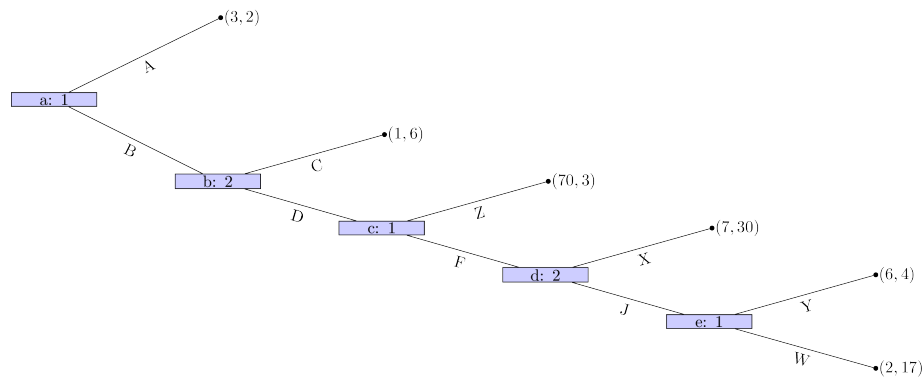


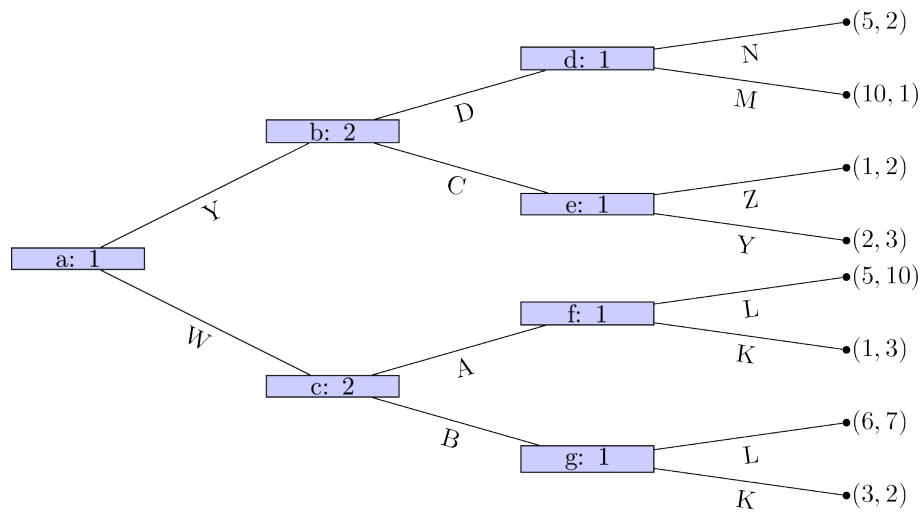
# 1 Homework sheet 3 - Extensive form games, subgame perfect equilibrium and repeated games

