

# 1 OR 3: Chapter 11 - Population Games and Evolutionary stable strategies

## 1.1 Recap

In the [previous chapter](#)

- We considered infinitely repeated games using a discount rate;
- We proved a powerful result stating that for a high enough discount rate player would cooperate.

In this chapter we'll start looking at a fascinating area of game theory.

## 1.2 Population Games

In this chapter (and the next) we will be looking at an area of game theory that looks at the evolution of strategic behaviour in a population.

### 1.2.1 Definition of a population vector

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Considering an infinite population of individuals each of which represents a strategy from  $\Delta S$ , we define the population profile as a vector  $\chi \in [0, 1]_{\mathbb{R}}^{|S|}$ . Note that:

$$\sum_{s \in S} \chi(s) = 1$$

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It is important to note that  $\chi$  does not necessarily correspond to any strategy adopted by particular any individual.

### 1.2.2 Example

Consider a population with  $S = \{s_1, s_2\}$ . If we assume that every individual plays  $\sigma = (.25, .75)$  then  $\chi = \sigma$ . However if we assume that .25 of the population play  $\sigma_1 = (1, 0)$  and .75 play  $\sigma_2 = (0, 1)$  then  $\chi = \sigma$ .

In evolutionary game theory we must consider the *utility* of a particular strategy when played in a particular population profile denoted by  $u(s, \chi)$  for  $s \in S$ .

Thus the utility to a player playing  $\sigma \in \Delta S$  in a population  $\chi$ :

$$u(s, \chi) = \sum_{s \in S} \sigma(s) u(s, \chi)$$

The interpretation of the above is that **these payoffs represent the number of descendants that each type of individual has.**

### 1.2.3 Example

If we consider a population of  $N$  individuals in which  $S = \{s_1, s_2\}$ . Assume that .5 of the population use each strategy so that  $\chi = (.5, .5)$  and assume that for the current population profile we have:

$$u(s_1, \chi) = 3 \text{ and } u(s_2, \chi) = 7$$

In the next generation we will have  $3N/2$  individuals using  $s_1$  and  $5N/2$  using  $s_2$  so that the strategy profile of the next generation will be  $(.3, .7)$ .

We are going to work towards understanding the evolutionary dynamics of given populations. If we consider  $\chi^*$  to be the strategy profile where all members of the population play  $\sigma^*$  then a population will be evolutionary stable only if:

$$\sigma^* \in \operatorname{argmax}_{\sigma \in \Delta S} u(\sigma, \chi)$$

Ie at equilibrium  $\sigma^*$  must be a best response to the population profile it generates.

### 1.2.4 Theorem for necessity of stability

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In a population game, consider  $\sigma^* \in \Delta S$  and the population profile  $\chi$  generated by  $\sigma^*$ . If the population is stable then:

$$u(s, \chi) = u(\sigma, \chi) \text{ for all } s \in \mathcal{S}(\sigma^*)$$


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(Recall that  $\mathcal{S}(s)$  denotes the support of  $s$ .)

### 1.2.5 Proof

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If  $|\mathcal{S}(\sigma^*)| = 1$  then the proof is trivial.

We assume that  $|\mathcal{S}(\sigma^*)| > 1$ . Let us assume that the theorem is not true so that there exists  $\bar{s} \in \mathcal{S}(\sigma^*)$  such that:

$$u(\sigma^*, \chi) \neq u(\bar{s}, \chi)$$

Without loss of generality let us assume that:

$$\bar{s} = \operatorname{argmax}_{s \in \mathcal{S}(\sigma^*)} u(s, \chi)$$

Thus we have:

$$\begin{aligned} u_i(\sigma^*, \chi) &= \sum_{s \in \mathcal{S}(\sigma^*)} \sigma^*(s) u(s, \chi) \\ &\leq \sum_{s \in \mathcal{S}(\sigma^*)} \sigma^*(s) u(\bar{s}, \chi) \\ &\leq u(\bar{s}, \chi) \sum_{s \in \mathcal{S}(\sigma^*)} \sigma^*(s) \\ &\leq u(\bar{s}, \chi) \end{aligned}$$

Which gives:

$$u(\sigma^*, \chi) < u(\bar{s}, \chi)$$

which implies that the population is not stable.

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## 1.3 Evolutionary Stable Strategies

### 1.3.1 Definition of an evolutionary stable strategy

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Consider a population where all individuals initially play  $\sigma^*$ . If we assume that a small proportion  $\epsilon$  start playing  $\sigma$ . The new population is called the **post entry population** and will be denoted by  $\chi_\epsilon$ .

