L17CharacteristicEqn

October 31, 2015

1 The Characteristic Equation

We continue our study of *linear dynamical systems*, systems that evolve according to the equation:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k.$$

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A =
 [[ 0.99500417 -0.09983342]
 [ 0.09983342  0.99500417]]
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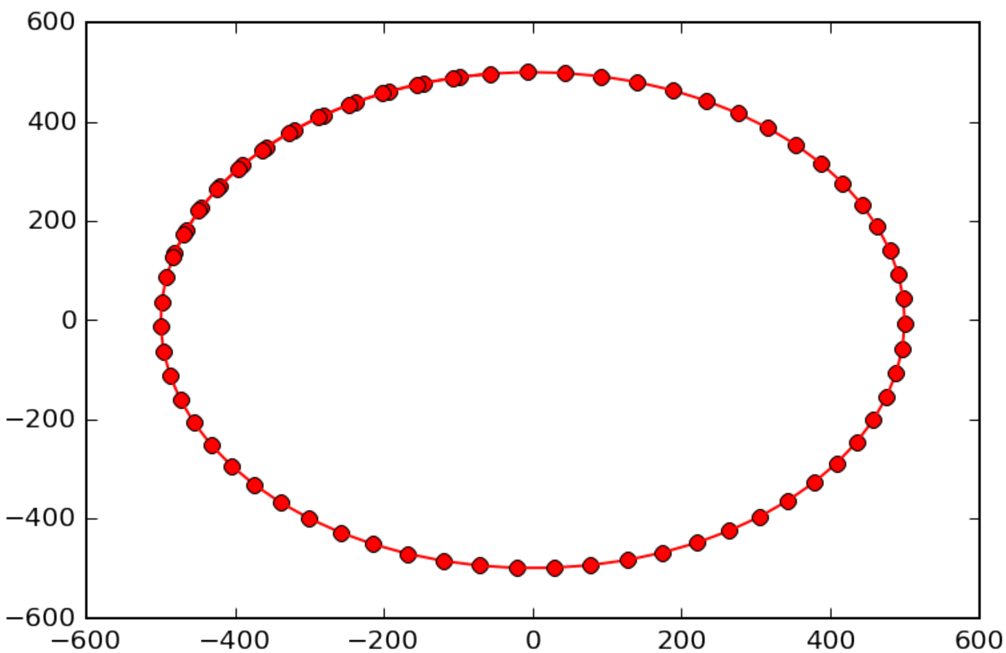


In the last lecture we saw that, if we know an eigenvalue λ of a matrix A, then computing the corresponding eigenspace can be done by constructing a basis for Nul $(A - \lambda I)$.

Today we'll discuss how to determine the eigenvalues of a matrix A.

The theory will make use of the *determinant* of a matrix.

Let's recall that the determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

We also have learned that A is invertible if and only if its determinant is not zero. (Recall that the inverse of of A is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$).

Let's use these facts to help us find the eigenvalues of a 2×2 matrix.

Example. Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Solution. We must find all scalars λ such that the matrix equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

has a nontrivial solution.

By the Invertible Matrix Theorem, this problem is equivalent to finding all λ such that the matrix $A - \lambda I$ is *not* invertible.

Now,

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}.$$

We know that A is not invertible exactly when its determinant is zero.

So the eigenvalues of A are the solutions of the equation

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = 0.$$

Since $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$, then

$$\det(A - \lambda I) = (2-\lambda)(-6-\lambda) - (3)(3)$$

$$= -12 + 6\lambda - 2\lambda + \lambda^2 - 9$$

$$= \lambda^2 + 4\lambda - 21$$

$$= (\lambda - 3)(\lambda + 7)$$

If $\det(A - \lambda I) = 0$, then $\lambda = 3$ or $\lambda = 7$. So the eigenvalues of A are 3 and -7.

1.1 Question Time! Q15.1

The same idea works for $n \times n$ matrices – but, for that, we need to define a *determinant* for larger matrices.

Determinants.

Previously, we've defined a determinant for a 2×2 matrix.

To find eigenvalues for larger matrices, we need to define the determinant for any sized (ie, $n \times n$) matrix.

Definition. Let A be an $n \times n$ matrix, and let U be any echelon form obtained from A by row replacements and row interchanges (no row scalings), and let r be the number of such row interchanges.

Then the **determinant** of A, written as $\det A$, is $(-1)^r$ times the product of the diagonal entries $u_{11}, \ldots, u_{nn}$ in U.

If A is invertible, then $u_{11}, \ldots, u_{nn}$ are all *pivots*.

If A is not invertible, then at least one diagonal entry is zero, and so the product $u_{11} \ldots u_{nn}$ is zero.

In other words:

$$\det\ A = \begin{cases} (-1)^r \cdot (\text{product of pivots in } U), & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is not invertible} \end{cases}$$

Example. Compute $\det A$ for $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

Solution. The following row reduction uses **one** row interchange:

$$A \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

So $\det A$ equals $(-1)^1(1)(-2)(-1) = (-2)$.

