Component-based Implementation of Relaxed Unscented Kalman-Filtering

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***Abstract*—Filtering and sensor fusion plays a critical role in advanced engineering applications in robotics and autonomous vehicles. The problem is that the high accuracy and low computational cost are mutually exclusive. The Unscented Transformation was proposed as a golden mean between EKF and Particle Filter and lately a generic computational relaxation was proposed by the authors. It provided options to decrease the computational cost exploiting the partially linear nature of the mappings in the system model considered. This paper introduces an an open source C++ implementation *RelaxedUnscentedTransformation* that covers all of the functionality of the method. Because of the optional parts of the method, components were defined with examples. The provided numerical example shows that the implementation can highly decrease the computational cost and even provide opportunity to increase the accuracy of filtering.**

***Index Terms*—Kalman-filter, Unscented Transformation, Unscented Kalman-filter, sensor fusion, computational relaxation**

I. Introduction

Advanced engineering systems can contain complex mechatronics, where different sensors are only able to reflect on the changes in the state of the system. Sensor fusion from the output of the different sensors and the prediction from the latest estimated state can only lead to the optimal estimation about the state of the system that is needed for control.

The state and output update via their nonlinear models are called prediction. The traditional low computational way (with limited accuracy) to do this was the Extended Kalman-filter and later a much more accurate but computationally much more demanding method was the Particle Filtering. The Unscented Transformation-based (UT) Unscented Kalman-filtering (UKF) provided a trade-off between accuracy and computational cost [1], [2].

The UT applies so-called sigma points around the expected value according to the distribution of the stochastic variable. By applying the nonlinear mapping on these sigma points, the expected value and the distribution can be estimated in a more accurate way than via the local linearization of EKF. The UT-based Kalman-filter is referred to as Unscented Kalman-filter (UKF) [2].

In the standard formulation of UT, the n dimensional ellipsoid is estimated by one sigma point at the expected value and further $2n$ sigma points selected symmetrically around it. There are other methods where more (e.g., $4n+1$) sigma points are used to achieve a better approximation of the uncertainty [3]. These sigma points are computed via Cholesky-factorization [4] of the covariance matrix. An other approach decreases the number of sigma points to $(n+1)$ [5], [6] via an optimization step but it is computationally more expensive than the Cholesky-factorization. For a systematic overview, see [7].

The key idea of UKF was extended with multivariate adaptive methods (see [8]–[11]) to estimate the necessary parameters of filtering. But there are robust extensions, see [12]–[15] to deal with varying parameters, sensor faults, etc. The method can be combined with the well known methods to take into account the time-delay of the measured signals, see [16].

Although after the symmetric (Cholesky-factorisation-based) UT a lot of more methods were proposed, it remained the most often used method. Different scaling methods, Taylor-series based derivations was proposed to improve the method by using different weights and sigma point distances.

The methodology is widespread applied in distributed filtering [17], consensus filter [18], finite-horizon extended Kalman-filter [19] and the robustness of distributed filters is also analyzed [20]. It is applied in remote state estimation with stochastic event-triggered sensor schedule [21], to test multi-agent communication schemes [22] and robust, adaptive and consensus aim are also formulated in the framework, see [23].

Paper [24] of the authors proposed more methods to decrease the computational cost of UT in cases where the function does not depend on all of the variables in a nonlinear way. In these cases, less sigma point can be used together avoiding the unnecessary operations with computation seen in original, linear Kalman-filtering.

This paper aims to introduce the functionality of an Eigen-based C++ library (available at [25]) developed to exploit advantages of Relaxed Unscented Transformation in Kalman-filtering. Because the method provides a lot of options that can be fitted to the considered model, components were defined from the steps of the method that can be combined according to the considered problem.

The numerical example shows that not only the computational cost can be reduced to almost 50% but the accuracy of the filtering can be also increased.

II. Notations

$a, b, \ldots$	scalar values
$\mathbf{a}, \mathbf{b}, \ldots$	vectors
$\mathbf{A}, \mathbf{B}, \ldots$	matrices
$\mathbf{0}^{a\times b}$, $\mathbf{I}^{a\times b}$	zero matrix, identity matrix of size $a \times b$
$\mathbf{a}(\mathbf{i})$	reindexed vector with $\mathbf{i}$ index vector, as $\mathbf{a}(\mathbf{i}) = \begin{bmatrix} a_{i_1} & a_{i_2} & \ldots \end{bmatrix}^T$
$\mathbf{A}(\mathbf{i}, :)$	matrix with reindexed rows using the $\mathbf{i}$ index vector, as $\mathbf{A}(\mathbf{i}, :) = \begin{bmatrix} A_{i_1,1} & A_{i_1,2} & \ldots \\ A_{i_2,1} & A_{i_2,2} & \ldots \\ \vdots & \vdots & \end{bmatrix}$
$\mathbf{A}(:, \mathbf{i})$	matrix with reindexed columns using the $\mathbf{i}$ index vector, as $\mathbf{A}(\mathbf{i}, :) = \begin{bmatrix} A_{1,i_1} & A_{1,i_2} & \ldots \\ A_{2,i_1} & A_{2,i_2} & \ldots \\ \vdots & \vdots & \end{bmatrix}$
$\mathbf{A}(\mathbf{i}, \mathbf{j})$	reindexed matrix with $\mathbf{i}$, $\mathbf{j}$ index vectors, as $\mathbf{A}(\mathbf{i}, \mathbf{j}) = \begin{bmatrix} A_{i_1,j_1} & A_{i_1,j_2} & \ldots \\ A_{i_2,j_1} & A_{i_2,j_2} & \ldots \\ \vdots & \vdots & \end{bmatrix}$
$\sqrt{\mathbf{A}}$	lower triangle Choleski-factorization of matrix $\mathbf{A}$ as $\mathbf{A} = \sqrt{\mathbf{A}}\sqrt{\mathbf{A}}^T$
$\hat{\mathbf{x}}$	estimated value of $\mathbf{x}$
$\tilde{\mathbf{x}}$	difference of $\mathbf{x}$ from the expected value
Σ_{xx}	estimated covariance matrix $\Sigma_{xx} = E(\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T)$
Σ_{xy}	estimated cross covariance matrix $\Sigma_{xy} = E(\tilde{\mathbf{x}}\tilde{\mathbf{y}}^T)$

