1 OR 3: Chapter 4 - Best responses

1.1 Recap

In the previous lecture we discussed:

- Predicting rational behaviour using dominated strategies;
- The CKR;

We did discover certain games that did not have any dominated strategies.

1.2 Best response functions

1.2.1 Definition of a best response

In an N player normal form game. A strategy s^* for player i is a best response to some strategy profile s_{-i} if and only if $u_i(s^*, s_{-i}) \geq u_i(s, s_{-i})$ for all $s \in S_i$.

We can now start to predict rational outcomes in pure strategies by identifying all best responses to a strategy.

$$\begin{pmatrix} (1,3) & (4,2) & (2,2) \\ (4,0) & (0,3) & (4,1) \\ (2,5) & (3,4) & (5,6) \end{pmatrix}$$

We will underline the best responses for each strategy giving (r_i is underlined if it is a best response to s_j and vice versa):

$$\begin{pmatrix} (1,\underline{3}) & (\underline{4},2) & (2,2) \\ (\underline{4},0) & (0,\underline{3}) & (4,1) \\ (2,5) & (3,4) & (\underline{5},\underline{6}) \end{pmatrix}$$

We see that (r_1, s_1) represented a pair of best responses. What can we say about the long term behaviour of this game?

1.3 Best responses against mixed strategies

We can identify best responses against mixed strategies. Let us take a look at the matching pennies game:

$$\begin{pmatrix}(1,-1) & (-1,1)\\(-1,1) & (1,-1)\end{pmatrix}$$

If we assume that player 2 plays a mixed strategy $\sigma_2 = (x, 1-x)$ we have:

$$u_1(r_1, \sigma_2) = 2x - 1$$

and

$$u_1(r_2, \sigma_2) = 1 - 2x$$

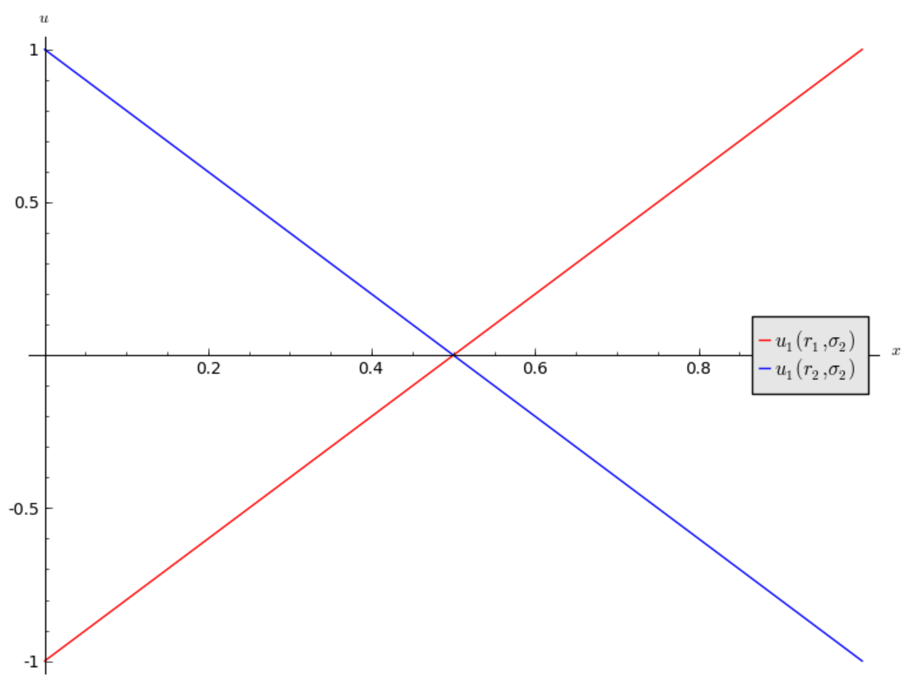

Figure 1: Mixed strategies fox the matching pennies game.

