L8MatrixInverse

February 22, 2015

1 The Inverse of a Matrix

Today we investigate the idea of the "reciprocal" of a matrix.
For reasons that will become clear, we will think about this way:
The reciprocal of any nonzero number r is its multiplicative inverse $1/r = r^{-1}$ such that $r \cdot r^{-1} = 1$.
This gives a way to define what is called the *inverse* of a matrix.
First, we have to recognize that this inverse does not exist for all matrices.

- It only exists for square matrices
- And not even for all square matrices – only those that are "invertible."

Definition. A matrix A is called **invertible** if there exists a matrix C such that

$$AC = I \text{ and } CA = I.$$

In that case C is called the *inverse* of A. Clearly, C must also be square and the same size as A.
The inverse of A is denoted A^{-1}.
A matrix that is not invertible is called a **singular** matrix.
Example.
If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then:

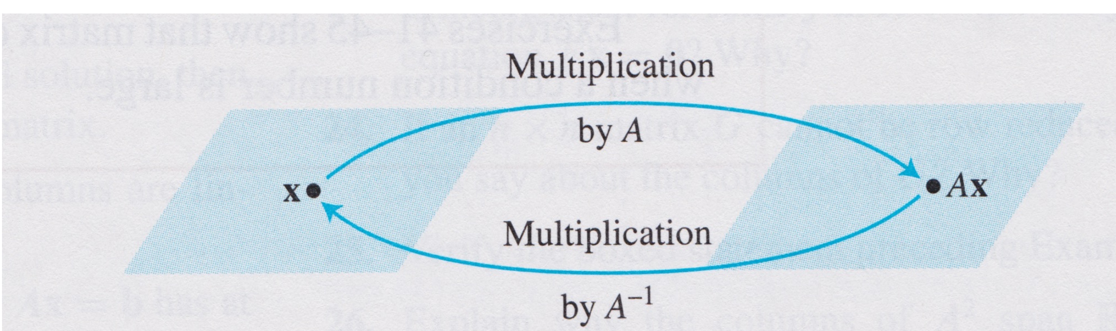
$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and:

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

1.1 As a Linear Transformation

Let's think about what a matrix inverse does as a linear transformation.



We have:

$$A\mathbf{x} = \mathbf{b}.$$

So:

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} = \mathbf{x}.$$

Theorem. If A is an invertible $n \times n$ matrix, the for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $A^{-1}\mathbf{b}$.

Proof. Follows directly from the definition of A^{-1}.

This very simple, powerful theorem gives us a new way to solve a linear system.

Furthermore, this theorem connects the matrix inverse to certain kinds of linear systems. We know that not all linear systems of n equations in n variables have a unique solution. Such systems may have no solutions (inconsistent) or an infinite number of solutions. But this theorem says that **if A is invertible, then the system has a unique solution.**

1.2 Computing the Matrix Inverse

Wonderful - so to solve a linear system, we simply need to compute the inverse of A (if it exists).

How do we do that?

Theorem. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Notice that this theorem tells us, for 2×2 matrices, exactly *which ones* are invertible: namely, those which have $ad - bc \neq 0$. This quantity is called the **determinant** of A.

Example. If a 2×2 matrix A implements a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$, is it invertible?

Solution. If $T : \mathbb{R}^2 \to \mathbb{R}$, we know that the span of the columns of A is $\mathbb{R}$. This means that at least one of the columns is a multiple of the other. Let the multiplier be m.

Then we can express A as: $\begin{bmatrix} a & ma \\ b & mb \end{bmatrix}$.

The determinant of A is $a(mb) - b(ma) = 0$.

So no linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$ is invertible.

Computing the inverse for matrices larger than 2×2.

Let's look at a general method for computing the inverse of A.

Recall our definition of matrix multiplication: AB is the matrix formed by multiplying A times each column of B.

Let's look at the equation

$$AA^{-1} = I.$$

Let's call the columns of $A^{-1} = [\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}]$. We know what the columns of I are: $[\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}]$. So:

$$AA^{-1} = A[\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}] = [\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}].$$

So here is a general way to compute the inverse of A:

- Solve the linear system $A\mathbf{x_1} = \mathbf{e_1}$ to get the first column of A^{-1}.
- Solve the linear system $A\mathbf{x_2} = \mathbf{e_2}$ to get the second column of A^{-1}.
- ...
- Solve the linear system $A\mathbf{x_n} = \mathbf{e_n}$ to get the last column of A^{-1}.

