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4.2 Vector spaces :-

Let V be a set on which two operations (vector additive & scalar multiplication) are defined for every vectors $u, v, w \in V$ and $c, d \in \mathbb{R}$ (scalars), then V is a vector space if the following axioms are satisfied

(I) For additive :

1. $u + v \in V$ closure
2. $u + v = v + u$ commutative
3. $u + (v + w) = (u + v) + w$ associative
4. V has a zero vector 0 s.t
   $u + 0 = 0 + u = u$ identity
5. for $\forall u \in V$ $\exists -u \in V$ s.t
   $u + (-u) = -u + u = 0$ inverse

(II) For scalar multiplication

6. $cu \in V$ closure
7. $c(u + v) = cu + cv$ distributive
8. $(c + d)u = cu + du$ "
9. $c(du) = (cd)u$ associative
10. $1(u) = u$ identity

Ex (1+2) $\mathbb{R}^2$ & $\mathbb{R}^n$ with additive & scalar multiplication are vector space.

Ex(4) Let P_2 be the set of all polynomials of the form $p(x) = a_0 + a_1 x + a_2 x^2$ (of deg. 2)

and $q(x) = b_0 + b_1 x + b_2 x^2$

show that P_2 is a vecter space.

Pf:- (I) <u>additive</u>

1. closure: $P(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \in P_2$

2. commutative:

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$
$$= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2$$
$$= q(x) + P(x)$$

associative

3. $$[P(x) + q(x)] + R(x) = [(a_0 + b_0) + c_0] + [(a_1 + b_1) + c_1]x + [(a_2 + b_2) + c_2]x^2$$
$$= a_0 + (b_0 + c_0) + [a_0 + (b_1 + c_1)]x + [a_2 + (b_2 + c_2)]x^2$$
$$= P(x) + [q(x) + R(x)]$$

4. Identity

$0(x) = 0 + 0x + 0x^2 \in P_2$ s.t $0 + P(x) = P(x)$

$\cancel{= 0(x) \in}$

5. inverse $-P(x) = -a_0 - a_1 x - a_2 x^2 \in P_2$

s.t $P(x) + (-P(x)) = 0(x) = 0$

II scalar multiplication :- for $c \in \mathbb{R}$

6. closure : for $c \in \mathbb{R}$ the

$$cP(x) = ca_0 + ca_1x + ca_2x^2 \in P_2$$

7. distributive

$$c[P(x) + q(x)] = c(a_0 + b_0) + c[a_1 + b_1]x + c[a_2 + b_2]x^2$$
$$= [ca_0 + ca_1x + ca_2x^2] + [cb_0 + cb_1x + cb_2x^2]$$
$$= cP(x) + cq(x)$$

8. Distributive :

$$(c+d)P(x) = (c+d)a_0 + (c+d)a_1x + (c+d)a_2x^2$$
$$= \underbrace{ca_0 + ca_1x + ca_2x^2} + \underbrace{da_0 + da_1x + da_2x^2}$$
$$= cP(x) + dP(x)$$

9. associative

$$c(dP(x)) = c[da_0 + da_1x + da_2x^2]$$
$$= cd[\underbrace{a_0 + a_1x + a_2x^2}_{P(x)}] = (cd)P(x)$$

10. $1 \cdot P(x) = P(x)$

$\therefore$ P_2 the set of all poly of degree 2 or less is vecter space.

Note : Similarly P_n the set of all polynomials of degree n or less is vector space.

Ex(6) Show that the set of all integers number under additive & multiplication is not a vecter space

sol Let $\frac{1}{2} \in \mathbb{R}$ & $x \in I = \{0, \pm 1, \pm 2, \dots\}$

$\Rightarrow \frac{1}{2} \notin I$ Ex($3 \in I \Rightarrow \frac{3}{2} \notin I$)

not closure under multiplication

Ex(6) The set of 2^{nd} degree polynomials is not a vecter space

Pf Let $p(x) = 1 + x - x^2$ and $q(x) = 1 + x + x^2$ are poly. of degree 2

but $p(x) + q(x) = 2 + 2x$ is poly of degree 1 not of degree 2.

not closure under additive

Ex(8) Let $V = \mathbb{R}^2$ under addition & scalar multiplication as $c(x_1, x_2) = (cx_1, 0)$ show that V is not vecter space.

V is satisfied all first nine axiom except 10. If $(x_1, x_2) \in V$

$\Rightarrow 1.(x_1, x_2) = (x_1, 0) \neq (x_1, x_2)$

Exercises

(1,5)

Describe the zero vector (additive identity) of the vectors space

1. R^4 is $(0,0,0,0)$

5. P_3 (poly. of degree 3 or less)

$p(x) = 0 = 0 + 0x + 0x^2 + 0x^3$

(7, 11) Discribe the additive inverse of a vector in the following vector space

7. $\mathbb{R}^3$ if $(x_1, x_2, x_3) \in \mathbb{R}^3$ the inverse is $(-x_1, -x_2, -x_3)$ s.t $(x_1, x_2, x_3) + (-x_1, -x_2, -x_3)$

$= (x_1 - x_1, x_2 - x_2, x_3 - x_3) = (0,0,0)$

11. P_4 if $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \in P_4$

then $-p(x) = -a_0 - a_1x - a_2x^2 - a_3x^3 - a_4x^4$ is the inverse of $p(x)$ s.t $p(x) + (-p(x)) = 0$

37. Let V be the set of all +ve real nos. ($V = \mathbb{R}^+$)

Determine whether V is a vector space with operations

$x + y = xy$ addition

$cx = x^c$ scalar multiplication

let $u, v, w \in \mathbb{R}^+ = V$ $\quad c, d \in \mathbb{R}$ (scaler)

I Additive: $u + v = uv$

1) closure $u + v = uv \in \mathbb{R}^+ = V$

2) commutative $u + v = uv = vu = v + u$

3) associative $(u+v) + w = (uv) + w = (uv)w = u(vw)$

$= u + (v + w)$

4). 1 is an identity of additive, $1 \in \mathbb{R}^+$ as

$1 + u = 1 . u = u$

5) for any $u \in \mathbb{R}^+$ there is $\frac{1}{u} \in \mathbb{R}^2$ s.t

$u + \frac{1}{u} = u . \frac{1}{u} = 1$

$\Rightarrow \frac{1}{u}$ is an additive inverse of u

(II) For scaler multiplication $cu = u^c$

6) $cu = u^c \in \mathbb{R}^+ = V$ closure

7) $c(u+v) = (u+v)^c = (uv)^c = u^c v^c = cu + cv$

(distributive)

8) $(c+d)u = u^{(c+d)} = u^c . u^d = cu + du$

(distributive)

9) $c(du) = c\underbrace{u^d}_{+ve\ number} = (u^d)^c = u^{dc} = (cd)u$

10) $1(u) = u^1 = u$

$\therefore V = \mathbb{R}^+$ is a vecter space under both

given operations