is none of a, b, c is zero.

The proof of Fermat's last theorem relied on some of the most intricate mathematics developed at the end of the 20th century. The central theorem necessary for the proof of Fermat's last theorem is as follows.

Theorem 1.1.1 (Modularity Theorem for Elliptic Curves)**.** *Every elliptic curve is modular.*

Fermat's last theorem follows from a special case of the modularity conjecture. The modularity conjecture for elliptic curves turns out to follow from a special case of a special case of the *Langlands conjectures*, originally formulated by Robert Langlands in 1970 [1]. More precisely, it is a corollary of the Langlands correspondence for $G = \mathrm{GL}_2$ over $\mathbb{Q}$ [1]. This part of the Langlands conjecture remains unproven as of May 2018 [3].

[1] In fact, the modularity theorem is strictly stronger than necessary. It was enough for Wiles et al. to prove that a special family of elliptic curves is modular. The case for general elliptic curves has since been proven by Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor [2].

We give a sketch of the statement of the number-theoretic Langlands correspondence, intended towards an audience with some background in *Galois theory* and the language of *adeles*.

Begin by considering the **absolute Galois group** of the rationals:

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

Here $\overline{\mathbb{Q}}$ is the algebraic completion of $\mathbb{Q}$, consisting of all algebraic numbers. This Galois group is tremendously large. As an example of its size, there is an open conjecture known as the *inverse Galois problem* that

Conjecture 1.1.2 (Inverse Galois)**.** *Every finite group is contained in* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The number theoretic Langlands correspondence considers the n-dimensional representations of the absolute Galois group (called *Galois representations*) and relates them to certain representations known as *automorphic representations*. To define these latter types of representations, we first make the definition

Definition 1.1.3 (Ring of adeles)**.** The **ring of adeles** of $\mathbb{Q}$ is defined as

$$\mathbb{A}_{\mathbb{Q}} := \mathbb{R} \times \prod_{p\,\text{prime}}^{res} \mathbb{Q}_p$$

where $\mathbb{Q}_p$ denotes the p-adic completion of the rationals [4] ($\mathbb{R}$ can be viewed as the completion at $p = \infty$) and the above product is *restricted* in the sense that:

$$\prod_{p\,\text{prime}}^{res} \mathbb{Q}_p := \left\{ (x_p) \in \prod_{p\,\text{prime}} \mathbb{Q}_p \ s.t.\ x_p \in \mathbb{Z}_p \text{ for all but finitely many } p \right\}.$$

Since $\mathbb{A}_{\mathbb{Q}}$ is a ring, we can define $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ as the set of $n \times n$ matrices with entries in $\mathbb{A}_{\mathbb{Q}}$. Further, because $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$, we also have

$$\mathrm{GL}_n(\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$$

which yields a left (and right) action[2]:

$$\mathrm{GL}_n(\mathbb{Q}) \circlearrowright \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}) \circlearrowleft \mathrm{GL}_n(\mathbb{Q}).$$

The left quotient space $\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ is well-defined in this case. Since $\mathrm{GL}_n(\mathbb{Q})$ still acts by right action on this space, functions of this space form a (left) representation of $\mathrm{GL}_n(\mathbb{Q})$

$$\mathrm{GL}_n(\mathbb{Q}) \circlearrowright \mathrm{Fun}\left(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})\right)$$

This can be decomposed into irreducible representations, which are known as the **automorphic representations** of $\mathrm{GL}_n(\mathbb{Q})$. Thought not absolutely precise [3], this is a good first-order description of what an automorphic representation is.

[2] In this paper we shall use $G \circlearrowright X$ to denote left action of G on X and $X \circlearrowleft G$ to denote right action.

Idea 1.1.4. *The Langlands correspondence associates to each n-dimensional representation of the absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *an automorphic representation of* $\mathrm{GL}_n(\mathbb{Q})$.

More than just an equivalence of sets, though, the Langlands correspondence states that a certain set of *eigenvalue data* must agree on both sides.

From the perspective of the absolute Galois group (henceforth referred to as the *Galois side*), this eigenvalue data is called the **Frobenius eigenvalues** of this representation. For p a prime, the Frobenius automorphism $x \to x^p$ is the generator of the Galois group of any finite extension $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. Given a finite-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as well as a conjugacy class, one can lift the Frobenius automorphism to a conjugacy class. The eigenvalues (well-defined for a given conjugacy class) of these elements are the Frobenius eigenvalues of that representation.

From the perspective of the automorphic representations (henceforth referred to as the *automorphic side*), the eigenvalue data is more difficult to see. **Finish this**

Conjecture 1.1.5 (Langlands)**.** *To each n-dimensional representation of the absolute Galois group, there is a corresponding automorphic representation of* $\mathrm{GL}_n(\mathbb{Q})$ *so that the Frobenius eigenvalues of the Galois representation agree with the Hecke eigenvalues of the automorphic representation.*

It is worth mentioning that the Langlands conjecture over $G = \mathrm{GL}_1$ is the same as what is known in number theory as *class field theory* [3].

Many questions in number theory can be formulated in terms of questions about the nature of the absolute Galois group. On the other hand, automorphic representations can be studied using analytic methods, which would imply that deep number-theoretic data can be made accessible by studying these analytic objects.

The eigenvalue data plays a particularly important role both in the Langlands correspondence and its geometric analogue. The study of this will become the study of the *geometric Satake* symmetries acting on both sides of the geometric Langlands equivalence, and this thesis will explore how ideas from physics can give a concrete realization of the eigenvalue data in terms of *operator insertions* in quantum field theory.

1.2 Weil's Rosetta Stone and Geometric Langlands

The following table [3] [5] captures the *function field analogy*, otherwise known as Weil's *Rosetta stone*.

It is the hope and goal of this correspondence that the extremely difficult number-theoretic Langlands program might become more accessible when phrased in the language of the second or third columns of Table **??**. A reason to believe this might be so is because the power of modern algebraic geometry, as developed by Grothendieck, Serre, Deligne, and others, becomes a prominent force in driving our understanding of columns two and three.

Number Theory	Curves over $\mathbb{F}_q$	Riemann Surfaces
$\mathbb{Z} \subset \mathbb{Q}$	$\mathbb{F}_q[t] \subset \mathbb{F}_q(t)$	$\mathcal{O}_{\mathbb{C}}^{hol} \subset \mathcal{O}_{\mathbb{C}}^{mer}$
$\operatorname{Spec} \mathbb{Z}$	$\mathbb{A}^1_{\mathbb{F}_q}$	$\mathbb{C}$
$\operatorname{Spec} \mathbb{Z} \cup \{\infty\}$	$\mathbb{P}^1_{\mathbb{F}_q}$ (projective line)	$\mathbb{CP}^1$ (Riemann sphere)
p prime number	$x \in \mathbb{F}_p$	$x \in \mathbb{C}$
$\mathbb{Z}_p$ (p-adic integers)	$\mathbb{F}_q[t-x]$ power series around x	$\mathbb{C}[[z-x]]$ holomorphic on formal disk around x
$\mathbb{Q}_p$ (p-adic numbers)	$\mathbb{F}_q((t-x))$ Laurent series around x	$\mathbb{C}((z-x))$ holomorphic on punctured formal disk around x
$\mathbb{A}_{\mathbb{Q}}$ (adeles)	$\mathbb{A}_{\mathbb{F}_q}$ function field adeles	$\prod_{x\in\mathbb{C}}^{res} \mathbb{C}((z-x))$ restricted product of functions on all punctured disks, with all but finitely many extending to the unpunctured disk
$F/\mathbb{Q}$ (number fields)	$F/\mathbb{F}_q(t)$ or $\mathbb{F}_q(C)/\mathbb{F}_q(\mathbb{P}^1)$ (function fields)	$C \to \mathbb{CP}^1$ (branched covers)
$\operatorname{Gal}(\overline{F}/F)$	$\operatorname{Gal}(\overline{F}/F) = \pi_1^{\text{ét}}(\operatorname{Spec} F, \operatorname{Spec} \overline{F})$ $\twoheadrightarrow \operatorname{Gal}(F^{\text{unr}}/F) = \pi_1^{\text{ét}}(C, x)$	$\pi_1(C, x)$

Table 1.1: Weil's *Rosetta stone*

1.3 The Fourier Transform, Pontryagin Duality, and Geometric Langlands

In this section, we will attempt to give an alternative motivation for the geometric Langlands program as a generalized non-abelian analogue of the Fourier transform.

First let us begin by working with a locally-compact abelian group G. Such groups have a unique normalized Haar measure. We make the following definition:

Definition 1.3.1 (Unitary Character)**.** For G locally-compact and abelian, a **unitary character** of G is a group homomorphism $\chi : G \to U(1)$.

From this definition, we define the following group, which plays a role as a *dual* to G. It is called the **Pontryagin dual**.

Definition 1.3.2. The set of all unitary characters χ together with multiplication given by $\chi_1 \cdot \chi_2 \in \operatorname{Hom}(G, U(1))$.

Example 1.3.3. We have the following examples:

1. Let $G = S^1$, then the space of unitary characters is precisely of the form $e^{inx} : G \to U(1)$. This makes $\widehat{G} = \mathbb{Z}$.

2. Let $G = \mathbb{Z}$, then $\chi(1)$ determines the representation uniquely, and so $\widehat{G} = U(1)$.

3. Let $G = \mathbb{R}$, then $e^{ikx} : \mathbb{R} \to U(1)$ is free to have k vary over $\mathbb{R}$ so $\widehat{G} = \mathbb{R}$.

Notice in all these cases that $\widehat{\widehat{G}} \cong G$. This is in fact true more general, and we have the following theorem:

Theorem 1.3.4 (Pontryagin Duality)**.** *$G \to \widehat{\widehat{G}}$ is an isomorphism of groups, given by sending $g \to g'$ which is given by $g'(\chi) = \chi(g)$.*

Observation 1.3.5. *The L^2-integrable functions on G have a basis given by characters.*

Example 1.3.6. We have the following examples:

1. $f : S^1 \to \mathbb{C}$ has $f(\theta) = \sum_n a_n e^{in\theta}$. This is known as the **Fourier series**.
2. $f : \mathbb{Z} \to \mathbb{C}$ has $f(n) = \int_0^{2\pi} F(\theta) e^{in\theta}$. This is known as the **discrete time series**.
3. $f : \mathbb{R} \to \mathbb{R}$ has $f(x) = \int_{-\infty}^{\infty} \widehat{f(k)} e^{ikx}$. This is known as the **Fourier transform**.

Let us now try to generalize the ideas of the Fourier transform to a more direct case. It is useful to view the Fourier transform as letting us see two different sides of the same object. Let that object be the direct product of the group G and $\hat{G}$. The reason this space is worth considering is by noting that there is a unique function on this space, which we can call the **kernel** $K : G \times \hat{G} \to \mathbb{C}$ defined by $K(g, \chi) = \chi(g)$. In the case of $G = \mathbb{R}$, this function is exactly $e^{ikx}, x \in \mathbb{R}, k \in \widehat{\mathbb{R}} = \mathbb{R}$, that is viewed as a function on *both* time and frequency space.

This space is also endowed with two obvious projections (namely to either factor of the product).

$$\begin{tikzcd} & G \times \hat{G} \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_{\hat{G}}"] & \\ G & & \hat{G} \end{tikzcd}$$

Any function f on G can be "pulled back" to a function on $G \times \hat{G}$, namely by ignoring the second component $f'(g, \hat{g}) = f(g)$. We will denote this pulled back function by $\pi_G^* f = f \circ \pi_G$.

Further, a suitable distribution on $G \times \hat{G}$ can be "pushed forward" to either G or $\hat{G}$ by integrating it over $\hat{G}$ or G respectively. We will denote these by $(\pi_G)_*$ and $(\pi_{\hat{G}})_*$, again respectively.

Now if $\hat{f}$ is a distribution on $\hat{G}$, we get that $\pi_{\hat{G}}^* \hat{f}$ is a distribution on $G \times \hat{G}$. This can be pushed forward to a function on G by integrating over the $\hat{G}$ coordinates, but because $\pi_{\hat{G}}^* \hat{f}$ is constant on the G-coordinate, this function will just be a constant independent of G.

On the other hand, if we look at:

$$f(g) := (\pi_G)_*([\pi_{\hat{G}}{}^* \hat{f}]K) = \int_{\chi \in \hat{G}} [(\hat{f} \circ \pi_{\hat{G}})(g, \chi)] K(g, \chi) \, d\chi \tag{1.1}$$

we obtain exactly the Fourier transform. For $G = \mathbb{R}$ this gives us:

$$f(x) = \int_{\mathbb{R}} \widehat{f(k)} e^{ikx} dk. \tag{1.2}$$

The reason that the Fourier transform finds so much use in practice is that it serves as an eigendecomposition for the derivative operator. More broadly, on $\mathbb{R}^n$, the eigenfunctions are plane waves $e^{i\vec{k}\cdot\vec{x}}$, which yield eigenvalues both under ∂_x and also under the translation operator more generally $\vec{x} \mapsto \vec{x} + \vec{y}$. Any abelian group acts on itself by translation[3]. Consequently, it acts on the functions living on it, $\mathrm{Fun}(G)$, by translation $f(x) \to f(x - y)$. Note however that the unitary characters satisfy:

$$y \cdot \chi(x) = \chi(x - y) = \chi(-y)\chi(x) \tag{1.3}$$

so that the characters *diagonalize* the translation operator as an eigenbasis, exactly as e^{ikx} did on the real line.

Fact 1.3.7. *The Fourier transform diagonalizes the action of G on the space of functions $L^2(G) \cong L^2(\hat{G})$.*

We have just treated Fourier analysis successfully for the category of locally-compact abelian groups. A natural next question is:

Question. How could we build upon the ideas Fourier analysis to generalize to non-abelian groups? That is, what could be the non-abelian analogue of the Fourier transform?

Already, one can see that the naive ideas from before will not hold up as well. For one, translation operators no longer commute, and so cannot be simultaneously diagonalizable with an eigenbasis of unitary characters. As we move to explore the non-abelian setting, the Pontryagin dual group $\hat{G}$ will be replaced by the Langlands dual group ${}^LG$, and of course Pontryagin duality will become a very special case of Langlands duality.

