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# 7.2 Diagonalization :-

Def: An $n \times n$ matrix $A$ is diagonalizable when $A$ is similar to diagonal matrix.

i.e. $A$ is diagonalizable if there exists an invertible matrix $P$ s.t. $P^{-1}AP$ is a diagonal matrix.

Ex (1) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is diagonalizable

because $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible

as $|P| = 1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \neq 0$

& $P^{-1}$ can find as

$R_2 - R_1$ $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ \boxed{1}^{0} & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $[-1 \quad -1 \quad 0 \mid -1 \quad 0 \quad 0]$

$\frac{R_2}{-2}$ $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{-2}^{1} & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_1 - R_2$ $\left[\begin{array}{ccc|ccc} 1 & \boxed{1}^{0} & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $[0 \quad -1 \quad 0 \mid -\frac{1}{2} \quad \frac{1}{2} \quad 0]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $P^{-1}$

$$P^{-1}AP = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{3}{2}+0 & \frac{3}{2}+\frac{1}{2}+0 & 0+0+0 \\ \frac{1}{2}-\frac{3}{2}+0 & \frac{3}{2}-\frac{1}{2}+0 & 0+0-0 \\ 0+0+0 & 0+0+0 & 0+0-2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2+0 & 2-2+0 & 0+0+0 \\ -1+1+0 & -1-1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0-2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \text{ diagonal.}$$

There is related between diagonalization Problem & eigenvalue problem :-

Theorem 1: if A & B are similar $n \times n$ matrices then they have the same eigenvalues

Ex(2) show that the two matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are similar

Since D is triangular matrix then eigen values of D is $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

to find eigenvalues of A:

$$|\lambda I - A| = \det \begin{bmatrix} \lambda-1 & 0 & 0 \\ 1 & \lambda-1 & -1 \\ 1 & 2 & \lambda-4 \end{bmatrix} = 0$$

$$\Rightarrow (\lambda-1)\left[(\lambda-1)(\lambda-4)+2\right] = 0$$

$$\Rightarrow \lambda - 1 = 0 \Rightarrow \boxed{\lambda_1 = 1}$$

or $(\lambda-1)(\lambda-4)+2=0$

$$\Rightarrow \lambda^2 - \lambda - 4\lambda + 4 + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda - 2 = 0 \Rightarrow \boxed{\lambda_2 = 2}$$

or $\lambda - 3 = 0 \Rightarrow \boxed{\lambda_3 = 3}$

$\therefore$ D & A have the same eigenvectors

$\therefore$ D & A is Similar.

Condition for diagonalization:-

Th(2) An $n \times n$ matrix $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

Ex (3) (a) in Ex (1) show That matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

is diagonalizable

pf: To find eigenvalues & eigenvectors :-

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{vmatrix} = 0$$

$$(\lambda+2) \begin{vmatrix} \lambda-1 & -3 \\ -3 & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+2)\left[(\lambda-1)^2 - 9\right] = 0$$

$$\lambda + 2 = 0 \Rightarrow \boxed{\lambda_1 = -2}$$

or $(\lambda-1)^2 - 9 = 0 \Rightarrow (\lambda-1)^2 = 9 \Rightarrow \lambda - 1 = \pm 3$

$$\Rightarrow \lambda_2 - 1 = 3 \Rightarrow \boxed{\lambda_2 = 4}$$

or $\lambda_3 - 1 = -3 \Rightarrow \boxed{\lambda_3 = -2}$

for $\lambda_1 = 4 \Rightarrow$ eigenvectors $\begin{bmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad 3x_1 - 3x_2 = 0$$

$$-3x_1 + 3x_2 = 0$$

$$\boxed{x_3 = 0}$$

$\Rightarrow \quad x_1 - x_2 = 0 \Rightarrow$ let $x_2 = t \Rightarrow x_1 - t = 0 \Rightarrow$ $x_1 = t$

$$\therefore X_1 = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = -2 \Rightarrow \begin{bmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 - 3x_2 = 0 \Rightarrow x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2$

let $x_3 = t \quad x_2 = s \Rightarrow x_1 = -s$

$$X_2 = \begin{bmatrix} -s \\ s \\ t \end{bmatrix} = -s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore P_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad P_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\therefore P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since $|P| = -2 \neq 0 \Rightarrow P_1, P_2$ & $P_3$ are L.I

$\therefore$ A is diagonalization matrix

with $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(b) in Ex(2) show that $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix}$

is diagonalizable matrix.

