4. (a) Provide definitions for the following terms:

- Normal form game.

  A N player **normal form game** consists of:
  - A finite set of N players;
  - Strategy spaces for the players: $S_1, S_2, S_3, \dots S_N$;
  - Payoff functions for the players: $u_i : S_1 \times S_2 \cdots \times S_N \to \mathbb{R}$

  [1]

- Strictly dominated strategy.

  In an N player normal form game. A pure strategy $s_i \in S_i$ is said to be **strictly dominated** if there is a strategy $\sigma_i \in \Delta S_i$ such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ of the other players.

  [1]

- Weakly dominated strategy.

  In an N player normal form game. A pure strategy $s_i \in S_i$ is said to be **weakly dominated** if there is a strategy $\sigma_i \in \Delta S_i$ such that $u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ of the other players and there exists a strategy profile $\bar{s} \in S_{-i}$ such that $u_i(\sigma_i, \bar{s}) > u_i(s_i, s_{-i})$.

  [1]

- Best response strategy.

  In an N player normal form game. A strategy s^* for player i is a best response to some strategy profile s_{-i} if and only if $u_i(s^*, s_{-i}) \geq u_i(s, s_{-i})$ for all $s \in S_i$.

  [1]

- Nash equilibrium.

  In an N player normal form game. A Nash equilibrium is a strategy profile $\tau = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_N)$ such that:

  $$u_i(\tilde{s}) \geq u_i(\bar{s}_i, \tilde{s}_{-i}) \text{ for all } i$$

  [1]

(b) Consider the following game:

$$\begin{pmatrix} (1,\alpha) & (0,2) \\ (0,0) & (\alpha,1) \end{pmatrix}$$

(i) Prove that a pure Nash equilibrium exists for all values of $\alpha \in \mathbb{R}$.
If $\alpha \leq 0$ then the best responses are given by:

$$\begin{pmatrix} (\underline{1},\alpha) & (\underline{0},\underline{2}) \\ (0,0) & (\alpha,\underline{1}) \end{pmatrix}$$

So (r_1, c_2) is a pure Nash equilibrium.

[2]

If $2 \geq \alpha \geq 0$ then the best responses are given by:

$$\begin{pmatrix} (\underline{1},\alpha) & (0,\underline{2}) \\ (0,0) & (\underline{\alpha},\underline{1}) \end{pmatrix}$$

So (r_2, c_2) is a pure Nash equilibrium.

[2]

If $2 \leq \alpha$ then the best responses are given by:

$$\begin{pmatrix} (\underline{1},\underline{\alpha}) & (0,2) \\ (0,0) & (\underline{\alpha},\underline{1}) \end{pmatrix}$$

So (r_1, c_1) and (r_2, c_2) are pure Nash equilibrium.

[3]

(ii) State the equality of payoffs theorem. Using this theorem obtain the value of α (if it exists) for which the following (σ_1, σ_2) are mixed Nash equilibria for the game.
The equality of payoffs theorem states:
In an N player normal form game if the strategy profile (σ_i, s_{-i}) is a Nash equilibria then:

$$u_i(\sigma_i, s_{-i}) = u_i(s, s_{-i}) \text{ for all } s \in \mathcal{S}(\sigma_i) \text{ for all } 1 \leq i \leq N$$

[1]

A. $(\sigma_1, \sigma_2) = ((1/2, 1/2), (1/2, 1/2))$
If $((1/2, 1/2), (1/2, 1/2))$ is a Nash equilibrium the equality of payoffs theorem states:

$$u_2((1/2, 1/2), c_1) = u_2((1/2, 1/2), c_2)$$

which implies:

$$\alpha/2 = 3/2$$

[2]

So $\alpha = 3$. If $\alpha = 3$ then

$$u_1(r_1, (1/2, 1/2)) = 1/2 \quad u_1(r_2, (1/2, 1/2)) = 3/2$$

Thus by the equality of payoffs theorem this is not a Nash equilibrium.

