

# 1 OR 3: Lecture 6 - Nash equilibria in mixed strategies

## 1.1 Recap

In the [previous lecture](#)

- The definition of Nash equilibria;
- Identifying Nash equilibria in pure strategies;
- Solving the duopoly game;

This brings us to a very important part of the course. We will now consider equilibria in mixed strategies.

## 1.2 Recall of expected utility calculation

In the matching pennies game discussed previously:

$$\begin{pmatrix} (1, -1) & (-1, 1) \\ (-1, 1) & (1, -1) \end{pmatrix}$$

Recalling [Chapter 2](#) a strategy profile of  $\sigma_1 = (.2, .8)$  and  $\sigma_2 = (.6, .4)$  implies that player 1 plays heads with probability .2 and player 2 plays heads with probability .6.

We can extend the utility function which maps from the set of pure strategies to  $\mathbb{R}$  using *expected payoffs*. For a two player game we have:

$$u_i(\sigma_1, \sigma_2) = \sum_{r \in S_1, s \in S_2} \sigma_1(r) \sigma_2(s) u_i(r, s)$$

## 1.3 Obtaining equilibria

Let us investigate the best response functions for the matching pennies game.

If we assume that player 2 plays a mixed strategy  $\sigma_2 = (y, 1 - y)$  we have:

$$u_1(r_1, \sigma_2) = 2y - 1$$

and

$$u_1(r_2, \sigma_2) = 1 - 2y$$

These utilities are shown in [Figure 1](#).

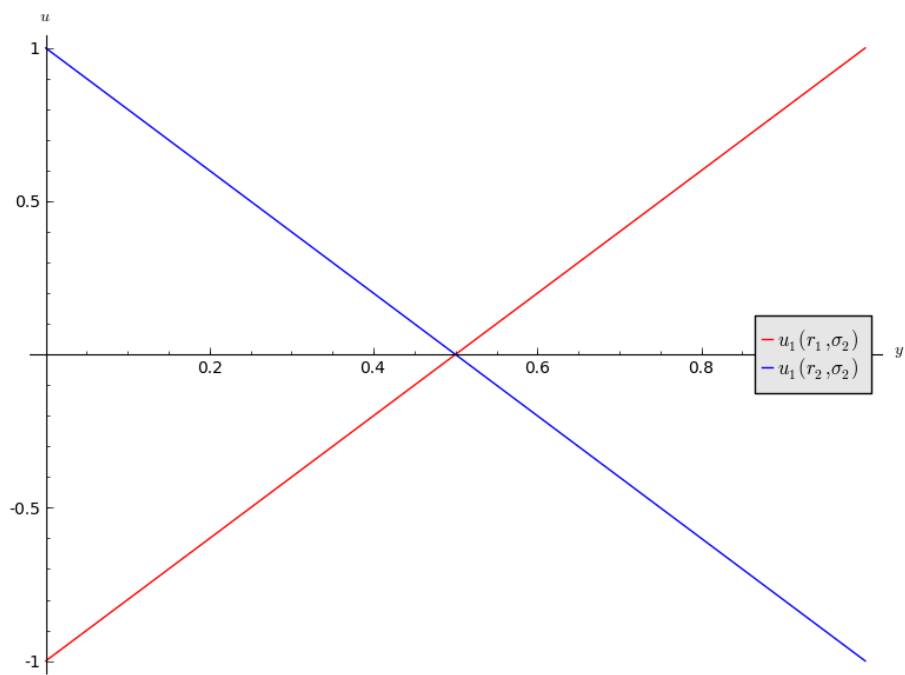


Figure 1: Utilities of player 1 in the matching pennies game.

1. If  $y < 1/2$  then  $r_2$  is a best response for player 1.
2. If  $y > 1/2$  then  $r_1$  is a best response for player 1.
3. If  $y = 1/2$  then player 1 is indifferent.

If we assume that player 1 plays a mixed strategy  $\sigma_1 = (x, 1 - x)$  we have:

$$u_2(\sigma_1, s_1) = 1 - 2x$$

and

$$u_2(\sigma_1, s_2) = 2x - 1$$

These utilities are shown in Figure 2.

