

# 1 Homework sheet 4 - Evolutionary games, games with incomplete information and stochastic games

1. Consider the pairwise contest games with the following associated two player games:

$$\begin{pmatrix} (2, 2) & (4, 5) \\ (5, 4) & (1, 1) \end{pmatrix}$$

## Solution

Using the Equality of payoffs theorem we obtain the Nash equilibria:

$$\{((1, 0), (0, 1)), ((0, 1), (1, 0)), ((1/2, 1/2), (1/2, 1/2))\}$$

The two pure Nash equilibria are ESS (because of the first condition of the theorem relating an evolutionary stable strategy to the Nash equilibrium of the associated game).

for  $\sigma = (\omega, 1 - \omega)$ :

$$u((1/2, 1/2), \sigma) = \omega + 2 - 2\omega + 5\omega/2 + (1 - \omega)/2 = \omega + 5/2$$

and:

$$u(\sigma, \sigma) = (2\omega^2 + 9(1 - \omega)\omega + (1 - \omega)^2)$$

thus (after some algebraic manipulation):

$$u((1/2, 1/2), \sigma) - u(\sigma, \sigma) = \frac{3}{2}(2\omega - 1)^2$$

which is positive for  $\omega \neq 1/2$  so this mixed strategy is an ESS.

$$\begin{pmatrix} (1, 1) & (0, 0) \\ (0, 0) & (1, 1) \end{pmatrix}$$

## Solution

Using the Equality of payoffs theorem we obtain the Nash equilibria:

$$\{((1, 0), (1, 0)), ((0, 1), (0, 1)), ((1/2, 1/2), (1/2, 1/2))\}$$

The two pure Nash equilibria are ESS (because of the first condition of the theorem relating an evolutionary stable strategy to the Nash equilibrium of the associated game).

As before:

$$u((1/2, 1/2), \sigma) = 1/2$$

and:

$$u((1/2, 1/2), \sigma) - u(\sigma, \sigma) = -1/2(2\omega - 1)^2$$

so not an ESS.

Identify all evolutionary stable strategies.

2. Consider the following game:

In a mathematics department, researchers can choose to use one of two systems for typesetting their research papers: LaTeX or Word. We will refer to these two strategies as  $L$  and  $W$  respectively. A user of  $W$  receives a basic utility of 1 and as  $L$  is more widely used by mathematicians out of the department and is in general considered to be a better system a user of  $L$  gets a basic utility of  $\alpha > 1$ . Members of the mathematics department often collaborate and as such it is beneficial for the researchers to use the same typesetting system. If we let  $\mu$  represent the proportion of users of  $L$  we let:

$$\begin{aligned} u(L, \chi) &= \alpha + 2\mu \\ u(W, \chi) &= 1 + 2(1 - \mu) \end{aligned}$$

What are the evolutionary stable strategies?

### Solutions

Using the theorem for necessity of stability we have the following candidate ESS:

1.  $\sigma_L$ : everyone uses  $L$ , thus  $\mu = 1$  (we have  $u(L, \chi) > u(W, \chi)$ ).
2.  $\sigma_W$ : everyone uses  $W$ , thus  $\mu = 0$  (we have  $u(L, \chi) < u(W, \chi)$ ).
3.  $\sigma_m$ : some use  $L$  and some use  $W$ , by the theorem we have  $u(L, \chi) = u(W, \chi)$  which implies  $\alpha + 2\mu = 1 + 2(1 - \mu)$  giving  $\mu = \frac{3-\alpha}{4}$ .

Now we consider the post entry population  $\chi_\epsilon = (1-\epsilon)\sigma^* + \epsilon\sigma$  (where  $\sigma^*$  is the base strategy and  $\sigma$  is the entry population). We denote  $\sigma = (\mu, 1-\mu)$  and  $\sigma^* = (\mu^*, 1-\mu^*)$  and  $\delta = u(\sigma^*, \chi_\epsilon) - u(\sigma, \chi_\epsilon)$ . We have:

$$\delta = \mu^* u(L, \chi_\epsilon) + (1-\mu^*) u(W, \chi_\epsilon) - \mu u(L, \chi_\epsilon) - (1-\mu) u(W, \chi_\epsilon) = (\mu^* - \mu)(u(L, \chi_\epsilon) - u(W, \chi_\epsilon))$$

which gives:

$$\delta = (\mu^* - \mu)(\alpha - 3 + 4((1 - \epsilon)\mu^* + \epsilon\mu)) = (\mu^* - \mu)(4\mu^* + \alpha - 3 + 4\epsilon(\mu - \mu^*))$$

