

# 1 OR 3: Chapter 15 - Matching games

## 1.1 Recap

In the [previous chapter](#):

- We defined stochastic games;
- We investigated approaches to obtaining Markov strategy Nash equilibria.

In this chapter we will take a look at a very different type of game.

## 1.2 Matching Games

Consider the following situation:

“In a population of  $N$  suitors and  $N$  reviewers. We allow the suitors and reviewers to rank their preferences and are now trying to match the suitors and reviewers in such a way as that every matching is stable.”

If we consider the following example with suitors:  $S = \{a, b, c\}$  and reviewers:  $R = \{A, B, C\}$  with preferences shown in Figure 1.

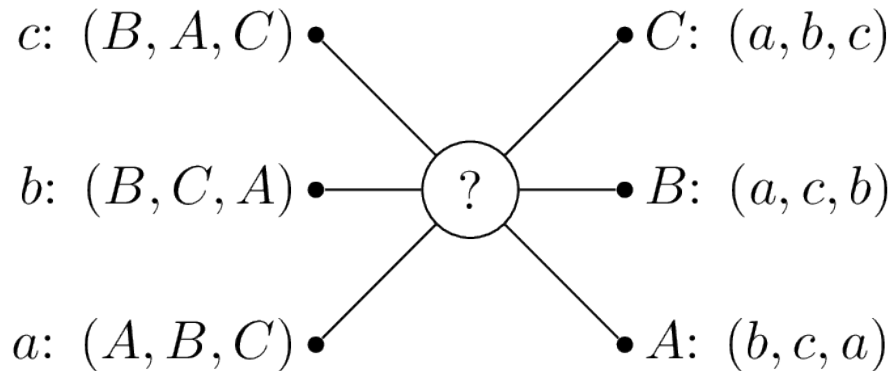


Figure 1: A matching game.

So that  $c$  would prefer to be matched with  $b$ , then  $c$  and lastly  $c$ . One possible matching would be is shown in Figure 2.

In this situation,  $a$  and  $b$  are getting their first choice and  $c$  their second choice. However  $B$  actually prefers  $c$  so that matching is unstable.

Let us write down some formal definitions:

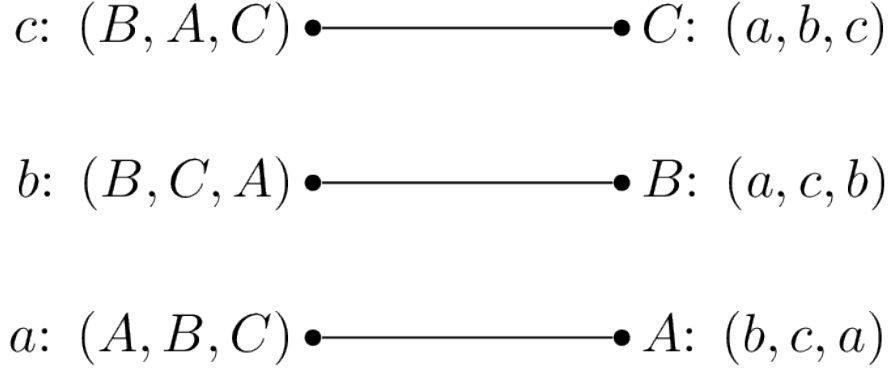


Figure 2: A simple matching.

### 1.2.1 Definition of a matching game

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A matching game of size  $N$  is defined by two disjoint sets  $S$  and  $R$  or suitors and reviewers of size  $N$ . Associated to each element of  $S$  and  $R$  is a preference list:

$$f : S \rightarrow R^N \text{ and } g : R \rightarrow S^N$$

A matching  $M$  is a any bijection between  $S$  and  $R$ . If  $s \in S$  and  $r \in R$  are matched by  $M$  we denote:

$$M(s) = r$$


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### 1.2.2 Definition of a blocking pair

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A pair  $(s, r)$  is said to **block** a matching  $M$  if  $M(s) \neq r$  but  $s$  prefers  $r$  to  $M(s)$  and  $r$  prefers  $s$  to  $M^{-1}(r)$ .

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In our previous example  $(c, B)$  blocks the proposed matching.

### 1.2.3 Definition of a stable matching

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A matching  $M$  with no blocking pair is said to be stable.

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A stable matching is shown in Figure 3.

