Probabilistic
Graphical
Models

Acting

Decision Making

Maximum Expected Utility

Simple Decision Making

A simple decision making situation $\mathcal{D}$:

- A set of possible actions $Val(A)=\{a^1,\ldots,a^K\}$
- A set of states $Val(\mathbf{X}) = \{\mathbf{x}^1,\ldots,\mathbf{x}^N\}$
- A distribution $P(\mathbf{X} \mid A)$
- A utility function $U(\mathbf{X}, A)$

Expected Utility

$$\mathrm{EU}[\mathcal{D}[a]] = \sum_{\boldsymbol{x}} P(\boldsymbol{x} \mid a) U(\boldsymbol{x}, a)$$

- Want to choose action a that maximizes the expected utility

Max. expected utility

$$\boxed{a^* = \mathrm{argmax}_a \mathrm{EU}[\mathcal{D}[a]]}$$

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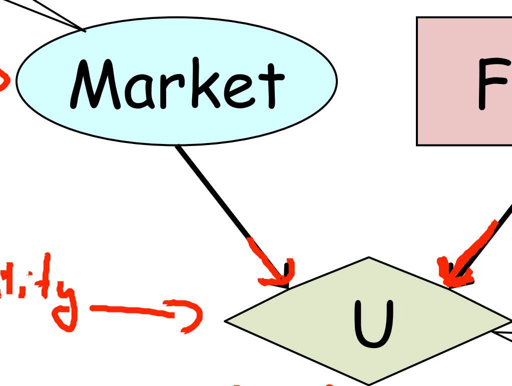
Simple Influence Diagram

poor	moderate	great
m^0	m^1	m^2
0.5	0.3	0.2

random variables



Action

Utility

$u(F, M)$

	do not found	found
	f^0	f^1
m^0	0	-7
m^1	0	5
m^2	0	20

$EU(f^0) = 0$

$\boxed{EU(f^1)} = 0.5 * -7 + 0.3 * 5 + 0.2 * 20 = 2$

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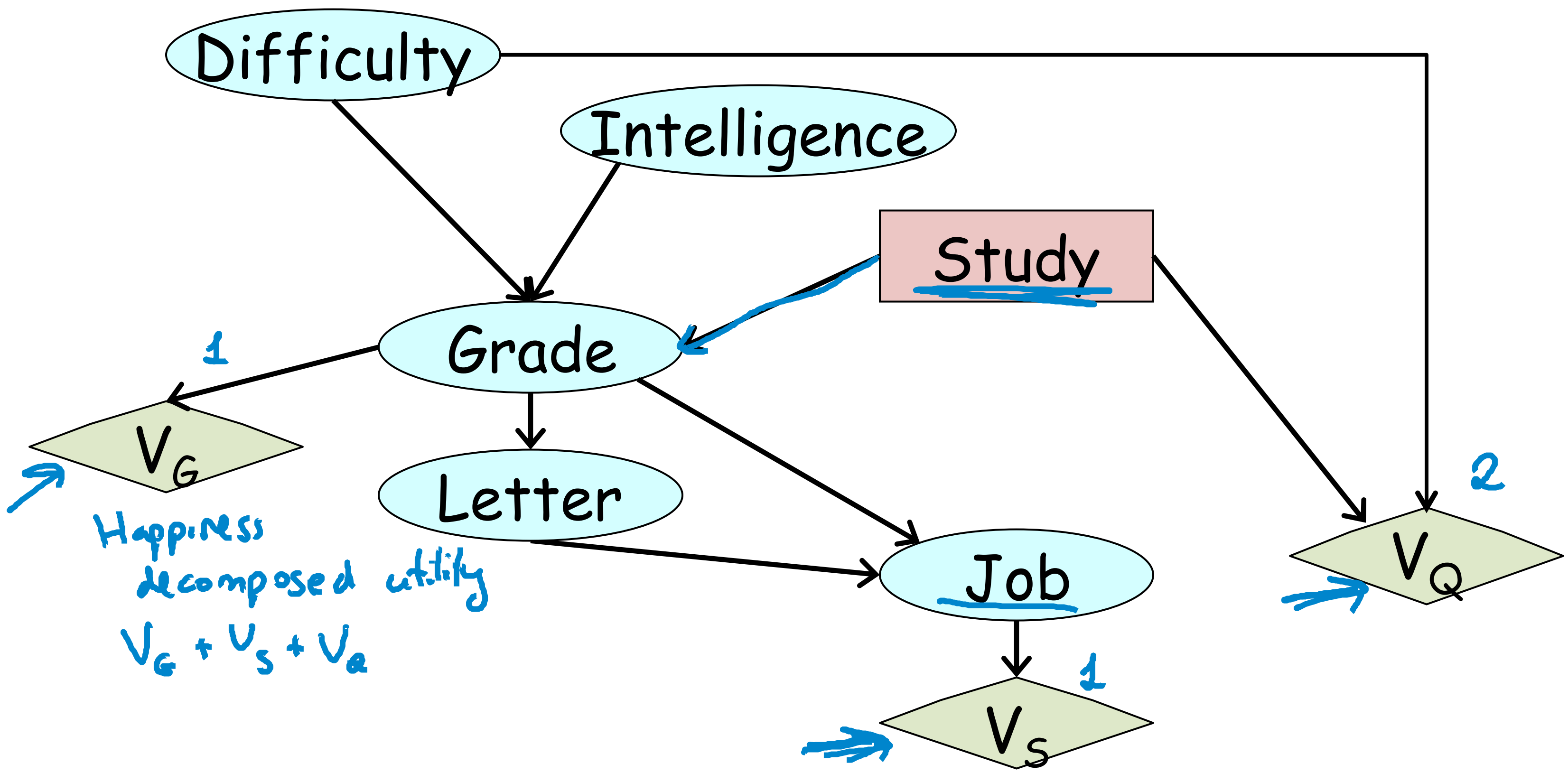
More Complex Influence Diagram



