

1 Homework sheet 4 - Evolutionary games, games with incomplete information and stochastic games

1. Consider the pairwise contest games with the following associated two player games:

$$\begin{pmatrix} (2, 2) & (4, 5) \\ (5, 4) & (1, 1) \end{pmatrix}$$

Solution

Using the Equality of payoffs theorem we obtain the Nash equilibria:

$$\{((1, 0), (0, 1)), ((0, 1), (1, 0)), ((1/2, 1/2), (1/2, 1/2))\}$$

The two pure Nash equilibria are ESS (because of the first condition of the theorem relating an evolutionary stable strategy to the Nash equilibrium of the associated game).

$$u((1/2, 1/2), \sigma) = 3$$

and for $\sigma = (\omega, 1 - \omega)$:

$$u(\sigma, \sigma) = (2\omega^2 + 9(1 - \omega)\omega + (1 - \omega)^2)$$

thus (after some algebraic manipulation):

$$u((1/2, 1/2), \sigma) - u(\sigma, \sigma) = (2\omega - 1)(3\omega - 2)$$

which is negative for $1/2 < \omega < 2/3$ so this mixed strategy is not an ESS.

$$\begin{pmatrix} (1, 1) & (0, 0) \\ (0, 0) & (1, 1) \end{pmatrix}$$

Solution

Using the Equality of payoffs theorem we obtain the Nash equilibria:

$$\{((1, 0), (1, 0)), ((0, 1), (0, 1)), ((1/2, 1/2), (1/2, 1/2))\}$$

The two pure Nash equilibria are ESS (because of the first condition of the theorem relating an evolutionary stable strategy to the Nash equilibrium of the associated game).

$$u((1/2, 1/2), \sigma) = 1/2$$

As before:

$$u((1/2, 1/2), \sigma) - u(\sigma, \sigma) = -1/2(2\omega - 1)^2$$

so not an ESS.

$$\begin{pmatrix} (\alpha, \alpha) & (1, \beta) \\ (\beta, 1) & (0, 0) \end{pmatrix}$$

(Assume $\alpha, \beta > 0$ and $\alpha \neq \beta$)

Solution

If $\beta < \alpha$ then we have a single pure Nash Equilibria $((1, 0), (1, 0))$ which is also an ESS.

If $\beta > \alpha$ the Nash Equilibria are $\{((1, 0), (1, 0)), ((0, 1), (0, 1)), ((1/(\beta - \alpha + 1), (\beta - \alpha)/(\beta - \alpha + 1)), (1/(\beta - \alpha + 1), (\beta - \alpha)/(\beta - \alpha + 1)))\}$. The pure are ESS.

$$u(\sigma^*, \sigma) = \frac{\beta}{\beta - \alpha + 1}$$

and

$$u(\sigma, \sigma) = -\alpha\omega^2 - \omega(1 - \omega)(1 + \beta)$$

thus (after some algebra):

$$u(\sigma^*, \sigma) - u(\sigma, \sigma) > 0 \Leftrightarrow \beta - (\beta - \alpha + 1)(\omega^2(\alpha - \beta - 1) + \omega(1 + \beta)) > 0$$

However the quadratic in ω has roots: $\left\{ \frac{\beta}{\beta - \alpha + 1}, \frac{1}{\beta - \alpha + 1} \right\}$ so $\exists \omega$ for which the difference is negative and thus this mixed strategy is not an ESS.

Identify all evolutionary stable strategies.

2. Consider the following game:

In a mathematics department, researchers can choose to use one of two systems for typesetting their research papers: LaTeX or Word. We will refer to these two strategies as L and W respectively. A user of W receives a basic utility of 1 and as L is more widely used by mathematicians out of the department and is in general considered to be a better system a user of L gets a basic utility of $\alpha > 1$. Members of the mathematics department often collaborate and as such it is beneficial for the researchers to use the same typesetting system. If we let μ represent the proportion of users of L we let:

$$u(L, \chi) = \alpha + 2\mu$$

$$u(W, \chi) = 1 + 2(1 - \mu)$$

What are the evolutionary stable strategies?

Solutions

Using the theorem for necessity of stability we have the following candidate ESS:

1. σ_L : everyone uses L , thus $\mu = 1$ (we have $u(L, \chi) > u(W, \chi)$).
2. σ_W : everyone uses W , thus $\mu = 0$ (we have $u(L, \chi) < u(W, \chi)$).
3. σ_m : some use L and some use W , by the theorem we have $u(L, \chi) = u(W, \chi)$ which implies $\alpha + 2\mu = 1 + 2(1 - \mu)$ giving $\mu = \frac{3-\alpha}{4}$.

