# OR 3: Chapter 15 - Matching games

## Recap

In the [previous chapter](Chapter_15_Matching_games.pdf):

* We defined matching games;
* We described the Gale-Shapley algorithm;
* We proved certain results regarding the Gale-Shapley algorithm.

In this Chapter we'll take a look at another type of game.

## Cooperative Games

In cooperative game theory the interest lies with understanding how coalitions form in competitive situations.

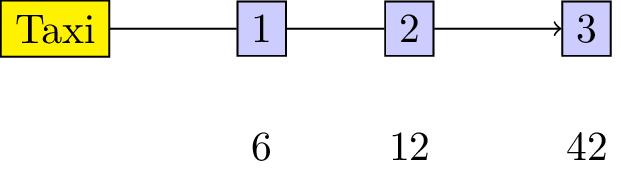
### Definition

A **characteristic function game** G is given by a pair where is the number of players and is a **characteristic function** which maps every coalition of players to a payoff.

Let's consider the following game:

"3 players must share a taxi. Here are the costs for each individual journey: - Player 1: 6 - Player 2: 12 - Player 3: 42 "

This is illustrated below:

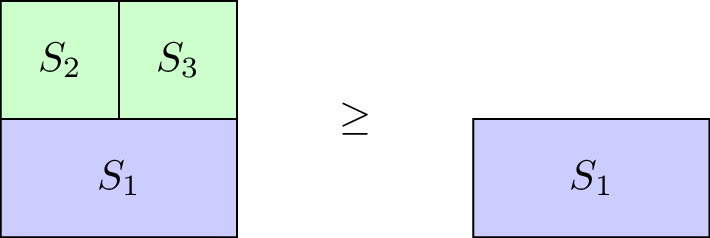


To construct the characteristic function we first obtain the power set (ie all possible coalitions) where denotes the set of all players ().

The characteristic function is given below:

### Definition

A characteristic function game is called **monotone** is it satisfies for all .

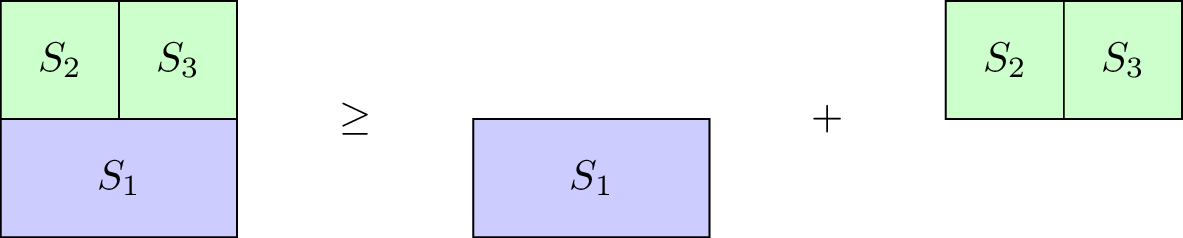


Our taxi example is monotone, however the with defined as:

is not.

### Definition

A characteristic function game is called **superadditive** if it satisfies



Our taxi example is not superadditive, however the with defined as:

is.

## Shapley Value

When talking about a solution to a characteristic function game we imply a payoff vector that divides the value of the grand coalition between the various players. Thus must satisfy:

Thus one potential solution to our taxi example would be . Obviously this is not ideal for player 1 and/or 2: they actually pay more than they would have paid without sharing the taxi!

Another potential solution would be , however at this point sharing the taxi is of no benefit to player 1. Similarly would have no incentive for player 2.

To find a "fair" distribution of the grand coalition we must define what is meant by "fair". We require four desirable properties:

* Efficiency;
* Null player;
* Symmetry;
* Additivity.

### Definition

For a payoff vector is **efficient** if:

### Definition

For a payoff vector possesses the **null player property** if for all then:

### Definition

For a payoff vector possesses the \**symmetry property* if for all then:

### Definition

For and and where for any . A payoff vector possesses the **additivity property** if:

We will not prove the following in this course but in fact there is a single payoff vector that satisfies these four properties. To define it we need two last definitions.

### Definition

If we consider any permutation of then we denote by the set of predecessors of in :

For example for we have .

### Definition

If we consider any permutation of then the marginal contribution of player with respect to is given by:

We can now define the **Shapley value** of any game .

### Definition

Given the **Shapley value** of player is denoted by and given by:

As an example here is the Shapley value calculation for our taxi sharing game:

For :

For :

For :

For :

For :

For :

Using this we obtain:

Thus the fair way of sharing the taxi fare is for player 1 to pay 1, player 2 to pay 5 and player 3 to pay 35.