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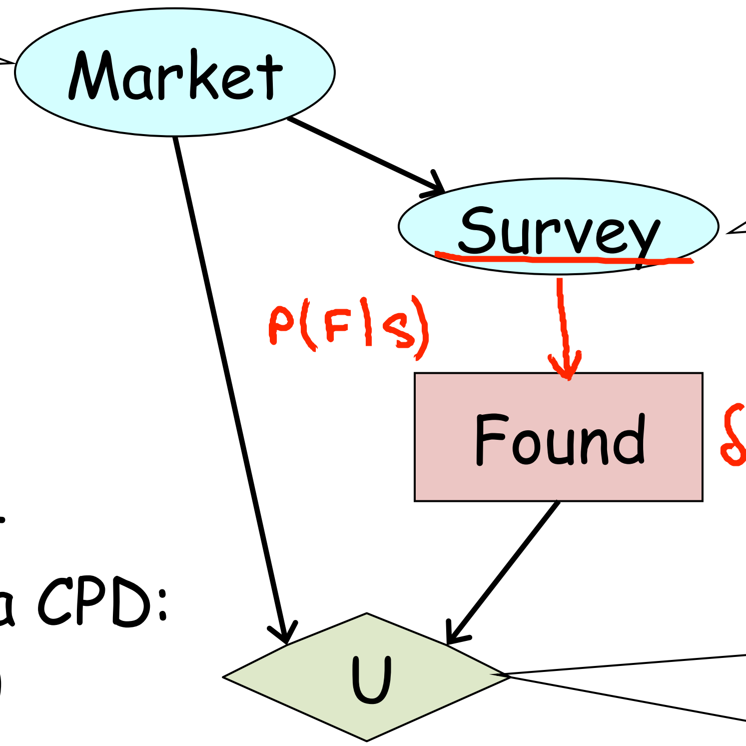
Information Edges

m^0	m^1	m^2
0.5	0.3	0.2



CPD P(S|M)

	s^0	s^1	s^2
m^0	0.6	0.3	0.1
m^1	0.3	0.4	0.3
m^2	0.1	0.4	0.5

	f^0	f^1
m^0	0	-7
m^1	0	5
m^2	0	20

Decision rule δ at action node A is a CPD: $P(A \mid \text{Parents}(A))$

Expected Utility with Information

joint prob. dist over $\bar{X} \cup \{A\}$

$$\mathrm{EU}[\mathcal{D}[\boxed{\delta_A}]] = \sum_{\boldsymbol{x},a} \underbrace{P_{\delta_A}(\boldsymbol{x},a)}\underbrace{U(\boldsymbol{x},a)}$$

- Want to choose the decision rule $\boxed{\delta_A}$ that maximizes the expected utility

$$\mathrm{argmax}_{\delta_A}\mathrm{EU}[\mathcal{D}[\delta_A]]$$

$$\mathrm{MEU}(\mathcal{D}) = \max_{\delta_A}\mathrm{EU}[\mathcal{D}[\delta_A]]$$

Finding MEU Decision Rules

optimize

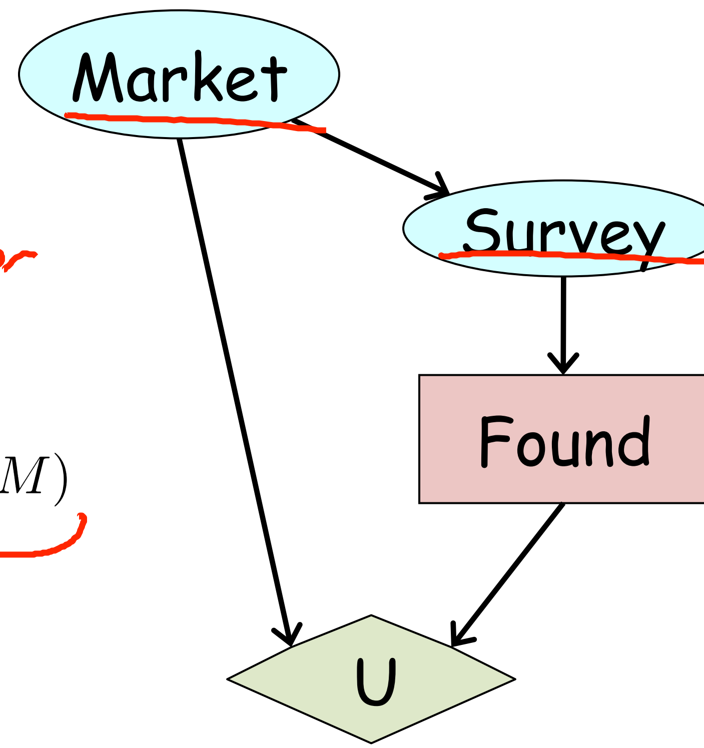
$$\mathrm{EU}[\mathcal{D}[\delta_A]] = \sum_{\boldsymbol{x},a} P_{\delta_A}(\boldsymbol{x},a)U(\boldsymbol{x},a)$$

