

A Problem-Solving Article on Stability for Linear ODE Systems

Lyapunov, LaSalle Invariance, Eigenvalue Classification, and Computation

What We Study

We analyze the planar linear system

$$\dot{x} = y, \quad \dot{y} = -x - y,$$

from three complementary viewpoints: (i) eigenvalues and explicit solution, (ii) Lyapunov $V = x^2 + y^2$ and the meaning of $\dot{V} \leq 0$, (iii) a rigorous upgrade to asymptotic stability using **LaSalle's invariance principle**. Finally, we discuss a common *consistency pitfall* involving $\ddot{x} = y$ and show how it forces a 3D phase space.

1 Preliminaries: Stability Notions and Two Key Theorems

Definition 1.1: Stability Notions

Consider the autonomous system $\dot{z} = f(z)$ with equilibrium z^* ($f(z^*) = 0$).

- **Stable (Lyapunov):** for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|z(0) - z^*\|_2 < \delta \Rightarrow \|z(t) - z^*\|_2 < \varepsilon$ for all $t \geq 0$.
- **Asymptotically stable:** stable and $z(t) \rightarrow z^*$ as $t \rightarrow \infty$.
- **Exponentially stable:** there exist $M, \alpha > 0$ such that $\|z(t) - z^*\|_2 \leq M e^{-\alpha t} \|z(0) - z^*\|_2$ for all $t \geq 0$.

Theorem 1.1: Lyapunov Direct Method

Let z^* be an equilibrium of $\dot{z} = f(z)$ and let $V \in C^1$ and constants $c_1, c_2 > 0$ such that

$$c_1 \|z - z^*\|_2^2 \leq V(z) \leq c_2 \|z - z^*\|_2^2$$

in a neighborhood of z^* , satisfy: (i) $V(z^*) = 0$, $V(z) > 0$ for $z \neq z^*$ (positive definite), (ii) $\dot{V}(z) = \nabla V(z) \cdot f(z) \leq 0$. Then z^* is stable. If additionally $\dot{V}(z) < 0$ for $z \neq z^*$ (negative definite), then z^* is asymptotically stable.

Theorem 1.2: LaSalle's Invariance Principle

Let $\dot{z} = f(z)$ be autonomous with f locally Lipschitz. Let $\Omega \subset \mathbb{R}^n$ be *compact and positively invariant*. Let $V \in C^1(\Omega)$ satisfy $\dot{V}(z) \leq 0$ on Ω . Define

$$E = \{z \in \Omega : \dot{V}(z) = 0\},$$

and let M be the largest invariant set contained in E . Then for any $z(0) \in \Omega$,

$$\text{dist}(z(t), M) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In particular, if $M = \{z^*\}$, then z^* is asymptotically stable (relative to Ω).

2 Planar System: Matrix Form, Eigenvalues, and Closed-Form Solution

Problem: Matrix Form and Classification

Consider

$$\dot{x} = y, \quad \dot{y} = -x - y.$$

- (a) Write the system in matrix form $\dot{z} = Az$ for $z = (x, y)^\top$.
- (b) Compute the eigenvalues of A and classify the equilibrium at the origin.
- (c) Solve the system explicitly in terms of initial data $(x(0), y(0)) = (x_0, y_0)$.

Solution

(a) **Matrix form.** Let $z = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x - y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

So

$$\dot{z} = Az, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

(b) **Eigenvalues.**

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -1 - \lambda \end{pmatrix} = (-\lambda)(-1 - \lambda) - (1)(-1) = \lambda^2 + \lambda + 1.$$

Thus

$$\lambda = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

The real part is $-1/2 < 0$ and the imaginary part is nonzero, so the origin is a **stable focus (spiral sink)**.