III. Original Unscented Transformation (UT) and the related components

The Unscented Transformation considers a continuous nonlinear mapping in general, as

$$\mathbf{y} = f(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is an (approximately) Gaussian stochastic variable with covariance matrix Σ_{xx}. The method approximates the expected value of $\mathbf{y}$, its covariance matrix Σ_{yy} and the cross covariance matrix Σ_{xy}.

The σ points are chosen around the expected value of $\mathbf{x}$, such that the expected values and covariance matrices computed from the σ points with appropriate weighting returns exact values for second order mappings.

In the Cholesky-factorization based method, the $(2m + 1)$ sigma points are chosen as

$$\mathcal{X}_0 = \hat{\mathbf{x}}, \quad \mathcal{X}_i = \hat{\mathbf{x}} - \sqrt{\kappa}\delta_i, \quad \mathcal{X}_{i+m} = \hat{\mathbf{x}} + \sqrt{\kappa}\delta_i, \tag{2}$$

where m denotes the rank of Σ_{xx}, δ_i is for the non-zero columns of matrix $\sqrt{\Sigma_{xx}}$ $(i = 1, ..., m)$, and κ is a parameter that covers various parameters of different approaches, e.g. the so-called scaling parameter that provides an extra degree of freedom to tune the method.

Then by mapping the sigma points as $\mathcal{Y}_i = f(\mathcal{X}_i)$, the expected value and the covariance matrices can be approximated as

$$\hat{\mathbf{y}} = W_0\mathcal{Y}_0 + W_1\sum_{i=1}^{2m}\mathcal{Y}_i,$$

$$\mathbf{\Sigma}_{yy} = V_0(\mathcal{Y}_0 - \hat{\mathbf{y}})(\mathcal{Y}_0 - \hat{\mathbf{y}})^T + V_1\sum_{i=1}^{2m}(\mathcal{Y}_i - \hat{\mathbf{y}})(\mathcal{Y}_i - \hat{\mathbf{y}})^T,$$

$$\mathbf{\Sigma}_{xy} = \sqrt{W_1V_1}\sum_{i=1}^{2m}(\mathcal{X}_i - \hat{\mathbf{x}})(\mathcal{Y}_i - \hat{\mathbf{y}})^T,$$

where weights W_0, W_1, V_0, V_1 and distance κ depends on the method to be applied.

The systematic investigations in [26], [27] showed that the accuracy of UT can be highly improved by tuning the scaling parameter in each step. A number of studies [27]–[30] proposed online optimisation for this purpose that performs the UT much more times in each sampling step. In these methods, the computational demand of UT is crucial.

From functional viewpoint, this method has two main parts:

1) The determination of vectors δ_i from the covariance matrix Σ_{xx}, their number will be equal to the rank of the matrix. This method is available in the repository as

```
std::vector<Eigen::VectorXd>
  GenSigmaDifferences(
  const Eigen::MatrixXd& S);
```

where the matrix `S` denotes the covariance matrix Σ_{xx} and the output is the `std::vector` of vectors δ_i $(i = 1, ..., m)$.

2) The second part is the computation of $\mathcal{X}_i$, then $\mathcal{Y}_i$ sigma points, and from the weighted summations the demanded values. It is available as

```
template<typename Func>
UT::ValWithCov UTCore(
  const Eigen::VectorXd& x,
  const std::vector<Eigen::VectorXd>&
  xdiffs, Func f,
  const UTSettings& settings);
```

where arbitrary weights and κ can be used (computed according to the value m) through the struct

```
struct UTSettings {
  double kappa, W0, W1, V0, V1;
  UTSettings(double kappa, double W0,
    double W1, double V0, double V1);
  UTSettings(double kappa, double W0,
    double W1); %Vi=Wi
};
```

that allows arbitrary UT parametrization like Scaled UT of [31], [32] but [INESKJ] proved relevance of original parameters for larger models.

The output of the method contains the expected value, the covariance matrix and the cross covariance matrix as

```
struct ValWithCov {
  Eigen::VectorXd y;
  Eigen::MatrixXd Sy, Sxy;
};
```

IV. Relaxed Unscented Transformation

The key feature of the method, that it can consider functions as

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x}(\mathbf{i}_l) + f(\mathbf{x}),$$

where f has a larger dimensional null space and, this way, less sigma point is enough to generate and perform the UT on. Furthermore, the linear part can also depend on a subset of the variables denoted by $\mathbf{i}_l$ indices.