As in Ex(2) the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$
$\lambda_3 = 3$. to find eigenvalues:-

$\lambda_1 = 1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ +1 & 0 & -1 \\ 1 & +2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$

$x_1 + 2x_2 - 3x_3 = 0 \Rightarrow x_1 + 2x_2 - 3x_1 = 0$

$\Rightarrow 2x_2 - 2x_1 = 0$

$\Rightarrow x_2 - x_1 = 0$

$\Rightarrow x_1 = x_2 = x_3 = t$

$X_1 = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{x_1 = 0}$

$x_1 + x_2 - x_3 = 0 \Rightarrow x_2 - x_3 = 0$

$x_1 + 2x_2 - 2x_3 = 0 \Rightarrow 2x_2 - 2x_3 = 0 \Rightarrow x_2 - x_3 = 0$

$\Rightarrow \boxed{x_2 = x_3 = t}$

$X_2 = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow P_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

for $\lambda_3 = 3 \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x_1 = 0 \Rightarrow \boxed{x_1 = 0}$

$2x_2 - x_3 = 0 \Rightarrow \boxed{x_3 = 2x_2}$

$x_1 + 2x_2 - x_3 = 0 \Rightarrow 2x_2 - x_3 = 0$

$\Rightarrow \boxed{x_3 = 2x_2}$

let $\boxed{x_2 = t} \Rightarrow \boxed{x_3 = 2t}$

$X_3 = \begin{bmatrix} 0 \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \Rightarrow P_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow |P| = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 \neq 0$

$\therefore P_1$ & $P_2$ & $P_3$ are L.I

& $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ Diagonalization.

Steps in Diagonalizing a square matrix:-
Let A an $n\times n$ matrix

1. Find L.I eigenvectors $P_1, P_2, \ldots P_n$ for A for corresponding eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_3$

If n linearly independent eigenvectors does not exist then A is not diagonalizable

2. Let P is $n\times n$ matrix whose column consists of these eigenvectors i.e.

$$P = [P_1 \quad P_2 \quad P_3]$$

3. The diagonal matrix $D = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

Ex(4) Show that $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$ is diagonalizable

To find eigenvalues & eigenvectors

[1] eigenvalues $|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix} = 0$

$$(\lambda-1)\begin{vmatrix}\lambda-3 & -1\\ -1 & \lambda+1\end{vmatrix} - \begin{vmatrix}-1 & -1\\ 3 & \lambda+1\end{vmatrix} + \begin{vmatrix}-1 & \lambda-3\\ 3 & -1\end{vmatrix} = 0$$

$$(\lambda-1)\left[(\lambda-3)(\lambda+1)-1\right] - \left[-(\lambda+1)+3\right] + 1-3(\lambda-3) = 0$$

$$(\lambda-1)\left[\lambda^2-2\lambda\underbrace{-3-1}_{-4}\right] - \left[-\lambda+2\right] + 1-3\lambda+9 = 0$$

$$\lambda^3-2\lambda^2-4\lambda-\lambda^2+2\lambda+\underbrace{4+\lambda-2}_{2}-3\lambda+10 = 0$$

$\lambda^3-3\lambda^2-4\lambda+12 = 0$ $\quad \lambda_1 = 2$ is a root

$$\begin{array}{c|cccc} 2 & 1 & -3 & -4 & 12 \\ & & 2 & -2 & -12 \\ \hline & 1 & -1 & -6 & 0 \\ & \lambda^2 & \lambda & 1 & \end{array}$$

$\Rightarrow \lambda^2-\lambda-6=0 \Rightarrow (\lambda-3)(\lambda+2)=0$

$\Rightarrow \lambda_2-3=0 \Rightarrow \boxed{\lambda_2=3}$

$\lambda_3+2=0 \Rightarrow \boxed{\lambda_3=-2}$

$$\lambda_1=2 \Rightarrow \begin{array}{c} \\ \\ R_1+R_2 \\ R_3-3R_1 \end{array}\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \end{array} \begin{bmatrix} -3 & -3 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -4 & 0 \end{bmatrix} \Rightarrow -4x_2 = 0 \Rightarrow x_2 = 0$$

$x_1+x_2+x_3=0 \Rightarrow x_1+x_3=0$

$x_1 = -x_3$

let $x_3 = t \Rightarrow x_1 = -t$

$$\therefore X_1 = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3 \Rightarrow \underset{R_1 \leftrightarrow R_2}{\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}} \begin{matrix} \\ +2R_1 \\ +3R_1 \end{matrix} \begin{bmatrix} -1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 4 \end{bmatrix} \begin{matrix} \\ [-2 \;\; 0 \;\; -2] \\ [-3 \;\; 0 \;\; -3] \end{matrix}$$

$$= \underset{R_2 + R_3}{\begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}} \rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_2 - x_3 = 0 \Rightarrow x_2 = x_3 = t$$

$$-x_1 - x_3 = 0 \Rightarrow -x_1 - t = 0 \Rightarrow x_1 = -t$$

$$\therefore X_2 = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow P_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -2 \Rightarrow \underset{R_1 \leftrightarrow R_2}{\begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix}} \rightarrow \underset{R_2 + R_3}{\begin{bmatrix} -1 & -5 & -1 \\ -3 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}}$$

$$R_2 - 3R_1 \begin{bmatrix} -1 & -5 & -1 \\ -3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} [3 \;\; 15 \;\; 3]$$

$$\frac{R_2}{4} \begin{bmatrix} -1 & -5 & -1 \\ 0 & 16 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$4x_2 + x_3 = 0 \Rightarrow \boxed{x_3 = -4x_2}$