[2]

B. $(\sigma_1, \sigma_2) = ((1/2, 1/2), (3/4, 1/4))$
If $((1/2, 1/2), (3/4, 1/4))$ is a Nash equilibrium the equality of payoffs theorem states:

$$u_2((1/2, 1/2), c_1) = u_2((1/2, 1/2), c_2)$$

which implies:

$$\alpha/2 = 3/2$$

[2]

So $\alpha = 3$. If $\alpha = 3$ then

$$u_1(r_1, (3/4, 1/4)) = 3/4 \quad u_1(r_2, (3/4, 1/4)) = 3/4$$

Thus by the equality of payoffs theorem this is a Nash equilibrium.

[2]

C. $(\sigma_1, \sigma_2) = ((1/5, 4/5), (3/4, 1/4))$
If $((1/5, 4/5), (3/4, 1/4))$ is a Nash equilibrium the equality of payoffs theorem states:

$$u_2((1/5, 4/5), c_1) = u_2((1/5, 4/5), c_2)$$

which implies:

$$\alpha/5 = 6/5$$

[2]

So $\alpha = 6$. If $\alpha = 6$ then

$$u_1(r_1, (1/4, 3/4)) = 1/4 \quad u_1(r_2, (1/4, 3/4)) = 3 \times 6/4$$

Thus by the equality of payoffs theorem this is not a Nash equilibrium.

[2]

5. (a) Define a (finitely) repeated game.

A repeated game is played over discrete time periods. Each time period is index by $0 < t \leq T$ where T is the total number of periods. In each period N players play a static game referred to as the **stage game** independently and simultaneously selecting actions. Players make decisions in full knowledge of the **history** of the game played so far (ie the actions chosen by each player in each previous time period). The payoff is defined as the sum of the utilities in each stage game for every time period.

[4]

(b) Define a strategy in a repeated game.

A repeated game strategy must specify the action of a player in a given stage game given the entire history of the repeated game.

[2]

(c) Prove that for any repeated game, any sequence of stage Nash profiles gives the outcome of a subgame perfect Nash equilibrium.

If we consider the strategy given by:

"Player i should play strategy $\tilde{s}_i^{(k)}$ regardless of the play of any previous strategy profiles."

[3]

where $\tilde{s}_i^{(k)}$ is the strategy played by player i in any stage Nash profile. The k is used to indicate that all players play strategies from the same stage Nash profile.

Using backwards induction we see that this strategy is a Nash equilibrium.

[2]

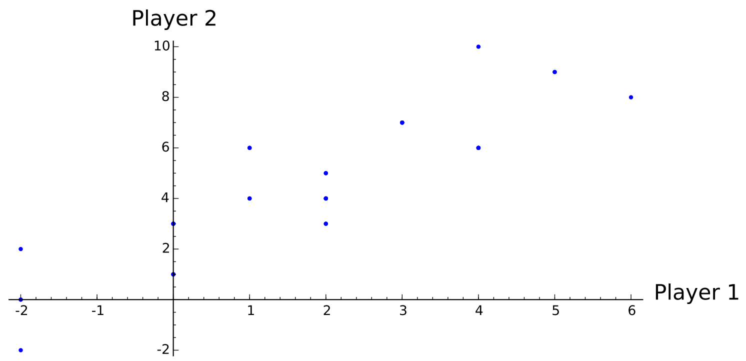
Furthermore it is a stage Nash profile so it is a Nash equilibria for the last stage game which is the last subgame. If we consider (in an inductive way) each subsequent subgame the result holds.

[2]

(d) For the following stage games, plot the possible outcomes for a repetition of $T = 2$ periods and obtain a Nash equilibria **that is not a sequence of stage Nash profiles**:

$$\begin{pmatrix} (3,4) & (1,2) & (2,5) \\ (-1,1) & (1,2) & (-1,-1) \end{pmatrix}$$

Here is the plot:



[1]

The following is a subgame perfect NE: play (r_1, c_1) in first round and (r_1, c_3) in second round. If not, play (r_2, c_2) in second.

