

# 1 OR 3: Chapter 16 - Cooperative games

## 1.1 Recap

In the [previous chapter](#):

- We defined matching games;
- We described the Gale-Shapley algorithm;
- We proved certain results regarding the Gale-Shapley algorithm.

In this Chapter we'll take a look at another type of game.

## 1.2 Cooperative Games

In cooperative game theory the interest lies with understanding how coalitions form in competitive situations.

### 1.2.1 Definition of a characteristic function game

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A **characteristic function game**  $G$  is given by a pair  $(N, v)$  where  $N$  is the number of players and  $v : 2^{[N]} \rightarrow \mathbb{R}$  is a **characteristic function** which maps every coalition of players to a payoff.

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Let us consider the following game:

“3 players share a taxi. Here are the costs for each individual journey: - Player 1: 6 - Player 2: 12 - Player 3: 42 How much should each individual contribute?”

This is illustrated in Figure 1.

To construct the characteristic function we first obtain the power set (ie all possible coalitions)  $2^{\{1,2,3\}} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \Omega\}$  where  $\Omega$  denotes the set of all players  $(\{1, 2, 3\})$ .

The characteristic function is given below:

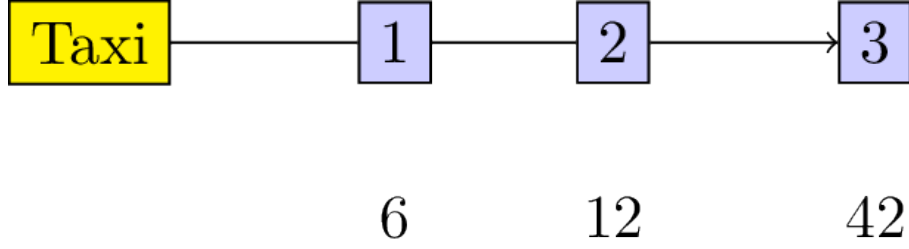


Figure 1: A taxi journey.

$$v(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 12, & \text{if } C = \{2\} \\ 42, & \text{if } C = \{3\} \\ 12, & \text{if } C = \{1, 2\} \\ 42, & \text{if } C = \{1, 3\} \\ 42, & \text{if } C = \{2, 3\} \\ 42, & \text{if } C = \{1, 2, 3\} \end{cases}$$

### 1.2.2 Definition of a monotone characteristic function game

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A characteristic function game  $G = (n, v)$  is called **monotone** if it satisfies  $v(S_2) \geq v(C_1)$  for all  $S_1 \subseteq C_2$ .

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This is illustrated in Figure 2.

Our taxi example is monotone, however the  $G = (3, v_1)$  with  $v_1$  defined as:

$$v_1(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 12, & \text{if } C = \{2\} \\ 42, & \text{if } C = \{3\} \\ 10, & \text{if } C = \{1, 2\} \\ 42, & \text{if } C = \{1, 3\} \\ 42, & \text{if } C = \{2, 3\} \\ 42, & \text{if } C = \{1, 2, 3\} \end{cases}$$

is not.

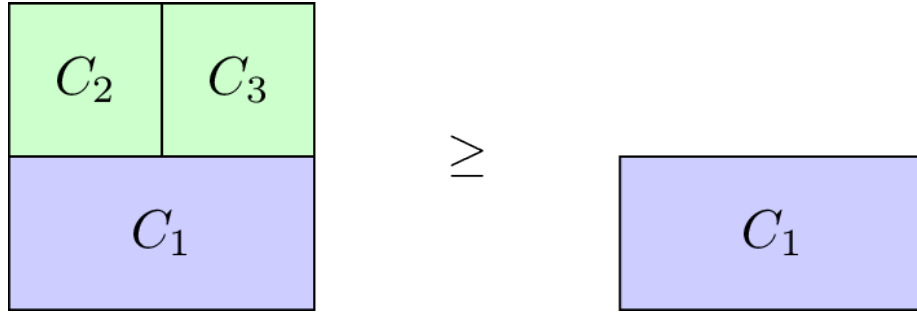


Figure 2: A diagrammatic representation of monotonicity.

### 1.2.3 Definition of a superadditive game

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A characteristic function game  $G = (n, v)$  is called **superadditive** if it satisfies  $v(C_1 \cup C_2) \geq v(C_1) + v(C_2)$ .

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This is illustrated in Figure 3.

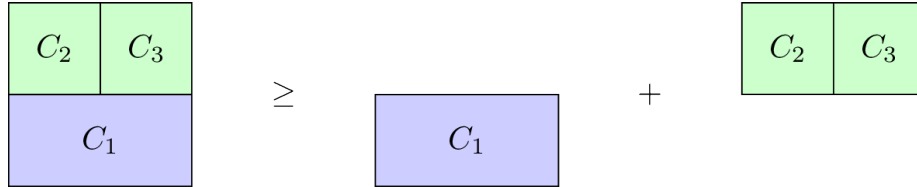


Figure 3: A diagrammatic representation of superadditivity.

