Linear Transformations

So far we've been treating the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

as simply another way of writing the vector equation

$$x_1\,\mathbf{a_1} + \cdots + x_n\,\mathbf{a_n} = \mathbf{b}.$$

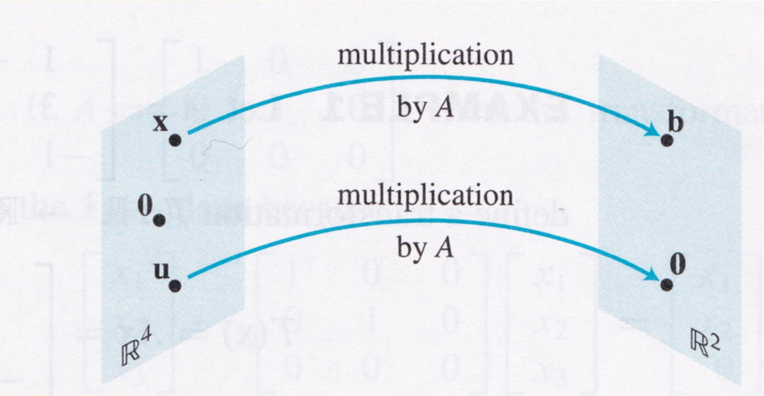
However, we'll now think of the matrix equation in a new way: we will think of A as "acting on" the vector $\mathbf{x}$ to form a new vector $\mathbf{b}$.

For example, let's let $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$. Then we find:

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words, if $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, then A *transforms* $\mathbf{x}$ into $\mathbf{b}$.

Likewise, if $\mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}$, then A transforms $\mathbf{u}$ into the $\mathbf{0}$ vector.



This gives a **new** way of thinking about solving $A\mathbf{x} = \mathbf{b}$. We are "searching" for the vectors $\mathbf{x}$ in $\mathbb{R}^4$ that are transformed into $\mathbf{b}$ in $\mathbb{R}^2$ under the "action" of A.

We have moved out of the familiar world of functions of one variable: we are now thinking about functions that transform a vector into a vector.

Or, put another way, functions that transform multiple variables into multiple variables.

Some terminology:

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T.

The notation:

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.

For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ (under T). The set of all images $T(\mathbf{x})$ is called the **range** of T.