

# L12Subspaces

March 22, 2015

## 1 Subspaces

So far have been working with vector spaces like  $\mathbb{R}^2, \mathbb{R}^3$ . But there are more vector spaces.

Today we'll define a **subspace** and show how the concept helps us understand the nature of matrices and their linear transformations.

**Definition.** A *subspace* is any set  $H$  in  $\mathbb{R}^n$  that has three properties:

1. The zero vector is in  $H$ .
2. For each  $\mathbf{u}$  and  $\mathbf{v}$  in  $H$ , the sum  $\mathbf{u} + \mathbf{v}$  is in  $H$ .
3. For each  $\mathbf{u}$  in  $H$  and each scalar  $c$ , the vector  $c\mathbf{u}$  is in  $H$ .

Another way of stating properties 2 and 3 is that  $H$  is *closed* under addition and scalar multiplication.

**Examples.** Many of the vector sets we've discussed so far are subspaces.

For example, if  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are in  $\mathbb{R}^n$  and  $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ , then  $H$  is a subspace of  $\mathbb{R}^n$ .

Let's check this:

- 1) The zero vector is in  $H$

because  $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$  is in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ .

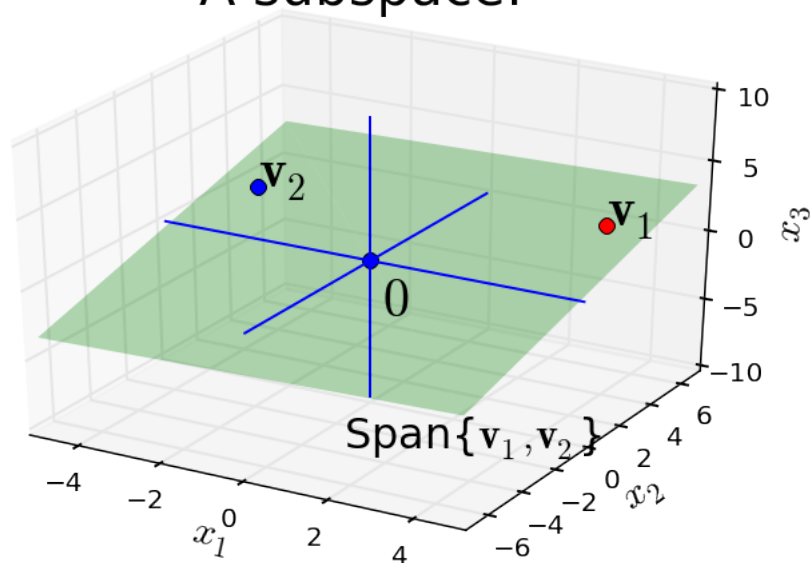
- 2) The sum of any two vectors in  $H$  is in  $H$

In other words, if  $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2$ , and  $\mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ ,  
... their sum  $\mathbf{u} + \mathbf{v}$  is  $(s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2$ ,  
... which is in  $H$ .

- 3) For any scalar  $c$ ,  $c\mathbf{u}$  is in  $H$

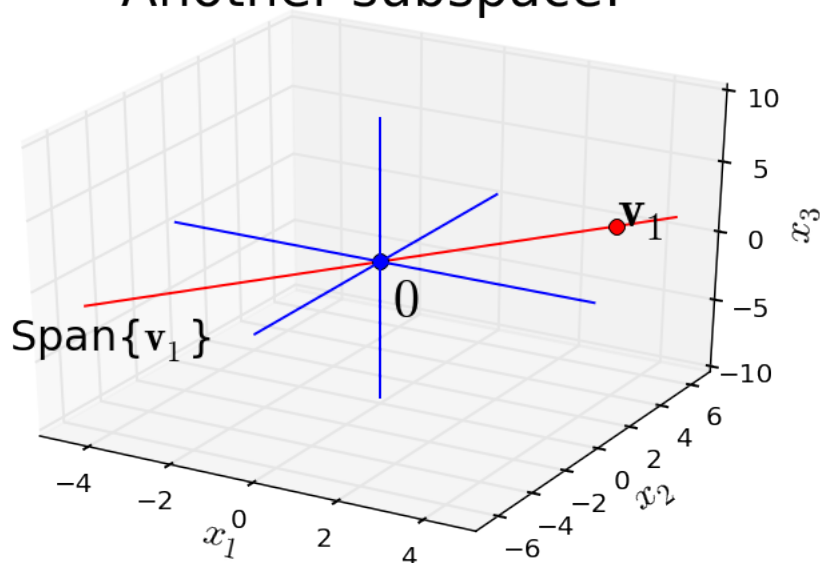
because  $c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1\mathbf{v}_1 + cs_2\mathbf{v}_2)$ .

