OR 3: Lecture 2 - Normal Form Games

Recap

In the previous lecture we discussed:

- Interactive decision making;
- Normal form games;
- Normal form games and representing information sets.

We did this looking at a game called "the battle of the sexes":

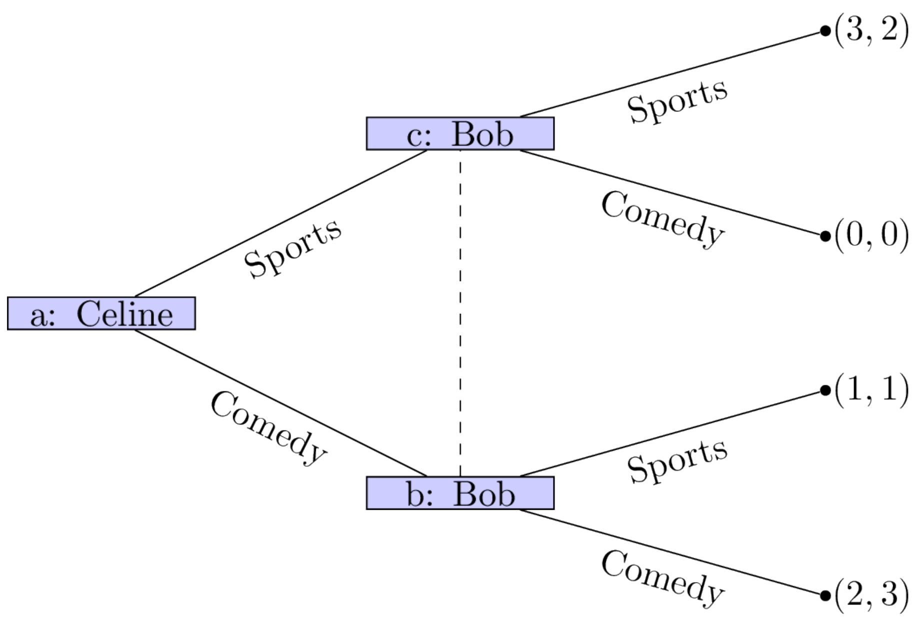

Figure 1: Celine and Bob with Information Set

Can we think of a better way of representing this game?

Normal form games

One other representation for a game is called the **normal form**.

Definition

A n player **normal form game** consists of:

1. A finite set of n players;
2. Strategy spaces for the players: $S_1, S_2, S_3, \dots S_n$;
3. Payoff functions for the players: $u_i : S_1 \times S_2 \dots \times S_n \to \mathbb{R}$

The convention used in this course (unless otherwise stated) is that all players aim to choose from their strategies in such a way as to maximise their utilities.

A natural way of representing a two player normal form game is using a **bi-matrix**. If we assume that $S_1 = \{r_i \mid 1 \leq i \leq m\}$ and $S_2 = \{s_j \mid 1 \leq j \leq n\}$ then the following is a **bi-matrix** representation of the game considered:

$$\begin{array}{cc} & \text{Column strategies (player 2)} \\ \text{Row strategies (player 1)} & \begin{pmatrix} (u_1(r_1, s_1), u_2(r_1, s_1)) & (u_1(r_1, s_2), u_2(r_1, s_2)) & \dots & (u_1(r_1, s_2), u_2(r_1, s_n)) \\ (u_1(r_2, s_1), u_2(r_2, s_1)) & (u_1(r_2, s_2), u_2(r_2, s_2)) & \dots & (u_1(r_2, s_2), u_2(r_2, s_n)) \\ \vdots & \dots & \dots & \vdots \\ (u_1(r_m, s_1), u_2(r_m, s_1)) & (u_1(r_m, s_2), u_2(r_m, s_2)) & \dots & (u_1(r_m, s_2), u_2(r_m, s_n)) \end{pmatrix} \end{array}$$

Figure 2: A bi matrix

Some examples

The battle of the sexes

This is the game we've been looking at between Bob and Celine:

$$\begin{pmatrix} (3,2) & (0,0) \\ (1,1) & (2,3) \end{pmatrix}$$

Prisoners' Dilemma

> Assume two thieves have been caught by the police and separated for questioning. If both thieves cooperate and don't divulge any information they will each get a short sentence. If one defects he/she is offered a deal while the other thief will get a long sentence. If they both defect they both get a medium sentence.

$$\begin{pmatrix} (2,2) & (0,3) \\ (3,0) & (1,1) \end{pmatrix}$$

Hawk-Dove/Chicken

> Suppose...

$$\begin{pmatrix} (0,0) & (3,1) \\ (1,3) & (2,2) \end{pmatrix}$$

Coordination

> Suppose...

$$\begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (1,1) \end{pmatrix}$$

Pareto Coordination

> Suppose...

$$\begin{pmatrix} (2,2) & (0,0) \\ (0,0) & (1,1) \end{pmatrix}$$

Pigs

> Suppose...

$$\begin{pmatrix} (4,2) & (2,3) \\ (6,-1) & (0,0) \end{pmatrix}$$

Mixed Strategies

So far we have only considered so called **pure strategies**. We will now allow players to play **mixed strategies**.

