OR 3: Lecture 6 - Nash equilibria in mixed strategies

Recap

In the previous lecture

- The definition of Nash equilibria;
- Identifying Nash equilibria in pure strategies;
- Solving the duopoly game;

This brings us to a very import part of the course. We will now consider equilibria in mixed strategies.

Recall of expected utility calculation

In the matching pennies game discussed previously:

$$\begin{pmatrix}(1,-1) & (-1,1)\\(-1,1) & (1,-1)\end{pmatrix}$$

A strategy profile of $\sigma_1 = (.2, .8)$ and $\sigma_2 = (.6, .4)$ implies that player 1 plays heads with probability .2 and player 2 plays heads with probability .6.

We can extend the utility function which maps from the set of pure strategies to $\mathbb{R}$ using *expected payoffs*. For a two player game we have:

$$u_i(\sigma_1, \sigma_2) = \sum_{r\in S_1, s\in S_2} \sigma_1(r)\sigma_2(s)u_i(r, s)$$

Obtaining equilibria

Let us investigate the best response functions for the matching pennies game.

If we assume that player 2 plays a mixed strategy $\sigma_2 = (y, 1-y)$ we have:

$$u_1(r_1, \sigma_2) = 2y - 1$$

and

$$u_1(r_2, \sigma_2) = 1 - 2y$$

Thus we have:

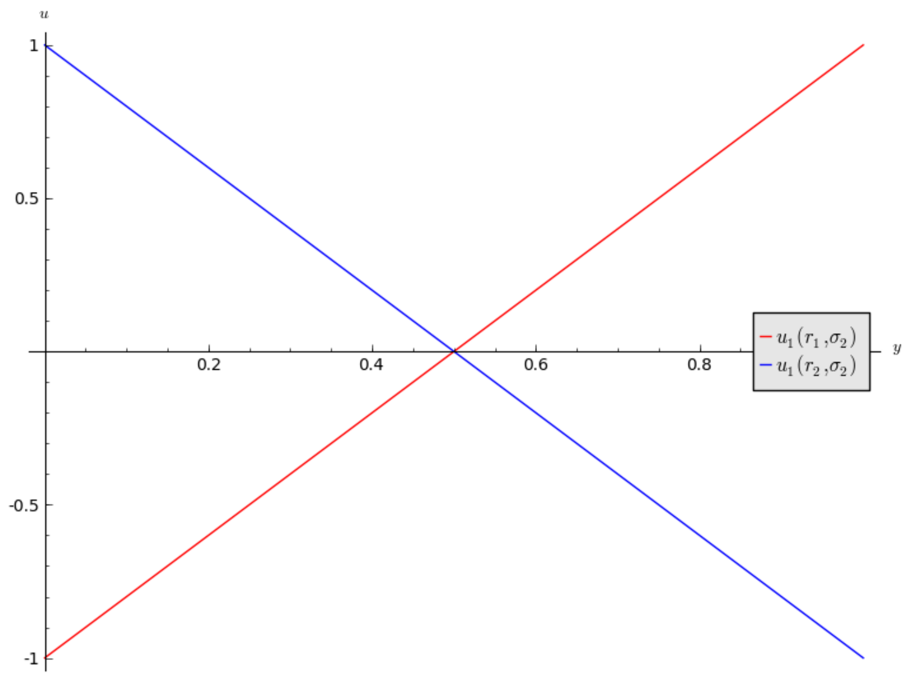


Figure 1:

1. If $y < 1/2$ then r_2 is a best response for player 1.
2. If $y > 1/2$ then r_1 is a best response for player 1.
3. If $y = 1/2$ then player 1 is indifferent.

If we assume that player 1 plays a mixed strategy $\sigma_1 = (x, 1-x)$ we have:

$$u_2(\sigma_1, s_1) = 1 - 2x$$

and

$$u_2(\sigma_1, s_2) = 2x - 1$$

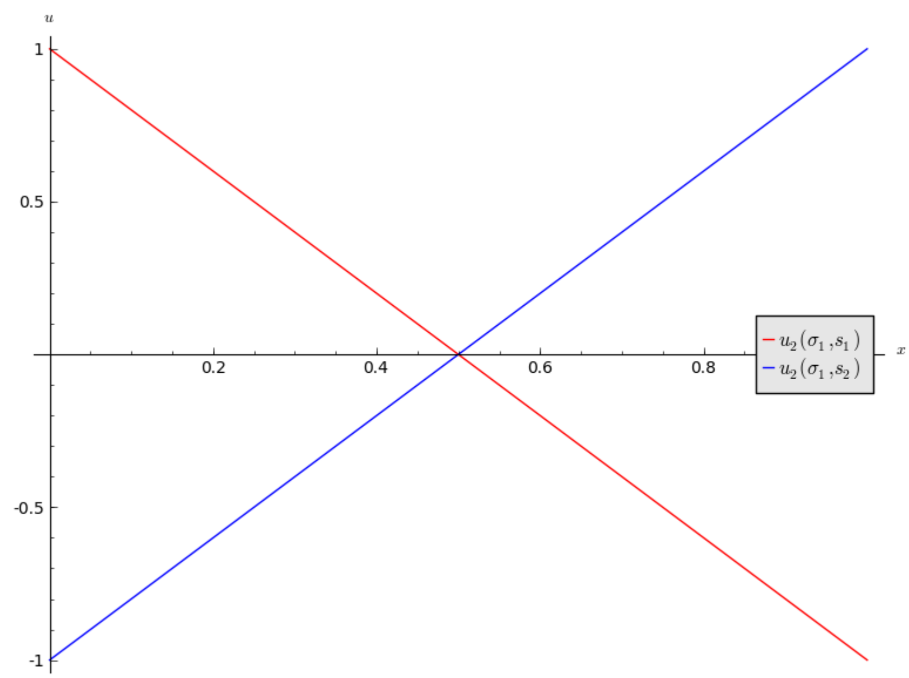
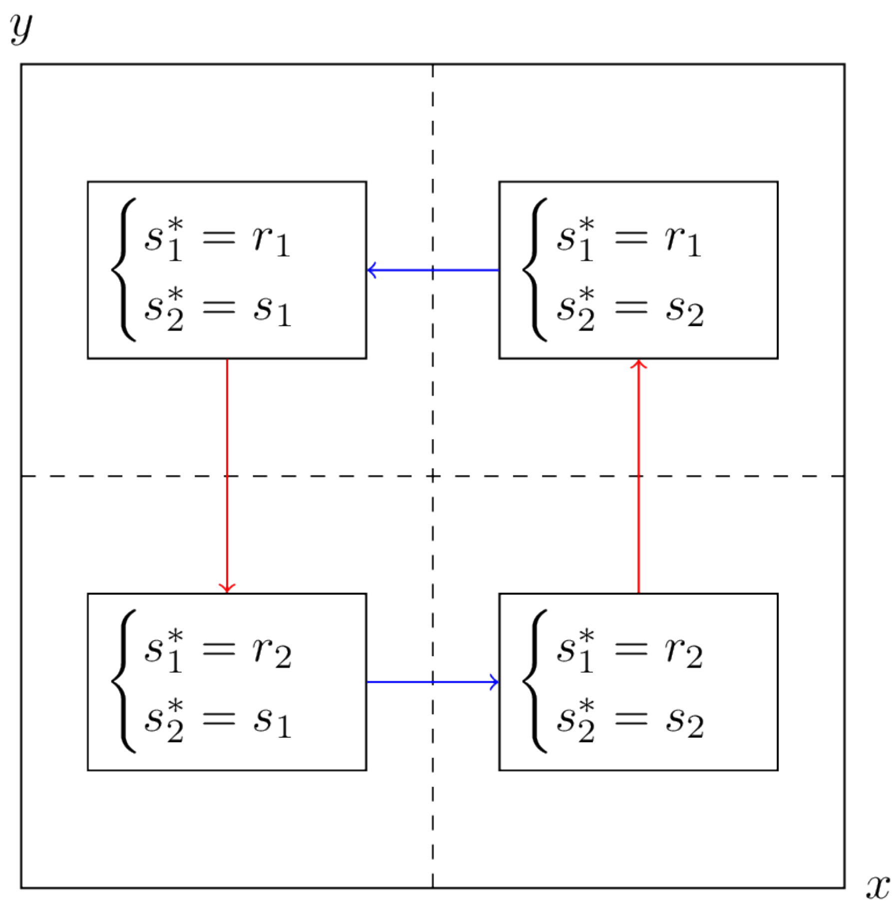


Figure 2:

Thus we have:

1. If $x < 1/2$ then s_1 is a best response for player 2.
2. If $x > 1/2$ then s_2 is a best response for player 2.
3. If $x = 1/2$ then player 2 is indifferent.



y
$\begin{cases} s_1^* = r_1 \\ s_2^* = s_1 \end{cases}$
$\begin{cases} s_1^* = r_1 \\ s_2^* = s_2 \end{cases}$
$\begin{cases} s_1^* = r_2 \\ s_2^* = s_1 \end{cases}$
$\begin{cases} s_1^* = r_2 \\ s_2^* = s_2 \end{cases}$
x

Figure 3:

Let us plot both best responses on a single plot, indicating the best responses in each quadrant. The arrows show the deviation indicated by the best responses.

If either player plays a mixed strategy other than $(1/2, 1/2)$ then the other player has an incentive to modify their strategy. Thus the Nash equilibria is:

$$((1/2, 1/2), (1/2, 1/2))$$

This notion of "indifference" is important and we will now prove an important theorem that will prove useful when calculating Nash Equilibria.

Equality of payoffs theorem

Definition

In an n player normal form game the **support** of a strategy $\sigma \in \Delta S_i$ is defined as:

$$\mathcal{S}(\sigma) = \{s \in S_i \mid \sigma(s) > 0\}$$

I.e. the support of a strategy is the set of pure strategies that are played with non zero probability.

For example, if the strategy set is $\{A, B, C\}$ and $\sigma = (1/3, 2/3, 0)$ then $\mathcal{S}(\sigma) = \{A, B\}$.

Theorem

In an n player normal form game if the strategy profile (σ_i, s_{-i}) is a Nash equilibria then:

$$u_i(\sigma_i, s_{-i}) = u_i(s, s_{-i}) \text{ for all } s \in \mathcal{S}(\sigma_i) \text{ for all } 1 \leq i \leq n$$

Proof

If $|\mathcal{S}(\sigma_i)| = 1$ then the proof is trivial.

We assume that $|\mathcal{S}(\sigma_i)| > 1$. Let us assume that the theorem is not true so that there exists $\bar{s} \in \mathcal{S}(\sigma)$ such that

$$u_i(\sigma_i, s_{-i}) \neq u_i(\bar{s}, s_{-i})$$

In particular as $u_i(\sigma_i, s_{-i}) = \sum_{s \in \mathcal{S}(\sigma_i)} \sigma_i(s) u_i(s, s_{-i})$ we can assume without loss of generality that

$$u_i(\sigma_i, s_{-i}) < u_i(\bar{s}, s_{-i})$$

which implies that (σ_i, s_{-i}) is not a Nash equilibrium.