## A subspace:



Indeed, by similar arguments, *any* span of a vector set is a subspace. For example,  $\text{Span}\{\mathbf{v}_1\}$ , which is a line through the origin, is a subspace.

## Another subspace:



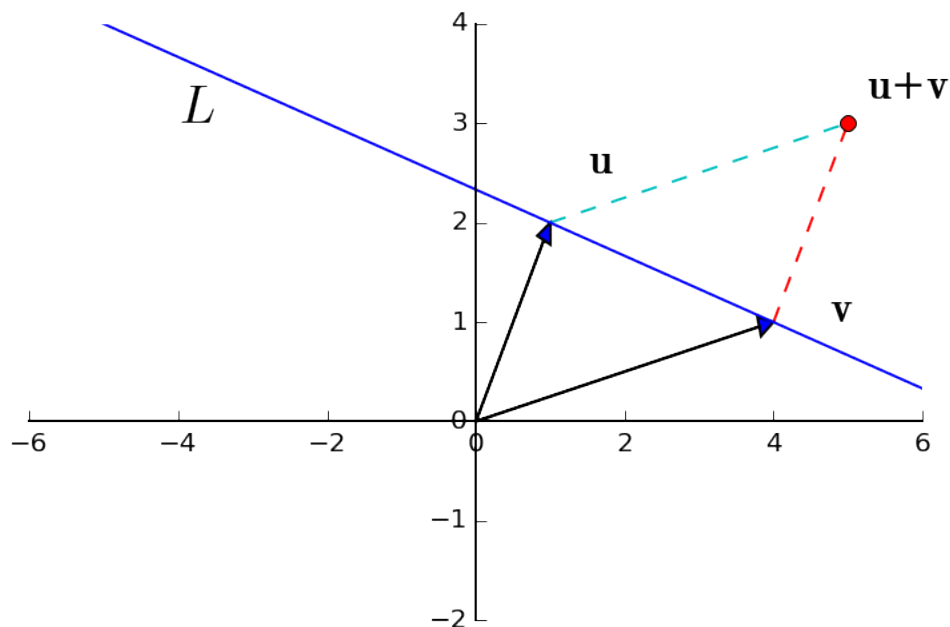
Because of this, we refer to  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  as **the subspace spanned by  $\mathbf{v}_1, \dots, \mathbf{v}_p$** .

Is **any** line a subspace? What about a line that is not through the origin?

In fact, a line  $L$  not through the origin **fails all three** requirements for a subspace:

- 1)  $L$  does not contain the zero vector.
- 2)  $L$  is not closed under addition.
- 3)  $L$  is not closed under scalar multiplication.

Let's just look at 2):



## 1.1 Column Space and Null Space of a Matrix

An important way to think about a matrix is in terms of two subspaces: **column space** and **null space**.

**Definition.** The **column space** of a matrix  $A$  is the set  $\text{Col } A$  of all linear combinations of the columns of  $A$ .

If  $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ , with columns in  $\mathbb{R}^m$ , then  $\text{Col } A$  is the same as  $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ .

The column space of an  $m \times n$  matrix is a subspace of  $\mathbb{R}^m$ .

In particular, note that  $\text{Col } A$  equals  $\mathbb{R}^m$  only when the columns of  $A$  span  $\mathbb{R}^m$ . Otherwise,  $\text{Col } A$  is only part of  $\mathbb{R}^m$ .

## 1.2 Question Time: Q 12.X

Let  $A =$  whatever. Let  $T(x)$  be  $Ax$ . Which of the following vectors are in  $\text{Col } A$ ? (include  $T(3,4,2)$ ).

When a system of linear equations is written in the form  $A\mathbf{x} = \mathbf{b}$ , the column space of  $A$  is the set of all  $\mathbf{b}$  for which the system has a solution.

Equivalently, when we consider the linear operator  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  that is implemented by the matrix  $A$ , the column space of  $A$  is the **range** of  $T$ .

**Definition.** The **null space** of a matrix  $A$  is the set  $\text{Nul } A$  of all solution of the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ .

When  $A$  has  $n$  columns, a solution of  $A\mathbf{x} = \mathbf{0}$  is a vector in  $\mathbb{R}^n$ . So the null space of  $A$  is a subset of  $\mathbb{R}^n$ . In fact,  $\text{Nul } A$  is a **subspace** of  $\mathbb{R}^n$ .

