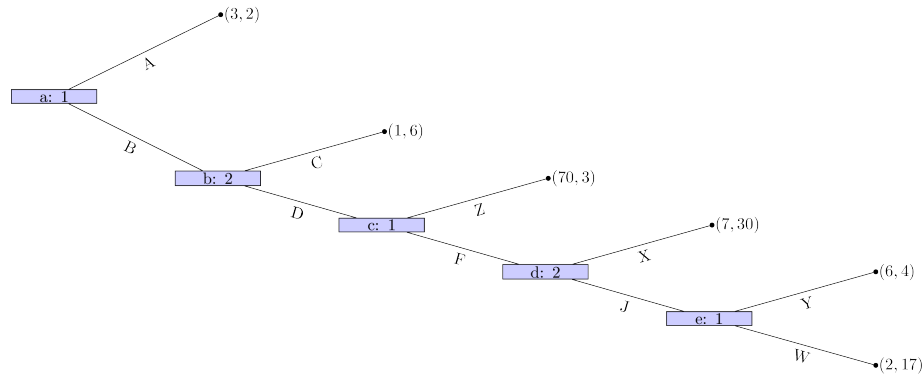


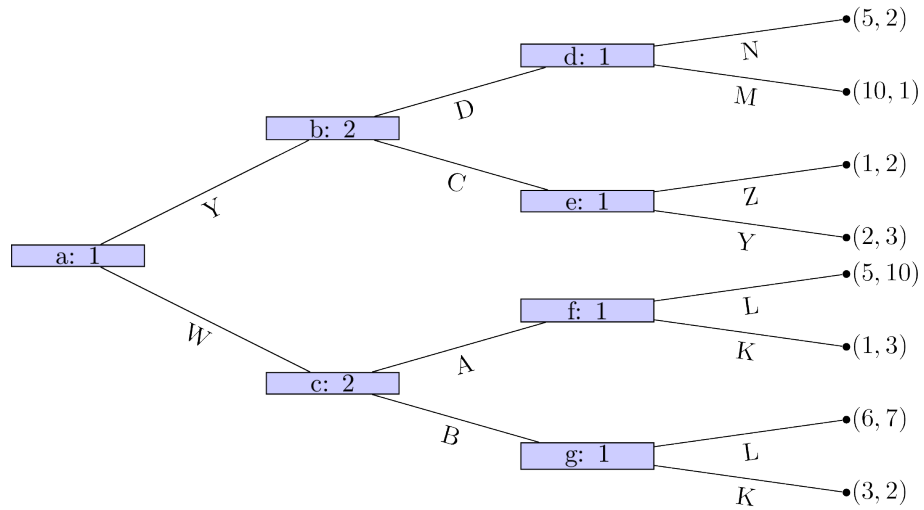
1 Homework sheet 3 - Extensive form games, subgame perfect equilibrium and repeated games

1. Obtain the Nash equilibrium for the following games using backward induction:



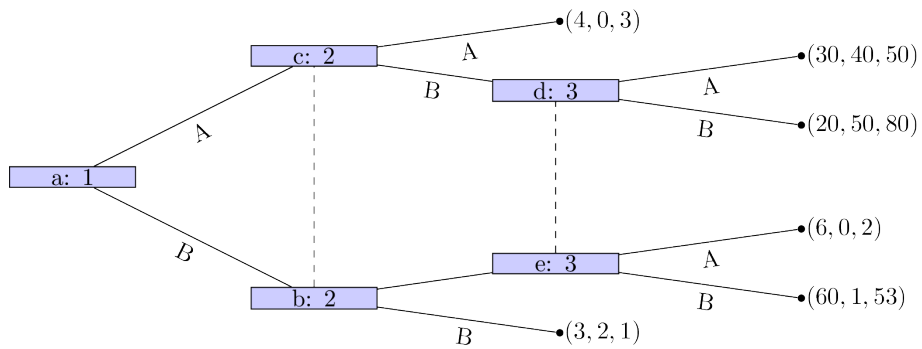
Solution

(AZY, CX)



