

L2RowReductions

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1 Gaussian Elimination



In the last lecture we described a method for solving linear systems, but our description was somewhat informal. Today we'll formally define *Gaussian Elimination*, sometimes called *Gauss-Jordan Elimination*.

Carl Gauss lived from 1777 to 1855, in Germany. He is often called “the greatest mathematician since antiquity.”

When Gauss was around 17 years old, he developed a method for working with inconsistent linear systems, called the method of *least squares*. A few years later (at the advanced age of 24) he turned his attention to a particular problem in astronomy. In 1801 the Sicilian astronomer Piazzi discovered a (dwarf) planet, which he named Ceres, in honor of the patron goddess of Sicily. Piazzi took measurements of Ceres' position for 40 nights, but then lost track of it when it passed behind the sun. Piazzi had only tracked Ceres through about 3 degrees of sky. Gauss however then succeeded in calculating the orbit of Ceres, even though the task seemed hopeless on the basis of so few observations. His computations were so accurate that the astronomer Olbers located Ceres again later the same year.

In the course of his computations Gauss had to solve systems of 17 linear equations. Since Gauss at first refused to reveal the methods that led to this amazing accomplishment, some even accused him of sorcery.

Eight years later, in 1809, Gauss revealed his methods of orbit computation in his book *Theoria Motus Corporum Coelestium*.

Although Gauss invented this method (which Jordan then popularized), it was a reinvention. As we mentioned in the previous lecture, linear systems were being solved by a similar method in China 2,000 years earlier.

Based on Bretscher, *Linear Algebra*, pp 17-18, and the Wikipedia article on Gauss.

1.1 Echelon Forms

An *echelon* is a term used in the military to describe an arrangement of rows (of troops, or ships, etc) in which each successive row extends further than the row in front of it.

Definition: A matrix is in **echelon form** (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

A “leading entry” is the first nonzero element in a row.

For example:

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In this diagram, the \blacksquare s are nonzero, and the *s can be any value.

This definition is a refinement of the notion of a *triangular* matrix (or system) that was introduced in the previous lecture.

The goal of the **first** step of Gaussian elimination is to convert the augmented matrix into echelon form.

Definition: A matrix is in **reduced echelon form** (or reduced row echelon form) if it is in echelon form, and furthermore:

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

For example:

$$\begin{bmatrix} 0 & \boxed{1} & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The goal of the **second** step of Gaussian elimination is to convert the matrix into reduced echelon form.

1.2 Properties of Echelon Forms

Any matrix may be row reduced to an echelon form. Echelon forms are *not* unique; depending on the sequence of row operations, different echelon forms may be produced from a given matrix.

However, the reduced echelon form of a matrix *is* unique.

Theorem: Each matrix is equivalent to one and only one reduced echelon matrix.

The positions of the leading entries of an echelon matrix and its reduced form are the same. So, by the Theorem, the leading entries of *any* echelon form of a given matrix are in the same positions.

Definition: A pivot position in a matrix A is the position of a leading 1 in the reduced echelon form of A .

1.3 Gaussian Elimination: The Algorithm

As suggested by the last lecture, Gaussian Elimination has two stages. Given an augmented matrix A representing a linear system: 1. Convert A to one of its echelon forms, say U . 2. Convert U to A 's reduced row echelon form.

Each stage iterates over the rows of A , starting with the first row.

Row Reduction Operations

Before stating the algorithm, let's recall the set of operations that we can perform on rows without changing the solution set:

1. Multiply a row by a nonzero value.
2. Add a multiple of a row to another row.
3. Swap two rows.

Stage 1 (Elimination):

Input: matrix A .

We will use i to denote the index of the current row. To start, let $i = 1$. Repeat the following steps:

1. Let j be the position of the **leftmost nonzero value** in row i or any row below it. If there is no such position, stop.
2. If the j th position in row i is zero, swap this row with a row below it to make the j th position nonzero. This creates a **pivot** in position i, j .
3. Use row reduction operations to create zeros in all positions below the pivot. If any operation creates a row that is all zeros *except the last element*, the system is inconsistent; stop.
4. Let $i = i + 1$. If i equals the number of rows in A , stop.

The output of this stage is an echelon form of A .

Stage 2 (Backsubstitution):

Input: an echelon form of A .

We start at the top again, so let $i = 1$. Repeat the following steps:

1. If row i is all zeros, or if i exceeds the number of rows in A , stop.
2. If row i has a nonzero pivot value, divide row i by its pivot value. This creates a 1 in the pivot position.
3. Use row reduction operations to create zeros in all positions above the pivot.
4. Let $i = i + 1$.

The output of this stage is the reduced echelon form of A .

Note: the book describes this stage as proceeding from the bottom of the matrix up; that procedure is equivalent to this one.

1.4 Example

The Gaussian Elimination process we've described is essentially equivalent to the process described in the last lecture, so we won't do a lengthy example. Let the input matrix A be

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Stage 1

Start with the first row ($i = 1$). The leftmost nonzero in row 1 and below is in position 1. But since it's not in row 1, we need to swap. We'll swap rows 1 and 3 (we could have swapped 1 and 2).

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

The pivot is shown in a box. Use row reduction operations to create zeros below the pivot. In this case, that means subtracting row 1 from row 2.

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Now $i = 2$. The pivot is boxed (no need to do any swaps). Use row reduction to create zeros below the pivot. To do so we subtract $3/2$ times row 2 from row 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Now $i = 3$. Since it is the last row, we are done with Stage 1. The pivots are marked:

$$\begin{bmatrix} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{2} & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

Stage 2

Starting again with the first row ($i = 1$). Divide row 1 by its pivot.

$$\begin{bmatrix} \boxed{1} & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Moving to the next row ($i = 2$). Divide row 2 by its pivot.

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & \boxed{1} & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

And use row reduction operations to create zeros in all elements above the pivot. In this case, that means adding 3 times row 2 to row 1.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & \boxed{1} & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Moving to the next row ($i = 3$). The pivot is already 1. So we subtract row 3 from row 2, and subtract 5 times row 3 from row 1.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

And we are done.

1.5 How Many Operations does Gaussian Elimination Require?