**Theorem.** The null space of an  $m \times n$  matrix  $A$  is a subspace of  $\mathbb{R}^n$ .

Equivalently, the set of all solutions of a system  $A\mathbf{x} = \mathbf{0}$  of  $m$  homogeneous linear equations in  $n$  unknowns is a subspace of  $\mathbb{R}^n$ .

**Proof.**

1) The zero vector is in  $\text{Nul } A$  because  $A\mathbf{0} = \mathbf{0}$ .

2) The sum of two vectors in  $\text{Nul } A$  is in  $\text{Nul } A$ .

Take two vectors  $\mathbf{u}$  and  $\mathbf{v}$  that are in  $\text{Nul } A$ . By definition  $A\mathbf{u} = \mathbf{0}$  and  $A\mathbf{v} = \mathbf{0}$ .

Then  $\mathbf{u} + \mathbf{v}$  is in  $\text{Nul } A$  because  $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ .

3) Any scalar multiple of a vector in  $\text{Nul } A$  is in  $\text{Nul } A$ .

Take a vector  $\mathbf{v}$  that is in  $\text{Nul } A$ . Then  $A(c\mathbf{v}) = cA\mathbf{v} = c\mathbf{0} = \mathbf{0}$ .

Testing whether a vector  $\mathbf{v}$  is in  $\text{Nul } A$  is easy: simply compute  $A\mathbf{v}$  and see if the result is zero.

### 1.3 Question Time Q12.Y

Which of the following vectors are in  $\text{Nul } A$ ?

**Comparing Col  $A$  and  $\text{Nul } A$ .**

What is the relationship between these two subspaces that are defined using  $A$ ?

Actually, there is no particular connection (at this point in the course).

The important thing to note at present is that these two subspaces live in different “universes”. For an  $m \times n$  matrix, the column space is a subset of  $\mathbb{R}^m$  (all its vectors have  $m$  components), while the null space is a subset of  $\mathbb{R}^n$  (all its vectors have  $n$  components).

(However: next lecture we will make a connection!)

### 1.4 Basis for a Subspace

A subspace usually contains an infinite number of vectors.

Often it is convenient to work with a small set of vectors that span the subspace. The smaller the set, the better.

It can be shown that the smallest possible spanning set must be linearly independent.

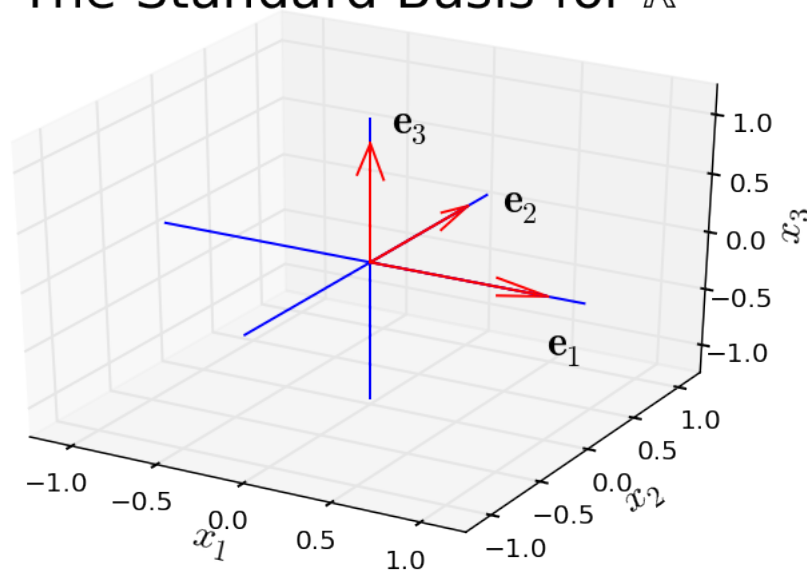
**Definition.** A **basis** for a subspace  $H$  of  $\mathbb{R}^n$  is a linearly independent set in  $H$  that spans  $H$ .

**Example.** The columns of **any** invertible  $n \times n$  matrix form a basis for  $\mathbb{R}^n$ . This is because, by the Invertible Matrix Theorem, they are linearly independent, and they span  $\mathbb{R}^n$ .

So, for example, we could use the identity matrix,  $I$ . Its columns are  $\mathbf{e}_1, \dots, \mathbf{e}_n$ .