Solution

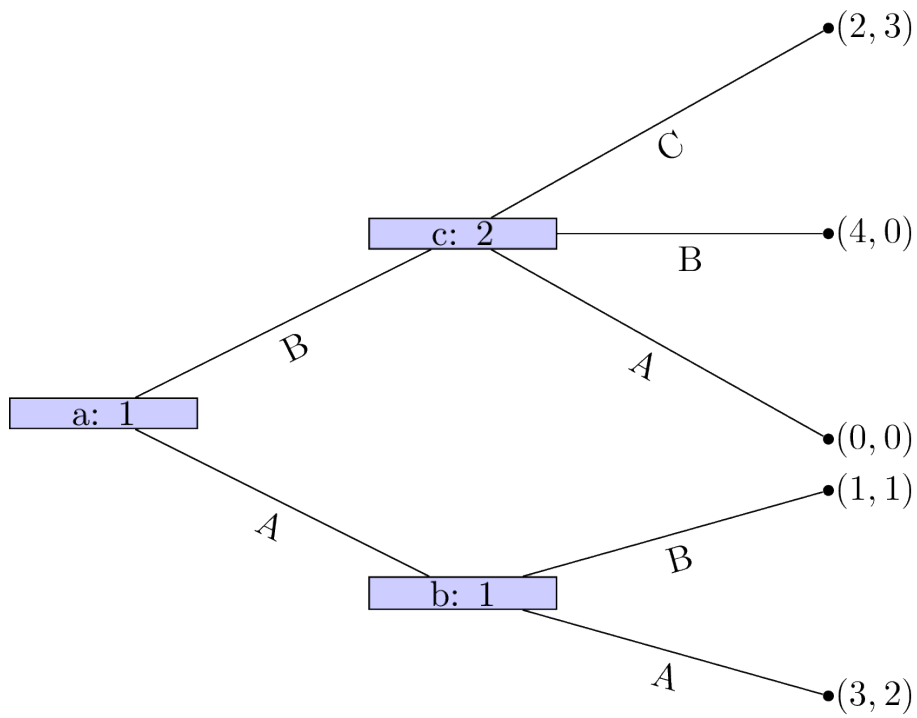
$(WNYLL, AC)$



For P_3 , B dominates A , for P_2 , B dominates A , finally A dominates B for P_1 :

Solution

(A, BB, BB)



Solution

(AA, C)

2. Obtain the Nash equilibrium for the following game:

Player 1 chooses a number $x \geq 0$, which player 2 observes. After this simultaneously and independatly player 1 and player 2 choose $y_1, y_2 \in \mathbb{R}$ respectively. The utility to player 1 is given by $2y_2y_1 + xy_1 - y_1^2 - x^3/3$ and the utility to player 2 is given by $-(y_1 - 2y_2)^2$.

Solution

For given x : y_1 maximises $2y_2y_1 + xy_1 - y_1^2 - x^3/3$, thus y_1^* is a solution to:

$$2y_2 + x - 2y_1 = 0$$

(As the function has a local maxima which is global)

so:

$$y_1^* = \frac{2y_2 + x}{2}$$

Similarly:

$$y_2^* = \frac{y_1}{2}$$

Thus $(\tilde{y}_1, \tilde{y}_2)$ solve the following system of equations:

$$\begin{cases} \tilde{y}_1 = \frac{\tilde{y}_1 + x}{2} \\ \tilde{y}_2 = \frac{\tilde{y}_1}{2} \end{cases}$$

Thus:

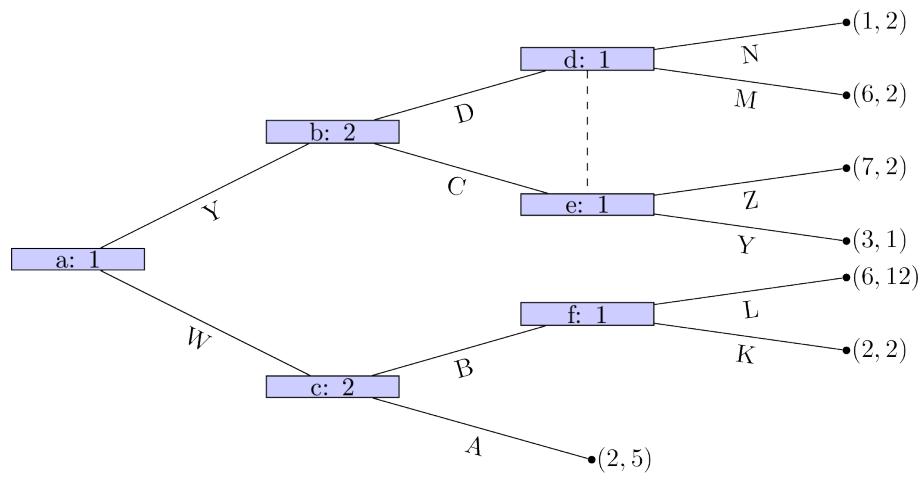
$$\begin{cases} \tilde{y}_1 = x \\ \tilde{y}_2 = \frac{x}{2} \end{cases}$$

