1 Homework sheet 2 - Nash equilibrium in normal form games

1. Compute the Nash equilibrium (if they exist) in pure strategies for the following games:

   Solution

$$\begin{pmatrix} (5,\underline{3}) & (70,-1) & (\underline{4},2) \\ (\underline{6},\underline{7}) & (\underline{71},2) & (2,1) \end{pmatrix}$$

$$\begin{pmatrix} (\underline{6},\underline{7}) & (2,1) & (4,6) \\ (0,4) & (\underline{3},\underline{8}) & (2,3) \\ (\underline{1},\underline{2}) & (\underline{1},\underline{5}) & (\underline{1},\underline{1}) \end{pmatrix}$$

$$\begin{pmatrix} (\underline{\pi},\underline{e}) & (1-\pi,\sqrt{(}e)) \\ (\sqrt{(}2),1/e) & (\underline{2},\underline{1}) \end{pmatrix}$$

2. For what values of α does a Nash equilibrium exist in pure strategies for the following game:

$$\begin{pmatrix} (3,5) & (2-\alpha,\alpha) \\ (4\alpha,6) & (\alpha,\alpha^2) \end{pmatrix}$$

   Solution

   - (r_1,s_1) is a pure strategy Nash equilibrium if:
     $3\geq 4\alpha$ and $5\geq\alpha$
     Thus (r_1,s_1) is a Nash equilibrium iff $\alpha\geq 3/4$.
   - (r_1,s_2) is a pure strategy Nash equilibrium if:
     $2-\alpha\geq\alpha$ and $\alpha\geq 5$
     This is not possible.
   - (r_2,s_1) is a pure strategy Nash equilibrium if:
     $4\alpha\geq 3$ and $6\geq\alpha^2$
     Thus (r_2,s_1) is a Nash equilibrium iff $4/3\leq\alpha\leq\sqrt{6}$
   - (r_2,s_2) is a pure strategy Nash equilibrium if:
     $\alpha\geq 2-\alpha$ and $6\leq\alpha^2$
     Thus (r_2,s_2) is a Nash equilibrium iff $\alpha\geq\sqrt{6}$

3. Consider the following game:

   Suppose two vendors (of an identical product) must choose their location along a busy street. It is anticipated that their profit is directly related to their position on the street.

If we allow their positions to be represented by a points x_1, x_2 on the $[0,1]_{\mathbb{R}}$ line segment then we have:

$$u_1(x_1,x_2) = \begin{cases} x_1 + (x_2 - x_1)/2, & \text{if } x_1 \leq x_2 \\ 1 - x_1 + (x_2 - x_1)/2, & \text{otherwise} \end{cases}$$

and

$$u_1(x_1,x_2) = \begin{cases} x_2 + (x_2 - x_1)/2, & \text{if } x_2 \leq x_1 \\ 1 - x_2 + (x_2 - x_1)/2, & \text{otherwise} \end{cases}$$

By considering best responses of each player, identify the Nash equilibrium for the game.

Solution

Consider $x_j < 1/2$, if $x_i = x_j$ then $u_i(x_i, x_j) = 1/2$. However $u_i(x_j + \epsilon, x_j) = 1 - x_2 - \epsilon/2 > 1/2 - \epsilon/2$ for some (arbitrarily) small $\epsilon > 0$. Thus for arbitrarily small ϵ, $x_i^* = x_j + \epsilon$. If $x_j > 1/2$ a similar argument gives $x_i^* = x_j - \epsilon$. If $x_j = 1/2$, considering $x_i = x_j$ we see that neither player has an incentive to move.

Thus we conclude:

$$x_i^* = \begin{cases} x_j + \epsilon, & \text{if } x_j < 1/2 \\ x_j - \epsilon & \text{if } x_j > 1/2 \\ x_j & \text{if } x_j = 1/2 \end{cases}$$

So the Nash equilibrium for this problem is $(\tilde{x}_1, \tilde{x}_2) = (1/2, 1/2)$.

4. Consider the following game:

$$\begin{pmatrix} (3,2) & (6,5) \\ (1,4) & (2,3) \end{pmatrix}$$

`Plot the expected utilities for each player against mixed strategies and use this to obtain t`

5. Assume a soccer player (player 1) is taking a penalty kick and has the option of shooting left or right: $S_1 = \{\text{SL}, \text{SR}\}$. A goalie (player 2) can either dive left or right: $S_2 = \{\text{DL}, \text{DR}\}$. The chances of a goal being scored are given below:

$$\begin{pmatrix} .8 & .15 \\ .2 & .95 \end{pmatrix}$$

   i. Assume the utility to player 1 if the probability of scoring and the utility to player 2 the probability of a goal not being scored. What is the Nash equilibrium for this game?

ii. Assume that player 1 now has a further strategy available: to shoot in the middle: $S_1 = \{\mathrm{SL}, \mathrm{SM}, \mathrm{SR}\}$ the probabilities of a goal being scored are now given:

$$\begin{pmatrix} .8 & .15 \\ .5 & .5 \\ .2 & .95 \end{pmatrix}$$

Obtain the new Nash equilibrium for the game.

6. In the notes the following theorem is given:

Every normal form game with a finite number of pure strategies for each player, has at least one Nash equilibrium.

Prove the theorem for 2 player games with $|S_1| = |S_2| = 2$. I.e. prove the above result in the special case of 2×2 games.