

## OR 3: Chapter 12 - Nash equilibrium and Evolutionary stable strategies

### Recap

In the [previous chapter](#):

- We considered population games;
- We proved a result concerning a necessary condition for a population to be evolutionary stable;
- We defined Evolutionary stable strategies and looked at an example in a game against the field.

In this chapter we'll take a look at pairwise contest games and look at the connection between Nash equilibrium and ESS.

### Pairwise contest games

In a population game when considering a pairwise contest game we assume that individuals are randomly matched. The utilities then depend just on what the individuals do:

$$u(\sigma, x) = \sum_{s \in S} \sum_{s' \in S} \sigma(s)x(s')u(s, s')$$

As an example we're going to consider the "Hawk-Dove" game: a model of predator interaction. We have  $S = \{H, D\}$  where:

- $H$ : Hawk represents being "aggressive";
- $D$ : Dove represents not being "aggressive".

At various times individuals come in to contact and must choose to act like a Hawk or like Dove over the sharing of some resource of value  $v$ . We assume that:

- If a Dove and Hawk meet the Hawk takes the resources;
- If two Doves meet they share the resources;
- If two Hawks meet there is a fight over the resources (with an equal chance of winning) and the winner takes the resources while the loser pays a cost  $c > v$ .

If we assume that  $\sigma = (\omega, 1 - \omega)$  and  $x = (h, 1 - h)$  the above gives:

$$u(\sigma, x) = \omega(1 - h)v + (1 - \omega)(1 - h)\frac{v}{2} + \omega h\frac{v - c}{2}$$

It is immediate to note that no pure strategy ESS exists. In a population of Doves ( $h = 0$ ):

$$u(\sigma, (0, 1)) = \omega v + (1 - \omega)\frac{v}{2} = (1 + \omega)\frac{v}{2}$$

thus the best response is setting  $\omega = 1$  i.e. to play Hawk.

In a population of Hawks ( $h = 1$ ):

$$u(\sigma, (1, 0)) = \omega h\frac{v - c}{2}$$

thus the best response is setting  $\omega = 0$  i.e. to play Dove.

So we will now try and find out if there is a mixed-strategy ESS:  $\sigma^* = (\omega^*, 1 - \omega^*)$ . For  $\sigma^*$  to be an ESS it must be a best response to the population it generates  $x^* = (\omega^*, 1 - \omega^*)$ . In this population the  $x^*$  the payoff to an arbitrary strategy  $\sigma$  is:

$$u(\sigma, x^*) = (1 - \omega^*)\frac{v}{2} + \left(\frac{v}{2} - \omega^*\right)\frac{\omega c}{2}$$

- If  $\omega^* < v/c$  then a best response is  $\omega = 1$ ;
- If  $\omega^* > v/c$  then a best response is  $\omega = 0$ ;
- If  $\omega^* = v/c$  then there is indifference.

So the only candidate for an ESS is  $\sigma^* = (v/c, 1 - v/c)$ . We now need to show that  $u(\sigma^*, x_\epsilon) > u(\sigma, x_\epsilon)$ .

We have:

$$x_\epsilon = (v/c + \epsilon(\omega - v/c), 1 - v/c + \epsilon(\omega - v/c))$$

and:

$$\begin{aligned} u(\sigma^*, x_\epsilon) &= \frac{v}{c} \left( \frac{v}{c} + \epsilon \left( \omega - \frac{v}{c} \right) \right) \frac{v - c}{2} + \frac{v}{c} \left( 1 - \frac{v}{c} + \epsilon \left( \frac{v}{c} - \omega \right) \right) v \\ &\quad + \left( 1 - \frac{v}{c} \right) \left( 1 - \frac{v}{c} + \epsilon \left( \frac{v}{c} - \omega \right) \right) \frac{v}{2} \end{aligned}$$

$$u(\sigma, x_\epsilon) = \omega \left( \frac{v}{c} + \epsilon \left( \omega - \frac{v}{c} \right) \right) \frac{v-c}{2} + \omega \left( 1 - \frac{v}{c} + \epsilon \left( \frac{v}{c} - \omega \right) \right) v \\ + (1-\omega) \left( 1 - \frac{v}{c} + \epsilon \left( \frac{v}{c} - \omega \right) \right) \frac{v}{2}$$

This gives:

$$u(\sigma^*, x_\epsilon) - u(\sigma, x_\epsilon) = \frac{\epsilon c}{2} \left( \frac{v}{c} - \omega \right)^2 > 0$$

which proves that  $\sigma^*$  is an ESS.