This gives $u_1 = x^2 + x^2 - x^2 - x^3/3 = x^2(1 - x/3)$ which has a maxima at $x = 2$ for $x \geq 0$. This gives:

$$(\tilde{x}, \tilde{y}_1, \tilde{y}_2) = (2, 2, 1)$$

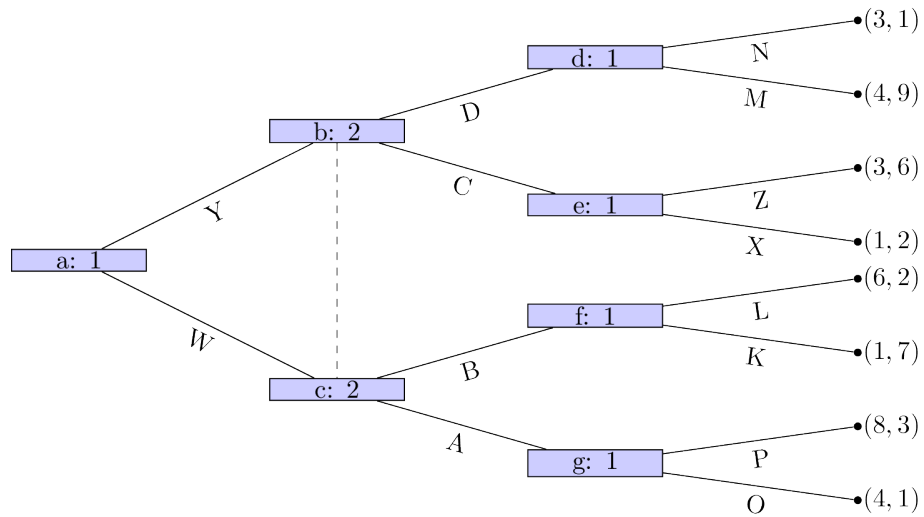
3. For each of the following games:

- i. Identify all subgames.
- ii. Identify the corresponding normal form representations and hence obtain all Nash equilibrium.
- iii. Identify which Nash equilibrium are also subgame perfect Nash equilibrium.



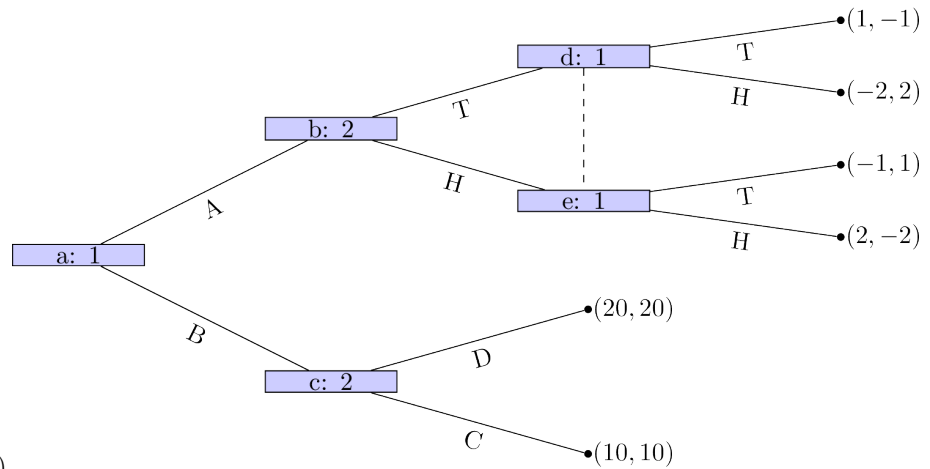
Solution

Not a valid game (node d and e are in same information set but have different action sets).