In Figure 1 we see that:

1. If $x < 1/2$ then r_2 is a best response for player 1.

2. If $x > 1/2$ then r_1 is a best response for player 1.
3. If $x = 1/2$ then player 1 is indifferent.

Let us repeat this exercise for the battle of the sexes game.

$$\begin{pmatrix}(3,2) & (0,0)\\(1,1) & (2,3)\end{pmatrix}$$

If we assume that player 2 plays a mixed strategy $\sigma_2 = (x, 1-x)$ we have:

$$u_1(r_1, \sigma_2) = 3x$$

and

$$u_1(r_2, \sigma_2) = 2 - x$$

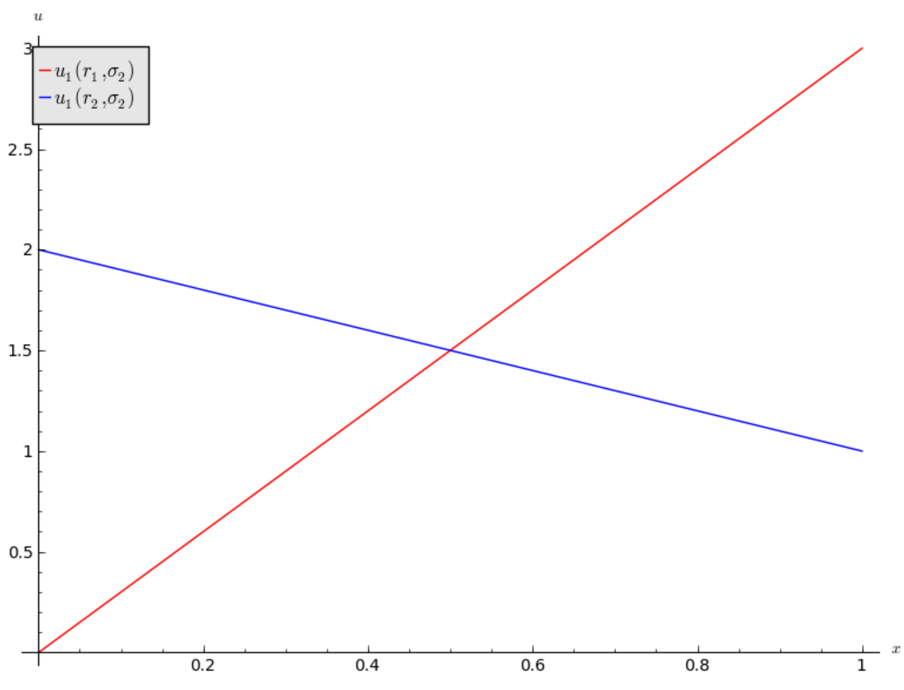

Figure 2: Mixed strategies for the battle of the sexes game.

In Figure 2 we see that:

1. If $x < 1/2$ then r_2 is a best response for player 1.

2. If $x > 1/2$ then r_1 is a best response for player 1.
3. If $x = 1/2$ then player 1 is indifferent.

1.4 Connection between best responses and dominance

1.4.1 Definition of the undominated strategy set

In an N player normal form game, let us define the undominated strategy set UD_i:

$$UD_i = \{s \in S_i \mid \text{s is not strictly dominated}\}$$

If we consider the following game:

$$\begin{pmatrix} (3,3) & (7,2) & (5,1) \\ (5,1) & (6,3) & (7,-1) \end{pmatrix}$$

We have:

$$UD_1 = \{r_1, r_2\}$$
$$UD_2 = \{s_1, s_2\}$$

1.4.2 Definition of the best responses strategy set

In an N player normal form game, let us define the best responses strategy set B_i:

$$B_i = \{s \in S_i \mid \exists\, \sigma \in \Delta S_{-i} \text{ such that } s \text{ is a best response to } \sigma\}$$

In other words B_i is the set of functions that are best responses to some strategy profile in S_{-i}.

Let us try to identify B_2 for the above game. Let us assume that player 1 plays $\sigma_1 = (x, 1-x)$. This gives:

$$u_2(\sigma_1, s_1) = 1 + 2x$$
$$u_2(\sigma_1, s_2) = 3 - x$$
$$u_2(\sigma_1, s_3) = 2x - 1$$

Figure 3 plots these utilities.