It will turn out that to understand the Fourier transform in the non-abelian case, we will have to appeal to *categorification*, one of the deepest aspects of twenty-first century mathematics.

1.4 The Geometric Langlands Correspondence

To begin with an illustration of the categorification mentioned in the previous chapter, we will illustrate the **Fourier-Mukai** transform, and generalize from there.

When viewing G as a topological category: a topological space equipped with Haar measure, the natural space of functions to study is $\mathrm{Fun}(G) = L^2(G)$. For an algebraic category H, the study of functions on H is often replaced by instead studying *sheaves* on H.

[3]Keep in mind that right and left action coincide for an abelian group.

	Abelian (classical)	Non-abelian (categorified)
Space of "functions"	$L^2(G) \cong L^2(\hat{G})$	$\mathcal{D}(\mathrm{Bun}_G) \cong ???$
Operators	$G \circlearrowleft L^2(G)$	$\mathrm{Sat}_G \circlearrowleft \mathcal{D}(\mathrm{Bun}_G)$
Eigenbasis	$\{e^{ikx}\}_{t\in\hat{G}}$	$\{???\}_{\star\in ???}$

Table 1.2: The analogy of Fourier analysis as an abelian case of the geometric Langlands correspondence

Chapter 2

The Basics of Field Theory

This chapter aims to give a background into the physical ideas needed to understand the remainder of this paper

2.1 Classical Field Theory

Here is a mathematical formulation of classical field theory:

Definition 2.1.1 (Classical Field Theory)**.** A classical field theory $\mathcal{E}$ is a collection of the following data:

- A manifold M known as the **spacetime** of the theory.
- A space $\text{Map}(M, X)$ of section maps $\Phi : M \to X$, where X is a Riemannian manifold called the "target space". Any such Φ is called a **field**.
- An action functional $S[\Phi]$ from the space of field configurations into $\mathbb{C}$ (or more generally some number field).

Classical field theory studies solutions to the **classical equations of motion**

$$\{\varphi \in \mathcal{F} \text{ s.t. } \delta S(\varphi) = 0\}.$$

Example 2.1.2. When $X = \mathbb{R}$, we get a single scalar field ϕ (here Φ is ϕ). An action for this field theory is often given by:

$$S[\phi] = \int_M |\partial_\mu \phi|^2.$$

Example 2.1.3. Classical electromagnetism is defined by $X = T^*M$ with an action given by:

$$S[A] = \int_M F \wedge \star F, \qquad F := \mathrm{d}A.$$

Here $F = dA$ is the *curvature form* or *electromagnetic field-strength tensor.*

More generally, Yang-Mills theory (to be more thoroughly defined and discussed in the next section) takes $X = T^*M \otimes \mathfrak{g}$ and given

$$S[A] = \int_M \mathrm{Tr}\left(F \wedge \star F\right), \qquad F := \mathrm{d}A + A \wedge A.$$

where the trace is taken over the Lie algebra using the Killing form.

2.2 Quantum Field Theory and the Operator-Product Expansion

Though we do not know how to make sense of quantum field theory, the intuitive picture that we have of it is given by the **Feynman Path Integral**. For a given quantum field theory, there is quantity known as the **partition function**, defined as:

$$\mathcal{Z} = \int \mathcal{D}\Phi \, e^{-S[\Phi]}. \tag{2.1}$$

This is an integral taken over the space of all fields. The measure on this space is mathematically ill-defined in general.

Definition 2.2.1 (Classical Observable)**.** A classical observable is a functional from the set of field configurations to the ground field $\mathbf{k}$.

Definition 2.2.2 (Observable)**.** A **quantum observable** (which we will refer to as just an *observable* in these lectures) is a functional from the a field theory into the ground field $\mathbf{k}$. In the Feynman picture, it can be seen as a statistical average of classical observables over all field configurations.

The partition function is an observable, as is the **1-point correlation function** at a point x_1:

$$\langle \Phi(x_1) \rangle := \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \, \Phi(x_1) e^{-S[\Phi]}.$$

In this example, the path integral over all configurations of Φ probes Φ at this single point, giving essentially an expectation value. We can take expectation values of many different operators, e.g. $\phi(x_1), \partial_\mu \phi(x_1), \mathbf{1}, \phi(x_1)\partial_\mu\phi(x_1)$ on X. We denote operators by $\mathcal{O}$. More generally, we define **correlation functions** as

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_g := \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \, \mathcal{O}_1 \ldots \mathcal{O}_n e^{-S[\Phi]}.$$

Definition 2.2.3 (TQFT)**.** If the correlation functions of a given quantum field theory are independent of the metric g, then the corresponding theory is called a **topological quantum field theory** (TQFT).

In fact metric independence implies diffeomorphism invariance.

Example 2.2.4 (Chern Simmons Theory)**.** It turns out the correlation functions of Chern-Simmons theory on a 3-manifold M with Φ being the field $A : M \to T^*M \otimes \mathfrak{g}$ and the action given by

$$S[A] \propto \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

This is clear because the metric has no role in defining the action.

Proposition 2.2.5 (Operator Product Expansion)**.** *Within the path integral, a product of two local fields can be replaced by a (possibly infinite) sum over individual fields. Namely, given two operators $\mathcal{O}_a, \mathcal{O}_b$ and evaluation points x_1, x_2, there is an open neighborhood U around x_2 such that*

$$\mathcal{O}_a(x_1)\mathcal{O}_b(x_2) = \sum_c C_{ab}^c(x_1 - x_2)\mathcal{O}_c(x_2) \tag{2.2}$$

where $\mathcal{O}_c$ are other operators in the quantum field theory, and the C_{ab}^c are analytic functions on $U \backslash \{x_2\}$ (that often become singular as $x_1 \to x_2$).

In the 2D case, this yields the (possibly familiar) Laurent series associated with CFT. The structure constants contain valuable information about the QFT that allow onw to view it *algebraically*. In particular, they satisfy **associativity conditions**. The philosophy of the OPE is as follows:

Idea 2.2.6. *The OPE coefficients, together with the 1-point correlation functions completely determine the n-point correlation functions in a quantum field theory.*

For example, a two-point function is simply given by:

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\rangle = \sum_c C_{ab}^c(x_1 - x_2)\langle \mathcal{O}_c(x_2)\rangle \tag{2.3}$$

2.3 Topological Quantum Field Theory

An understanding of topological quantum field theory.

In categorical language, we say:

Definition 2.3.1. A **n-dimensional topological quantum field theory** is a symmetric monoidal functor:

$$\mathcal{Z} : \mathrm{Bord_n} \to \mathrm{Vect}_{\mathbf{k}}$$

Theorem 2.3.2. *The category of 2-dimensional topological quantum field theories is the same as the category of Frobenius algebras.*

In general, besides just considering n-bordisms between $n-1$ manifolds, one might also be inclined to consider the **extended** topological quantum field theory in n-dimensions. These are difficult to define, and would in principle rely on the language of n-categories to give a satisfactory definition. W

2.4 Supersymmetry

Definition 2.4.1. A **Lie superalgebra** is a $\mathbb{Z}_2$-graded Lie algebra with a commutator bracket satisfying:

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

In our case, we will be extending the familiar *Poincare algebra* of Lie$\{\mathrm{SO}(3,1) \ltimes \mathbb{R}^4\}$ by $\mathcal{N}$ "odd" vectors, which transform in the fundamental representation of $\mathrm{SL}(2,\mathbb{C})$, which is a projective *spinor* representation of the Lorentz group.

The brackets between the odd vectors $\{Q^A_\alpha, Q^B_\beta\}$ give rise to various central elements Z^{AB} in the algebra. These are called *supercharges*:

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB}$$

and they satisfy

$$Z^{AB} = -Z^{BA}$$

So that there are a total of $\mathcal{N}(\mathcal{N}-1)/2$ distinct supercharges in a theory with $\mathcal{N}$ supersymmetry generators.

Definition 2.4.2 (R-symmetry group)**.** The group of transformations exchanging the supercharges. For the case of $\mathcal{N} = 4$ this is Spin(6)

Definition 2.4.3 (Subsector)**.** Given a supersymmetry operator Q s.t. $Q^2 = \frac{1}{2}[Q, Q] = 0$, we define the subsector of our theory $\mathcal{E}$ by the set of Q invariants, and denote this as $(\mathcal{E}, [Q, -])$.

Slightly more precisely, $[Q, -]$ defines a differential operator, and the "observables" become exactly those gauge-invariant quantities annihilated by Q modulo those that are Q-exact.

Chapter 3

Gauge Theory

Gauge theory will play a central role in understanding the geometric Langlands correspondence physically. The role of the group G in the Langlands correspondence is played by the gauge group in the physical theory.

3.1 Fiber Bundles

3.1.1 Definitions and Examples

We will be working on a manifold M (not necessarily Riemannian). In the first definition, we can assume M is just a topological space.

Definition 3.1.1 (Fiber Bundle)**.** We define a **fiber bundle** on a topological space

- A topological space E called the **total space**
- A topological space M called the **base space**
- A topological space F called the **fiber**
- A **projection map** $\pi : E \to M$ that is surjective so that $\pi^{-1}(p) := E_p$ is homeomorphic to F. This is the fiber over p.
- For each $x \in E$ there is an open neighborhood $U \subseteq M$ of $p = \pi(x)$ so that there is a homeomorphism ψ from $U \times F$ to $\pi^{-1}(U)$ in such a way that projection p_1 onto the first factor of $U \times F$ gives π

$$\begin{tikzcd} U \times F \arrow[rr, "\psi"] \arrow[dr, "p_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

Fiber bundles generalize the notion of cartesian products of two spaces M and F by allowing for the same local product structure but much more interesting global "twisted structure". For $p \in M$, the space $E_p := \pi^{-1}(p)$ is called the **fiber over p of E** and is homeomorphic to the fiber F.

In physics, especially when calculations are to be performed, manifolds are often described in terms of a set of coordinate charts U_α that are homeomorphic to $\mathbb{R}^n$ with $n = \dim M$ and $\alpha \in I$ is an index in some indexing set, not necessarily finite[1]. A covering of M in terms of coordinate charts

$$M = \bigcup_{\alpha \in A} U_\alpha.$$

together with homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is called an **atlas** for M. In order to make sense of M in terms of an atlas, we define **transition maps** between different U_α that intersect.

Definition 3.1.2 (Transition Map)**.** Given an atlas $\{U_\alpha\}_{\alpha \in I}$, the transition maps $\tau_{\alpha\to\beta} : U_\alpha \to U_\beta$ defined by $\varphi_\beta \circ \varphi_\alpha^{-1}$

By using transition maps, we can transport data locally defined on U_α to other parts of M by "moving it across" other U_β. This data often comes from the fiber bundles over M.

In physics, this perspective is particularly important, as it gives us an ability to "glue together" locally trivial bundles on the U_α to construct a globally nontrivial fiber bundle. In the cases of interest to us, fiber bundles will have a group of automorphism that will act on the fibers when comparing the data across different U_α. We define this more precisely:

Definition 3.1.3 (Coordinate Bundle)**.** A **coordinate bundle** consists of

- A fiber bundle, defined as before

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

- A group G, called the **structure group** of E acting freely on each fiber[2].

[1] But in the case of M compact, I can always be made finite.

[2] A G-action is free iff $\forall f \in F\, gf = f \Rightarrow g = 1$. The reason we consider free actions is that if G did not act freely, then the stabilizer of F would give a normal subgroup N. Upon passing to the quotient we would get a free action of G/N.

- A set of open coverings $\{U_\alpha\}_{\alpha\in I}$ of M with diffeomorphisms $\phi_\alpha : U_\alpha \times F \to \pi^{-1}(U_i)$ called **local trivializations** so that the following diagram commutes.

$$\begin{tikzcd}
U_\alpha \times F \arrow[rr, "\psi_\alpha"] \arrow[dr, "p_1"'] & & \pi^{-1}(U_\alpha) \arrow[dl, "\pi"] \\
& U_\alpha &
\end{tikzcd}$$

- On $U_{\alpha\beta} := U_\alpha \cap U_\beta$, $\psi_\beta^{-1}\psi_\alpha$ acts as a diffeomorphism[3] coinciding with the action of an element of G on each E_p (we say "fiberwise").

In gauge theory, G is taken to be a **Lie group** called the **structure group** of E.

Definition 3.1.4. A Lie group is a group that is also a differentiable manifold so that the group operations of multiplication and inversion are compatible with the differentiable structure.

A basic working knowledge of Lie theory is assumed, however we will go over relevant aspects of Lie groups in the following sections of this chapter.

Note. In the above, we described φ_α, τ_α as *homeomorphisms*, which are indeed morphisms in the category of topological spaces. If we wish to work in other categories, such as C^r-differentiable, smooth, analytic, or complex manifolds, then the transition functions would have to be C^r-differentiable, smooth, convergent Taylor series, or holomorphic respectively. If we were working in the category of algebraic varieties, the corresponding maps we consider would have to be *regular*.

We can also identify $\psi_{\beta,p}^{-1} \circ \psi_{\alpha,p}$ with an element in G by $g_{\alpha,\beta} : U_{\alpha\beta} \to G$ acting fiberwise on the overlap of the two bundles over U_α, U_β to translate data from one coordinate patch into the other.

Proposition 3.1.5. *$g_{\alpha\beta}$ satisfies*

- $g_{\alpha\alpha} = 1$
- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$
- $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$

Moreover

1. $g_{\alpha\beta}f_\beta = f_\alpha$
   That is, $g_{\alpha\beta}$ maps the fiber corresponding to U_β to the fiber corresponding to U_α.

[3] This is a diffeomorphism in this case because we are considering a C^∞ fiber bundle. If we were considering continuous, differentiable, or holomorphic fiber bundles this map would be continuous, differentiable, or holomorphic respectively.

2. $\psi_j(p, f) = \psi_i(p, g_{ij} f)$

Proof. These are all easy to check just by the definition of $g_{\alpha\beta}$ as a composition of the ψ_α and by invoking the cartesian properties of local trivialization. $\square$

The equivalence class of a set of coordinate bundles on M is called a corresponding fiber bundle over M.