1. Obtain the Nash equilibrium for the following games using backward induction:



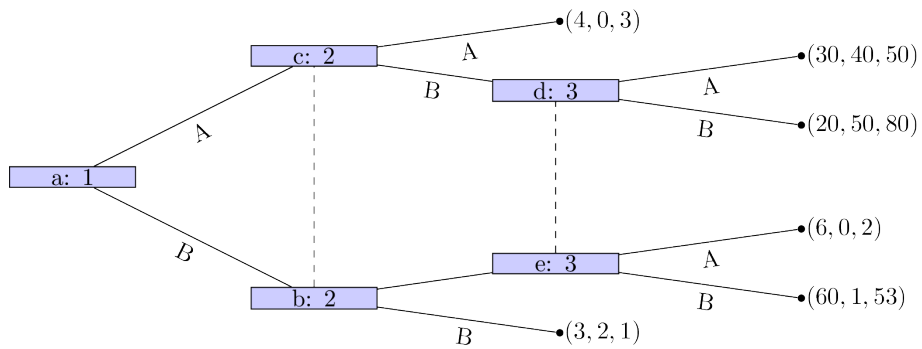
**Solution**

$(AZY, CX)$



**Solution**

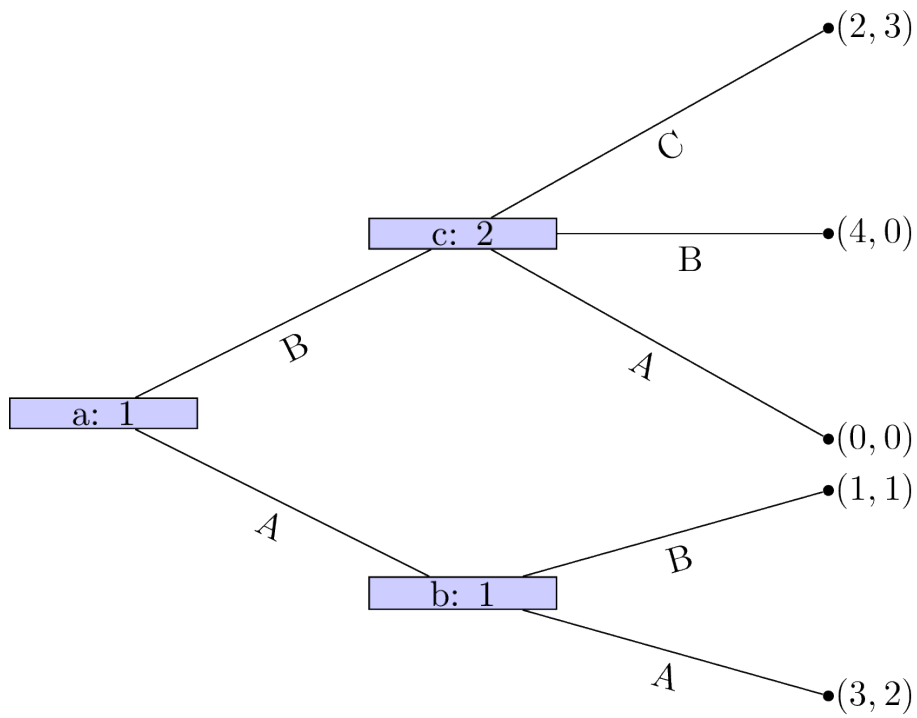
$(WMYLL, AC)$



For  $P_3$ ,  $B$  dominates  $A$ , for  $P_2$ ,  $B$  dominates  $A$ , finally  $A$  dominates  $B$  for  $P_1$ :

**Solution**

$(A, B, B)$



**Solution**

$(AA, C)$

2. Obtain the Nash equilibrium for the following game:

Player 1 chooses a number  $x \geq 0$ , which player 2 observes. After this simultaneously and independatly player 1 and player 2 choose  $y_1, y_2 \in \mathbb{R}$  respectively. The utility to player 1 is given by  $2y_2y_1 + xy_1 - y_1^2 - x^3/3$  and the utility to player 2 is given by  $-(y_1 - 2y_2)^2$ .

### Solution

For given  $x$ :  $y_1$  maximises  $2y_2y_1 + xy_1 - y_1^2 - x^3/3$ , thus  $y_1^*$  is a solution to:

$$2y_2 + x - 2y_1 = 0$$

(As the function has a local maxima which is global)

so:

$$y_1^* = \frac{2y_2 + x}{2}$$

Similarly:

$$y_2^* = \frac{y_1}{2}$$

Thus  $(\tilde{y}_1, \tilde{y}_2)$  solve the following system of equations:

$$\begin{cases} \tilde{y}_1 = \frac{\tilde{y}_1 + x}{2} \\ \tilde{y}_2 = \frac{\tilde{y}_1}{2} \end{cases}$$

Thus:

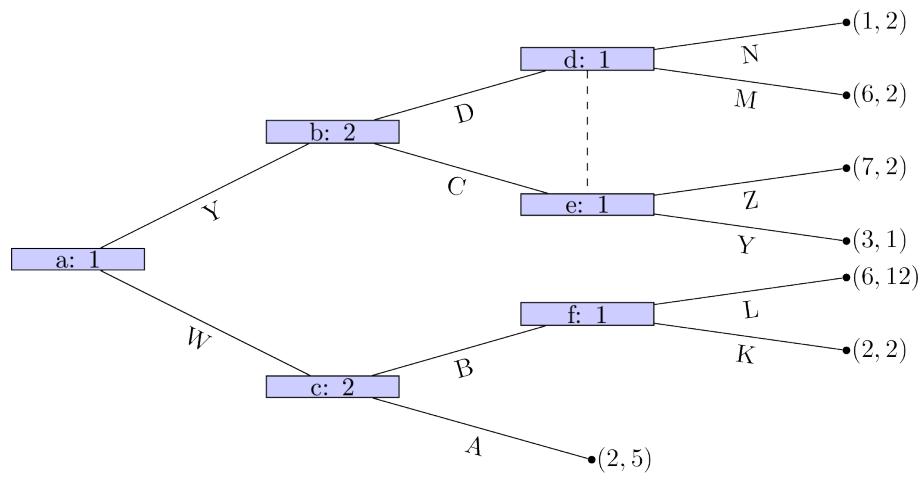
$$\begin{cases} \tilde{y}_1 = x \\ \tilde{y}_2 = \frac{x}{2} \end{cases}$$