$-x_1 - 5x_2 - x_3 = 0$

$-x_1 - 5x_2 + 4x_2 = 0 \Rightarrow -x_1 - x_2 = 0$

$\Rightarrow \boxed{x_1 = -x_2}$

let $x_2 = t \Rightarrow x_1 = -t \quad x_3 = -4t$

$\therefore x_3 = \begin{bmatrix} -t \\ t \\ -4t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \Rightarrow P_3 = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}$

$\therefore P = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -4 \end{bmatrix} \;\&\; D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Ex 6 show that $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable

$\because$ A is triangular then $\lambda_1 = 1, \lambda_2 = 1$

for $\lambda = 1 \Rightarrow \begin{bmatrix} \lambda-1 & -2 \\ 0 & \lambda-1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$

$-2x_2 = 0$
$\Rightarrow x_2 = 0$ $\Rightarrow x_1 = t \Rightarrow x = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So A does not have two L-I eigen vector

$\Rightarrow$ A is not diagonalable

Theorem 3 [sufficient condition for Diagonalizable]

if an $n \times n$ matrix $A$ has $n$ distinct eigenvalues then $A$ is diagonalizable.

Ex (7) Determine whether $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

$\because$ $A$ is triangular then eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = -3$ are distinct values then by Th. (3) $A$ is diagonalizable.

Ex (1-6) Verify that $A$ is diagonalizable by & ~~find~~ eigenvalues.

1) $A = \begin{bmatrix} -11 & 36 \\ -3 & 10 \end{bmatrix}$ $P = \begin{bmatrix} -3 & -4 \\ -1 & -1 \end{bmatrix}$

sol if $P^{-1}AP = D$ then $A$ is diagonalizable

$\Delta = |P| = 3 - 4 = -1 \neq 0$

$\therefore P^{-1} = \begin{bmatrix} \frac{-1}{\Delta} & +\frac{4}{\Delta} \\ \frac{+1}{\Delta} & \frac{-3}{\Delta} \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -11 & 36 \\ -3 & 10 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= D$$

$\therefore \lambda_1 = 1, \lambda_2 = 2$ are eigenvalues of $A$

Ex (7-14) Find if possible a matrix $P$ s.t $P^{-1}AP = D$ (diagonal) s.t eigenvalue in the main diagonal.

7. $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

eigenvalues $\begin{vmatrix} \lambda-6 & 3 \\ 2 & \lambda-1 \end{vmatrix} = 0$

$(\lambda-6)(\lambda-1) - 6 = 0$

$\lambda^2 - 7\lambda + \cancel{6} \cancel{-6} = 0 \Rightarrow \lambda^2 - 7\lambda = 0$

$\lambda(\lambda - 7) = 0 \Rightarrow \lambda = 0, \lambda = 7$

$\lambda = 0 \Rightarrow \frac{R_2}{3}\begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \xrightarrow[R_1+R_2]{} \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 = 0 \Rightarrow 2x_1 = x_2$ let $x_1 = t$

$\Rightarrow x_2 = 2t$

$X = \begin{bmatrix} t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

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$$\lambda = 7 \Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \xrightarrow{} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$R_2 - 2R_1$ $[-2 \ \ -6]$

$$x_1 + 3x_2 = 0 \Rightarrow x_1 = -3x_2$$

$$x_2 = t \Rightarrow x_1 = -3t$$

$$x_2 = \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \Rightarrow P_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} \qquad \Delta = 1 + 6 = 7$$

للتحقق

Verification :- $P^{-1} \begin{bmatrix} \frac{1}{7} & +\frac{3}{7} \\ -\frac{2}{7} & \frac{1}{7} \end{bmatrix}$

Now $P^{-1}AP = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ -\frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & +1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{6}{7} - \frac{6}{7} & -\frac{3}{7} + \frac{3}{7} \\ -\frac{12}{7} - \frac{2}{7} & \frac{6}{7} + \frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & +1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & +1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$$

$= D$

with $\lambda_1 = 0$, $\lambda_2 = 7$

(15-22) Show that the matrix is not diagonalizable

19) $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

since A is triangular then $\lambda_1 = 1, \lambda_2 = 1$
$\lambda_3 = 2$

$\lambda_1 = 1 \Rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & -4 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow -x_3 = 0 \Rightarrow x_3 = 0$
$-4x_2 = 0 \Rightarrow x_2 = 0$

$x_1 = t$

$X_1 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 2 \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$x_2 - 4x_3 = 0 \Rightarrow x_2 = 4x_3 \quad x_3 = t$

$\Rightarrow x_2 = 4t$

$x_1 + 2x_2 - x_3 = 0 \Rightarrow x_1 + 8t - t = 0 \Rightarrow x_1 = -7t$

$X_2 = \begin{bmatrix} -7t \\ 4t \\ t \end{bmatrix} = t \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix} \Rightarrow P_2 = \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$

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there is no 3 L.I eigenvector

∴ A is not diagonalizable.