# OR 3: Chapter 12 - Nash equilibrium and Evolutionary stable strategies

## Recap

In the previous chapter:

* We considered population games;
* We proved a result concerning a necessary condition for a population to be evolutionary stable;
* We defined Evolutionary stable strategies and looked at an example in a game against the field.

In this chapter we'll take a look at pairwise contest games and look at the connection between Nash equilibrium and ESS.

## Pairwise contest games

In a population game when considering a pairwise contest game we assume that individuals are randomly matched. The utilities then depend just on what the individuals do:

As an example we're going to consider the "Hawk-Dove" game: a model of predator interaction. We have were:

* : Hawk represents being "aggressive";
* : Dove represents not being "aggressive".

At various times individuals come in to contact and must choose to act like a Hawk or like Dove over the sharing of some resource of value . We assume that:

* If a Dove and Hawk meet the Hawk takes the resources;
* If two Doves meet they share the resources;
* If two Hawks meet there is a fight over the resources (with an equal chance of winning) and the winner takes the resources while the loser pays a cost .

If we assume that and the above gives:

It is immediate to note that no pure strategy ESS exists. In a population of Doves ():

thus the best response is setting i.e. to play Hawk.

In a population of Hawks ():

thus the best response is setting i.e. to play Dove.

So we will now try and find out if there is a mixed-strategy ESS: . For to be an ESS it must be a best response to the population it generates . In this population the payoff to an arbitrary strategy is:

* If then a best response is ;
* If then a best response is ;
* If then there is indifference.

So the only candidate for an ESS is . We now need to show that .

We have:

and:

This gives:

which proves that is an ESS.

We will now take a closer look the connection between ESS and Nash equilibria.

## ESS and Nash equilibria

When considering pairwise contest population games there is a natural way to associate a normal form game.

### Definition

The **associated two player game** for a pairwise contest population game is the normal form game with payoffs given by: .

Note that the resulting game is symmetric (other contexts would give non symmetric games but we won't consider them here).

Using this we have the powerful result:

### Theorem relating an evolutionary stable strategy to the Nash equilibrium of the associated game

If is an ESS in a pairwise contest population game then for all :

1. OR
2. and

Conversely, if either (1) or (2) holds for all in a two player normal form game then is an ESS.

### Proof

If is an ESS, then by definition:

which corresponds to:

* If condition 1 of the theorem holds then the above inequality can be satisfied for sufficiently small. If condition 2 holds then the inequality is satisfied.
* Let us assume that :
  + If then we can find sufficiently small such that the inequality is violated. Thus the inequality implies .
  + If then : as required.

This result gives us an efficient way of computing ESS. The first condition is in fact almost a condition for Nash Equilibrium (with a strict inequality), the second is thus a stronger condition that removes certain Nash equilibria from consideration. This becomes particularly relevant when considering Nash equilibrium in mixed strategies.

To find ESS in a pairwise context population game we:

1. Write down the associated two-player game;
2. Identify all symmetric Nash equilibria of the game;
3. Test the Nash equilibrium against the two conditions of the above Theorem.

### Example

Let us consider the Hawk-Dove game. The associated two-player game is:

Recalling that we have so we can use the Equality of payoffs theorem to obtain the Nash equilibrium:

Thus we will test using the above theorem.

**Importantly** from the equality of payoffs theorem we immediately see that condition 1 does not hold as . Thus we need to prove that:

We have:

After some algebra:

Giving the required result.