1 Homework sheet 5 - Matching games, cooperative games and routing games

1. Obtain stable suitor optimal and reviewer optimal matchings for the matching games shown in Figures **??** to **??**.

   - Game 1:

     $c\colon (B, A, C)$ •   • $C\colon (a, b, c)$

     $b\colon (A, C, B)$ •   • $B\colon (b, a, c)$

     $a\colon (A, C, B)$ •   • $A\colon (b, c, a)$

     Solution

     Following the algorithm:

     Suitor optimal: $\{a : C, b : A, c : B\}$ Reviewer optimal: $\{'A' :' b',' B' :' c',' C' :' a'\}$

   - Game 2:

     $c\colon (A, C, B)$ •   • $C\colon (a, b, c)$

     $b\colon (B, C, A)$ •   • $B\colon (b, c, a)$

     $a\colon (A, C, B)$ •   • $A\colon (b, c, a)$

     Solution

     Following the algorithm:

     Suitor optimal: $\{a : C, b : B, c : A\}$ Reviewer optimal: $\{A : c, B : b, C : a\}$

   - Game 3:

d: (A, D, B, C) • • D: (a, d, b, c)

c: (B, A, C, D) • • C: (a, c, d, b)

b: (D, A, C, B) • • B: (d, a, c, b)

a: (A, D, C, B) • • A: (b, d, a, c)

Solution

Following the algorithm:

Suitor optimal: $\{a : D, b : A, c : C, d : B\}$ Reviewer optimal: $\{A : b, B : d, C : c, D : a\}$

- Game 4:

d: (B, A, D, C) • • D: (a, b, d, c)

c: (B, C, A, D) • • C: (d, b, a, c)

b: (A, D, C, B) • • B: (d, a, c, b)

a: (A, D, C, B) • • A: (c, b, d, a)

Solution

Following the algorithm:

Suitor optimal: $\{a : D, b : A, c : C, d : B\}$ Reviewer optimal: $\{A : c, B : d, C : b, D : a\}$

2. Consider a matching game where all reviewers have the same preference list. Prove that there is a single stable matching.

Solution

Let M be the suitor optimal matching (given by the Gale-Shapley algorithm).

Assume $\exists\ M' \neq M$. As M is reviewer sub-optimal $\exists$ a subset $\bar{R} \subseteq R$ such that: For all $r \in \bar{R}$: $M^{-1}(r)$ is worse than $M'^{-1}(r)$. For $r \in R \setminus \bar{R}$ $M^{-1}(r) = M'^{-1}(r)$.

Consider $\bar{r} \in \bar{R}$, as all reviewers have same reference list, let r be the reviewer with ''best'' suitor under matching M (the matching given by the Gale Shapley algorithm).

When considering M', reviewers outside of $\bar{R}$ have same matching as in M. All reviewers in $\bar{R}$ must have a ''better'' matching.

As all reviewers have the same preference list, $\bar{r}$ cannot be matched thus M' is not a matching.

3. For the following cooperative games:

   i. Verify if the game is monotonic.
   ii. Verify if the game is super additive.
   iii. Obtain the Shapley value.

$$v_1(C) = \begin{cases} 5, & \text{if } C = \{1\} \\ 3, & \text{if } C = \{2\} \\ 2, & \text{if } C = \{3\} \\ 12, & \text{if } C = \{1,2\} \\ 5, & \text{if } C = \{1,3\} \\ 4, & \text{if } C = \{2,3\} \\ 13, & \text{if } C = \{1,2,3\} \end{cases}$$

Solution

Game is monotone but is not super additive: $v_1(\{1,3\}) = 5$ and $v_1(\{1\}) + v_1(\{3\}) = 5 + 2 = 7$.

The Shapley value is $\phi = (20/3, 31/6, 7/6)$.

$$v_2(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 0, & \text{if } C = \{2\} \\ 5, & \text{if } C = \{1,2\} \end{cases}$$

Solution

Game is not monotone: $v_2(\{1\}) = 6 \geq v_2(\{1,2\}) = 5$. Game is not super additive: $v_2(\{1,2\}) = 5 \leq v_2(\{1\}) + v_2(\{2\}) = 6$.

The Shapley value is $\phi = (11/2, -1/2)$.

$$v_3(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 6, & \text{if } C = \{2\} \\ 13, & \text{if } C = \{3\} \\ 6, & \text{if } C = \{1,2\} \\ 13, & \text{if } C = \{1,3\} \\ 13, & \text{if } C = \{2,3\} \\ 26, & \text{if } C = \{1,2,3\} \end{cases}$$

Game is monotone but not super additive: $v_3(\{1,2\}) = 6 \leq v_3(\{1\}) + v_3(\{2\}) = 12$

The Shapley value is $\phi = (19/3, 19/3, 40/3)$.