Fiber bundles whose fibers are are vector spaces are called **vector bundles**. A rank n vector bundle over a field $\mathbf{k}$ will have structure group $G \subseteq \mathrm{GL}_n(\mathbf{k})$. Examples are the tangent/cotangent bundles of a manifold, and any tensor/symmetric/exterior powers thereof. We will see that we can view vector fields, p-forms, and many other interesting and physically-relevant objects as **sections** of fiber bundles, to be described in the later sections.

3.1.2 Morphisms and Extensions

The morphisms in the category of fiber bundles are called **bundle maps**:

Definition 3.1.6 (Bundle Map)**.** For two fiber bundles $\pi : E \to M, \pi' : E' \to M'$ a bundle map is a smooth map $\overline{f} : E \to E'$ that naturally induces a smooth map on the base spaces so that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\overline{f}} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{f} & M \end{array}$$

From this we obtain the way which we will identify two bundles as identical.

Definition 3.1.7 (Equivalence of fiber bundles)**.** Two bundles are equivalent if there is a bundle map so that both $\overline{f}$ and f are diffeomorphisms.

If we have a fiber bundle $\pi : E \to M$ and $\varphi : N \to M$ for another manifold N, then we can pull back E to form a bundle over N

$$\varphi^* E = \{(y, [f, p]) \in N \times E \ s.t. \ \varphi(y) = p\}. \tag{3.1}$$

We have projection on the second factor of $\varphi^* E$ as a map $g : \varphi^* E \to E$. This is the **pullback bundle** $\varphi^* E$.

Definition 3.1.8 (Pullback Bundle)**.** For a map $\varphi : N \to M$ and E a fiber bundle over M so that $\pi : E \to M$, we define the pullback bundle $\varphi^* E$ so that the following diagram commutes:

$$\begin{array}{ccc} \varphi^* E & \xrightarrow{g} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\varphi} & M \end{array}$$

Let us consider an example which will appear later in the context of studying a monopole placed at the origin of $\mathbb{R}^3$.

Example 3.1.9. Consider a vector bundle over $\mathbb{R}^3\backslash\{0\}$. The restriction of this vector bundle to the sphere S^2 gives rise to a vector bundle on S^2 which is the same as the pullback bundle induced from $\iota : S^2 \to \mathbb{R}^3$

We can take products of fiber bundles as topological spaces in the obvious way to obtain a fiber bundle over $M \times M'$,

$$E \times E' \xrightarrow{\pi\times\pi'} M \times M'.$$

In the special case where $M = M'$ we can also define

Definition 3.1.10 (Direct Sum of Vector Bundles)**.** For E, E' vector bundles over M with structure groups G, G' respectively, we can define their sum as $E \oplus E'$ to be pullback bundle $E \times E'$ along the diagonal map $\Delta : M \to M \times M$.

More explicitly, this is a fiber bundle over M with $F \oplus F'$ fibered over every point. The structure group of $E \oplus E'$ is the product $G \times G'$ of the structure groups of the original bundles and it acts diagonally on their sum.

$$G^{E\oplus E'} = \left\{ \begin{pmatrix} g^E & 0 \\ 0 & g^{E'} \end{pmatrix} : g^E \in G, g^{E'} \in G' \right\} \tag{3.2}$$

and the transition functions act diagonally in the same way.

Similarly, we can define arbitrary direct sums of bundles recursively:

$$E_1 \oplus \cdots \oplus E_r \tag{3.3}$$

For some intuition about when fiber bundles are *nontrivial*, consider the following theorem which we state without proof but refer to [6] chapter 3. Stated simply: taking the pullback of a bundle along a map is topologically invariant under homotopy of the map.

Theorem 3.1.11. *Let $\pi : E \to M$ be a fiber bundle over M and consider maps f, g from $N \to M$ so that f, g are homotopic, then the pullback bundles are equivalent: $f^*E \cong g^*E$ over N.*

An important fact is the following corollary:

Corollary 3.1.12. *If M is contractible, every fiber bundle $\pi : E \to M$ is topologically trivial*[4]*.*

Proof. Let $f : pt \to M$ and $g : M \to pt$ be such that $f \circ g \sim id|_M$ and $g \circ f \sim id|_{pt}$. Then because pullback respects homotopy equivalence, we will have that $E \sim (f \circ g)^*E \sim f^*(g^*E)$ but g^*E is the (necessarily trivial) bundle on a point, so this will pull back along f to the trivial bundle along f. □

[4]In the language of classifying spaces, M being trivial implies there is only one homotopy class of map $M \to BG$, so that consequently the only fiber bundle over M is the trivial one

3.1.3 Principal Bundles

When the structure group acts freely and transitively on the fiber, we can identify F with G. In this case, we get a **principal G-bundle**. This will be an object of central interest in what follows. In general, unlike G itself, F need not have a canonical choice of identity element. Indeed, if it did then the bundle would necessarily have to be the trivial $M \times G$. We give a an example for motivation:

Example 3.1.13 (Frame Bundle)**.** The fiber bundle of all **frames**, namely choices of bases in an n-dimensional space is a principal GL_n bundle.

The frame bundle is generally nontrivial.

Proposition 3.1.14. *A principal bundle is equipped with a natural right action of G, R_g so that $R_g : \pi^{-1}(U_\alpha) \to \pi^{-1}(U_\alpha)$ by acting on the fiber appropriately $R_g(p, h) = (p, hg)$.*

We state the following theorem without proof (c.f. Chapter 9 of [7])

Theorem 3.1.15. *When G is a compact Lie Group acting smoothly and freely on a manifold M, the orbit space M/G is a topological manifold with dimension $\dim M - \dim G$ and a unique smooth structure so that $\pi : M \to M/G$ is a smooth submersion (differential is locally surjective).*

Corollary 3.1.16. *For a principal bundle $P(M, G)$ we get $\dim P = \dim M + \dim G$*

If M, F are two manifolds and G has an action $G \times F \to F$, then for an open cover $\{U_\alpha\}$ of M with a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ we can construct a fiber bundle by first building the set

$$X = \bigcup_\alpha U_\alpha \times F \tag{3.4}$$

and quotienting out by the relation

$$(x, f) \in U_\alpha \times F \sim (x', f') \in U_\beta \times F \Longleftrightarrow x = x', f = g_{\alpha\beta}(x) f' \tag{3.5}$$

Then $E = X/\sim$ is a fiber bundle over M. We can locally denote elements of E by $[x, f]$ so that

$$\pi(x, f) = x, \ \psi_\alpha(x, f) = [x, f]. \tag{3.6}$$

Proposition 3.1.17. *For a fiber bundle $\pi : E \to M$ with overlap functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ between charts, we can form a principal bundle $P(M, G)$ so that*

$$P = X/\sim, \ X = \bigcup_\alpha U_\alpha \times G \tag{3.7}$$

In certain contexts that we will encounter later, the $g_{\alpha\beta}$ are referred to as **clutching functions**. Note that there was no requirement here that G be compact.

Example 3.1.18. Take $M = \mathbb{CP}^1$ the Riemann sphere and consider constructing a G-bundle over it. The Riemann sphere can be decomposed as a union of two copies of $\mathbb{C}$ with overlap exactly on the cylinder $\mathbb{C}^\times$. On each copy of $\mathbb{C}$ the G-bundle is trivializable since $\mathbb{C}$ is contractible. A clutching function would be a map $\rho : \mathbb{C}^\times \to G$, and this gives rise to a principal G-bundle on M.

3.1.4 Sections and Lifts

As mentioned before, any specific smooth vector field on a manifold M can be viewed as a smooth "slice" of the vector bundle of the tangent spaces of M: TM. This motivates the notion of a **section** of a fiber bundle that associates to each base point $p \in M$ an element f in the fiber F_p, giving together $(p, f) \in E$.

A **global section** of the fiber bundle $\pi : E \to M$ is a map $s : M \to E$ so that $\pi \circ s = \mathrm{id}$. When it's restricted, $s : U \subseteq M \to E$, we call s a **local section**. The set of smooth global sections is denoted by $\Gamma^\infty(M, E)$.

Example 3.1.19. The set of all smooth r-forms on M is $\Gamma^\infty(M, \Lambda^r(T^*M))$ on which the structure group acts on each component of M.

When the group is fibered over the manifold, then on the local cartesian structure, we can easily pick the section $p \mapsto [p, s(p)]$.

Proposition 3.1.20. *For a principal bundle $P(M, G)$, any local trivialization $\psi : U \times G \to \pi^{-1}(U)$ defines a local section by $s : p \mapsto \psi(p, e)$ and conversely any local section defines a trivialization by $\psi(p, g) = s(p)g$*

By using sections, we can prove the existence of lifts. That is, for a principal bundle $P(M, G)$ over M, and a map $\varphi : M \to N$ we can get a principal bundle over N by forming the projection $\varphi \circ \pi$.

Proposition 3.1.21. *For a principal bundle $P(M, G)$ and $\varphi : M \to N$, then φ is smooth iff $\varphi \circ \pi$ is smooth according to the following diagram.*

$$\begin{tikzcd} P(M,G) \arrow[d, "\pi"] \arrow[dr, "\varphi\circ\pi"] & \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Proof. If φ is smooth, then $\varphi \circ \pi$ is a composition of smooth maps. On the other hand, if $\varphi \circ \pi$ is smooth, then for each point p there is a coordinate neighborhood U_α on which we have trivial fiber structure. Take a local section s_α so that $\varphi \circ \pi \circ s = \varphi|_{U_\alpha}$. □

Proposition 3.1.22. *For $P(M, G)$ principal and $\tilde{\varphi} : P(M, G) \to N$ a smooth G-invariant map so that*

$$\tilde{\varphi}(xg) = \tilde{\phi}(x), \; x \in P(M, G) \tag{3.8}$$

then there is a unique map ϕ induced on the base space so that the following diagram commutes:

$$\begin{tikzcd} P(M,G) \arrow[d, "\pi"] \arrow[dr, "\tilde{\varphi}"] & \\ M \arrow[r, dashed, "\varphi"] & N \end{tikzcd}$$

and is given by $\tilde{\phi}([x,g]) = \varphi(x)$. This is well-defined.

3.1.5 Associated Bundles

Take a principal bundle $P(M,G)$ and let F be a space with associated automorphism $\mathrm{Aut}(F)$ so that $\rho : G \to \mathrm{Aut}(F)$ is a faithful representation. Then $g \cdot f$ is a well-defined notion, with free action, and we can consider the (right) action of G on $P(M,G) \times F$ by

$$g \cdot ([p,h], f) = ([p, hg], \rho(g)^{-1} f) \tag{3.9}$$

This is a free action as well. Then if G is compact (important) we have the orbit space

$$E_\rho = P(M,G) \times F/G \tag{3.10}$$

is a manifold

Theorem 3.1.23. *The space E_ρ can be made into a fiber bundle over M with fiber F called the **associated fiber bundle** of $P(M,G)$.*

Proof. (Following [8]) We make $P \times F$ into a bundle by defining the projection

$$\pi_\rho([p,h], f) = p \tag{3.11}$$

and trivializations $\psi_\alpha : U_\alpha \times F \to \pi^{-1}(U)\alpha$ by

$$(\psi_\rho)_\alpha(p, f) = ([p, s_\alpha(p)], f) \tag{3.12}$$

and inverse

$$(\psi_\rho)_\alpha^{-1}([p,g], f) = [p, \rho(g) f] \tag{3.13}$$

□

From this, if F is a group then we can make $\pi_\rho^{-1}(p)$ into a group at each fiber in the obvious way, defining $[(p,v)][p,w] = [p, vw]$. And if F is a vector space then we can do the same construction to make each fiber have vector space structure. Two associated bundles that we'll care about are $P(M,G) \times_{\mathrm{Ad}} G$ and $P(M,G) \times_{\mathrm{ad}} \mathfrak{g}$.

The study of equivalence classes of G-bundles can be equivalently cast as a study of their associated bundles.

ELABORATE HERE

3.2 Lie Theory

Although standard knowledge on the definition of a Lie Group/Algebra is assumed, let's try to motivate the ideas within this field in a more geometric way than is often done.

Consider a manifold M, and consider $\mathrm{Vect}(M)$, the space of all smooth vector fields on M. For a map $\varphi : M \to N$ we have a notion of **pushforward** $\varphi_* : \mathrm{Vect}(M) \to \mathrm{Vect}(N)$ on vector fields given by their actions on functions as

$$[\varphi_*(v)](f) = v(\varphi^*(f)) \tag{3.14}$$

A smooth vector field X on M gives rise to **flows** that are solutions to the differential equation of motion

$$\frac{d}{dt} f(\gamma(t)) = Xf. \tag{3.15}$$

One could argue, more strongly, that in fact the *entire field* of ordinary differential equations has an interpretation as equations of motion along flows of vector fields. Such a viewpoint has brought forward the lucrative insights of symplectic geometry.

The motion along this flow is expressed as the exponential:

$$f(\gamma(t)) = e^{tX} f(p),\ p = \gamma(0) \tag{3.16}$$

Now consider two vector fields X, Y on M. Let Y flow along X so we move along X giving:

$$e^{tX} Y = Y(\gamma(t)) \in T_{\gamma(t)} M \tag{3.17}$$

Note that the reverse flow e^{-tX} maps $T_{\gamma(t)}M \to T_{\gamma(0)}M = T_pM$, so acts by pushforward on $e^{tX}Y$ equivalent to:

$$e^{tX} Y e^{-tX} \in T_p \tag{3.18}$$

We can compare this to Y and take the local change by dividing through by t as $t \to 0$, giving the Lie derivative

$$\mathcal{L}_X Y := \frac{e^{tX} Y e^{-tX} - Y}{t} \tag{3.19}$$

It is easy to check that this is in fact antisymmetric and gives rise to a bilinear form on $\mathrm{Vect}(M)$

$$[X, Y] := L_X Y \tag{3.20}$$

A vector space endowed with such a bilinear form and satisfying the Jacobi identity is a **Lie algebra**.

Most important is when M itself has group structure, so is a **Lie group**, which we will denote by G. Then the vector fields on G of course also form a Lie algebra, just by virtue of the manifold structure of G.