The set  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is called the **standard basis** for  $\mathbb{R}^n$ .

# The Standard Basis for $\mathbb{R}^3$



## Finding a basis for the subspaces associated with $A$ .

We will often want to find a basis for  $\text{Col } A$  or for  $\text{Nul } A$ .

We'll start with finding a basis for the null space of a matrix.

**Example.** Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

**Solution.** We would like to describe the set of all solutions of  $A\mathbf{x} = \mathbf{0}$ .

We start by writing the solution of  $A\mathbf{x} = \mathbf{0}$  in parametric form:

$$[A \ \mathbf{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{array}{rrcrcl} x_1 & -2x_2 & & -x_4 & +3x_5 & = & 0 \\ & & x_3 & +2x_4 & -2x_5 & = & 0 \\ & & & & & 0 & = & 0 \end{array}$$

So  $x_1$  and  $x_3$  are basic, and  $x_2, x_4$ , and  $x_5$  are free.

So the general solution is:

$$\begin{aligned} x_1 &= 2x_2 + x_4 - 3x_5, \\ x_3 &= -2x_4 + 2x_5. \end{aligned}$$

Now, what we want to do is write the solution set as a weighted combination of vectors. The free variables will become the weights.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{aligned}
&= x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \\
&= x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w}.
\end{aligned}$$

Now what we have is an expression that describes the entire solution set of  $A\mathbf{x} = \mathbf{0}$ .

So  $\text{Nul } A$  is the set of all linear combinations of  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ . That is,  $\text{Nul } A$  is the subspace spanned by  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ .

Furthermore, this construction automatically makes  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  linearly independent.

Since each weight appears by itself in one position, the only way for the whole weighted sum to be zero is if every weight is zero – which is the definition of linear independence.

So  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is a **basis** for  $\text{Nul } A$ .

Conclusion: by finding a parametric description of the solution of the equation  $A\mathbf{x} = \mathbf{0}$ , we can construct a basis for the nullspace of  $A$ .

**Finding a basis for the column space.**

**Warmup.** We start with a warmup example. Suppose we have a matrix  $B$  that happens to be in reduced echelon form:

$$B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Denote the columns of  $B$  by  $\mathbf{b}_1, \dots, \mathbf{b}_5$  and note that  $\mathbf{b}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_2$  and  $\mathbf{b}_4 = 5\mathbf{b}_1 - \mathbf{b}_2$ .

So any combination of  $\mathbf{b}_1, \dots, \mathbf{b}_5$  is actually just a combination of  $\mathbf{b}_1, \mathbf{b}_2$ , and  $\mathbf{b}_5$ .

So  $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_5\}$  spans  $\text{Col } B$ .

Also,  $\mathbf{b}_1, \mathbf{b}_2$ , and  $\mathbf{b}_5$  are linearly independent, because they are columns from an identity matrix.

So: the pivot columns of  $B$  form a basis for  $\text{Col } B$ .

**The general case.** Now I'll show that the pivot columns of  $A$  form a basis for  $\text{Col } A$  for **any**  $A$ .

Consider the case where  $A\mathbf{x} = \mathbf{0}$  for some nonzero  $\mathbf{x}$ .

This says that there is a linear dependence relation between some of the columns of  $A$ .

If any of the entries in  $\mathbf{x}$  are zero, then those columns do not participate in the linear dependence relation.

When we row-reduce  $A$  to its reduced echelon form  $B$ , the columns are changed, but the equations  $A\mathbf{x} = \mathbf{0}$  and  $B\mathbf{x} = \mathbf{0}$  have the same solution set.

So this means that the columns of  $A$  and the columns of  $B$  have exactly the same dependence relationships as the columns of  $B$ .

**Example.** Consider the matrix  $A$ :

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

It is row equivalent to the matrix  $B$  that we considered above. So to find its basis, we simply need to look at the basis for its reduced row echelon form. We already computed that a basis for  $B$  was columns 1, 2, and 5.

Therefore we can immediately conclude that a basis for  $\text{Col } A$  is  $A$ 's columns 1, 2, and 5.

So a basis for  $A$  is:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -9 \\ 2 \\ 1 \\ -8 \end{bmatrix} \right\}$$

**Theorem.** The pivot columns of a matrix  $A$  form a basis for the column space of  $A$ .

Be careful here – note that you compute the reduced row echelon form of  $A$  to find which columns are pivot columns, but you used the columns of  $A$  itself as the basis for  $\text{Col } A$ !