Solution

Not a valid game (node b and c are in same information set but have different ac-



tion sets)

There are two subgames:

- Generated by node c (trivial)
- Generated by node b

The strategy sets are:

$$S_1 = \{AT, AH, BT, BH\}$$

$$S_2 = \{TD, TC, HD, HC\}$$

Here is the corresponding normal form representation:

$$\begin{pmatrix} (-1, 1) & (-1, 1) & (1, -1) & (1, -1) \\ (2, -2) & (2, -2) & (2, -2) & (2, -2) \\ (20, 20) & (10, 10) & (20, 20) & (10, 10) \\ (20, 20) & (10, 10) & (20, 20) & (10, 10) \end{pmatrix}$$

By examining best responses we identify the following 4 pure Nash equilibria:

$$\{(BT, TD), (BT, TD), (BH, TD), (BH, HD)\}$$

Looking at the subgame initiated at b (with $S_1 = \{H, T\}$ and $S_2 = \{H, T\}$):

$$\begin{pmatrix} (2, -2) & (-2, 2) \\ (-1, 1) & (1, -1) \end{pmatrix}$$

None of the above strategy pairs are Nash equilibria!

However (using the Equality of Payoffs theorem) we see that:

$$\sigma_1 = (0, 0, 2/3, 1/3)$$

and

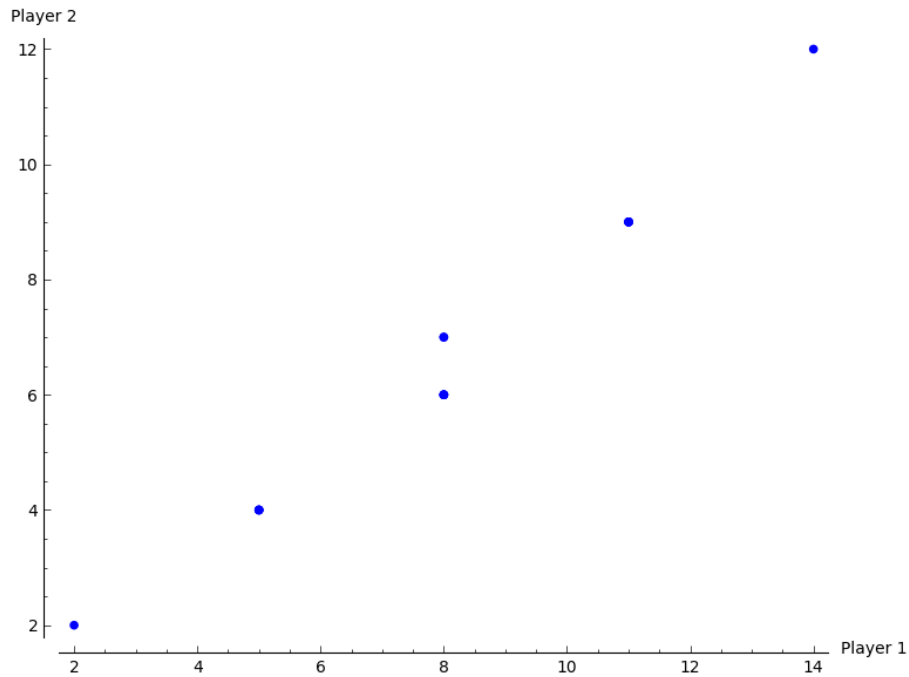
$$\sigma_2 = (1/2, 0, 1/2, 0)$$

is a NE, and in fact is also a NE for the entire game (again using the Equality of Payoffs theorem). 5. For the following stage games:

- i. Plot all possible utility pairs for $T = 2$;
- ii. Recalling that subgame perfect equilibrium for the repeated game must play a stage Nash equilibrium in the final stage attempt to identify a Nash equilibrium for the repeated game that is not a sequence of stage Nash profiles.

