1 OR 3: Chapter 12 - Nash equilibrium and Evolutionary stable strategies

1.1 Recap

In the previous chapter:

- We considered population games;
- We proved a result concerning a necessary condition for a population to evolutionary stable;
- We defined Evolutionary stable strategies and looked at an example in a game against the field.

In this chapter we'll take a look at pairwise contest games and look at the connection between Nash equilibrium and ESS.

1.2 Pairwise contest games

In a population game when considering a pairwise contest game we assume that individuals are randomly matched. The utilities then depend just on what the individuals do:

$$u(\sigma, x) = \sum_{s\in S}\sum_{s'\in S}\sigma(s)x(s')u(s, s')$$

As an example we're going to consider the "Hawk-Dove" game: a model of predator interaction. We have $S = \{H, D\}$ were:

- H: Hawk represents being "aggressive";
- D: Dove represents not being "aggressive".

At various times individuals come in to contact and must choose to act like a Hawk or like Dove over the sharing of some resource of value v. We assume that:

- If a Dove and Hawk meet the Hawk takes the resources;
- If two Doves meet they share the resources;
- If two Hawks meet there is a fight over the resources (with an equal chance of winning) and the winner takes the resources while the loser pays a cost $c > v$.

If we assume that $\sigma = (\omega, 1-\omega)$ and $x = (h, 1-h)$ the above gives:

$$u(\sigma, x) = \omega(1-h)v + (1-\omega)(1-h)\frac{v}{2} + \omega h\frac{v-c}{2}$$

It is immediate to note that no pure strategy ESS exists. In a population of Doves ($h = 0$):

$$u(\sigma, (0,1)) = \omega v + (1-\omega)\frac{v}{2} = (1+\omega)\frac{v}{2}$$

thus the best response is setting $\omega = 1$ i.e. to play Hawk.

In a population of Hawks ($h = 1$):

$$u(\sigma, (1,0)) = \omega h\frac{v-c}{2}$$

thus the best response is setting $\omega = 0$ i.e. to play Dove.

So we will now try and find out if there is a mixed-strategy ESS: $\sigma^* = (\omega^*, 1-\omega^*)$. For σ^* to be an ESS it must be a best response to the population it generates $x^* = (\omega^*, 1-\omega^*)$. In this population the x^* the payoff to an arbitrary strategy σ is:

$$u(\sigma, x^*) = (1-\omega^*)\frac{v}{2} + \left(\frac{v}{2} - \omega^*\right)\frac{\omega c}{2}$$

- If $\omega^* < v/c$ then a best response is $\omega = 1$;
- If $\omega^* > v/c$ then a best response is $\omega = 0$;
- If $\omega^* = v/c$ then there is indifference.

So the only candidate for an ESS is $\sigma^* = (v/c, 1 - v/c)$. We now need to show that $u(\sigma^*, x_\epsilon) > u(\sigma, x_\epsilon)$.

We have:

$$x_\epsilon = (v/c + \epsilon(\omega - v/c), 1 - v/c + \epsilon(\omega - v/c)$$

and:

$$\begin{aligned} u(\sigma^*, x_\epsilon) = \frac{v}{c}\left(\frac{v}{c} + \epsilon\left(\omega - \frac{v}{c}\right)\right)\frac{v-c}{2} &+ \frac{v}{c}\left(1 - \frac{v}{c} + \epsilon\left(\frac{v}{c} - \omega\right)\right)v \\ &+ \left(1 - \frac{v}{c}\right)\left(1 - \frac{v}{c} + \epsilon\left(\frac{v}{c} - \omega\right)\right)\frac{v}{2} \end{aligned}$$

$$
\begin{aligned}
u(\sigma, x_\epsilon) = \omega\left(\frac{v}{c}+\epsilon\left(\omega-\frac{v}{c}\right)\right)\frac{v-c}{2} &+ \omega\left(1-\frac{v}{c}+\epsilon\left(\frac{v}{c}-\omega\right)\right)v \\
&+ (1-\omega)\left(1-\frac{v}{c}+\epsilon\left(\frac{v}{c}-\omega\right)\right)\frac{v}{2}
\end{aligned}
$$

This gives:

$$u(\sigma^*, x_\epsilon) - u(\sigma, x_\epsilon) = \frac{\epsilon c}{2}\left(\frac{v}{c}-\omega\right)^2 > 0$$

which proves that σ^* is an ESS.