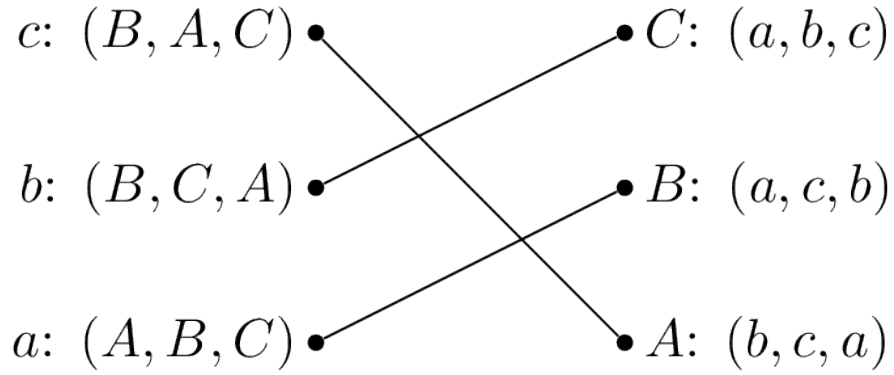


Figure 3: A stable matching.

The stable matching is not unique, the matching shown in Figure 4 is also stable:

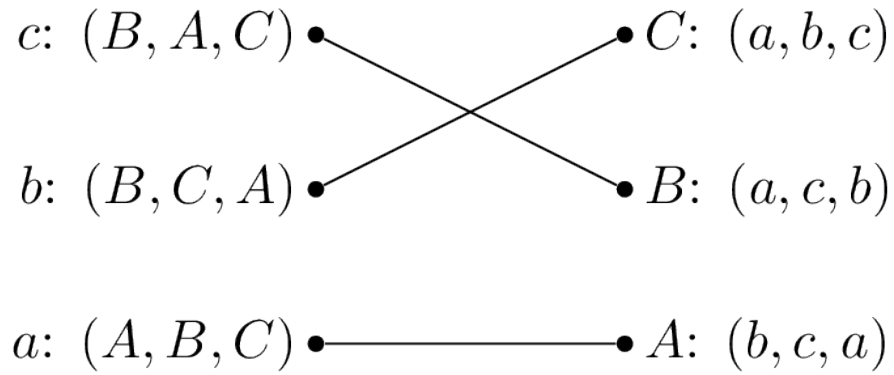


Figure 4: Another stable matching.

### 1.3 The Gale-Shapley Algorithm

Here is the Gale-Shapley algorithm, which gives a stable matching for a matching game:

1. Assign every  $s \in S$  and  $r \in R$  to be unmatched
2. Pick some unmatched  $s \in S$ , let  $r$  be the top of  $s$ 's preference list:
  - If  $r$  is unmatched set  $M(s) = r$
  - If  $r$  is matched:
    - If  $r$  prefers  $s$  to  $M^{-1}(r)$  then set  $M(r) = s$
    - Otherwise  $s$  remains unmatched and remove  $r$  from  $s$ 's preference list.
3. Repeat step 2 until all  $s \in S$  are matched.

Let us illustrate this algorithm with the above example (shown again in Figure 10).

$$\begin{array}{ll}
 c: (B, A, C) \bullet & \bullet C: (a, b, c) \\
 b: (B, C, A) \bullet & \bullet B: (a, c, b) \\
 a: (A, B, C) \bullet & \bullet A: (b, c, a)
 \end{array}$$

Figure 5: Base example.

We pick  $c$  and as all the reviewers are unmatched set  $M(c) = B$  as shown in Figure 6.

We pick  $b$  and as  $B$  is matched but prefers  $c$  to  $b$  we cross out  $B$  from  $b$ 's preferences as shown in Figure 7.

We pick  $b$  again and set  $M(b) = C$  as shown in Figure 8.

We pick  $a$  and set  $M(a) = A$  as shown in Figure 9.

**Let us repeat the algorithm but pick  $b$  as our first suitor.**

**We initialise the game as shown in Figure 10.**

We pick  $b$  and as all the reviewers are unmatched set  $M(b) = B$  as shown in Figure 11.

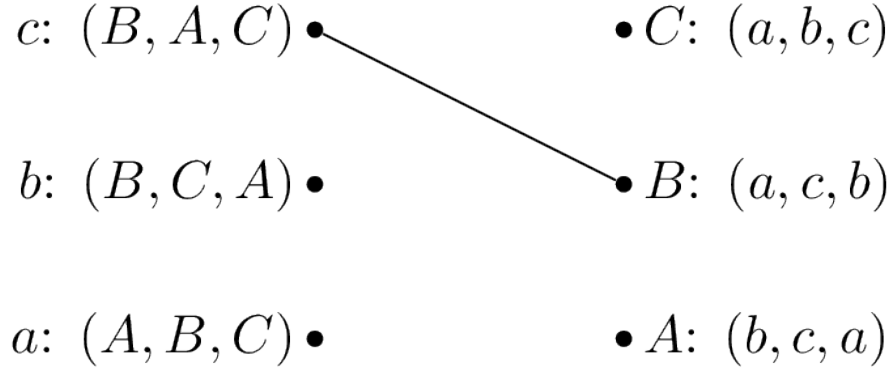


Figure 6: Setting  $M(c) = B$ .