This gives  $u_1 = x^2 + x^2 - x^2 - x^3/3 = x^2(1 - x/3)$  which has a maxima at  $x = 2$  for  $x \geq 0$ . This gives:

$$(\tilde{x}, \tilde{y}_1, \tilde{y}_2) = (2, 2, 1)$$

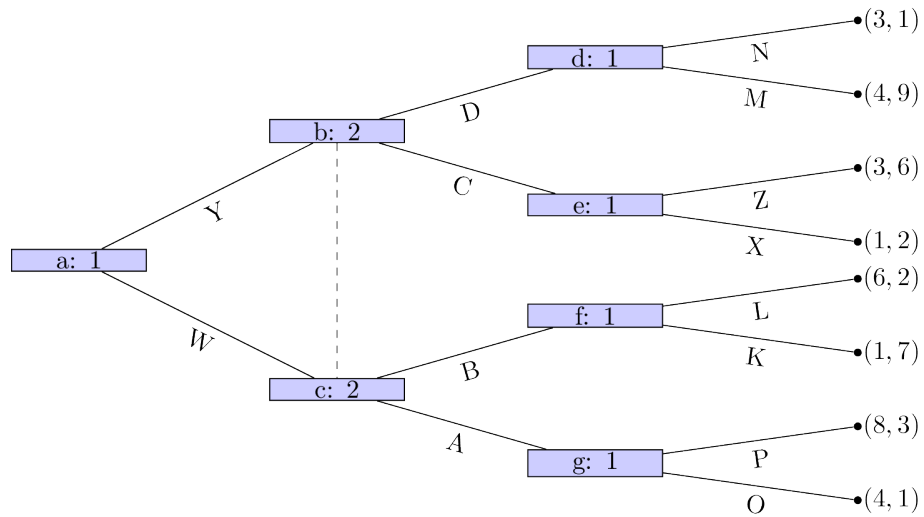
3. For each of the following games:

- i. Identify all subgames.
- ii. Identify the corresponding normal form representations and hence obtain all Nash equilibrium.
- iii. Identify which Nash equilibrium are also subgame perfect Nash equilibrium.



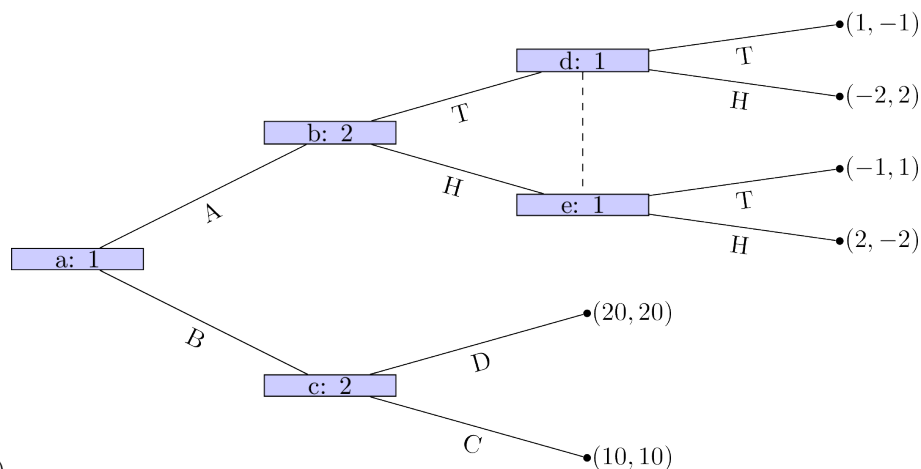
### Solution

Not a valid game (node d and e are in same information set but have different action sets).