(c) **Explicit solution.** From $\dot{x} = y$, differentiate to get $\ddot{x} = \dot{y} = -x - y = -x - \dot{x}$, hence

$$\ddot{x} + \dot{x} + x = 0.$$

Let $\beta = \frac{\sqrt{3}}{2}$. Then

$$x(t) = e^{-t/2} \left(C_1 \cos(\beta t) + C_2 \sin(\beta t) \right).$$

Since $y = \dot{x}$, differentiating gives

$$y(t) = e^{-t/2} \left[-\frac{1}{2}(C_1 \cos \beta t + C_2 \sin \beta t) + \beta(-C_1 \sin \beta t + C_2 \cos \beta t) \right].$$

Impose $x(0) = x_0$: $C_1 = x_0$. Impose $y(0) = y_0$:

$$y_0 = -\frac{1}{2}x_0 + \beta C_2 \quad \Rightarrow \quad C_2 = \frac{y_0 + \frac{1}{2}x_0}{\beta} = \frac{2}{\sqrt{3}} \left(y_0 + \frac{1}{2}x_0 \right).$$

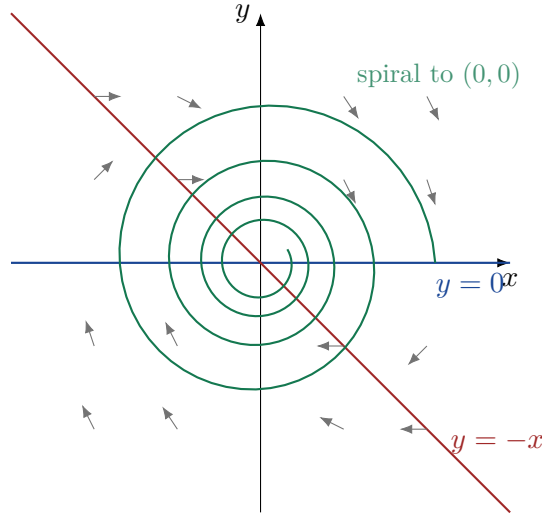
So

$$x(t) = e^{-t/2} \left[x_0 \cos(\beta t) + \frac{2}{\sqrt{3}} \left(y_0 + \frac{1}{2}x_0 \right) \sin(\beta t) \right], \quad \beta = \frac{\sqrt{3}}{2},$$

and $y(t) = \dot{x}(t)$ from the formula above. This also shows exponential decay by the factor $e^{-t/2}$.

A Clean Phase-Plane Schematic

Nullclines: $\dot{x} = 0 \Rightarrow y = 0$ and $\dot{y} = 0 \Rightarrow y = -x$. Trajectories spiral into the origin.



3 Lyapunov Function and a Rigorous LaSalle Upgrade

Problem: Lyapunov and LaSalle

For the same system, define

$$V(x, y) = x^2 + y^2.$$

- Compute \dot{V} along solutions and explain why $\dot{V} \leq 0$ is not (by itself) enough to conclude asymptotic stability from Lyapunov's basic theorem.
- Use **LaSalle's invariance principle** to prove that the origin is asymptotically stable (in fact, globally).

Solution

(a) **Compute \dot{V} .** Along solutions,

$$\dot{V} = \frac{d}{dt}(x^2 + y^2) = 2x\dot{x} + 2y\dot{y}.$$

Substitute $\dot{x} = y$ and $\dot{y} = -x - y$:

$$\dot{V} = 2xy + 2y(-x - y) = 2xy - 2xy - 2y^2 = \boxed{-2y^2 \leq 0}.$$

This is *negative semidefinite*, not negative definite: $\dot{V} = 0$ on the whole line $y = 0$. Therefore Lyapunov's basic theorem (Theorem 1.1) gives stability, but does not automatically give asymptotic stability.