$$v_4(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 7, & \text{if } C = \{2\} \\ 0, & \text{if } C = \{3\} \\ 8, & \text{if } C = \{4\} \\ 7, & \text{if } C = \{1,2\} \\ 6, & \text{if } C = \{1,3\} \\ 12, & \text{if } C = \{1,4\} \\ 7, & \text{if } C = \{2,3\} \\ 12, & \text{if } C = \{2,4\} \\ 8 & \text{if } C = \{3,4\} \\ 7, & \text{if } C = \{1,2,3\} \\ 24, & \text{if } C = \{1,2,4\} \\ 12, & \text{if } C = \{1,3,4\} \\ 12, & \text{if } C = \{2,3,4\} \\ 25, & \text{if } C = \{1,2,3,4\} \end{cases}$$

Game is monotone but not super additive: $v_4(\{1,2\}) = 7 \leq v_4(\{1\}) + v_4(\{2\}) = 13$

The Shapley value is $\phi = (83/12, 89/12, 1/4, 125/12)$.

4. Prove that the Shapley value has the following properties:

   - Efficiency
     Solution
     For every permutation π we have:

$$\sum_{i=1}^{N} \Delta_{\pi}^{G}(i) = v(S_\pi(1) \cup \{1\}) - v(S_\pi(1)) + v(S_\pi(2) \cup \{2\}) - v(S_\pi(2)) \ldots v(S_\pi(N) \cup N) - v(S_\pi(N)) = v($$

taking the mean over all permutations (which is by definition the Shapley value) we have the required result.

- Null player

  Solution

  Consider any permutation π and a null player i. We have $v(S_\pi(i)) \cup \{i\} = v(S_\pi)$. Thus, $\Delta_\pi^G(i) = 0$, as this holds for all π the result follows.

- Symmetry

  Solution

  Assume that i and j are symmetric. Given a permutation π, let π' denote the permutation obtained by swapping i and j.

  – Assume that i precedes j in π, this gives $S_\pi(i) = S_{\pi'}(j)$, if we let $C = S_{\pi'}(j)$:

  $$\Delta_\pi^G(i) = v(C \cup \{i\}) - v(C)$$

  and

  $$\Delta_{\pi'}^G(j) = v(C \cup \{j\}) - v(C)$$

  By symmetry $\Delta_\pi^G(i) = \Delta_{\pi'}^G(j)$.

  – Assume that i does not precede j in π, let $C = S_\pi(i) \setminus \{j\}$. We have:

  $$\Delta_\pi^G(i) = v(C \cup \{i\} \cup \{j\}) - v(C \cup \{j\})$$

  and

  $$\Delta_{\pi'}^G(j) = v(C \cup \{j\} \cup \{i\}) - v(C \cup \{i\})$$

  Since $C \subseteq N$ and $i, j \notin C$ we have by symmetry $v(C \cup \{i\}) = v(C \cup \{j\})$ and therefore $\Delta_\pi^G(i) = \Delta_{\pi'}^G(j)$.

  We have that $\Delta_\pi^G(i) = \Delta_{\pi'}^G(j)$ for all $\pi \in \Pi_N$, there is an abvious bijection between all π and corresponding π' thus:

  $$\phi_i(G) = 1/n! \sum_{\pi \in \Pi_N} \Delta_\pi^G(i) = \sum_{\pi \in \Pi_N} \Delta_{\pi'}^G(j) = \phi_j(G)$$

  as required.

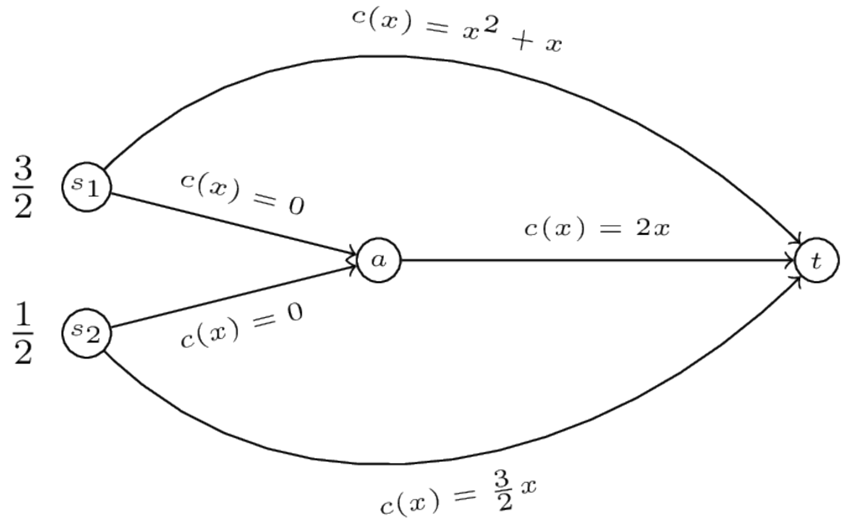
- Additivity

Solution

Let v^+ be the characteristic function of the game $G^1 + G^2$. Following from the definition of additivity it is immediate to note that we have $\Delta_\pi^+(i) = \Delta_\pi^{v_1}(i) + \Delta_\pi^{v_2}(i)$. The result follows.