[1]

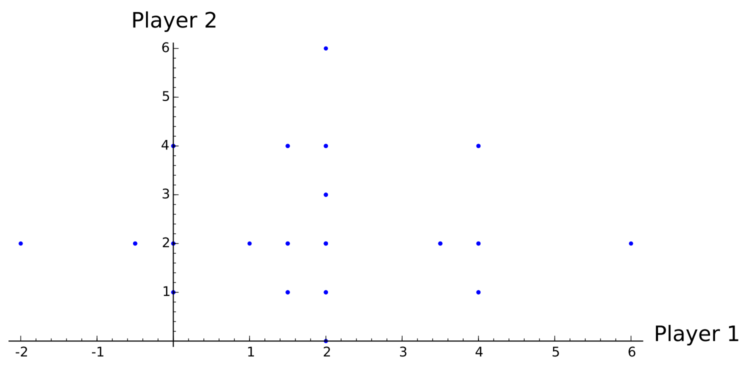
This is subgame perfect:

- First player has no incentive to deviate;
- Second player can deviate in first to gain 1, but will lose 3 in second round.
- First subgame: playing stage NE.

[1]

$$\begin{pmatrix} (3,1) & (1,1) \\ (-1,1) & (1,0) \\ (1,3) & (.5,1) \end{pmatrix}$$

Here is the plot:



[1]

The following is a subgame perfect NE: play (r_3, c_1) in first round and (r_1, c_1) in second round. If not, play (r_1, c_2) in second.

[1]

This is subgame perfect:

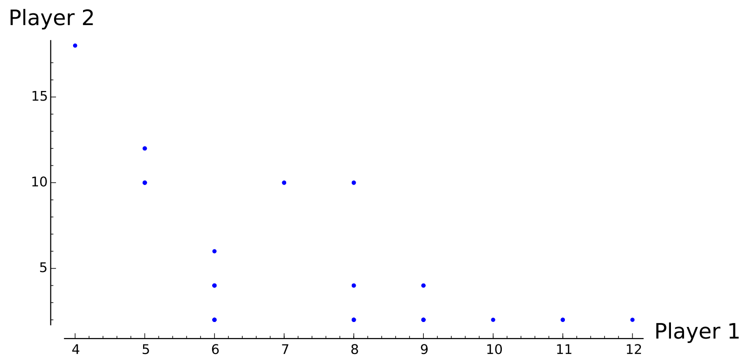
- Second player has no incentive to deviate;

- First player can deviate in first to gain 2, but will lose 2 in second round so no incentive.
- First subgame: playing stage NE.

[1]

$$\begin{pmatrix}(2,9) & (3,1)\\(3,1) & (6,1)\\(3,3) & (5,1)\end{pmatrix}$$

Here is the plot:



[1]

The following is a subgame perfect NE: play (r_1, c_1) in first round and (r_2, c_2) in second round. If not, play (r_3, c_1) in second.

[1]

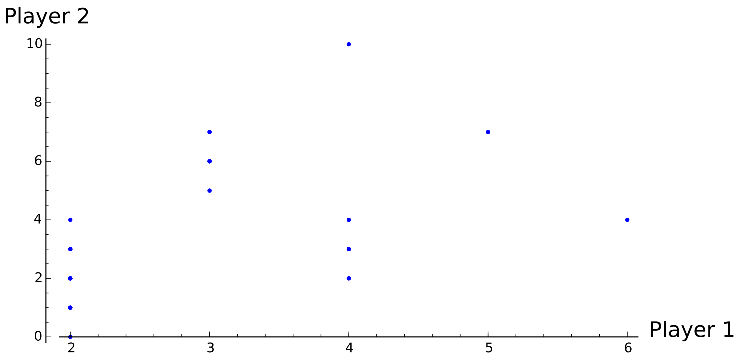
This is subgame perfect:

- Second player has no incentive to deviate;
- First player can deviate in first to gain 1, but will lose 3 in second round so no incentive.
- First subgame: playing stage NE.

[1]

$$\begin{pmatrix}(1,1) & (1,0) & (1,1)\\(1,2) & (3,2) & (2,5)\end{pmatrix}$$

Here is the plot:



[1]

The following is a subgame perfect NE: play (r_2, c_2) in first round and (r_2, c_3) in second round. If not, play (r_1, c_1) in second.

[1]

This is subgame perfect:

- First player has no incentive to deviate;
- Second player can deviate in first to gain 3, but will lose 4 in second round so no incentive.
- First subgame: playing stage NE.

[1]

6. (a) Define a stochastic game.