We will now take a closer look the connection between ESS and Nash equilibria.

1.3 ESS and Nash equilibria

When considering pairwise contest population games there is a natural way to associate a normal form game.

1.3.1 Definition

The **associated two player game** for a pairwise contest population game is the normal form game with payoffs given by: $u_1(s, s') = u(s, s') = u_2(s', s)$.

Note that the resulting game is symmetric (other contexts would give non symmetric games but we won't consider them here).

Using this we have the powerful result:

1.3.2 Theorem

If σ^* is an ESS in a pairwise contest population game then for all $\sigma \neq \sigma^*$:

1. $u(\sigma^*, \sigma^*) > u(\sigma, \sigma^*)$ OR
2. $u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*)$ and $u(\sigma^*, \sigma) > u(\sigma, \sigma)$

Conversely, if either (1) or (2) holds for all $\sigma \neq \sigma^*$ in a two player normal form game then σ is an ESS.

1.3.3 Proof

If σ^* is an ESS, then by definition:

$$u(\sigma^*, x_\epsilon) > u(\sigma, x_\epsilon)$$

which corresponds to:

$$(1-\epsilon)u(\sigma^*, \sigma^*) + \epsilon u(\sigma^*, \sigma) > (1-\epsilon)u(\sigma, \sigma^*) + \epsilon u(\sigma, \sigma)$$

- If the condition 1 of the theorem holds then the above inequality can be satisfied for ϵ sufficiently small. If condition 2 holds then the inequality is satisfied.
- Let us assume that $u(\sigma^*, \sigma^*) < u(\sigma, \sigma^*)$:
  - If $u(\sigma^*, \sigma^*) < u(\sigma, \sigma^*)$ then we can find ϵ sufficiently small such that the inequality is violated. Thus the inequality implies $u(\sigma^*, \sigma^*) \geq u(\sigma, \sigma^*)$.
  - If $u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*)$ then :$u(\sigma^*, \sigma^*) > u(\sigma, \sigma^*)$ as required.

This result gives us a more a efficient way of computing ESS. The first condition is in fact almost a condition for Nash Equilibrium (with a strict inequality), the second is thus a stronger condition that removes certain Nash equilibria from consideration. This becomes particularly relevant when considering Nash equilibrium in mixed strategies.

To find ESS in a pairwise context population game we:

1. Write down the associated two-player game;
2. Identify all symmetric Nash equilibria of the game;
3. Test the Nash equilibrium against the two conditions of the above Theorem.

1.3.4 Example

Let us consider the Hawk-Dove game. The associated two-player game is:

$$\begin{pmatrix} \left(\frac{v-c}{2}, \frac{v-c}{2}\right), (v, 0) \\ (0, v), \left(\frac{v}{2}, \frac{v}{2}\right) \end{pmatrix}$$

Recalling that we have $v < c$ so we can use the Equality of payoffs theorem to obtain the Nash equilibrium:

$$u_2(\sigma^*, H) = u_2(\sigma^*, D)$$
$$q^* = \frac{v}{c}$$

Thus we will test $\sigma^* = (\frac{v}{c}, 1 - \frac{v}{c})$ using the above theorem.

Importatny from the equality of payoffs theorem we immediately see that condition 1 does not hold as $u(\sigma^*, H) = u(\sigma^*, D)$. Thus we need to prove that:

$$u(\sigma^*, \sigma) > u(\sigma, \sigma)$$

We have:

$$u(\sigma^*, \sigma) = \frac{v}{c}\omega\frac{v-c}{2} + \frac{v}{c}(1-\omega)v + (1-\frac{v}{c})(1-\omega)\frac{v}{c}$$
$$u(\sigma, \sigma) = \omega^2\frac{v-c}{2} + \omega(1-\omega)v + (1-\omega)^2\frac{v}{c}$$

After some Algebra:

$$u(\sigma^*, \sigma) - u(\sigma, \sigma) = \frac{c}{2}(\frac{v}{c} - \omega)^2 > 0$$

Giving the required result.