Chapter 6
Oscillations

Abstract *Chapter 6* deals with simple harmonic motion and its application to various problems, physical pendulums, coupled systems of masses and springs, the normal coordinates and damped vibrations.

6.1 Basic Concepts and Formulae

Simple Harmonic Motion (SHM)

In SHM the restoring force (F) is proportional to the displacement but is oppositely directed.

$$F = -kx \tag{6.1}$$

where k is a constant, known as force constant or spring constant. The negative sign in (6.1) implies that the force is opposite to the displacement.

When the mass is released, the force produces acceleration a given by

$$a = F/m = -k/m = -\omega^2 x \tag{6.2}$$

$$\text{where } \omega^2 = k/m \tag{6.3}$$

$$\text{and } \omega = 2\pi f \tag{6.4}$$

is the angular frequency.

Differential equation for SHM:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{6.5}$$

Most general solution for (6.5) is

$$x = A\sin(\omega t + \varepsilon) \tag{6.6}$$

where A is the amplitude, $(\omega t + \varepsilon)$ is called the phase and ε is called the phase difference.

The velocity v is given by

$$v = \pm\omega\sqrt{A^2 - x^2} \tag{6.7}$$

The acceleration is given by

$$a = -\omega^2 x \tag{6.8}$$

The frequency of oscillation is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{6.9}$$

where m is the mass of the particle.

The time period is given by

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \tag{6.10}$$

Total energy (E) of the oscillator:

$$E = {}^1\!/\!_2\, mA^2\omega^2 \tag{6.11}$$

$$K_{\mathrm{av}} = U_{\mathrm{av}} = {}^1\!/\!_4\, mA^2\omega^2 \tag{6.12}$$

Loaded spring:

$$T = 2\pi\sqrt{\frac{\left(M + \frac{m}{3}\right)}{k}} \tag{6.13}$$

where M is the load and m is the mass of the spring.

If v_1 and v_2 are the velocities of a particle at x_1 and x_2, respectively, then

$$T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}} \tag{6.14}$$

$$A = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}} \tag{6.15}$$

Pendulums

Simple Pendulum (Small Amplitudes)

$$T = 2\pi \sqrt{\frac{L}{g}} \tag{6.16}$$

T is independent of the mass of the bob. It is also independent of the amplitude for small amplitudes.

Seconds pendulum is a simple pendulum whose time period is 2 s.

Simple Pendulum (Large Amplitude)

For large amplitude θ_0, the time period of a simple pendulum is given by

$$T = 2\pi \sqrt{\frac{L}{g}} \left[1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\theta_0}{2}\right) + \left(\frac{1.3}{2.4}\right)^2 \sin^4\left(\frac{\theta_0}{2}\right) + \left(\frac{1.3.5}{2.4.6}\right)^2 \sin^6\left(\frac{\theta_0}{2}\right) \right] \tag{6.17}$$

where we have dropped higher order terms.

Simple pendulum on an elevator/trolley moving with acceleration a. Time period of the stationary pendulum is T and that of moving pendulum T'.

(a) Elevator has upward acceleration a

$$T' = T\sqrt{\frac{g}{g+a}} \tag{6.18}$$

(b) Elevator has downward acceleration a

$$T' = T\sqrt{\frac{g}{g-a}} \tag{6.19}$$

(c) Elevator has constant velocity, i.e. $a = 0$

$$T' = T \tag{6.20}$$

(d) Elevator falls freely or is kept in a satellite, $a = g$

$$T' = \infty \tag{6.21}$$

The bob does not oscillate at all but assumes a fixed position.

(a) Trolley moving horizontally with acceleration a

$$T' = T\sqrt{\frac{g}{\sqrt{g^2 + a^2}}} \tag{6.22}$$

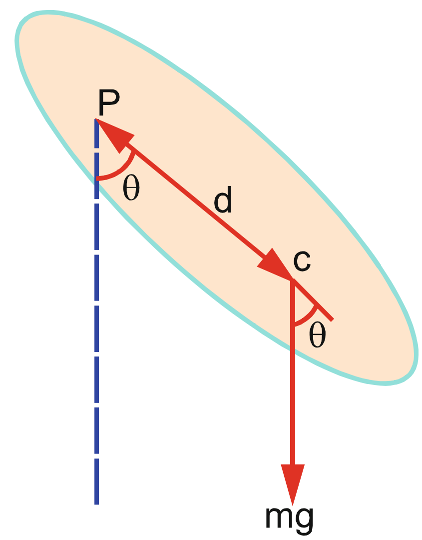
(b) Trolley rolls down on a frictionless incline at an angle θ to the horizontal plane

$$T' = T/\cos\theta \tag{6.23}$$

Physical Pendulum

Any rigid body mounted such that it can swing in a vertical plane about some axis passing through it is called a physical pendulum, Fig. 6.1.

Fig. 6.1



The body is pivoted to a horizontal frictionless axis through P and displaced from the equilibrium position by an angle θ. In the equilibrium position the centre of mass C lies vertically below the pivot P. If the distance from the pivot to the centre of mass be d, the mass of the body M and the moment of inertia of the body about an axis through the pivot I, the time period of oscillations is given by

$$T = 2\pi\sqrt{\frac{1}{Mgd}} \tag{6.24}$$

The equivalent length of simple pendulum is

$$L_{\text{eq}} = I/Md \tag{6.25}$$

The torsional oscillator consists of a flat metal disc suspended by a wire from a clamp and attached to the centre of the disc. When displaced through a small angle about the vertical wire and released the oscillator would execute oscillations in the horizontal plane. For small twists the restoring torque will be proportional to the angular displacement

$$\tau = -C\theta \tag{6.26}$$

where C is known as torsional constant. The time period of oscillations is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \tag{6.27}$$

Coupled Harmonic Oscillators

Two equal masses connected by a spring and two other identical springs fixed to rigid supports on either side, Fig. 6.2, permit the masses to jointly undergo SHM along a straight line, so that the system corresponds to two coupled oscillators. The equation of motion for mass m_1 is

$$m\ddot{x}_1 + k(2x_1 - x_2) = 0 \tag{6.28}$$

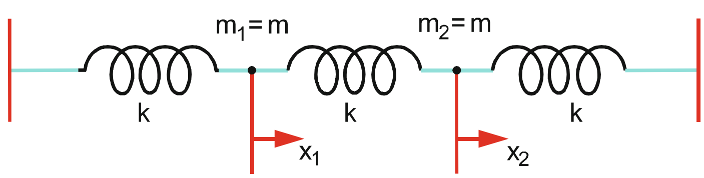


Fig. 6.2

and that for m_2 is

$$m\ddot{x}_2 + k(2x_2 - x_1) = 0 \tag{6.29}$$

Equations (6.28) and (6.29) are coupled equations.

Assuming $x_1 = A_1 \sin\omega t$ and $x_2 = A_2 \sin\omega t$
(6.28) and (6.29) become

$$\ddot{x}_1 = -\omega^2 A_1 \sin\omega t = -\omega^2 x_1 \tag{6.30}$$

$$\ddot{x}_2 = -\omega^2 A_2 \sin\omega t = -\omega^2 x_2 \tag{6.31}$$

Inserting (6.30) and (6.31) in (6.28) and (6.29), we get on rearrangement

$$(2k - m\omega^2)x_1 - kx_2 = 0 \tag{6.32}$$

$$-kx_1 + (2k - m\omega^2)x_2 = 0 \tag{6.33}$$

For a non-trivial solution, the determinant formed from the coefficients of x_1 and x_2 must vanish.

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0$$

The expansion of the determinant gives a quadratic equation in ω whose solutions are

$$\omega_1 = \sqrt{k/m} \tag{6.35}$$

$$\omega_2 = \sqrt{3k/m} \tag{6.36}$$

Normal coordinates: It is always possible to define a new set of coordinates called normal coordinates which have a simple time dependence and correspond to the excitation of various oscillation modes of the system. Consider a pair of coordinates defined by

$$\eta_1 = x_1 - x_2, \eta_2 = x_1 + x_2 \tag{6.37}$$

$$\text{or} \quad x_1 = \frac{1}{2}(\eta_1 + \eta_2), x_2 = \frac{1}{2}(\eta_2 - \eta_1) \tag{6.38}$$

Substituting (6.38) in (6.28) and (6.29) we get

$$m(\ddot{\eta}_1 + \ddot{\eta}_2) + k(3\eta_1 + \eta_2) = 0$$
$$m(\ddot{\eta}_1 - \ddot{\eta}_2) + k(3\eta_1 - \eta_2) = 0$$

which can be solved to yield

$$m\ddot{\eta}_1 + 3k\eta_1 = 0$$
$$m\ddot{\eta}_2 + k\eta_2 = 0 \tag{6.39}$$

The coordinates η_1 and η_2 are now uncoupled and are therefore independent unlike the old coordinates x_1 and x_2 which were coupled.

The solutions of (6.39) are

$$\eta_1(t) = B_1 \sin \omega_1 t, \quad \eta_2(t) = B_2 \sin \omega_2 t \tag{6.40}$$

where the frequencies are given by (6.35) and (6.36).

A deeper insight is obtained from the energies expressed in normal coordinates as opposed to the old coordinates. The potential energy of the system

$$U = \frac{1}{2}k{x_1}^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k{x_2}^2$$
$$= k({x_1}^2 - x_1x_2 + {x_2}^2) \tag{6.41}$$

The term proportional to the cross-product x_1x_2 is the one which expresses the coupling of the system. The kinetic energy of the system is

$$K = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 \tag{6.42}$$

In terms of normal coordinates defined by (6.38)

$$U = \frac{k}{4}(\eta_1^2 + 3\eta_2^2) \tag{6.43}$$

$$K = \frac{m}{4}(\dot{\eta}_1^2 + \dot{\eta}_2^2) \tag{6.44}$$

Thus, the cross-product term has disappeared and the kinetic and potential energies appear in quadratic form. Each normal coordinate corresponds to an independent mode of vibration of the system, with its own characteristic frequency and the general vibratory motion may be regarded as the superposition of some or all of the independent normal vibrations.

Damped Vibrations

For small velocities the resisting force f_r (friction) is proportional to the velocity:

$$f_\mathrm{r} = -r\frac{\mathrm{d}x}{\mathrm{d}t} \tag{6.45}$$

where r is known as the resistance constant or damping constant. The presence of the dissipative forces results in the loss of energy in heat motion leading to a gradual decrease of amplitude. The equation of motion is written as

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + r\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0 \tag{6.46}$$

where m is the mass of the body and k is the spring constant.

Putting $r/m = 2b$ and $k/m = {\omega_0}^2$, (6.46) becomes on dividing by m

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2b\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2x = 0 \tag{6.47}$$

Let $x = \mathrm{e}^{\lambda t}$ so that $\mathrm{d}x/\mathrm{d}t = \lambda\mathrm{e}^{\lambda t}$ and $\mathrm{d}^2x/\mathrm{d}t^2 = \lambda^2\mathrm{e}^{\lambda t}$

The corresponding characteristic equation is

$$\lambda^2 + 2b\lambda + {\omega_0}^2 = 0 \tag{6.48}$$

The roots are

$$\lambda = -b \pm \sqrt{b^2 - \omega_0^2} \tag{6.49}$$

Calling $R = \sqrt{b^2 - \omega_0^2}$

$$\lambda_1 = -b + R \quad \lambda_2 = -b - R$$

Using the boundary conditions, at $t = 0$, $x = x_0$ and $\mathrm{d}x/\mathrm{d}t = 0$ the solution to (6.47) is found to be

$$x = \frac{1}{2}x_0 \mathrm{e}^{-bt}\left[(1 + b/R)\mathrm{e}^{Rt} + (1 - b/R)\mathrm{e}^{-Rt}\right] \tag{6.50}$$

The physical solution depends on the degree of damping.