In this case, the computation can be decomposed to three main steps: first the sigma points must be generated. Two different methods will be presented for this purpose. Then the properties of the output of the nonlinear part will be presented based on the determined sigma points. Then the properties of $\mathbf{y}$ will be computed by exploiting the properties of linear mappings.

A. Generation of sigma points

Denote the result of the nonlinear part as $\mathbf{b} = \mathbf{f}(\mathbf{x})$. The first challenge is to generate only as much sigma points as necessary. Paper [24] proposed two methods to obtain them.

1) Less sigma points by considering only a subset of variables: The nonlinear part depend on the variables with indices $\mathbf{i}_{nl}$. In this case, the Choleski-factorization must be performed only in the given columns.

The result can also be described as performing Cholesky-factorization on the rearranged Σ_x as

$$\Delta\mathbf{A} = \sqrt{\Sigma_x(\begin{bmatrix}\mathbf{i}_{nl} & \bar{\mathbf{i}}_{nl}\end{bmatrix}, \begin{bmatrix}\mathbf{i}_{nl} & \bar{\mathbf{i}}_{nl}\end{bmatrix})}, \tag{3}$$

and then $\Delta\mathbf{X}$ is constructed from its first m columns, by rearranging the rows as

$$\Delta\mathbf{X}(\begin{bmatrix}\mathbf{i}_{nl} & \bar{\mathbf{i}}_{nl}\end{bmatrix}, :) = \Delta\mathbf{A}(:, \begin{bmatrix}1 & \dots & m\end{bmatrix}), \tag{4}$$

where index vector $\bar{\mathbf{i}}_{nl}$ is constructed from the complementer set of indices in $\mathbf{i}_{nl}$ and the δ_i $(i = 1, ..., m)$ vectors are its columns.

This method is available with function

```
std::vector<Eigen::VectorXd>
  GenSigmaDifferences(
  const Eigen::MatrixXd& S,
  const Eigen::VectorXi& inl);
```

2) Less sigma points by considering a subspace of the domain: Assume that the nonlinear part depends on the linear combinations of the variables. Describe these values as $(\mathbf{m}_1 \cdot \mathbf{x}(\mathbf{i}_1))$, $(\mathbf{m}_2 \cdot \mathbf{x}(\mathbf{i}_2))$, ..., $(\mathbf{m}_K \cdot \mathbf{x}(\mathbf{i}_K))$, where $\mathbf{m}_k$ is the vector of weights for values with indices $\mathbf{i}_k$ and K denotes the number of combinations.

In this case, the function depends on values of $\mathbf{M} \cdot \mathbf{x}$, where

$$\mathbf{M}(k, \mathbf{i}_k) = \mathbf{m}_k, \quad k = 1, ..., K,$$

denote its rank by m.

Example 1: Considering a mapping, e.g.,

$$f(\mathbf{x}) = \sin(x_1 + 0.1x_3) - \cos(0.5x_2 + x_3),$$

two linear combinations of the variables can be seen in nonlinear functions. In the first case, the index vector is $\mathbf{i}_1 = \begin{bmatrix}1 & 3\end{bmatrix}$ *and the corresponding weight vector is* $\mathbf{m}_1 = \begin{bmatrix}1 & 0.1\end{bmatrix}$. *In the second linear combination, the index vector is* $\mathbf{i}_2 = \begin{bmatrix}2 & 3\end{bmatrix}$ *and the weight vector is* $\mathbf{m}_2 = \begin{bmatrix}0.5 & 1\end{bmatrix}$. *This way, the matrix* $\mathbf{M}$ *can be written in this case as*

$$\mathbf{M} = \begin{bmatrix}1 & 0 & 0.1\\ 0 & 0.5 & 1\end{bmatrix}.$$

Denote the matrix constructed from an orthonormal basis of the rowspace of $\mathbf{M}$ by $\mathbf{Q}_1 \in \mathbb{R}^{m\times n}$ and the matrix constructed from an orthonormal basis of nullspace of $\mathbf{M}$ by $\mathbf{Q}_2 \in \mathbb{R}^{(n-m)\times n}$. The matrix $\mathbf{Q}$ is constructed of them as

$$\mathbf{Q} = \begin{bmatrix}\mathbf{Q}_1\\ \mathbf{Q}_2\end{bmatrix}, \tag{5}$$

and can be easily computed as RQ factorization of $\mathbf{M}$.

The Choleski-factorization of rearranged Σ_x must be computed as

$$\Delta\mathbf{A} = \sqrt{\mathbf{Q}\Sigma_x\mathbf{Q}^T}, \tag{6}$$

and the $\Delta\mathbf{X}$ is constructed from its first m columns, by rearranging the rows as

$$\Delta\mathbf{X} = \mathbf{Q}^T\Delta\mathbf{A}(:, \begin{bmatrix}1 & \dots & m\end{bmatrix}) \tag{7}$$

and the vectors δ_i are its columns.

The matrix $\mathbf{Q}$ and value m can be determined via the constructor of the following struct:

```
struct ExactSubspace {
  struct MixedNonlin {
    Eigen::VectorXi i;
    Eigen::VectorXd M;
  };
  typedef std::vector<MixedNonlin>
  MixedNonlinearityList;
  Eigen::SparseMatrix<double> Q;
  int m;
  ExactSubspace(int n, const
    Eigen::VectorXi& inl, const
    MixedNonlinearityList& mix);
};
```

This method must be used only at initialization of the program.