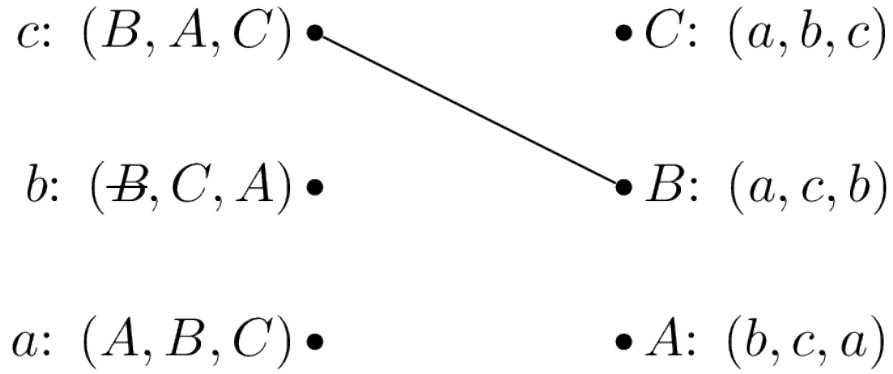


Figure 7: Removing  $B$  from  $b$ 's preference list.

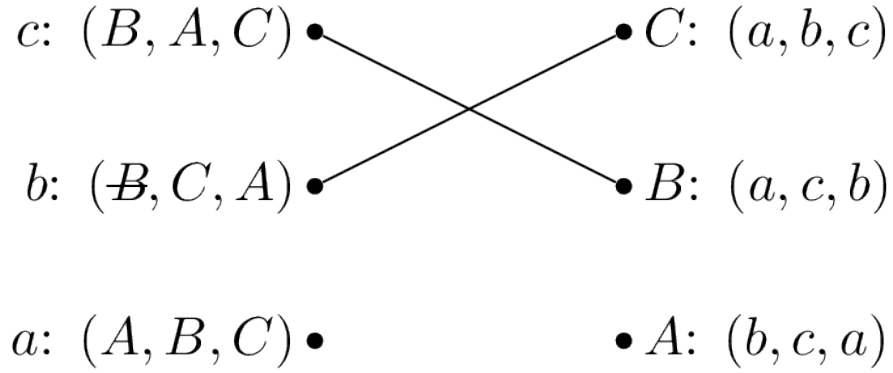


Figure 8: Setting  $M(b) = C$ .

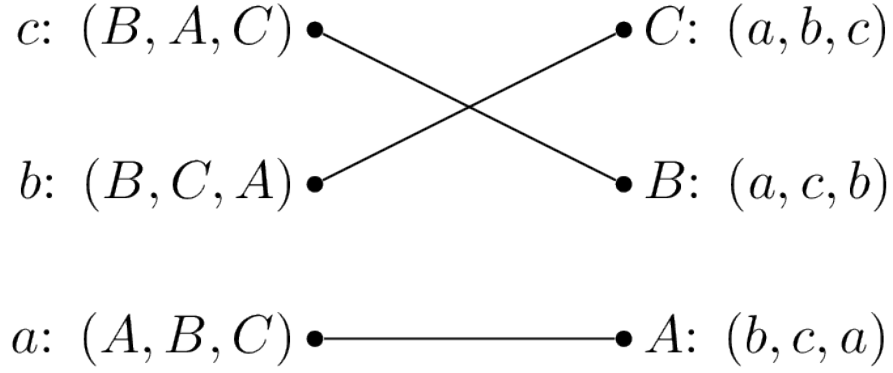


Figure 9: Setting  $M(a) = A$ .