Case 1: Small frictional forces: $b < \omega_0$ (underdamping)
$b^2 < k/m$ or $(r/2m)^2 < k/m$

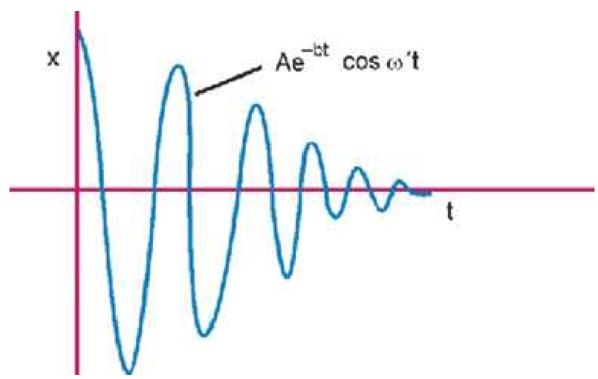
R is imaginary. $R = j\omega'$, where $j = \sqrt{-1}$

$$\omega'^2 = \omega_0^2 - b^2 \tag{6.51}$$

$$x = A\mathrm{e}^{-bt}\cos(\omega' t + \varepsilon) \tag{6.52}$$

$$\text{where } A = \omega_0 x_0/\omega' \text{ and } \varepsilon = \tan^{-1}(-b/\omega') \tag{6.53}$$

Fig. 6.3 Underdamped motion



Equation (6.52) represents damped harmonic motion of period

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} \tag{6.54}$$

$T = 1/b$ is the time in which the amplitude is reduced to $1/e$.

The logarithmic decrement Δ is

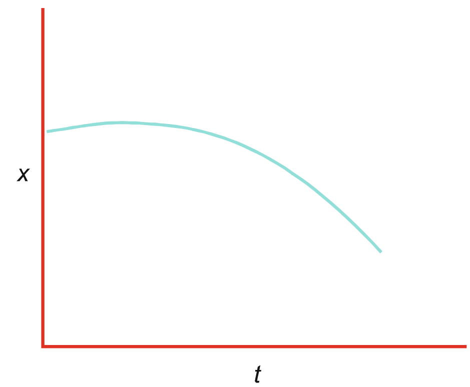
$$\Delta = \ln\left(\frac{A'}{Ae^{-bT'}}\right) = bT' \tag{6.55}$$

Case 2: Large frictional forces (overdamping)
$b > \omega_0$. Distinct real roots.

Both the exponential terms in (6.50) are negative and they correspond to exponential decrease. The motion is not oscillatory. The general solution is of the form

$$x = e^{-bt}(Ae^{Rt} + Be^{-Rt}) \tag{6.56}$$

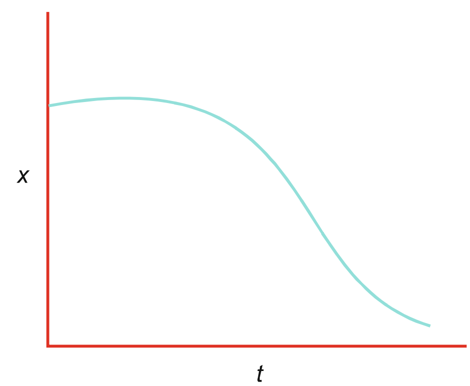
Fig. 6.4 Overdamped motion



Case 3: Critical damping

$$b = \omega,\ \ R = 0$$

Fig. 6.5 Criticallydamped motion



The exponentials in the square bracket may be expanded to terms linear in Rt. The solution is of the form

$$x = x_0\, e^{-bt}(1 + bt) \tag{6.57}$$

The motion is not oscillatory and is said to be critically damped. It is a transition case and the motion is just aperiodic or non-oscillatory. There is an initial rise in the displacement due to the factor $(1 + bt)$ but subsequently the exponential term dominates.

Energy and Amplitude of a Damped Oscillator

$$E(t) = E_0 e^{-t/t_c} \tag{6.58}$$

where $t_c = m/r$

$$A(t) = A_0 e^{-t/2t_c} \tag{6.59}$$

Quality factor

$$Q = \omega t_c = \omega m/r \tag{6.60}$$

$$\omega' = \omega_0 \sqrt{1 - \frac{1}{4Q^2}} \tag{6.61}$$

The value of quality factor indicates the sharpness of resonance.

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} \tag{6.62}$$

where ω_0 is the resonance angular frequency and ω_2 and ω_1 are, respectively, the two angular frequencies above and below resonance at which the average power has dropped to one-half its resonance value. (Fig. 6.6).

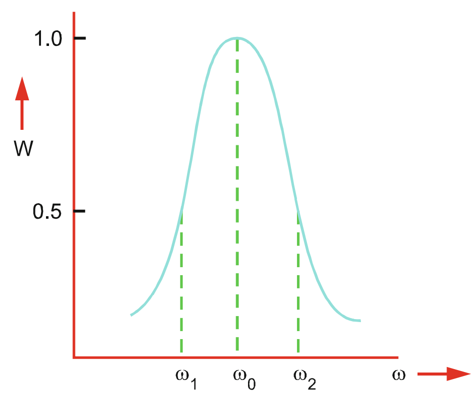


Fig. 6.6 Resonance frequency curve, ω_0 is the resonance angular frequency. ω_1 and ω_2 are defined in the text

Forced vibrations are set up by a periodic force $F \cos \omega t$.

Equation of motion of a particle of mass m

$$\frac{m\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{r\mathrm{d}x}{\mathrm{d}t} + kx = F\cos\omega t \tag{6.63}$$

or

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2b\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = p\cos\omega t \tag{6.64}$$

where

$$k/m = \omega_0^2,\ r/m = 2b \text{ and } F/m = p \tag{6.65}$$

ω_0 being the resonance frequency.

$$x = A\cos(\omega t - \varepsilon) \tag{6.66}$$

$$\tan\varepsilon = \frac{2b\omega}{\omega_0^2 - \omega^2} \tag{6.67}$$

Mechanical impedance

$$Z_m = \sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2} \tag{6.68}$$

$$A = \frac{p}{Z_m} \tag{6.69}$$

$$Q = \frac{\omega_0}{2b} \tag{6.70}$$

Power

$$W = \frac{F^2 - \sin\varepsilon}{2Z_m} \tag{6.71}$$

6.2 Problems

6.2.1 Simple Harmonic Motion (SHM)

6.1 The total energy of a particle executing SHM of period 2π s is 0.256 J. The displacement of the particle at $\pi/4$ s is $8\sqrt{2}$ cm. Calculate the amplitude of motion and mass of the particle.

6.2 A particle makes SHM along a straight line and its velocity when passing through points 3 and 4 cm from the centre of its path is 16 and 12 cm/s, respectively. Find **(a)** the amplitude; **(b)** the time period of motion.

[Northern Universities of UK]

6.3 A small bob of mass 50 g oscillates as a simple pendulum, with amplitude 5 cm and period 2 s. Find the velocity of the bob and the tension in the supporting thread when velocity of the bob is maximum.

[University of Aberystwyth, Wales]

6.4 A particle performs SHM with a period of 16 s. At time $t = 2$ s, the particle passes through the origin while at $t = 4$ s, its velocity is 4 m/s. Show that the amplitude of the motion is $32\sqrt{2}/\pi$.

[University of Dublin]

6.5 Show that given a small vertical displacement from its equilibrium position a floating body subsequently performs simple harmonic motion of period $2\pi\sqrt{V/Ag}$ where V is the volume of displaced liquid and A is the area of the plane of floatation. Ignore the viscous forces.

6.6 Imagine a tunnel bored along the diameter of the earth assumed to have constant density. A box is thrown into the tunnel (chute). **(a)** Show that the box executes SHM inside the tunnel about the centre of the earth. **(b)** Find the time period of oscillations.

6.7 A particle which executes SHM along a straight line has its motion represented by $x = 4\sin(\pi t/3 + \pi/6)$. Find **(a)** the amplitude; **(b)** time period; **(c)** frequency; **(d)** phase difference; **(e)** velocity; **(f)** acceleration, at $t = 1$ s, x being in cm.

6.8 **(a)** At what distance from the equilibrium position is the kinetic energy equal to the potential energy for a SHM?

(b) In SHM if the displacement is one-half of the amplitude show that the kinetic energy and potential energy are in the ratio 3:1.

6.9 A mass M attached to a spring oscillates with a period 2 s. If the mass is increased by 2 kg, the period increases by 1 s. Assuming that Hooke's law is obeyed, find the initial mass M.

6.10 A particle vibrates with SHM along a straight line, its greatest acceleration is $5\pi^2$ cm/s^2, and when its distance from the equilibrium is 4 cm the velocity of the particle is 3π cm/s. Find the amplitude and the period of oscillation of the particle.

6.11 If the maximum acceleration of a SHM is α and the maximum velocity is β, show that the amplitude of vibration is given by β^2/α and the period of oscillation by $2\pi\beta/\alpha$.

6.12 If the tension along the string of a simple pendulum at the lowest position is 1% higher than the weight of the bob, show that the angular amplitude of the pendulum is 0.1 rad.

6.13 A particle executes SHM and is located at $x = a$, b and c at time t_0, $2t_0$ and $3t_0$, respectively. Show that the frequency of oscillation is $\dfrac{1}{2\pi t_0}\cos^{-1}\dfrac{a+c}{2b}$.

6.14 A 4 kg mass at the end of a spring moves with SHM on a horizontal frictionless table with period 2 s and amplitude 2 m. Determine **(a)** the spring constant; **(b)** maximum force exerted on the spring.

6.15 A particle moves in the xy-plane according to the equations $x = a \sin \omega t$; $y = b \cos \omega t$. Determine the path of the particle.

6.16 **(a)** Prove that the force $\boldsymbol{F} = -kx\hat{i}$ acting in a SHO is conservative. **(b)** Find the potential energy of an SHO.

6.17 A 2 kg weight placed on a vertical spring stretches it 5 cm. The weight is pulled down a distance of 10 cm and released. Find **(a)** the spring constant; **(b)** the amplitude; **(c)** the frequency of oscillations.

6.18 A mass m is dropped from a height h on to a scale-pan of negligible weight, suspended from a spring of spring constant k. The collision may be considered to be completely inelastic in that the mass sticks to the pan and the pan begins to oscillate. Find the amplitude of the pan's oscillations.

6.19 A particle executes SHM along the x-axis according to the law $x = A \sin \omega t$. Find the probability $\mathrm{d}p(x)$ of finding the particle between x and $x + \mathrm{d}x$.

6.20 Using the probability density distribution for the SHO, calculate the mean potential energy and the mean kinetic energy over an oscillation.

6.21 A cylinder of mass m is allowed to roll on a smooth horizontal table with a spring of spring constant k attached to it so that it executes SHM about the equilibrium position. Find the time period of oscillations.

6.22 Two simple pendulums of length 60 and 63 cm, respectively, hang vertically one in front of the other. If they are set in motion simultaneously, find the time taken for one to gain a complete oscillation on the other.

[Northern Universities of UK]

6.23 A pendulum that beats seconds and gives correct time on ground at a certain place is moved to the top of a tower 320 m high. How much time will the pendulum lose in 1 day? Assume earth's radius to be 6400 km.

6.24 Taking the earth's radius as 6400 km and assuming that the value of g inside the earth is proportional to the distance from the earth's centre, at what depth below the earth's surface would a pendulum which beats seconds at the earth's surface lose 5 min in a day?

[University of London]

6.25 A U-tube is filled with a liquid, the total length of the liquid column being h. If the liquid on one side is slightly depressed by blowing gently down, the levels of the liquid will oscillate about the equilibrium position before finally coming to rest. **(a)** Show that the oscillations are SHM. **(b)** Find the period of oscillations.

6.26 A gas of mass m is enclosed in a cylinder of cross-section A by means of a frictionless piston. The gas occupies a length l in the equilibrium position and is at pressure P. **(a)** If the piston is slightly depressed, show that it will execute SHM. **(b)** Find the period of oscillations (assume isothermal conditions).

