المحاضرة العاشرة
(88)

3.4 Applications of Determinants :-

1) The Adjoint of a matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{n3} \end{bmatrix}$

then we know the cofactor of the Matrix A is

$$\text{Cofactor } A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

where C_{ij} of A is $(-1)^{i+j}$ [det of matrix obtained by deleting the i^{th} row and j^{th} column

the transpose of cofactor matrix A is called the adjoint of A. that is

$$adj(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & & C_{nn} \end{bmatrix}$$

Ex (1) Find the adjoint of $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$

$$\text{Cof.}(A) = \begin{bmatrix} +\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} & +\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

$$\text{Adj}\,A = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$\boxed{2}$ inverse of A matrix by using adjoint

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A$$

Ex(2) Use the adjoint to find A^{-1} ~~&~~ $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$

sol As Ex(1) $\text{Adj}\,A = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$

$$|A| = -\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} = -(4-0)+(3+4)$$
$$= -4+7 = 3$$

$$\therefore A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = \begin{bmatrix} \frac{4}{3} & \frac{6}{3} & \frac{7}{3} \\ \frac{1}{3} & \frac{0}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{3}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

3 Cramer's Rule:

depends on determinants.

If the system of eq. is

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

We can write as

$$Ax = b$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} ; \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

the solution of the system if $|A| \neq 0$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{|A|} , \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{|A|}$$

Ex (3) Use Cramer's Rule to solve the system

$$4x_1 - 2x_2 = 10$$
$$3x_1 - 5x_2 = 11$$

Sol^n $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ $b = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$|A| = 4(-5) - (-2)(3) = -20 + 6 = -14 \neq 0$$

$$\therefore x_1 = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} = \frac{-50 + 22}{-14} = \frac{-28}{-14} = \boxed{2}$$

$$x_2 = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} = \frac{44 - 30}{-14} = \frac{14}{-14} = \boxed{-1}$$

For system of 3 linear eqs with 3 unknown

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ if $|A| \neq 0 \Rightarrow$ one solution

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The solution is

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{33} & b_3 \end{vmatrix}}{|A|}$$

Ex(4) Use Cramer's Rule to solve

$$-x + 2y - 3z = 1$$
$$2x + z = 0$$
$$3x - 4y + 4z = 2$$

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$|A| = -2\begin{vmatrix} 2 & -3 \\ -4 & 4 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} = -2(8-12)-(4-6)$$
$$= -2(-4)-(-2)$$
$$= 8+2 = 10$$

Sol^n is

$$x = \frac{\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = -\frac{\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix}}{10} = \frac{-(-4-4)}{10} = \frac{8}{10} = \boxed{\frac{4}{5}}$$

$$y = \frac{\begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix}}{10} = \frac{-2\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix}}{10} = \frac{-2(10)-(-5)}{10}$$
$$= \frac{-15}{10} = \boxed{-\frac{3}{2}}$$

$$z = \frac{\begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix}}{10} = \frac{-2\begin{vmatrix} 2 & 1 \\ -4 & 2 \end{vmatrix}}{10} = \frac{-2(4+4)}{10} = -\frac{16}{10} = \boxed{-\frac{8}{5}}$$

(1 – 3) Find adjoint of A for

(1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\operatorname{cof}(A) = \begin{bmatrix} +4 & -3 \\ -2 & +1 \end{bmatrix} \Rightarrow \operatorname{adj} A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

(3) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{bmatrix}$

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$$\operatorname{Cof}(A) = \begin{bmatrix} +\begin{vmatrix} 2 & 6 \\ -4 & -12 \end{vmatrix} & -\begin{vmatrix} 0 & 6 \\ 0 & -12 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 0 & -4 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ -4 & -12 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & -12 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & -4 \end{vmatrix} \\ +\begin{vmatrix} 0 & 0 \\ 2 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 6 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 0 & 0 \\ 0 & -12 & 4 \\ 0 & -6 & 2 \end{bmatrix} \Rightarrow \operatorname{Adj} A = \begin{bmatrix} -14 & 0 & 0 \\ 0 & -12 & -6 \\ 0 & 4 & 2 \end{bmatrix}$$