We state the following proposition without proof

Proposition 3.2.1. *Let $\varphi : G_1 \to G_2$ be a diffeomorphism of Lie groups, then $\varphi_* : \mathrm{Vect}(G_1) \to \mathrm{Vect}(G_2)$ is a homomorphism of Lie algebras.*

For a Lie group, group elements induce automorphisms on the manifold by left multiplication, denoted L_g and by right multiplication R_g:

$$\begin{aligned} R_g : G \to G,\ g : h \mapsto gh \\ L_g : G \to G,\ g : h \mapsto hg \end{aligned} \tag{3.21}$$

We have that each group element induces (by pushforward) a map between tangent spaces

$$\begin{aligned} (L_g)_* : T_h G \to T_{gh} G \\ (R_g)_* : T_h G \to T_{hg} G \end{aligned} \tag{3.22}$$

A vector field X is left-invariant if $(L_g)_* X(h) = X(gh)$.

By the proposition, we get that $(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y]$ so these left-invariant vector fields in fact form a Lie algebra for the group. Physically, this is the set of vector fields corresponding to the isometries of G.

In local coordinates, the commutator can be written as:

$$\begin{aligned} X =& X^\mu \partial_\mu,\ Y = Y^\mu \partial_\mu \\ [X, Y] =& (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu) \partial_\mu \end{aligned} \tag{3.23}$$

Left-invariant vectors flow in a way that is consistent with the group action:

$$(L_g)_* X(e) = X(g) \tag{3.24}$$

The set of all left-invariant vector fields can be uniquely extracted from their value at the identity by this rule, and in fact for any vector $x \in T_e G$, there is a corresponding left-invariant vector field $X(g) = (L_g)_* x$. Therefore the tangent space to the identity gives rise to a Lie algebra which we will call the Lie algebra of G and denote by $\mathfrak{g}$. This Lie algebra (often referred to as *the* Lie algebra $\mathfrak{g}$ associated to the group G) is finite dimensional when G is.

Now because we define the Lie algebra as the “tangent space to the identity”, it is worth asking “how does the Lie algebra appear at a generic point, g, on the group?”. The idea is to bring that vector back to the identity using G and see what it looks like.

This is accomplished by using the **Maurer-Cartan form** Θ, which is a $\mathfrak{g}$-valed 1-form on G so that

$$\Theta(g) = (L_{g^{-1}})_* \tag{3.25}$$

Note that this maps from $\mathrm{Vect}(G) \to \mathfrak{g}$. It takes a vector v at point g and traces it back to the natural vector at the identity that would have gotten pushed forward to v under g.

Proposition 3.2.2 (Properties of exp)**.** *For G a compact and connected Lie group, with Lie algebra $\mathfrak{g}$, we have a map* $\exp : \mathfrak{g} \to G$.

1. $[X, Y] = 0 \Leftrightarrow e^X e^Y = e^Y e^X$

2. *The map* $t \to \exp(tX)$ *is a homomorphism from* $\mathbb{R}$ *to* G.

3. *If* G *is connected then* exp *generates* G *as a group, meaning all elements can be written as some product* $\exp(X_1)\dots\exp(X_n)$ *for* $X_i \in \mathfrak{g}$

4. *If* G *is connected and compact then* exp *is surjective. It is almost never injective.*

Example 3.2.3. The Lie algebra associated to the Lie group $\mathrm{U}(n)$ of unitary matrices is $\mathfrak{u}(n)$ of antihermitian matrices. This is the same as the Lie algebra for the group $\mathrm{SU}(n)$

Definition 3.2.4 (Adjoint Action on G)**.** For each g we can consider the homomorphism $\mathrm{Ad}_g : h \mapsto ghg^{-1}$ or $\mathrm{Ad}_g = L_g \circ R_{g^{-1}}$. This defines a representation

$$\mathrm{Ad} : g \to \mathrm{Diff}(G) \tag{3.26}$$

Definition 3.2.5 (Adjoint Representation of $\mathfrak{g}$)**.** The pushforward of this action gives rise to the **adjoint representation** of the Lie group $\mathfrak{g}$ by

$$(\mathrm{Ad}_g)_* = (L_g \circ R_{g^{-1}})_* \tag{3.27}$$

From the product rule, this acts as $[g, -]$ at the identity. We denote this as

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\ \mathfrak{g} \tag{3.28}$$

The Jacobi identity ensures that ad is a homomorphism. If the center of G is zero then ad is faithful and we have an embedding into $\mathrm{GL}(n)$. This is nice because it also shows that after a central extension, every Lie algebra can be represented into $\mathrm{GL}(n)$, a weaker form of Ado's theorem.

Moreover the adjoint representation gives rise to a natural metric on $\mathfrak{g}$ called the **Killing Form** given by

$$\kappa(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) \tag{3.29}$$

Proposition 3.2.6. *For* $\mathfrak{g}$ *a semisimple Lie algebra, the above gives rise to a non-degenerate bilinear form.*

For a proof see [9].

3.3 Connections on Principal Bundles

3.3.1 The Ehresman Connection

Take a G-principal bundle $\pi : P \to M$. Just like $\xi \in \mathfrak{g}$ gives rise to a vector field X_ξ on G, it also canonically gives rise to a vector field $\sigma(\xi)$ on P.

Definition 3.3.1 (**Fundamental Vector Field** of ξ)**.** Let $\xi \in \mathfrak{g}$ and consider $\exp(t\xi) \in G$ so that for $p \in P(M, G)$ we get $c_p(t) = R_{\exp(t\xi)}p$ which depends smoothly on p. Note $c_p'(0) \in T_pP(M, G)$ at each point.

$$\sigma : \mathfrak{g} \to \text{Vect}(P(G, M)),\ [\sigma(\xi)](p) \mapsto \left[\frac{d}{dt}pe^{t\xi}\right]_{t=0} \tag{3.30}$$

The **vertical subspace** V_pP at a point p of a fiber bundle is the tangent space at p restricted to the fiber over x, i.e. $T_p(\pi^{-1}(x))|_{F_x}$. Equivalently, this is $\ker \pi_*$. Note

$$\pi_* \circ \sigma(x) = \frac{d}{dt}(\pi \circ c_p(t))|_{t=0} = \frac{d}{dt}(p) = 0 \tag{3.31}$$

so $\sigma(x) \in V_pP$. Since E is a manifold of dimension $\dim M + \dim G$, $\pi_* : T_pE \to T_{\pi(p)}M$ has a kernel of dimension $\dim G = \dim \mathfrak{g}$ In fact:

Proposition 3.3.2. *σ_p is a Lie algebra isomorphism between $\mathfrak{g}$ and V_pP*

Proof. Since G acts freely on principal bundles, σ is injective, so in fact it must be an isomorphism. □

Lemma 3.3.3 (Properties of σ)**.** *We get that σ satisfies:*

1. $[R_g]_*\sigma(x) = \sigma(ad_{g^{-1}}x)$
2. $[g_i]_*\sigma(x) = g_i(p)x$

Proof. 1. We have

$$\begin{aligned} [R_g]_* [\sigma(x)](p) &= \frac{d}{dt}(R_gpe^{tx}) \\ &= \frac{d}{dt}pg\text{Ad}_{g^{-1}}e^{tx} \\ &= \frac{d}{dt}pg \exp[t(\text{ad}_{g^{-1}}x)] \\ &= [\sigma(\text{ad}_{g^{-1}}x)](pg) \end{aligned} \tag{3.32}$$

2. And

$$\begin{aligned} [g_i]_* [\sigma(x)](p) &= \frac{d}{dt}g_ipe^{tx} \\ &= g_i(p)x \end{aligned} \tag{3.33}$$

□

Now σ respects the Lie algebra structure and forms a homomorphism from $\mathfrak{g}$ to $\text{Vect}(P(M, G))$ so that in fact

Corollary 3.3.4. *$(R_g)_*V_p = V_{pg}$: pushforward acts equivariantly on vertical subspaces.*

Proof. Let $X(p) \in V_p$ pick $A \in \mathfrak{g}$ s.t. the corresponding fundamental vector field $\sigma(A)(p) = X(p)$. Then we just look at

$$(R_g)_* \sigma(A)(p) = \sigma(\mathrm{ad}_{g^{-1}} A)(pg) \tag{3.34}$$

which is vertical. It's easy to go back from pg to g as well by picking $A \in \mathfrak{g}$ so that $X(pg) = \mathrm{ad}_{g^{-1}} A$. $\square$

Now note:

$$0 \longrightarrow V_p P \longrightarrow T_p P \xrightarrow{\pi_*} T_{\pi(p)} M \longrightarrow 0$$

An injection of $T_{\pi(p)} P$ into P to make the above sequence split is called a **horizontal subspace** $H_p P$.

Definition 3.3.5 (Horizontal Subspace)**.** A horizontal subspace is a subspace $H_p P$ of $T_p P$ s.t.

$$T_p P = V_p P \oplus H_p P \tag{3.35}$$

We'll abbreviate this by H_p and the vertical subspace by V_p when our principal bundle is unambiguous.

Crucially, there is *no canonical choice of* H_p, reflecting the physical fact there is no "god-given" way to compare local gauges between different points. For a gauge g at x, a vector on $T_x M$ should lift to a vector on $T_{[x,g]} P$ given by lifting to a horizontal subspace. A choice of horizontal gives rise to the following:

Definition 3.3.6. An **Ehresmann connection** is a choice of horizontal subspace at each point $p \in P(M, G)$ so that

1. Any smooth vector field X splits as a sum of two smooth vector fields: a **vertical field** X_V and a **horizontal field** X_H so that at each point $p \in P(M, G)$ we have $X_V \in V_p$, $X_H \in H_p$. That is, the choice of H_p varies smoothly.

2. G acts equivariantly on H_{pg}:

$$H_{pg} = (R_g)_* H_p \tag{3.36}$$

We will denote the collection of our choice of $H_p P$ by HP and similarly define VP to be the (always canonical) collection of vertical subspaces. We say any vector field can be split into a vector field $X^H \in HP$ and $X^V \in VP$.

Naturally, for any choice of HP, we have a corresponding projection operator π_H on vector fields $\pi_H : \mathrm{Vect}(P(M, G)) \to HP$ and similarly $\pi_V = id - \pi_H$, both with corresponding equivariance conditions.

Proposition 3.3.7. *We have the following correspondence:*

$$\begin{array}{ccccc} \text{Ehresman} & & \text{Horizontal/Vertical} & & \mathfrak{g}\text{-valued} \\ \text{Connections } HP & \longleftrightarrow & \text{Projection Operators } H/V & \longleftrightarrow & \text{1-forms } \omega \end{array}$$

Each of the above are smooth on E, and have appropriate equivariance conditions:

- $R_g H_p = H_{pg}$*: Horizontal subspaces are G-equivariant*
- $[R_g]_* H = H[R_g]$*: Horizontal projection commutes with G action of "changing gauge"*
- $\omega(pg) = R_g^* \omega = g^{-1}\omega(p)g$*: The 1-form is G-covariant*

3.3.2 The Group of Gauge Transformations

A diffeomorphism $\Phi : P \to P$ is a **gauge transformation** if it satisfies

1. $\pi \circ \Phi = \pi$, so Φ acts fiberwise
2. $R_g \circ \Phi = \Phi \circ R_g$, so Φ is G-equivariant.

the group of all such diffeomorphisms is called the **gauge group** of P and denoted by $\mathcal{G}$.

3.4 Chern-Weil Theory

3.4.1 Symmetric Invariant Polynomials on $\mathfrak{g}$

Consider $\mathfrak{g}$ as an affine algebraic variety ($\cong \mathbb{C}^{\dim \mathfrak{g}}$), and consider the ring of functions $\mathbb{C}[\mathfrak{g}]$. Since $G \circlearrowleft \mathfrak{g}$ by Ad_G, we naturally have a G-action on this space of polynomials

$$\mathbb{C}[\mathfrak{g}] \circlearrowright G$$

Taking $f(x) \to f(\mathrm{Ad}_g x)$. Polynomials that are fixed by this action are called **invariant polynomials** on $\mathfrak{g}$.

Chapter 4

Instantons and the ADHM Construction

Instantons are objects of significant interest to both physicists and mathematicians. For physicists, they represent *classical solutions to the equations of motion*. In the context of field theory, and more specifically *Yang-Mills Field Theory*, instantons correspond to nontrivial field configurations on a given spacetime manifold.

Donaldson used the interesting mathematical properties of Yang-Mills instantons on $\mathbb{R}^4$ to prove novel and extremely surprising statements about the nontrivial smooth structures that can be associated to $\mathbb{R}^4$ uniquely among all Euclidean spaces[10].

A useful picture comes from quantum mechanics, of a particle in a double-well potential. Having a particle localized at the bottom of either well gives rise to a classical solution. Perturbative corrections around this minimum due to the quantum theory may give rise to harmonic-oscillator-type structure within the well, but is completely unable to account for the possibility of *quantum tunneling* across the barrier into the second well of the potential. To account for this, we must understand the space of classical solutions in addition to performing perturbation theory.

Mathematically, this often manifests itself in the fact that $e^{-1/x}$ has every higher derivative vanish as $x \to 0^+$. It is also the same phenomenon that allows for the existence of *bump functions* in real analysis and also for *asymptotic expansions* in various areas of physics and engineering.

4.1 Instantons in Classical Yang-Mills Field Theory

Yang-Mills gauge theory is a gauge theory with gauge group $G = \mathrm{SU}(n)$. In four dimensions, the objects of study are G-bundles and associated G-bundles on Euclidean 4-space $M = \mathbb{R}^4$. $\mathbb{R}^4$ has a Riemannian metric, so we have a Hodge-star operator giving a (metric-dependent) canonical isomorphism:

$$\star : \Omega^k \to \Omega^{n-k}.$$

From the prior section, gauge theory on $\mathbb{R}^4$ involves a connection 1-form A transforming in the $\mathrm{ad}\,\mathfrak{g}$ representation. From this, we obtain the curvature form F, again transforming in the adjoint action, by applying the covariant exterior derivative:

$$F = \mathrm{d}_A A = \mathrm{d}A + [A, A] \tag{4.1}$$

Note that

In this case, the action of the theory is given by:

$$S_E[A] := \frac{1}{8\pi} \int \mathrm{Tr}\,(F \wedge \star F)$$

Proposition 4.1.1. $\mathrm{Tr}\,(F \wedge \star F)$ *is gauge independent and globally defined.*

Proof. Since F transforms in the adjoint representation, the cyclic property of the trace gives:

$$\mathrm{Tr}\,(F \wedge \star F) \to \mathrm{Tr}\,(gFg^{-1} \wedge g \star Fg^{-1}) = \mathrm{Tr}\,(F \wedge \star F)$$

□

We aim to find A so that $S_E[A]$ is a minimum. To do this, we use standard calculus of variations. Consider a local perturbation A + t

Definition 4.1.2. A Hermitian vector bundle $\pi : E \to M$ over a base space M is a complex vector bundle E over M equipped with a Hermitian inner product on each fiber.