6.27 A SHM is given by $y = 8\sin\left(\frac{2\pi t}{\tau} + \varphi\right)$, the time period being 24 s. At $t = 0$, the displacement is 4 cm. Find the displacement at $t = 6$ s.

6.28 In a vertical spring-mass system, the period of oscillation is 0.89 s when the mass is 1.5 kg and the period becomes 1.13 s when a mass of 1.0 kg is added. Calculate the mass of the spring.

6.29 Consider two springs A and B with spring constants k_A and k_B, respectively, A being stiffer than B, that is, $k_A > k_B$. Show that

(a) when two springs are stretched by the same amount, more work will be done on the stiffer spring.
(b) when two springs are stretched by the same force, less work will be done on the stiffer spring.

6.30 A solid uniform cylinder of radius r rolls without sliding along the inside surface of a hollow cylinder of radius R, performing small oscillations. Determine the time period.

6.2.2 Physical Pendulums

6.31 Consider the rigid plane object of weight Mg shown in Fig. 6.7, pivoted about a point at a distance D from its centre of mass and displaced from equilibrium by a small angle φ. Such a system is called a physical pendulum. Show that the oscillatory motion of the object is simple harmonic with a period given by $T = 2\pi\sqrt{\frac{I}{MgD}}$ where I is the moment of inertia about the pivot point.

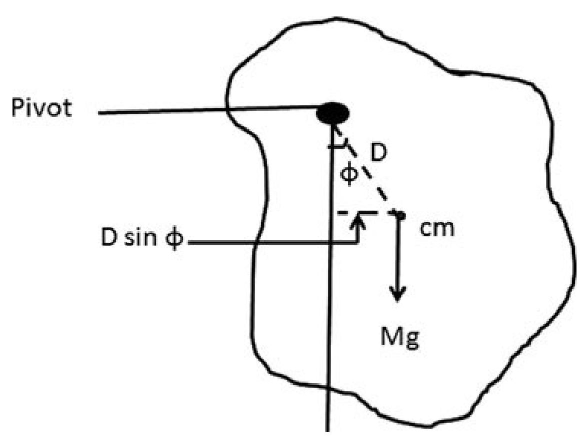
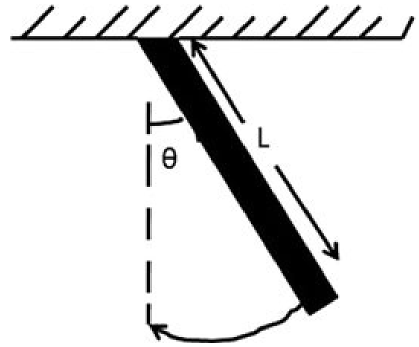


Fig. 6.7

6.32 A thin, uniform rod of mass M and length L swings from one of its ends as a physical pendulum (see Fig. 6.8). Given that the moment of inertia of a

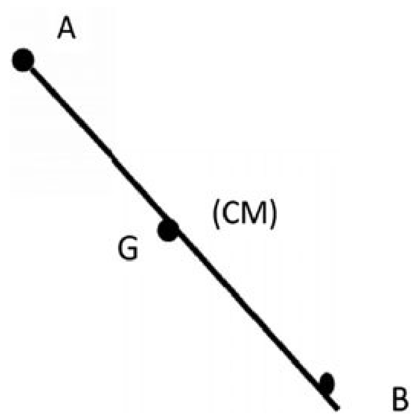
Fig. 6.8



uniform rod about one end is $I = \frac{1}{3}ML^2$, obtain an equation for the period of the oscillatory motion for small angles. What would be the length l of a simple pendulum that has the same period as the swinging rod?

6.33 The physical pendulum has two possible pivot points A and B, distance L apart, such that the period of oscillations is the same (Fig. 6.9). Show that the acceleration due to gravity at the pendulum's location is given by $g = 4\pi^2 L/T^2$.

Fig. 6.9



6.34 A semi-circular homogeneous disc of radius R and mass m is pivoted freely about the centre. If slightly tilted through a small angle and released, find the angular frequency of oscillations.

6.35 A ring is suspended on a nail. It can oscillate in its plane with time period T_1 or it can oscillate back and forth in a direction perpendicular to the plane of the ring with time period T_2. Find the ratio T_1/T_2.

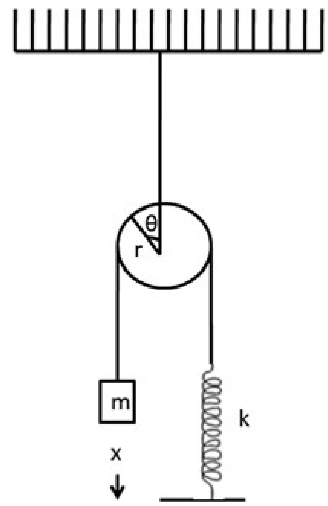
6.36 A torsional oscillator consists of a flat metal disc suspended by a wire. For small angular displacements show that time period is given by

$$T = 2\pi\sqrt{\frac{I}{C}}$$

where I is the moment of inertia about its axis and C is known as torsional constant given by $\tau = -C\theta$, where τ is the torque.

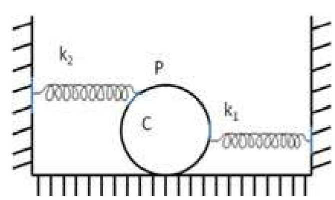
6.37 In the arrangement shown in Fig. 6.10, the radius of the pulley is r, its moment of inertia about the rotation axis is I and k is the spring constant. Assuming that the mass of the thread and the spring is negligible and that the thread does not slide over the frictionless pulley, calculate the angular frequency of small oscillations.

Fig. 6.10



6.38 Two unstretched springs with spring constants k_1 and k_2 are attached to a solid cylinder of mass m as in Fig. 6.11. When the cylinder is slightly displaced and released it will perform small oscillations about the equilibrium position. Assuming that the cylinder rolls without sliding, find the time period.

Fig. 6.11



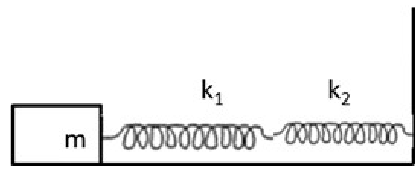
6.39 A particle of mass m is located in a one-dimensional potential field $U(x) = \dfrac{a}{x^2} - \dfrac{b}{x}$ where a and b are positive constants. Show that the period of small oscillations that the particle performs about the equilibrium position will be $T = 4\pi\sqrt{\dfrac{2a^3 m}{b^4}}$

[Osmania University 1999]

6.2.3 Coupled Systems of Masses and Springs

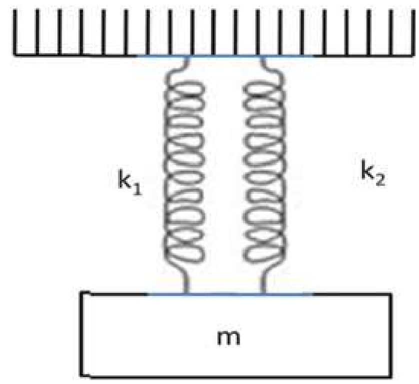
6.40 Two springs of constants k_1 and k_2 are connected in series, Fig. 6.12. Calculate the effective spring constant.

Fig. 6.12



6.41 A mass m is connected to two springs of constants k_1 and k_2 in parallel, Fig. 6.13. Calculate the effective (equivalent) spring constant.

Fig. 6.13



6.42 A mass m is placed on a frictionless horizontal table and is connected to fixed points A and B by two springs of negligible mass and of equal natural length with spring constants k_1 and k_2, Fig. 6.14. The mass is displaced along x-axis and released. Calculate the period of oscillation.

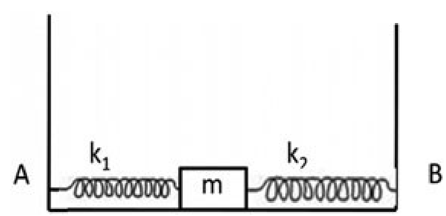


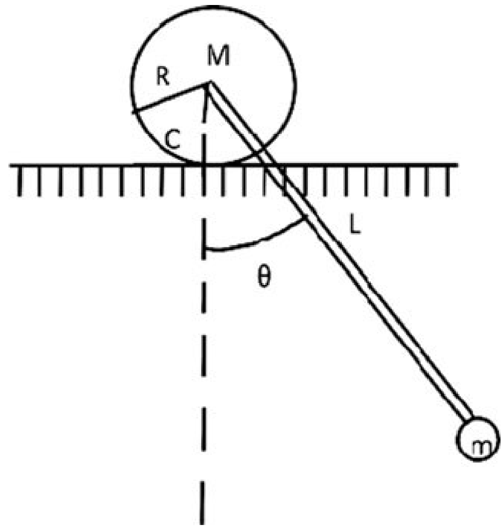
Fig. 6.14

6.43 One end of a long metallic wire of length L is tied to the ceiling. The other end is tied to a massless spring of spring constant k. A mass m hangs freely from the free end of the spring. The area of cross-section and the Young's modulus of the wire are A and Y respectively. The mass is displaced down and released. Show that it will oscillate with time period $T = 2\pi\sqrt{\dfrac{m(YA + kL)}{YAk}}$.

[Adapted from Indian Institute of Technology 1993]

6.44 The mass m is attached to one end of a weightless stiff rod which is rigidly connected to the centre of a uniform cylinder of radius R, Fig. 6.15. Assuming that the cylinder rolls without slipping, calculate the natural frequency of oscillation of the system.

Fig. 6.15



6.45 Find the natural frequency of a semi-circular disc of mass m and radius r which rolls from side to side without slipping.

6.46 Determine the eigenfrequencies and describe the normal mode motion for two pendula of equal lengths b and equal masses m connected by a spring of force constant k as shown in Fig. 6.16. The spring is unstretched in the equilibrium position.

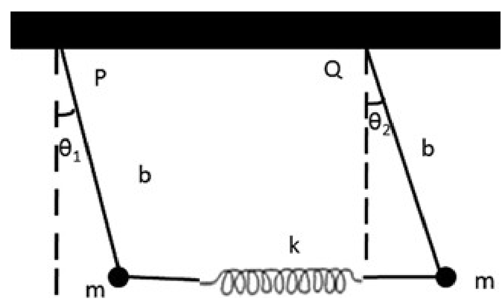


Fig. 6.16

6.47 In prob. (6.46) express the equations of motion and the energy in terms of normal coordinates. What are the characteristics of normal coordinates?

6.48 The superposition of two harmonic oscillations in the same direction leads to the resultant displacement $y = A\cos 6\pi t \sin 90\pi$, where t is expressed in seconds. Find the frequency of the component vibrations and the beat frequency.

6.49 Find the fundamental frequency of vibration of the HCl molecule. The masses of H and Cl may be assumed to be 1.0 and 36.46 amu.

$1\,\text{amu} = 1.66 \times 10^{-27}$ kg and $k = 480\,\text{N/m}$

6.50 Find the resultant of the vibrations $y_1 = \cos\omega t$, $y_2 = \frac{1}{2}\cos(\omega t + \pi/2)$ and $y_3 = \dfrac{1}{3}\cos(\omega t + \pi)$, acting in the same straight line.

6.2.4 Damped Vibrations

6.51 A mass attached to a spring vibrates with a natural frequency of 20 c/s while its frequency for damped vibrations is 16 c/s. Determine the logarithmic decrement.

6.52 The equation of motion for a damped oscillator is given by

$$4\mathrm{d}^2x/\mathrm{d}t^2 + r\mathrm{d}x/\mathrm{d}t + 32x = 0$$

For what range of values for the damping constant will the motion be **(a)** underdamped; **(b)** overdamped; **(c)** critically damped?