### Solution

Not a valid game (node b and c are in same information set but have different ac-



tion sets)

There are two subgames:

- Generated by node  $c$  (trivial)
- Generated by node  $b$

The strategy sets are:

$$S_1 = \{AT, AH, BT, BH\}$$

$$S_2 = \{TD, TC, HD, HC\}$$

Here is the corresponding normal form representation:

$$\begin{pmatrix} (-1, 1) & (-1, 1) & (1, -1) & (1, -1) \\ (-2, 2) & (-2, 2) & (2, -2) & (2, -2) \\ (20, 20) & (10, 10) & (20, 20) & (10, 10) \\ (20, 20) & (10, 10) & (20, 20) & (10, 10) \end{pmatrix}$$

By examining best responses we identify the following 4 pure Nash equilibria:

$$\{(BT, TD), (BT, HD), (BH, TD), (BH, HD)\}$$

Looking at the subgame initiated at  $b$  (with  $S_1 = \{H, T\}$  and  $S_2 = \{H, T\}$ ):

$$\begin{pmatrix} (2, -2) & (-2, 2) \\ (-1, 1) & (1, -1) \end{pmatrix}$$

None of the above strategy pairs are Nash equilibria!

However (using the Equality of Payoffs theorem) we see that:

$$\sigma_1 = (0, 0, 2/3, 1/3)$$

and

$$\sigma_2 = (1/2, 0, 1/2, 0)$$

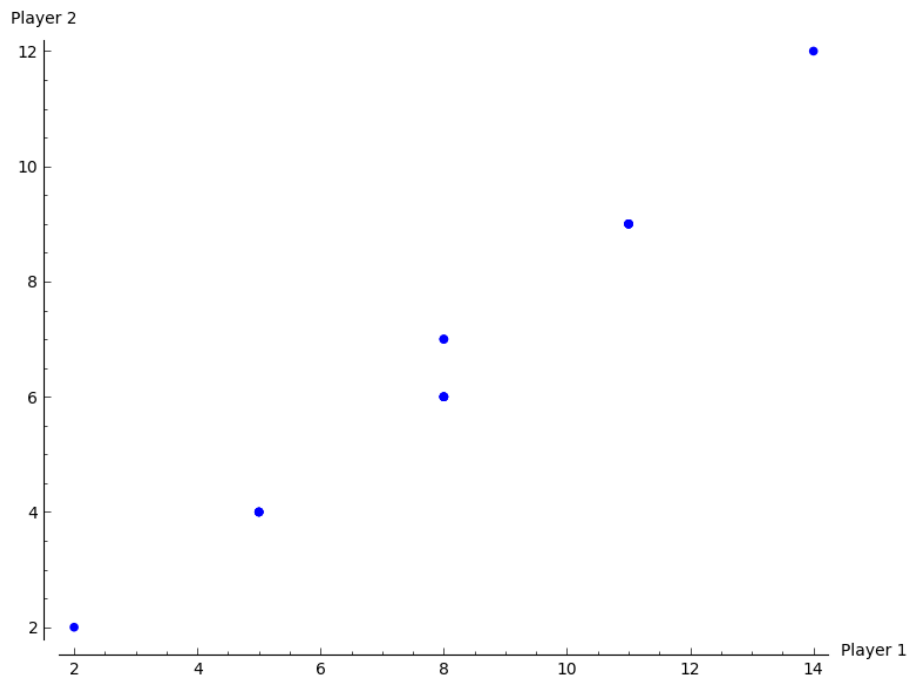
is a NE, and in fact is also a NE for the entire game (again using the Equality of Payoffs theorem).

4. For the following stage games:

- i. Plot all possible utility pairs for  $T = 2$ ;
- ii. Recalling that subgame perfect equilibrium for the repeated game must play a stage Nash equilibrium in the final stage attempt to identify a Nash equilibrium for the repeated game that is not a sequence of stage Nash profiles.

