OR 3: Chapter 4 - Normal Form Games

Recap

In the previous lecture we discussed:

- Predicting rational behaviour using dominated strategies;
- The CKR;

We did discover certain games that did not have any dominated strategies.

Best response functions

Definition

In an n player normal form game. A strategy s^* for player i is a best response to some strategy profile s_{-i} if and only if $u_i(s^*,s_{-i})\geq u_i(s,s_{-i})$ for all $s\in S_i$.

We can now start to predict rational outcomes in pure strategies by identifying all best responses to a strategy.

$$\begin{pmatrix}(1,3)&(4,2)&(2,2)\\(4,0)&(0,3)&(4,1)\\(2,5)&(3,4)&(5,6)\end{pmatrix}$$

We will underline the best responses for each strategy giving:

$$\begin{pmatrix}(1,\underline{3})&(\underline{4},2)&(2,2)\\(\underline{4},0)&(0,\underline{3})&(4,1)\\(2,5)&(3,4)&(\underline{5},\underline{6})\end{pmatrix}$$

We see that (r_1,s_1) represented a pair of best responses. What can we say about the long term behaviour of this game?

Best responses against mixed strategies

We can identify best responses against mixed strategies. Let us take a look at the matching pennies game:

$$\begin{pmatrix}(1,-1) & (-1,1)\\(-1,1) & (1,-1)\end{pmatrix}$$

If we assume that player 2 plays a mixed strategy $\sigma_2 = (x, 1-x)$ we have:

$$u_1(r_1, \sigma_2) = 1 - 2x$$

and

$$u_1(r_2, \sigma_2) = 2x - 1$$

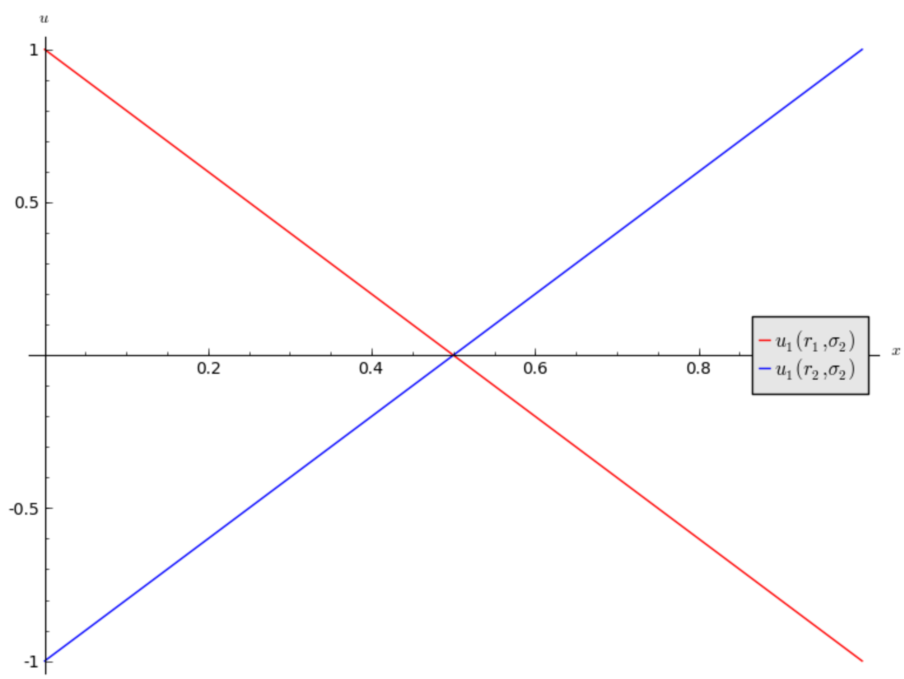

Figure 1:

1. If $x < 1/2$ then r_1 is a best response for player 1.
2. If $x > 1/2$ then r_1 is a best response for player 1.

3. If $x = 1/2$ then player 1 is indifferent.

Let us repeat this exercise for the battle of the sexes game.