6.53 A mass of 4 kg attached to the lower end of a vertical spring of constant 20 N/m oscillates with a period of 10 s. Find **(a)** the natural period; **(b)** the damping constant; **(c)** the logarithmic decrement.

6.54 Solve the equation of motion for the damped oscillator $\mathrm{d}^2x/\mathrm{d}t^2 + 2\mathrm{d}x/\mathrm{d}t + 5x = 0$, subject to the condition $x = 5$, $\mathrm{d}x/\mathrm{d}t = -3$ at $t = 0$.

6.55 A 1 kg weight attached to a vertical spring stretches it 0.2 m. The weight is then pulled down 1.5 m and released. **(a)** Is the motion underdamped, overdamped or critically damped? **(b)** Find the position of the weight at any time if a damping force numerically equal to 14 times the instantaneous speed is acting.

6.56 A periodic force acts on a 6 kg mass suspended from the lower end of a vertical spring of constant 150 N/m. The damping force is proportional to the instantaneous speed of the mass and is 80 N when $v = 2\,\text{m/s}$. find the resonance frequency.

6.57 The equation of motion for forced oscillations is $2\ \mathrm{d}^2x/\mathrm{d}t^2 + 1.5\mathrm{d}x/\mathrm{d}t + 40x = 12\cos 4t$. Find **(a)** amplitude; **(b)** phase lag; **(c)** Q factor; **(d)** power dissipation.

6.58 An electric bell has a frequency 100 Hz. If its time constant is 2 s, determine the Q factor for the bell.

6.59 An oscillator has a time period of 3 s. Its amplitude decreases by 5% each cycle **(a)** By how much does its energy decrease in each cycle? **(b)** Find the time constant **(c)** Find the Q factor.

6.60 A damped oscillator loses 3% of its energy in each cycle. **(a)** How many cycles elapse before half its original energy is dissipated? **(b)** What is the Q factor?

6.61 A damped oscillator has frequency which is 9/10 of its natural frequency. By what factor is its amplitude decreased in each cycle?

6.62 Show that for small damping $\omega' \approx (1 - r^2/8mk)\omega_0$ where ω_0 is the natural angular frequency, ω' the damped angular frequency, r the resistance constant, k the spring constant and m the particle mass.

6.63 Show that the time elapsed between successive maximum displacements of a damped harmonic oscillator is constant and equal to $4\pi m/\sqrt{4km - r^2}$, where m is the mass of the vibrating body, k is the spring constant, $2b = r/m$, r being the resistance constant.

6.64 A dead weight attached to a light spring extends it by 9.8 cm. It is then slightly pulled down and released. Assuming that the logarithmic decrement is equal to 3.1, find the period of oscillation.

6.65 The position of a particle moving along x-axis is determined by the equation $\mathrm{d}^2x/\mathrm{d}t^2 + 2\mathrm{d}x/\mathrm{d}t + 8x = 16\cos 2t$.

(a) What is the natural frequency of the vibrator?
(b) What is the frequency of the driving force?

6.66 Show that the time $t_{1/2}$ for the energy to decrease to half its initial value is related to the time constant by $t_{1/2} = t_c \ln 2$.

6.67 The amplitude of a swing drops by a factor $1/e$ in 8 periods when no energy is pumped into the swing. Find the Q factor.

6.3 Solutions

6.3.1 Simple Harmonic Motion (SHM)

6.1 $x = A\sin\omega t$ (SHM)

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1\,\mathrm{rad/s}$$

$$8\sqrt{2} = A\sin\left(\frac{1\cdot\pi}{4}\right)$$

$$A = 16\,\mathrm{cm} = 0.16\,\mathrm{m}$$

$$E = \frac{1}{2}mA^2\omega^2$$

$$\therefore \quad m = \frac{2E}{A^2\omega^2} = \frac{2\times 0.256}{(0.16)^2\times 1^2} = 20.0\,\mathrm{kg}$$

6.2 **(a)** $$v = \omega\sqrt{A^2 - x^2} \tag{1}$$

$$16 = \omega\sqrt{A^2 - 3^2} \tag{2}$$

$$12 = \omega\sqrt{A^2 - 4^2} \tag{3}$$

Solving (2) and (3) A = 5 cm and $\omega = 4$ rad/s

(b) Therefore $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{4} = 1.57\,\text{s}$

6.3 $x = A\sin\omega t$

$$v = \frac{dx}{dt} = \omega A\cos\omega\text{t}$$

$$v_{\max} = A\omega = \frac{2\pi A}{T} = \frac{2\pi \times 5}{2} = 5\pi\ \text{cm/s}$$

At the equilibrium position the weight of the bob and the tension act in the same direction

$$\text{Tension} = mg + \frac{mv_{\max}^2}{L}$$

Now the length of the simple pendulum is calculated from its period T.

$$L = \frac{gT^2}{4\pi^2} = \frac{980 \times 2^2}{4\pi^2} = 99.29\,\text{cm}$$

$$\text{Tension} = m\left(1 + \frac{v_{\max}^2}{gL}\right)g = 50\left(1 + \frac{25\pi^2}{980 \times 99.29}\right)g$$

$$= 50.13\ \text{g dynes} = 50.13\ \text{g wt}$$

6.4 The general equation of SHM is

$$x = A\sin(\omega t + \varepsilon)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{16} = \frac{\pi}{8}$$

When $t = 2$ s, $x = 0$.

$$0 = A\sin\left(\frac{\pi}{8} \times 2 + \varepsilon\right)$$

Since $A \neq 0$, $\sin\left(\dfrac{\pi}{4} + \varepsilon\right) = 0$

$$\therefore \quad \frac{\pi}{4} + \varepsilon = 0 \quad \varepsilon = -\frac{\pi}{4}$$

Now $v = \dfrac{\mathrm{d}x}{\mathrm{d}t} = A\omega \cos(\omega t + \varepsilon)$

When $t = 4$, $v = 4$.

$$\therefore \quad 4 = \frac{A\pi}{8} \cos\left(\frac{\pi}{8}4 - \frac{\pi}{4}\right)$$

$$\therefore \quad A = \frac{32\sqrt{2}}{\pi}$$

6.5 Let the body with uniform cross-section A be immersed to a depth h in a liquid of density D. Volume of the liquid displaced is $V = Ah$. Weight of the liquid displaced is equal to VDg or $AhDg$. According to Archimedes principle, the weight of the liquid displaced is equal to the weight of the floating body Mg.

$Mg = Ahdg$ or $\mathrm{M} = AhD$

The body occupies a certain equilibrium position. Let the body be further depressed by a small amount x. The body now experiences an additional upward thrust in the direction of the equilibrium position. When the body is released it moves up with acceleration

$$a = -\frac{AxDg}{M} = -\frac{AxDg}{AhD} = -\frac{gx}{h} = -\omega^2 x$$

with $\omega^2 = \dfrac{g}{h}$

Time period $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{h}{g}} = 2\pi\sqrt{\dfrac{V}{Ag}}$

6.6 The acceleration due to gravity g at a depth d from the surface is given by

$$g = g_0\left(1 - \frac{d}{R}\right) \tag{1}$$

where g_0 is the value of g at the surface of the earth of radius R.

Writing $x = R - d$ (2)

Equation (1) becomes $g = g_0 \dfrac{x}{R}$ (3)

where x measures the distance from the centre. The acceleration g points opposite to the displacement x. We can therefore write

$$a = g = -\frac{g_0 x}{R} = -\omega^2 x \tag{4}$$

with $\omega^2 = \dfrac{g_0}{R}$

Equation (4) shows that the box performs SHM. The period is calculated from

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g_0}} = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}} = 5074\,\text{s or } 84.6\,\text{min}$$

6.7 Standard equation for SHM is

$x = A\sin(\omega t + \varepsilon)$

$x = 4\sin\left(\dfrac{\pi t}{3} + \dfrac{\pi}{6}\right)$

(a) A = 4 cm

(b) $\omega = \dfrac{\pi}{3}$. Therefore $T = \dfrac{2\pi}{\omega} = 6\,\text{s}$

(c) $f = \dfrac{1}{T} = \dfrac{1}{6}/\text{s}$

(d) $\varepsilon = \dfrac{\pi}{6}$

(e) $v = \dfrac{dx}{dt} = \dfrac{4\pi}{3}\cos\left(\dfrac{\pi t}{3} + \dfrac{\pi}{6}\right) = \dfrac{4\pi}{3}\cos\left(\dfrac{\pi}{3} \times 1 + \dfrac{\pi}{6}\right) = 0$

(f) $a = \dfrac{dv}{dt} = -\dfrac{4\pi^2}{9}\sin\left(\dfrac{\pi}{3} \times 1 + \dfrac{\pi}{6}\right) = -\dfrac{4\pi^2}{9}$

6.8 **(a)** $K = \dfrac{1}{2}m\omega^2(A^2 - x^2) \quad U = \dfrac{1}{2}m\omega^2 x^2 \quad K = U$

$$\therefore \quad \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}m\omega^2 x^2$$

$$\therefore \quad x = \frac{A}{\sqrt{2}}$$

(b) $K = \dfrac{1}{2}m\omega^2\left(A^2 - \dfrac{A^2}{4}\right) = \dfrac{1}{2}m\omega^2\dfrac{3}{4}A^2$

$$U = \frac{1}{2}m\omega^2\frac{A^2}{4}$$

$$\therefore \quad K : U = 3 : 1$$

6.9 $$T = 2\pi\sqrt{\frac{M}{k}} \tag{1}$$

$$2 = 2\pi\sqrt{\frac{M}{k}} \tag{2}$$

$$3 = 2\pi\sqrt{\frac{M+2}{k}} \tag{3}$$

Dividing (2) by (3) and solving for M, we get $M = 1.6\,\text{kg}$.

6.10 $$a_{\text{max}} = \omega^2 A$$

$$5\pi^2 = \omega^2 A \tag{1}$$

$$v = \omega\sqrt{A^2 - x^2}$$

$$3\pi = \omega\sqrt{A^2 - 16} \tag{2}$$

Solving (1) and (2), we get $A = 5\,\text{cm}$ and $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2\,\text{s}$.

6.11 $$\alpha = \omega^2 A \tag{1}$$

$$\beta = \omega A \tag{2}$$

$$\therefore \quad \beta^2 = \omega^2 A^2 = \alpha A$$

$$\text{or} \quad A = \frac{\beta^2}{\alpha}$$

Dividing (2) by (1)

$$\frac{\beta}{\alpha} = \frac{1}{\omega}$$

$$\text{or} \quad T = \frac{2\pi}{\omega} = \frac{2\pi\beta}{\alpha}$$

6.12 By problem $\frac{mg + mv^2/L}{mg} = 1.01$

$$\therefore \quad \frac{v^2}{gL} = 0.01$$

Conservation of energy gives

$$\frac{1}{2}mv^2 = mgh = mgL(1 - \cos\theta) \simeq mgL\frac{\theta^2}{2} \quad \text{for small } \theta$$

$$\theta^2 = \frac{v^2}{gL} = 0.01$$
$$\therefore \quad \theta = \sqrt{0.01} = 0.1 \text{ rad}$$

6.13 $a = A \sin \omega t_0$

$$b = A \sin 2\omega t_0$$
$$c = A \sin 3\omega t_0$$
$$a + c = 2A \sin 2\omega t_0 \cos \omega t_0$$
$$\frac{a+c}{2b} = \cos \omega t_0$$
$$\omega = \frac{1}{t_0} \cos^{-1}\left(\frac{a+c}{2b}\right)$$
$$f = \frac{1}{2\pi t_0} \cos^{-1}\left(\frac{a+c}{2b}\right)$$

6.14 **(a)** $\omega = \sqrt{\dfrac{k}{m}}$

$$k = m\omega^2 = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2 \times 4}{2^2} = 39.478\,\text{N/m}$$

(b) $F_{\max} = m\omega^2 A = kA = 39.478 \times 2 = 78.96\,\text{N}$

6.15 $x = a \sin \omega t$

$$y = b \cos \omega t$$
$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2 \omega t + \cos^2 \omega t = 1$$

Thus the path of the particle is an ellipse.