(b) **LaSalle upgrade to asymptotic stability.** Fix any $c > 0$ and consider the sublevel set

$$\Omega_c = \{(x, y) \in \mathbb{R}^2 : V(x, y) \leq c\} = \{x^2 + y^2 \leq c\}.$$

It is closed and bounded, hence compact (Heine–Borel). Also it is positively invariant because $\dot{V} \leq 0$ implies $V(t)$ is nonincreasing: if $V(0) \leq c$ then $V(t) \leq V(0) \leq c$ for all $t \geq 0$. Now define

$$E = \{(x, y) \in \Omega_c : \dot{V} = 0\}.$$

Since $\dot{V} = -2y^2$, we have $\dot{V} = 0 \iff y = 0$. Hence

$$E = \{(x, 0) : x^2 \leq c\}.$$

We seek the largest invariant set $M \subset E$. If $y = 0$ but $x \neq 0$, then $\dot{y} = -x - y = -x \neq 0$, so the trajectory immediately leaves E . Thus the only point that stays in E for all future time is $(0, 0)$. Therefore

$$M = \{(0, 0)\}.$$

By LaSalle (Theorem 1.2), any trajectory starting in Ω_c converges to M , i.e.

$$(x(t), y(t)) \rightarrow (0, 0) \quad \text{as } t \rightarrow \infty.$$

Because every initial condition lies in some Ω_c , the origin is **globally asymptotically stable**.

4 A Common Pitfall: If the Notes Say $\ddot{x} = y$

Problem: Consistency and Dimension

Suppose one writes

$$\ddot{x} = y, \quad \dot{y} = -x - y.$$

- Explain why this is not a first-order system on \mathbb{R}^2 in variables (x, y) .
- Convert it to a first-order system and state the phase-space dimension.
- Derive the scalar ODE satisfied by $x(t)$.

Solution

(a) A first-order system on \mathbb{R}^2 must have the form $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$. But \ddot{x} appears, so knowing (x, y) at time t does not determine \dot{x} . One must introduce \dot{x} as an additional state variable.

(b) Let $v = \dot{x}$ and $Z = (x, v, y) \in \mathbb{R}^3$. Then $\dot{x} = v$, $\dot{v} = \ddot{x} = y$, and $\dot{y} = -x - y$. Thus

$$\dot{Z} = \begin{pmatrix} v \\ y \\ -x - y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ v \\ y \end{pmatrix} =: BZ.$$

Hence phase space is \mathbb{R}^3 .

(c) Since $v = \dot{x}$ and $y = \dot{v} = \ddot{x}$, we have $\dot{y} = \ddot{v} = \ddot{\ddot{x}}$. But $\dot{y} = -x - y = -x - \ddot{x}$, so

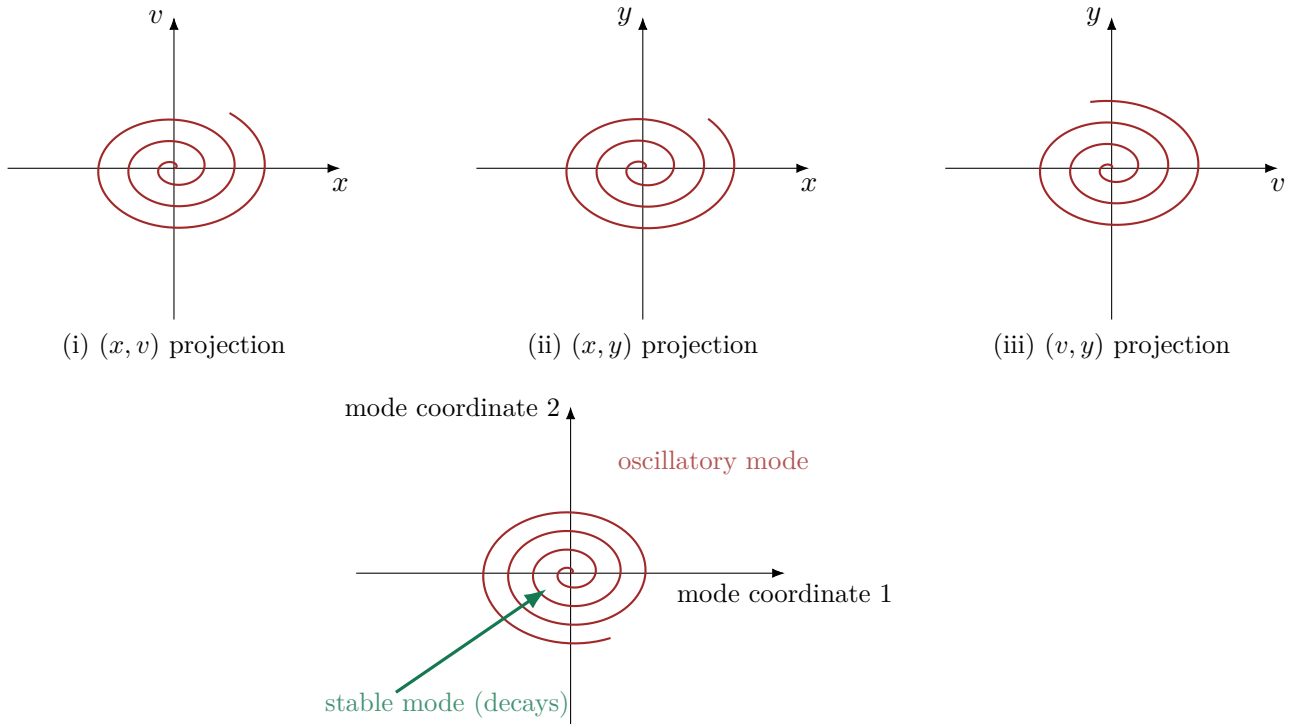
$$\boxed{\ddot{\ddot{x}} + \ddot{x} + x = 0.}$$

