1 OR 3: Chapter 16 - Cooperative games

1.1 Recap

In the previous chapter:

- We defined matching games;
- We described the Gale-Shapley algorithm;
- We proved certain results regarding the Gale-Shapley algorithm.

In this Chapter we'll take a look at another type of game.

1.2 Cooperative Games

In cooperative game theory the interest lies with understanding how coalitions form in competitive situations.

1.2.1 Definition of a characteristic function game

A **characteristic function game** G is given by a pair (N, v) where N is the number of players and $v : 2^{[N]} \to \mathbb{R}$ is a **characteristic function** which maps every coalition of players to a payoff.

Let us consider the following game:

> "3 players share a taxi. Here are the costs for each individual journey:
> - Player 1: 6 - Player 2: 12 - Player 3: 42 How much should each individual contribute?"

This is illustrated in Figure 1.

To construct the characteristic function we first obtain the power set (ie all possible coalitions) $2^{\{1,2,3\}} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \Omega\}$ where Ω denotes the set of all players ($\{1, 2, 3\}$).

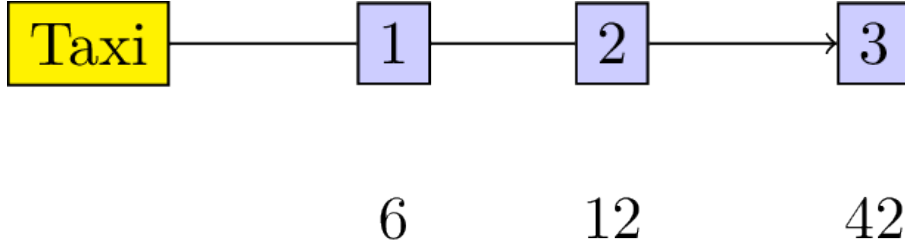The characteristic function is given below:

Figure 1: A taxi journey.

$$v(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 12, & \text{if } C = \{2\} \\ 42, & \text{if } C = \{3\} \\ 12, & \text{if } C = \{1,2\} \\ 42, & \text{if } C = \{1,3\} \\ 42, & \text{if } C = \{2,3\} \\ 42, & \text{if } C = \{1,2,3\} \end{cases}$$

1.2.2 Definition of a monotone characteristic function game

A characteristic function game $G = (N, v)$ is called **monotone** if it satisfies $v(C_2) \geq v(C_1)$ for all $C_1 \subseteq C_2$.

This is illustrated in Figure 2.

Our taxi example is monotone, however the $G = (3, v_1)$ with v_1 defined as:

$$v_1(C) = \begin{cases} 6, & \text{if } C = \{1\} \\ 12, & \text{if } C = \{2\} \\ 42, & \text{if } C = \{3\} \\ 10, & \text{if } C = \{1,2\} \\ 42, & \text{if } C = \{1,3\} \\ 42, & \text{if } C = \{2,3\} \\ 42, & \text{if } C = \{1,2,3\} \end{cases}$$

is not.

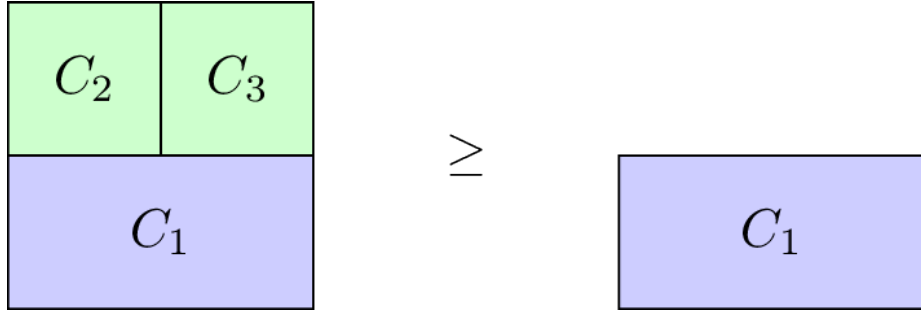Figure 2: A diagrammatic representation of monotonicity.

1.2.3 Definition of a superaddtive game

A characteristic function game $G = (N, v)$ is called **superadditive** if it satisfies $v(C_1 \cup C_2) \geq v(C_1) + v(C_2)$.
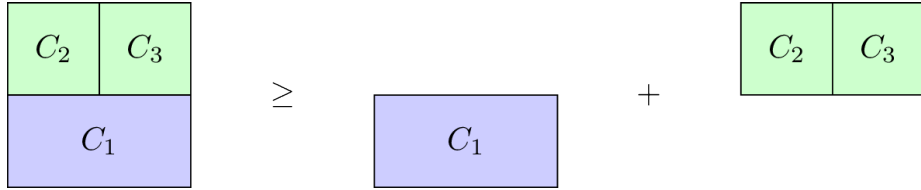

This is illustrated in Figure 3.

Figure 3: A diagrammatic representation of superadditivity.

