اثنا عشر بنفقة

3.1 Determinants of matrix

Definition: The determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ is}$$

$$\det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$$

Ex(1) Find $|A|$ and $|B|$ for:-

(i) $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

sol^n $|A| = 2\times 2 - (-3)(1) = 4+3 = 7$

(ii) $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \Rightarrow |B| = 2\times 2 - 1\times 4$
$= 4-4 = 0$

Minor & cofactor of a square matrix:-

If A is a square matrix, then the Minor M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting ith row and jth column of A, the cofacter C_{ij} of a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Find minor and cofactor of a_{21} &

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = (-1)^3 M_{21} = -M_{23}$$

Ex(2) Find all the miners & cofactor of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$a_{11} = 0 \Rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1-0 = -1 \Rightarrow C_{11} = (-1)^2(-1) = -1$

$a_{12} = 2 \Rightarrow M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3-8 = -5 \Rightarrow C_{12} = (-1)^3(-5) = 5$

$a_{13} = 1 \Rightarrow M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 0+4 = 4 \Rightarrow C_{13} = (-1)^4(4) = 4$

$a_{21} = 3 \Rightarrow M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2-0 = 2 \Rightarrow C_{21} = (-1)^3(2) = -2$

$a_{22} = -1 \Rightarrow M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = 0-4 = -4 \Rightarrow C_{22} = (-1)^4(-4) = -4$

$a_{23} = 2 \Rightarrow M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix} = 0-8 = -8 \Rightarrow C_{23} = (-1)^5(-8) = 8$

$a_{31} = 4 \Rightarrow M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 4+1 = 5 \Rightarrow C_{31} = (-1)^4(5) = 5$

$a_{32} = 0 \Rightarrow M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 0-3 = -3 \Rightarrow C_{32} = (-1)^5(-3) = 3$

$a_{33} = 1 \Rightarrow M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = 0-6 = -6 \Rightarrow C_{33} = (-1)^6(-6) = -6$

$$\text{cofactor } C = \begin{bmatrix} -1 & 5 & 4 \\ -2 & -4 & 8 \\ 5 & 3 & -6 \end{bmatrix} = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then we can

find $\det(A) = |A|$ by several way.

by 1^{st} row $= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

" 2^{nd} row $= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

" 3^{rd} row $= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

or same as column.

by 1^{st} row $= +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Ex(4) Find the determinant of

$$A = \begin{bmatrix} \overset{+}{1} & \overset{-}{2} & \boxed{\overset{+}{3}} & 0 \\ -1 & 1 & \overset{-}{0} & 2 \\ 0 & 2 & \overset{+}{0} & 3 \\ 3 & 4 & \overset{-}{0} & -2 \end{bmatrix}$$

sol^n: نختار الصف او العمود الأكثر اصفار

$$|A| = 3\,C_{13} + \cancel{0\,C_{23}} + \cancel{0\,C_{33}} + \cancel{0\,C_{43}}$$

$$= 3\begin{vmatrix} -1 & 1 & 2 \\ \boxed{0 \quad 2 \quad 3} \\ 3 & 4 & -2 \end{vmatrix}$$

نختار الصف او العمود اللي فيه اكثر اصفار

$$= 3\left[\cancel{0\,C_{21}} + 2\,C_{22} + 3\,C_{23}\right]$$

$$= 3\left[2\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} - 3\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}\right] = 3\left[2\overset{-4}{(2-6)} - 3\overset{-7}{(-4-3)}\right]$$

$$= 3\left[-8+21\right] = 3\left[13\right]$$

$$= 39$$

There is another method to find $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$\begin{array}{ccccc} \overset{+}{a_{11}} & \overset{+}{a_{12}} & \overset{+}{a_{13}} & \overset{-}{a_{11}} & \overset{-}{a_{12}} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33}$$
$$- a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$$

Ex(5) Find the det. of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}$$

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$$\begin{array}{ccccc} 0 & 2 & 1 & 0 & 2 \\ 3 & -1 & 2 & 3 & -1 \\ 4 & -4 & 1 & 4 & -4 \\ -4 & 0 & 6 & 0 & 16 & -12 \end{array}$$

$$\therefore |A| = \underbrace{0 + 16 - 12}_{4} - \underbrace{(-4)}_{4} - (0) - \underbrace{(6)}_{-6}$$
$$= 4 + 4 - 6 = 2$$

* Determinant of a triangular matrix

either $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ or $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

upper triangular — lower triangular

$\Rightarrow \det = a_{11}a_{22}a_{33}$ Product of the entries on the main diagonal main diagonal.

Ex(6) Find $|A|$ for $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$

$|A| = (2)(-2)(1)(3) = -12$ Lower triangular

Exercises.

(1 - 12) Find the det. for

(2) $[-3] \Rightarrow \det = -3$

(3) $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \Rightarrow \det = (2)(4) - (1)(3) = 5$

(11) $\begin{bmatrix} \lambda-3 & 2 \\ 4 & \lambda-1 \end{bmatrix} \Rightarrow \det = (\lambda-3)(\lambda-1) - 8$

$= \lambda^2 - 4\lambda + 3 - 8 = \lambda^2 - 4\lambda - 5$

(19-22) Use expansion by cofactors to find det.

(20) $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 4 \\ -2 & 0 & 1 \end{bmatrix}$ by 3^{rd} row

$\det = -2\,C_{31} + \cancel{0\,C_{32}} + 1\,C_{33}$

$= -2(-1)^4 \begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} + 1(-1)^6 \begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix}$

$= -2(-6) + (3+4)$

$= 12 + 7 = 19$

(21) $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$ $\det = 2(3)(-5) = -30$

upper triangle

by cofactor

$\det = -5\,C_{33} = -5(-1)^6 \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix}$

$= -5(6-0) = -30$

(49 - 50) Find the value of λ for which the det = 0

49) $\begin{bmatrix} \lambda+2 & 2 \\ 1 & \lambda \end{bmatrix} \Rightarrow \det = 0$

$\Rightarrow (\lambda+2)(\lambda) - 2 = 0$

$\Rightarrow \lambda^2 + 2\lambda - 2 = 0$ $\quad a = 1$, $b = 2$, $c = -2$

$\Rightarrow \lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2(a)}$

$\Rightarrow \lambda = \dfrac{-2 \pm \sqrt{4 - 4(1)(-2)}}{2(1)} = \dfrac{-2 \pm \sqrt{12}}{2}$

$\Rightarrow \lambda = \dfrac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$

51) $\begin{bmatrix} \boxed{\begin{matrix} \lambda \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 2 & 0 \\ \lambda+1 & 2 \\ 1 & 1 \end{matrix} \end{bmatrix}$ $\det = 0 \Rightarrow \overset{1^{st}\ \text{colum}}{\lambda C_{11}} = \lambda(-1)^2 \begin{vmatrix} \lambda+1 & 2 \\ 1 & 1 \end{vmatrix} = 0$

$\Rightarrow \lambda\left[\lambda + 1 - 2\right] = 0$

$\Rightarrow \lambda[\lambda - 1] = 0$

either $\lambda = 0$ or $\lambda - 1 = 0 \Rightarrow \lambda = 0$

$\Rightarrow \lambda = 0, 1$