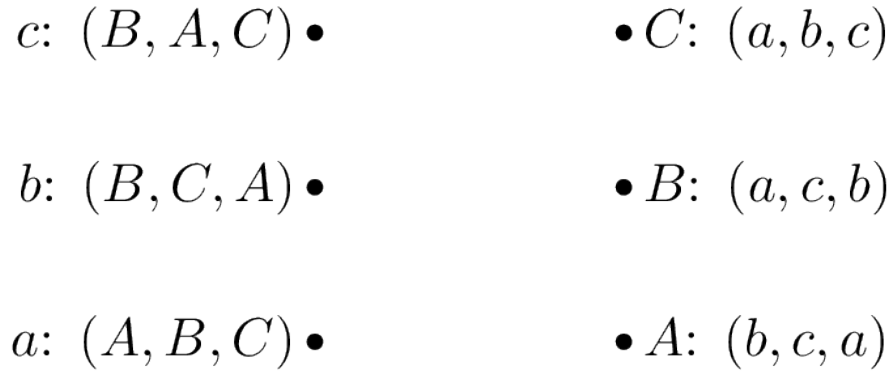


Figure 10: Base example.

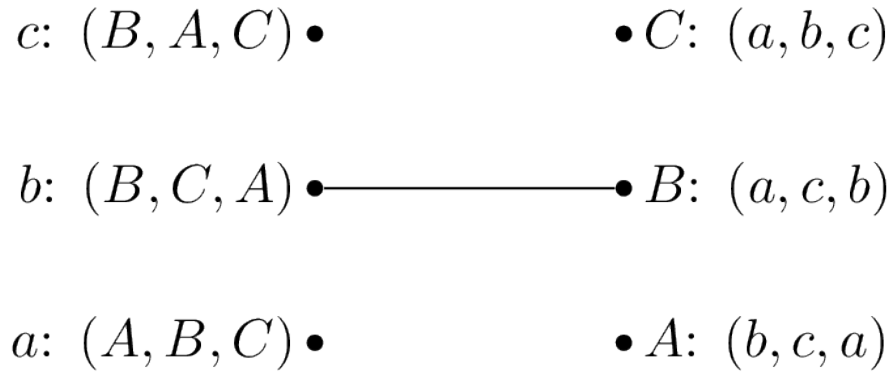


Figure 11: Setting  $M(b) = B$ .

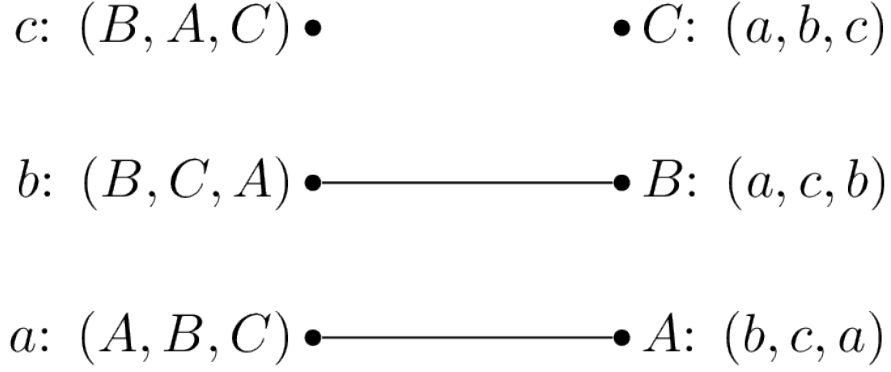


Figure 12: Setting  $M(a) = A$ .

We pick  $a$  and as  $A$  is unmatched set  $M(a) = A$  as shown in Figure 12.

We pick  $c$  and  $b$  is matched but prefers  $c$  to  $M^{-1}(B) = b$ , we set  $M(c) = B$  as shown in Figure 13.

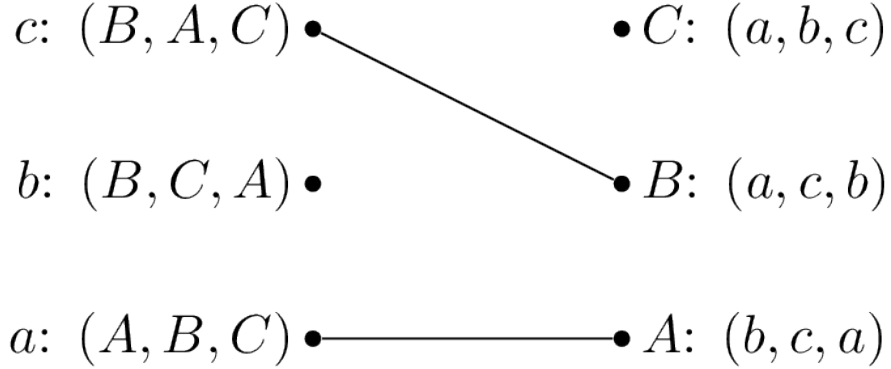


Figure 13: Setting  $M(c) = B$ .