Now we consider the post entry population $\chi_\epsilon = (1-\epsilon)\sigma^* + \epsilon\sigma$ (where σ^* is the base strategy and σ is the entry population). We denote $\sigma = (\mu, 1-\mu)$ and $\sigma^* = (\mu^*, 1-\mu^*)$ and $\delta = u(\sigma^*, \chi_\epsilon) - u(\sigma, \chi_\epsilon)$. We have:

$$\delta = \mu^*u(L, \chi_\epsilon) + (1-\mu^*)u(W, \chi_\epsilon) - \mu u(L, \chi_\epsilon) + (1-\mu)u(W, \chi_\epsilon) = (\mu^* - \mu)(u(L, \chi_\epsilon) - u(W, \chi_\epsilon))$$

which gives:

$$\delta = (\mu^* - \mu)(\alpha - 3 + 4((1-\epsilon)\mu^* + \epsilon\mu)) = (\mu^* - \mu)(4\mu^* + \alpha - 3 + 4\epsilon(\mu - \mu^*))$$

We now consider each potential ESS in turn, if $\delta > 0$ for all $\epsilon < \bar{\epsilon}$ for some $\bar{\epsilon}$ then we have an ESS (this is by definition):

1. $\mu^* = 1$: $\delta = (1-\mu)(1 + \alpha + 4\epsilon(\mu - 1)) > (1-\mu)(1 + \alpha - 4\epsilon) > 0$ for all $\mu \neq 1$ and $\epsilon < \bar{\epsilon} = \frac{1+\alpha}{4}$. Thus σ_L is an ESS.
2. $\mu^* = 0$: $\delta = -\mu(\alpha - 3 + 4\epsilon\mu)$. If $\alpha \geq 3$ then $\delta \leq 0$ for all values of μ, ϵ , thus if L is 3 times better than W σ_W is not an ESS. If $\alpha < 3$ $\delta > 0 \Leftrightarrow \alpha - 3 + 4\epsilon\mu < 0 \Rightarrow \alpha - 3 + 4\epsilon\mu < \alpha - 3 + 4\epsilon < 0$ for all $\mu \neq 0$ $\epsilon < \bar{\epsilon} = \frac{3-\alpha}{4}$. Thus σ_W is an ESS for $\alpha < 3$.
3. $\mu^* = \frac{3-\alpha}{4}$: $\delta = -4\epsilon\left(\frac{3-\alpha}{4} - \mu\right)^2 < 0$ for all $\mu \neq \frac{3-\alpha}{4}$ and for all $\epsilon > 0$ so σ_m is not an ESS.

3. Consider the following two normal form games:

$$A = \begin{pmatrix} (3, 0) & (-1, -1) & (1, 2) \\ (1, 0) & (-1, 1) & (2, 0) \end{pmatrix}$$

$$B = \begin{pmatrix} (2, 2) & (1, 1) & (1, 3) \\ (1, 3) & (-2, -3) & (4, 2) \end{pmatrix}$$

Assume both players play either game A or game B with probability $1/2$, neither player knows which game is played. Obtain the Nash equilibrium for this game.

Solution

The described game is akin to the following game:

$$\begin{pmatrix} (5/2, 1) & (0, 0) & (1, 5/2) \\ (1, 3/2) & (-3/2, -1) & (3/2, 1) \end{pmatrix}$$

We see that s_2 is dominated and solve the game using the equality of payoffs theorem to give the following Nash equilibrium:

$$((1/4, 3/4), (1/4, 3/4))$$

4. Repeat the analysis of the principal agent game assuming that p is the probability of the project being successful in case of a high level of effort by the employee.
 - i. What are the expected utilities to the employer and the employee?

Solution

Repeating the analysis, we see that the employee will carry out a high effort iff:

$$p(\omega + B - 1)^\alpha + (1 - p)(\omega - 1)^\alpha \geq \omega^\alpha$$

Following the same argument as in the notes we arrive at:

$$p(\omega + B - 1)^\alpha + (1 - p)(\omega - 1)^\alpha = 1 = \omega$$

thus:

$$\beta = (1/p)^{(1/\alpha)}$$

The utilities are then:

Employer:

$$p(K - 1 - (1/p)^{(1/\alpha)}) + (1 - p)(\kappa - 1)$$

Employee:

$$1$$

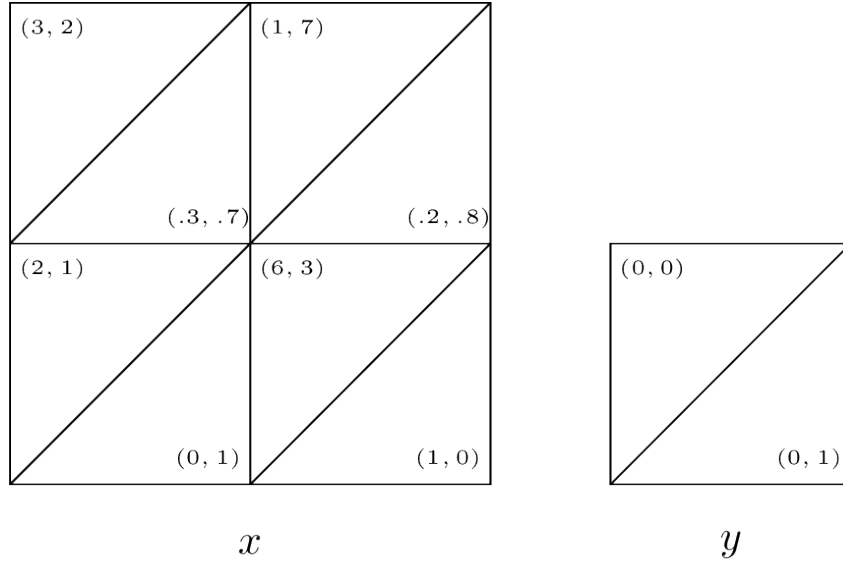
- ii. Obtain a condition for which the employer should offer a bonus.

Solution

If no bonus is offered the employee has no incentive for a high effort thus $\kappa = \omega = 1$, thus the employer should offer a bonus iff:

$$p(K - 1 - (1/p)(1/\alpha)) + (1 - p)(\kappa - 1) \geq 0$$

5. Obtain the Markov Nash equilibrium (in pure strategies if it exists) for the following games assuming $\delta = 1/4$.



Solution

State y gives no value to either player so we only need to consider state x . Let the future gains to player 1 in state x be u and the future gains to player 2 in state x be v . Thus the players are facing the following game:

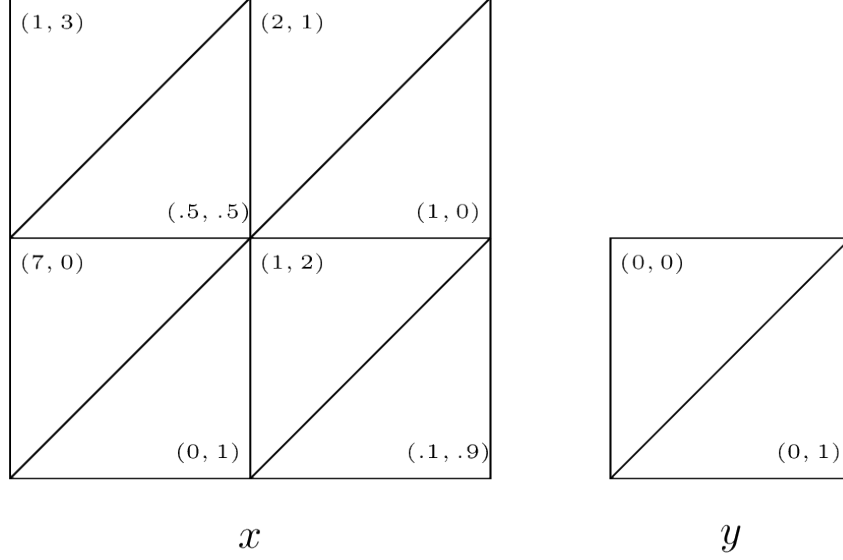
$$\begin{pmatrix} (3 + 3u/40, 2 + 3v/40) & (1 + u/20, 7 + v/20) \\ (2, 1) & (6 + u/4, 3 + v/4) \end{pmatrix}$$

There are four possible equilibria:

1. (a, b) which requires: $3 + 3u/40 \geq 2$ and $2 + 3v/40 \geq 7 + v/20 \Rightarrow u \geq -40/3$ and $v \geq 200$. However if this is the equilibria then $u = 120/37$ and $v = 80/37$ which contradicts the constraints.
2. (a, d) which requires: $1 + u/20 \geq 6 + u/4$ and $2 + 3v/40 \leq 7 + v/20 \Rightarrow u \leq -25$ and $v \leq 200$. However if this is the equilibria then $u = 20/19$ and $v = 140/19$ which contradicts the constraints.
3. (b, c) which requires: $2 \geq 3 + 3u/40$ and $1 \geq 3 + v/4 \Rightarrow u \leq -40/3$ and $v \leq -8$. However if this is the equilibria then $u = 2$ and $v = 1$ which contradicts the constraints.

4. (b, d) which requires: $2 \leq 3 + 3u/40$ and $1 \leq 3 + v/4 \Rightarrow u \geq -40/3$ and $v \geq -8$. However if this is the equilibria then $u = 8$ and $v = 4$ which **does not** contradict the constraints.

Thus (b, d) is the unique pure strategy equilibrium.