factor

$$\sum_{M,S,F} P(M)P(S \mid M)\delta_F(F \mid S)U(F,M) =$$

$$= \sum_{S,F} \delta_F(F \mid S) \sum_{M} P(M)P(S \mid M)U(F,M)$$

$$= \sum_{S,F} \delta_F(F \mid S)\mu(F,S)$$



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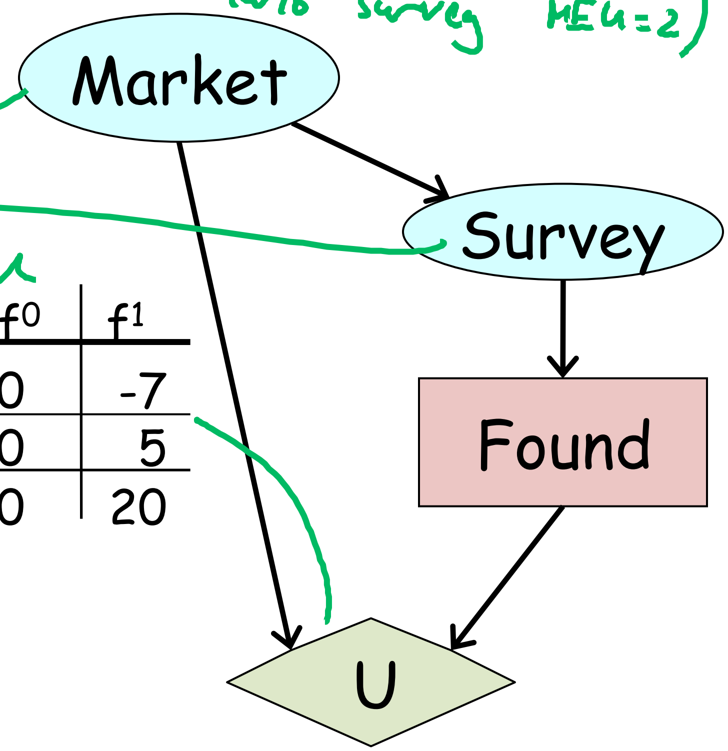
Finding MEU Decision Rules

(w/o survey MEU=2)

$$\sum_{S,F} \delta_F(F \mid S) \boxed{\sum_M P(M) P(S \mid M) U(F,M)}$$

$$= \boxed{\sum_{S,F} \delta_F(F \mid S) \mu(F,S)}$$



m^0	m^1	m^2
0.5	0.3	0.2

	s^0	s^1	s^2
m^0	0.6	0.3	0.1
m^1	0.3	0.4	0.3
m^2	0.1	0.4	0.5

u

	f^0	f^1
m^0	0	-7
m^1	0	5
m^2	0	20

	f^0	f^1
s^0	0	-1.25
s^1	0	1.15
s^2	0	2.1

$s_0 \to f_0$

$s_1 \to f_1$

$s_2 \to f_1$

$0 + 1.15 + 2.1 = 3.25$

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More Generally

$$\mathrm{EU}[\mathcal{D}[\delta_A]] = \sum_{\boldsymbol{x},a} \overbrace{P_{\delta_A}(\boldsymbol{x},a)}^{\text{joint dist.}} U(\boldsymbol{x},a)$$

$\boldsymbol{Z} = \mathbf{Pa}_A$ observations prior to A

$\boldsymbol{W} = \{X_1,\ldots,X_n\} - \boldsymbol{Z}$

$$= \sum_{X_1,\ldots,X_n,A} \left(\left(\prod_i P(X_i \mid \mathbf{Pa}_{X_i}) \right) U(\mathbf{Pa}_U)\delta_A(A \mid \boldsymbol{Z}) \right)$$

$$= \sum_{\boldsymbol{Z},A} \delta_A(A \mid \boldsymbol{Z}) \underbrace{\sum_{\boldsymbol{W}} \left(\left(\prod_i P(X_i \mid \mathbf{Pa}_{X_i}) \right) U(\mathbf{Pa}_U) \right)}_{\text{prob}}$$

$$= \sum_{\boldsymbol{Z},A} \delta_A(A \mid \boldsymbol{Z}) \mu(A,\boldsymbol{Z})$$

$$\delta^*_A(a \mid \boldsymbol{z}) = \begin{cases} 1 & a = \mathrm{argmax}_A \mu(A,\boldsymbol{z}) \\ 0 & \text{otherwise} \end{cases}$$

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MEU Algorithm Summary

- To compute MEU & optimize decision at A:
  - Treat A as random variable with arbitrary CPD
  - Introduce utility factor with scope Pa_U
  - VE – Eliminate all variables except A, $\mathbf{Z}$ (A's parents) to produce factor $\mu(A, \mathbf{Z})$
  - For each $\mathbf{z}$, set:

$$\delta_A^*(a \mid \boldsymbol{z}) = \begin{cases} 1 & a = \text{argmax}_A \mu(A, \boldsymbol{z}) \\ 0 & \text{otherwise} \end{cases}$$