From this data, the sigma points can be computed via the method:

```
std::vector<Eigen::VectorXd>
  GenSigmaDifferences(const
  Eigen::MatrixXd& S,
  const ExactSubspace& sp);
```

B. Computing Sigma points and approximations from them

The computation of output of this nonlinear $\mathbf{b} = f(\mathbf{x})$ is the same as before: the `UTCore(...)` function can be used to do it, but in this case it results in $\hat{\mathbf{b}}$, Σ_{bb}, Σ_{xb} values.

C. Merging linear and nonlinear results

It considers the problem of

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x}(\mathbf{i}_l) + \mathbf{b} \tag{8}$$

where $\hat{\mathbf{x}}$, Σ_{xx}, $\hat{\mathbf{b}}$, Σ_{bb}, Σ_{xb} are given from the previous computations. In order to solve it, the library implements optimized version of computation:

$$\begin{aligned}\hat{\mathbf{y}} &= \mathbf{A}\cdot\hat{\mathbf{x}}(\mathbf{i}_l) + \hat{\mathbf{b}}, \\ \Sigma_y &= \Sigma_b + \mathbf{A}\Sigma_x(\mathbf{i}_l,\mathbf{i}_l)\mathbf{A}^T + Sym(\mathbf{A}\Sigma_{xb}(\mathbf{i}_l,:)), \\ \Sigma_{xy} &= \Sigma_{xb} + \Sigma_x(:,\mathbf{i}_l)\mathbf{A}^T.\end{aligned} \tag{9}$$

That is available as:

```
UT::ValWithCov MixedLinSources(
  const UT::ValWithCov& x,
  const UT::ValWithCov& b,
  const Eigen::VectorXi& il,
  const Eigen::MatrixXd& A);
```

D. Reduced size output of the nonlinear function

The previous components can be enough to perform the Relaxed Unscented Transformation with smaller computational cost than the original UT.

The following to component can decrease the computational cost further, in case the values in the nonlinear function are not linearly independent.

The first one is that the function can be defined as

$$\mathbf{b} = \begin{bmatrix}\mathbf{b}_0 \\ \mathbf{F}\cdot\mathbf{b}_0\end{bmatrix}, \quad \mathbf{b}_0 = f_0(\mathbf{x}). \tag{10}$$

In this case, the $\mathbf{b}$ related quantities can be computed as

$$\hat{\mathbf{b}} = \begin{bmatrix}\hat{\mathbf{b}}_0 \\ \mathbf{F}\hat{\mathbf{b}}_0\end{bmatrix}, \qquad \Sigma_{xb} = \begin{bmatrix}\Sigma_{xb_0} & \Sigma_{xb_0}\mathbf{F}^T\end{bmatrix}, \tag{11}$$
$$\Sigma_b = \begin{bmatrix}\Sigma_{b_0} & \Sigma_{b_0}\mathbf{F}^T \\ \mathbf{F}\Sigma_{b_0} & \mathbf{F}\Sigma_{b_0}\mathbf{F}^T\end{bmatrix}.$$

that is available in the component

```
ValWithCov LinearMappingOnb(
  const ValWithCov& b0,
  const Eigen::MatrixXd& F);
```

A similar case, where

$$\mathbf{b} = \begin{bmatrix}0 \\ \mathbf{b}_0 \\ \mathbf{F}\cdot\mathbf{b}_0\end{bmatrix}, \quad \mathbf{b}_0 = f_0(\mathbf{x}), \tag{12}$$

is also implemented and is available as

```
ValWithCov LinearMappingOnbWith0(
  const ValWithCov& b0,
  const Eigen::MatrixXd& F);
```

If there are multiple entries of this expression, or only the order must be changed reordering can be applied as

$$\mathbf{b} = \mathbf{b}_0(\mathbf{g}), \tag{13}$$

then

$$\hat{\mathbf{b}} = \hat{\mathbf{b}}_0(\mathbf{g}), \quad \Sigma_b = \Sigma_{b_0b_0}(\mathbf{g},\mathbf{g}), \quad \Sigma_{xb} = \Sigma_{xb_0}(:,\mathbf{g}). \tag{14}$$

where $\mathbf{g}\in\mathbb{N}^b$ is an index vector and $f_0:\mathbb{R}^n\to\mathbb{R}^b$.

It is implemented as

```
ValWithCov Reordering(const ValWithCov&
  b0, const Eigen::VectorXi& g);
```

V. Multiscaled Unscented Transformation

In Multiscaled Unscented Transformation instead of $(2m+1)$ sigma points, $(2m\cdot N_{max}+1)$ sigma points are used, defined as

$$\mathcal{X}_0 = \hat{\mathbf{x}}, \quad \mathcal{X}_i^{(N)} = \hat{\mathbf{x}} - \sqrt{\kappa_N}\delta_i, \quad \mathcal{X}_{i+m}^{(N)} = \hat{\mathbf{x}} + \sqrt{\kappa_N}\delta_i, \tag{15}$$

where δ_i denotes the non-zero columns of matrix $\sqrt{\Sigma_x}$, m their numbers, $i=1,...,m$, and κ_N is the parameter for $N=1,...,N_{max}$ circle of sigma points.

