In-Class Exercise: The tank is 3/4 full:

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In 2002 there was caller in the radioshow CarTalk, who drove an 18 wheeler with saddle tanks, these are cylindrical tanks with the axis mounted horizontally. Since his fuel gage was broken, he wanted to mark a stick that he would insert vertically into the tank. Marking the half full position on the stick is trivial, but he asked where to mark the 3/4 full position on the stick (which of course also gives the 1/4 full location) due to symmetry.

We can reduce the problem to a single quadrant of a circle with a radius of 1 as shown in Figure 1. The total area of this circle would be π, and therefore the area of the quadrant is $\frac{\pi}{4}$. To mark the quarter full point, we need to find the value of $y = z$ where the area above the curve is equal to half the area of the quadrant which is $\frac{\pi}{8}$.

Before we find out what z is let's calculate the area of the quadrant itself where y ranges from $[0, 1]$.

$$A = \int_{y=0}^{y=1} x dy \tag{1}$$

We know the equation for a circle which is:

$$x^2 + y^2 = 1 \tag{2}$$

Substitution gives:

$$A = \int_{y=0}^{y=1} \sqrt{1 - y^2} dy \tag{3}$$

This is not an easy integral to solve; the trick is to replace y by $\sin(p)$. Now we have to evaluate the limits as follows:

$$y = 0 \Rightarrow p = 0 \tag{4}$$

$$y = 1 \Rightarrow p = \frac{\pi}{2} \tag{5}$$

$$dy \Rightarrow \cos(p) dp \tag{6}$$

Substitution gives:

$$\int_{y=0}^{y=1} \sqrt{1 - y^2} dy = \int_{p=0}^{p=\frac{\pi}{2}} \sqrt{1 - \sin^2(p)} \cos(p) dp = \int_{p=0}^{p=\frac{\pi}{2}} \sqrt{\cos^2(p)} \cos(p) dp = \int_{p=0}^{p=\frac{\pi}{2}} cos^2(p) dp \tag{7}$$

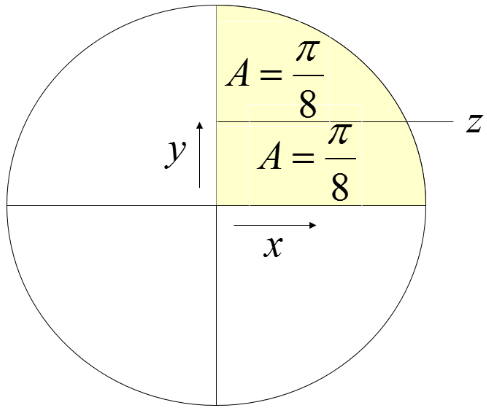

Figure 1: The question is where to mark the dipstick such that a cylindrical saddle tank on an 18 wheeler shows 1/4 full.

Let's use Euler's formula to simplify the $\cos^2(t)$ term as follows:

$$\cos^2(p) = \left(\frac{e^{jp} + e^{-jp}}{2}\right)^2 = \frac{e^{2jp} + 2 + e^{-2jp}}{4} = \frac{1}{2} + \frac{1}{2}\left(\frac{e^{2jp} + e^{-2jp}}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos(2p) \quad (8)$$

Substitution gives:

$$\int_{p=0}^{p=\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos(2p)dp = \left|\frac{1}{2}p + \frac{1}{4}\sin(2p)\right|_{p=0}^{p=\frac{\pi}{2}} = \frac{\pi}{4} \quad (9)$$

This all makes sense, but now we need to find the value of $z = y$ where the area is equal to $\frac{\pi}{8}$ or:

$$\left|\frac{1}{2}p + \frac{1}{4}\sin(2p)\right|_{p=0}^{p=z} = \frac{\pi}{8} \quad (10)$$

Substituting the limits gives:

$$\boxed{\frac{1}{2}z + \frac{1}{4}\sin(2z) = \frac{\pi}{8} \Rightarrow z + \frac{1}{2}\sin(2z) = \frac{\pi}{4}} \quad (11)$$

This is called a transcendental equation because it has the transcendental function sine in it. Believe it or not, but we cannot solve this simple equation analytically; therefore, we have to resort to numerical methods. The standard method is expression of the sine function in a MacLaurin form, where in the first order approximation we can say that $\sin(x) \approx x$ and therefore:

$$\frac{1}{2}z + \frac{1}{4}2z = \frac{\pi}{8} \Rightarrow z \approx \frac{\pi}{8} = 0.392699081698724 \quad (12)$$

Wolfram's calculator gives an approximate answer of $z \approx 0.415856$ (I assume a higher order MacLaurin approximation.)