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### 1.3.2 Example

Consider a population with  $S = \{s_1, s_2\}$  and initial population profile  $\chi = (1/2, 1/2)$ . If we assume that  $\sigma = (1/3, 2/3)$  is introduced in to the population then:

$$\begin{aligned}\chi_\epsilon &= (1 - \epsilon)(1/2, 1/2) + \epsilon\sigma \\ &= (1/2 - \epsilon/3, 1/2 + 2\epsilon/3)\end{aligned}$$

### 1.3.3 Definition

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A strategy  $\sigma^* \in \Delta S$  is called an **Evolutionary Stable Strategy** if there exists an  $0 < \bar{\epsilon} < 1$  such that for every  $0 < \epsilon < \bar{\epsilon}$  and every  $\sigma \neq \sigma^*$ :

$$u(\sigma^*, \chi) > u(\sigma, \chi)$$

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Before we go any further we must consider two types of population games:

1. Games against the field. In this setting we assume that no individual has a specific adversary but has a utility that depends on what the rest of the individuals are doing.
2. Pairwise contest games. In this setting we assume that every individual is continuously assigned to *play* against another individual.

The differences between these two types of population games will hopefully become clear.

## 1.4 Games against the field

We will now consider an example of a game against the field. The main difficulty with these games is that the utility  $u(s, \chi)$  is not generally linear in  $\chi$ .

Let us consider the Male to Female ratio in a population, and make the following assumptions:

1. The proportion of Males in the population is  $\alpha$  and the proportion of females is  $1 - \alpha$ ;

2. Each female has a single mate and has  $K$  offspring;
3. Males have on average  $(1 - \alpha)/\alpha$  mates;
4. Females are solely responsible for the sex of the offspring.

We assume that  $S = \{M, F\}$  so that females can either only produce Males or only produce Females. Thus, a general mixed strategy  $\sigma = (\omega, 1 - \omega)$  produces a population with a proportion of  $\omega$  males. Furthermore we can write  $\chi = (\alpha, 1 - \alpha)$ .

The females are the decision makers so that we consider them as the individuals in our population. The immediate offspring of the females  $n$  is constant and so cannot be used as a utility. We use the second generation offspring:

$$\begin{aligned} u(M, \chi) &= K^2 \frac{1 - \alpha}{\alpha} \\ u(F, \chi) &= K^2 \end{aligned}$$

Thus:

$$u(\sigma, \chi) = K^2 \left( \omega \frac{(1 - \alpha)}{\alpha} + (1 - \omega) \right)$$

Let us try and find an ESS for this game:

1. If  $\alpha \neq 1/2$  then  $u(M, \chi) \neq u(F, \chi)$  so that any mixed strategy  $\sigma$  with support  $\{M, F\}$  would not give a stable population.
2. Thus we need to check if  $\sigma^* = (1/2, 1/2)$  is an ESS (this is the only candidate).

We consider some mutation  $\sigma = (p, 1 - p)$ :

$$\chi_\epsilon = (1 - \epsilon)\sigma^* + \epsilon\sigma$$

which implies:

$$\alpha\epsilon = (1 - \epsilon)1/2 + p\epsilon = 1/2 + \epsilon(p - 1/2)$$

We have:

$$u(\sigma^*, \chi_\epsilon) = 1/2 + \frac{1 - \alpha_\epsilon}{2\alpha_\epsilon}$$

and:

$$u(\sigma, \chi_\epsilon) = (1 - p) + p \frac{1 - \alpha_\epsilon}{\alpha_\epsilon}$$

The difference:

$$\begin{aligned} u(\sigma^*, \chi_\epsilon) - u(\sigma, \chi_\epsilon) &= p - 1/2 + (1/2 - p) \frac{1 - \alpha_\epsilon}{\alpha_\epsilon} \\ &= (1/2 - p) \frac{1 - 2\alpha_\epsilon}{\alpha_\epsilon} \end{aligned}$$

We note that if  $p < 1/2$  then  $\alpha_\epsilon < 2$  which implies that  $u(\sigma^*, \chi_\epsilon) - u(\sigma, \chi_\epsilon) > 0$ . Similarly if  $p > 1/2$  then  $\alpha_\epsilon < 2$  which implies that  $u(\sigma^*, \chi_\epsilon) - u(\sigma, \chi_\epsilon) > 0$ . Thus  $\sigma^* = (1/2, 1/2)$  is a ESS.