6.16 **(a)** To show that $\nabla \times \boldsymbol{F} = 0$.

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -Kx & 0 & 0 \end{vmatrix} = 0$$

(b) $U = -\int F \mathrm{d}x = -\int (K\ ix)(-\hat{i}\ \mathrm{d}x) = \dfrac{1}{2}Kx^2$

6.17 **(a)** $F = kx$

$$\therefore \quad k = \frac{F}{x} = \frac{2 \times 9.8}{5 \times 10^{-2}} = 392\,\text{N/m}$$

(b) 10 cm

(c) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{392}{2 \times 9.8}} = 0.712/\,\text{s}$

6.18 Let x_0 be the extension of the spring. Deformation energy = gravitational potential energy

$$\frac{1}{2}kx_0^2 = mgh + mgx_0$$

Rearranging

$$x_0^2 - \frac{2mg}{k}x_0 - mgh = 0$$

The quadratic equation has the solutions

$$x_{01} = \frac{mg}{k} + \sqrt{\frac{m^2g^2}{k^2} + \frac{2mgh}{k}}$$

$$x_{02} = \frac{mg}{k} - \sqrt{\frac{m^2g^2}{k^2} + \frac{2mgh}{k}}$$

The equilibrium position is depressed by $x_0 = \dfrac{mg}{k}$ below the initial position. The amplitude of the oscillations as measured from the equilibrium position is equal to $\sqrt{\dfrac{m^2g^2}{k^2} + \dfrac{2mgh}{k}}$.

6.19 It is reasonable to assume that the probability density $\dfrac{\mathrm{d}p(x)}{\mathrm{d}x}$ for finding the particle is proportional to the time spent at a given point and is therefore inversely proportional to its speed v.

$$\frac{\mathrm{d}p(x)}{\mathrm{d}x} = \frac{C}{v} \tag{1}$$

where C = constant of proportionality.

$$\text{But } v = \omega\sqrt{A^2 - x^2} \tag{2}$$

The probability density

$$\frac{dp(x)}{dx} = \frac{C}{\omega\sqrt{A^2 - x^2}} \tag{3}$$

C can be found by normalization of distribution

$$\int_{-A}^{A} dp(x) = \frac{C}{\omega}\int_{-A}^{A}\frac{dx}{\sqrt{A^2 - x^2}} = 1$$

or $\dfrac{C\pi}{\omega} = 1 \to \dfrac{C}{\omega} = \dfrac{1}{\pi}$

$$\therefore \quad \frac{dp(x)}{dx} = \frac{1}{\pi\sqrt{A^2 - x^2}}$$

6.20 $U = \frac{1}{2}kx^2$

Using the result of prob. (6.19)

$$\langle U\rangle = \int U\,dp(x) = \int_{-A}^{A}\frac{1}{2}kx^2\frac{dx}{\pi\sqrt{A^2 - x^2}}$$

Put $x = A\sin\theta$, $\quad dx = A\cos\theta\, d\theta$

$$\langle U\rangle = \left(\frac{kA^2}{2\pi}\right)\int_{-\pi/2}^{\pi/2}\sin^2\theta\, d\theta = \frac{1}{4}kA^2$$

Also, $\langle K\rangle = \langle E - U\rangle = \frac{1}{2}kA^2 - \frac{1}{4}kA^2 = \frac{1}{4}kA^2$

6.21 $K_{\text{trans}} + K_{\text{rot}} + U = \text{constant}$

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 = \text{constant}$$

But $I = \frac{1}{2}mR^2$ and $\omega = \frac{v}{R}$

$$\therefore \quad \frac{3}{4}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = 0 \text{ constant}$$

Differentiating

$$\frac{3}{2}m\frac{d^2x}{dt^2}\frac{dx}{dt} + kx\frac{dx}{dt} = 0$$

Cancelling $\mathrm{d}x/\mathrm{d}t$ throughout and simplifying

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\frac{2k}{3m}\right)x = 0$$

This is the equation for SHM

with $\omega^2 = \left(\dfrac{2k}{3m}\right)$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3m}{2k}}$$

6.22 The time period of the pendulums is

$$T_1 = 2\pi\sqrt{\frac{60}{g}} \tag{1}$$

$$T_2 = 2\pi\sqrt{\frac{63}{g}} \tag{2}$$

Let the time be t in which the longer length pendulum makes n oscillations while the shorter one makes $(n+1)$ oscillations. Then

$$t = (n+1)T_1 = nT_2 \tag{3}$$

Using (1) and (2) in (3), we find $n = 40.5$ and $t = 64.49$ s.

6.23 Let g_0 be the acceleration due to gravity on the ground and g at height above the ground. Then

$$g = \frac{g_0 R^2}{(R+h)^2}$$

At the ground, $T_0 = 2\pi\sqrt{\dfrac{L}{g_0}}$. At height h, $T = 2\pi\sqrt{\dfrac{L}{g}}$

$$T = T_0\sqrt{\frac{g_0}{g}} = T_0\left(1 + \frac{h}{R}\right) = 2\left(1 + \frac{320}{6.4 \times 10^6}\right) = 2.0001\ \mathrm{s}$$

Time lost in one oscillation on the top of the tower $= 2.0001 - 2.0000 = 0.0001$ s. Number of oscillations in a day for the pendulum which beats seconds on the ground

$$= \frac{86400}{2.0} = 43{,}200$$

Therefore, time lost in 43,200 oscillations

$= 42{,}300 \times 0.0001 = 4.32\,\text{s}$

6.24 $$g = g_0\left(1 - \frac{d}{R}\right) \tag{1}$$

where g and g_0 are the acceleration due to gravity at depth d and surface, respectively, and R is the radius of the earth.

$$T = T_0\sqrt{\frac{g_0}{g}} = T_0\left(1 - \frac{d}{R}\right)^{-1/2} = T_0\left(1 + \frac{d}{2R}\right)$$

Time registered for the whole day will be proportional to the time period. Thus

$$\frac{T}{T_0} = \frac{t}{t_0} = 1 + \frac{d}{2R}$$
$$\frac{86{,}400}{86{,}400 - 300} = 1 + \frac{d}{2R}$$

Substituting $R = 6400\,\text{km}$, we find $d = 44.6\,\text{km}$.

6.25 **(a)** Let the liquid level in the left limb be depressed by x, so that it is elevated by the same height in the right limb (Fig. 6.17). If ρ is the density of the liquid, A the cross-section of the tube, M the total mass, and m the mass of liquid corresponding to the length $2x$, which provides the unbalanced force,

$$\frac{M\mathrm{d}^2x}{\mathrm{d}t^2} = -mg = -(2xA\rho)g$$
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{2A\rho g}{M}x = -\frac{2A\rho gx}{hA\rho} = -\frac{2gx}{h} = -\omega^2 x$$

This is the equation of SHM.

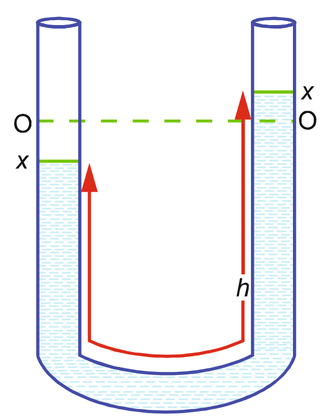


Fig. 6.17

(b) The time period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{2g}}$$

6.26 **(a)** Let the gas at pressure P and volume V be compressed by a small length x, the new pressure being p' and new volume V' (Fig. 6.18) under isothermal conditions.

$$P'V' = PV$$
$$\text{or} \quad P'(l - x)A = PlA$$

where A is the cross-sectional area.

$$P' = \frac{Pl}{l - x} - P\left(1 - \frac{x}{l}\right)^{-1} \simeq P\left(1 + \frac{x}{l}\right)$$

where we have expanded binomially up to two terms since $x << l$. The change in pressure is

$$\Delta P = P' - P = \frac{Px}{h}$$

The unbalanced force

$$F = -\Delta PA = -\frac{APx}{l}$$

and the acceleration

$$a = \frac{F}{m} = -\frac{APx}{ml} = -\omega^2 x$$

which is the equation for SHM.

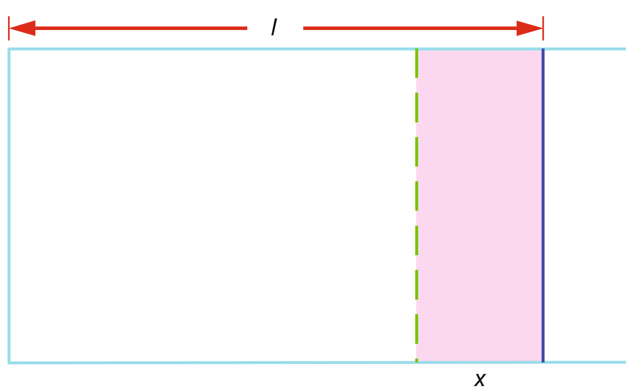


Fig. 6.18

(b) The time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{AP}}$$

6.27 $y = 8\sin\left(\frac{2\pi t}{T} + \phi\right)$

At $t = 0$; $\quad 4 = 8\sin\phi$

$\therefore \quad \phi = 30° = \frac{\pi}{6}$

$y = 8\sin\left(\frac{2\pi \times 6}{24} + \frac{\pi}{6}\right) = 8\sin 120 = 4\sqrt{3}\,\text{cm}$

6.28 Time period of a loaded spring

$$T = 2\pi\sqrt{\frac{M + \frac{m}{3}}{k}} \tag{1}$$

where M is the suspended mass, m is the mass of the spring and k is the spring constant

$$0.89 = 2\pi\sqrt{\frac{1.5 + \frac{m}{3}}{k}} \tag{2}$$

$$1.13 = 2\pi\sqrt{\frac{2.5 + \frac{m}{3}}{k}} \tag{3}$$

Dividing the two equations and solving for m, we get $m = 0.39\,\text{kg}$.

6.29 **(a)** $k_A > k_B$

Let the springs be stretched by the same amount. Then the work done on the two springs will be

$$W_A = \frac{1}{2}k_A x^2 \quad W_B = \frac{1}{2}k_B x^2$$

$$\frac{W_A}{W_B} = \frac{k_A}{k_B}$$

Thus $W_A > W_B$, i.e. when two springs are stretched by the same amount, more work will be done on the stiffer spring.

(b) Let the two springs be stretched by equal force. Thus the work done

$$W_A = \frac{1}{2}k_A x^2 = \frac{1}{2}k_A\left(\frac{F}{k_A}\right)^2 = \frac{1}{2}\frac{F^2}{k_A}$$

$$W_B = \frac{1}{2}\frac{F^2}{k_B}$$

$$\therefore \quad \frac{W_A}{W_B} = \frac{k_B}{k_A}$$

Thus when two springs are stretched by the same force, less work will be done on the stiffer spring.