We now consider each potential ESS in turn, if  $\delta > 0$  for all  $\epsilon < \bar{\epsilon}$  for some  $\bar{\epsilon}$  then we have an ESS (this is by definition):

1.  $\mu^* = 1$ :  $\delta = (1 - \mu)(1 + \alpha + 4\epsilon(\mu - 1)) > (1 - \mu)(1 + \alpha - 4\epsilon) > 0$  for all  $\mu \neq 1$  and  $\epsilon < \bar{\epsilon} = \frac{1+\alpha}{4}$ . Thus  $\sigma_L$  is an ESS.
2.  $\mu^* = 0$ :  $\delta = -\mu(\alpha - 3 + 4\epsilon\mu)$ . If  $\alpha \geq 3$  then  $\delta \leq 0$  for all values of  $\mu, \epsilon$ , thus if  $L$  is 3 times better than  $W$   $\sigma_W$  is not an ESS. If  $\alpha < 3$   $\delta > 0 \Leftrightarrow \alpha - 3 + 4\epsilon\mu < 0 \Rightarrow \alpha - 3 + 4\epsilon\mu < \alpha - 3 + 4\epsilon < 0$  for all  $\mu \neq 0$   $\epsilon < \bar{\epsilon} = \frac{3-\alpha}{4}$ . Thus  $\sigma_W$  is an ESS for  $\alpha < 3$ .
3.  $\mu^* = \frac{3-\alpha}{4}$ :  $\delta = -4\epsilon \left( \frac{3-\alpha}{4} - \mu \right)^2 < 0$  for all  $\mu \neq \frac{3-\alpha}{4}$  and for all  $\epsilon > 0$  so  $\sigma_m$  is not an ESS.

3. Consider the following two normal form games:

$$A = \begin{pmatrix} (3, 0) & (-1, -1) & (1, 2) \\ (1, 0) & (-1, 1) & (2, 0) \end{pmatrix}$$

$$B = \begin{pmatrix} (2, 2) & (1, 1) & (1, 3) \\ (1, 3) & (-2, -3) & (4, 2) \end{pmatrix}$$

Assume both players play either game  $A$  or game  $B$  with probability  $1/2$ , neither player knows which game is played. Obtain the Nash equilibrium for this game.

### Solution

The described game is akin to the following game:

$$\begin{pmatrix} (5/2, 1) & (0, 0) & (1, 5/2) \\ (1, 3/2) & (-3/2, -1) & (3/2, 1) \end{pmatrix}$$

We see that  $s_2$  is dominated and solve the game using the equality of payoffs theorem to give the following Nash equilibrium:

$$((1/4, 3/4), (1/4, 3/4))$$

4. Repeat the analysis of the principal agent game assuming that  $p$  is the probability of the project being successful in case of a high level of effort by the employee.
  - i. What are the expected utilities to the employer and the employee?

**Solution**

Repeating the analysis, we see that the employee will carry out a high effort iff:

$$p(\omega + B - 1)^\alpha + (1 - p)(\omega - 1)^\alpha \geq \omega^\alpha$$

Following the same argument as in the notes we arrive at:

$$p(\omega + B - 1)^\alpha + (1 - p)(\omega - 1)^\alpha = 1 = \omega^\alpha$$

thus:

$$\beta = (1/p)^{1/\alpha}$$

The utilities are then:

Employer:

$$p(K - 1 - (1/p)^{1/\alpha}) + (1 - p)(\kappa - 1)$$

Employee:

$$1$$

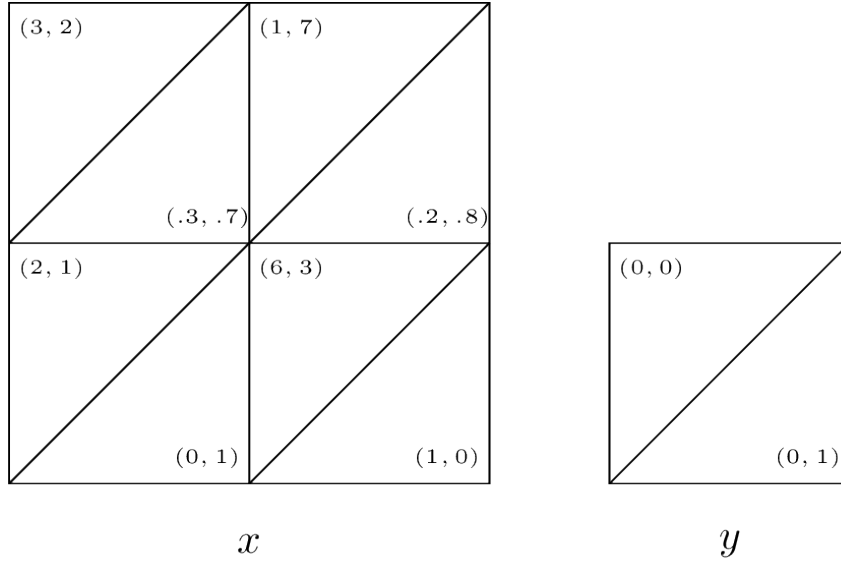
- ii. Obtain a condition for which the employer should offer a bonus.

