

# 1 OR 3: Chapter 4 - Normal Form Games

## 1.1 Recap

In the [previous lecture](#) we discussed:

- Predicting rational behaviour using dominated strategies;
- The CKR;

We did discover certain games that did not have any dominated strategies.

## 1.2 Best response functions

### 1.2.1 Definition

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In an  $n$  player normal form game. A strategy  $s^*$  for player  $i$  is a best response to some strategy profile  $s_{-i}$  if and only if  $u_i(s^*, s_{-i}) \geq u_i(s, s_{-i})$  for all  $s \in S_i$ .

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We can now start to predict rational outcomes in pure strategies by identifying all best responses to a strategy.

$$\begin{pmatrix} (1, 3) & (4, 2) & (2, 2) \\ (4, 0) & (0, 3) & (4, 1) \\ (2, 5) & (3, 4) & (5, 6) \end{pmatrix}$$

We will underline the best responses for each strategy giving:

$$\begin{pmatrix} (1, \underline{3}) & (\underline{4}, 2) & (2, 2) \\ (\underline{4}, 0) & (0, \underline{3}) & (4, 1) \\ (2, 5) & (3, 4) & (\underline{5}, \underline{6}) \end{pmatrix}$$

We see that  $(r_1, s_1)$  represented a pair of best responses. What can we say about the long term behaviour of this game?

### 1.3 Best responses against mixed strategies

We can identify best responses against mixed strategies. Let us take a look at the matching pennies game:

$$\begin{pmatrix} (1, -1) & (-1, 1) \\ (-1, 1) & (1, -1) \end{pmatrix}$$

If we assume that player 2 plays a mixed strategy  $\sigma_2 = (x, 1 - x)$  we have:

$$u_1(r_1, \sigma_2) = 1 - 2x$$

and

$$u_1(r_2, \sigma_2) = 2x - 1$$

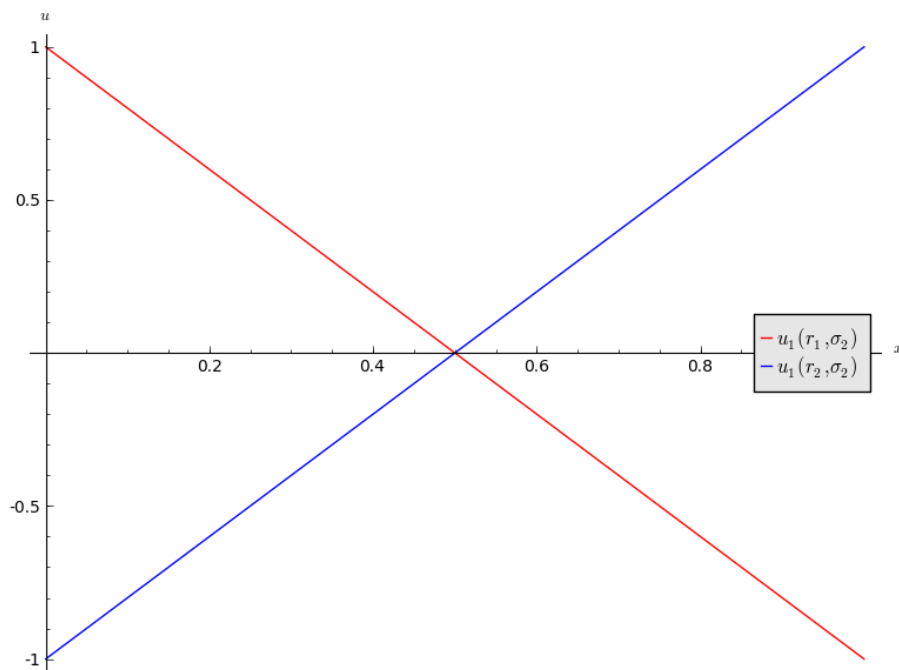


Figure 1:

1. If  $x < 1/2$  then  $r_1$  is a best response for player 1.
2. If  $x > 1/2$  then  $r_1$  is a best response for player 1.

