المحاضرة السادسة

3.3 Properties of determinants

Ex(1) The Det. of Matrix Products:
Find $|A|$, $|B|$ & $|AB|$ for:-

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

To find $|A| = \begin{vmatrix} \overset{+}{1} & -2 & 2 \\ \overset{-}{0} & 3 & 2 \\ \overset{+}{1} & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} -2 & 2 \\ 3 & 2 \end{vmatrix}$

$= (3-0) + (-4-6)$

$= 3 - 10 = -7$

$$\begin{array}{c} \;\;+ \quad - \quad + \\ |B| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} \end{array} = 2 \begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix}$$

$= 2(2+2) + (0+3) = 8 + 3 = 11$

$$AB = \begin{bmatrix} 2+0+6 & 0+2+2 & 1+4-4 \\ 0+0+6 & 0-3+2 & 0-6-4 \\ 2+0+3 & 0+0+1 & 1+0-2 \end{bmatrix} = \begin{array}{c} \;+ \quad - \quad + \\ \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix} \end{array}$$

$|AB| = 8 \begin{vmatrix} -1 & -10 \\ 1 & -1 \end{vmatrix} - 4 \begin{vmatrix} 6 & -10 \\ 5 & -1 \end{vmatrix} + \begin{vmatrix} 6 & -1 \\ 5 & 1 \end{vmatrix}$

$= 8(1+10) - 4(-6+50) + (6+5)$

$= 88 - 4(44) + 11 = 99 - 176 = -77 = |A||B|$

$\therefore |AB| = |A||B|$

- If A is a square matrix of order n, c is a scalar $c \in \mathbb{R}$ then $|cA| = c^n |A|$

Ex(2) Find $|A|$ for $A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}$

$$A = 10 \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix}$$

$$= 10^3 \left(-3 \begin{vmatrix} -2 & 4 \\ -3 & 1 \end{vmatrix} - 5 \begin{vmatrix} 1 & -2 \\ -2 & -3 \end{vmatrix} \right) = 10^3 \left[-3(-2+12) - 5(-3-4) \right]$$

$$= 10^3 \left[-30 + 35 \right] = 10^3 [5] = 5000$$

Note $|A+B| \neq |A| + |B|$

Theorem: A square matrix A is invertible (nonsingular) if and only if $|A| \neq 0$

Ex(3) Determine whether each matrix has an inverse

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

sol $$|A| = \begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix} \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 3 & 0 & 1 \\ 3 & 0 & -1 \end{vmatrix} = 0$$

$C_2 + 2C_3$

$\Rightarrow A$ is singular $\Rightarrow A$ has no inverse.

$$|B| = \begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{vmatrix} \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 3 & 0 & 1 \\ 3 & 4 & 1 \end{vmatrix} = -1 \begin{vmatrix} 3 & 0 \\ 3 & 4 \end{vmatrix}$$

$C_2 + 2C_3$

$= -1(12-0)$

$= -12 \neq 0$

A has an inverse (is nonsingular)

Theorem: If A is $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Ex (4) if $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ find $|A^{-1}|$

$$|A| = \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} + 3 \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = (0-2) + 3(0+2)$$
$$= -2 + 6 = 4$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|} = \frac{1}{4}.$$

theorem: If A is $n \times n$ matrix then the statements below are equivalent $[1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 1]$

$1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5 \Leftrightarrow 6$

1. A is invertible
2. $Ax = b$ has a unique solution
3. $Ax = 0$ has only the trivial solution i.e. $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$
4. A is row-equivalent to I_n
   $[A : I_n] \to [I_n : A^{-1}]$
5. $|A| \neq 0$

Ex(5) which of the systems has a unique sol^n

a. $$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$ $Ax = b$

coeff matrix $A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix}$

$$\Rightarrow |A| = \begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix} \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 3 & 0 & 1 \\ 3 & 0 & -1 \end{vmatrix} = 0$$

$C_2 + 2C_3$

$\Rightarrow$ has no unique sol^n either no sol^n or infinitly many solution

b. $$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$ $Ax = b$

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{vmatrix} \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

$C_2 + 2C_3$

$$= \begin{vmatrix} 0 & 0 & -1 \\ 3 & 0 & 1 \\ 3 & 4 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & 0 \\ 3 & 4 \end{vmatrix} = -1 \neq 0$$

$\Rightarrow$ has a unique sol^n.