**Solution**

If no bonus is offered the employee has no incentive for a high effort thus  $\kappa = \omega = 1$ , thus the employer should offer a bonus iff:

$$p(K - 1 - (1/p)^{1/\alpha}) + (1 - p)(\kappa - 1) \geq 0$$

5. Obtain the Markov Nash equilibrium (in pure strategies if it exists) for the following games assuming  $\delta = 1/4$ .



### Solution

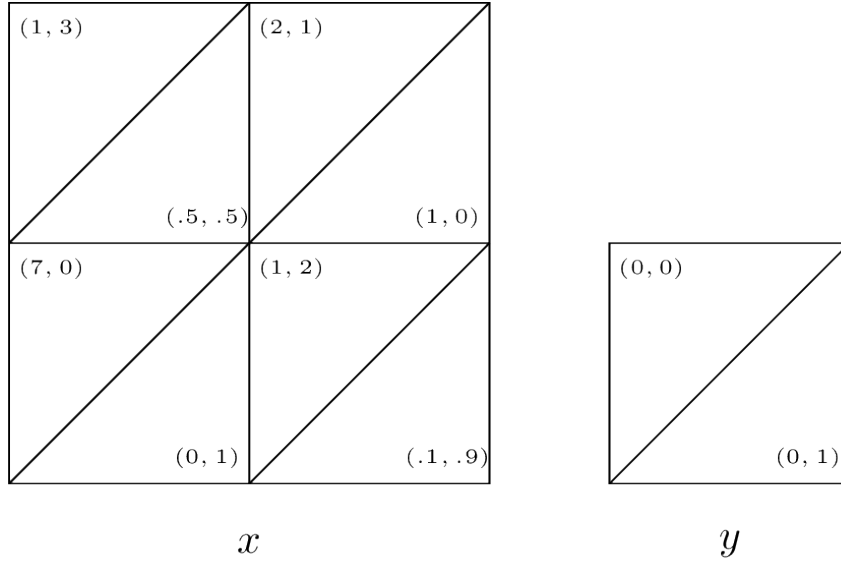
State  $y$  gives no value to either player so we only need to consider state  $x$ . Let the future gains to player 1 in state  $x$  be  $u$  and the future gains to player 2 in state  $x$  be  $v$ . Thus the players are facing the following game:

$$\begin{pmatrix} (3 + 3u/40, 2 + 3v/40) & (1 + u/20, 7 + v/20) \\ (2, 1) & (6 + u/4, 3 + v/4) \end{pmatrix}$$

There are four possible equilibria:

1.  $(a, c)$  which requires:  $3 + 3u/40 \geq 2$  and  $2 + 3v/40 \geq 7 + v/20 \Rightarrow u \geq -40/3$  and  $v \geq 200$ . However if this is the equilibria then  $u = 120/37$  and  $v = 80/37$  which contradicts the constraints.
2.  $(a, d)$  which requires:  $1 + u/20 \geq 6 + u/4$  and  $2 + 3v/40 \leq 7 + v/20 \Rightarrow u \leq -25$  and  $v \leq 200$ . However if this is the equilibria then  $u = 20/19$  and  $v = 140/19$  which contradicts the constraints.
3.  $(b, c)$  which requires:  $2 \geq 3 + 3u/40$  and  $1 \geq 3 + v/4 \Rightarrow u \leq -40/3$  and  $v \leq -8$ . However if this is the equilibria then  $u = 2$  and  $v = 1$  which contradicts the constraints.
4.  $(b, d)$  which requires:  $2 \leq 3 + 3u/40$  and  $1 \leq 3 + v/4 \Rightarrow u \geq -40/3$  and  $v \geq -8$ . However if this is the equilibria then  $u = 8$  and  $v = 4$  which **does not** contradict the constraints.

Thus  $(b, d)$  is the unique pure strategy equilibrium.