Definition 4.1.3 (Connection on a Vector Bundle)**.**

Chapter 5

Magnetic Monopoles and the Equations of Bogomolny and Nahm

The goal of this chapter is to give the reader a gentle introduction to the notable discoveries in the study of monopoles in $\mathbb{R}^3$.

In section 1 we give two derivations of the Bogomolny equations. The first approach derives the equations directly from the anti-self-duality (ASD) conditions for instanton solutions in $\mathbb{R}^4$ by treating the fourth component of the connection 1-form, A_4, as a scalar field ϕ and ignoring translations ∂_4 along the x_4 direction. The second approach works directly with the action to derive not only the Bogomolny equations but also an integrality condition on the asymptotics of ϕ that allow $\mathfrak{su}(2)$ monopole solutions, much like instantons, to be characterized by a single number k: the magnetic charge[1].

In section 2, we then study the (moduli) space of directed lines on $\mathbb{R}^3$ and make the identification between this space and the (holomorphic) tangent bundle of the Riemann sphere $T\mathbb{CP}^1$. From here, we motivate Hitchin's use of a 1-dimensional scattering equation along a line $(D_t - i\phi)s = 0$ to characterize monopole solutions to the Bogomolny equations as giving rise to a holomorphic vector bundle $\tilde{E}$ over $T\mathbb{CP}^1$ corresponding to the solution space of the scattering equation for a given line. An asymptotic analysis of the solutions to this equation naturally leads to both Hitchin's spectral curve Γ and Donaldson's rational map theorem.

In section 3, we motivate the Nahm transform by analogy to the ADHM construction for instantons from the prior chapter. The story is a little bit more complicated here, since rather than a reduction to linear data, we have a reduction to a Sobolev space of functions on the line segment $(0, 2)$. The Nahm equations are the related to the spectral curve Γ. We finally show how a solution of Nahm's equation gives rise to a monopole solution (A, ϕ) on $\mathbb{R}^3$.

The main ideas relating to understanding the Bogomolny equations can be simply diagrammed in the triangle of Figure 5.1.

[1] For general $\mathfrak{su}(n)$ instantons, $n-1$ numbers are required, associated to the Cartan subalgebra of $\mathfrak{g}$. We restrict to the $\mathfrak{su}(2)$ case, as most authors do, although the generalization of many of these statements to other real Lie groups is not difficult. For the purposes of the Langlands program $\mathfrak{su}(2)$ will play a special role [11].

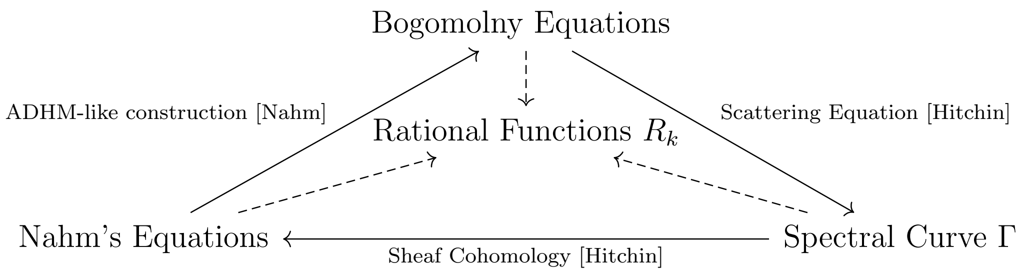

Figure 5.1: The triangle of ideas in the construction of monopoles.

Historically, the Bogomolny equations were first introduced by Bogomolny [12] together with Prasad and Sommerfield [13] in their studies of spherically-symmetric single-monopole solutions to nonabelian gauge theories. Explicitly, the $\mathfrak{su}(2)$ single-monopole solution takes the form

$$A = \left(\frac{1}{\sinh|x|} - \frac{1}{|x|}\right)\epsilon_{ijk}\frac{x_j}{|x|}\sigma_k dx^i$$
$$\phi = \left(\frac{1}{\tanh|x|} - \frac{1}{|x|}\right)\frac{x_i}{|x|}\sigma_i$$

where σ_i are the generators of $\mathfrak{su}(2)$ and we are using Eisntein summation convention.

In [14], Hitchin considered the complex structure of geodesics (i.e. directed lines) in $\mathbb{R}^3$ and used this together with the previous scattering ideas in the Atiyah-Ward $\mathcal{A}_k$ ansatz [15] to develop his approch using the spectral curve (righthand arrow in Figure 5.1). In a separate approach, Nahm [16] made use of the ADHM ansatz to formulate the solutions to the Bogomolny equations for $\mathfrak{su}(2)$ in terms of solutions to a coupled system of differential equations, now known as the Nahm equations:

$$\frac{dT_j}{ds}(s) = \epsilon_{ijk}[T_j(s), T_k(s)]$$

where T_i for $i \in \{1, 2, 3\}$ are $k \times k$-matrix valued functions of s on the interval $(0, 2)$, subject to certain conditions. This is the lefthand arrow of Figure 5.1.

The equivalence of these two approaches, corresponding to the bottom arrow in Figure 5.1 was demonstrated by Hitchin in [17]. Hitchin considered the spectral curve of a monopole and constructed a set of Nahm data associated to it, from which one could obtain Nahm's equations. This construction involved methods from sheaf cohomology for the construction of a necessary set of bundles $\mathcal{L}^s$ over $T\mathbb{CP}^1$. This general circle of ideas for $\mathrm{SU}(n)$ monopoles was completed in [18].

Remarkably, these three various descriptions of monopoles can all be related using relatively straightforward constructions to a fourth object: the space of rational functions of a complex variable z with denominator of degree k. This is the rational map constructed by Donaldson [19].

In general, the role of the Nahm transform in understanding the moduli space instanton-like solutions in $\mathbb{R}^4/\Lambda$ for Λ a subgroup of translations in $\mathbb{R}^4$ is as follows:

$$\text{Yang-Mills(-Higgs) on } \mathbb{R}^4/\Lambda \underset{\text{Nahm Transform}}{\Longleftrightarrow} \text{Nahm Equations on } (\mathbb{R}^4)^*/\Lambda^*$$

5.1 Monopoles on $\mathbb{R}^3$

We give here an exposition to magnetic monopoles, following the book of Atiyah and Hitchin [20].

5.1.1 From the Reduction of the ASD Equations

Taking the source-free Yang-Mills equations on $\mathbb{R}^4$, consider solutions that are translation invariant under one coordinate, say x_4. There are two ways forward: either by immediately considering the ASD connections together with translation invariance or by building up the action and seeing how the 3D analogue of the ASD connections emerges.

Observation 5.1.1 (ASD Connection)**.** *The ASD conditions for instantons on $\mathbb{R}^4$ can be explicitly written as*

$$F_{14} = -F_{32}, \quad F_{24} = -F_{13}, \quad F_{34} = -F_{21} \tag{5.1}$$

For F translation invariant w.r.t. x_4, we get

$$\partial_2 A_3 - \partial_3 A_2 + [A_2, A_3] = \partial_1 A_4 + [A_1, A_4] \tag{5.2}$$

and the two other permutations. Taking $A_4 = \phi$ gives that all three of these equations can be written as

$$\star F = \mathrm{d}_A \phi. \tag{5.3}$$

*These are the **Bogomolny equations**. Any solution to this gives us a translation-invariant instanton in $\mathbb{R}^4$. Note that these do not satisfy the decay conditions necessary for the instantons of the ADHM construction.*

5.1.2 From the Yang-Mills-Higgs Action on $\mathbb{R}^3$

To derive an effective action for the $\mathbb{R}^3$ field theory from translation invariance in $\mathbb{R}^4$ we first write:

$$A_{4D} = A_1\, dx^1 + A_2\, dx^2 + A_3\, dx^3 + \phi dx^4.$$

Under the translation assumption, the spatial symmetry group of 4D Euclidean transformations $\mathrm{ISO}(4) = \mathbb{R}^4 \rtimes SO(4)$ reduces down to the 3D group $\mathrm{ISO}(3) = \mathbb{R}^3 \rtimes SO(3)$. With this reduced symmetry, the x^4 component of A (namely ϕ) remains invariant under SO(3) transformations and does not mix with the other three components.

Thus, we have a reduction of A from lying in $\Omega^1(\mathbb{R}^4)$, as a fundamental representation of $\mathrm{SO}(4,\mathbb{R})$ fiberwise to lying in an inhomogeneous direct sum $\Omega^1(\mathbb{R}^3)\oplus\Omega^0(\mathbb{R}^3)$ of the fundamental $\mathrm{SO}(3,\mathbb{R})$ representation of $\mathrm{SO}(3)$ with the trivial one.

Note that both A and ϕ are still valued in $\mathfrak{g}$ and transform in the adjoint representation. The covariant derivative becomes $(\mathrm{d}_A)_{3D} = \mathrm{d}_{3D} + A$, since $\phi\, dx^4 = 0$ on any vector in $\mathbb{R}^3$. Now note that the 4D curvature form becomes

$$(\mathrm{d}_A)_{3D}(A_{3D}+\phi) = F_{3D} + (\mathrm{d}_A)_{3D}\phi. \tag{5.4}$$

From now on we write F for F_{3D} and d_A for $(\mathrm{d}_A)_{3D}$. The associated action is then

$$S = \frac{1}{8\pi}\int \mathrm{Tr}\left[F\wedge\star F + (\mathrm{d}_A\phi)\wedge\star(\mathrm{d}_A\phi)\right] = \frac{1}{8\pi}\int \left[(F,F) + (\mathrm{d}_A\phi,\mathrm{d}_A\phi)\right]. \tag{5.5}$$

where $(\Omega,\Omega) := \mathrm{Tr}[\Omega\wedge\star\Omega]$ denotes the inner product on p-forms induced by the metric on $\mathbb{R}^3$. From now on, we restrict to the case $\mathfrak{g}=\mathfrak{su}(2)$, though many of the more general results for $\mathfrak{su}(n)$ follow analogously.

Letting B_R be ball of radius R centered at the origin in $\mathbb{R}^3$, we recover the action as the limit of the integral:

$$\lim_{R\to\infty}\frac{1}{8\pi}\int_{B_R}\left[(F-\star\mathrm{d}_A\phi, F-\star\mathrm{d}_A\phi) + 2\,(\star\mathrm{d}_A\phi, F)\right]$$

Before tackling this last term, make the following observations:

Observation 5.1.2. *For the above action to be well-defined, we require $|F(\vec{x})| = O(|x|^{-2})$ and $|\mathrm{d}\phi(\vec{x})| = O(|x|^{-2})$. This implies that the killing norm of ϕ, $|\phi|$, tends to a constant value as $|x|\to\infty$.*

Observation 5.1.3. *If $(A(\vec{x}),\phi(\vec{x}))$ is solution to the equations of motion, then $(cA(\vec{x}/c), c\phi(\vec{x}/c))$ is also a solution.*

For this reason, without loss of generality we may assume $|\phi(\vec{x})|\to 1$ as $|x|\to\infty$. For R large, this makes $\phi|_{S_R} : S^2_R \to S^2$ map from the sphere of radius R in $\mathbb{R}^3$ to the unit sphere S^2 in $\mathfrak{su}(2)$.

Let's make one more observation before tackling the second term

$$\begin{aligned}
\mathrm{d}(\phi,\star F) &= \mathrm{d}\mathrm{Tr}[\phi F]\\
&= \mathrm{Tr}[\mathrm{d}\phi\wedge F - \phi\,\mathrm{d}F]\\
&= \mathrm{Tr}[\mathrm{d}_A\phi\wedge F - \phi A\wedge F + \phi\, A\wedge F]\\
&= (\mathrm{d}_A\phi, \star F)\\
&= (\star\mathrm{d}_A\phi, F)
\end{aligned} \tag{5.6}$$

This implies that the second term can be written as a boundary term:

$$\int_{B_R}(\star\mathrm{d}_A\phi, F) = \int_{S^2_R}\mathrm{Tr}[F\phi]$$

Note ϕ acting on a bundle E transforming in the fundamental representation of $\mathfrak{su}(2)$ has two eigenspaces of opposite imaginary eigenvalues, and by assumption that $|\phi| \to 1$, these eigenvalues cannot both be zero. Thus, they cannot cross and this gives us two well-defined line bundles L_+, L_- over S^2_R corresponding to the positive and the negative eigenvalues.