Then by mapping the sigma points as $\mathcal{Y}_i^{(N)} = f(\mathcal{X}_i^{(N)})$, the expected value and the covariance matrices can be approximated as

$$\begin{aligned}\hat{\mathbf{y}} &= W_0\mathcal{Y}_0 + \sum_{N=1}^{N_{max}} W_N \sum_{i=1}^{2n}\mathcal{Y}_i^{(N)}, \\ \mathbf{\Sigma}_{yy} &= V_0(\mathcal{Y}_0-\hat{\mathbf{y}})(\mathcal{Y}_0-\hat{\mathbf{y}})^T + \\ &\quad + \sum_{N=1}^{N_{max}} V_N \sum_{i=1}^{2n}(\mathcal{Y}_i^{(N)}-\hat{\mathbf{y}})(\mathcal{Y}_i^{(N)}-\hat{\mathbf{y}})^T, \\ \mathbf{\Sigma}_{xy} &= \sum_{N=1}^{N_{max}}\sqrt{W_N V_N}\sum_{i=1}^{2n}(\mathcal{X}_i^{(N)}-\hat{\mathbf{x}})(\mathcal{Y}_i^{(N)}-\hat{\mathbf{y}})^T,\end{aligned}$$

where weights W_N, V_N and distances κ_N depend on the method to be applied.

This computation is available as

```
template<typename Func>
UT::ValWithCov MultiScaledUTCore(const
  Eigen::VectorXd& x, const
  std::vector<Eigen::VectorXd>& xdiffs,
  Func f,
  const MultiScaledUTSettings& settings);
```

where the necessary constants are provided in the struct

```
struct MultiScaledUTSettings {
  std::vector<double> kappa, W, V;
  double W0, V0;
};
```

VI. Kalman-filtering

For the sake of convenience, the well-known equations of the Kalman-filtering

$$\begin{aligned}\hat{\mathbf{x}} &= \overline{\mathbf{x}} - \mathbf{K}(\overline{\mathbf{y}} - \mathbf{y}_{meas}), \\ \hat{\Sigma}_{xx} &= \overline{\Sigma}_{xx} - \mathbf{K}\overline{\Sigma}_{yx},\end{aligned}$$

where $\mathbf{K} = \overline{\Sigma}_{xy}\overline{\Sigma}_{yy}^{-1}(1-\epsilon)$, are also available as

```
ValWithCov KalmanFilter(const ValWithCov&
  x, const ValWithCov& y, const
  Eigen::VectorXd& ymeas,
  double eps = 1e-5);
```

where ϵ is a small number to avoid negative eigen values, but also highly depends on the parity of weight W_0.

VII. UKF FROM THE GIVEN COMPONENTS

A. UT of a function

1) Original UT: If there is a nonlinear function, a struct can be built around it like

```
struct StateUpdate {
  static VectorXd f(const VectorXd& x) {
    VectorXd out = VectorXd::Zero(11);
    out(0) = ....;
    ...
    return out;
  }
  ValWithCov UT(const ValWithCov& x) {
    auto xdiffs =
      GenSigmaDifferences(x.Sy);
    int m = xdiffs.size();
    return UTCore(x.y, xdiffs, f,
      UTSettings(m, 0, 0.5 / double(m)));
  }
};
```

2) Relaxed UT: in this case the necessary values can be initialized in the constructor and only used in the UT subroutine, as

```
struct StateUpdate {
  Eigen::MatrixXd A, F;
  Eigen::VectorXi il, inl;
  StateUpdate() {
    // Deriving matrix A
    A = Eigen::MatrixXd::Zero(11, 6);
    for (int i = 0; i < 6; i++)
    A(i, i) = 1;
    A(6, 2) = Ts;
    // il, inl
    il = Eigen::VectorXi(6);
    for (int n = 0; n < 6; n++)
      il(n) = n;
    inl = Eigen::VectorXi(5);
    for (int n = 0; n < 5; n++)
      inl(n) = n + 6;
    F = Eigen::MatrixXd::Zero(0, 11);
  }
  static VectorXd f(const VectorXd& x) {
    VectorXd out = VectorXd::Zero(11);
    out(0) = ....;
    ...
    return out;
  }
  ValWithCov UT(const ValWithCov& x) {
    auto xdiffs = GenSigmaDifferences(
      x.Sy, inl);
    auto b0 = UTCore(x.y, xdiffs, f,
      UTSettings(m, 0, 0.5 / double(m)));
    auto b = LinearMappingOnb(b0, F);
    return MixedLinSources(x, b, il, A);
  }
};
```

where $\kappa = m$, $W_0 = V_0 = 0$, $W_1 = V_1 = 1/2m$ weights were applied.

(Similarly, struct `ExactSubspace` can be also initialized and used in function `GenSigmaDifferences(...)` or if reordering is needed, the function `StateUpdate::UT` must return with

```
return MixedLinSources(x,
  Reordering(b, g), il, A);
```

where vector `g` is initialized in the constructor.)

B. Implementation of Unscented Kalman-Filter

If the standard UT is applied, the system model is traditionally defined as

$$\mathbf{x}_k = f_k(\mathbf{x}_{k-1}) + \mathbf{w}_k, \tag{16}$$

$$\mathbf{y}_k = g_k(\mathbf{x}_k) + \mathbf{v}_k, \tag{17}$$

or it can depend on the disturbance and noise signals in a nonlinear way and the $\mathbf{x}_0$ value is considered an initial value. In this case, the output of the nonlinear function can be computed via UT, and the covariance must be increased with Σ_{ww} and Σ_{vv} (and the values with $\hat{\mathbf{w}}$ and $\hat{\mathbf{v}}$ if they are not zero).