### Solution

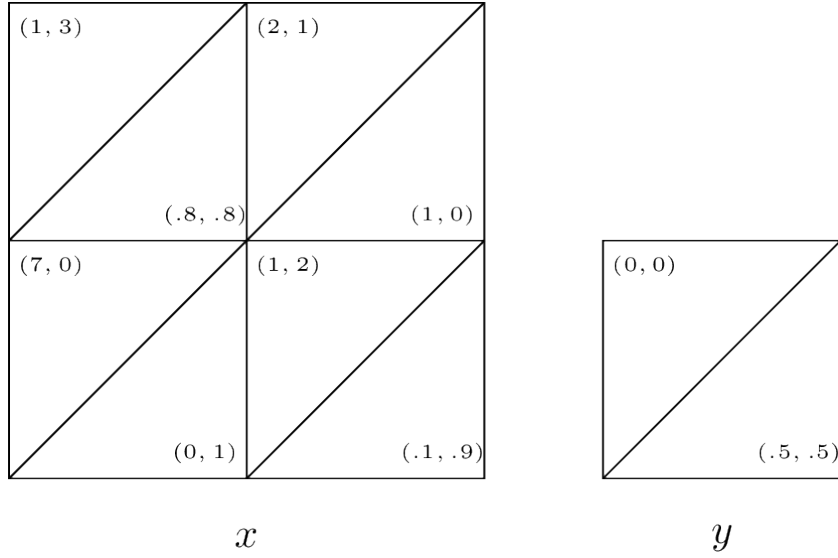
State  $y$  gives no value to either player so we only need to consider state  $x$ . Let the future gains to player 1 in state  $x$  be  $u$  and the future gains to player 2 in state  $x$  be  $v$ . Thus the players are facing the following game:

$$\begin{pmatrix} (1 + u/8, 3 + v/8) & (2 + u/4, 1 + v/4) \\ (7, 0) & (1 + u/40, 2 + v/40) \end{pmatrix}$$

There are four possible equilibria:

1.  $(a, c)$  which requires:  $1 + u/8 \geq 7$  and  $3 + v/8 \geq 1 + v/4 \Rightarrow u \geq 48$  and  $v \leq 16$ . However if this is the equilibria then  $u = 8/7$  and  $v = 24/7$  which contradicts the constraints.
2.  $(a, b)$  which requires:  $2 + u/4 \geq 1 + u/40$  and  $3 + v/8 \leq 1 + v/4 \Rightarrow u \geq -40/9$  and  $v \geq 16$ . However if this is the equilibria then  $u = 8/3$  and  $v = 4/3$  which contradicts the constraints.
3.  $(b, c)$  which requires:  $7 \geq 1 + u/8$  and  $0 \geq 2 + v/40 \Rightarrow u \leq 48$  and  $v \leq -80$ . However if this is the equilibria then  $u = 7$  and  $v = 0$  which contradicts the constraints.
4.  $(a, b)$  which requires:  $2 + u/4 \leq 1 + u/40$  and  $2 + v/40 \geq 0 \Rightarrow u \leq -40/9$  and  $v \geq -80$ . However if this is the equilibria then  $u = 40/39$  and  $v = 80/39$  which contradicts the constraints.

Thus no Nash equilibrium exists in pure strategies.



### Solution

State  $y$  gives no value to either player so we only need to consider state  $x$ . Let the future gains to player 1 in state  $x$  be  $u$  and the future gains to player 2 in state  $x$  be  $v$ . Thus the players are facing the following game:

$$\begin{pmatrix} (1 + u/20, 3 + v/20) & (2 + 7u/40, 1 + 7v/40) \\ (7, 0) & (1 + u/40, 2 + 9v/40) \end{pmatrix}$$

There are four possible equilibria:

1.  $(a, c)$  which requires:  $1 + u/20 \geq 7$  and  $3 + v/20 \geq 1 + 7v/40 \Rightarrow u \geq 120$  and  $v \leq 16$ . However if this is the equilibria then  $u = 20/19$  and  $v = 60/19$  which contradicts the constraints.
2.  $(a, b)$  which requires:  $2 + 7u/40 \geq 1 + u/40$  and  $1 + 7v/40 \geq 3 + v/20 \Rightarrow u \geq -20/3$  and  $v \leq 16$ . However if this is the equilibria then  $u = 80/33$  and  $v = 40/33$  which does not contradict any constraints.
3.  $(b, c)$  which requires:  $7 \geq 1 + u/20$  and  $0 \geq 2 + 9v/40 \Rightarrow u \leq 120$  and  $v \leq -80/9$ . However if this is the equilibria then  $u = 7$  and  $v = 0$  which contradicts the constraints.
4.  $(a, b)$  which requires:  $2 + u/4 \leq 1 + u/40$  and  $2 + 9v/40 \geq 0 \Rightarrow u \leq -40/9$  and  $v \geq -80/9$ . However if this is the equilibria then  $u = 40/39$  and  $v = 80/31$  which contradicts the constraints.

Thus  $(a, b)$  is the unique pure strategy equilibrium.