

1 Homework sheet 4 - Evolutionary games, games with incomplete information and stochastic games

1. Consider the pairwise contest games with the following associated two player games:

$$\begin{pmatrix} (2, 2) & (4, 5) \\ (5, 4) & (1, 1) \end{pmatrix}$$

Solution

Using the Equality of payoffs theorem we obtain the Nash equilibria:

$$\{((1, 0), (0, 1)), ((0, 1), (1, 0)), ((1/2, 1/2), (1/2, 1/2))\}$$

The two pure Nash equilibria are ESS (because of the first condition of the theorem relating an evolutionary stable strategy to the Nash equilibrium of the associated game).

$$u((1/2, 1/2), \sigma) = 3$$

and for $\sigma = (\omega, 1 - \omega)$:

$$u(\sigma, \sigma) = (2\omega^2 + 9(1 - \omega)\omega + (1 - \omega)^2)$$

thus (after some algebraic manipulation):

$$u((1/2, 1/2), \sigma) - u(\sigma, \sigma) = (2\omega - 1)(3\omega - 2)$$

which is negative for $1/2 < \omega < 2/3$ so this mixed strategy is not an ESS.

$$\begin{pmatrix} (1, 1) & (0, 0) \\ (0, 0) & (1, 1) \end{pmatrix}$$

Solution

Using the Equality of payoffs theorem we obtain the Nash equilibria:

$$\{((1, 0), (1, 0)), ((0, 1), (0, 1)), ((1/2, 1/2), (1/2, 1/2))\}$$

The two pure Nash equilibria are ESS (because of the first condition of the theorem relating an evolutionary stable strategy to the Nash equilibrium of the associated game).

$$u((1/2, 1/2), \sigma) = 1/2$$

As before:

$$u((1/2, 1/2), \sigma) - u(\sigma, \sigma) = -1/2(2\omega - 1)^2$$

so not an ESS.

$$\begin{pmatrix} (\alpha, \alpha) & (1, \beta) \\ (\beta, 1) & (0, 0) \end{pmatrix}$$

(Assume $\alpha, \beta > 0$ and $\alpha \neq \beta$)

Solution

If $\beta < \alpha$ then we have a single pure Nash Equilibria $((1, 0), (1, 0))$ which is also an ESS.

If $\beta > \alpha$ the Nash Equilibria are $\{((1, 0), (1, 0)), ((0, 1), (0, 1)), ((1/(\beta - \alpha + 1), (\beta - \alpha)/(\beta - \alpha + 1)), (1/(\beta - \alpha + 1), (\beta - \alpha)/(\beta - \alpha + 1)))\}$. The pure are ESS.

$$u(\sigma^*, \sigma) = \frac{\beta}{\beta - \alpha + 1}$$

and

$$u(\sigma, \sigma) = -\alpha\omega^2 - \omega(1 - \omega)(1 + \beta)$$

thus (after some algebra):

$$u(\sigma^*, \sigma) - u(\sigma, \sigma) > 0 \Leftrightarrow \beta - (\beta - \alpha + 1)(\omega^2(\alpha - \beta - 1) + \omega(1 + \beta)) > 0$$

However the quadratic in ω has roots: $\left\{ \frac{\beta}{\beta - \alpha + 1}, \frac{1}{\beta - \alpha + 1} \right\}$ so $\exists \omega$ for which the difference is negative and thus this mixed strategy is not an ESS.

Identify all evolutionary stable strategies.

2. Consider the following game:

In a mathematics department, researchers can choose to use one of two systems for typesetting their research papers: LaTeX or Word. We will refer to these two strategies as L and W respectively. A user of W receives a basic utility of 1 and as L is more widely used by mathematicians out of the department and is in general considered to be a better system a user of L gets a basic utility of $\alpha > 1$. Members of the mathematics department often collaborate and as such it is beneficial for the researchers to use the same typesetting system. If we let μ represent the proportion of users of L we let:

$$u(L, \chi) = \alpha + 2\mu$$

$$u(W, \chi) = 1 + 2(1 - \mu)$$

What are the evolutionary stable strategies?

