4. (a) Provide definitions for the following terms:

- Normal form game. [1]
- Strictly dominated strategy. [1]
- Weakly dominated strategy. [1]
- Best response strategy. [1]
- Nash equilibrium. [1]

For the remainder of this question consider the battle of the sexes game:

$$\begin{pmatrix} (3,2) & (0,0) \\ (1,1) & (2,3) \end{pmatrix}$$

(b) By clearly stating the techniques used, obtain all (if any) pure Nash equilibria.
[4]

(c) Plot the utilities to player 1 (the row player) assuming that the 2nd player (the column player) plays a mixed strategy: $\sigma_2 = (y, 1-y)$.
[2]

(d) Plot the utilities to player 2 (the column player) assuming that the 1st player (the row player) plays a mixed strategy: $\sigma_1 = (x, 1-x)$.
[2]

(e) Assuming that player 1 plays the mixed strategy $\sigma_1 = (x, 1-x)$, show that player 1's best response x^* to a mixed strategy $\sigma_2 = (y, 1-y)$ is given by:

$$x^* = \begin{cases} 0, & \text{if } y < 1/2 \\ 1, & \text{if } y > 1/2 \\ \text{indifferent}, & \text{otherwise} \end{cases}$$

Similarly show that player 2's best response y^* is given by:

$$y^* = \begin{cases} 0, & \text{if } x < 1/2 \\ 1, & \text{if } x > 1/2 \\ \text{indifferent}, & \text{otherwise} \end{cases}$$

[4]

(f) Use the above to obtain all Nash equilibria for the game.
[2]

(g) Confirm this result by stating, proving and using the Equality of Payoffs theorem.
[6]

5. Consider the following stage game:

$$\begin{pmatrix}(2,2) & (5,0)\\(0,5) & (4,4)\end{pmatrix}$$

This game shall be referred to as the Prisoner's Dilemma. The first strategy for both players will be referred to as 'Cooperate' (C) and the second strategy will be referred to as 'Defect' (D). **Players aim to minimise their payoffs.**

Consider the following strategies:

- s_C: Always cooperate;
- s_D: Always defect;
- s_G: Start by cooperating until your opponent defects at which point defect in all future stages.

Assume $S_1 = S_2 = \{s_C, s_D, s_G\}$.

(a) Assuming a discounting factor of δ, obtain the utility to both players if the strategy pair (s_C, s_C) is played.

[2]

(b) Assuming a discounting factor of δ, obtain the utility to both players if the strategy pair (s_D, s_D) is played.

[2]

(c) For what values of δ is (s_G, s_G) a Nash equilibrium? **Recall that players aim to minimise their payoffs.**

[5]

(d) Define the average payoff in an infinitely repeated game.

[1]

(e) Plot the feasible average payoffs and the individually rational payoffs for the Prisoner's Dilemma. **Recall that players aim to minimise their payoffs.**

[4]

(f) Prove the following theorem (**for games where players aim to minimise their payoffs**):

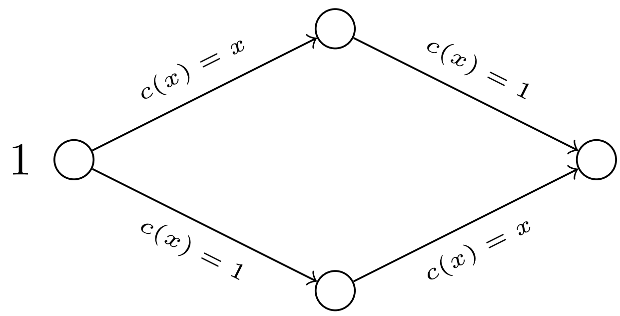
"Let u_1^*, u_2^* be a pair of Nash equilibrium payoffs for a stage game. For every individually rational pair v_1, v_2 there exists $\bar{\delta}$ such that for all $1 > \delta > \bar{\delta} > 0$ there is a subgame perfect Nash equilibrium with payoffs v_1, v_2."

[11]

6. (a) Define a routing game (G, r, c).

[2]

(b) Define a Nash flow and using this definition obtain the Nash flow for the following game:



[3]

(c) Define an optimal flow and using this definition obtain the optimal flow for the above game.

[3]

(d) State the theorem connecting the following function Φ to the Nash flow of a routing game:

$$\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x)dx$$

[2]

(e) Using the theorem from (d) confirm the Nash flow previously found in (b).

[2]

(f) State the theorem connecting the marginal cost $c^*(x) = \frac{d(xc(x))}{dx}$ to the optimal flow of a routing game.

[2]

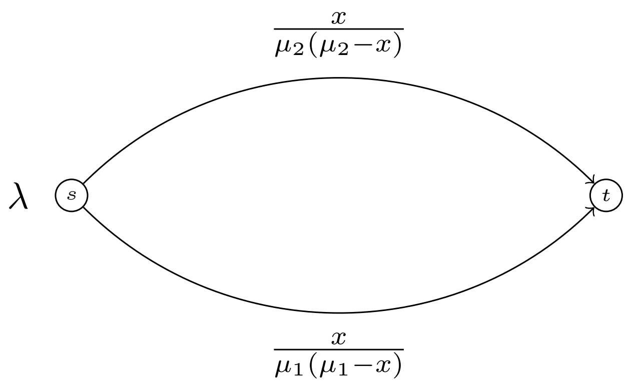
(g) Using the theorem from (f) confirm the optimal flow previously found in (c) .

[2]

(h) The expected time spent in an $M/M/1$ queue at steady state is given by:

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

where μ, λ are the mean service and inter arrival rates and $\lambda < \mu$ respectively. Explain how a system with two $M/M/1$ queues and players choosing which queue to join can be studied using the following routing game:



[2]

(i) Obtain the Nash and Optimal flows for the game in (h) with $\mu_1 = 4, \mu_2 = 3$ and $\lambda = 2$.

You might find it useful to know that the equation:

$$x^4 - 2x^3 + x^2 - 420x + 324 = 0$$

has a single solution in the range $0 \leq x < 2$ given by $x \approx 0.7715$.

[7]