OR 3: Chapter 3 - Dominance

Recap

In the previous lecture we discussed:

- Normal form games;
- Mixed strategies and expected utilities.

We spent some time talking about predicting rational behaviour in the above games but we will now look a particular tool in a formal way.

Dominant strategies

In certain games it is evident that certain strategies should never be used by a rational player. To formalise this we need a couple of definitions.

Definition

In an n player normal form game when considering player i we denotes by s_{-i} a fixed strategy profile for all other players in the game.

For example in a 3 player game where $S_i = \{A, B\}$ for all i a valid strategy profile is $s = (A, A, B)$ while $s_{-2} = (A, B)$ denotes the incomplete strategy profile where player 1 and 3 are playing A and B (respectively).

This notation now allows us to define an important notion in game theory.

Definition

In an n player normal form game. A pure strategy $s_i \in S_i$ is said to be **strictly dominated** if there is a strategy (pure or mixed) $\sigma_i \in \Delta S_i$ such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ of the other players.

When attempting to predict rational behaviour we can elimate dominated strategies.

$$\begin{pmatrix} (0,1) & (3,5) \\ (5,10) & (-1,34) \end{pmatrix}$$

If we let $S_1 = \{r_1, r_2\}$ and $S_2 = \{s_1, s_2\}$ we see that:

$$u_2(s_2, r_1) > u_2(s_1, r_1)$$

and

$$u_2(s_2, r_2) > u_2(s_1, r_2)$$

so s_1 is a strictly dominated strategy for player 2. As such we can elimante it from the game when attempting to predict rational behaviour. This gives the following game:

$$\begin{pmatrix} (3,5) \\ (-1,34) \end{pmatrix}$$

At this point it is straightforward to see that r_2 is a strictly dominated strategy for player 1 giving the following predicted strategy profile: $s = (r_1, s_2)$.

Definition

In an n player normal form game. A pure strategy $s_i \in S_i$ is said to be **weakly dominated** if there is a strategy (pure or mixed) $\sigma_i \in \Delta S_i$ such that $u_i(\sigma_i, s_{-i}) \leq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ of the other players and there exists a strategy profile $\bar{s} \in S_{-i}$ such that $u_i(\sigma_i, \bar{s}) < u_i(s_i, s_{-i})$.

We can once again predict rational behaviour by eliminating weakly dominated strategies.

As an example consider the following two player game:

$$\begin{pmatrix} (3,3) & (2,2) \\ (2,1) & (2,1) \end{pmatrix}$$

Using the same convention as before for player 2, s_1 weakly dominates s_2 and for player 1, r_1 weakly dominates r_2 giving the following predicted strategy profile (r_1, s_1).

Common knowledge of rationality

An important aspect of game theory and the tool that we have in fact been using so far is to assume that players are rational. However we can (and need) to go further:

- The players are rational;
- The players all know that the other players are rational;
- The players all know that the other players know that they are rationals;
- ...

This chain of assumptions is called **Common knowledge of rationality** (CKR). By applying the CKR assumption we can attempt to predict rational behaviour through the iterated elimination of dominated strategies (as we have been doing above).

Example

Les us try to predict rational behaviour in the following game using iterated elimination of dominated strategies:

$$\begin{pmatrix}(1,0) & (1,2) & (0,1)\\ (0,3) & (0,1) & (2,0)\end{pmatrix}$$

Initially player 1 has no dominated strategies. For player 2, s_3 is dominated by s_2. Now r_2 is dominated by r_1 for player 1. Finally, s_1 is dominated by s_2. Thus (r_1, s_2) is a predicted rational outcome.

Example

Les us try to predict rational behaviour in the following game using iterated elimination of dominated strategies:

$$\begin{pmatrix}(10,1) & (5,1) & (4,-2)\\ (10,1) & (5,0) & (1,1)\end{pmatrix}$$

- r_1 weakly dominated by r_2
- s_1 strictly dominated by s_3
- s_1 weakly dominated by s_2

Thus (r_1, s_1) is a predicted rational outcome.

Not all games can be solved using dominance

Consider the following two games:

$$\begin{pmatrix} (3,2) & (0,0) \\ (1,1) & (2,3) \end{pmatrix}$$

$$\begin{pmatrix} (1,3) & (4,2) & (2,2) \\ (4,0) & (0,3) & (4,1) \\ (2,5) & (3,4) & (5,6) \end{pmatrix}$$

Why can't we predict rational behaviour using dominance?