الباب الخامس

5.1 length and dot product of $\mathbb{R}^n$:-

Definition: The length or norm of a vector $v = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

length is called also magnitude

If $\|v\| = 1$ then the vector v is a unit vector.

Ex(1) Find the length of

(a) $v = (0, -2, 1, 4, -2) \in \mathbb{R}^5$

$$\Rightarrow \|v\| = \sqrt{0^2 + (-2)^2 + (1)^2 + 4^2 + (-2)^2}$$

$$= \sqrt{0 + 4 + 1 + 16 + 4} = \sqrt{25} = 5$$

(b) $v = \left(\frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right) \in \mathbb{R}^3$

$$\|v\| = \sqrt{\left(\frac{2}{\sqrt{17}}\right)^2 + \left(\frac{-2}{\sqrt{17}}\right)^2 + \left(\frac{3}{\sqrt{17}}\right)^2}$$

$$= \sqrt{\frac{4}{17} + \frac{4}{17} + \frac{9}{17}} = \sqrt{\frac{17}{17}} = 1$$

$\Rightarrow \left(\frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$ is called unit vector

Note: Each vector of standard basis for $\mathbb{R}^n$ has length 1 and is a standard unit vector

- Two vecter u & $v \in \mathbb{R}^n$ are parallel
  if $u = cv$ $\quad c \in \mathbb{R}$
  if $c > 0$ then u & v have the same directin
  if $c < 0$ " " " " " opposite "

Th① Let $v \in \mathbb{R}^n$, $c \in \mathbb{R}$ (scalor) Then
$$\|cv\| = |c|\,\|v\|$$

Th② if $v \neq 0$, $v \in \mathbb{R}^n$ then the vecter
$u = \frac{v}{\|v\|}$ has length $= 1$ and has the same direction of v & u is called a unit vecter of v

Ex(2) Find the unit vector in the direction of $v = (3, -1, 2)$

sol $u = \frac{v}{\|v\|} = \frac{(3,-1,2)}{\sqrt{3^2+(-1)^2+2^2}} = \frac{(3,-1,2)}{\sqrt{14}}$

$u = \left(\frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$

$\|u\| = 1$ (check)

Definition [Dot product]

If $u = (u_1, u_2, \dots, u_n)$ $v = (v_1, v_2, \dots, v_n)$
then the dot product between u & v is

$$u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Ex(4) If $u = (1, 2, 0, -3)$, $v = (3, -2, 4, 2)$
find $u \cdot v$

$$u \cdot v = (1)(3) + (2)(-2) + (0)(4) + (-3)(2)$$
$$= 3 - 4 + 0 - 6 = -7$$

Properties of dot product:

If $u, v, w \in \mathbb{R}^n$ (vectors) $c \in \mathbb{R}$ (scalar)
then:

1) $u \cdot v = v \cdot u$
2) $u \cdot (v + w) = u \cdot v + u \cdot w$
3) $c(u \cdot v) = (cu) \cdot v = u \cdot (cv)$
4) $v \cdot v = \|v\|^2$
5) $v \cdot v \geq 0$ and $v \cdot v = 0 \Leftrightarrow v = 0$

Ex(5) Let $u = (2, -2)$ $v = (5, 8)$ $w = (-4, 3)$
Find

a) $u \cdot v = (2)(5) + (-2)(8) = 10 - 16 = -6$
b) $(u \cdot v)w = -6(-4, 3) = (24, -18)$
c) $u \cdot 2v = u \cdot (10, 16) = 2(10) + (-2)(16)$
$= 20 - 32 = -12$

d) $\|W\|^2 = W \cdot W = (-4)(-4) + (3)(3) = 16+9 = 25$

e) $u \cdot (v - 2w) = u \cdot [(5,8) - 2(-4,3)]$

$= (2,-2) \cdot [(5+8, 8-6)]$

$= (2,-2) \cdot (13,2)$

$= (2)(13) + (-2)(2) = 26 - 4 = 22$

Ex(6) Let $u, v \in \mathbb{R}^n$ s.t $u \cdot u = 39$, $u \cdot v = -3$ and $v \cdot v = 79$ Evaluate $(u+2v) \cdot (3u+v)$

Sol $(u+2v) \cdot (3u+v)$

$= u \cdot 3u + u \cdot v + 2v \cdot 3u + 2v \cdot v$

$= 3(u \cdot u) + u \cdot v + 6(v \cdot u) + 2(v \cdot v)$

$= 3(39) + (-3) + 6(-3) + 2(79)$

$= \underline{117} - 21 + \underline{158} = 275 - 21 = 254$

definition of orthogonal vectors :-

Two vecter u & $v \in \mathbb{R}^n$ are orthogonal when $u \cdot v = 0$

Ex(9) show that u & v are orthogonal

(a) $u = (1,0,0)$ $v = (0,1,0)$

$u \cdot v = (1)(0) + (0)(1) + (0)(0) = 0 \Rightarrow$ orthogonal

(b) $u = (3,2,-1,4)$, $v = (1,-1,1,0)$

$u \cdot v = (3)(1) + (2)(-1) + (-1)(1) + 4(0) = 3-2-1 = 0$

orthogonal.