If any of the systems are inconsistent or have an infinite solution set, then A^{-1} does not exist.

The Computational View.

This general strategy leads to an algorithm, described in the book, for inverting any matrix. However, in this course I will not ask you invert matrices larger than 2×2 by hand. Any time you need to invert a matrix larger than 2×2, you may use a calculator or computer.

To invert a matrix in `Python/numpy`, use the function `np.linalg.inv()`. For example:

```
In [42]: import numpy as np
         A = np.array([[2.0,5.0],[-3.0,-7.0]])
         print 'A ='; print A
         B = np.linalg.inv(A)
         print 'B = '; print B
```

```
A =
[[ 2.  5.]
 [-3. -7.]]
B =
[[-7. -5.]
 [ 3.  2.]]
```

What do you think happens if you call `np.linalg.inv()` on a matrix that is not invertible?

```
In [27]: A = np.array([[2.,5.],[2.,5.]])
         np.linalg.inv(A)
```

```
        ---------------------------------------------------------------------------
    LinAlgError                               Traceback (most recent call last)

        <ipython-input-27-68ca4cda2b89> in <module>()
          1 A = np.array([[2.,5.],[2.,5.]])
    ----> 2 np.linalg.inv(A)


        /Users/markcrovella/canopy/lib/python2.7/site-packages/numpy/linalg/linalg.pyc in inv(a)
        518     signature = 'D->D' if isComplexType(t) else 'd->d'
        519     extobj = get_linalg_error_extobj(_raise_linalgerror_singular)
    --> 520     ainv = _umath_linalg.inv(a, signature=signature, extobj=extobj)
        521     return wrap(ainv.astype(result_t))
        522


        /Users/markcrovella/canopy/lib/python2.7/site-packages/numpy/linalg/linalg.pyc in _raise_linalge
         88
         89 def _raise_linalgerror_singular(err, flag):
    ---> 90     raise LinAlgError("Singular matrix")
         91
         92 def _raise_linalgerror_nonposdef(err, flag):


        LinAlgError: Singular matrix
```

The right way to handle this is:

```
In [28]: A = np.array([[2.0,5.0],[2,5]])
         try:
             np.linalg.inv(A)
         except np.linalg.LinAlgError:
             print 'Oops, looks like A is singular!'
```

```
Oops, looks like A is singular!
```

Using the matrix inverse to solve a linear system.
Solve the system:

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7 \end{aligned}$$

Rewrite this system as $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

The determinant of A is $3(6) - 4(5) = -2$, which is nonzero, so A has an inverse. The inverse of A is:

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}.$$

So the solution is:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Theorem.

- If A is an invertible matrix, then A^{-1} is invertible, and

$$(A^{-1})^{-1} = A.$$

- If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}.

$$(A^T)^{-1} = (A^{-1})^T.$$

- If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order.

$$(AB)^{-1} = B^{-1}A^{-1}.$$

The first two are straightforward. Let's verify the last one because it shows some common calculation patterns:

$$(AB)(B^{-1}A^{-1})$$

$$= A(BB^{-1})A^{-1}$$

$$= AIA^{-1}$$

$$= AA^{-1}$$

$$= I.$$

1.3 The Invertible Matrix Theorem

Earlier we saw that if a matrix A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$.

This suggests a deep connection between the invertibility of A and the linear system $A\mathbf{x} = \mathbf{b}$.

In fact, we are now at the point where we can collect together in a fairly complete way much of what we have learned about matrices and linear systems. This remarkable collection of ten interrelated properties is called the **Invertible Matrix Theorem (IMT).**

Invertible Matrix Theorem. Let A by a square $n \times n$ matrix. Then the following statements are equivalent; that is, they are either **all true** or **all false**:

- A is an invertible matrix.
- A^T is an invertible matrix.
  - Proof by direct construction: $(A^T)^{-1} = (A^{-1})^T$.
- The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
  - As already mentioned, we proved this above.
- A is row equivalent to the identity matrix.
  - If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$, then the reduced row echelon form of A is I.
- A has n pivot positions.
  - Follows directly from the previous statement.
- The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$, then the unique solution for $\mathbf{b} = \mathbf{0}$ must be $\mathbf{0}$.
- The columns of A form a linearly independent set.
  - Follows directly the previous statement and the definition of linear independence.
- The columns of A span $\mathbb{R}^n$.
  - For any $\mathbf{b} \in \mathbb{R}^n$, there is a set of coefficients $\mathbf{x}$ which can be used to construct $\mathbf{b}$ from the columns of A.
- The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
  - "maps onto" means that every element in the codomain is in the range.
  - Follows directly from the previous statement.
- The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
  - "one-to-one" means that there is a unique $\mathbf{x}$ for each $A\mathbf{x}$.
  - Follows directly from the fact that $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$.

The arguments above show that if A is invertible, then all the other statements are true. In fact, the converse holds as well: if A is not invertible, then all the other statements are false. (We will skip the proof of the converse, but it's not difficult.)

This theorem has wide-ranging implications. It divides the set of all $n \times n$ matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices.

The power of the IMT lies in the conections it provides among so many important concepts, such as linear idenpendence of the columns of a matrix A and the existence of solutions to equations of the form $A\mathbf{x} = \mathbf{b}$.

This allows us to bring many tools to bear as needed to solve a problem.

Example. Decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}.$$

Solution.

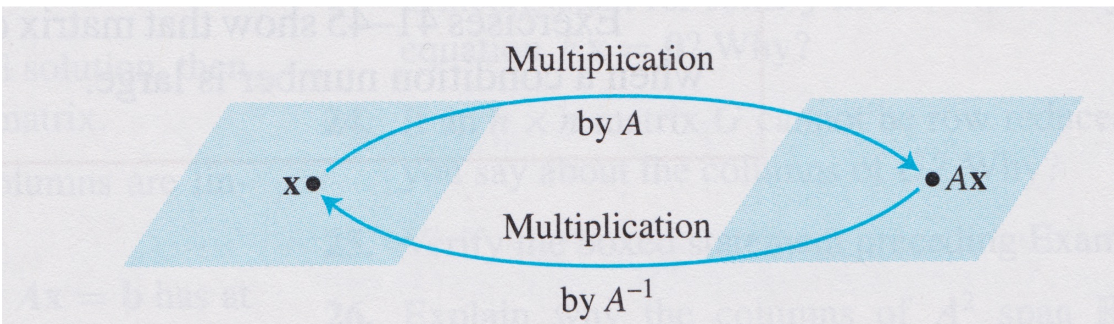
$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

So A has three pivot positions and hence is invertible, by the IMT.

Note. While the IMT is quite powerful, it does not completely settle issues that arise with respect to $A\mathbf{x} = \mathbf{b}$. This is because **it only applies to square matrices.**

So if A is nonsquare, then we can't use the IMT to conclude anything about the existence or nonexistence of solutions to $A\mathbf{x} = \mathbf{b}$.

1.4 Invertible Linear Transformations



A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

and

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying the definition.

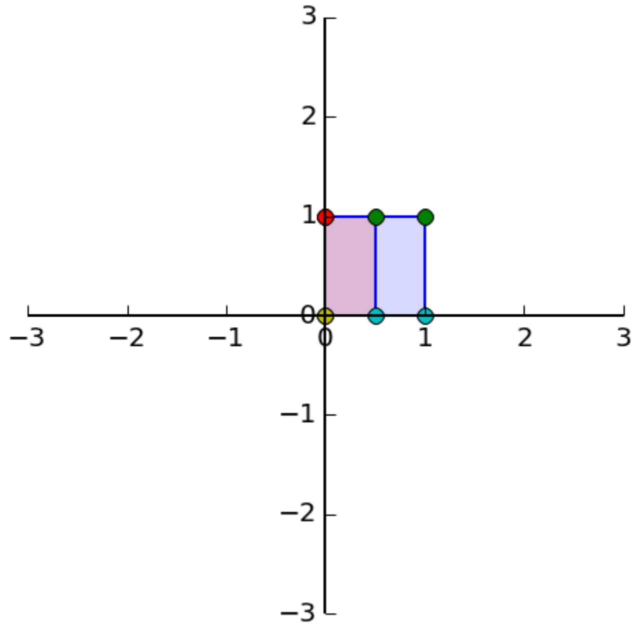
Let's look at some invertible and non-invertible linear transformations.

```
In [39]: square = np.array([[0.0,1,1,0],[1,1,0,0]])
         A = np.array([[0.5,0],[0,1]])
         print A
         ax = dm.plotSetup(-3,3,-3,3)
         dm.plotSquare(square)
         dm.plotSquare(A.dot(square),'r')
         Latex(r'Horizontal Contraction')
```

```
[[ 0.5  0. ]
 [ 0.   1. ]]
```