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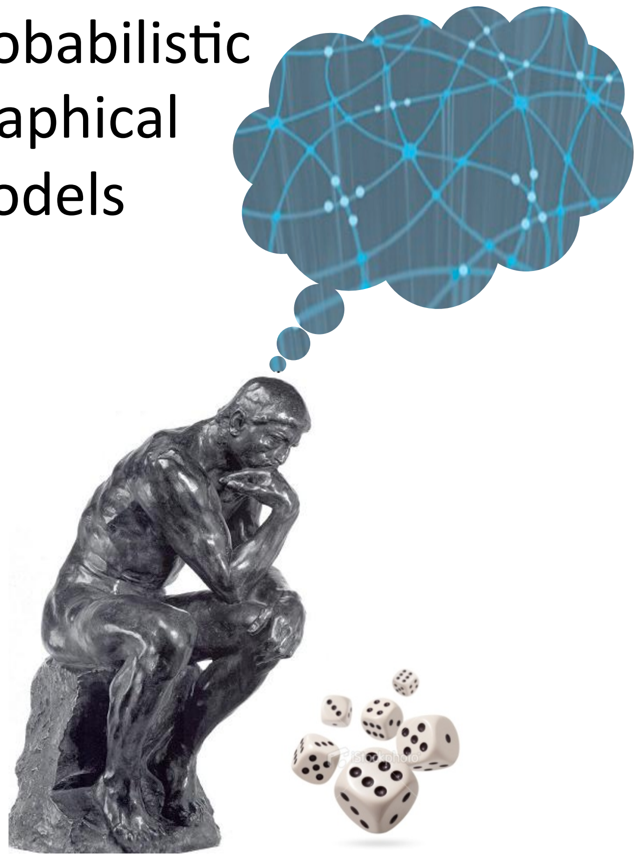
Decision Making under Uncertainty

- MEU principle provides rigorous foundation
- PGMs provide structured representation for probabilities, actions, and utilities
- PGM inference methods (VE) can be used for
  - Finding the optimal strategy
  - Determining overall value of the decision situation
- Efficient methods also exist for:
  - Multiple utility components
  - Multiple decisions

END END END

Probabilistic
Graphical
Models



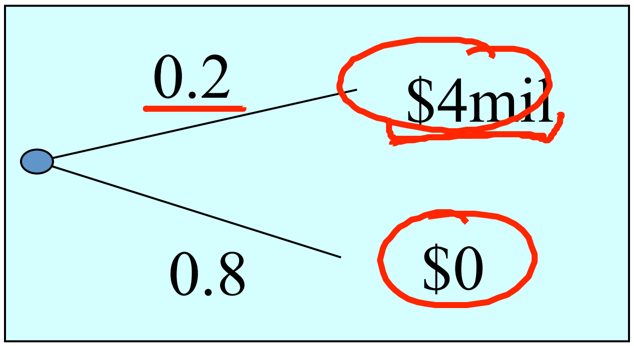
Acting

Decision Making

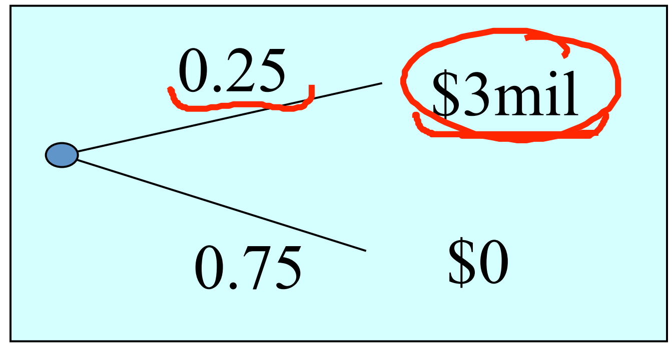
Utility Functions

Utilities and Preferences

lotteries



$\preceq \quad \succeq \quad \approx$



$0.2 \times u(4) + 0.8\, u(0)$

$0.25\, u(3) + 0.75\, u(0)$

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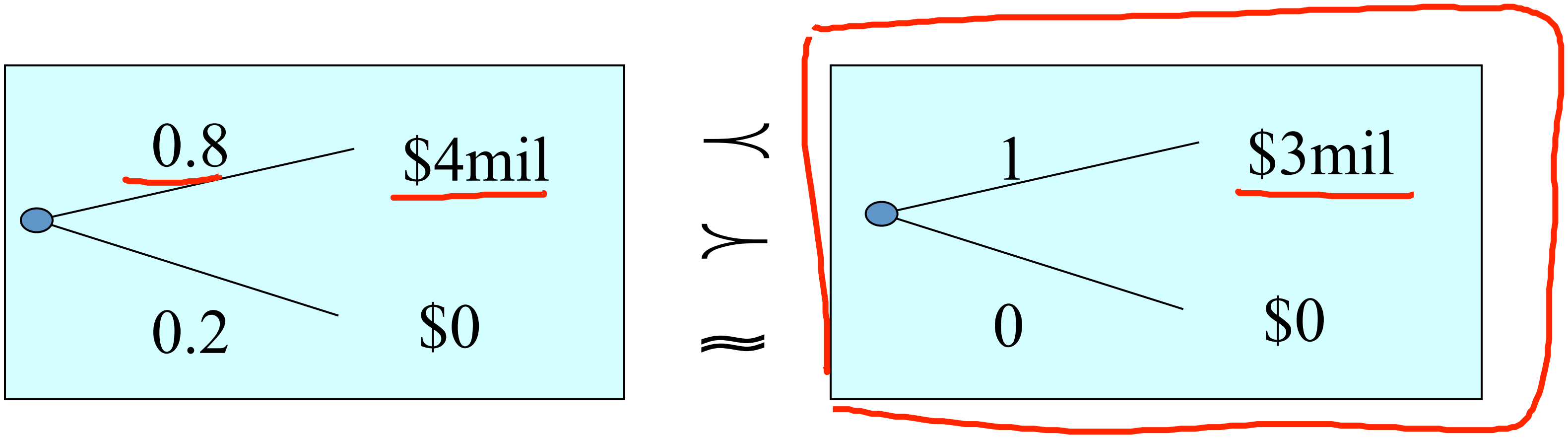
Utility = Payoff?