Solution

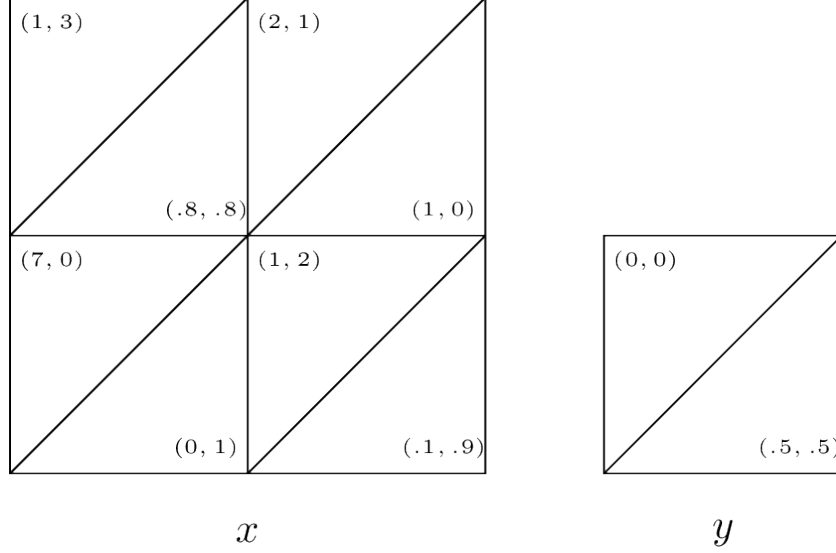
State y gives no value to either player so we only need to consider state x . Let the future gains to player 1 in state x be u and the future gains to player 2 in state x be v . Thus the players are facing the following game:

$$\begin{pmatrix} (1 + u/8, 3 + v/8) & (2 + u/4, 1 + v/4) \\ (7, 0) & (1 + u/40, 2 + 9v/40) \end{pmatrix}$$

There are four possible equilibria:

1. (a, b) which requires: $1 + u/8 \geq 7$ and $3 + v/8 \geq 1 + v/4 \Rightarrow u \geq 48$ and $v \leq 16$. However if this is the equilibria then $u = 8/7$ and $v = 24/7$ which contradicts the constraints.
2. (a, b) which requires: $2 + u/4 \geq 1 + u/40$ and $3 + v/8 \leq 1 + v/4 \Rightarrow u \geq -40/9$ and $v \geq 16$. However if this is the equilibria then $u = 8/3$ and $v = 4/3$ which contradicts the constraints.
3. (b, c) which requires: $7 \geq 1 + u/8$ and $0 \geq 2 + 9v/40 \Rightarrow u \leq 48$ and $v \leq -80/9$. However if this is the equilibria then $u = 7$ and $v = 0$ which contradicts the constraints.
4. (a, b) which requires: $2 + u/4 \leq 1 + u/40$ and $2 + 9v/40 \geq 0 \Rightarrow u \leq -40/9$ and $v \geq -80/9$. However if this is the equilibria then $u = 40/39$ and $v = 80/31$ which contradicts the constraints.

Thus no Nash equilibrium exists in pure strategies.



Solution

State y gives no value to either player so we only need to consider state x . Let the future gains to player 1 in state x be u and the future gains to player 2 in state x be v . Thus the players are facing the following game:

$$\begin{pmatrix} (1 + u/20, 3 + v/20) & (2 + 7u/40, 1 + 7v/40) \\ (7, 0) & (1 + u/40, 2 + 9v/40) \end{pmatrix}$$

There are four possible equilibria:

1. (a, b) which requires: $1 + u/20 \geq 7$ and $3 + v/20 \geq 1 + 7v/40 \Rightarrow u \geq 120$ and $v \leq 16$. However if this is the equilibria then $u = 20/19$ and $v = 60/19$ which contradicts the constraints.
2. (a, b) which requires: $2 + 7u/40 \geq 1 + u/40$ and $1 + 7v/40 \geq 3 + v/20 \Rightarrow u \geq -20/3$ and $v \leq 16$. However if this is the equilibria then $u = 80/33$ and $v = 40/33$ which does not contradict any constraints.
3. (b, c) which requires: $7 \geq 1 + u/20$ and $0 \geq 2 + 9v/40 \Rightarrow u \leq 120$ and $v \leq -80/9$. However if this is the equilibria then $u = 7$ and $v = 0$ which contradicts the constraints.
4. (a, b) which requires: $2 + u/4 \leq 1 + u/40$ and $2 + 9v/40 \geq 0 \Rightarrow u \leq -40/9$ and $v \geq -80/9$. However if this is the equilibria then $u = 40/39$ and $v = 80/31$ which contradicts the constraints.

Thus (a, b) is the unique pure strategy equilibrium.