(c) Determine all vectors in $\mathbb{R}^2$ that are orthogonal to $u=(4,2)$

sol let $v=(v_1,v_2)$ $\perp$ to u

$\Rightarrow u \cdot v = 0 \Rightarrow 4v_1 + 2v_2 = 0$

$\Rightarrow 2v_2 = -4v_1$

$\Rightarrow v_2 = -2v_1$

$v = (v_1, -2v_1) = v_1(1,-2)$

every vector of $c(1,-2)$ is $\perp$ to u

Exercises (3,4) Find the length of

(3) $v=(5,-3,-4) \Rightarrow \|v\| = \sqrt{25+9+16}$
$= \sqrt{50} = 5\sqrt{2}$

(4) $v=(2,0,-5,5) \Rightarrow \|v\| = \sqrt{4+0+25+25}$
$= \sqrt{54} = 3\sqrt{6}$

(10-11) Find a unit vector u
(a) in direction of v (b) opposite direction of v

10) $v=(2,-2)$

unit vector
(a) in the direction of v is $\frac{v}{\|v\|} = \frac{(2,-2)}{\sqrt{4+4}} = \frac{(2,-2)}{\sqrt{8}}$

$u = \frac{(2,-2)}{2\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

(b) unit vector opposite of v is $u = -\frac{v}{\|v\|} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

11) $v=(3,2,-5)$

(a) unit vecter in the direction of v is

$$u=\frac{v}{\|v\|}=\frac{(3,2,-5)}{\sqrt{9+4+25}}=\frac{(3,2,-5)}{\sqrt{38}}$$

$$u=\left(\frac{3}{\sqrt{38}},\frac{2}{\sqrt{38}},\frac{-5}{\sqrt{38}}\right)$$

(b) unit vecter opossite to the direction of v is

$$-u=-\frac{v}{\|v\|}=\left(\frac{-3}{\sqrt{38}},\frac{-2}{\sqrt{38}},\frac{5}{\sqrt{38}}\right)$$

25) Let $u=(2,-2,1)$ $v=(2,-1,-6)$

Find

(a) $u\cdot v=(2)(2)+(-2)(-1)+(1)(-6)=0$

(b) $v\cdot v=(2)(2)+(-1)(-1)+(-6)(-6)=41$

(c) $\|u\|^2=u\cdot u=(2)(2)+(-2)(-2)+(1)(1)=9$

(d) $(u\cdot v)v=(0)v=0=(0,0,0)$

(e) $u\cdot 5v=5(u\cdot v)=5(0)=0$

Determine whether u & v are Parallel orthogonal or neither

$\left[\begin{array}{ll} u=cv & \parallel \\ u\cdot v=0 & \perp \end{array}\right.$

47) $u(2,18)$ $v=\left(\frac{3}{2},-\frac{1}{6}\right)$

$u\cdot v=(2)\left(\frac{3}{2}\right)+18\left(-\frac{1}{6}\right)=3-3=0\Rightarrow u\perp v$

49) $u=\left(-\frac{1}{3},\frac{2}{3}\right)$ $v=(2,-4)$

$u\cdot v=-\frac{2}{3}-\frac{8}{3}=-\frac{10}{3}\neq 0$ not orthogonal

$-6u=\left(\frac{6}{3},-\frac{12}{3}\right)=(2,-4)=v\Rightarrow v=-6u$

$u\parallel v$

51) $u = (0,1,0)$ $v = (1,-2,0)$

$u \cdot v = -2 \neq 0$ v not orthogonal to u

$u \neq cv \Rightarrow u \not\parallel v \Rightarrow$ neither

Determine all vectors $v \perp$ to u

(56) $u = (11,2)$ let $v = (v_1, v_2)$

$u \perp v \Rightarrow u \cdot v = 0 \Rightarrow 11v_1 + 2v_2 = 0$

$\Rightarrow 11v_1 = -2v_2$

$\Rightarrow v_1 = -\frac{2}{11} v_2$

$v = (-\frac{2}{11} v_2, v_2) = v_2(-\frac{2}{11}, 1) \Rightarrow c(-\frac{2}{11}, 1) \perp u$

(57) $(2,-1,1)$ $v = (v_1, v_2, v_3)$

$u \perp v \Rightarrow u \cdot v = 0 \Rightarrow 2v_1 - v_2 + v_3 = 0$

let $v_3 = t$, $v_2 = r$

$2v_1 = r - t \Rightarrow v_1 = \frac{r-t}{2}$

$v = (\frac{r-t}{2}, r, t) =$

or $v_1 = t$ $v_2 = r \Rightarrow v_3 = v_2 - 2v_1 = r - 2t$

$v = (t, r, r-2t)$

or $v_1 = t$ $v_3 = r \Rightarrow v_2 = 2v_1 + v_3 = 2t + r$

$v = (t, 2t+r, r)$