Solutions

Using the theorem for necessity of stability we have the following candidate ESS:

1. σ_L : everyone uses L , thus $\mu = 1$ (we have $u(L, \chi) > u(W, \chi)$).
2. σ_W : everyone uses W , thus $\mu = 0$ (we have $u(L, \chi) < u(W, \chi)$).
3. σ_m : some use L and some use W , by the theorem we have $u(L, \chi) = u(W, \chi)$ which implies $\alpha + 2\mu = 1 + 2(1 - \mu)$ giving $\mu = \frac{3-\alpha}{4}$.

Now we consider the post entry population $\chi_\epsilon = (1-\epsilon)\sigma^* + \epsilon\sigma$ (where σ^* is the base strategy and σ is the entry population). We denote $\sigma = (\mu, 1-\mu)$ and $\sigma^* = (\mu^*, 1-\mu^*)$ and $\delta = u(\sigma^*, \chi_\epsilon) - u(\sigma, \chi_\epsilon)$. We have:

$$\delta = \mu^* u(L, \chi_\epsilon) + (1-\mu^*) u(W, \chi_\epsilon) - \mu u(L, \chi_\epsilon) - (1-\mu) u(W, \chi_\epsilon) = (\mu^* - \mu)(u(L, \chi_\epsilon) - u(W, \chi_\epsilon))$$

which gives:

$$\delta = (\mu^* - \mu)(\alpha - 3 + 4((1-\epsilon)\mu^* + \epsilon\mu)) = (\mu^* - \mu)(4\mu^* + \alpha - 3 + 4\epsilon(\mu - \mu^*))$$

We now consider each potential ESS in turn, if $\delta > 0$ for all $\epsilon < \bar{\epsilon}$ for some $\bar{\epsilon}$ then we have an ESS (this is by definition):

1. $\mu^* = 1$: $\delta = (1-\mu)(1 + \alpha + 4\epsilon(\mu - 1)) > (1-\mu)(1 + \alpha - 4\epsilon) > 0$ for all $\mu \neq 1$ and $\epsilon < \bar{\epsilon} = \frac{1+\alpha}{4}$. Thus σ_L is an ESS.
2. $\mu^* = 0$: $\delta = -\mu(\alpha - 3 + 4\epsilon\mu)$. If $\alpha \geq 3$ then $\delta \leq 0$ for all values of μ, ϵ , thus if L is 3 times better than W σ_W is not an ESS. If $\alpha < 3$ $\delta > 0 \Leftrightarrow \alpha - 3 + 4\epsilon\mu < 0 \Rightarrow \alpha - 3 + 4\epsilon\mu < \alpha - 3 + 4\epsilon < 0$ for all $\mu \neq 0$ $\epsilon < \bar{\epsilon} = \frac{3-\alpha}{4}$. Thus σ_W is an ESS for $\alpha < 3$.
3. $\mu^* = \frac{3-\alpha}{4}$: $\delta = -4\epsilon\left(\frac{3-\alpha}{4} - \mu\right)^2 < 0$ for all $\mu \neq \frac{3-\alpha}{4}$ and for all $\epsilon > 0$ so σ_m is not an ESS.

