1 Homework sheet 5 - Matching games, cooperative games and routing games

1. Obtain stable suitor optimal and reviewer optimal matchings for the matching games shown in Figures **??** to **??**.

   - Game 1:

     $c\colon\ (B, A, C)\bullet \qquad \bullet C\colon\ (a, b, c)$

     $b\colon\ (A, C, B)\bullet \qquad \bullet B\colon\ (b, a, c)$

     $a\colon\ (A, C, B)\bullet \qquad \bullet A\colon\ (b, c, a)$

     Solution

     Following the algorithm:

     Suitor optimal: ({a: C, b: A, c: B}) Reviewer optimal: ({'A': 'b', 'B': 'c', 'C': 'a'})

   - Game 2:

     $c\colon\ (A, C, B)\bullet \qquad \bullet C\colon\ (a, b, c)$

     $b\colon\ (B, C, A)\bullet \qquad \bullet B\colon\ (b, c, a)$

     $a\colon\ (A, C, B)\bullet \qquad \bullet A\colon\ (b, c, a)$

     Solution

     Following the algorithm:

     Suitor optimal: ({a: C, b: B, c: A}) Reviewer optimal: ({A: c, B: b, C: a})

   - Game 3:

d: (A, D, B, C) • • D: (a, d, b, c)

c: (B, A, C, D) • • C: (a, c, d, b)

b: (D, A, C, B) • • B: (d, a, c, b)

a: (A, D, C, B) • • A: (b, d, a, c)

Solution

Following the algorithm:

Suitor optimal: ({a: D, b: A, c: C, d: B}) Reviewer optimal: ({A: b, B: d, C: c, D: a})

- Game 4:

d: (B, A, D, C) • • D: (a, b, d, c)

c: (B, C, A, D) • • C: (d, b, a, c)

b: (A, D, C, B) • • B: (d, a, c, b)

a: (A, D, C, B) • • A: (c, b, d, a)

Solution

Following the algorithm:

Suitor optimal: ({a: D, b: A, c: C, d: B}) Reviewer optimal: ({A: c, B: d, C: b, D: a})

2. Consider a matching game where all reviewers have the same preference list. Prove that there is a single stable matching.

Solution

Let (M) be the suitor optimal matching (given by the Gale-Shapley algorithm).

Assume $\exists\ M' \neq M$. As M is reviewer sub-optimal $\exists$ a subset $\bar{R} \subseteq R$ such that: For all $r \in \bar{R}$: $M^{-1}(r)$ is worse than $M'^{-1}(r)$. For $r \in R \setminus \bar{R}$ $M^{-1}(r) = M'^{-1}(r)$.

Consider $\bar{r} \in \bar{R}$, as all reviewers have same reference list, let r be the reviewer with ''best'' suitor under matching M (the matching given by the Gale Shapley algorithm).

When considering M', reviewers outside of $\bar{R}$ have same matching as in M. All reviewers in $\bar{R}$ must have a ''better'' matching.

As all reviewers have the same preference list, $\bar{r}$ cannot be matched thus M' is not a matching.

3. For the following cooperative games:

i. Verify if the game is monotonic.

ii. Verify if the game is super additive.

iii. Obtain the Shapley value.

$$v_1(C) = \begin{cases} 5, & \text{if } C = \{1\} \\ 3, & \text{if } C = \{2\} \\ 2, & \text{if } C = \{3\} \\ 12, & \text{if } C = \{1,2\} \\ 5, & \text{if } C = \{1,3\} \\ 4, & \text{if } C = \{2,3\} \\ 13, & \text{if } C = \{1,2,3\} \end{cases}$$

$$v_2(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 0, & \text{if } C = \{2\} \\ 5, & \text{if } C = \{1,2\} \end{cases}$$

$$v_4(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 6, & \text{if } C = \{2\} \\ 13, & \text{if } C = \{3\} \\ 6, & \text{if } C = \{1,2\} \\ 13, & \text{if } C = \{1,3\} \\ 13, & \text{if } C = \{2,3\} \\ 26, & \text{if } C = \{1,2,3\} \end{cases}$$

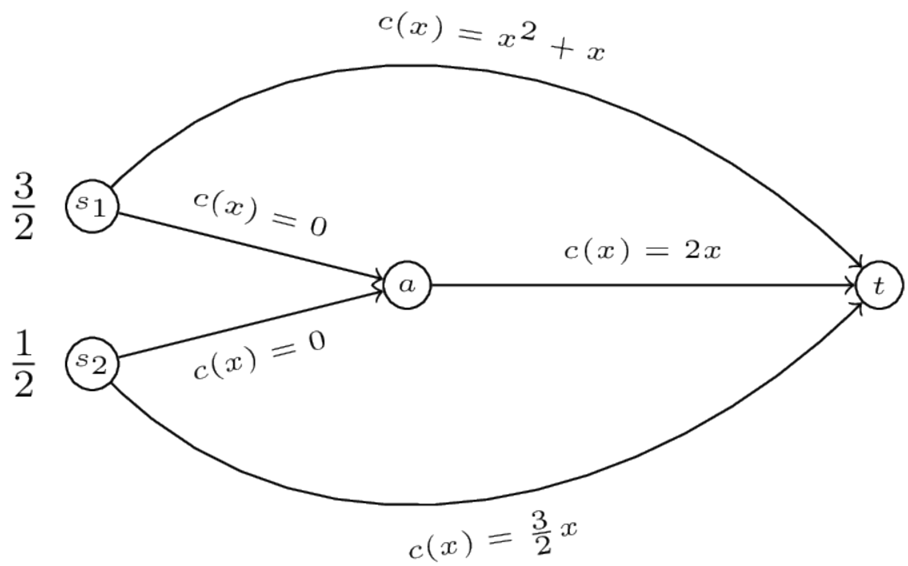
$$v_4(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 7, & \text{if } C = \{2\} \\ 0, & \text{if } C = \{3\} \\ 8, & \text{if } C = \{4\} \\ 7, & \text{if } C = \{1,2\} \\ 6, & \text{if } C = \{1,3\} \\ 12, & \text{if } C = \{1,4\} \\ 7, & \text{if } C = \{2,3\} \\ 12, & \text{if } C = \{2,4\} \\ 8 & \text{if } C = \{3,4\} \\ 7, & \text{if } C = \{1,2,3\} \\ 24, & \text{if } C = \{1,2,4\} \\ 12, & \text{if } C = \{1,3,4\} \\ 12, & \text{if } C = \{2,3,4\} \\ 25, & \text{if } C = \{1,2,3,4\} \end{cases}$$