$$\begin{pmatrix} (4, 3) & (7, 6) \\ (1, 1) & (4, 3) \end{pmatrix}$$

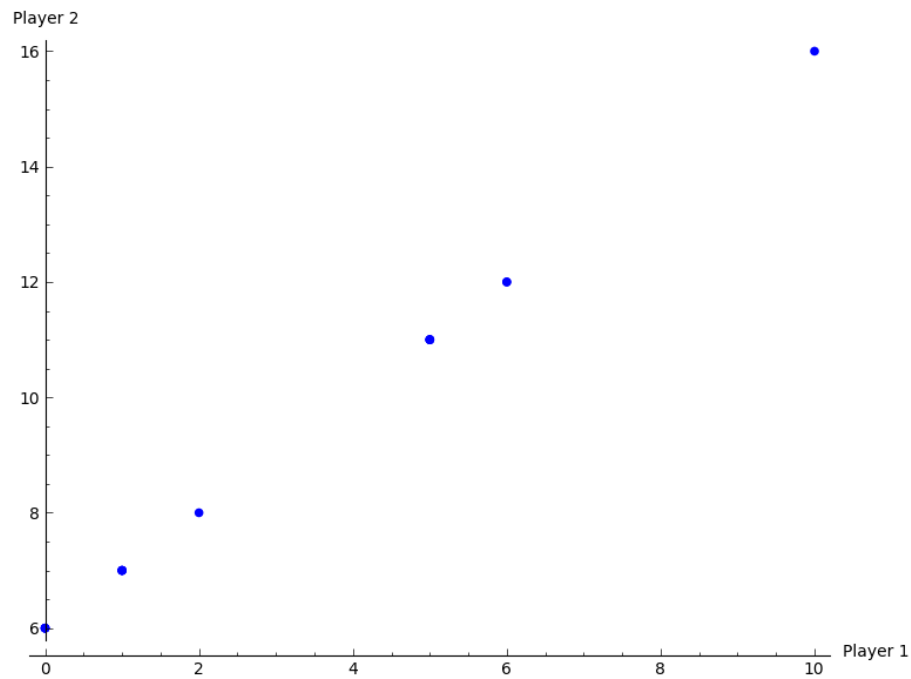
Solution



No other strategy is subgame perfect.

$$\begin{pmatrix} (5, 8) & (0, 3) \\ (0, 3) & (1, 4) \\ (1, 6) & (0, 3) \end{pmatrix}$$

Solution



“Play (r_3, s_1) in first round and (r_2, s_1) in second round unless P1 deviates in which case play (r_2, s_2) .”

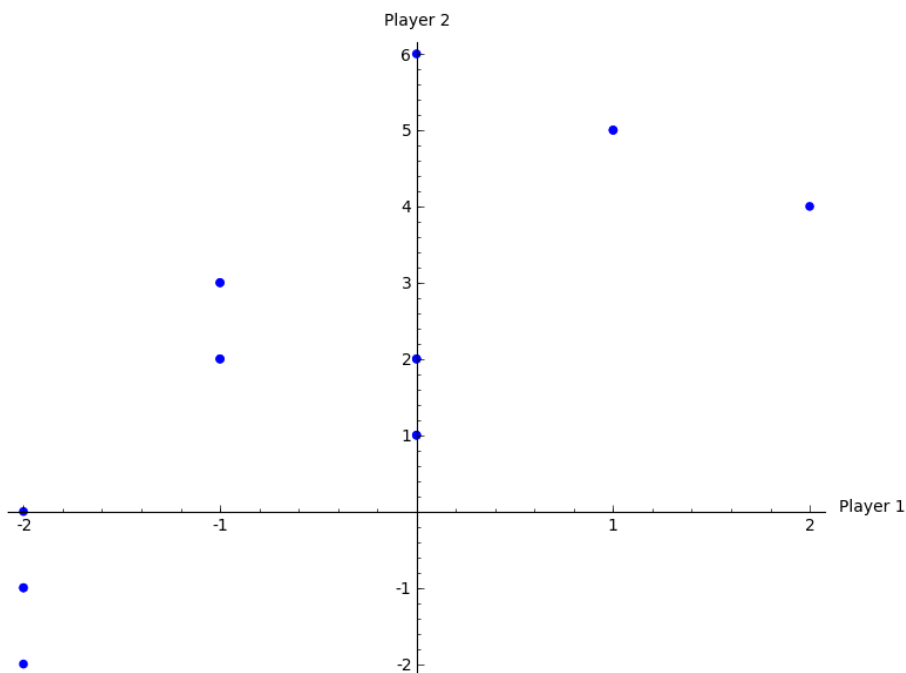
Outcome: $(8, 10)$.

Deviate?

- P2: No incentive;
- P1: Gain 2 in 1st round but lose 4 in second round.

$$\begin{pmatrix} (5, 2) & (2, 0) & (6, 3) \\ (5, 2) & (1, 3) & (7, 1) \end{pmatrix}$$

Solution



“Play (r_1, s_1) in first round and (r_1, s_2) in second round unless P2 deviates in which case play (r_2, s_3) .”

