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المحاضرة الثالثة عشر

4.3 Subspaces of vectorspaces

Def^n: A non empty subset W of a vector space V is a subspace of V when W is a vector space under the operations of addition & scalar multiplication defined in V

Ex (1) Show that $W = \{(x_1, 0, x_3) : x_1 \,\&\, x_3 \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$ with standard operations

Let $X, Y, Z \in W$, $c, d \in \mathbb{R}$ (scalars)

(I) additive :- $X = (x_1, 0, x_3)$, $Y = (y_1, 0, y_3)$, $Z = (z_1, 0, z_3)$

1. closure: $X + Y = (x_1 + y_1, 0, x_3 + y_3) \in W$

2. Commutative: $X + Y = (x_1 + y_1, 0, x_3 + y_3) = (y_1 + x_1, 0, y_3 + x_3)$
$= (y_1, 0, y_3) + (x_1, 0, x_3) = Y + X$

3. associative $(X+Y)+Z = ((x_1+y_1)+z_1, 0, (x_3+y_3)+z_3)$
$= (x_1 + (y_1 + z_1), 0, (x_3 + (y_3 + z_3)))$
$= (x_1, 0, x_3) + (y_1 + z_1, 0, y_3 + z_3)$
$= X + (Y + Z)$

4. $0 = (0, 0, 0) \in W$ is an Identity of additive
as $X + 0 = (x_1, 0, x_2) + (0, 0, 0) = (x_1, 0, x_2) = X$

5. $-X = (-x_1, 0, -x_3) \in W$ is an additive inverse of $X = (x_1, 0, x_3)$ as
$X + (-X) = (x_1 - x_1, 0, x_3 - x_3) = (0, 0, 0) = 0$

II scalar multiplication :

6. $cX = (cx_1, 0, cx_2) \in W$ closure

7. $c(X+Y) = c(x_1+y_1, 0, x_3+y_3) = (cx_1+cy_1, 0, cx_3+cy_3)$

$= (cx_1, 0, cx_3) + (cy_1, 0, cy_3)$

$= c(x_1, 0, x_3) + c(y_1, 0, y_3)$

$= cX + cY$ distributive

8. $(c+d)X = (c+d)(x_1, 0, x_3) = ((c+d)x_1, 0, (c+d)x_3)$

$= (cx_1+dx_1, 0, cx_3+dx_3)$

$= (cx_1, 0, cx_3) + (dx_1, 0, dx_3)$

$= c(x_1, 0, x_3) + d(x_1, 0, x_3)$

$= cX + dX$ distributive

9. $c(dX) = c(dx_1, 0, dx_3) = (cdx_1, 0, cdx_3)$

$= cd(x_1, 0, x_3) = (cd)X$

10. $1.X = (1.x_1, 0, 1.x_3) = (x_1, 0, x_3) = X$

تعريف بديل

Theorem : If W is a nonempty subset of a vector space of V, then W is a subspace of V iff

1. $u, v \in W \Rightarrow u+v \in W$
2. if $u \in W$, c any scalar then $cu \in W$

Ex(4) Show that $W = \{(x_1, x_2) : x_1, x_2 \geq 0\}$ with standard operations is not subspace of $\mathbb{R}^2$

W is closed under additive but not closed under scalor multiplication as

let $(2,3) \in W \quad -2 \in \mathbb{R}$

$\Rightarrow -2(2,3) = (-4,-6) \notin W$

$\therefore$ W is not subspace of $\mathbb{R}^2$

<u>Theorem</u> :- If V & W are both subspaces of a vector space U then $V \cap W$ is also a subspace of U

Ex(6) Determine whether each subsets of $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$

a. $U = \{(x,y) : x + 2y = 0\}$

$\because x + 2y = 0 \quad$ put $y = t \Rightarrow x + 2t = 0$

$\Rightarrow x = -2t$

$\Rightarrow (-2t, t) \in U$

1. additive closure : if $X, Y \in U$

$\Rightarrow X = (-2t_1, t_1) \quad , Y = (-2t_2, t_2)$

$s = t_1 + t_2$

$(-2s, s)$

$X + Y = (-2t_1 - 2t_2, t_1 + t_2) = (-2(t_1 + t_2), t_1 + t_2) \in U$

2. closure under scalor multiplication : $c \in \mathbb{R}, X \in U$

$cX = c(-2t_1, t_1) = (-2ct_1, ct_1) \in U$

$\therefore$ U is a subspace of $\mathbb{R}^2$

b. $U = \{(x, y) : x + 2y = 1\}$

$\because$ $x + 2y = 1$ not passes through the origin then $(0,0) \in U$ & not closure under additive