But to exploit the benefits of the Relaxed UT, one or both of them can be written in a partially linear form as

$$\mathbf{x}_k = \mathbf{A}_k \mathbf{x}_{k-1,l} + f_k(\mathbf{x}_{k-1}) + \mathbf{w}_k, \tag{18}$$

or in a more relaxed form

$$\mathbf{x}_k = \mathbf{A}_k \mathbf{x}_{k-1,l} + \mathbf{b}(\mathbf{g}) + \mathbf{w}_k, \tag{19}$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{F}\mathbf{b}_0 \end{bmatrix}, \qquad \mathbf{b}_0 = f_k(\mathbf{x}_{k-1}). \tag{20}$$

If the function depends on the $\mathbf{w}$ disturbance or $\mathbf{v}$ noise in a more sophisticated way, the function can be considered on the variable

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}$$

and the same formalisms can be used.

In all of the cases, a simple struct can wrap the specifics of the UT for either the state update or the output update. (Via so-called lambda functions the varying sampling time T_s or other parameters can be also injected into the functions.)

Then the implementation of the simple Kalman-filter can be implemented as

```
(Inputs: ValWithCov xold;
  Eigen::VectorXd ymeas, w, v;
  Eigen::MatrixXd Sw, Sv;
  StateUpdate stateUpdate;
  OutputUpdate outputUpdate;)
auto xnew = stateUpdate.UT(xold);
xnew.Sy += Sw;
xnew.y += w;
auto ynew = outputUpdate.UT(xnew);
ynew.Sy += Sv;
ynew.y += v;
auto xnewfiltered = KalmanFilter(
  xnew, ynew, ymeas);
```

It is easily extendable to achieve an adaptive UT method like [Straka]...

C. Implemented compact methods

The repository provides compact methods for the original UT or some types of the relaxed UT as

```
template <typename Func>
ValWithCov FullUT(Func f,
  const ValWithCov& x);
```

```
template <typename Func>
ValWithCov RelaxedUT(
  const Eigen::MatrixXd& A,
  const Eigen::VectorXi& il, Func f,
  const Eigen::MatrixXd& F,
  const Eigen::VectorXi& g,
  const Eigen::VectorXi& inl,
  const ValWithCov& x);
```

But they cover only a small subset of the functionality, and the large number of arguments should be also wrapped for the sake of compactness. For these reasons, the building from the given components are recommended and they provide only examples for that.

VIII. Comparison of functions and methods

A. Compared approaches

In this section, the following function

$$f(\mathbf{x}) = \begin{bmatrix} \sin(x_1 + 4x_2 - 0.5x_3) \\ \cos(x_1 + 4x_2 - 0.5x_3) \\ x_4 + x_5 \\ x_4 + x_6 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.$$

Here the following implementations will be compared:

1) Original UT on the nonlinear function using $2 \cdot 6 + 1$ sigma points.
2) Relaxed UT considering the function as

$$f(\mathbf{x}) \;=\; \mathbf{A} \cdot \mathbf{x}(\mathbf{i}_l) + \begin{bmatrix} \sin(x_1 + 4x_2 - 0.5x_3) \\ \cos(x_1 + 4x_2 - 0.5x_3) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

where $\mathbf{i}_l = [4, 5, 6]$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

applying
   a) using $2 \cdot 3 + 1$ sigma points,
   b) using $2 \cdot 1 + 1$ sigma points applying exact subspace.
3) Relaxed UT considering the function as

$$f(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}(\mathbf{i}_l) + \mathbf{b}([2, 3, 1, 1, 1, 1, 1]),$$

where

$$\mathbf{b} = \begin{bmatrix} 0 \\ f_0(\mathbf{x}) \end{bmatrix}, \quad f_0(\mathbf{x}) = \begin{bmatrix} \sin(x_1 + 4x_2 - 0.5x_3) \\ x_1 \cos(x_1 + 4x_2 - 0.5x_3) \end{bmatrix}.$$

applying
   a) using $2 \cdot 3 + 1$ sigma points,
   b) using $2 \cdot 1 + 1$ sigma points applying exact subspace.

B. Comparison of computational cost

First the computational cost of the different implementations are compared. All of them was called 10^7 times on an i7-9750H (2.6GHz x64) processor. The measured average computational times are discussed in Table I.

The first column in Table I refers to the original UT. It can be seen the reducing the sigma points from $2 \cdot 6 + 1$ to $2 \cdot 3 + 1$ in approach 2a does reduce the computational cost 80% and by reducing it to $2 \cdot 1 + 1$ in approach 2b reduces further it to 53%.

	1	2/a	2/b	3/a	3/b
Comp. time[μs]	9.436	7.512	5.075	7.772	6.027

TABLE I
Comparison of average computational cost of different relaxed approaches with the original UT

The next columns show that the reduced output size did not decreased the computational time because it is not crucial if there is only a few sigma points as in these cases.

C. Comparison of accuracy

To compare the accuracy of the relaxed UT approach with the original UT method considering random inputs (number of 10^4 with given $Tr(\Sigma_{xx})$) with values resulted by Monte-Carlo simulations of number 10^5 samples.