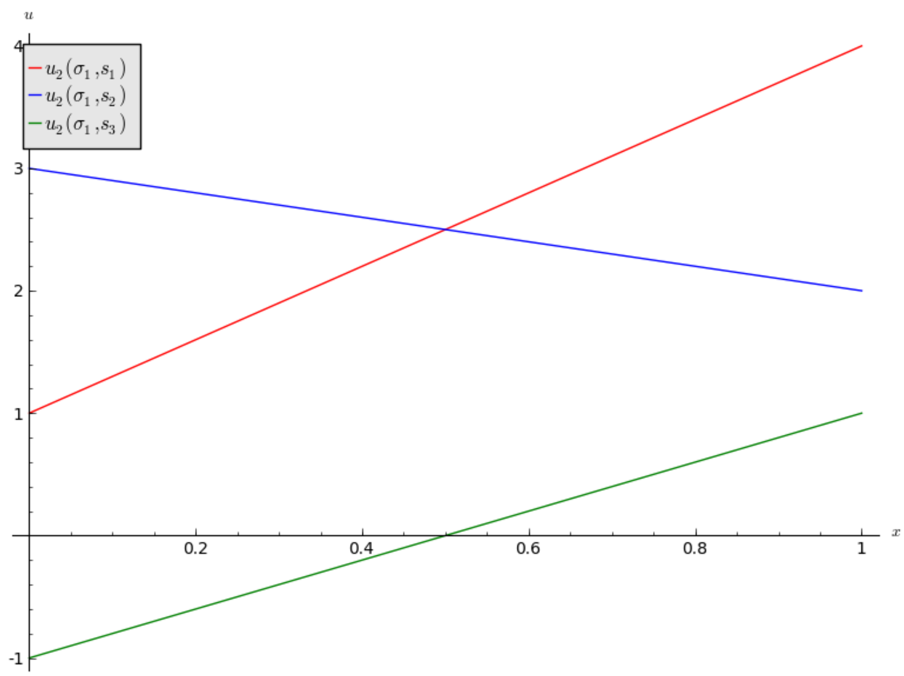

Figure 3: Utilities of player 2 in our example.

We see that s_3 is never a best response for player 2:

$$B_2 = \{s_1, s_2\}$$

We will now attempt to identify B_1 for the above game. Let us assume that player two plays $\sigma_2 = (x, y, 1-x-y)$. This gives:

$$u_1(r_1, \sigma_2) = xu_1(r_1, s_1) + yu_1(r_1, s_2) + (1-x-y)u_1(r_1, s_3) = 3x + 7y + 5 - 5y - 5x$$

$$u_1(r_2, \sigma_2) = xu_1(r_2, s_1) + yu_1(r_2, s_2) + (1-x-y)u_1(r_2, s_3) = 5x + 6y + 7 - 7y - 7x$$

$$u_1(r_1, \sigma_2) - u_2(r_2, \sigma_2) = 3y - 2$$

If we can find values of y that give valid $\sigma_2 = (x, y, 1-x-y)$ and that make the above difference both positive and negative then:

$$B_1 = \{r_1, r_2\}$$

$y = 1$ gives $u_1(r_1, \sigma_2) - u_2(r_2, \sigma_2) = 1 > 0$ (thus r_1 is best response to $\sigma_2 = (0, 1, 0)$). Similarly, $y = 0$ gives $u_1(r_1, \sigma_2) - u_2(r_2, \sigma_2) = -2 < 0$ (thus r_2 is best response to $\sigma_2 = (x, 0, 1-x)$ for any $0 \leq x \leq 1$) as required.

We have seen in our example that $B_i = UD_i$. This leads us to two Theorems (the proofs are omitted).

1.4.3 Theorem of equality in 2 player games

In a 2 player normal form game $B_i = UD_i$ for all $i \in \{1, 2\}$.

This is however not always the case:

1.4.4 Theorem of inclusion in N player games

In an N player normal form game $B_i \subseteq UD_i$ for all $1 \leq i \leq n$.