Outcome: $(1, 5)$.

Deviate?

- P1: No incentive;
- P2: Gain 1 in 1st round but lose 2 in second round.

6. Consider the following stage game:

$$\begin{pmatrix} (-1, 1) & (3, -7) \\ (-2, 6) & (2, 2) \end{pmatrix}$$

- For $\delta = 1/3$ obtain the utilities for the infinitely repeated game for the strategies S_D : “play the first strategy throughout” and S_C : “play the second strategy throughout”.
- Plot the space of feasible average payoffs and the space of individually rational payoffs.
- State whether or not it is possible according to the Folk theorem to obtain δ that ensures that a strategy profile exists that would give a subgame perfect Nash equilibrium with average payoffs: $(3/2, 3/2)$, $(0, 3)$, $(2, 6)$ and $(2, 0)$.

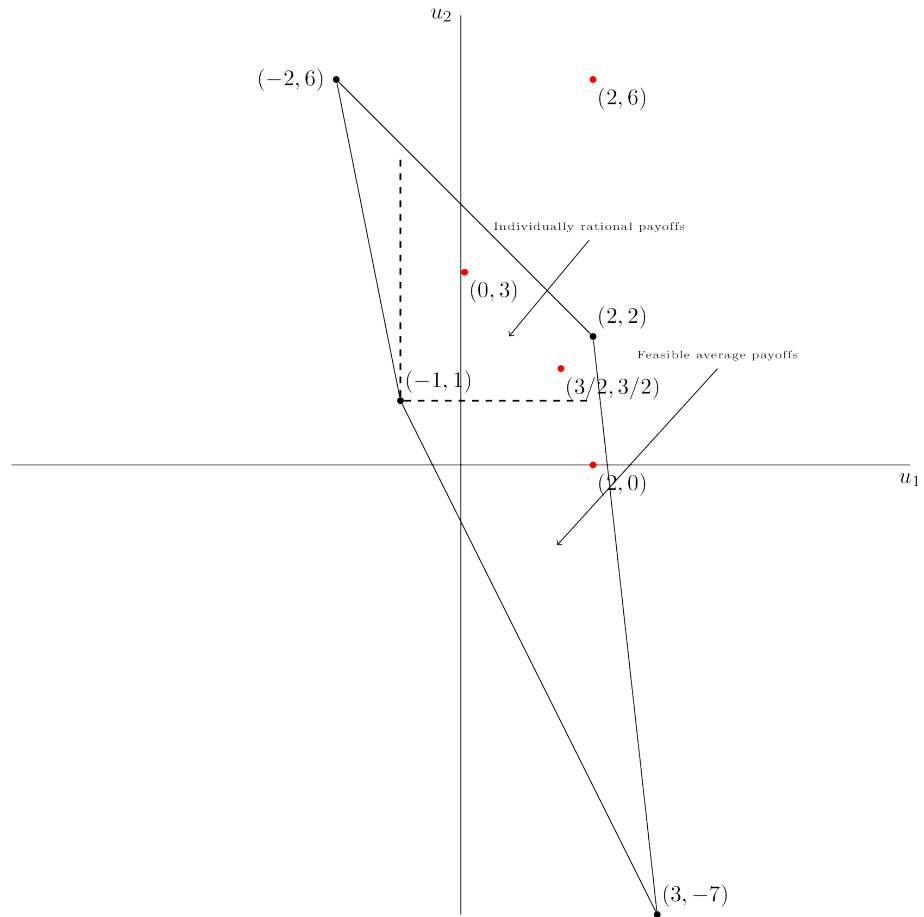
Solution

$$\begin{cases} u_1(S_D, S_D) &= \sum_{i=0}^{\infty} \delta^i (-1) = -\frac{1}{1-\delta} = -3/2 \\ u_2(S_D, S_D) &= 3/2 \end{cases}$$

$$\begin{cases} u_1(S_D, S_C) &= 9/2 \\ u_2(S_D, S_C) &= -21/3 \end{cases}$$

$$\begin{cases} u_1(S_C, S_D) &= -3 \\ u_2(S_C, S_D) &= 9 \end{cases}$$

$$\begin{cases} u_1(S_C, S_C) &= 3 \\ u_2(S_C, S_C) &= 3 \end{cases}$$



We see that it is possible to find a δ for $(3/2, 3/2)$ and $(0, 3)$.