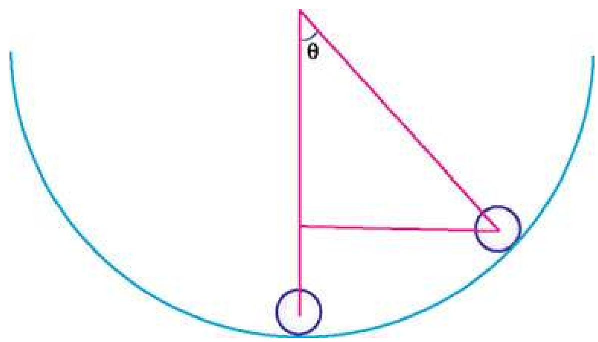


Fig. 6.19

6.30 $K_{\text{trans}} + K_{\text{rot}} + U = C = \text{constant}$

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mg(R-r)(1-\cos\theta) = C$$

Now $\quad I = \frac{1}{2}mr^2 \quad \omega = \frac{v}{r}$

$$\frac{3}{4}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + mg(R-r)\frac{\theta^2}{2} = C$$

Differentiating with respect to time

$$\frac{3}{2}m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\frac{\mathrm{d}x}{\mathrm{d}t} + mg(R-r)\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$$

Now $x = (R-r)\theta$

$$\therefore \quad \frac{3}{2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(R-r)\frac{\mathrm{d}\theta}{\mathrm{d}t} + gx\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$$

Cancelling $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ throughout

$$\frac{d^2x}{dt^2} + \frac{2}{3}\frac{gx}{(R-r)} = 0$$

which is the equation for SHM, with

$$\omega^2 = \frac{2}{3}\frac{g}{R-r}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3(R-r)}{2g}}$$

6.3.2 Physical Pendulums

6.31 If α is the angular acceleration, the torque τ is given by

$$\tau = I\alpha = I\frac{d^2\phi}{dt^2} \tag{1}$$

The restoring torque for an angular displacement ϕ is

$$\tau = -MgD\sin\phi \tag{2}$$

which arises due to the tangential component of the weight. Equating the two torques for small ϕ,

$$I\frac{d^2\phi}{dt^2} = -MgD\sin\phi = -MgD\,\phi$$

$$\text{or} \quad \frac{d^2\phi}{dt^2} + \frac{MgD}{I}\phi = 0 \tag{3}$$

which is the equation for SHM with

$$\omega^2 = \frac{MgD}{I} \tag{4}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{MgD}}$$

6.32 Equation for the oscillatory motion is obtained by putting $I = \frac{1}{3}ML^2$ and $D = \frac{L}{2}$ in (3) of prob. (6.31).

$$\frac{d^2\theta}{dt^2} + \frac{MgD}{I}\theta = 0 \tag{3}$$

$$\frac{d^2\theta}{dt^2} + \frac{3}{2}\frac{g}{L}\theta = 0$$

$$\omega^2 = \frac{3}{2}\frac{g}{L}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2L}{3g}} \tag{4}$$

For a simple pendulum

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{5}$$

Comparing (4) and (5), the equivalent length of a simple pendulum is $l = \frac{2}{3}L$.

6.33 From the results of prob. (6.31) the time period of a physical pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{MgD}} \tag{1}$$

where I is the moment of inertia about the pivot A, Fig. 6.9.

Now $\quad I = I_C + MD^2$ and $I_C = Mk^2$ (2)

where k is the radius of gyration. Formula (1) then becomes

$$T = 2\pi\sqrt{\frac{k^2 + D^2}{gD}} \tag{3}$$

and the length of the simple equivalent pendulum is $D + \frac{k^2}{D}$.

If a point B be taken on AG such that $AB = D + \frac{k^2}{D}$, A and B are known as the centres of suspension and oscillation, respectively. Here G is the centre of mass (CM) of the physical pendulum.
Suppose now the body is suspended at B, then the time of oscillation is obtained by substituting $\frac{k^2}{D}$ for D in the expression

$$2\pi\sqrt{\frac{k^2+D^2}{gD}} \text{ and is therefore } 2\pi\sqrt{\frac{k^2+\dfrac{k^4}{D^2}}{g\dfrac{k^2}{D}}} \text{ i.e. } 2\pi\sqrt{\frac{D^2+k^2}{gD}}$$

Thus the centres of suspension and oscillation are convertible, for if the body be suspended from either it will make small vibrations in the same time as a simple pendulum whose length L is the distance between these centres.

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \text{or} \quad g = \frac{4\pi^2 L}{T^2}$$

6.34 $$\omega = \sqrt{\frac{mgd}{I}} \tag{1}$$

$$d = \frac{4R}{3\pi} \tag{2}$$

the distance of the point of suspension from the centre of mass

$$I = \frac{mR^2}{2} \tag{3}$$

Substituting (2) and (3) in (1) and simplifying

$$\omega = \sqrt{\frac{8g}{3\pi R}}$$

6.35 $$T = 2\pi\sqrt{\frac{I}{mgd}}$$

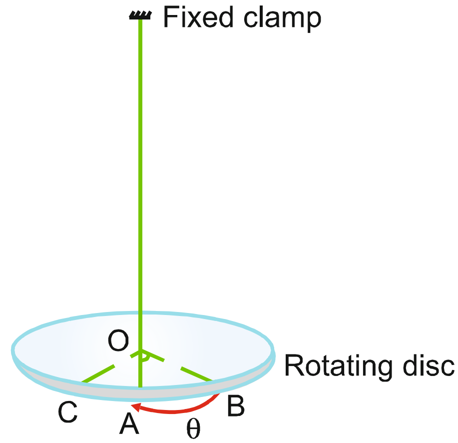
$$T_1 = 2\pi\sqrt{\frac{mr^2+mr^2}{mgr}} = 2\pi\sqrt{\frac{2r}{g}}$$

$$T_2 = 2\pi\sqrt{\frac{\frac{1}{2}mr^2+mr^2}{mgr}} = 2\pi\sqrt{\frac{3}{2}\frac{r}{g}}$$

$$\therefore \quad \frac{T_1}{T_2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

6.36 In Fig. 6.20 OA is the reference line or the disc in the equilibrium position. If the disc is rotated in the horizontal plane so that the reference line occupies the line OB, the wire would have twisted through an angle θ. The twisted wire will exert a restoring torque on the disc causing the reference line to move to

Fig. 6.20



its original position. For small twists the restoring torque will be proportional to the angular displacement in accordance with Hooke's law.

$$\tau = -C\theta \tag{1}$$

where C is known as torsional constant. If I is the moment of inertia of the disc about its axis, α the angular acceleration, the torque τ is given by

$$\tau = I\alpha = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{2}$$

Comparing (1) and (2)

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -C\theta$$

$$\text{or} \quad \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{C}{I}\theta = 0 \tag{3}$$

which is the equation for angular SHM with $\omega^2 = \frac{C}{I}$. Time period for small oscillations is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \tag{4}$$

6.37 Total kinetic energy of the system

$$K = K(\text{mass}) + K(\text{pulley}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$$

Replacing x by $r\theta$ and $\dot{x}$ by $r\dot{\theta}$

$$K = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}(mr^2 + I)\dot{\theta}^2$$

Potential energy of the spring

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kr^2\theta^2$$

Total energy

$$E = K + U = \frac{1}{2}(mr^2 + I)\dot{\theta}^2 + \frac{1}{2}kr^2\theta^2 = \text{constant}$$

Differentiating with respect to time

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (mr^2 + I)\dot{\theta} \cdot \ddot{\theta} + kr^2\theta \cdot \dot{\theta} = 0$$

Cancelling $\dot{\theta}$

$$\ddot{\theta} + \frac{kr^2\theta}{mr^2 + I} = 0$$

which is the equation for angular SHM with

$\omega^2 = \dfrac{kr^2}{mr^2 + I}$. Therefore

$$\omega = \sqrt{\frac{kr^2}{mr^2 + I}}$$

6.38 Let at any instant the centre of the cylinder be displaced by x towards right. Then the spring at C is compressed by x while the spring at P is elongated by $2x$. If $v = \dot{x}$ is the velocity of the centre of mass of the cylinder and $\omega = \dot{\theta}$ its angular velocity, the total energy in the displaced position will be

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_C\dot{\theta}^2 + \frac{1}{2}k_1x^2 + \frac{1}{2}k_2(2x)^2 \tag{1}$$

Substituting $x = r\theta$, $\dot{x} = r\dot{\theta}$, and $I_\mathrm{C} = \dfrac{1}{2}mr^2$, where r is the radius of the cylinder, (1) becomes

$$E = \frac{3}{4}mr^2\dot{\theta}^2 + \frac{1}{2}r^2(k_1 + 4k_2)\theta^2 = \text{constant}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{3}{2}mr^2\dot{\theta}\ddot{\theta} + r^2(k_1 + 4k_2)\theta\dot{\theta} = 0$$

$$\therefore \quad \ddot{\theta} + \frac{2}{3m}(k_1 + 4k_2)\theta = 0$$

which is the equation for angular SHM with $\omega^2 = \dfrac{2}{3m}(k_1 + 4k_2)$.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3m}{2(k_1 + 4k_2)}}$$

6.39 $U(x) = \dfrac{a}{x^2} - \dfrac{b}{x}$

Equilibrium position is obtained by minimizing the function $U(x)$.

$$\frac{\mathrm{d}U}{\mathrm{d}x} = -\frac{2a}{x^3} + \frac{b}{x^2} = 0$$
$$x = x_0 = \frac{2a}{b}$$

Measuring distances from the equilibrium position and replacing x by $x + \dfrac{2a}{b}$

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x} = \frac{2a}{x^3} - \frac{b}{x^2}$$
$$F = \frac{2a}{(x + 2a/b)^3} - \frac{b}{(x + 2a/b)^2}$$
$$= \frac{2a}{(2a/b)^3}\left(1 + \frac{bx}{2a}\right)^{-3} - \frac{b}{(2a/b)^2}\left(1 + \frac{bx}{2a}\right)^{-2}$$

Since the quantity $bx/2a$ is assumed to be small, use binomial expansion retaining terms up to linear in x.

$$F = -\frac{b^4 x}{8a^3}$$

Acceleration $a = \dfrac{F}{m} = -\dfrac{b^4 x}{8a^3 m} = -\omega^2 x$

where $\omega = \sqrt{\dfrac{b^4}{8a^3 m}}$

$$T = \frac{2\pi}{\omega} = 4\pi\sqrt{\frac{2ma^2}{b^4}}$$

6.3.3 Coupled Systems of Masses and Springs

6.40 Let spring 1 undergo an extension x_1 due to force F. Then $x_1 = \frac{F}{k_1}$. Similarly, for spring 2, $x_2 = \frac{F}{k_2}$.

The force is the same in each spring, but the total displacement x is the sum of individual displacements:

$$x = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2}$$

$$k_{\mathrm{eq}} = \frac{F}{x} = \frac{F}{x_1 + x_2} = \frac{F}{\frac{F}{k_1} + \frac{F}{k_2}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = \frac{k_1 k_2}{k_1 + k_2}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k_{\mathrm{eq}}}} = 2\pi\sqrt{\frac{(k_1 + k_2)m}{k_1 k_2}}$$

6.41 The displacement is the same for both the springs and the total force is the sum of individual forces.

$$F_1 = k_1 x, \;\; F_2 = k_2 x$$
$$F = F_1 + F_2 = (k_1 + k_2)x$$
$$k_{\mathrm{eq}} = \frac{F}{x} = k_1 + k_2$$
$$T = 2\pi\sqrt{\frac{m}{k_{\mathrm{eq}}}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

6.42 Let the centre of mass be displaced by x. Then the net force

$$F = -k_1 x - k_2 x = -(k_1 + k_2)x$$

$$\text{Acceleration } a = \frac{F}{m} = -(k_1 + k_2)\frac{x}{m} = -\omega^2 x$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

6.43 Spring constant of the wire is given by

$$k' = \frac{YA}{L} \tag{1}$$

Since the spring and the wire are in series, the effective spring constant k_{eff} is given by

$$k_{\mathrm{eff}} = \frac{k'k}{k + k} \tag{2}$$

The time period of oscillations is given by

$$T = 2\pi\sqrt{\frac{m}{k_{\mathrm{eff}}}} \tag{3}$$

Combining (1), (2) and (3)

$$T = 2\pi\sqrt{\frac{m(YA + kL)}{YAk}}$$

6.44 In Fig. 6.15, C is the point of contact around which the masses M and m rotate. As it is the instantaneous centre of zero velocity, the equation of motion is of the form $\Sigma\tau_{\mathrm{c}} = I_{\mathrm{c}}\ddot{\theta}$, where I_{c} is the moment of inertia of masses M and m with respect to point C. Now

$$I_{\mathrm{c}} = \left(\frac{1}{2}MR^2 + MR^2\right) + md^2 \tag{1}$$

where $d^2 = L^2 + R^2 - 2RL\cos\theta$. (2)

For small oscillations, $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$ and

$$I_{\mathrm{c}} = \frac{3MR^2}{2} + m(L - d)^2 \tag{3}$$

Therefore the equation of motion become

$$\left[\frac{3MR^2}{2} + m(L - d)^2\right]\ddot{\theta} = -mgL\sin\theta = -mgL\theta$$

$$\text{or} \quad \ddot{\theta} + \frac{mgL}{3MR^2/2 + m(L - d)^2}\theta = 0$$

$$\therefore \quad \omega = \sqrt{\frac{mgL}{3MR^2/2 + m(L - d)^2}}\ \mathrm{rad/s}$$

6.45 Figure 6.21 shows the semicircular disc tilted through an angle θ compared to the equilibrium position (b). G is the centre of mass such that $a = \mathrm{OG} = \dfrac{4r}{3\pi}$, where r is the radius.