$$\begin{pmatrix} (4, 3) & (7, 6) \\ (1, 1) & (4, 3) \end{pmatrix}$$

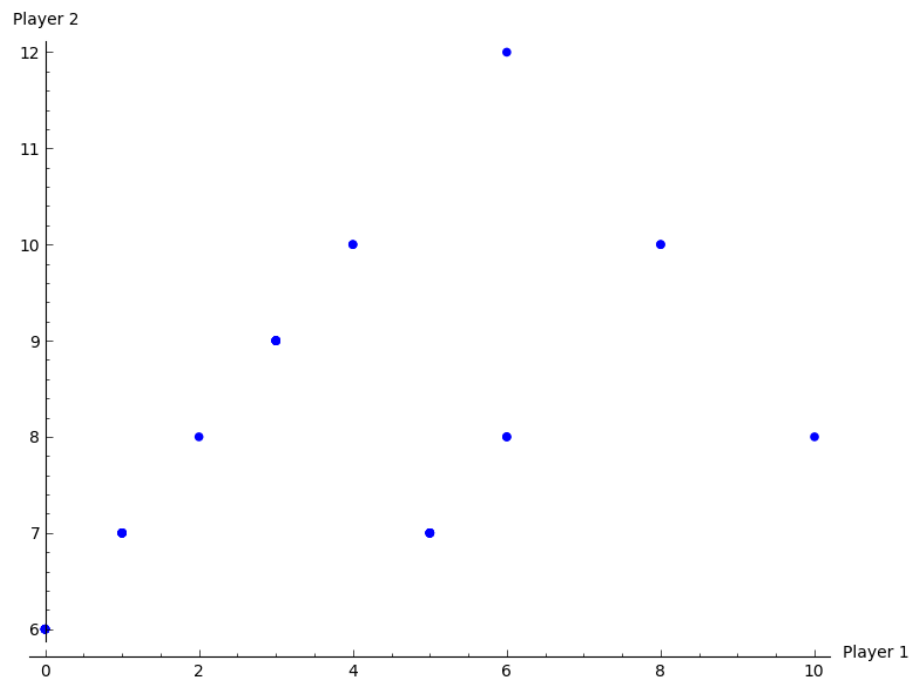
### Solution



No other strategy is subgame perfect.

$$\begin{pmatrix} (5,4) & (0,3) \\ (0,3) & (1,4) \\ (3,6) & (0,3) \end{pmatrix}$$

### Solution



“Play  $(r_3, s_1)$  in first round and  $(r_2, s_1)$  in second round unless P1 deviates in which case play  $(r_2, s_2)$ .”

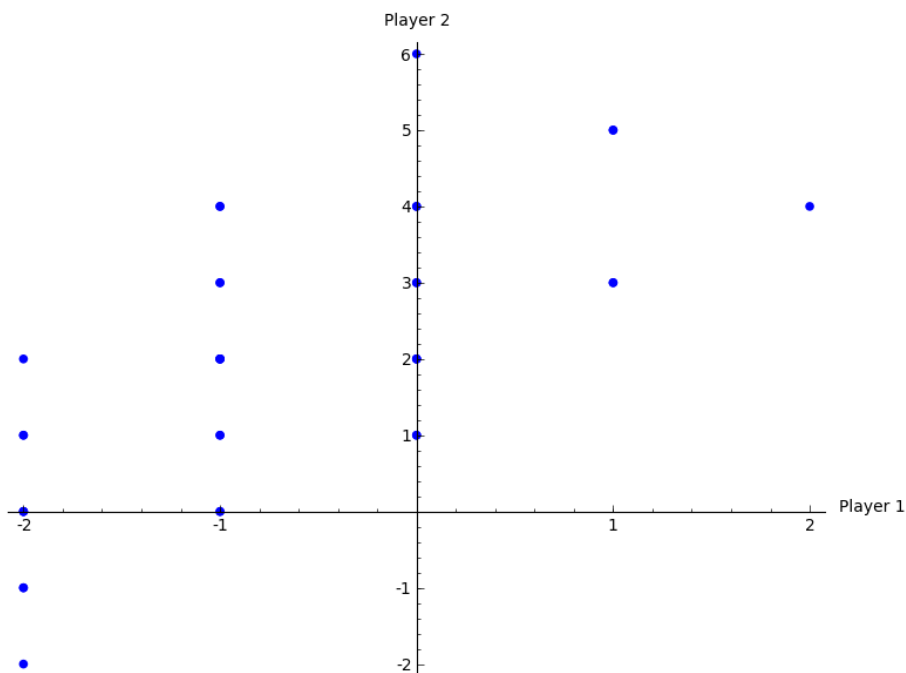
Outcome:  $(8, 10)$ .

Deviate?

- P2: No incentive;
- P1: Gain 2 in 1st round but lose 4 in second round.

$$\begin{pmatrix} (1,2) & (0,3) & (-1,1) \\ (-1,0) & (-1,-1) & (0,1) \end{pmatrix}$$

### Solution



“Play  $(r_1, s_1)$  in first round and  $(r_1, s_2)$  in second round unless P2 deviates in which case play  $(r_2, s_3)$ .”

Outcome:  $(1, 5)$ .