$\Rightarrow$ U is not subspace of $\mathbb{R}^2$ because every subspace of $\mathbb{R}^2$ must contains an additive identity $(0,0)$

Ex(7) Show that $W = \{(x, y) : x^2 + y^2 = 1\}$ is not subspace of $\mathbb{R}^2$

sol $(1,0), (0,1) \in W$ but $(1,0) + (0,1) = (1,1) \notin W$

$\therefore$ additive is not closure. and also $(0,0) \notin W$

Ex(8) Determine whether each subset of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$

a. $W = \{(x_1, x_2, 1) ; x_1, x_2 \in \mathbb{R}\}$

$0 = (0, 0, 0) \in W$ so W is not a subspace of $\mathbb{R}^3$

& also is not closure under additive operation

b. $W = \{(x_1, x_1 + x_3, x_3) : x_1, x_3 \in \mathbb{R}\}$

let $u = (u_1, u_1 + u_3, u_3)$ $\quad v = (v_1, v_1 + v_3, v_3)$

$c \in \mathbb{R}$ then W is a subspace of $\mathbb{R}^3$ as

W

1- closed under additive $u + v = (u_1 + v_1, u_1 + v_1 + u_3 + v_3, u_3 + v_3)$

$= (w_1, w_1 + w_3, w_3) \in W$

where $w_1 = u_1 + v_1$, $w_3 = u_3 + v_3$

2. W is closed under scalar multiplication as

$$cu = c(u_1, u_1+u_3, u_3) = (cu_1, cu_1+cu_3, cu_3)$$
$$= (x_1, x_1+x_3, x_3) \in W$$

where $x_1 = cu_1$, $x_3 = cu_3$

W is a subspace of $\mathbb{R}^3$

Ex(1+2) verify that W is a subspace of V
assume that V has the standard operations

1. $W = \{(x_1, x_2, x_3, 0) : x_1, x_2, x_3 \in \mathbb{R}\}$; $V = \mathbb{R}^4$

Let $X = (x_1, x_2, x_3, 0)$, $Y = (y_1, y_2, y_3, 0) \in W$
$c \in \mathbb{R}$

1.) closed under additive :-
$X + Y = (x_1+y_1, x_2+y_2, x_3+y_3, 0) \in W$

2) $cX = (cx_1, cx_2, cx_3, 0) = (z_1, z_2, z_3, 0) \in W$
where $cx_1 = z_1$, $cx_2 = z_2$, $cx_3 = z_3$
$\Rightarrow$ closed under scalar multiplication.
$\Rightarrow$ W is a subspace of $\mathbb{R}^4$

2. $W = \{(x, y, 4x-5y) : x, y \in \mathbb{R}\}$ $V = \mathbb{R}^3$

1) Let $x, y \in W$, $x = (x_1, x_2, 4x_1 - 5x_2)$
$y = (y_1, y_2, 4y_1 - 5y_2)$

$\Rightarrow x + y = (x_1+y_1, x_2+y_2, 4(x_1+y_1) - 5(x_2+y_2))$
$= (z_1, z_2, 4z_1 - 5z_2) \in W$
$\Rightarrow$ closed under addition

2) let $c \in \mathbb{R} \Rightarrow cx = (cx_1, cx_2, c(4x_1 - 5x_2))$

$= (cx_1, cx_2, 4cx_1 - 5cx_2)$

$= (z_1, z_2, 4z_1 - 5z_2) \in W$

$z_1 = cx_1, \; z_2 = cx_2$

W is a subspace of $\mathbb{R}^3$.

