1 Homework sheet 4 - Evolutionary games, games with incomplete information and stochastic games

1. Consider the pairwise contest games with the following associated two player games:

$$\begin{pmatrix} (2,2) & (4,5) \\ (5,4) & (1,1) \end{pmatrix}$$

Solution

Using the Equality of payoffs theorem we obtain the Nash equilibria:

$$\{((1,0),(0,1)),((0,1),(1,0)),((1/2,1/2),(1/2,1/2))\}$$

The two pure Nash equilibria are ESS (because of the first condition of the theorem relating an evolutionary stable strategy to the Nash equilibrium of the associated game).

$$u((1/2, 1/2), \sigma) = 3$$

and for $\sigma = (\omega, 1 - \omega)$:

$$u(\sigma, \sigma) = (2\omega^2 + 9(1 - \omega)\omega + (1 - \omega)^2)$$

thus (after some algebraic manipulation):

$$u((1/2, 1/2), \sigma) - u(\sigma, \sigma) = (2\omega - 1)(3\omega - 2)$$

which is negative for $1/2 < \omega < 2/3$ so this mixed strategy is not an ESS.

$$\begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (1,1) \end{pmatrix}$$

Solution

Using the Equality of payoffs theorem we obtain the Nash equilibria:

$$\{((1,0),(1,0)),((0,1),(0,1)),((1/2,1/2),(1/2,1/2))\}$$

The two pure Nash equilibria are ESS (because of the first condition of the theorem relating an evolutionary stable strategy to the Nash equilibrium of the associated game).

$$u((1/2, 1/2), \sigma) = 1/2$$

As before:

$$u((1/2, 1/2), \sigma) - u(\sigma, \sigma) = -1/2(2\omega - 1)^2$$

so not an ESS.

$$\begin{pmatrix} (\alpha, \alpha) & (1, \beta) \\ (\beta, 1) & (0, 0) \end{pmatrix}$$

(Assume $\alpha, \beta > 0$ and $\alpha \neq \beta$)

Solution

If $\beta < \alpha$ then we have a single pure Nash Equilibria ((1,0),(1,0)) which is also an ESS.

If $\beta > \alpha$ the Nash Equilbria are $\{((1,0),(1,0)),((0,1),(0,1)),((1/(\beta-\alpha+1),(\beta-\alpha)/(\beta-\alpha+1)),(1/(\beta-\alpha+1),(\beta-\alpha)/(\beta-\alpha+1)),(1/(\beta-\alpha+1),(\beta-\alpha)/(\beta-\alpha+1)$

$$u(\sigma^*, \sigma) = \frac{beta}{\beta - \alpha + 1}$$

and

$$u(\sigma, \sigma) = -\alpha\omega^2 - \omega(1 - \omega)(1 + \beta)$$

thus (after some algebra):

$$u(\sigma^*, \sigma) - u(\sigma, \sigma) > 0 \Leftrightarrow \beta - (\beta - \alpha + 1)(\omega^2(\alpha - \beta - 1) + \omega(1 + \beta)) > 0$$

However the quadratic in ω has roots: $\left\{\frac{\beta}{\beta-\alpha+1}, \frac{1}{\beta-\alpha+1}\right\}$ so \exists ω for which the difference is negative and thus this mixed strategy is not an ESS. Identify all evolutionary stable strategies.

2. Consider the following game:

In a mathematics department, researchers can choose to use one of two systems for type setting their research papers: LaTeX or Word. We will refer to these two strategies as L and W respectively. A user of W receives a basic utility of 1 and as L is more widely used by mathematicians out of the department and is in general considered to be a better system a user of L gets a basic utility of $\alpha>1$. Members of the mathematics department of ten collaborate and as such it is beneficial for the researchers to use the same type setting system. If we let μ represent the proportion of users of L we let:

$$u(L,\chi) = \alpha + 2\mu$$
$$u(W,\chi) = 1 + 2(1 - \mu)$$

What are the evolutionary stable strategies?

Solutions

Using the theorem for necessity of stability we have the following candidate ESS:

- 1. σ_L : everyone uses L, thus $\mu = 1$ (we have $u(L, \chi) > u(W, \chi)$).
- 2. σ_W : everyone uses W, thus $\mu = 0$ (we have $u(L, \chi) < u(W, \chi)$).
- 3. σ_m : some use L and some use W, by the theorem we have $u(L,\chi) = u(W,\chi)$ which implies $\alpha + 2\mu = 1 + 2(1-\mu)$ giving $\mu = \frac{3-\alpha}{4}$.