The results of the method highly depends on the applied κ W_i and V_i values. In order to ensure positive definiteness of matrices

$$\Sigma_{yy}, \quad (\Sigma_{xx} - \Sigma_{xy}(\Sigma_{yy} + \Sigma_{vv})^{-1}\Sigma_{yx})$$

where $\Sigma_{vv} \to \mathbf{0}$ is assumed as a worst case scenario, all of the W_i weights must be positive. By applying

$$W_1 = V_1 = \frac{1}{2\kappa}, \quad W_0 = V_0 = 1 - \frac{m}{\kappa},$$

where $2m + 1$ is the number of sigma points, it can be seen that $\kappa \geq m$ must be used.

From this condition, and Taylor-series-based derivations that prefer $\kappa = 2, 3$ values, the following situations will be considered in the next comparison:

- (1) $\kappa = m = 6$,
- (2/a) and (3/a) $\kappa = m = 3$,
- (2/b) and (3/b) where $m = 1$: $\kappa = 1, 2, 3$.

To characterize the results, the following values were computed

$$\epsilon(\mathbf{y}) = mean(||E(\mathbf{y}) - \hat{\mathbf{y}}||),$$
$$\epsilon(\Sigma_{yy}) = mean(||E(\Sigma_{yy}) - \hat{\Sigma}_{yy}||),$$

where $E(\mathbf{y})$ and $E(\Sigma_{yy})$ are computed from the Monte-Carlo simulations and mean is computed on the 10^4 random data.

The results can be seen in Table II for $Tr(\Sigma_{xx}) = 0.1$ and $Tr(\Sigma_{xx}) = 1$ cases. It can be seen that the relaxed UT approaches overwhelm the original UT by allowing larger variety of κ values and this increase in accuracy is significant.

IX. Conclusion

The paper introduced the functionality of the C++ library for Relaxed Unscented Transformation. It has showed the implemented components, their mathematical role and practical usage. Finally, a numerical example has showed that the appropriate setup in the considered case was able to reduce the computational cost to 53% and to increase the accuracy of filtering.

	$Tr(\Sigma_{xx}) = 0.1$		$Tr(\Sigma_{xx}) = 1$	
	$\epsilon(\mathbf{y})$	$\epsilon(\Sigma_{yy})$	$\epsilon(\mathbf{y})$	$\epsilon(\Sigma_{yy})$
1($\kappa = 6$)	5.53e-5	3.26e-3	1.19e-1	0.283
2/a($\kappa = 3$)	2.27e-5	8.48e-4	2.03e-2	0.153
2/b($\kappa = 1$)	8.28e-5	4.55e-3	1.60e-1	0.472
2/b($\kappa = 2$)	2.12e-5	6.51e-4	8.73e-3	0.197
2/b($\kappa = 3$)	3.79e-6	1.55e-4	2.30e-2	0.336
3/a($\kappa = 3$)	2.27e-5	8.48e-4	2.03e-2	0.153
3/b($\kappa = 1$)	8.28e-5	4.55e-3	1.60e-1	0.472
3/b($\kappa = 2$)	2.12e-5	6.51e-4	8.73e-3	0.197
3/b($\kappa = 3$)	3.79e-6	1.55e-4	2.30e-2	0.336

TABLE II
COMPARISON OF ACCURACY OF ORIGINAL UT WITH RELAXED UT APPROACHES WITH DIFFERENT WEIGHTS AND UNDER DIFFERENT $Tr(\Sigma_{xx})$ CIRCUMSTANCES

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REFERENCES

[1] S. J. Julier, J. K. Uhlmann, and H. F. Durrant-Whyte, "A new approach for filtering nonlinear systems," in *Proceedings of 1995 American Control Conference-ACC'95*, vol. 3. IEEE, 1995, pp. 1628–1632.

[2] S. J. Julier and J. K. Uhlmann, "Unscented filtering and nonlinear estimation," *Proceedings of the IEEE*, vol. 92, no. 3, pp. 401–422, 2004.

[3] R. Radhakrishnan, A. Yadav, P. Date, and S. Bhaumik, "A new method for generating sigma points and weights for nonlinear filtering," *IEEE Control Systems Letters*, vol. 2, no. 3, pp. 519–524, 2018.

[4] N. J. Higham, "Cholesky factorization," *Wiley Interdisciplinary Reviews: Computational Statistics*, vol. 1, no. 2, pp. 251–254, 2009.

[5] S. J. Julier, "The spherical simplex unscented transformation," in *Proceedings of the 2003 American Control Conference, 2003.*, vol. 3. IEEE, 2003, pp. 2430–2434.

[6] S. J. Julier and J. K. Uhlmann, "Reduced sigma point filters for the propagation of means and covariances through nonlinear transformations," in *Proceedings of the 2002 American Control Conference (IEEE Cat. No. CH37301)*, vol. 2. IEEE, 2002, pp. 887–892.

[7] H. M. T. Menegaz, J. Y. Ishihara, G. A. Borges, and A. N. Vargas, "A Systematization of the Unscented Kalman Filter Theory," *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2583–2598, 2015.

[8] W. Khalaf, I. Chouaib, and M. Wainakh, "Novel adaptive UKF for tightly-coupled INS/GPS integration with experimental validation on an UAV," *Gyroscopy and Navigation*, vol. 8, no. 4, pp. 259–269, 2017.