th:- If A is a square matrix then $|A| = |A^T|$

Ex(6) show that $|A| = |A^T|$ for

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & 0 \\ -4 & -1 & 5 \end{bmatrix}$$

$$|A| = -2\begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix} = -2(5-2) = -2(3) = -6$$

$$|A^T| = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 0 & -1 \\ -2 & 0 & 5 \end{vmatrix} = -2\begin{vmatrix} 1 & -1 \\ -2 & 5 \end{vmatrix} = -2(5-2)$$
$$= -2(3) = -6$$

$$\therefore |A| = |A^T|$$

(2-3) verify $|AB| = |A|\,|B|$

2. $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}$

$|A| = 9 - 16 = -7$ $|B| = 0 + 5 = 5$

$$|AB| = \begin{vmatrix} 26 & -3 \\ 23 & -4 \end{vmatrix} = -104 + 69 = -35$$

$$\Rightarrow |AB| = -35 = (-7)(5) = |A|\,|B|$$

3. $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$|A| = -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-1-1) = 2$ $|B| = (-1)(2)(3) = -6$

$$|AB| = \begin{vmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} = -2(3+3)$$

$$= -12$$

$$\therefore |AB| = -12 = (2)(-6) = |A||B|$$

(8 - 9) Use the fact $|cA| = c^n|A|$ if A is $n \times n$ to find $|A|$

8) $A = \begin{bmatrix} 21 & 7 \\ 28 & -56 \end{bmatrix}$

$$A = 7\begin{bmatrix} 3 & 1 \\ 4 & -8 \end{bmatrix} \Rightarrow |A| = 7^2 \begin{vmatrix} 3 & 1 \\ 4 & -8 \end{vmatrix}$$

$$= 49(-24-4)$$

$$= 49(-28) = -1372$$

9) $A = \begin{bmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{bmatrix} = 3\begin{bmatrix} -1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

$$\therefore |A| = 3^3 \begin{vmatrix} -1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 27\left[-\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}\right]$$

$$= 27\left[-(15-16) - 2(10-12) + 3(8-9)\right]$$

$$= 27[1+4-3] = 27(2) = 54$$

16-17 verify $|A+B| = |A| + |B|$

16. $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$

$|A| = 0 + 2 = 2 \quad |B| = 0$

$|A+B| = \begin{vmatrix} 4 & -4 \\ 1 & 0 \end{vmatrix} = 0 + 4 = 4$

$|A| + |B| = 2 + 0 \neq 4 = |A+B|$

17. $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$|A| = 1\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (1-2) - (-1-1) = -1 + 2 = 1$

$|B| = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = \cancel{(2-2)} + (-1-0) = -1$

$|A+B| = \begin{vmatrix} 0 & 1 & 3 \\ -1 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix} = -(-1)\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}$

$= 1(1-6) + (3-6)$

$= -5 + (-3) = -8$

$|A| + |B| = 1 + (-1) = 0 \neq -8 = |A+B|$

(20, 22) Use det to decide whether matrix is singular or nonsingular

(22) $A = \begin{bmatrix} 3 & -6 \\ 4 & 2 \end{bmatrix} \Rightarrow |A| = 6 + 24 = 30 \neq 0$

$\Rightarrow$ non singular

(23) $A = \begin{bmatrix} 1 & 0 & -8 & 2 \\ 0 & 8 & -1 & 10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ upper triangular

$\Rightarrow |A| = (1)(8)(0)(2)$

$= 0$

$\Rightarrow$ singular

Ex (25, 28) Find $|A^{-1}|$

(25) $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ $|A| = (8-3) = 5$

$\therefore |A^{-1}| = \frac{1}{|A|} = \frac{1}{5}$

(28) $A = \begin{bmatrix} \overset{+}{1} & \overset{-}{0} & \overset{+}{1} \\ 2 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ $|A| = \begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$

$= (-3+4) + (-4+1)$

$= 1 - 3 = -2$

$\therefore |A^{-1}| = \frac{1}{|A|} = -\frac{1}{3}$

(37, 41) Find k s.t A is singular

37) $A = \begin{bmatrix} k-1 & 3 \\ 2 & k-2 \end{bmatrix}$ A is singular $\Leftrightarrow |A| = 0$

$\Rightarrow (k-1)(k-2) - 6 = 0 \Rightarrow k^2 - 3k + 2 - 6 = 0$

$\Rightarrow k^2 - 3k - 4 = 0$

$(k-4)(k+1) = 0$

either $k - 4 = 0 \Rightarrow \boxed{k = 4}$

or $k + 1 = 0 \Rightarrow \boxed{k = -1}$

(87)

41) $A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} 0 & k & 1 \\ k & 1 & k \\ 1 & k & 0 \end{bmatrix} \end{array}$ A is singular if $|A| = 0$

$\Rightarrow -k\begin{vmatrix} k & k \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} k & 1 \\ 1 & k \end{vmatrix} = 0$

$\Rightarrow -k(0-k) + (k^2-1) = 0$

$\Rightarrow k^2 + k^2 - 1 = 0$

$\Rightarrow 2k^2 - 1 = 0 \Rightarrow 2k^2 = 1 \Rightarrow k^2 = \frac{1}{2}$

$\Rightarrow k = \pm \frac{1}{\sqrt{2}}$