We pick  $b$  and as  $B$  is matched but prefers  $c$  to  $b$  we cross out  $B$  from  $b$ 's preferences:

We pick  $b$  again and set  $M(b) = C$  as shown in Figure 15.

Both these have given the same matching.

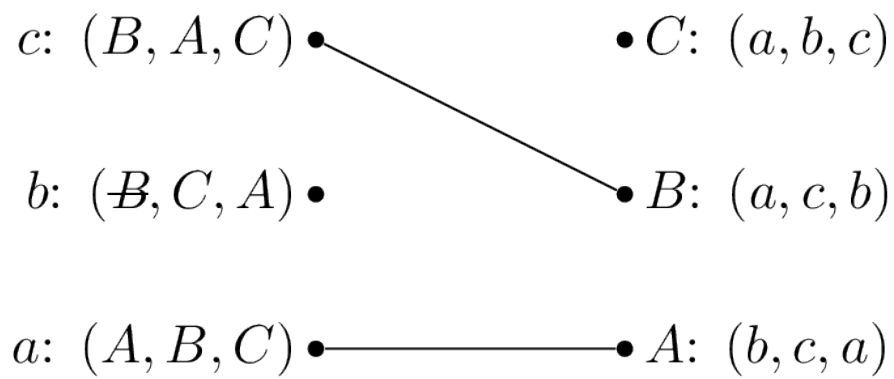


Figure 14: Removing  $B$  from  $b$ 's preference list.

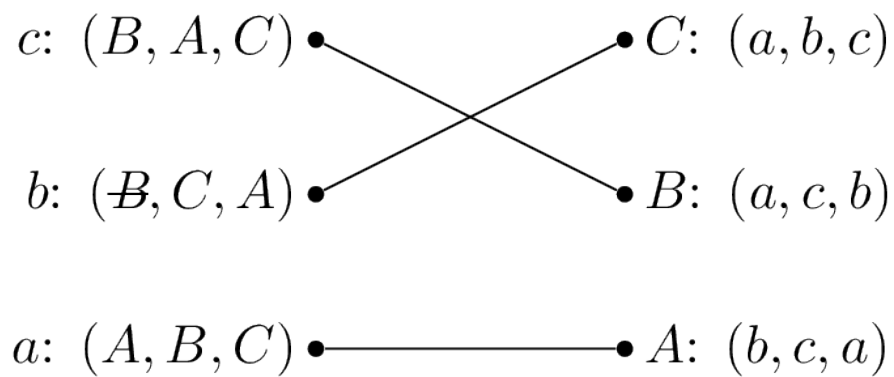


Figure 15: Setting  $M(b) = C$ .



### 1.3.1 Theorem guaranteeing a unique matching as output of the Gale Shapley algorithm.

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All possible executions of the Gale-Shapley algorithm yield the same stable matching **and** in this stable matching every suitor has the best possible partner in any stable matching.

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### 1.3.2 Proof

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Suppose that an arbitrary execution  $\alpha$  of the algorithm gives  $M$  and that another execution  $\beta$  gives  $M'$  such that  $\exists s \in S$  such that  $s$  prefers  $r' = M'(s)$  to  $r = M(s)$ .

Without loss of generality this implies that during  $\alpha$   $r'$  must have rejected  $s$ . Suppose, again without loss of generality that this was the first occasion that a rejection occurred during  $\alpha$  and assume that this rejection occurred because  $r' = M'(s')$ . This implies that  $s'$  has no stable match that is higher in  $s'$ 's preference list than  $r'$  (as we have assumed that this is the first rejection).

Thus  $s'$  prefers  $r'$  to  $M'(s')$  so that  $(s', r')$  blocks  $M'$ . Each suitor is therefore matched in  $M$  with his favorite stable reviewer and since  $\alpha$  was arbitrary it follows that all possible executions give the same matching.

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We call a matching obtained from the Gale Shapley algorithm *suitor-optimal* because of the previous theorem. The next theorem shows another important property of the algorithm.

### 1.3.3 Theorem of reviewer sub optimality

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In a suitor-optimal stable matching each reviewer has the worst possible matching.

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#### 1.3.4 Proof

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Assume that the result is not true. Let  $M_0$  be a suitor-optimal matching and assume that there is a stable matching  $M'$  such that  $\exists r$  such that  $r$  prefers  $s = M_0^{-1}(r)$  to  $s' = M'^{-1}(r)$ . This implies that  $(r, s)$  blocks  $M'$  unless  $s$  prefers  $M'(s)$  to  $s$  which contradicts the fact the  $s$  has no stable match that he prefers in  $M_0$ .

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