Note that this does not prove that the Shapley value is the only vector that has those properties (it in fact is though).

5. Calculate the Nash flow and the optimal flow for the following routing game:



Solution

For the **Nash flow**:

Let α be the traffic along the top arc and β the traffic along the bottom arc.

By definition for commodity 1 we have:

$$\alpha^2 + \alpha = 2(2/3 - \alpha + 1/2 - \beta)$$

By definition for commodity 2 we have:

$$3/2\beta = 2(2/3 - \alpha + 1/2 - \beta)$$

Solving this later equation gives:

$$\beta = (8 - 4\alpha)/7$$

Substituting this in to the first equation gives:

$$\alpha^2 + 13/7\alpha - 12/7 = 0$$

which has solution $\alpha = \frac{1}{14}\sqrt{505} - \frac{13}{14}, -\frac{2}{49}$, substituting this in to our expression for β gives: $\beta = \sqrt{505} + \frac{82}{49}$.

For the **Optimal flow** we use the marginal costs:

$$c^*(x) = (2x+1)x + x^2 + x$$
$$c^*(x) = 4x$$
$$c^*(x) = 3x$$

We now repeat the above:

By definition for commodity 1 we have:

$$(2\,\alpha + 1)\alpha + \alpha^2 + \alpha = 4(2/3 - \alpha + 1/2 - \beta)$$

By definition for commodity 2 we have:

$$3\beta = 4(2/3 - \alpha + 1/2 - \beta)$$

Solving this later equation gives:

$$\beta = (8 - 4\alpha)/7$$

Substituting this in to the first equation gives:

$$(2 * \alpha + 1) * \alpha + \alpha^2 + 19/7 * \alpha - 24/7 = 0$$

which has solution $\alpha = \frac{1}{21}\sqrt{673} - \frac{13}{21}$, substituting this in to our expression for β gives: $\beta = \sqrt{673} + \frac{220}{147}$.

6. For a routing game the 'Price of Anarchy' is defined as:

$$\text{PoA} = \frac{C(\tilde{f})}{C(f^*)}$$

For the game shown (a generalisation of "Pigou's example") obtain the PoA as a function of α.

Solution

Let x be the flow along the bottom arc. The Nash flow $\tilde{x}$ is immediate:

$$\tilde{x} = 1$$

giving $C(\tilde{f}) = 1$

The optimal flow is x^* solves:

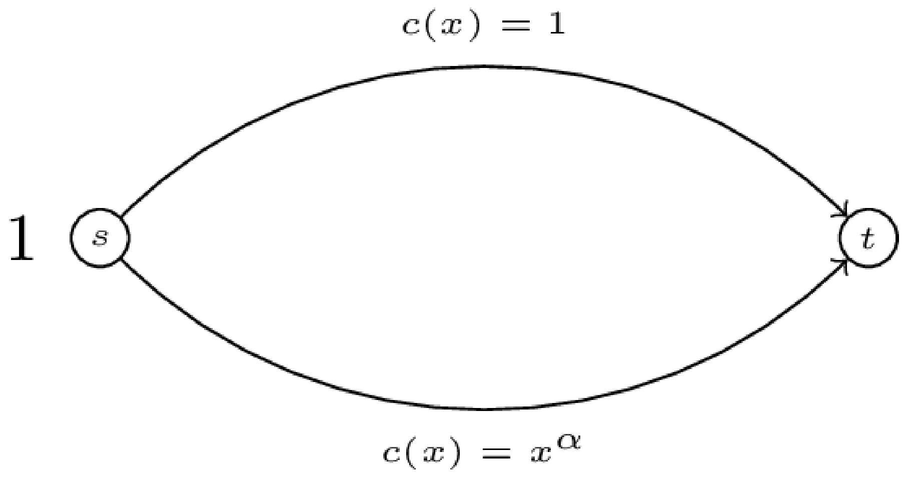


Figure 1: A generalization of Pigou's example

$$(\alpha + 1)x^{\alpha} = 1$$

thus

$$x^* = \left(\frac{1}{\alpha+1}\right)^{1/\alpha}$$

giving $C(f^*) = (1-x^*)+x^{*\alpha}x^* = \left(\left(\frac{1}{\alpha+1}\right)^{1/\alpha}\right)^{\alpha}\left(\frac{1}{\alpha+1}\right)^{1/\alpha}+1-\left(\frac{1}{\alpha+1}\right)^{1/\alpha}$

Thus:

$$\text{PoA} = \frac{(\alpha+1)^{1/\alpha+1}}{\alpha+2}$$

It can be shown that the above is a decreasing function in α, this implies that as the 'shortcut' gets 'better' (recall that $x \leq 1$) the negative effect of selfish behaviour increases.