3. If  $x = 1/2$  then player 1 is indifferent.

Let us repeat this exercise for the battle of the sexes game.

$$\begin{pmatrix} (3, 2) & (0, 0) \\ (1, 1) & (2, 3) \end{pmatrix}$$

If we assume that player 2 plays a mixed strategy  $\sigma_2 = (x, 1 - x)$  we have:

$$u_1(r_1, \sigma_2) = 3x$$

and

$$u_1(r_2, \sigma_2) = 2 - x$$

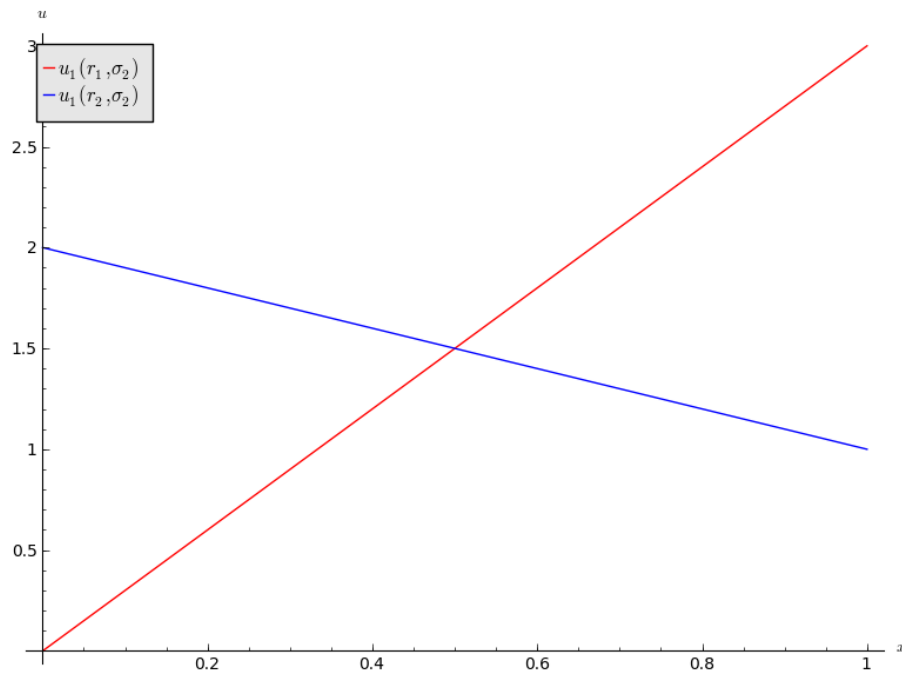


Figure 2:

1. If  $x < 1/2$  then  $r_2$  is a best response for player 1.
2. If  $x > 1/2$  then  $r_1$  is a best response for player 1.
3. If  $x = 1/2$  then player 1 is indifferent.

## 1.4 Connection between best responses and dominance

### 1.4.1 Definition

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In an  $n$  player normal form game, let us define the set  $UD_i$ :

$$UD_i = \{s \in S_i \mid s \text{ is not strictly dominated}\}$$

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If we consider the following game:

$$\begin{pmatrix} (3, 2) & (7, 2) & (5, 1) \\ (5, 1) & (6, 3) & (7, -1) \end{pmatrix}$$

We have:

$$UD_1 = \{r_1, r_2\}$$

$$UD_2 = \{s_1, s_2\}$$

### 1.4.2 Definition

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In an  $n$  player normal form game, let us define the set  $B_i$ :

$$B_i = \{s \in S_i \mid \exists \sigma \in \Delta S_{-i} \text{ such that } s \text{ is a best response to } \sigma\}$$

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In other words  $B_i$  is the set of functions that are best responses to some strategy profile in  $S_{-i}$ .

Let us try to identify  $B_2$  for the above game. Let us assume that player 1 plays  $\sigma_1 = (x, 1 - x)$ . This gives:

$$u_2(\sigma, s_1) = 1 + 3x$$

$$u_2(\sigma, s_2) = 3 - x$$

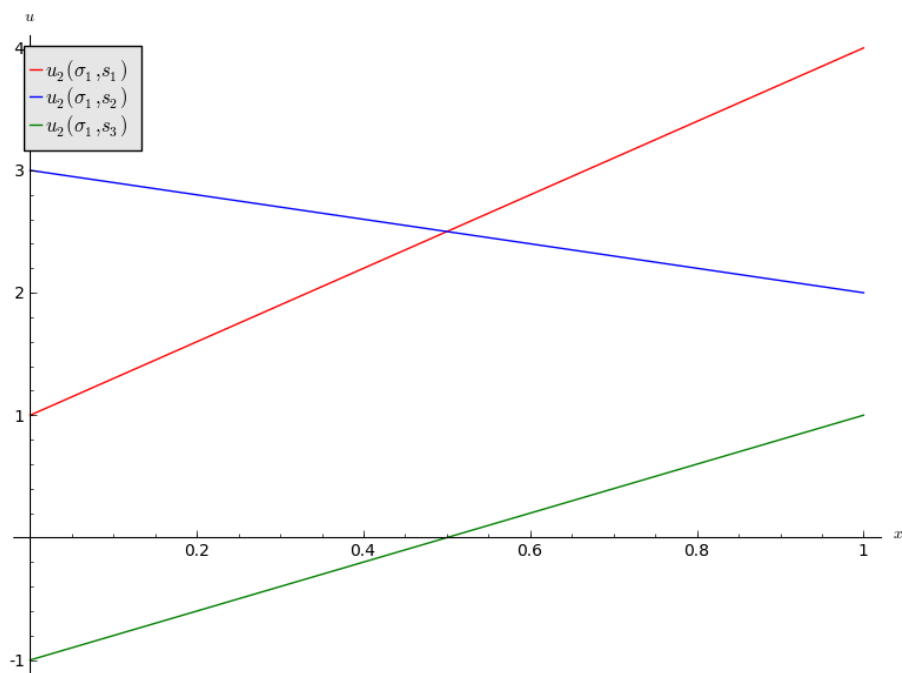


Figure 3:

$$u_2(\sigma, s_3) = 2x - 1$$

We see that  $s_3$  is never a best response for player 2:

$$B_2 = \{s_1, s_2\}$$

We will now attempt to identify  $B_1$  for the above game. Let us assume that player two plays  $\sigma_2 = (x, y, 1 - x - y)$ . This gives:

$$u_1(r_1, \sigma_2) = xu_1(r_1, s_1) + yu_2(r_1, s_2) + (1 - x - y)u_3(r_1, s_3)$$

$$u_1(r_2, \sigma_2) = xu_1(r_2, s_1) + yu_2(r_2, s_2) + (1 - x - y)u_3(r_2, s_3)$$

However as noted earlier  $s_3$  is dominated by  $s_2$  so:

$$u_1(r_1, \sigma_2) < xu_1(r_1, s_1) + u_2(r_1, s_2)(1 - x) = 7 - 4x$$

$$u_1(r_2, \sigma_2) < xu_1(r_2, s_1) + u_2(r_2, s_2)(1 - x) = 6 - x$$

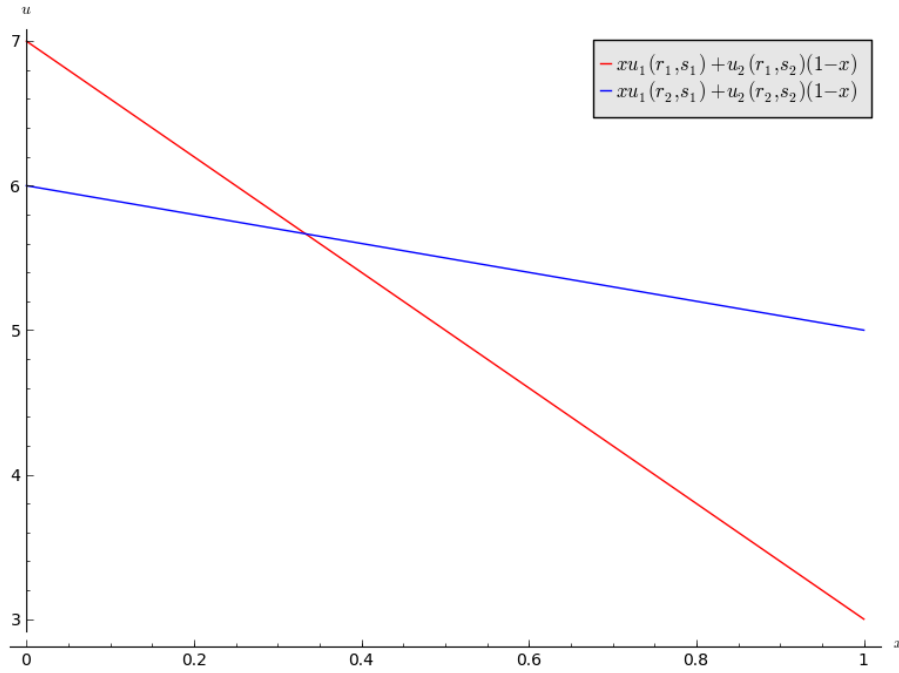


Figure 4:

We see that  $r_1$  and  $r_2$  are best responses for player 1:

$$B_1 = \{r_1, r_2\}$$

We have seen in our little example that  $B_i = UD_i$ . This leads us to two Theorems (the proofs are omitted).

### 1.4.3 Theorem

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In a 2 player normal form game  $B_i = UD_i$  for all  $i \in \{1, 2\}$ .

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This is however not always the case:

### 1.4.4

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In an  $n$  player normal form game  $B_i \subseteq ID_i$  for all  $1 \leq i \leq n$ .

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