Proposition 5.1.4. *$E = L_+ \oplus L_-$ has vanishing first Chern class $c_1(E) = 0$.*

Proof. This follows from the fact that $\mathfrak{su}(2)$ is traceless □

Corollary 5.1.5. *The first Chern class of L_+ is $c_1(L_+) = +k$ and L_- is $c_1(L_-) = -k$ for an integer k* [2]*.*

Proof. After picking an orientation so that the first Chern class of L_+ is positive, the corollary immediately follows upon observing that the Chern classes of complex line bundles over the sphere are always integral, and the first Chern class of a direct sum is the sum of the individual first Chern classes. □

Proposition 5.1.6. $\lim_{R\to\infty} \int_{S^2_R} (F, \phi) = \pm 4\pi k$*.*

Proof. By definition, the first Chern class of a vector bundle E is $\frac{i}{2\pi}\int_{S^R} \mathrm{Tr}(\Omega)$ for Ω the curvature two-form associated to E. Now note that on the eigenbundles of ϕ, we have that since $|\phi| \to 1$, it acts as $\pm i$ (σ_3 up to gauge) so that we must have (from before)

$$\lim_{R\to\infty} i \int_{S^2_R} \mathrm{Tr}(F_{L_+}) - i \int_{S^2_R} \mathrm{Tr}(F_{L_+}) = \pm(2\pi k c_1(L_+) + 2\pi k c_1(L_-)) = \pm 4\pi k. \quad (5.7)$$

□

As we take $R \to \infty$, this proposition gives us an action of

$$S = \frac{1}{8\pi} \int_{B_R} ||F - \star \mathrm{d}_A \phi||^2 \pm k. \quad (5.8)$$

In this case, the absolute minimum is achieved when (A, ϕ) satisfy the following:

Proposition 5.1.7 (Bogomolny Equations). *The monopole solutions for Yang-Mills theory on $\mathbb{R}^3$ satisfy*

$$\star F(\vec{x}) = \mathrm{d}_A \phi(\vec{x}) \quad (5.9)$$

subject to the constraints (after rescaling of axes and fields) that:

1. $|\phi(\vec{x})| \to 1 - \frac{k}{2r}$ *as* $|x| = r \to \infty$,

[2] It should be noted that (besides the non-monopole case of $k = 0$), this makes the bundle E nontrivial. This means that E cannot just be the restriction of a (necessarily trivial) vector bundle over $\mathbb{R}^3$. To understand this: the non-triviality of E can be seen to come from singularities induced on the vector bundle by the insertion of monopole. In the $k = 1$ BPS case, this corresponds to E being a nontrivial bundle on $\mathbb{R}^3 \backslash \{0\}$

2. $\partial|\phi(\vec{x})|/\partial\Omega = O(r^{-2})$, *where* Ω *denotes any angular variable in polar coordinates,*

3. $|\mathrm{d}_A\phi(\vec{x})| = O(r^{-2})$.

The norm $|\phi|$ *is the standard killing norm on* $\mathfrak{g} = \mathfrak{su}(2)$*. These equations can also describe* $\mathfrak{su}(n)$ *monopoles, with adapted decay conditions.*

Note under $\phi \to -\phi$ we get that the Bogomolny equations with $k \leq 0$ become the anti-Bogomolny equations and $F = -\star\,\mathrm{d}_A\phi$ and $k \geq 0$. Further, spatial inversion together with $A \to -A$ can flip these to the Bogomolny equations with $k \geq 0$. Therefore, it is enough look at solutions to the Bogomolny equations for $k \geq 0$.

Definition 5.1.8 (Magnetic Charge)**.** The positive integer k is called the **monopole number** or **magnetic charge** of the monopole solution.

Though our analysis has been for $\mathfrak{su}(2)$, the $\mathfrak{u}(1)$ case has the same equations characterizing a monopole solution.

Observation 5.1.9. *Note when* $\mathfrak{g} = \mathfrak{u}(1)$*, and using the notation* $B_k = \epsilon_{ijk}F_{ij}$ *the Bogomolny equation becomes* $B = \nabla\phi$*, giving the first known magnetic monopole, the* ***Dirac Monopole****:*

$$\phi = \frac{k}{2r}.$$

Note. We aim to study the solutions of the Bogomolny equations modulo the action of the gauge group $\mathcal{G}$. However, not all gauge transformations preserve the decay conditions on $\mathrm{d}_A\phi$ and $|\partial\phi/\partial\Omega|$. Consequently, we study the Bogomolny equations modulo the restricted gauge group $\tilde{\mathcal{G}}$ of transformations that tend to a constant element g as $|x| \to \infty$.

5.2 Hitchin's Scattering Equation, Donaldson's Rational Map, and the Spectral Curve

5.2.1 The moduli spaces N_k and M_k

We make the following notational definition

Definition 5.2.1. Let N_k be the space of gauge-equivalent $\mathfrak{su}(2)$ monopoles of magnetic charge k.

This is our main object of study in what follows.

This section involves studying the solutions of "scattering-type" equations along directed lines in $\mathbb{R}^3$. Consequently, the covariant derivative operator when restricted to a line, say along a line parallel to the x_1 axis, becomes:

$$\mathrm{d}_A \to \frac{d}{dx_1} + A_1 \tag{5.10}$$

In this case, we can make a gauge transformation

$$A \to gAg^{-1} + g^{-1}dg$$

so as to make $A_1 = 0$. This simplifies the covariant derivative along lines parallel to the x_1 axis to become just $\mathrm{d}_A \to \frac{d}{dx_1}$.

A copy of $U(1)$ still remains to act on A_2 and A_3. Thus, as $x_1 \to \infty$, because the decay conditions on ϕ, we have that any gauge transformation tends to a constant element in this $U(1)$ subgroup. In this context, define:

Definition 5.2.2 (Framing)**.** Define a **framed gauge transformation** [17, 21] to be one that tends to the identity as $x_1 \to \infty$.

If we only identify solutions modulo *framed* gauge, then the asymptotic $U(1)$ element as $x_1 \to \infty$ will differentiate between solutions that are otherwise equivalent modulo the full gauge group. We thus make a definition

Definition 5.2.3. Define M_k to the the space of solutions to the Bogomolny equations modulo framed gauge. This is fibered over N_k with fiber S_1

$$S_1 \hookrightarrow M_k \twoheadrightarrow N_k$$

Proof. We have seen that upon choosing $A_1 = 0$, gauge transformations can still have an asymptotic value in a $U(1) \cong S^1$ subgroup. Thus, quotienting out by only *framed* gauge transformations to get M_k leaves a piece of S^1 information that N_k does not have. We will call this S^1 element the *phase* of a given monopole solution. $\square$

Note. M_k depends on a choice of oriented x_1-axis in $\mathbb{R}^3$. A more coordinate-free way of defining this extension M_k of N_k is given in [20]. It relies on a simple observation from the previous section that asymptotically the restriction of E over S^2_R is a direct sum of k-twisted bundles: $E_k = L_{-k} \oplus L_k$. The automorphism group in SU(2) fixing this direct sum is exactly the $U(1)$ diagonal action:

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Thus, up to this $U(1)$ automorphism determining phase, every k-monopole solution is asymptotically equivalent to a fixed E_k. Informally: restricting the gauge transformation group so as to retain this automorphism information gives us $M_k = N_k \times S^1$.

5.2.2 Hitchin's Scattering Transform

In [14] Hitchin made use of a scattering method to show the following equivalence:

Theorem 5.2.4 (Hitchin)**.** *Given a solution (A, ϕ) to the Bogomolny equations satisfying the criteria of 5.1.7, then let ℓ be a directed line in $\mathbb{R}^3$ pointing along a direction $\hat{n}$ with distance parameterized by t and consider the following **scattering equation** along ℓ*

$$(D_{\hat{n}} - i\phi)\psi = 0. \tag{5.11}$$

Here $D_{\hat{n}}$ is a restriction of the covariant derivative d_A to act along ℓ, ϕ is the scalar field restricted to ℓ, and ψ is a section of the restriction of the vector bundle E associated to the fundamental representation $\mathbb{C}^2$ to the line ℓ.

The solutions to this equation form a complex two-dimensional space $\tilde{E}_\ell$ of sections. If A, ϕ satisfy the Bogomolny equations, then $\tilde{E}_\ell$ is a holomorphic vector bundle over the space of directed lines in $\mathbb{R}^3$.

There are several propositions that need to be developed before this theorem can be made sense of. Firstly,

Proposition 5.2.5. *The space of directed lines in $\mathbb{R}^3$ forms a complex variety isomorphic to the tangent bundle to the Riemann sphere $T\mathbb{CP}^1$ with a real structure σ.*

Proof. Once a normal direction $\hat{n}$ is chosen, a directed line ℓ in $\mathbb{R}^3$ is uniquely determined by a vector $\vec{v} \perp \hat{n}$. Thus our space is

$$\{(n, v) : |n| = 1, u \cdot v = 0\} \tag{5.12}$$

Clearly $\hat{n}$ sits on a sphere S^2 and $(\hat{n}, v)$ form TS^2. It is sufficient to find a complex structure to make this into the complex variety $T\mathbb{CP}^1$. We will form a complex structure on $\mathbb{CP}^1$ and then this lifts to one on the tangent bundle. The complex structure J acting on a point (n, v) is given by taking $J(v) = \hat{n} \times v$. This corresponds exactly to the complex structure on the holomorphic tangent bundle of the Riemann sphere.

The real structure σ comes from reversing the orientation of a line $(\hat{n}, v) \to (-\hat{n}, v)$. It is easy to see $\sigma^2 = 0$, and since it reverses orientation in $\mathbb{R}^3$ is takes $J \to -J$. □

Example 5.2.6. To make this picture clearer for the reader, let's note that given a point (x_1, x_2, x_3), each direction $\hat{n}$ has a unique line $(\hat{n}, v)$ passing through this point. Thus, a point $\vec{x} \in \mathbb{R}^3$ determines a section $s : \mathbb{CP}^1 \to T\mathbb{CP}^1$. Explicitly, picking a local coordinate ζ on $\mathbb{CP}^1$ we get:

$$s(\zeta) = ((x_1 + ix_2) - 2x_3\zeta - (x_1 - ix_2)\zeta^2)\frac{d}{d\zeta}. \tag{5.13}$$

The fact that the coefficient is a degree 2 polynomial in ζ is a consequence of the tangent bundle being a bundle of degree 2 over $\mathbb{CP}^1$. Note further that this corresponds to describing $\mathbb{R}^3$ as the space of real holomorphic vector fields on the Riemann sphere, namely $\mathfrak{so}(3, \mathbb{R})$.

Next, let us try to study this scattering equation. It will be useful to restrict, without loss of generality, to lines parallel to the x_1 axis.

Proposition 5.2.7. *The solutions to the scattering equation on a line form a two dimensional space.*

Proof. In the gauge $A_1 = 0$ developed before, this is an easy consequence of the fact that E is rank two and so upon decomposing E into eigenspaces of ϕ, $L_+ \oplus L_-$, the scattering equation decouples into two linear differential equations:

$$\left[\frac{d}{dx} - i\lambda_j(x_1)\right] s_j = 0, \qquad j = 1, 2. \tag{5.14}$$

Because these equations are both linear and first-order, they each have a one-dimensional space of solutions. □

We can now understand the vector bundle that Hitchin constructed on $T\mathbb{CP}^1$.

Observation 5.2.8. *Let $\tilde{E} \to T\mathbb{CP}^1$ denote the two-dimensional space of solutions to the scattering equation associated to a given line in $\mathbb{R}^3$. This forms a vector bundle.*

We are now ready to prove Hitchin's theorem.

Proposition 5.2.9 (Construction of a Holomorphic Vector Bundle)**.** *If (A, ϕ) satisfy the Bogomolny equations, then $\tilde{E}$ is holomorphic.*

Proof. Hitchin appeals to a theorem of Nirenberg [22]: that it is sufficient to construct an operator

$$\bar{\partial} : \Gamma(T\mathbb{CP}^1, \tilde{E}) \to \Gamma(T\mathbb{CP}^1, \Omega^{(0,1)}(\tilde{E})).$$

The existence of $\bar{\partial}$ on $\tilde{E}$ would give $\tilde{E}$ a holomorphic structure for which $\bar{\partial}$ plays the role of the anti-holomorphic differential. Let s be a section of $\tilde{E}$ for a given directed line ℓ in $\mathbb{R}^3$. Let t be the coordinate alone this line an x, y be orthogonal coordinates in the plane perpendicular to ℓ. In this case, define:

$$\bar{\partial} s = [D_x + iD_y]\, s(dx - idy). \tag{5.15}$$

Where D_x, D_y are shorthand for the x and y components of the covariant derivative d_A.

It is easy to show that this operator satisfies the Leibniz rule together with $(\bar{\partial})^2 = 0$, but we must show that it is *well-defined* as an operator from $\Gamma(T\mathbb{CP}^1, \tilde{E}) \to \Gamma(T\mathbb{CP}^1, \Omega^{(0,1)}(\tilde{E}))$. Namely, we must show that it fixes $\tilde{E}$, meaning that:

$$\left(\frac{d}{dt} - i\phi\right)(D_x + iD_y) = 0. \tag{5.16}$$

But this can be written as the requirement that the following commutator vanishes:

$$\begin{aligned} 0 = \left[\frac{d}{dt} - i\phi, D_x + iD_y\right] &= F_{12} + iF_{13} - D_y\phi + iD_x\phi \\ \Rightarrow F_{12} = D_y\phi \quad F_{31} &= D_x\phi. \end{aligned} \tag{5.17}$$

These are exactly the Bogomolny equations, as desired. We have thus shown that Hitchin's construction works. □

5.2.3 The Spectral Curve

Given the above discussion, it is worth trying to understand what the solutions of this scattering equation mean. We know from before that the null space of the scattering operator consists of two linearly independent solutions, s_0 and s_1. Let us look at their asymptotics. Again, let ℓ be a line parallel to the x_1 axis with $A_1 = 0$. Then

Proposition 5.2.10. *As $t \to \infty$, the two solutions to Hitchin's scattering equation are combinations of the following two solutions:*

$$s_0(t) = t^{k/2}e^{-t}\, e_0, \qquad s_1(t) = t^{-k/2}e^{t}\, e_1 \tag{5.18}$$

where e_0 and e_1 are constant vectors in E in the asymptotic gauge.

Proof. Since $A_1 = 0$, the scattering equation becomes

$$\frac{d}{dt} - i\phi = 0. \tag{5.19}$$

Using asymptotics on ϕ from the prior section, we get

$$\frac{d}{dt} - i\left(1 - \frac{k}{2t}\right)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + O(1/t^2) = 0. \tag{5.20}$$

This yields two differential equations:

$$\frac{d}{dt} + \left(1 - \frac{k}{2t}\right) + O(1/t^2) = 0, \qquad \frac{d}{dt} - \left(1 - \frac{k}{2t}\right) + O(1/t^2) = 0, \tag{5.21}$$

which in turn yield two solutions as $t \to \infty$:

$$s_0(t) \to t^{k/2}e^{-t}\, e_0, \qquad s_1(t) \to t^{-k/2}e^{t}\, e_1. \tag{5.22}$$

□

Note that (by t-reversal symmetry) we must have the same type of solutions as $t \to -\infty$. Namely, there is a basis where one solution blows up as $t \to -\infty$ and the other decays to zero. The solution that decays to zero, s', must necessarily be some linear combination of the $t \to \infty$ solutions s_0 and s_1. We thus have:

$$s' = as_0 + bs_1. \tag{5.23}$$

In the special case that $b = 0$, we get that s' decays not only as $t \to -\infty$ but also as $t \to \infty$. Physically, this is called a **bound state**.