Deviate?

- P1: No incentive;
- P2: Gain 1 in 1st round but lose 2 in second round.

5. Consider the following stage game:

$$\begin{pmatrix} (-1, 1) & (3, -7) \\ (-2, 6) & (2, 2) \end{pmatrix}$$

- For  $\delta = 1/3$  obtain the utilities for the infinitely repeated game for the strategies  $S_D$ : “play the first strategy throughout” and  $S_C$ : “play the second strategy throughout”.
- Plot the space of feasible average payoffs and the space of individually rational payoffs.
- State whether or not it is possible according to the Folk theorem to obtain  $\delta$  that ensures that a strategy profile exists that would give a subgame perfect Nash equilibrium with average payoffs:  $(3/2, 3/2)$ ,  $(0, 3)$ ,  $(2, 6)$  and  $(2, 0)$ .



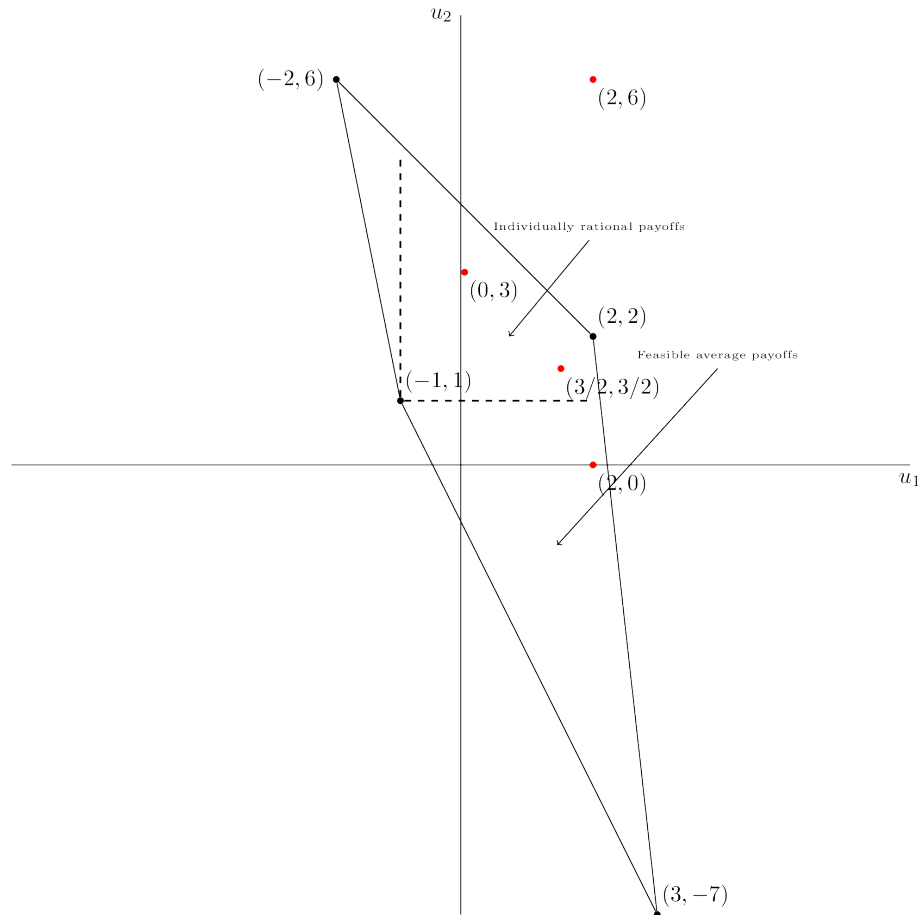
### Solution

$$\begin{cases} u_1(S_D, S_D) &= \sum_{i=0}^{\infty} \delta^i (-1) = -\frac{1}{1-\delta} = -3/2 \\ u_2(S_D, S_D) &= 3/2 \end{cases}$$

$$\begin{cases} u_1(S_D, S_C) &= 9/2 \\ u_2(S_D, S_C) &= -21/2 \end{cases}$$

$$\begin{cases} u_1(S_C, S_D) &= -3 \\ u_2(S_C, S_D) &= 9 \end{cases}$$

$$\begin{cases} u_1(S_C, S_C) &= 3 \\ u_2(S_C, S_C) &= 3 \end{cases}$$



We see that it is possible to find a  $\delta$  for  $(3/2, 3/2)$  and  $(0, 3)$ .