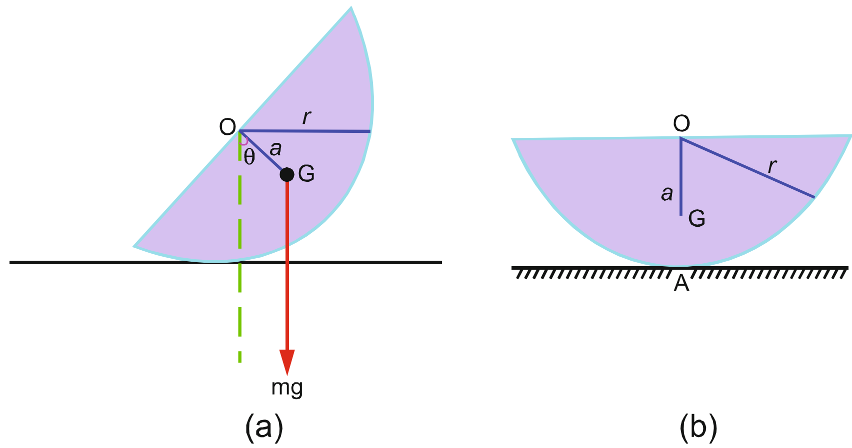


Fig. 6.21

We use the energy method.

$$K(\max) = \frac{1}{2} I_A \omega^2$$
$$= \frac{1}{2}(I_G + \overline{GA}^2)\omega^2$$
$$= \frac{1}{2}[I_0 - ma^2 + m(r-a)^2]\omega^2 = \frac{1}{2}\left[\frac{1}{2}mr^2 + mr(r-2a)\right]\omega^2$$
$$= mr\left(\frac{3}{4}r - a\right)\omega^2$$

$$K_{\max} = U_{\max}$$
$$mr\left(\frac{3}{4}r - a\right)\omega^2 = mga(1-\cos\theta)$$

But $a = \dfrac{4r}{3\pi}$

$$\omega = 4\sqrt{\frac{(1-\cos\theta)g}{(9\pi - 16)r}}$$

6.46 Referring to Fig. 6.16, take torques about the two hinged points P and Q.

$$mb^2\ddot{\theta}_1 = -mgb\theta_1 - kb^2(\theta_1 - \theta_2)$$

The left side gives the net torque which is the product of moment of inertia about P and the angular acceleration. The first term on the right side gives the torque of the force mg, which is force times the perpendicular distance from the vertical through P. The second term on the right side is the torque produced by the spring which is $k(x_1 - x_2)$ times the perpendicular distance from P, that is, $k(x_1 - x_2)b$ or $k(\theta_1 - \theta_2)b^2$. The second equation of motion can be similarly written. Thus, the two equations of motion are

$$mb\ddot{\theta}_1 + mg\theta_1 + kb(\theta_1 - \theta_2) = 0 \tag{1}$$

$$mb\ddot{\theta}_2 + mg\theta_2 + kb(\theta_2 - \theta_1) = 0 \tag{2}$$

The harmonic solutions are

$$\theta_1 = A\sin\omega t, \quad \theta_2 = B\sin\omega t \tag{3}$$

$$\ddot{\theta}_1 = -A\omega^2\sin\omega t, \quad \ddot{\theta}_2 = -B\omega^2\sin\omega t \tag{4}$$

Substituting (3) and (4) in (1) and (2) and simplifying

$$(mg + kb - mb\omega^2)A - kb\ B = 0 \tag{5}$$

$$-kbA + (mg + kb - mb\omega^2)B = 0 \tag{6}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B.

$$\begin{vmatrix} (mg + kb - mb\omega^2) & -kb \\ -kb & (mg + kb - mb\omega^2) \end{vmatrix} = 0$$

Expanding the determinant and solving for ω we obtain

$$\omega_1 = \sqrt{\frac{g}{b}}, \quad \omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}}$$

6.47 In prob. (6.46) equations of motion (1) and (2) can be re-written in terms of Cartesian coordinates x_1 and x_2 since $x_1 = b\theta_1$ and $x_2 = b\theta_2$.

$$m\ddot{x}_1 + \frac{mgx_1}{b} + k(x_1 - x_2) = 0 \tag{1}$$

$$m\ddot{x}_2 + \frac{mgx_2}{b} + k(x_2 - x_1) = 0 \tag{2}$$

It is possible to make linear combinations of x_1 and x_2 such that a combination involves but a single frequency. These new coordinates X_1 and X_2, called normal coordinates, vary harmonically with but a single frequency. No energy transfer occurs from one normal coordinate to another. They are completely independent.

$$x_1 = \frac{X_1 + X_2}{2}, \quad x_2 = \frac{X_1 - X_2}{2} \tag{3}$$

Substituting (3) in (1) and (2)

$$\frac{m}{2}(\ddot{X}_1 + \ddot{X}_2) + \frac{mg}{2b}(X_1 + X_2) + kX_2 = 0 \tag{4}$$
$$\frac{m}{2}(\ddot{X}_1 - \ddot{X}_2) + \frac{mg}{2b}(X_1 - X_2) - kX_2 = 0 \tag{5}$$

Adding (4) and (5)

$$m\ddot{X}_1 + \frac{mg}{b}X_1 = 0 \tag{6}$$

which is a linear equation in X_1 alone with constant coefficients.
Subtracting (5) from (4), we obtain

$$m\ddot{X}_2 + \left(\frac{mg}{b} + 2k\right) X_2 = 0 \tag{7}$$

This is again a linear equation in X_2 as the single dependent variable. Since the coefficients of X_1 and X_2 are positive, both (6) and (7) are differential equations of simple harmonic motion having frequencies $\omega_1 = \sqrt{\frac{g}{b}}$ and $\omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}}$. Thus when equations of motion are expressed in normal coordinates, the equations are linear with constant coefficients and each contains only one dependent variable.
We now calculate the energy in normal coordinates. The potential energy arises due to the energy stored in the spring and due to the position of the body.

$$V = \frac{1}{2}k(x_1 - x_2)^2 + mgb(1 - \cos\theta_1) + mgb(1 - \cos\theta_2) \tag{8}$$

Now $b(1 - \cos\theta_1) = b\frac{\theta_1^2}{2} = \frac{x_1^2}{2b}$

Similarly $b(1 - \cos\theta_2) = \frac{x_2^2}{2b}$

Hence $V = \frac{k}{2}(x_1 - x_2)^2 + \frac{mgx_1^2}{2b} + \frac{mgx_2^2}{2b}$ (9)

Kinetic energy $T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right)$ (10)

Although there is no cross-product term in (10) for the kinetic energy, there is one in the potential energy of the spring in (9). The presence of the cross-product term means coupling between the components of the vibrating system. However, in normal coordinates the cross-product terms are avoided. Using (3) in (9) and (10)

$$V = \frac{mg}{4b}X_1^2 + \left(\frac{mg}{4b} + \frac{k}{2}\right)X_2^2 \tag{11}$$

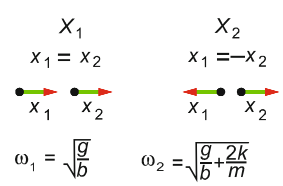
$$T = \frac{m}{4}(\dot{X}_1^2 + \dot{X}_2^2) \tag{12}$$

Thus the cross terms have now disappeared. The potential energy V is now expressed as a sum of squares of normal coordinates multiplied by constant coefficients and kinetic energy. T is expressed in the form of a sum of squares of the time derivatives of the normal coordination.

We can now describe the mode of oscillation associated with a given normal coordinate. Suppose $X_2 = 0$, then $0 = x_1 - x_2$, which implies $x_1 = x_2$. The mode X_1 is shown in Fig. 6.22, where the particles oscillate in phase with frequency $\omega_1 = \sqrt{g/b}$ which is identical for a simple pendulum of length b. Here the spring plays no role because it remains unstretched throughout the motion.

If we put $X_1 = 0$, then we get $x_1 = -x_2$. Here the pendulums are out of phase. The X_2 mode is also illustrated in Fig. 6.22, the associated frequency being $\omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}}$. Note that $\omega_2 > \omega_1$, because greater potential energy is now available due to the spring.

Fig. 6.22



6.48 $y = A \cos 6\pi t \sin 90\pi$

Now $\sin C + \sin D = 2 \sin \frac{1}{2}(C + D) \cos \frac{1}{2}(C - D)$

Comparing the two equations we get

$$\frac{C + D}{2} = 90\pi \quad \frac{C - D}{2} = 6\pi$$

$$\therefore \quad C = 96\pi \text{ and } D = 84\pi$$

$$\omega_1 = 2\pi f_1 = 96\pi \quad \text{or} \quad f_1 = 48\,\text{Hz}$$

$$\omega_2 = 2\pi f_2 = 84\pi \quad \text{or} \quad f_2 = 42\,\text{Hz}$$

Thus the frequency of the component vibrations are 48 Hz and 42 Hz. The beat frequency is $f_1 - f_2 = 48 - 42 = 6$ beats/s.

6.49 The frequency is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

where μ is the reduced mass given by

$$\mu = \frac{m_H m_{Cl}}{m_H + m_{Cl}} = \frac{10. \times 36.46}{1.0 + 36.46}$$
$$= 0.9733\,\text{amu} = 0.9733 \times 1.66 \times 10^{-27}\,\text{kg} = 1.6157 \times 10^{-27}\,\text{kg}$$
$$f = \frac{1}{2\pi}\sqrt{\frac{480}{1.6157 \times 10^{-27}}} = 8.68 \times 10^{13}\,\text{Hz}$$

6.50 Each vibration is plotted as a vector of magnitude which is proportional to the amplitude of the vibration and in a direction which is determined by the phase angle. Each phase angle is measured with respect to the x-axis. The vectors are placed in the head-to-tail fashion and the resultant is obtained by the vector joining the tail of the first vector with the head of the last vector, Fig. 6.23. $y_1 = \text{OA} = 1$ unit, parallel to x-axis in the positive direction, $y_2 = \text{AB} = \frac{1}{2}$ unit parallel to y-axis and $y_3 = \text{BC} = \frac{1}{3}$ unit parallel to the x-axis in the negative direction.