Our taxi example is not superadditive, however the $G = (3, v_2)$ with v_2 defined as:

$$
v_2(C) = \begin{cases}
6, & \text{if } C = \{1\} \\
12, & \text{if } C = \{2\} \\
42, & \text{if } C = \{3\} \\
18, & \text{if } C = \{1, 2\} \\
48, & \text{if } C = \{1, 3\} \\
55, & \text{if } C = \{2, 3\} \\
80, & \text{if } C = \{1, 2, 3\}
\end{cases}
$$

is.

1.3 Shapley Value

When talking about a solution to a characteristic function game we imply a payoff vector $\lambda \in \mathbb{R}_{\geq 0}^N$ that divides the value of the grand coalition between the various players. Thus λ must satisfy:

$$\sum_{i=1}^{N} \lambda_i = v(\Omega)$$

Thus one potential solution to our taxi example would be $\lambda = (14, 14, 14)$. Obviously this is not ideal for player 1 and/or 2: they actually pay more than they would have paid without sharing the taxi!

Another potential solution would be $\lambda = (6, 6, 30)$, however at this point sharing the taxi is of no benefit to player 1. Similarly $(0, 12, 30)$ would have no incentive for player 2.

To find a "fair" distribution of the grand coalition we must define what is meant by "fair". We require four desirable properties:

- Efficiency;
- Null player;
- Symmetry;
- Additivity.

1.3.1 Definition of efficiency

For $G = (N, v)$ a payoff vector λ is **efficient** if:

$$\sum_{i=1}^{N} \lambda_i = v(\Omega)$$

1.3.2 Definition of null players

For $G(N, v)$ a payoff vector possesses the **null player property** if $v(C \cup i) = v(C)$ for all $C \in 2^\Omega$ then:

$$x_i = 0$$

1.3.3 Definition of symmetry

For $G(N, v)$ a payoff vector possesses the **symmetry property** if $v(C \cup i) = v(C \cup j)$ for all $C \in 2^{\Omega} \setminus \{i, j\}$ then:

$$x_i = x_j$$

1.3.4 Definition of additivity

For $G_1 = (N, v_1)$ and $G_2 = (N, v_2)$ and $G^+ = (N, v^+)$ where $v^+(C) = v_1(C) + v_2(C)$ for any $C \in 2^{\Omega}$. A payoff vector possesses the **additivity property** if:

$$x_i^{(G^+)} = x_i^{(G_1)} + x_i^{(G_2)}$$

We will not prove in this course but in fact there is a single payoff vector that satisfies these four properties. To define it we need two last definitions.

1.3.5 Definition of predecessors

If we consider any permutation π of $[N]$ then we denote by $S_{\pi}(i)$ the set of **predecessors** of i in π:

$$S_{\pi}(i) = \{j \in [N] \mid \pi(j) < \pi(i)\}$$

For example for $\pi = (1, 3, 4, 2)$ we have $S_{\pi}(4) = \{1, 3\}$.

1.3.6 Definition of marginal contribution

If we consider any permutation π of $[N]$ then the **marginal contribution** of player i with respect to π is given by:

$$\Delta_\pi^G(i) = v(S_\pi(i) \cup i) - v(S_\pi(i))$$

We can now define the **Shapley value** of any game $G = (N, v)$.

1.3.7 Definition of the Shapley value

Given $G = (N, v)$ the **Shapley value** of player i is denoted by $\phi_i(G)$ and given by:

$$\phi_i(G) = \frac{1}{N!} \sum_{\pi \in \Pi_n} \Delta_\pi^G(i)$$

As an example here is the Shapley value calculation for our taxi sharing game:

For $\pi = (1, 2, 3)$:

$$\Delta_\pi^G(1) = 6$$
$$\Delta_\pi^G(2) = 6$$
$$\Delta_\pi^G(3) = 30$$

For $\pi = (1, 3, 2)$:

$$\Delta_\pi^G(1) = 6$$
$$\Delta_\pi^G(2) = 0$$
$$\Delta_\pi^G(3) = 36$$

For $\pi = (2, 1, 3)$:

$$\Delta_\pi^G(1) = 0$$
$$\Delta_\pi^G(2) = 12$$
$$\Delta_\pi^G(3) = 30$$

For $\pi = (2, 3, 1)$:

$$\Delta_\pi^G(1) = 0$$
$$\Delta_\pi^G(2) = 12$$
$$\Delta_\pi^G(3) = 30$$

For $\pi = (3, 1, 2)$:

$$\Delta_\pi^G(1) = 0$$
$$\Delta_\pi^G(2) = 0$$
$$\Delta_\pi^G(3) = 42$$

For $\pi = (3, 2, 1)$:

$$\Delta_\pi^G(1) = 0$$
$$\Delta_\pi^G(2) = 12$$
$$\Delta_\pi^G(3) = 42$$

Using this we obtain:

$$\phi(G) = (2, 5, 35)$$

Thus the fair way of sharing the taxi fare is for player 1 to pay 2, player 2 to pay 5 and player 3 to pay 35.