The remarkable thing is that **any other** way of computing the echelon form gives the same determinant. For example, this row reduction does not use a row interchange:

$$A \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

Using this echelon form to compute the determinant yields $(-1)^0(1)(-6)(1/3) = -2$, the same as before.

1.2 Question Time! Q15.2

Invertibility.

The formula for the determinant shows that A is invertible if and only if $\det A$ is nonzero.

We have **yet another** part to add to the Invertible Matrix Theorem:

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

1. The number 0 is *not* an eigenvalue of A.
2. The determinant of A is *not* zero.

Some facts about determinants (proved in the book):

1. $\det AB = (\det A)(\det B)$.
2. $\det A^T = \det A$.
3. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A.

1.3 The Characteristic Equation

So, A is invertible if and only if $\det A$ is not zero.

To return to the question of how to compute eigenvalues of A, recall that λ is an eigenvalue if and only if $(A - \lambda I)$ is *not* invertible.

We capture this fact using the **characteristic equation:**

$$\det(A - \lambda I) = 0.$$

We can conclude that λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation $\det(A - \lambda I) = 0$.

Example. Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution. Form $A - \lambda I$, and note that $\det A$ is the product of the entries on the diagonal of A, if A is triangular.

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda).$$

So the characteristic equation is:

$$(\lambda-5)^2(\lambda-3)(\lambda-1) = 0.$$

Expanding this out we get:

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0.$$

Notice that, once again, $\det(A-\lambda I)$ is a polynomial in λ.

In fact, for any $n \times n$ matrix, $\det(A-\lambda I)$ is a polynomial of degree n, called the **characteristic polynomial** of A.

We say that the eigenvalue 5 in this example has **multiplicity** 2, because $(\lambda-5)$ occures two times as a factor of the characteristic polynomial. In general, the mutiplicity fo an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Example. The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues and their multiplicity.

Solution Factor the polynomial

$$\lambda^6 - 4\lambda^5 - 12\lambda^4 = \lambda^4(\lambda^2 - 4\lambda - 12) = \lambda^4(\lambda-6)(\lambda+2)$$

So the eigenvalues are 0 (with multiplicity 4), 6, and -2.

Since the characteristic polynomial for an $n \times n$ matrix has degree n, the equation has n roots, counting multiplicities – provided complex numbers are allowed.

Note that even for a real matrix, eigenvalues may sometimes be complex.

Practical Issues.

These facts show that there is, in principle, a way to find eigenvalues of any matrix. However, you need not compute eigenvalues for matrices larger than 2×2 by hand. For any matrix 3×3 or larger, you should use a computer.

1.4 Similarity

An important concept for things that come later is the notion of **similar** matrices.

Definition. If A and B are $n \times n$ matrices, then A **is similar to** B if there is an invertible matrix P sch that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.

Similarity is symmetric, so if A is similar to B, then B is similar to A. Hence we just say that A and B **are similar.**

Changing A into B is called a **similarity transformation.**

An important way to think of similarity between A and B is that they **have the same eigenvalues.**

Theorem. IF $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial, and hence the same eigenvalues (with the same multiplicities.)

Proof. If $B = P^{-1}AP$, then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}(AP - \lambda P)$$

$$= P^{-1}(A - \lambda I)P$$

Now let's construct the characteristic polynomial by taking the determinant:

$$\det(B - \lambda I) = \det[P^{-1}(A - \lambda I)P]$$

Using the properties of determinants we discussed earlier, we compute:

$$= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P).$$

Since $\det(P^{-1}) \cdot \det(P) = \det(P^{-1}P) = \det I = 1$, we can see that

$$\det(B - \lambda I) = \det(A - \lambda I).$$

1.5 Question Time! Q15.3

1.6 Markov Chains

Let's return to the problem of solving a Markov Chain.

At this point, we can place the theory of Markov Chains into the broader context of eigenvalues and eigenvectors.

Theorem. The largest eigenvalue of a Markov Chain is 1.

Proof. It is obvious that 1 is an eigenvalue of a Markov chain since we know that every Markov Chain A has a steady-state vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{v}$.|

To prove that 1 is the largest eigenvalue, recall that each column of a Markov Chain sums to 1.

Then, consider the sum of the values in the vector $A\mathbf{x}$.

$$A\mathbf{x} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n \end{bmatrix}.$$

Let's just sum the first terms in each component of $A\mathbf{x}$:

$$a_{11}x_1 + a_{21}x_1 + \dots + a_{n1}x_1 = x_1 \sum_i a_{i1} = x_1.$$

So we can see that the sum of all terms in $A\mathbf{x}$ is equal to $x_1 + x_2 + \dots + x_n$ – i.e., the sum of all terms in $\mathbf{x}$.

