OR 3: Chapter 15 - Matching games

Recap

In the previous chapter:

- We defined matching games;
- We described the Gale-Shapley algorithm;
- We proved certain results regarding the Gale-Shapley algorithm.

In this Chapter we'll take a look at another type of game.

Cooperative Games

In cooperative game theory the interest lies with understanding how coalitions form in competitive situations.

Definition

A **characteristic function game** G is given by a pair (n, v) where n is the number of players and $v : 2^{[n]} \to \mathbb{R}$ is a **characteristic function** which maps every coalition of players to a payoff.

Let's consider the following game:

> "3 players must share a taxi. Here are the costs for each individual journey: - Player 1: 6 - Player 2: 12 - Player 3: 42"

This is illustrated below:

To construct the characteristic function we first obtain the power set (ie all possible coalitions) $2^{\{1,2,3\}} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \Omega\}$ where Ω denotes the set of all players ($\{1,2,3\}$).

The characteristic function is given below:

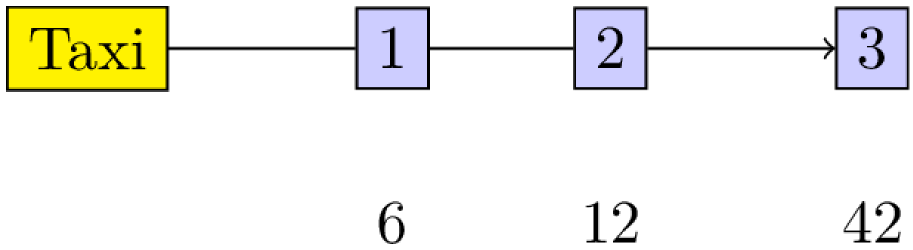

Figure 1:

$$v(S) = \begin{cases} 6, & \text{if } S = \{1\} \\ 12, & \text{if } S = \{2\} \\ 42, & \text{if } S = \{3\} \\ 12, & \text{if } S = \{1,2\} \\ 42, & \text{if } S = \{1,3\} \\ 42, & \text{if } S = \{2,3\} \\ 42, & \text{if } S = \{1,2,3\} \end{cases}$$

Definition

A characteristic function game $G = (n, v)$ is called **monotone** is it satisfies $v(S_2) \geq v(S_1)$ for all $S_1 \subseteq S_2$.

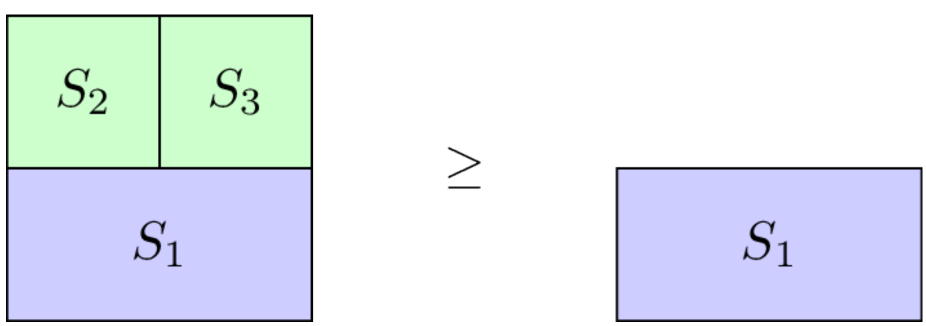

Figure 2:

Our taxi example is monotone, however the $G = (3, v_1)$ with v_1 defined as:

$$v_1(S) = \begin{cases} 6, & \text{if } S = \{1\} \\ 12, & \text{if } S = \{2\} \\ 42, & \text{if } S = \{3\} \\ 10, & \text{if } S = \{1,2\} \\ 42, & \text{if } S = \{1,3\} \\ 42, & \text{if } S = \{2,3\} \\ 42, & \text{if } S = \{1,2,3\} \end{cases}$$

is not.

Definition

A characteristic function game $G = (n, v)$ is called **superadditive** if it satisfies $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$.

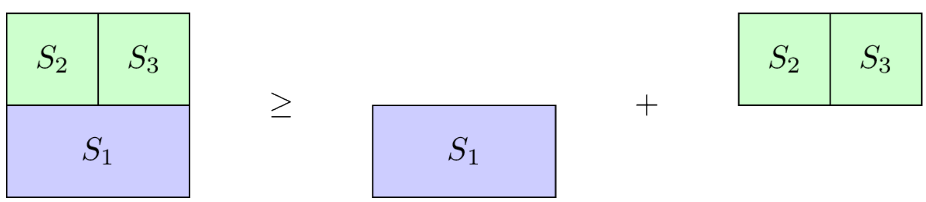

Figure 3:

Our taxi example is not superadditive, however the $G = (3, v_2)$ with v_2 defined as:

$$v_2(S) = \begin{cases} 6, & \text{if } S = \{1\} \\ 12, & \text{if } S = \{2\} \\ 42, & \text{if } S = \{3\} \\ 18, & \text{if } S = \{1,2\} \\ 48, & \text{if } S = \{1,3\} \\ 55, & \text{if } S = \{2,3\} \\ 80, & \text{if } S = \{1,2,3\} \end{cases}$$

is.