Our taxi example is not superadditive, however the  $G = (3, v_2)$  with  $v_2$  defined as:

$$v_2(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 12, & \text{if } C = \{2\} \\ 42, & \text{if } C = \{3\} \\ 18, & \text{if } C = \{1, 2\} \\ 48, & \text{if } C = \{1, 3\} \\ 55, & \text{if } C = \{2, 3\} \\ 80, & \text{if } C = \{1, 2, 3\} \end{cases}$$

is.

### 1.3 Shapley Value

When talking about a solution to a characteristic function game we imply a payoff vector  $\lambda \in \mathbb{R}_{\geq 0}^N$  that divides the value of the grand coalition between the various players. Thus  $\lambda$  must satisfy:

$$\sum_{i=1}^n \lambda_i = v(\Omega)$$

Thus one potential solution to our taxi example would be  $\lambda = (14, 14, 14)$ . Obviously this is not ideal for player 1 and/or 2: they actually pay more than they would have paid without sharing the taxi!

Another potential solution would be  $\lambda = (6, 6, 30)$ , however at this point sharing the taxi is of no benefit to player 1. Similarly  $(0, 12, 30)$  would have no incentive for player 2.

To find a “fair” distribution of the grand coalition we must define what is meant by “fair”. We require four desirable properties:

- Efficiency;
- Null player;
- Symmetry;
- Additivity.

#### 1.3.1 Definition of efficiency

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For  $G = (N, v)$  a payoff vector  $\lambda$  is **efficient** if:

$$\sum_{i=1}^N \lambda_i = v(\Omega)$$

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#### 1.3.2 Definition of null players

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For  $G(N, v)$  a payoff vector possesses the **null player property** if  $v(C \cup i) = v(C)$  for all  $C \in 2^\Omega$  then:

$$x_i = 0$$


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### 1.3.3 Definition of symmetry

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For  $G(N, v)$  a payoff vector possesses the **symmetry property** if  $v(C \cup i) = v(C \cup j)$  for all  $C \in 2^\Omega \setminus \{i, j\}$  then:

$$x_i = x_j$$


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### 1.3.4 Definition of additivity

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For  $G_1 = (N, v_1)$  and  $G_2 = (N, v_2)$  and  $G^+ = (N, v^+)$  where  $v^+(C) = v_1(C) + v_2(C)$  for any  $C \in 2^\Omega$ . A payoff vector possesses the **additivity property** if:

$$x_i^{(G^+)} = x_i^{(G_1)} + x_i^{(G_2)}$$


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We will not prove in this course but in fact there is a single payoff vector that satisfies these four properties. To define it we need two last definitions.

### 1.3.5 Definition of predecessors

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If we consider any permutation  $\pi$  of  $[N]$  then we denote by  $S_\pi(i)$  the set of **predecessors** of  $i$  in  $\pi$ :

$$S_\pi(i) = \{j \in [n] \mid \pi(j) < \pi(i)\}$$


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For example for  $\pi = (1, 3, 4, 2)$  we have  $S_\pi(4) = \{1, 3\}$ .

### 1.3.6 Definition of marginal contribution

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If we consider any permutation  $\pi$  of  $[N]$  then the **marginal contribution** of player  $i$  with respect to  $\pi$  is given by:

$$\Delta_{\pi}^G(i) = v(S_{\pi}(i) \cup i) - v(S_{\pi}(i))$$

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We can now define the **Shapley value** of any game  $G = (N, v)$ .

### 1.3.7 Definition of the Shapley value

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Given  $G = (N, v)$  the **Shapley value** of player  $i$  is denoted by  $\phi_i(G)$  and given by:

$$\phi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi_n} \Delta_{\pi}^G(i)$$

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As an example here is the Shapley value calculation for our taxi sharing game:

For  $\pi = (1, 2, 3)$ :

$$\begin{aligned}\Delta_{\pi}^G(1) &= 6 \\ \Delta_{\pi}^G(2) &= 6 \\ \Delta_{\pi}^G(3) &= 30\end{aligned}$$

For  $\pi = (1, 3, 2)$ :

$$\begin{aligned}\Delta_{\pi}^G(1) &= 6 \\ \Delta_{\pi}^G(2) &= 0 \\ \Delta_{\pi}^G(3) &= 36\end{aligned}$$

For  $\pi = (2, 1, 3)$ :

$$\Delta_{\pi}^G(1) = 0$$

$$\Delta_{\pi}^G(2) = 12$$

$$\Delta_{\pi}^G(3) = 30$$

For  $\pi = (2, 3, 1)$ :

$$\Delta_{\pi}^G(1) = 0$$

$$\Delta_{\pi}^G(2) = 12$$

$$\Delta_{\pi}^G(3) = 30$$

For  $\pi = (3, 1, 2)$ :

$$\Delta_{\pi}^G(1) = 0$$

$$\Delta_{\pi}^G(2) = 0$$

$$\Delta_{\pi}^G(3) = 42$$

For  $\pi = (3, 2, 1)$ :

$$\Delta_{\pi}^G(1) = 0$$

$$\Delta_{\pi}^G(2) = 12$$

$$\Delta_{\pi}^G(3) = 42$$

Using this we obtain:

$$\phi(G) = (2, 5, 35)$$

Thus the fair way of sharing the taxi fare is for player 1 to pay 2, player 2 to pay 5 and player 3 to pay 35.