$4mill × 0.8 = $3.2mill

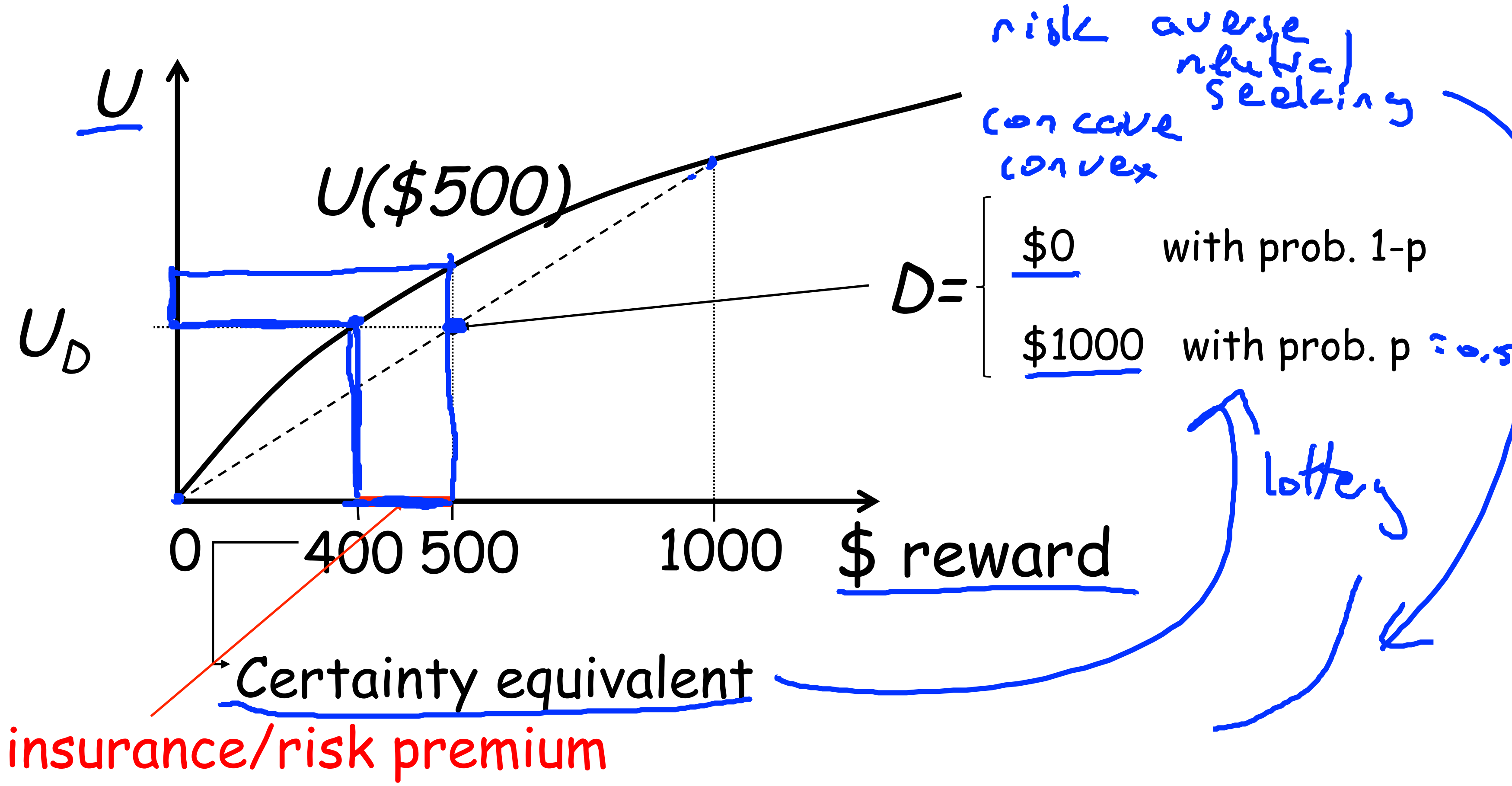
$3mill

St. Petersburg Paradox

- Fair coin is tossed repeatedly until it comes up heads, say on the n^{th} toss
- Payoff = $\$2^n$

$$\frac{1}{2} \times 2 + \frac{1}{4} \times 4 + \frac{1}{8} \times 8 + \ldots = \infty$$