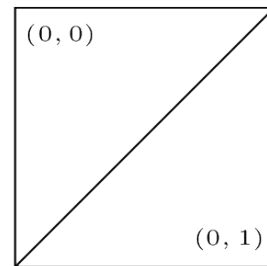
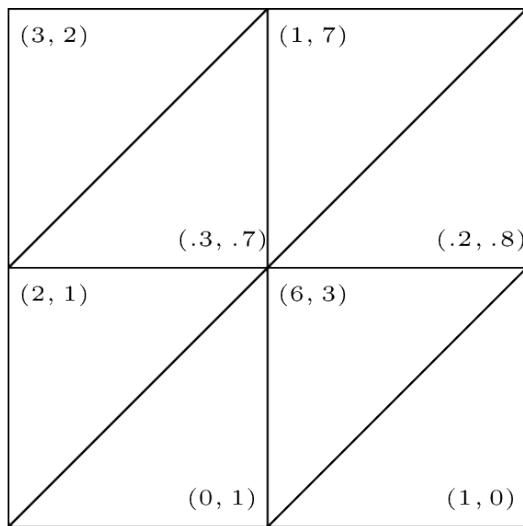
3. Consider the following two normal form games:

$$A = \begin{pmatrix} (3, 0) & (-1, -1) & (1, 2) \\ (1, 0) & (-1, 1) & (2, 0) \end{pmatrix}$$

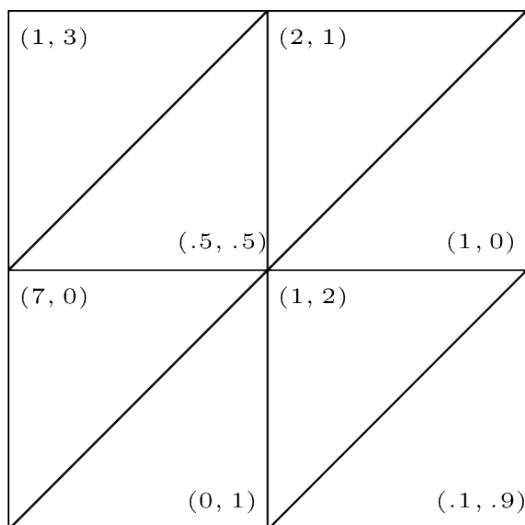
$$B = \begin{pmatrix} (2, 2) & (1, 1) & (1, 3) \\ (1, 3) & (-2, -3) & (4, 2) \end{pmatrix}$$

Assume both players play either game A or game B with probability $1/2$, neither player knows which game is played. Obtain the Nash equilibrium for this game.

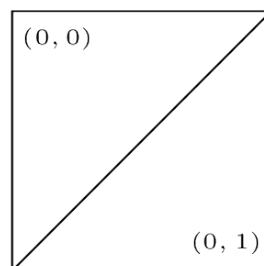
4. Repeat the analysis of the principal agent game assuming that p is the probability of the project being successful in case of a high level of effort by the employee.
 - i. What are the expected utilities to the employer and the employee?
 - ii. Obtain a condition for which the employer should offer a bonus.
5. Obtain the Markov Nash equilibrium for the following games assuming $\delta = 1/4$.



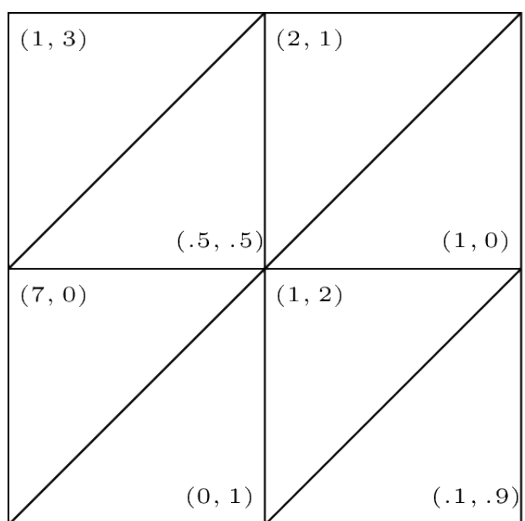
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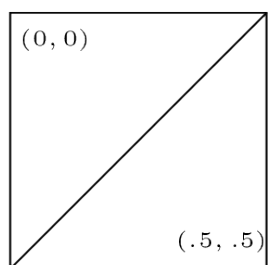
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