(10 + 17) Use Cramer's Rule to solve

10) $2x - y = -10$

$3x + 2y = -1$

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} -10 \\ -1 \end{bmatrix}$$

$|A| = (2)(2) - (-1)(3) = 4 + 3 = 7 \neq 0$ one sol.

$$x = \frac{\begin{vmatrix} -10 & -1 \\ -1 & 2 \end{vmatrix}}{7} = \frac{-20-1}{7} = -\frac{21}{7} = -3$$

$$y = \frac{\begin{vmatrix} 2 & -10 \\ 3 & -1 \end{vmatrix}}{7} = \frac{-2+30}{7} = \frac{28}{7} = 4$$

17) $4x - y - z = 1$

$2x + 2y + 3z = 10$

$5x - 2y - 2z = -1$

$$A = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 2 & 3 \\ 5 & -2 & -2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 10 \\ -1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$|A| = 4\begin{vmatrix}2 & 3\\ -2 & -2\end{vmatrix} - (-1)\begin{vmatrix}2 & 3\\ 5 & -2\end{vmatrix} - \begin{vmatrix}2 & 2\\ 5 & -2\end{vmatrix}$$

$$= 4(2) + (-19) - (-14) = 8 - 19 + 14 = 3 \neq 0$$

$$x = \frac{\begin{vmatrix}1 & -1 & -1\\ 10 & 2 & 3\\ -1 & -2 & -2\end{vmatrix}}{3} = \frac{\begin{vmatrix}2 & 3\\ -2 & -2\end{vmatrix} - (-1)\begin{vmatrix}10 & 3\\ -1 & -2\end{vmatrix} - \begin{vmatrix}10 & 2\\ -1 & -2\end{vmatrix}}{3}$$

$$= \frac{(-4+6) + (-20+3) - (-20+2)}{3}$$

$$= \frac{2 - 17 + 18}{3} = \frac{2+1}{3} = \frac{3}{3} = \boxed{1}$$

$$\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ y = \frac{\begin{vmatrix}4 & 1 & -1\\ 2 & 10 & 3\\ 5 & -1 & -2\end{vmatrix}}{3} \end{array} = \frac{4\begin{vmatrix}10 & 3\\ -1 & -2\end{vmatrix} - \begin{vmatrix}2 & 3\\ 5 & -2\end{vmatrix} - \begin{vmatrix}2 & 10\\ 5 & -1\end{vmatrix}}{3}$$

$$= \frac{4(-20+3) - (-4-15) - (-2-50)}{3}$$

$$= \frac{4(-17) + 19 + 52}{3} = \frac{3}{3} = 1$$

$$\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ z = \frac{\begin{vmatrix}4 & -1 & 1\\ 2 & 2 & 10\\ 5 & -2 & -1\end{vmatrix}}{3} \end{array} = \frac{4\begin{vmatrix}2 & 10\\ -2 & -1\end{vmatrix} - (-1)\begin{vmatrix}2 & 10\\ 5 & -1\end{vmatrix} + \begin{vmatrix}2 & 2\\ 5 & -2\end{vmatrix}}{3}$$

$$= \frac{4(-2+20) + (-2-50) + (-4-10)}{3}$$

$$= \frac{4(18) - 52 - 14}{3} = \frac{72 - 66}{3} = \frac{6}{3} = \boxed{2}$$

27) Use Cramer's Rule to solve

$$kx + (1-k)y = 1$$

$$(1-k)x + ky = 3$$

For what value(s) of k will the system be
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inconsistent

$$A = \begin{bmatrix} k & 1-k \\ 1-k & k \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$|A| = k^2 - (1-k)^2 = \cancel{k^2} - (1 - 2k + \cancel{k^2}) = 2k - 1$$

$$x = \frac{\begin{vmatrix} 1 & 1-k \\ 3 & k \end{vmatrix}}{2k-1} = \frac{k - 3(1-k)}{2k-1} = \frac{k - 3 + 3k}{2k-1}$$

$$= \frac{4k - 3}{2k-1}$$

$$y = \frac{\begin{vmatrix} k & 1 \\ 1-k & 3 \end{vmatrix}}{2k-1} = \frac{3k - (1-k)}{2k-1} = \frac{4k-1}{2k-1}$$

The system will be inconsistent if $2k - 1 = 0$

$$\Rightarrow k = \frac{1}{2}$$