most people value $\approx$ \$2



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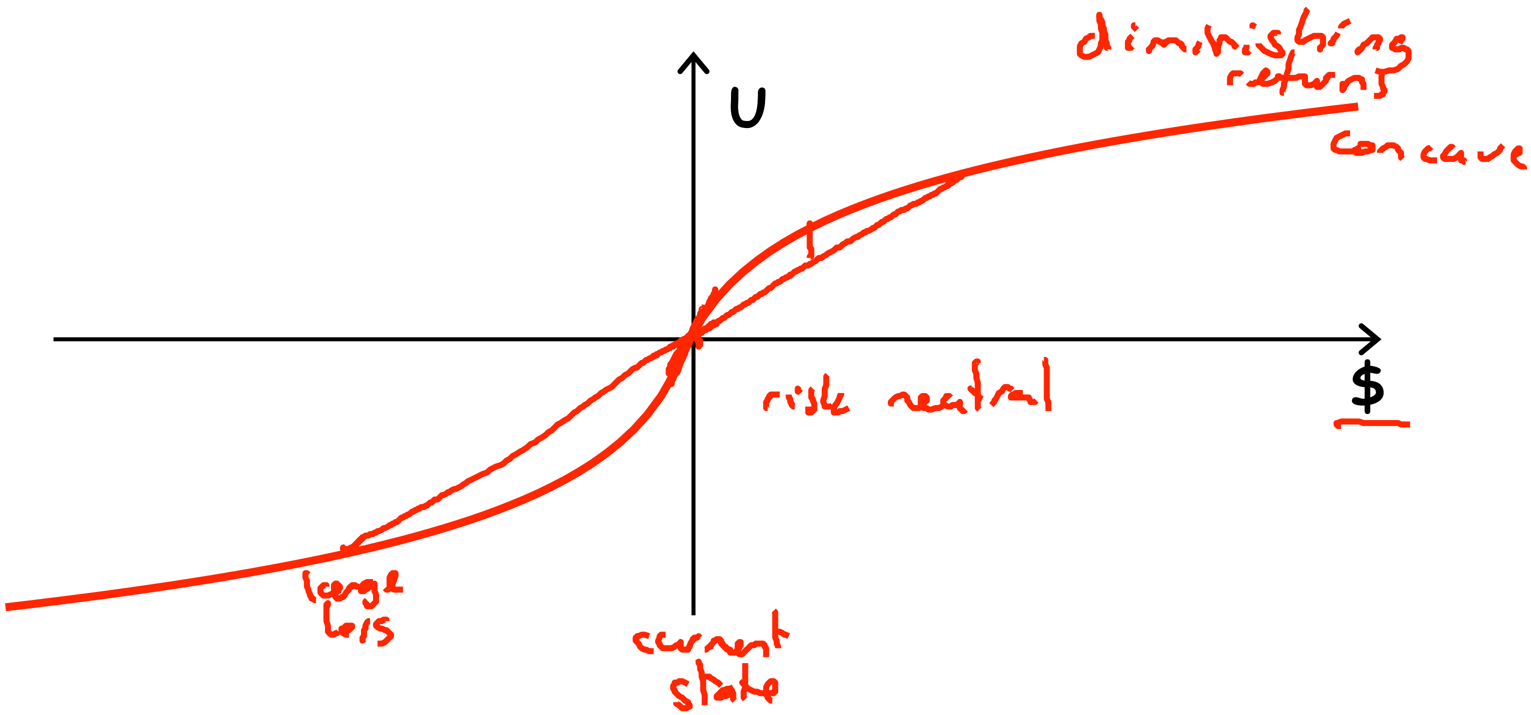
Typical Utility Curve



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Multi-Attribute Utilities

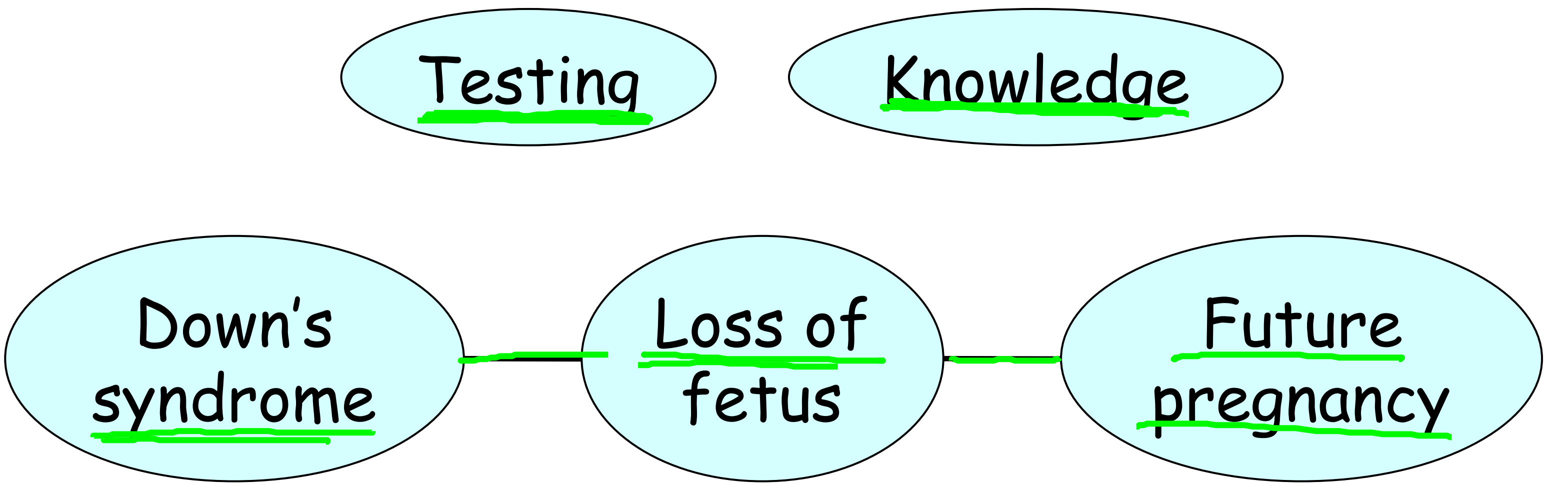
- All attributes affecting preferences must be integrated into one utility function

  money, time, pleasure, ...