$$\begin{pmatrix} (3,2) & (0,0) \\ (1,1) & (2,3) \end{pmatrix}$$

If we assume that player 2 plays a mixed strategy $\sigma_2 = (x, 1-x)$ we have:

$$u_1(r_1, \sigma_2) = 3x$$

and

$$u_1(r_2, \sigma_2) = 2 - x$$

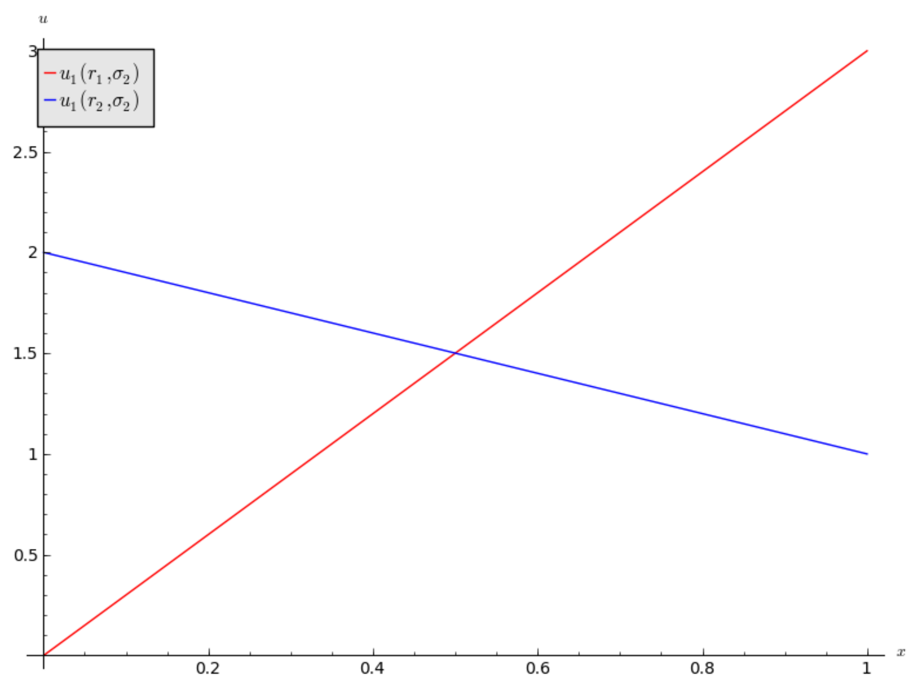

Figure 2:

1. If $x < 1/2$ then r_2 is a best response for player 1.
2. If $x > 1/2$ then r_1 is a best response for player 1.
3. If $x = 1/2$ then player 1 is indifferent.

Connection between best responses and dominance

Definition

In an n player normal form game, let us define the set UD_i:

$$UD_i = \{s \in S_i \mid \text{s is not strictly dominated}\}$$

If we consider the following game:

$$\begin{pmatrix} (3,2) & (7,2) & (5,1) \\ (5,1) & (6,3) & (7,-1) \end{pmatrix}$$

We have:

$$UD_1 = \{r_1, r_2\}$$
$$UD_2 = \{s_1, s_2\}$$

Definition

In an n player normal form game, let us define the set B_i:

$$B_i = \{s \in S_i \mid \exists\ \sigma \in \Delta S_{-i} \text{ such that } s \text{ is a best response to } \sigma\}$$

In other words B_i is the set of functions that are best responses to some strategy profile in S_{-i}.

Let us try to identify B_2 for the above game. Let us assume that player 1 plays $\sigma_1 = (x, 1-x)$. This gives:

$$u_2(\sigma, s_1) = 1 + 3x$$
$$u_2(\sigma, s_2) = 3 - x$$

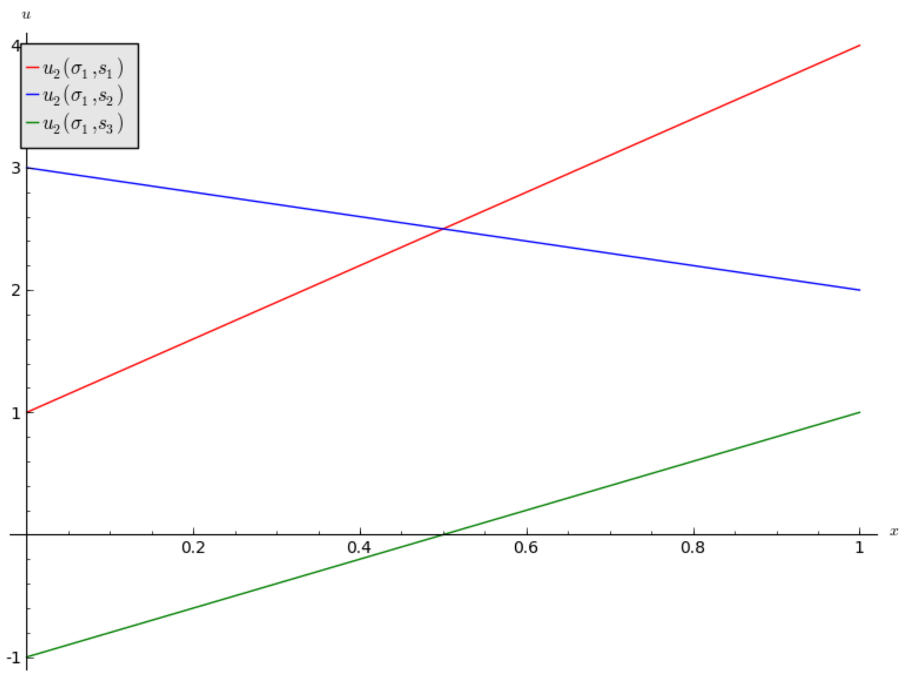

Figure 3:

$$u_2(\sigma, s_3) = 2x - 1$$

We see that s_3 is never a best response for player 2:

$$B_2 = \{s_1, s_2\}$$

We will now attempt to identify B_1 for the above game. Let us assume that player two plays $\sigma_2 = (x, y, 1 - x - y)$. This gives:

$$u_1(r_1, \sigma_2) = xu_1(r_1, s_1) + yu_2(r_1, s_2) + (1 - x - y)u_3(r_1, s_3)$$
$$u_1(r_2, \sigma_2) = xu_1(r_2, s_1) + yu_2(r_2, s_2) + (1 - x - y)u_3(r_2, s_3)$$

However as noted earlier s_3 is dominated by s_2 so:

$$u_1(r_1, \sigma_2) < xu_1(r_1, s_1) + u_2(r_1, s_2)(1 - x) = 7 - 4x$$
$$u_1(r_2, \sigma_2) < xu_1(r_2, s_1) + u_2(r_2, s_2)(1 - x) = 6 - x$$

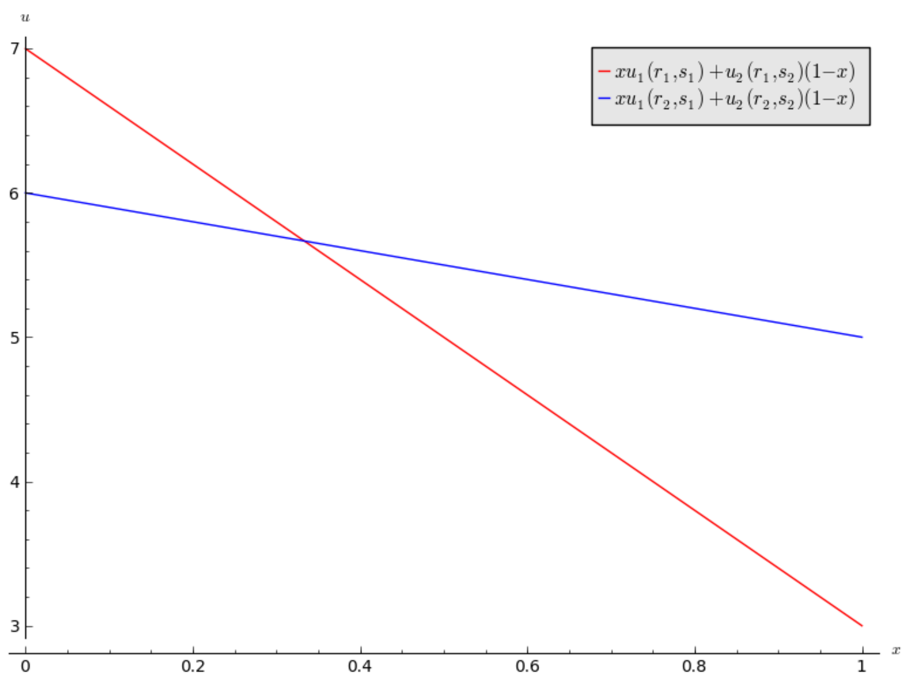

Figure 4:

We see that r_1 and r_2 are best responses for player 1:

$$B_1 = \{r_1, r_2\}$$

We have seen in our little example that $B_i = UD_i$. This leads us to two Theorems (the proofs are omitted).

Theorem

In a 2 player normal form game $B_i = UD_i$ for all $i \in \{1, 2\}$.

This is however not always the case:

In an n player normal form game $B_i \subseteq ID_i$ for all $1 \leq i \leq n$.