Now we consider the post entry population $\chi_{\epsilon} = (1-\epsilon)\sigma^* + \epsilon\sigma$ (where σ^* is the base strategy and σ is the entry population). We denote $\sigma = (\mu, 1-\mu)$ and $\sigma^* = (\mu^*, 1-\mu^*)$ and $\delta = u(\sigma^*, \chi_{\epsilon}) - u(\sigma, \chi_{\epsilon})$. We have:

$$\delta = \mu^* u(L, \chi_{\epsilon}) + (1 - \mu^*) u(W, \chi_{\epsilon}) - \mu u(L, \chi_{\epsilon}) + (1 - \mu) u(W, \chi_{\epsilon}) = (\mu^* - \mu) (u(L, \chi_{\epsilon}) - u(W, \chi_{\epsilon}))$$

which gives:

$$\delta = (\mu^* - \mu)(\alpha - 3 + 4((1 - \epsilon)\mu^* + \epsilon\mu)) = (\mu^* - \mu)(4\mu^* + \alpha - 3 + 4\epsilon(\mu - \mu^*))$$

We now consider each potential ESS in turn, if $\delta > 0$ for all $\epsilon < \bar{\epsilon}$ for some $\bar{\epsilon}$ then we have an ESS (this is by definition):

- 1. $\mu^* = 1$: $\delta = (1 \mu)(1 + \alpha + 4\epsilon(\mu 1)) > (1 \mu)(1 + \alpha 4\epsilon) > 0$ for all $\mu \neq 1$ and $\epsilon < \bar{\epsilon} = \frac{1+\alpha}{4}$. Thus σ_L is an ESS.
- 2. $\mu^* = 0$: $\delta = -\mu(\alpha 3 + 4\epsilon\mu)$. If $\alpha \geq 3$ then $\delta \leq 0$ for all values of μ, ϵ , thus if L is 3 times better than W σ_W is not an ESS. If $\alpha < 3$ $\delta > 0 \Leftrightarrow \alpha 3 + 4\epsilon\mu < 0 \Rightarrow \alpha 3 + 4\epsilon\mu < \alpha 3 + 4\epsilon < 0$ for all $\mu \neq 0$ $\epsilon < \bar{\epsilon} = \frac{3-\alpha}{4}$. Thus σ_W is an ESS for $\alpha < 3$.
- 3. $\mu^* = \frac{3-\alpha}{4}$: $\delta = -4\epsilon \left(\frac{3-\alpha}{4} \mu\right)^2 < 0$ for all $\mu \neq \frac{3-\alpha}{4}$ and for all $\epsilon > 0$ so σ_m is not an ESS.
- 3. Consider the following two normal form games:

$$A = \begin{pmatrix} (3,0) & (-1,-1) & (1,2) \\ (1,0) & (-1,1) & (2,0) \end{pmatrix}$$

$$B = \begin{pmatrix} (2,2) & (1,1) & (1,3) \\ (1,3) & (-2,-3) & (4,2) \end{pmatrix}$$

Assume both players play either game A or game B with probability 1/2, neither player knows which game is played. Obtain the Nash equilibrium for this game.

Solution

The described game is akin to the following game:

$$\begin{pmatrix} (5/2,1) & (0,0) & (1,5/2) \\ (1,3/2) & (-3/2,-1) & (3/2,1) \end{pmatrix}$$

We see that s_2 is dominated and sole the game using the equality of payoffs theorem to give the following Nash equilibrium:

- 4. Repeat the analysis of the principal agent game assuming that p is the probability of the project being successful in case of a high level of effort by the employee.
 - i. What are the expected utilities to the employer and the employee?

Solution

Repeating the analysis, we see that the employee will carry out a high effort iff:

$$p(\omega + B - 1)^{\alpha} + (1 - p)(\omega - 1)^{\alpha} \ge \omega^{\alpha}$$

Following the same argument as in the notes we arrive at:

$$p(\omega + B - 1)^{\alpha} + (1 - p)(\omega - 1)^{\alpha} = 1 = \omega$$

thus:

$$\beta = (1/p)^{(1/\alpha)}$$

The utilities are then:

Employer:

$$p(K-1-(1/p)^{(1/\alpha)})+(1-p)(\kappa-1)$$

Employee:

1

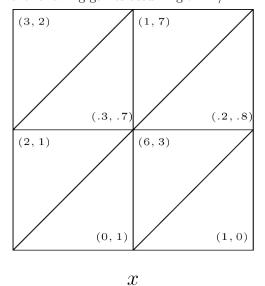
ii. Obtain a condition for which the employer should offer a bonus.

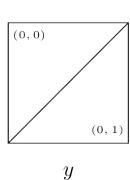
Solution

If no bonus is offered the employee has no incentive for a high effort thus $\kappa = \omega = 1$, thus the employer should offer a bonus iff:

$$p(K-1-(1/p)^{(1/\alpha)})+(1-p)(\kappa-1)\geq 0$$

5. Obtain the Markov Nash equilibrium (in pure strategies if it exists) for the following games assuming $\delta = 1/4$.





Solution

State y gives no value to either player so we only need to consider state x. Let the future gains to player 1 in state x be u and the future gains to player 2 in state x be v. Thus the players are facing the following game:

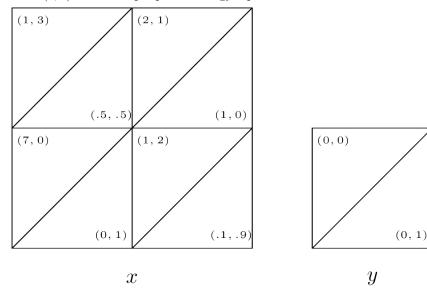
$$\begin{pmatrix} (3+3u/40,2+3v/40) & (1+u/20,7+v/20) \\ (2,1) & (6+u/4,3+v/4) \end{pmatrix}$$

There are four possible equilibria:

- 1. (a,b) which requires: $3+3u/40 \ge 2$ and $2+3v/40 \ge 7+v/20$ $\Rightarrow u \ge -40/3$ and $v \ge 200$. However if this is the equilibria then u=120/37 and v=80/37 which contradicts the constraints.
- 2. (a,d) which requires: $1+u/20 \ge 6+u/4$ and $2+3v/40 \le 7+v/20$ $\Rightarrow u \le -25$ and $v \le 200$. However if this is the equilibria then u=20/19 and v=140/19 which contradicts the constraints.
- 3. (b,c) which requires: $2 \ge 3 + 3u/40$ and $1 \ge 3 + v/4 \Rightarrow u \le -40/3$ and $v \le -8$. However if this is the equilibria then u=2 and v=1 which contradicts the constraints.

4. (b,d) which requires: $2 \le 3 + 3u/40$ and $1 \le 3 + v/4 \Rightarrow u \ge -40/3$ and $v \ge -8$. However if this is the equilibria then u = 8 and v = 4 which **does not** contradict the constraints.

Thus (b,d) is the unique pure strategy equilibrium.



Solution

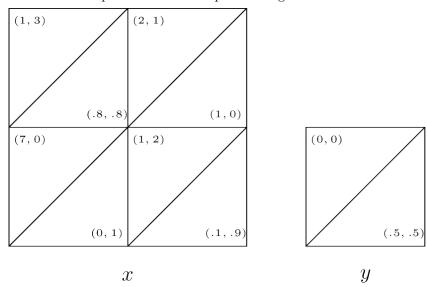
State y gives no value to either player so we only need to consider state x. Let the future gains to player 1 in state x be u and the future gains to player 2 in state x be v. Thus the players are facing the following game:

$$\begin{pmatrix} (1+u/8, 3+v/8) & (2+u/4, 1+v/4) \\ (7,0) & (1+u/40, 2+9v/40) \end{pmatrix}$$

There are four possible equilibria:

- 1. (a,b) which requires: $1+u/8 \ge 7$ and $3+v/8 \ge 1+v/4 \Rightarrow u \ge 48$ and $v \le 16$. However if this is the equilibria then u=8/7 and v=24/7 which contradicts the constraints.
- 2. (a,b) which requires: $2+u/4 \ge 1+u/40$ and $3+v/8 \le 1+v/4 \Rightarrow u \ge -40/9$ and $v \ge 16$. However if this is the equilibria then u=8/3 and v=4/3 which contradicts the constraints.
- 3. (b,c) which requires: $7 \ge 1 + u/8$ and $0 \ge 2 + 9v/40 \Rightarrow u \le 48$ and $v \le -80/9$. However if this is the equilibria then u = 7 and v = 0 which contradicts the constraints.
- 4. (a,b) which requires: $2 + u/4 \le 1 + u/40$ and $2 + 9v/40 \ge 0 \Rightarrow u \le -40/9$ and $v \ge -80/9$. However if this is the equilibria then u = 40/39 and v = 80/31 which contradicts the constraints.

Thus no Nash equilibrium exists in pure strategies.



Solution

State y gives no value to either player so we only need to consider state x. Let the future gains to player 1 in state x be u and the future gains to player 2 in state x be v. Thus the players are facing the following game:

$$\begin{pmatrix} (1+u/20, 3+v/20) & (2+7u/40, 1+7v/40) \\ (7,0) & (1+u/40, 2+9v/40) \end{pmatrix}$$

There are four possible equilibria:

- 1. (a,b) which requires: $1+u/20 \ge 7$ and $3+v/20 \ge 1+7v/40 \Rightarrow u \ge 120$ and $v \le 16$. However if this is the equilibria then u=20/19 and v=60/19 which contradicts the constraints.
- 2. (a, b) which requires: $2 + 7u/40 \ge 1 + u/40$ and $1 + 7v/40 \ge 3 + v/20$ $\Rightarrow u \ge -20/3$ and $v \le 16$. However if this is the equilibria then u = 80/33 and v = 40/33 which does not contradict any constraints.
- 3. (b,c) which requires: $7 \ge 1 + u/20$ and $0 \ge 2 + 9v/40 \Rightarrow u \le 120$ and $v \le -80/9$. However if this is the equilibria then u = 7 and v = 0 which contradicts the constraints.
- 4. (a,b) which requires: $2 + u/4 \le 1 + u/40$ and $2 + 9v/40 \ge 0 \Rightarrow u \le -40/9$ and $v \ge -80/9$. However if this is the equilibria then u = 40/39 and v = 80/31 which contradicts the constraints.

Thus (a, b) is the unique pure strategy equilibrium.