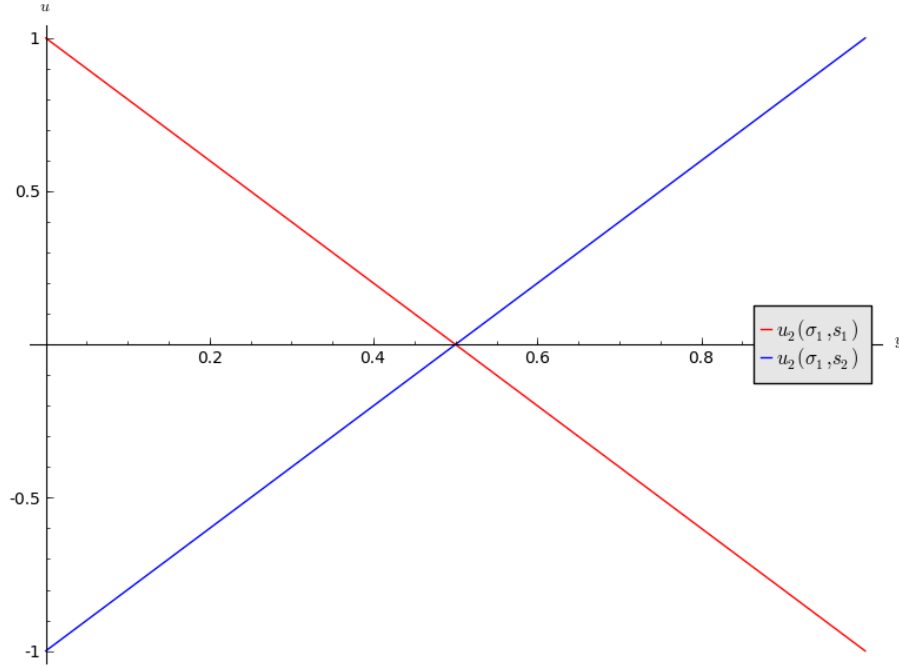


Figure 2: Utilities of player 2 in the matching pennies game.

Thus we have:

1. If  $x < 1/2$  then  $s_1$  is a best response for player 2.
2. If  $x > 1/2$  then  $s_2$  is a best response for player 2.
3. If  $x = 1/2$  then player 2 is indifferent.

Let us draw both best responses on a single diagram, indicating the best responses in each quadrant as shown in Figure 3. The arrows show the deviation indicated by the best responses.

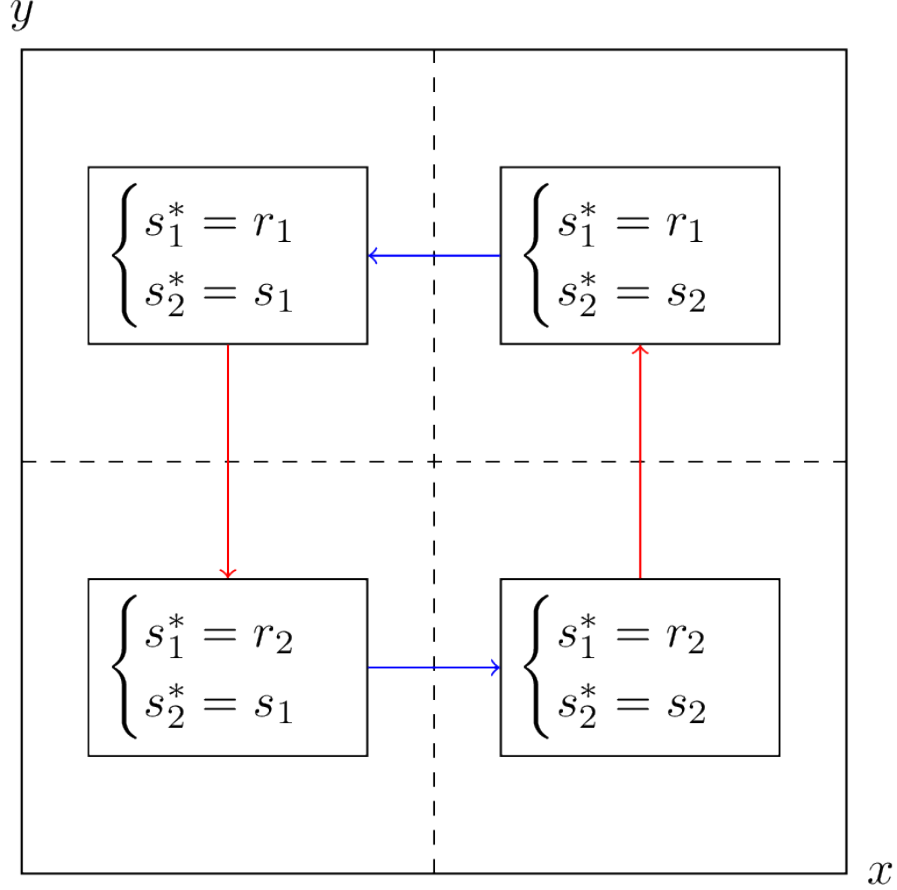


Figure 3: Best response moves based on current strategy.