Out[39]:

Horizontal Contraction



Here $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$. Its determinant is $1(0.5) - 0(0) = 0.5$, so this linear transformation is invertible. Its inverse is:

$$\frac{1}{0.5}\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

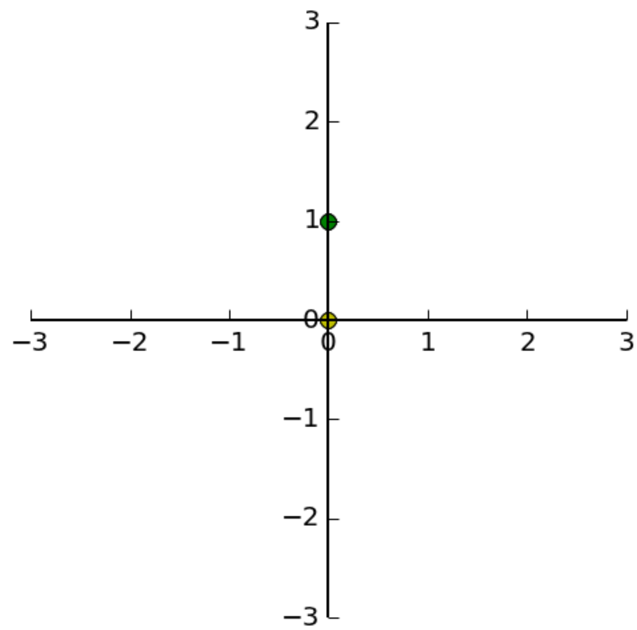
Clearly, just as A contracted the x_1 direction by 0.5, A^{-1} will expand the x_1 direction by 2.

In [40]:
```
A = np.array([[0,0],[0,1]])
print A
ax = dm.plotSetup(-3,3,-3,3)
dm.plotSquare(A.dot(square))
Latex(r'Projection onto the $x_2$ axis')
```

```
[[0 0]
 [0 1]]
```

Out[40]:

Projection onto the x_2 axis



Here $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Its determinant is zero, so this linear transformation is **not** invertible.

By the IMT, there are many equivalent ways to look at this:

- The mapping T is not from $\mathbb{R}^2$ to $\mathbb{R}^2$.
- There are many values $\mathbf{x}$ that give the same $A\mathbf{x}$.
- A does not have 2 pivots.
- etc.

1.5 Ill-Conditioned Matrices

The notion of a matrix inverse has some complications when used in practice. As we've noted, numerical computations are not exact, and in particular, we often find that `a - b(a/b)` does not evaluate to exactly zero on a computer.

For similar reasons, a matrix which is actually singular may not appear to be so when used in a computation. Conversely, a matrix which is not singular may appear to be singular when used in a computation. This happens because, for example, the determinant does not evaluate to exactly zero, even though it should.

You recall that when we were implementing Gaussian elimination, we established a rule that if `a - b(a/b)` $<$ `epsilon` for some `epsilon`, we would treat that quantity as if it were zero.

We need an equivalent rule for matrices, so that we recognize when matrices are "nearly singular," and we don't try to solve $A\mathbf{x} = \mathbf{b}$ when that is the case.

This value is called the **condition number.** The larger the condition number, the closer the matrix is to being singular.

The condition number of the identity matrix is 1, which is the smallest possible value. A singular matrix has an infinite condition number.

The point is that a matrix with a very large condition number, say, bigger than 10^8, will behave much like a singular matrix in practice. One should not try to solve linear systems by computer when the matrix A has a very large condition number.

To compute the condition number of a matrix `A` in `Python/numpy`, use `np.linalg.cond(A)`