(7 - 10) verify that W is not subspace of the given vector space (give an specific example)

7) $W = \{(x, y, -1) : x, y \in \mathbb{R}\}$ vector space $\mathbb{R}^3$

$\because (0,0,0) \notin W$ or also not closed under additive as $x = (1,1,-1)$, $y = (2,2,-1)$

$\Rightarrow x + y = (3, 3, -2) \notin W$

8) $W = \{(2, y) : y \in \mathbb{R}\}$; $\mathbb{R}^2$

$(0,0) \notin W$ or not closed under addition

let $x = (2, x_2)$, $y = (2, y_2) \Rightarrow x, y \in W$

but $x + y = (2+2, x_2 + y_2) = (4, x_2 + y_2) \notin W$

9) $W = \{(x, y) : x, y \in \mathbb{Q}$ (rational number$)\}$ $\mathbb{R}^2$

let $x = (2, 5) \in W$ $\quad c = \sqrt{3} \in \mathbb{R}$

$\Rightarrow c \cdot x = \sqrt{3}\, x = (2\sqrt{3}, 5\sqrt{3}) \notin W$

as $2\sqrt{3}, 5\sqrt{3} \notin \mathbb{Q}$

not closed under scaler multiplication

10) $W = \{(x,y) : x, y \in \mathbb{Z} \text{ (integers)}\}$ $\mathbb{R}^2$

let $x = (1,-1) \in W$ $c = \frac{1}{2} \in \mathbb{R}$

$\Rightarrow cx = \frac{1}{2}x = \left(\frac{1}{2}, -\frac{1}{2}\right) \notin W$ as $\frac{1}{2}, -\frac{1}{2} \notin \mathbb{Z}$

not closed under scalor multiplication.

(37-42) Determine whether the set W is a subspace of $\mathbb{R}^3$ with standard operations. Justify your answer.

37. $W = \{(0, x_2, x_3) : x_2, x_3 \in \mathbb{R}\}$

W is a subspace of $\mathbb{R}^3$ because for $x, y \in W$

$\Rightarrow x = (0, x_2, x_3)$ $y = (0, y_2, y_3)$ $c \in \mathbb{R}$

1. additive closed: $x + y = (0, x_2 + y_2, x_3 + y_3)$

$= (0, z_2, z_3) \in W$

$z_2 = x_2 + y_2$, $z_3 = x_3 + y_3$

2. $cx = (0, cx_2, cx_3) = (0, z_2, z_3) \in W$

$z_2 = cx_2$, $z_3 = c\,x_3$

38) $W = \{(x_1, x_2, 4) : x_1, x_2 \in \mathbb{R}\}$

W not subspace of $\mathbb{R}^3$ as $(0,0,0) \notin W$

or also W is not closed under aditive

let $(1,-1,4)$, $(-1,2,4) \in W$ but

$(1,-1,4) + (-1,2,4) = (0,1,8) \notin W$

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39) $W = \{(a, a-3b, b) : a, b \in \mathbb{R}\}$

W is subspace of $\mathbb{R}^3$ as

$X = (x_1, x_1 - 3x_3, x_3)$, $Y = (y_1, y_1 - 3y_3, y_3)$

1.) W is closed under addition :-

$X + Y = (x_1 + y_1, x_1 + y_1 - 3(x_3 + y_3), x_3 + y_3)$

$= (z_1, z_1 - 3z_3, z_3) \in W$

where $z_1 = (x_1 + y_1)$ $z_3 = x_3 + y_3$

2) W is closed under scalar multiplication . as

$cX = (cx_1, cx_1 - 3c\,x_3, cx_3)$

$= (z_1, z_1 - 3z_3, z_3) \in W$

where $z_1 = cx_1$ $z_3 = cx_3$

41) $W = \{(x_1, x_2, x_1 x_2) : x_1, x_2 \in \mathbb{R}\}$

W not subspace of $\mathbb{R}^3$

let $(1, 2, 2)$ & $(2, 1, 2) \in W$

but $(1, 2, 2) + (2, 1, 2) = (3, 3, 4) \notin W$

as $(3, 3, 3 \times 3) = (3, 3, 9) \in W$

W not closed under addition.

42) $W = \{(x_1, \frac{1}{x_1}, x_3) : x_1 \neq 0, x_1, x_3 \in \mathbb{R}\}$

$0 = (0, 0, 0) \notin W \Rightarrow W$ is not subspace of $\mathbb{R}^3$