Definition

In an n player normal form game a **mixed strategy** for player i denoted by $\sigma_i \in [0,1]_{\mathbb{R}}^{|S_i|}$ is a probability distribution over the pure strategies of player i.

For example in the matching pennies game discussed previously. A strategy profile of $\sigma_1 = (.2, .8)$ and $\sigma_2 = (.6, .4)$ implies that player 1 plays heads with probability .2 and player 2 plays heads with probability .6.

We can extend the utility function which maps from the set of pure strategies to $\mathbb{R}$ using *expected payoffs*. For a two player game we have:

$$u_i(\sigma_1, \sigma_2) = \sum_{r \in S_1, s \in S_2} \sigma_1(r)\sigma_2(s)u_i(r, s)$$

(where we relax our notation to allow $\sigma_i : S_i \to [0,1]_{\mathbb{R}}$)

Matching pennies revisited.

In the previously discussed strategy profile of $\sigma_1 = (.2, .8)$ and $\sigma_2 = (.6, .4)$ the expected utilities can be calculated as follows:

$$u_1(\sigma_1, \sigma_2) = \sum_{r \in S_1, s \in S_2} \sigma_1(r)\sigma_2(s)u_1(r, s) = .2\times.6\times1 + .2\times.4\times(-1) + .8\times.6\times(-1) + .8\times.4\times1 = -.12$$

$$u_2(\sigma_1, \sigma_2) = \sum_{r \in S_1, s \in S_2} \sigma_1(r)\sigma_2(s)u_2(r, s) = .2\times.6\times(-1) + .2\times.4\times1 + .8\times.6\times1 + .8\times.4\times(-1) = .12$$

Example

If we assume that player 2 always plays tails, what is the expected utility to player 1?

Let $\sigma_1 = (x, 1-x)$ and we have $\sigma_2 = (0, 1)$ which gives:

$$u_1(\sigma_1, \sigma_2) = \sum_{r\in S_1, s\in S_2} \sigma_1(r)\sigma_2(s)u_1(r, s) = -x + (1-x) = 1 - 2x$$

Similarly if player 1 always plays tails the expected utility to player 2 is:

$$u_2(\sigma_1, \sigma_2) = \sum_{r\in S_1, s\in S_2} \sigma_1(r)\sigma_2(s)u_2(r, s) = x + (x-1) = 2x - 1$$

A plot of this is shown here:

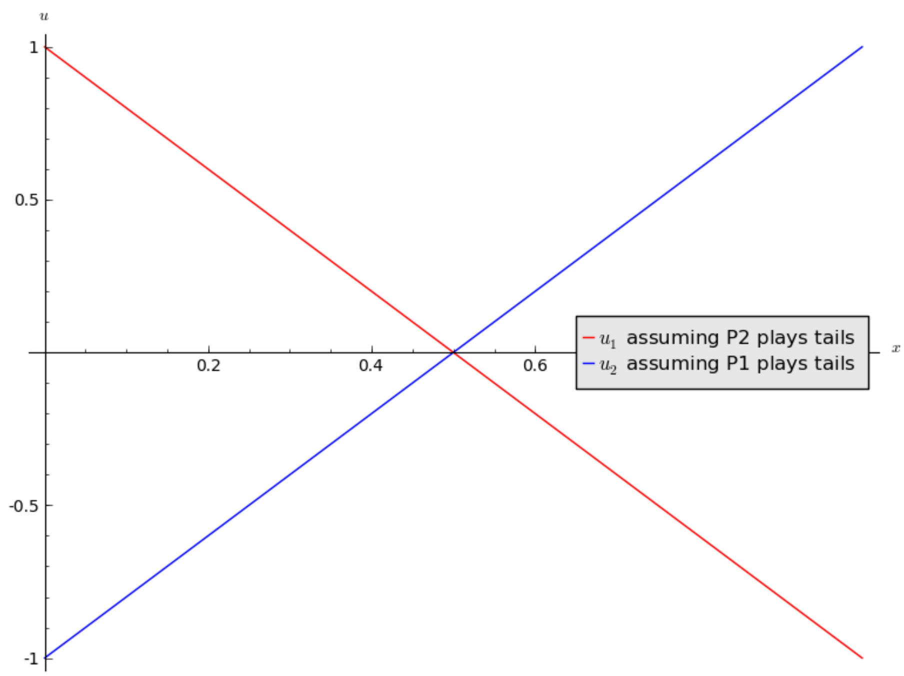

Figure 3:

Add to this plot by assuming that the players independently both play heads.