So there can be no $\lambda > 1$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

A complete solution for the evolution of a Markov Chain.

Previously, we were only able to ask about the "eventual" steady state of a Markov Chain.

But a crucial question is: **how long does it take** for a particular Markov Chain to reach steady state from some initial starting condition?

Let's use an example: we previously studied the Markov Chain defined by $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$.

Let's ask how long until it reaches steady state, from the starting point defined as $\mathbf{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$.

Using the methods we studied today, we can find the characteristic equation:

$$\lambda^2 - 1.92\lambda + 0.92$$

Using the quadratic formula, we find the roots of this equation to be 1 and 0.92. (Note that, as expected, 1 is the largest eigenvalue.)

Next, using the methods in the previous lecture, we find a basis for each eigenspace of A (each nullspace of $A - \lambda I$).

For $\lambda = 1$, a corresponding eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

For $\lambda = 0.92$, a corresponding eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Next, we write $\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. This can be done because $\{\mathbf{v}_1, \mathbf{v}_2\}$ is obviously a basis for $\mathbb{R}^2$.

To write $\mathbf{x}_0$ this way, we want to solve the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}_0$$

In other words:

$$[\mathbf{v}_1\ \mathbf{v}_2]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{x}_0.$$

The matrix $[\mathbf{v}_1\ \mathbf{v}_2]$ is invertible, so,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\mathbf{v}_1\ \mathbf{v}_2]^{-1}\mathbf{x}_0 = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

$$= \frac{1}{-8}\begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.225 \end{bmatrix}.$$

So, now we can put it all together.
Let's compute each $\mathbf{x}_k$:

$$\mathbf{x}_1 = A\mathbf{x}_0 = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2$$

$$= c_1\mathbf{v}_1 + c_2(0.92)\mathbf{v}_2.$$

Now note the power of the eigenvalue approach:

$$\mathbf{x}_2 = A\mathbf{x}_1 = c_1A\mathbf{v}_1 + c_2(0.92)A\mathbf{v}_2$$

$$= c_1\mathbf{v}_2 + c_2(0.92)^2\mathbf{v}_2.$$

And so in general:

$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(0.92)^k\mathbf{v}_2 \quad (k = 0, 1, 2, \dots)$$

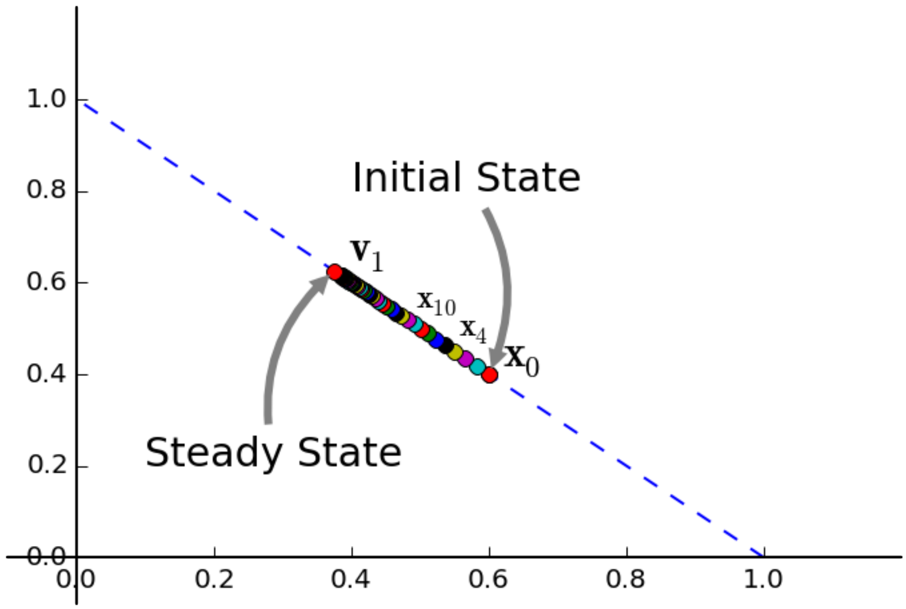
And using the c_1 and c_2 and $\mathbf{v}_1$, $\mathbf{v}_2$ we computed above:

$$\mathbf{x}_k = 0.125\begin{bmatrix} 3 \\ 5 \end{bmatrix} + 0.225(0.92)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (k = 0, 1, 2, \dots)$$

This explicit formula for $\mathbf{x}_k$ gives the solution of the Markov Chain $\mathbf{x}_{k+1} = A\mathbf{x}_k$ starting from the initial state $\mathbf{x}_0$.

As $k \to \infty$, $(0.92)^k \to 0$.

Thus $\mathbf{x}_k \to 0.125\mathbf{v}_1 = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$.



Initial State
Steady State
$\mathbf{v}_1$
$\mathbf{x}_{10}$
$\mathbf{x}_4$
$\mathbf{x}_0$
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0