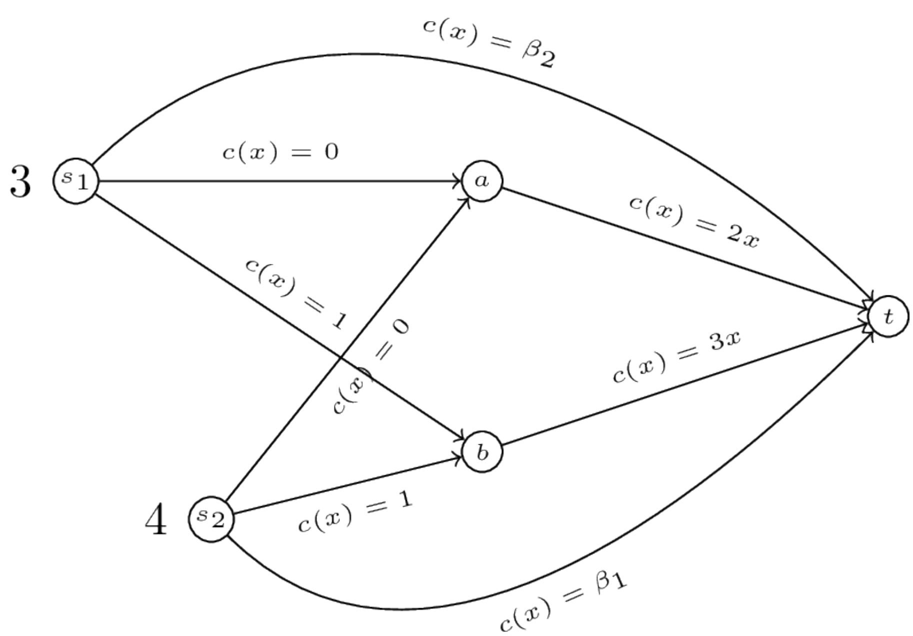
4. Prove that the Shapley value has the following properties:

- Efficiency
- Null player
- Symmetry
- Additivity

Note that this does not prove that the Shapley value is the only vector that has those properties (it in fact is though).

5. Calculate the Nash flow and the optimal flow for the routing games shown in Figures **??** and **??**.





6. For a routing game the 'Price of Anarchy' is defined as:

$$\mathrm{PoA} = \frac{C(\tilde{f})}{C(f^*)}$$

For the game shown in Figure 1 (a generalisation of "Pigou's example") obtain the PoA as a function of α.

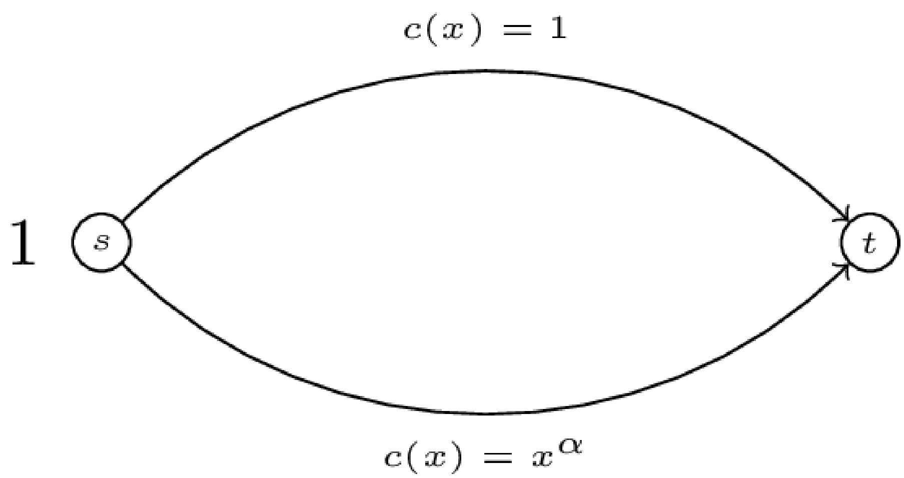


Figure 1: A generalization of Pigou's example

Now obtain the PoA for the game shown in Figure **??** as a function of Λ, α and β. For what value of Λ is the PoA at it's maximum?