[9] M. Bozorg, M. S. Bahraini, and A. B. Rad, "New adaptive UKF algorithm to improve the accuracy of SLAM," *International Journal of Robotics, Theory and Applications*, vol. 5, no. 1, pp. 35–46, 2019.

[10] S. Gao, G. Hu, and Y. Zhong, "Windowing and random weighting-based adaptive unscented Kalman filter," *International Journal of Adaptive Control and Signal Processing*, vol. 29, no. 2, pp. 201–223, 2015.

[11] G.-g. Hu, S.-s. Gao, and Y. Zhao, "Novel adaptive UKF and its application in integrated navigation [j]," *Journal of Chinese Inertial Technology*, vol. 22, no. 3, pp. 357–361, 2014.

[12] S. Ishihara and M. Yamakita, "Gain constrained robust UKF for nonlinear systems with parameter uncertainties," in *2016 European Control Conference (ECC)*. IEEE, 2016, pp. 1709–1714.

[13] B. Gao, S. Gao, Y. Zhong, G. Hu, and C. Gu, "Interacting multiple model estimation-based adaptive robust unscented Kalman filter," *International Journal of Control, Automation and Systems*, vol. 15, no. 5, pp. 2013–2025, 2017.

[14] J. Zhao, M. Netto, and L. Mili, "A robust iterated extended Kalman filter for power system dynamic state estimation," *IEEE Transactions on Power Systems*, vol. 32, no. 4, pp. 3205–3216, 2016.

[15] W. Li, S. Sun, Y. Jia, and J. Du, "Robust unscented Kalman filter with adaptation of process and measurement noise covariances," *Digital Signal Processing*, vol. 48, pp. 93–103, 2016.

[16] A. Hermoso-Carazo and J. Linares-Pérez, "Unscented filtering from delayed observations with correlated noises," *Mathematical Problems in Engineering*, 2009.

[17] Y. Zhang, B. Chen, L. Yu, and D. W. C. Ho, "Distributed kalman filtering for interconnected dynamic systems," *IEEE Transactions on Cybernetics*, pp. 1–10, 2021.

[18] B. Lian, Y. Wan, Y. Zhang, M. Liu, F. L. Lewis, and T. Chai, "Distributed kalman consensus filter for estimation with moving targets," *IEEE Transactions on Cybernetics*, pp. 1–13, 2020.

[19] P. Duan, Z. Duan, Y. Lv, and G. Chen, "Distributed finite-horizon extended kalman filtering for uncertain nonlinear systems," *IEEE Transactions on Cybernetics*, vol. 51, no. 2, pp. 512–520, 2021.

[20] B. Lian, F. L. Lewis, G. A. Hewer, K. Estabridis, and T. Chai, "Robustness analysis of distributed kalman filter for estimation in sensor networks," *IEEE Transactions on Cybernetics*, pp. 1–12, 2021.

[21] L. Li, D. Yu, Y. Xia, and H. Yang, "Remote nonlinear state estimation with stochastic event-triggered sensor schedule," *IEEE Transactions on Cybernetics*, vol. 49, no. 3, pp. 734–745, March 2019.

[22] M. C. Fowler, T. C. Clancy, and R. K. Williams, "Intelligent knowledge distribution: Constrained-action pomdps for resource-aware multiagent communication," *IEEE Transactions on Cybernetics*, 2020.

[23] W. Li, G. Wei, F. Han, and Y. Liu, "Weighted average consensus-based unscented kalman filtering," *IEEE Transactions on Cybernetics*, vol. 46, no. 2, pp. 558–567, 2016.

[24] J. Kuti, I. J. Rudas, H. Gao, and P. Galambos, "Computationally Relaxed Unscented Kalman Filter," *IEEE Transactions on Cybernetics*, 2022, accepted.

[25] J. Kuti, "C++ library for relaxed unscented transformation," https://github.com/ABC-iRobotics/RelaxedUnscentedTransformation.git, 2021.

[26] O. Straka, J. Dunik, and M. Simandl, "Scaling parameter in unscented transform: Analysis and specification," in *2012 American Control Conference (ACC)*. IEEE, 2012, pp. 5550–5555.

[27] O. Straka, J. Duník, and M. Šimandl, "Unscented Kalman filter with advanced adaptation of scaling parameter," *Automatica*, vol. 50, no. 10, pp. 2657–2664, 2014.

[28] J. Dunik, M. Simandl, and O. Straka, "Unscented Kalman filter: aspects and adaptive setting of scaling parameter," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2411–2416, 2012.

[29] L. A. Scardua and J. J. da Cruz, "Adaptively tuning the scaling parameter of the unscented Kalman filter," in *CONTROLO'2014–Proceedings of the 11th Portuguese Conference on Automatic Control*. Springer, 2015, pp. 429–438.

[30] M. S. Bahraini, M. Bozorg, and A. B. Rad, "A new adaptive UKF algorithm to improve the accuracy of slam," *Int J Robot*, 2018.

[31] S. J. Julier, "The scaled unscented transformation," in *Proceedings of the 2002 American Control Conference (IEEE Cat. No. CH37301)*, vol. 6. IEEE, 2002, pp. 4555–4559.

[32] E. A. Wan and R. Van Der Merwe, "The unscented Kalman filter for nonlinear estimation," in *Proceedings of the IEEE 2000 Adaptive Systems for Signal Processing, Communications, and Control Symposium (Cat. No. 00EX373)*. Ieee, 2000, pp. 153–158.