If either player plays a mixed strategy other than  $(1/2, 1/2)$  then the other player has an incentive to modify their strategy. Thus the Nash equilibria is:

$$((1/2, 1/2), (1/2, 1/2))$$

This notion of “indifference” is important and we will now prove an important theorem that will prove useful when calculating Nash Equilibria.

## 1.4 Equality of payoffs theorem

### 1.4.1 Definition of the support of a strategy

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In an  $N$  player normal form game the **support** of a strategy  $\sigma \in \Delta S_i$  is defined as:

$$\mathcal{S}(\sigma) = \{s \in S_i \mid \sigma(s) > 0\}$$

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I.e. the support of a strategy is the set of pure strategies that are played with non zero probability.

For example, if the strategy set is  $\{A, B, C\}$  and  $\sigma = (1/3, 2/3, 0)$  then  $\mathcal{S}(\sigma) = \{A, B\}$ .

### 1.4.2 Theorem of equality of payoffs

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In an  $N$  player normal form game if the strategy profile  $(\sigma_i, s_{-i})$  is a Nash equilibria then:

$$u_i(\sigma_i, s_{-i}) = u_i(s, s_{-i}) \text{ for all } s \in \mathcal{S}(\sigma_i) \text{ for all } 1 \leq i \leq N$$

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### 1.4.3 Proof

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If  $|\mathcal{S}(\sigma_i)| = 1$  then the proof is trivial.

We assume that  $|\mathcal{S}(\sigma_i)| > 1$ . Let us assume that the theorem is not true so that there exists  $\bar{s} \in \mathcal{S}(\sigma)$  such that

$$u_i(\sigma_i, s_{-i}) \neq u_i(\bar{s}, s_{-i})$$

Without loss of generality let us assume that:

$$\bar{s} = \operatorname{argmax}_{s \in \mathcal{S}(\sigma)} u_i(s, s_{-i})$$

Thus we have:

$$\begin{aligned}
u_i(\sigma_i, s_{-i}) &= \sum_{s \in \mathcal{S}(\sigma_i)} \sigma_i(s) u(s, s_{-i}) \\
&\leq \sum_{s \in \mathcal{S}(\sigma_i)} \sigma_i(s) u(\bar{s}, s_{-i}) \\
&\leq u(\bar{s}, s_{-i}) \sum_{s \in \mathcal{S}(\sigma_i)} \sigma_i(s) \\
&\leq u(\bar{s}, s_{-i})
\end{aligned}$$

Giving:

$$u_i(\sigma_i, s_{-i}) < u_i(\bar{s}, s_{-i})$$

which implies that  $(\sigma_i, s_{-i})$  is not a Nash equilibrium.

#### 1.4.4 Example

Let's consider the matching pennies game yet again. To use the equality of payoffs theorem we identify the various supports we need to try out. As this is a  $2 \times 2$  game we can take  $\sigma_1 = (x, 1 - x)$  and  $\sigma_2 = (y, 1 - y)$  and assume that  $(\sigma_1, \sigma_2)$  is a Nash equilibrium.

from the theorem we have that  $u_1(\sigma_1, \sigma_2) = u_1(r_1, \sigma_2) = u_1(r_2, \sigma_2)$

$$\begin{aligned}
u_1(r_1, \sigma_2) &= u_2(r_2, \sigma_2) \\
y - (1 - y) &= -y + (1 - y) \\
y &= 1/2
\end{aligned}$$

Thus we have found player 2's Nash equilibrium strategy by finding the strategy that makes player 1 indifferent. Similarly for player 1:

$$\begin{aligned}
u_2(\sigma_1, s_1) &= u_2(\sigma_1, s_2) \\
-x + (1 - x) &= x - (1 - x) \\
x &= 1/2
\end{aligned}$$

Thus the Nash equilibria is:

$$((1/2, 1/2), (1/2, 1/2))$$

To finish this chapter we state a famous result in game theory:

#### **1.4.5 Nash's Theorem**

Every normal form game with a finite number of pure strategies for each player, has at least one Nash equilibrium.