- Human life
  - Micromorts $1/1000000$ chance of death $\approx$ \$20 1980
  - QALY (quality-adjusted life year)

Example: Prenatal diagnosis

$$U_1(T) + U_2(K) + U_3(D,L) + U_4(L,F)$$



Summary

- Our utility function determines our preferences about decisions that involve uncertainty
- Utility generally depends on multiple factors
  - Money, time, chances of death, …
- Relationship is usually nonlinear
  - Shape of utility curve determines attitude to risk
- Multi-attribute utilities can help decompose high-dimensional function into tractable pieces

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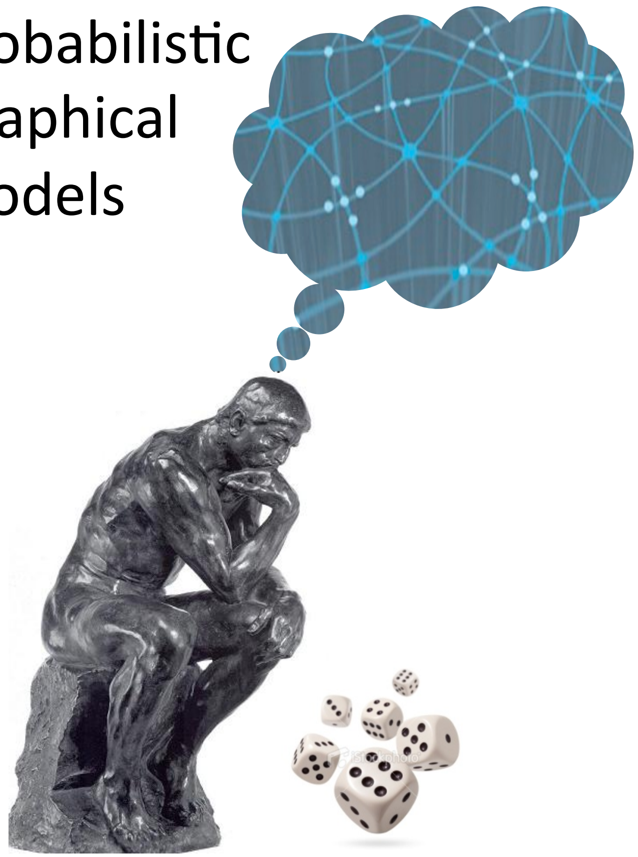


Acting

Decision Making

Value of Perfect Information

Value of Information

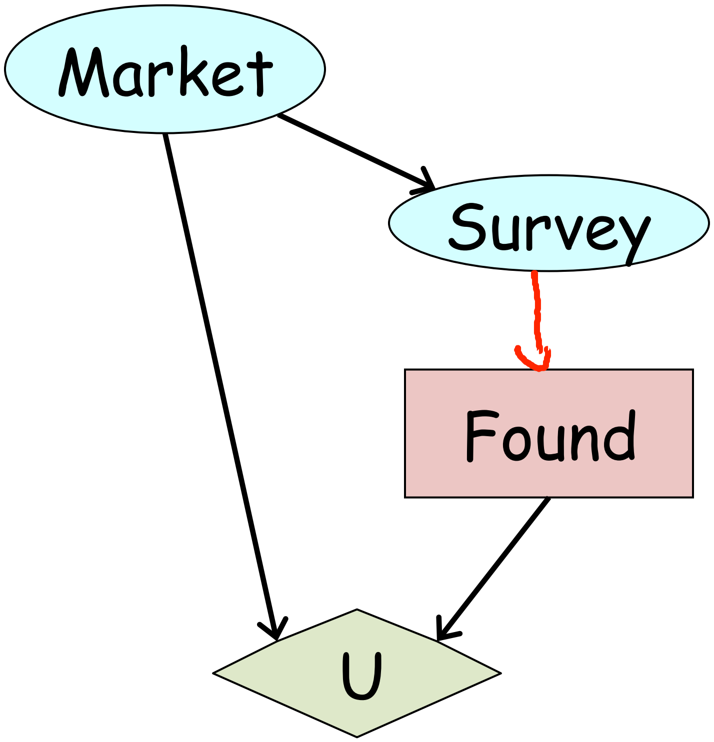
- value of perfect information
  VPI(A | X) is the value of observing X before choosing an action at A
- $\mathcal{D}$ = original influence diagram
- $\mathcal{D}_{X \to A}$ = influence diagram with edge $X \to A$

$$\mathrm{VPI}(A \mid X) := \underline{\mathrm{MEU}(\mathcal{D}_{X \to A})} - \mathrm{MEU}(\mathcal{D})$$

Finding MEU Decision Rules

$MEU(\mathcal{D}_{S \to F}) - MEU(\mathcal{D})$

$3.25 \qquad 2 \qquad = 1.25$



Value of Information

$$\underbrace{\mathrm{VPI}(A \mid X) := \mathrm{MEU}(\mathcal{D}_{X \to A})}_{\text{optimizing } \delta(A \mid \bar{Z}, x)} - \underbrace{\mathrm{MEU}(\mathcal{D})}_{\text{optimizing } \delta(A \mid \bar{Z})}$$

- Theorem:
  - VPI(A | X) ≥ 0
  - VPI(A | X) = 0 if and only if the optimal decision rule for $\mathcal{D}$ is still optimal for $\mathcal{D}_{X \to A}$

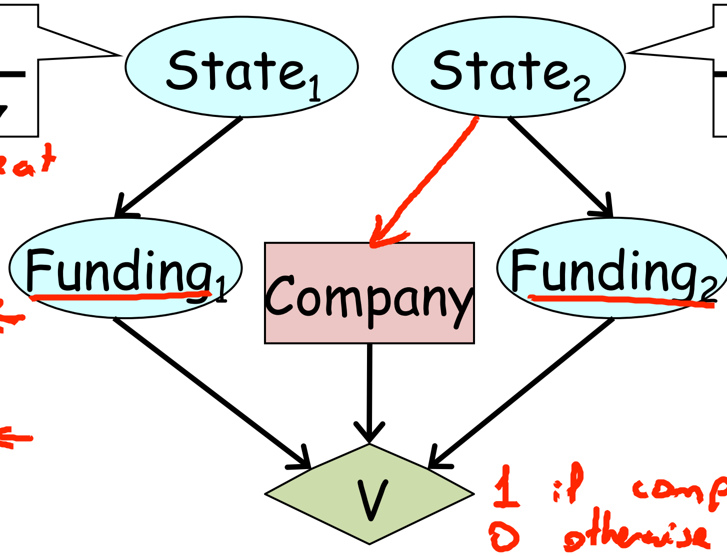
Any CPD $\delta(A \mid \bar{Z})$ is also a CPD $\delta(A \mid \bar{Z}, x)$