Definition 5.2.11 (Bound state)**.** A bound state $\psi(\vec{x})$ is a state of a physical system that decays "sufficiently quickly" (i.e. as $e^{-|x|}$) as $|x| \to \infty$). It captures the notion of a localized particle.

Since the linear combination for s' is a relationship between sections of a holomorphic line bundle, the ratio a/b is a well-defined meromorphic function on $T\mathbb{CP}^1$. Fixing $\hat{n}$, the poles of this function generically give k points on $T_{\hat{n}}\mathbb{CP}^1$. Letting $\hat{n}$ vary gives Hitchin's **spectral curve** Γ on $T\mathbb{CP}^1$. Note this is a k-fold cover of $\mathbb{CP}^1$, and an application of the Riemann-Hurwitz formula would yield that Γ in fact has genus $k-1$. We will illustrate more on why this curve deserves its name using the Nahm transform in section 4.

Hitchin gives the following theorem, which we will state without proof:

Theorem 5.2.12 (Hitchin)**.** *If two monopole solutions $(A, \phi), (A', \phi')$ have spectral curves Γ', Γ, then (A, ϕ) is a gauge transform of (A', ϕ').*

Note that here there is no assumption on framing. The spectral curve itself does not carry information about the phase of the monopole solution. On the other hand, the section s' associated to a given line for a monopole solution gives rise to a distinguished line bundle $\mathcal{L}$ over Γ, alongside the standard restriction of the vector bundle $\widetilde{E}$ to Γ.

Note that Γ is holomorphic and *real* in the sense that it is preserved by the real structure σ on $T\mathbb{CP}^1$.

The proof that a spectral curve satisfying the conditions imposed on Γ will give rise to a monopole solution is done by going through the Nahm equations. As mentioned before, Hitchin [17] showed using ideas from sheaf cohomology that a spectral curve on $T\mathbb{CP}^1$ naturally gives rise to a set of Nahm data from which the Nahm equations can be constructed. In this way, the construction of monopoles goes in the direction of Figure 5.1.

5.2.4 The Rational Map

Let $x_1 = t$ and $z = x_2 + ix_3$. Let ℓ be a line parallel to the x_1 axis. Note it is determined by its intersection z with the x_2, x_3 plane. a and b are as before: the linear combination of $s' = as_0 + bs_1$, the solution decaying as $t \to -\infty$.

It is a powerful result of Donaldson [19] that tells us: for a fixed direction x_1 we not only obtain a meromorphic function of the lines ℓ parallel to x_1, namely $S(z) = a(z)/b(z)$, but that in fact *any* meromorphic function on $\mathbb{CP}^1$ with denominator degree k has an interpretation as a k-monopole solution. This rational function depends on the point of M_k specifying the monopole. In this sense it is *almost* gauge invariant, except for the S^1 phase associated to it. The poles of this rational function correspond to when the solution has $s' = s_0$ from before, namely a bound state.

We state Donaldson's result:

Theorem 5.2.13 (Donaldson)**.** *For any $m \in M_k$, the scattering function S_m is a rational function of degree k with $S_m(\infty) = 0$. Denote this space of rational functions by R_k. The identification of $m \to S_m$ gives a scattering map diffeomorphism $M_k \to R_k$.*

Example 5.2.14. For $k = 1$ we have R_k takes functions of the form $\frac{\alpha}{z-\beta}$, which turns out to correspond to a monopole at $(\log 1/\sqrt{|\alpha|}, \mathrm{Re}(\beta), \mathrm{Im}(\beta))$. The argument of α describes the $U(1)$ phase at $t \to \infty$. This means M_1 has complex structure $\mathbb{C} \times \mathbb{C}^\times$.

Example 5.2.15. For higher k, in the generic case a rational function in R_k will split as a sum of simple poles

$$\sum_i \frac{\alpha_i}{z - \beta_i}.$$

This has the interpretation of monopoles having centers at positions $(\log 1/\sqrt{|\alpha_i|}, \mathrm{Re}(\beta_i), \mathrm{Im}(\beta_i))$ and phases described by the arguments of the α_i.

5.3 The Nahm Equations

5.3.1 Motivation

By adopting the monad construction of ADHM, Nahm succeeded in adapting their formalism to solving the 3D Bogomolny equation. The idea of Nahm (and indeed, the idea behind the Nahm transform more broadly) was to recognize monopoles on $\mathbb{R}^3$ as solutions to the anti-self-duality equations in $\mathbb{R}^4$ that were invariant under translation along one direction, and then appropriately modify ADHM to account for the different decay conditions and symmetries of the configuration.

We present a review of the ADHM construction from the prior section. In what follows, a **quaternionic vector space of dimension** k is taken to mean k copies of $\mathbb{C}^2$, $\mathbb{C}^{2k}$, where each copy has quaternionic structure.

Review. The ADHM construction for $\mathfrak{su}(2)$ starts with W a real vector space of dimension k and V a quaternionic vector space of dimension $k+1$ with inner product respecting the quaternionic structure. Then, for a given $x \in \mathbb{R}^4$ it forms the operator:

$$\Delta(x) : W \to V. \tag{5.24}$$

The operator $\Delta(x)$ is written as $Cx + D$ where C, D are constant matrices and $x \in \mathbb{H}$ is viewed a quaternionic variable once a correspondence is made $\mathbb{R}^4 \cong \mathbb{H}$.

If Δ is of maximal rank, then the adjoint $\Delta^*(x) : V \to W$ has a one-dimensional quaternionic subspace E_x that, as x varies, can be described as a bundle over $\mathbb{H} \cong \mathbb{R}^4$. The orthogonal projection to E_x (viewed as a horizontal subspace) in V defines the (Ehresman) connection on the vector bundle $E \to \mathbb{R}^4$. [17]

Here, we will use the zero-indexed (x_0, x_1, x_2, x_3) to label the coordinates so that the imaginary quaternionic structure of the latter three becomes more clear. Nahm's approach [16] was to seek vector spaces W, V fulfilling the same function, and look for the following conditions:

1. $\Delta(x)^*\Delta(x)$ is real and invertible (as before).

2. $\ker \Delta(x)^*\Delta(x)$ has quaternionic dimension 1 (as before).

3. $\Delta(x+x_0) = U(x_0)^{-1}\Delta(x)U(x_0)$.

This last point is equivalent to the translation invariance of the connection in x_0, up to gauge transformation.

Because of this new condition, unlike the case of ADHM, V and W turn out to be infinite dimensional. Consequently, Δ, Δ^* become differential (Dirac) operators.

5.3.2 Construction

To construct V, first consider the space of all complex-valued L^2 integrable functions on the interval $(0, 2)$. Denote this space by H^0 (this notation coming from the fact that this is the zeroth Sobolev space). This space has a real structure coming not only from $f(s) \to \bar{f}(s)$ but also from $f(s) \to \bar{f}(2-s)$. Define $V = H^0 \otimes \mathbb{C}^k \otimes \mathbb{H}$, where $\mathbb{C}^k$ is taken to have a real structure.

Similarly, we define W by considering the space of functions whose derivatives are L^2 integrable. This will be denoted by H^1 (again with motivation deriving from a corresponding Sobolev space concept). Define

$$W = \{H^1 \otimes \mathbb{C}^k : f(0) = f(1) = 0\}.$$

Now define $\Delta : W \to V$ by

$$\Delta(x)f = i\frac{df}{ds} + x_0 f + \sum_{i=1}^{3}(x_i e_i + iT_i(s)e_i)f, \tag{5.25}$$

where e_i denote multiplication by the quaternions i, j, k respectively and $T_i(s)$ are $k \times k$ matrices. It is clear that this operator is the form $Cx + D$ with $C = 1$ and $D = i\frac{d}{ds} + i\sum T_j e_j$.

Using the language of [17] we make the following proposition

Proposition 5.3.1. *The following hold:*

1. *The requirement that Δ is quaternionic implies $T_i(s) = T_i(2-s)^*$.*

2. *The requirement that Δ is real implies $T_i(s)$ are anti-hermitian and also that $[T_i, T_j] = \epsilon_{ijk}\frac{dT_k}{dt}$.*

3. *The requirement that Δ is invariant under x_0 translation is automatically satisfied*

4. *The requirement that Δ^* has kernel of quaternionic dimension 1 comes from requiring that the residues of T_i at $s = 0, 2$ form a representation of* SU(2)

Proof. The first two are relatively straightforward to see. The new condition follows immediately from

$$
\begin{aligned}
e^{ix_0(s-1)}[\Delta(x)]e^{-ix_0(s-1)}f &= e^{ix_0(s-1)}\left[i\frac{d}{ds}+\dots\right](e^{-ix_0(s-1)}f) \\
&= \Delta(x)f + x_0 f \\
&= \Delta(x+x_0)f.
\end{aligned}
\tag{5.26}
$$

The last item states that since the residues of a $k \times k$ matrix valued functions are themselves $k \times k$ matrices, that in fact the commutation relations of these residue matrices at $s = 0$ and 2 form k-dimensional representations of SU(2). This requires a bit of work, and can be found in [17]. □

We thus have the following data:

$T_1(s), T_2(s), T_3(s)$ $k \times k$ matrix-valued functions for $s \in (0, 2)$ satisfying

$$
\frac{dT_i}{ds} + \epsilon_{ijk}[T_j, T_k] = 0. \tag{5.27}
$$

together with the requirements

1. $T_i(s)^* = -T_i(s)$
2. $T_i(2-s) = -T_i(s)$
3. T_i has simple poles at 0 and 2 and is otherwise analytic
4. At each pole, the residues T_1, T_2, T_3 define an irreducible representation of $\mathfrak{su}(2)$.

These are **Nahm's equations**.

For a given solution of Nahm's equations, the associated Dirac operator $\Delta^*(x)$, depending on a chosen $\vec{x}$, can be shown to again yield a 1-dimensional quaternionic (2-dimensional complex) kernel E_x. Here, though, it does not specify a connection on $\mathbb{R}^4$ but instead gives rise to A and ϕ through the following way construction:

Construction 5.3.2 (3D Monopole from Nahm's Equations)**.** Pick an orthonormal basis of $E_x = \ker \Delta^*(x) \cong C^2$. Call this v_1, v_2. We view E_x as a fiber at x corresponding to a $\mathbb{C}^2$ bundle, and construct ϕ and A by their actions on a given v_a at x.

$$
\begin{aligned}
\phi(\vec{x})(v_a) &= i\frac{v_1}{||v_1||_{L^2}}\int_0^2 (v_1, (1-s)v_a)ds + i\frac{v_2}{||v_2||_{L^2}}\int_0^2 (v_2, (1-s)v_a)ds, \\
A(\vec{x})(v_a) &= \frac{v_1}{||v_1||_{L^2}}\int_0^2 (v_1, \partial_i v_a)ds + \frac{v_2}{||v_2||_{L^2}}\int_0^2 (v_2, \partial_i v_a)ds.
\end{aligned}
\tag{5.28}
$$

5.3.3 The Spectral Curve in Nahm's Equations

For any complex number ζ we can make a definition:

$$\begin{aligned} A(\zeta) &= (T_1 + iT_2) + 2T_3\zeta - (T_1 - iT_2)\zeta^2, \\ A_+ &= iT_3 - (iT_1 + T_2)\zeta. \end{aligned} \tag{5.29}$$

Nahm's equations can then be recast as:

$$\frac{dA}{ds} = [A_+, A]. \tag{5.30}$$

This is the **Lax Form** of Nahm's equations. This can be solved by considering the curve $\mathbf{S}$ in $\mathbb{C}^2$ with coordinates (η, ζ) defined by

$$\det\left(\eta - A(\zeta)\right).$$

Proposition 5.3.3. *The above equation is independent of s.*

Proof. Let v be an eigenvector of A and let it evolve as $\frac{dv}{ds} = A_+ v$. Then

$$\frac{d(Av)}{ds} = [A_+, A]v + AA_+v = A_+Av = \lambda A_+ v, \tag{5.31}$$

so this gives

$$\frac{d}{ds}(A - \lambda v) = 0. \tag{5.32}$$

Since $A - \lambda v = 0$ at $s = 0$, it is always zero. Thus, this curve of eigenvalues is independent of s. □

It is in fact a remarkable result that:

Proposition 5.3.4. *The curve* $\mathbf{S}$ *constructed above is the same as the spectral curve* Γ *constructed previously.*

Hitchin showed this by associating to a given spectral curve Γ a set of Nahm data in [17].

5.4 The Nahm Transform and Periodic Monopoles

The Nahm transform is a nonlinear generalization of the Fourier transform, related to the Fourier-Mukai transform. It allows for the construction of instantons on $\mathbb{R}^4/\Lambda$. Some examples are below:

1. $\Lambda = 0$: ADHM Construction of Instantons on $\mathbb{R}^4$,
2. $\Lambda = \mathbb{R}$: The monopole construction that this paper has described,
3. $\Lambda = \mathbb{R} \times \mathbb{Z}$: Periodic monopoles on $\mathbb{R}^3$ (calorons, c.f. [23]),
4. $\Lambda = (\mathbb{R} \times \mathbb{Z})^2$: Hitchin system on a torus.

Chapter 6

The Physical Picture

The aim of this chapter is to first develop for the reader a picture of $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory together with its topological twists. With this, we bring together the ideas of the previous chapters and study the actions of line defects on the categories of boundary conditions of two topological twists of $\mathcal{N} = 4$ SYM.