A stochastic game is defined by:

- X a set of states with a stage game defined for each state;
- A set of strategies $S_i(x)$ for each player for each state $x \in X$;
- A set of rewards dependant on the state and the actions of the other players: $u_i(x, s_1, s_2)$;
- A set of probabilities of transitioning to a future state: $\pi(x' \| x, s_1, s_2)$;
- Each stage game is played at a set of discrete times t.

[4]

(b) Define a Markov strategy.

A strategy is call a **Markov strategy** if the behaviour dictated is not time dependent.

[2]

(c) Give the conditions for Nash equilibrium in a stochastic game.

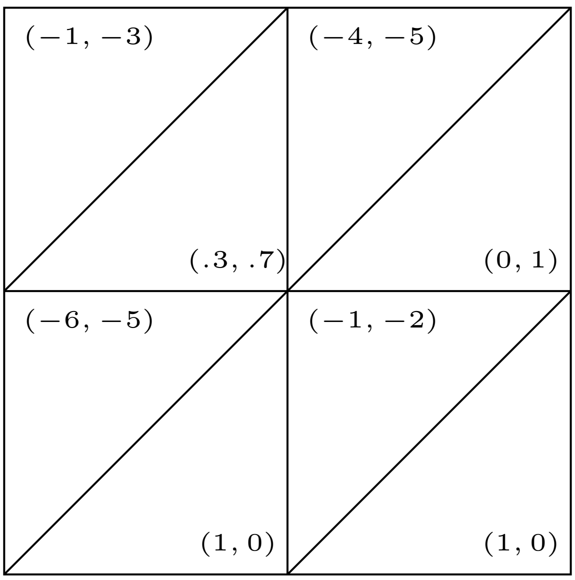
A Nash equilibrium satisfies:

$$U_1^*(x) = \max_{r \in S_1(x)} (u_i(x, r, s^*) + \delta \sum_{x' \in X} \pi(x'|x, r, s^*) U_1^*(x')$$

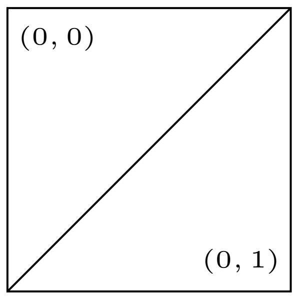
$$U_2^*(x) = \max_{s \in S_2(x)} (u_i(x, r^*, s) + \delta \sum_{x' \in X} \pi(x'|x, r^*, s) U_1^*(x')$$

[3]

(d) Obtain the pure strategy Nash equilibria (if any exist) for the following game with $\delta = .3$:



x



y

State y gives no value to either player so we only need to consider state x. Let the future gains to 1 in state x by u and the future gains to player 2 in state x be v. Thus the players are facing the following game:

[2]

$$\begin{pmatrix} (-1+9/100u, -3+9/100v) & (-4,-5) \\ (-6+3/10u, -5+3/10v) & (-1+3/10u, -2+3/10v) \end{pmatrix}$$

[2]

There are 4 potential pure strategy equilibrium:

- (a,c) which requires $-1+9/100u \geq -6+3/10u$ and $-3+9/100v \geq -5 \Rightarrow u \leq 500/21$ and $v \geq -200/9$. If this is the equilibria then $u = -1+9/100u$ which gives $u = -100/91$ and $-3+9/100v = v$ which gives $v = -300/91$. This contradicts no constraints.

  [3]
- (a,d) which requires $-4 \geq -1+3/10u$ and $-3+9/100v \leq -5 \Rightarrow u \leq -10$ and $v \leq -200/9$. If this is the equilibria then $u = -4$ which gives $u = -5$. This contradicts both constraints.

  [3]
- (b,c) which requires $-1+9/100u \leq -6+3/10u$ and $-5+3/10v \geq -2+3/10v$, this later inequality has no solution.

  [3]
- (b,d) which requires $-4 \leq -1+3/10u$ and $-5+3/10v \leq -2+3/10v) \Rightarrow u \geq -10$ and $v \in \mathbb{R}$. If this is the equilibria then $u = -1+3/10u$ which gives $u = -10/7$ and $-2+3/10v = v$ which gives $v = -20/7$. This contradicts no constraints.

  [3]

Thus (a,c) and (b,d) are the Nash Equilibria for this stochastic game.