Shapley Value

When talking about a solution to a characteristic function game we imply a payoff vector $x \in \mathbb{R}^n_{\geq 0}$ that divides the value of the grand coalition between the various players. Thus x must satisfy:

$$\sum_{i=1}^{n} x_i = v([n])$$

Thus one potential solution to our taxi example would be $x = (14, 14, 14)$. Obviously this is not ideal for player 1 and/or 2: they actually pay more than they would have paid without sharing the taxi!

Another potential solution would be $x = (6, 6, 30)$, however at this point sharing the taxi is of no benefit to player 1. Similarly $(0, 12, 30)$ would have no incentive for player 2.

To find a "fair" distribution of the grand coalition we must define what is meant by "fair". We require four desirable properties:

- Efficiency;
- Null player;
- Symmetry;
- Additivity.

Definition

For $G = (n, v)$ a payoff vector x is **efficient** if:

$$\sum_{i=1}^{n} x_i = v([n])$$

Definition

For $G(n, v)$ a payoff vector possesses the **null player property** if $v(S \cup i) = v(S)$ for all $S \in 2^{[n]}$ then:

$$x_i = 0$$

Definition

For $G(n, v)$ a payoff vector possesses the **symmetry property* if $v(S \cup i) = v(S \cup j)$ for all $S \in 2^{[n]} \setminus \{i, j\}$ then:

$$x_i = x_j$$

Definition

For $G_1 = (n, v_1)$ and $G_2 = (n, v_2)$ and $G^+ = (n, v^+)$ where $v^+(S) = v_1(S) + v_2(S)$ for any $S \in 2^{[n]}$. A payoff vector possesses the **additivity property** if:

$$x_i^{(G^+)} = x_i^{(G_1)} + x_i^{(G_2)}$$

We will not prove the following in this course but in fact there is a single payoff vector that satisfies these four properties. To define it we need two last definitions.

Definition

If we consider any permutation π of $[n]$ then we denote by $S_\pi(i)$ the set of predecessors of i in π:

$$S_\pi(i) = \{j \in [n] \mid \pi(j) < \pi(i)\}$$

For example for $\pi = (1, 3, 4, 2)$ we have $S_\pi(4) = \{1, 3\}$.

Definition

If we consider any permutation π of $[n]$ then the marginal contribution of player i with respect to π is given by:

$$\Delta_\pi^G(i) = v(S_\pi(i) \cup i) - v(S_\pi(i))$$

We can now define the **Shapley value** of any game $G = (n, v)$.

Definition

Given $G = (n, v)$ the **Shapley value** of player i is denoted by $\phi_i(G)$ and given by:

$$\phi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi_n} \Delta_\pi^G(i)$$

As an example here is the Shapley value calculation for our taxi sharing game:

For $\pi = (1, 2, 3)$:

$$\begin{aligned}\Delta_\pi^G(1) &= 6\\ \Delta_\pi^G(2) &= 6\\ \Delta_\pi^G(3) &= 30\end{aligned}$$

For $\pi = (1, 3, 2)$:

$$\begin{aligned}\Delta_\pi^G(1) &= 6\\ \Delta_\pi^G(2) &= 0\\ \Delta_\pi^G(3) &= 36\end{aligned}$$

For $\pi = (2, 1, 3)$:

$$\Delta_\pi^G(1) = 0$$
$$\Delta_\pi^G(2) = 12$$
$$\Delta_\pi^G(3) = 30$$

For $\pi = (2, 3, 1)$:

$$\Delta_\pi^G(1) = 0$$
$$\Delta_\pi^G(2) = 12$$
$$\Delta_\pi^G(3) = 30$$

For $\pi = (3, 1, 2)$:

$$\Delta_\pi^G(1) = 0$$
$$\Delta_\pi^G(2) = 0$$
$$\Delta_\pi^G(3) = 42$$

For $\pi = (3, 2, 1)$:

$$\Delta_\pi^G(1) = 0$$
$$\Delta_\pi^G(2) = 12$$
$$\Delta_\pi^G(3) = 42$$

Using this we obtain:

$$\phi(G) = (2, 5, 35)$$

Thus the fair way of sharing the taxi fare is for player 1 to pay 1, player 2 to pay 5 and player 3 to pay 35.