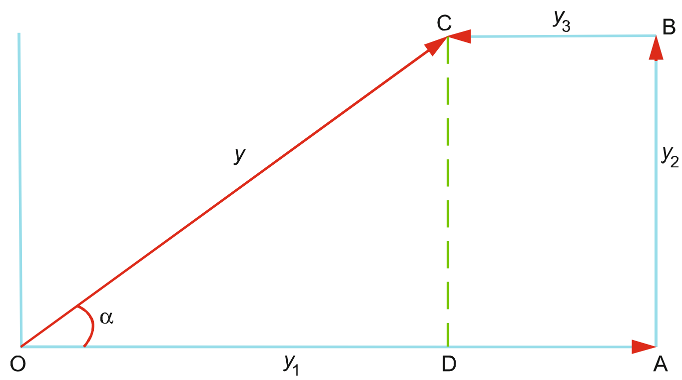


Fig. 6.23

The resultant is given by OC both in magnitude and in direction. From the geometry of the diagram

$$y = \text{OC} = \sqrt{\text{OD}^2 + \text{DC}^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2} = 5/6$$
$$\alpha = \tan^{-1}(\text{CD}/\text{OD}) = \tan^{-1}\left(\frac{1/2}{2/3}\right) = \tan^{-1}(3/4) = 37^\circ$$

6.3.4 Damped Vibrations

6.51 The logarithmic decrement Δ is given by

$$\Delta = bT' \tag{1}$$

where $T' = \dfrac{2\pi}{\omega'}$ is the time period for damped vibration and $b = \sqrt{\omega_0^2 - \omega'^2}$, where ω_0 and ω' are the angular frequencies for natural and damped vibrations, respectively.

$$\Delta = 2\pi\sqrt{\frac{\omega_0^2}{\omega'^2} - 1} = 2\pi\sqrt{\frac{f^2}{f'^2} - 1} = 2\pi\sqrt{\left(\frac{20}{16}\right)^2 - 1} = \frac{3\pi}{2}$$

6.52 The equation for damped oscillations is $4\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + \dfrac{r\,\mathrm{d}x}{\mathrm{d}t} + 32x = 0$
Dividing the equation by 4

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{r}{4}\frac{\mathrm{d}x}{\mathrm{d}t} + 8x = 0$$

Comparing the equation with the standard equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{r}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = 0$$

$$m = 4, \quad \frac{k}{m} = 8 \rightarrow k = 32$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{8} = 2\sqrt{2}$$

The quantity $b = \dfrac{r}{2m}$ represents the decay rate of oscillation where r is the resistance constant.

(a) The motion will be underdamped if

$$b < \omega_0 \text{ or } \frac{r}{2m} < \sqrt{\frac{k}{m}} \text{ or } r < 2\sqrt{km}$$

$$\text{i.e.} \quad r < 2\sqrt{32 \times 4} \text{ or } r < 16\sqrt{2}$$

(b) The motion is overdamped if $r > 16\sqrt{2}$.
(c) The motion is critically damped if $r = 16\sqrt{2}$.

6.53 **(a)** $\omega_0 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{20}{4}} = 2.23\,\text{rad/s}$

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2.23} = 2.8\,\text{s}$$

(b) $\omega' = \dfrac{2\pi}{T'} = \dfrac{2\pi}{10} = 0.628\,\text{rad/s}$

$b = \sqrt{\omega_0^2 - \omega'^2} = \sqrt{2.236^2 - 0.628^2} = 2.146$

$\dfrac{r}{2m} = b$ or $r = 2mb = 2 \times 4 \times 2.146 = 17.17\,\text{Ns/m}$

(c) $\Delta = bT' = 2.146 \times 10 = 21.46$

6.54 $\dfrac{d^2x}{dt^2} + \dfrac{2dx}{dt} + 5x = 0$

Let $x = e^{\lambda t}$. The characteristic equation then becomes $\lambda^2 + 2\lambda + 5 = 0$ with the roots $\lambda = -1 \pm 2i$

$x = Ae^{-(1-2i)t} + Be^{-(1+2i)t}$

or $\quad x = e^{-t}[C\cos 2t + D\sin 2t]$

where A, B, C and D are constants.

C and D can be determined from initial conditions. At $t = 0$, $x = 5$. Therefore $C = 5$.

Also $\dfrac{dx}{dt} = -e^{-t}(C\cos 2t + D\sin 2t) + e^{-t}(-2C\sin 2t + 2D\cos 2t)$

At $t = 0$, $\dfrac{dx}{dt} = -3$

$\therefore \quad -3 = -C + 2D = -5 + 2D$

$\therefore \quad D = 1$

The complete solution is

$x = e^{-t}(5\cos 2t + \sin 2t)$

6.55 $F = mg = kx$

$k = \dfrac{mg}{x} = \dfrac{(1.0)(9.8)}{0.2} = 49\,\text{N/m}$

Equation of motion is

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0 \tag{1}$$

Substituting $m = 1.0$, $r = 14$, $k = 49$, (1) becomes

$$\frac{d^2x}{dt^2} + 14\frac{dx}{dt} + 49x = 0 \tag{2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{49}{1}} = 7\,\text{rad/s}$$

$$b = \frac{r}{2m} = \frac{14}{2 \times 1} = 7$$

(a) Therefore the motion is critically damped.
(b) For critically damped motion, the equation is

$$x = x_0 e^{-bt}(1 + bt) \tag{3}$$

With $b = 7$ and $x_0 = 1.5$, (3) becomes

$$x = 1.5\,e^{-7t}(1 + 7t)$$

6.56 $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{150}{60}} = 5$

Damping force $f_r = r \cdot v$

$$\text{or} \quad r = \frac{f_r}{v} = \frac{80}{2} = 40$$

$$b = \frac{r}{2m} = \frac{40}{2 \times 6} = 3.33\,\text{rad/s}$$

$$\omega(\text{res}) = \sqrt{\omega_0^2 - 2b^2} = \sqrt{5^2 - 2 \times (3.33)^2} = 1.66\,\text{rad/s}$$

$$f(\text{res}) = \frac{\omega(\text{res})}{2\pi} = 0.265\,\text{vib/s}$$

6.57 Equation of motion is

$$\frac{2d^2x}{dt^2} + 1.5\frac{dx}{dt} + 40x = 12\cos 4t$$

Dividing throughout by 2

$$\frac{d^2x}{dt^2} + 0.75\frac{dx}{dt} + 20x = 6\cos 4t$$

Comparing this with the standard equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = p\cos\omega t$$

$$b = 0.375;\ \omega_0 = \sqrt{20},\ \ p = 6,\ \ \omega = 4$$

$$Z_M = \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2} = \sqrt{(20-16)^2 + 4 \times 0.375^2 \times 4^2} = 5$$

(a) $A = \dfrac{p}{Z_m} = \dfrac{6}{5} = 1.2$

(b) $\tan\varepsilon = \dfrac{2b\omega}{\omega_0^2 - \omega^2} = \dfrac{2 \times 0.375 \times 4}{(20-16)} = 0.75 \to \varepsilon = 37°$

(c) $Q = \dfrac{\omega_0 m}{r} = \dfrac{\omega_0}{2b} = \dfrac{\sqrt{20}}{2 \times 0.375} = 5.96$

(d) $F = pm = 6 \times 2 = 12$

$$W = \frac{F^2}{2Z_m}\sin\varepsilon = \frac{12^2}{2 \times 5}\sin 37° = 8.64\,\text{W}$$

6.58 $Q = \dfrac{2\pi t_c}{T} = 2\pi t_c f = 2\pi \times 2 \times 100 = 1256$

6.59 **(a)** Energy is proportional to the square of amplitude

$$E = \text{const.}A^2$$

$$\frac{dE}{E} = \frac{2dA}{A} = \frac{2 \times 5}{100} = 10\%$$

(b) $E = E_0 e^{-t/t_c}$

$$\therefore\quad \frac{E}{E_0} = \frac{A^2}{A_0^2} = e^{-t/t_c}$$

$$\therefore\quad \frac{A}{A_0} = \frac{95}{100} = e^{-t/2t_c}$$

$$\frac{t}{2t_c} = \ln\left(\frac{100}{95}\right) = 0.05126$$

$$t_c = \frac{3}{2 \times 0.05126} = 29.26\,\text{s}$$

(c) $Q = \dfrac{2\pi t_c}{T} = \dfrac{(2\pi)(29.26)}{3.0} = 61.25$

6.60 **(a)** $E = E_0 e^{-t/t_c}$

$$\therefore \quad \frac{t}{t_c} = \ln\left(\frac{E_0}{E}\right) = \ln 2 = 0.693$$

Put $\quad t = nT$

$$\therefore \quad n = 0.693 \frac{t_c}{T}$$

But $\quad -\dfrac{\Delta E}{E} = \dfrac{3}{100} = \dfrac{T}{t_c}$

$$\therefore \quad n = 0.693 \times \frac{100}{3} = 23.1$$

(b) $Q = \dfrac{2\pi t_c}{T} = 2\pi \times \dfrac{100}{3} = 209.3$

6.61 $\omega' = \omega_0 \sqrt{1 - \dfrac{1}{4Q^2}} = \dfrac{9\omega_0}{10}$

$$\therefore \quad Q = 1.147$$

$$Q = \frac{2\pi t_c}{T}$$

or $\dfrac{T}{2t_c} = \dfrac{\pi}{Q} = \dfrac{3.14}{1.147} = 2.737$

$$\frac{A}{A_0} = e^{-T/2t_c} = e^{-2.737} = 0.065$$

6.62 $\omega'^2 = \omega_0^2 - b^2 \quad (1)$

where $b = \dfrac{r}{2m} \quad (2)$

$$\omega' = \omega_0 \left(1 - \frac{b^2}{\omega_0^2}\right)^{1/2} \approx \omega_0 \left(1 - \frac{b^2}{2\omega_0^2}\right) \quad (3)$$

where we have expanded the radical binomially, assuming that $b/\omega_0 << 1$.

Now $\quad \omega_0^2 = \dfrac{k}{m} \quad (4)$

$$\therefore \quad \frac{b^2}{2\omega_0^2} = \frac{r^2}{8mk} \quad (5)$$

Substituting (5) in (3)

$$\omega' = \omega_0 \left(1 - \frac{r^2}{8mk}\right) \quad \text{(for small damping)}$$

6.63 The time elapsed between successive maximum displacements of a damped harmonic oscillator is represented by T', the period.

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}} = \frac{4\pi m}{\sqrt{4km - r^2}} = \text{constant}$$

6.64 Force $= mg = kx$

$$\therefore \quad \frac{k}{m} = \frac{g}{x} = \frac{980}{9.8} = 100$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{100} = 10\,\text{rad/s}$$

$$\Delta = bT' = \frac{2\pi b}{\sqrt{\omega_0^2 - b^2}} \tag{1}$$

Substituting $\Delta = 3.1$ and $\omega_0 = 10$ in (1), $b = 4.428$

$$T' = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = \frac{2\pi}{\sqrt{10^2 - (4.428)^2}} = 0.7\,\text{s}$$

6.65 $$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{2\mathrm{d}x}{\mathrm{d}t} + 8x = 16\cos 2t \tag{1}$$

This is the equation for the forced oscillations, the standard equation being

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + r\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F\cos\omega t \tag{2}$$

Comparing (1) and (2) we find

$$m = 1\,\text{kg},\ r = 2,\ k = 8,\ F = 16\,N,\ \omega = 2$$

(a) $\omega_0 = 2\pi f_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1}} = 2\sqrt{2}$

$$\therefore \quad f_0 = \frac{2\sqrt{2}}{2\pi} = \frac{\sqrt{2}}{\pi}/\text{s}$$

(b) $\omega = 2\pi f = 2$

$$\therefore \quad f = \frac{2}{2\pi} = \frac{1}{\pi}/\text{s}$$

6.66 $E(t) = E_0 e^{-t/t_c}$

$$\therefore \quad \frac{E(t_{1/2})}{E_0} = \frac{1}{2} = e^{-t_{1/2}/t_c}$$

or $t_{1/2} = t_c \ln 2$

6.67 $A(t) = A_0 e^{-t/2t_c}$

$$\frac{A(t)}{A_0} = \frac{1}{e}$$

If $t = 2t_c = 8T$

$$\therefore \quad t_c = 4T$$

$$Q = 2\pi \frac{t_c}{T} = 2\pi \times 4 = 25.1$$