Clear notion of when information useful

if changes my decision

Value of Information Example

s^1	s^2	s^3
0.1	0.2	0.7
poor	med	great



s^1	s^2	s^3
0.4	0.5	0.1

	f^0	f^1
s^1	0.9	0.1
s^2	0.6	0.4
s^3	0.1	0.9

1 if company gets funded
0 otherwise

$$\delta^*(C \mid S_2) = \begin{cases} P(c^2)=1 & \text{if } S_2 = s^3 \\ P(c^1)=1 & \text{otherwise} \end{cases}$$

$EU(\mathcal{D}[c_1]) = 0.72$

$EU(\mathcal{D}[c_2]) = 0.33$

EU if $c=c_2$ $S_2=s^1$ 0.1 $\Rightarrow$ Prefer c_1
EU if $c=c_2$ $S_2=s^2$ 0.4
EU if $c=c_2$ $S_2=s^3$ 0.9 $\Rightarrow$ prefer c_2

$MEU(\mathcal{D}_{S_1 \rightarrow C}) = 0.743$

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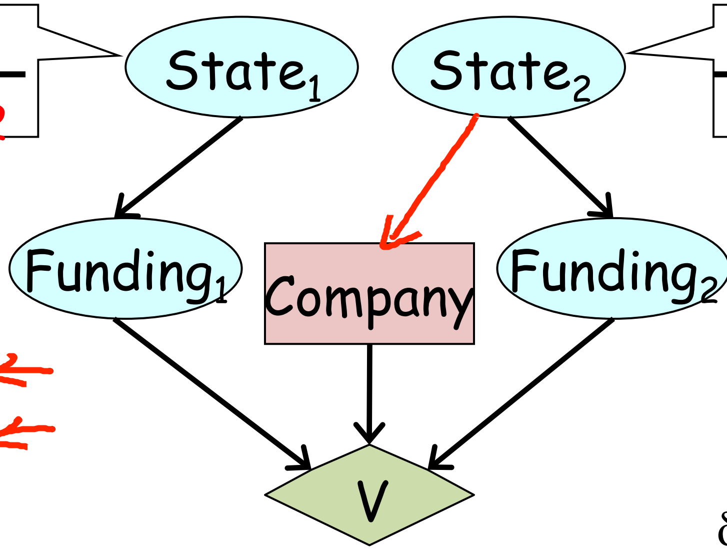
Value of Information Example

s^1	s^2	s^3
0.5	0.3	0.2



s^1	s^2	s^3
0.4	0.5	0.1

	f^0	f^1
s^1	0.9	0.1
s^2	0.6	0.4
s^3	0.1	0.9

$$\delta^*(C \mid S_2) = \begin{cases} P(c^2)=1 & \text{if } S_2 = s^2, s^3 \\ P(c^1)=1 & \text{otherwise} \end{cases}$$

$$EU(\mathcal{D}[c_1]) = 0.35$$

$$EU(\mathcal{D}[c_2]) = 0.33$$

$$MEU(\mathcal{D}_{S_1 \to C}) = 0.43$$

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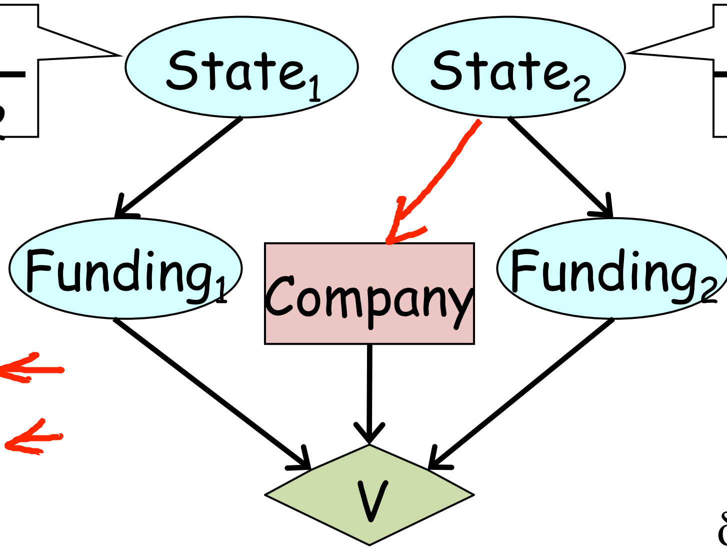
Value of Information Example

s^1	s^2	s^3
0.5	0.3	0.2



s^1	s^2	s^3
0.4	0.5	0.1

	f^0	f^1
s^1	0.3	0.7
s^2	0.2	0.8
s^3	0.01	0.99

$$\delta^*(C \mid S_2) = \begin{cases} P(c^2)=1 & \text{if } S_2 = s^2, s^3 \\ P(c^1)=1 & \text{otherwise} \end{cases}$$

$EU(\mathcal{D}[c_1]) = 0.788$

$EU(\mathcal{D}[c_2]) = 0.779$

$MEU(\mathcal{D}_{S_1 \to C}) = 0.8142$

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Summary

- Influence diagrams provide clear and coherent semantics for the value of making an observation
  - Difference between values of two IDs
- Information is valuable if and only if it induces a change in action in at least one context

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