We will now take a closer look the connection between ESS and Nash equilibria.

## ESS and Nash equilibria

When considering pairwise contest population games there is a natural way to associate a normal form game.

### Definition

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The **associated two player game** for a pairwise contest population game is the normal form game with payoffs given by:  $u_1(s, s') = u(s, s') = u_2(s', s)$ .

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Note that the resulting game is symmetric (other contexts would give non symmetric games but we won't consider them here).

Using this we have the powerful result:

### Theorem

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If  $\sigma^*$  is an ESS in a pairwise contest population game then for all  $\sigma \neq \sigma^*$ :

1.  $u(\sigma^*, \sigma^*) > u(\sigma, \sigma^*)$  OR
2.  $u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*)$  and  $u(\sigma^*, \sigma) > u(\sigma, \sigma)$

Conversely, if either (1) or (2) holds for all  $\sigma \neq \sigma^*$  in a two player normal form game then  $\sigma$  is an ESS.

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## Proof

If  $\sigma^*$  is an ESS, then by definition:

$$u(\sigma^*, x_\epsilon) > u(\sigma, x_\epsilon)$$

which corresponds to:

$$(1 - \epsilon)u(\sigma^*, \sigma^*) + \epsilon u(\sigma^*, \sigma) > (1 - \epsilon)u(\sigma, \sigma^*) + \epsilon u(\sigma, \sigma)$$

- If the condition 1 of the theorem holds then the above inequality can be satisfied for  $\epsilon$  sufficiently small. If condition 2 holds then the inequality is satisfied.
- Let us assume that  $u(\sigma^*, \sigma^*) < u(\sigma, \sigma^*)$ :
  - If  $u(\sigma^*, \sigma^*) < u(\sigma, \sigma^*)$  then we can find  $\epsilon$  sufficiently small such that the inequality is violated. Thus the inequality implies  $u(\sigma^*, \sigma^*) \geq u(\sigma, \sigma^*)$ .
  - If  $u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*)$  then  $u(\sigma^*, \sigma^*) > u(\sigma, \sigma^*)$  as required.

This result gives us a more efficient way of computing ESS. The first condition is in fact almost a condition for Nash Equilibrium (with a strict inequality), the second is thus a stronger condition that removes certain Nash equilibria from consideration. This becomes particularly relevant when considering Nash equilibrium in mixed strategies.

To find ESS in a pairwise context population game we:

1. Write down the associated two-player game;
2. Identify all symmetric Nash equilibria of the game;
3. Test the Nash equilibrium against the two conditions of the above Theorem.

## Example

Let us consider the Hawk-Dove game. The associated two-player game is:

$$\left( \left( \frac{v-c}{2}, \frac{v-c}{2} \right), (v, 0) \right) \\ \left( 0, v \right), \left( \frac{v}{2}, \frac{v}{2} \right)$$

Recalling that we have  $v < c$  so we can use the Equality of payoffs theorem to obtain the Nash equilibrium:

$$u_2(\sigma^*, H) = u_2(\sigma^*, D)$$

$$q^* = \frac{v}{c}$$

Thus we will test  $\sigma^* = (\frac{v}{c}, 1 - \frac{v}{c})$  using the above theorem.

**Important** from the equality of payoffs theorem we immediately see that condition 1 does not hold as  $u(\sigma^*, H) = u(\sigma^*, D)$ . Thus we need to prove that:

$$u(\sigma^*, \sigma) > u(\sigma, \sigma)$$

We have:

$$u(\sigma^*, \sigma) = \frac{v}{c}\omega\frac{v-c}{2} + \frac{v}{c}(1-\omega)v + (1-\frac{v}{c})(1-\omega)\frac{v}{c}$$

$$u(\sigma, \sigma) = \omega^2\frac{v-c}{2} + \omega(1-\omega)v + (1-\omega)^2\frac{v}{c}$$

After some Algebra:

$$u(\sigma^*, \sigma) - u(\sigma, \sigma) = \frac{c}{2}(\frac{v}{c} - \omega)^2 > 0$$

Giving the required result.