6.1 Reduction from Ten Dimensions

One of the simplest ways to arrive at 4D $\mathcal{N} = 4$ SYM is to begin with gauge theory in 10 dimensions with gauge group G [24]. The action here is:

$$S = \int \mathrm{Tr}\,\left(F_{IJ}F^{IJ}\right) \tag{6.1}$$

6.2 Topological Twisting

First recall from Chapter 2 Section 2.4 the following definition:

Definition 6.2.1 (Subsector)**.** Given a supersymmetry operator Q s.t. $Q^2 = \frac{1}{2}[Q, Q] = 0$, we define the subsector of our theory $\mathcal{E}$ by the set of Q invariants, and denote this as $(\mathcal{E}, [Q, -])$.

Slightly more precisely, $[Q, -]$ defines a differential operator, and the "observables" become exactly those gauge-invariant quantities annihilated by Q modulo those that are Q-exact.

Definition 6.2.2 (Topological Twist)**.** Given a supersymmetric (SUSY) field theory $\mathcal{E}$, a topological twist is a procedure for extracting a sector of $\mathcal{E}$ that depends only on the topology of the spacetime manifold. The resulting field theory is **topological** in the definition of Section 2.3

In general this involves a homomorphism from the universal cover of the structure group of the spacetime tangent space TM to the R-symmetry group. For our four-dimensional $\mathcal{N} = 4$ case this is

$$\rho : \mathrm{Spin}(4) \to \mathrm{Spin}(6)$$

This redefines how the fields transform under the cover of the Lorentz group, Spin(4). We have an equivalence-class of obvious embeddings.

$$\mathrm{Spin}(4) \hookrightarrow \mathrm{Spin}(6)$$

given by:

$$\begin{pmatrix} * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and this is what Geometric Langlands will be concerned with.

After twisting by ρ, the group Spin(4) acts differently on the supersymmetry operators. In particular one of the left-handed and one of the right-handed supersymmetries become scalars under Spin(4). We thus get scalars Q_l, Q_r, and any linear combination of these gives rise to a different "sector" of invariants. Clearly overall scaling does not matter, so we have $\mathbb{P}^1(\mathbb{C})$ of subsectors to chose from.

Proposition 6.2.3. *Any such subsector defines a theory that is independent of the Riemannian metric (i.e. diffeomorphism invariant). The path integral localizes to Q-invariant configurations.*

6.3 Montonen-Olive Duality

6.4 Wilson Lines

In general, the connection 1-form, A, gives a way to transport data along any given vector bundle E associated to a representation R of G. This allows us to compare the values of fields operators at different points by integrating along E using our connection. The result is:

$$W_R(\gamma) = \exp\left(\int_\gamma A\right) \tag{6.2}$$

This classical operator is called a **Wilson line**. Wilson lines transform (under a general transformation $g \in \mathcal{G}$), as:

$$W_R(\gamma) = g(\gamma(1))W_R(\gamma)g(\gamma(0))^{-1} \tag{6.3}$$

in the special case of γ closed, we see this is gauge-invariant. In this case, it called a **Wilson loop**. It can be viewed as yielding an element of the group G in the representation R. In this case, the trace of this element gives an invariant scalar quantity (known in physics as a c-number), and so for γ closed we further add a trace.

Definition 6.4.1 (Wilson Loop)**.** Given a field theory with gauge group G and a finite-dimensional representation R of G together with a closed loop γ, we define the Wilson loop operator:

$$\mathcal{W}_R(\gamma) := \operatorname{Tr} R(\operatorname{Hol}(A, \gamma)). \tag{6.4}$$

The algebra of Wilson loops is simple. For $\gamma \to \gamma'$ the operator product expansion gives us that

$$\lim_{\gamma \to \gamma'} W_R(\gamma) W_{R'}(\gamma') = \sum_\alpha n_\alpha W_{R_\alpha}(L'). \tag{6.5}$$

In our picture, let M be a 4-manifold and let $L \subset M$ be an oriented 1-manifold embedded in M. On the $\hat{B}$-twist, we can consider taking the holonomy of the connection $\mathcal{A}$ along L, when L is closed, giving us a Wilson loop.

Proposition 6.4.2. *The $\hat{B}$ model condition on the flatness of $\mathcal{A}$ implies that the holonomy of the Wilson loop only depends on the homotopy class of L*

If M has boundaries, we can let L be an open 1-manifold connecting two ends of M. Then, the Wilson operator will give us matrix elements between the initial and final states of the theory. Because Wilson operators geometrize $\operatorname{Rep}(\check{G})$, the space of physical states living on the boundary of M is exactly $\check{R}$ for some $\check{R} \in \operatorname{Rep}(\check{G})$. A Wilson loop connecting boundary components gives us a matrix element between initial and final vectors in $\check{R}$.

In the G theory: the $\hat{A}$-twist, A and ϕ instead obey a different equation:

$$F - \phi \wedge \phi = \star D_A \phi. \tag{6.6}$$

This equation is analogous to the equation of motion for the 2D A models. We will see how the Bogomolny equations for magnetic monopoles arise as a special restriction of this equation in the next section.

From the above discussion, we should ask

Question. What is the dual operator to a Wilson line?

From the physics viewpoint, ‘t Hooft showed in the 1980s that MO duality will exchange a Wilson line (a type of “order operator”) on one side with something known as a ‘t Hooft line (a type of “disorder operator”) on the other side.

We can intuitively understand the insertions of ‘t Hooft lines in the path integral as imposing divergence conditions on the curvature form F so that in local coordinates $x^1 \dots x^3$ perpendicular to the line we have

$$F(\vec{x}) \sim \star_3 d\left(\frac{\mu}{2r}\right) \tag{6.7}$$

where μ is an element of the lie algebra $\mathfrak{g}$. It turns out that for us to be able to find a gauge field A whose curvature F satisfies this condition, we must have that μ

is a Lie algebra homomorphism $\mathbb{R} \to \mathfrak{g}$ obtained as the pushforward of a Lie group homomorphism $U(1) \to G$.

Another way to say this is (after using gauge freedom to conjugate μ to a particular Cartan subalgebra) that μ must lie in the coweight lattice Λ_{cw}. In fact the 't Hooft operator remains the same after the action of the Weyl group $\mathcal{W}$ on μ so we have that 't Hooft operators are classified by the space:

$$\Lambda_{cw}(G)/\mathcal{W}.$$

But this is also the same as

$$\Lambda_w(\check{G})/\mathcal{W}.$$

We know that this is the space of representations of the Langlands dual group.

Proposition 6.4.3. *'t Hooft operators in gauge group G are classified by irreducible representations of $\check{G}$.*

The operator product expansion of Wilson lines captures the monoidal category structure of $\mathrm{Rep}(\check{G})$. By duality, this category must also be capturing the OPE of 't Hooft lines. Can we say anything about the OPE of 't Hooft lines in terms of G?

6.5 Operator Product Expansion of 't Hooft Lines

6.5.1 Reduction to 3D

Because the operator product expansion is a local process, we can assume our base manifold looks like anything. It turns out to be fruitful to take $X = I \times C \times \mathbb{R}$. Here, I is the unit interval $(0, 1)$, C is a Riemann surface (which we can take to be $\mathbb{CP}^1$ WLOG) and $\mathbb{R}$ is regarded as the "time" direction and adopt a Hamiltonian point of view on $W = I \times C$.

The boundary conditions on I matter here, and it turns out that in the $\hat{A}$ model we should consider *Dirichlet* boundary conditions on one end and *Neumann* boundary conditions on the other. In the language of gauge theory, Dirichlet boundary conditions demand the bundle to be trivial on that boundary, while Neumann boundary conditions allow for it to be arbitrary.

Now 't Hooft lines look like points on the 3-manifold $W = I \times C$. We can locally take $\phi = \phi_4 dx^4$ so that on W, ϕ behaves as a scalar. Then, on W, Equation (6.7) reduces exactly to the Bogomolny equations for monopoles:

$$F = \star_3 D_A \phi.$$

Let's write a local coordinate $z \in \mathbb{C}$ parameterizing C and $\sigma \in \mathbb{R}$ parameterizing I. Gauging away $A_\sigma = 0$, these equations reduce to the following:

$$\partial_\sigma A_{\bar{z}} = -i D_{\bar{z}} \phi.$$

This condition can be interpreted as stating that the isomorphism class of the holomorphic G-bundle corresponding to the connection $A_{\bar{z}}$ is independent of y. This is because the right hand side corresponds to changing A by a gauge transformation generated by $-i\phi$. Thus, gauge transforming $A \to A + i\phi$ gives us a holomorphic connection on the new G-bundle, putting it in the same holomorphic class.

The only place where this is violated is at the values σ where the Bogomolny equations become singular. This is where we have the insertion of a 't Hooft operator.

It is worth noting that this construction follows very closely the inverse scattering approach of Hitchin[14][20]. In that case, the curve C corresponded to the (non-compact) Riemann surface $\mathbb{C}$ parameterizing the $x_1 - x_2$ plane, and lines along the x_3 direction take the place of our s variable along the unit interval I.

6.5.2 The Affine Grassmannian

The Langlands dual is defined to have the property that any highest weight representation $\hat{\rho} : \hat{G} \to U(1)$ is dual to a morphism $\rho : U(1) \to G$ which can be viewed as a *clutching function* for a G bundle on the Riemann sphere $\mathbb{CP}^1$. Complexifying this gives $\rho : G \to \mathbb{C}^\times \cong \mathbb{CP}^1 \backslash \{p, q\}$, AKA gluing a trivial bundle over $\mathbb{CP}^1 \backslash \{p\}$ to a trivial bundle over $\mathbb{CP}^1 \backslash \{q\}$. This is exactly what we call a Hecke modification of type ρ. Every holomorphic G-bundle over $\mathbb{CP}^1$ arises in this way. We can recognize this space of Hecke modifications as the affine Grassmannian $Gr_G = G((z))/G[[z]]$.

6.5.3 The Space of Physical States

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It turns out that for $\mathcal{N} = 4$ supersymmetric Yang Mills, the space of physical states is the (intersection) cohomology of the space of solutions to the Bogomolny equations with prescribed singularities labeled by $\check{R}_i, p_i$[1]. We denote this space by $\mathcal{Z}(\check{R}_1, p_1, \ldots, \check{R}_k, p_k)$. Because the underlying field theory is topological, and because the space of n-tuples on W is simply connected (so no monodromy can occur), we have that $\mathcal{Z}$ does not depend on the explicit positions of any of the p_i. Thus we can write $\mathcal{H}(\check{R}_1, \ldots \check{R}_k) = H^*(\mathcal{Z})$ and define this as the *space of physical states* for this given set of line defect insertions.

Further, $\mathcal{Z}(\check{R}_1, \ldots \check{R}_k)$ turns out to topologically be a simple product $\prod_{i=1}^k \mathcal{Z}(\check{R}_i)$ where $\mathcal{Z}(\check{R}_i)$ is the same as the compactified space $\mathcal{N}(\check{R}_i)$ of Hecke modifications of type $\check{R}_i$, then by using the fact that *the product of cohomologies is the cohomology of the product* we obtain:

$$\mathcal{H}(\check{R}_1, \ldots \check{R}_k) = \bigotimes_{i=1}^{k} \mathcal{H}(\check{R}_i) \tag{6.8}$$

This suggests that there is an isomorphism of $\check{R}_i$ and $\mathcal{H}(\check{R}_i)$ as vector spaces. Indeed, it can be shown that such an isomorphism is the only way for these categories of (finite dimensional) vector spaces to have the same monoidal structure.

[1]In general, there are so-called "instanton corrections" to this space of states, but they are absent in this situation for reasons relating to supersymmetry.

6.6 The Action of Wilson Loops on Boundary Conditions

If we assume that $M = \Sigma \times C$ for C a compact Riemann surface and Σ a (not necessarily compact) surface with boundary, we can study loop insertions more naturally. The following is a simplified picture of the general case:

Definition 6.6.1 (Hitchin's Moduli Space)**.** $\mathcal{M}_H(G, C)$ is the space of solutions to the Hitchin equations on a curve C.

If we consider C to be "small" relative to Σ, for each point in Σ, the additional data for the field configurations on the space C must give us a point in this moduli space. That is, we get a nonlinear sigma model on $\Sigma \to \mathcal{M}(G, C)$.

Let the curve defining a (Wilson or 't Hooft) operator be $\gamma = \gamma_0 \times p$ in $\Sigma \times C$ with p a point on C and γ_0 a curve on Σ. Let $\partial\Sigma_0$ be a connected component of $\partial\Sigma$. A boundary condition for the field theory on Σ_0 is called a **brane**.

Let γ_0 approach this boundary. On the $\hat{B}$ side, the insertion of a Wilson loop acts as an associative endofunctor for the category of boundary conditions on the topological sigma model on Σ with target $\mathcal{M}_H(G, C)$. This target space, with choice of complex structure J, can be identified with $\mathcal{M}_{flat}(G, C)$.

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This functor will depend on the point $p \in C$ corresponding to the Wilson line. Consider the product $\mathcal{M}_{flat}(G, C) \times C$. There is a universal G-bundle $\mathcal{E}$ over this space, given by taking a point in $\mathcal{M}_{flat}$ and restricting the corresponding bundle to a point in C.

Given any coherent sheaf on $\mathcal{M}_{flat}$, we can tensor this with $R(\mathcal{E})$. This is the action of the Wilson loop insertion on the space.

Consider the structure sheaf $\mathcal{O}_x$ of a point $x \in \mathcal{M}_{flat}(\check{G}, C)$. For any representation $\check{R}$, the Wilson loop maps $\mathcal{O}_x$ to $\mathcal{O}_x \otimes \check{R}$. Thus $\mathcal{O}_x$ is an eigenobject for the functor $W_{\check{R}}(p)$, which acts on it by tensoring it with the vector space $\check{R}(\mathcal{E}_p)_x$. In fact, letting p vary we see that it is an eigenobject for all $W_{\check{R}}(p)$. Another way of saying this is that the eigenvalue is the flat $\check{G}$-bundle $\check{R}(\mathcal{E})_x$ on C.

More directly, this flat bundle is obtained by taking the flat principle bundle on C corresponding to x and forming the associated bundle via $\check{R}$.

The action of the 't Hooft operators is more difficult to see. They will end up acting by Hecke transformations on the space of boundary conditions. By Monotonen-Olive duality, it turns out that the brane corresponding to a fiber of the Hitchin fibration in $\mathcal{M}_H(G, C)$ is a common eigenobject for all operators.

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