Qualitative Mode Decomposition (3D)

The 3D linear system $\dot{Z} = BZ$ typically decomposes into:

- a **stable real mode** (decaying direction) associated with a real eigenvalue with negative real part;
- an **oscillatory mode** associated with a complex conjugate pair. If the real part is positive, trajectories exhibit *growing spirals* in the corresponding invariant 2D subspace.

The diagrams below are schematic: they illustrate *projections* and *qualitative* behavior (not exact solutions).



5 Computational Experiments and Algorithms (Python)

Algorithm: RK4 for $\dot{z} = f(t, z)$

Algorithm 1 Classical Runge–Kutta (RK4) one step

Require: step $h > 0$, time t_n , state z_n , vector field $f(t, z)$

- 1: $k_1 = f(t_n, z_n)$
- 2: $k_2 = f(t_n + \frac{h}{2}, z_n + \frac{h}{2}k_1)$
- 3: $k_3 = f(t_n + \frac{h}{2}, z_n + \frac{h}{2}k_2)$
- 4: $k_4 = f(t_n + h, z_n + hk_3)$
- 5: $z_{n+1} = z_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
- 6: $t_{n+1} = t_n + h$

Python Experiment: 2D: phase portrait, $V(t)$ monotonicity, and decay-rate estimate

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 # 2D system:  $x' = y, y' = -x - y$ 
5 def f(t, z):
6     x, y = z
7     return np.array([y, -x - y], dtype=float)
8
9 def rk4(f, t0, z0, h, nsteps):

```

```

10     t = np.zeros(nsteps + 1)
11     Z = np.zeros((nsteps + 1, len(z0)))
12     t[0], Z[0] = t0, np.array(z0, dtype=float)
13     for k in range(nsteps):
14         tk, zk = t[k], Z[k]
15         k1 = f(tk, zk)
16         k2 = f(tk + 0.5*h, zk + 0.5*h*k1)
17         k3 = f(tk + 0.5*h, zk + 0.5*h*k2)
18         k4 = f(tk + h, zk + h*k3)
19         Z[k+1] = zk + (h/6.0)*(k1 + 2*k2 + 2*k3 + k4)
20         t[k+1] = tk + h
21     return t, Z
22
23 # --- Phase portrait ---
24 inits = [(2,0), (0,2), (-2,1), (1,-2), (2,2), (-2,-2)]
25 T, h = 25.0, 0.01
26 nsteps = int(T/h)
27
28 plt.figure()
29 for z0 in inits:
30     t, Z = rk4(f, 0.0, z0, h, nsteps)
31     plt.plot(Z[:,0], Z[:,1], linewidth=1)
32
33 xx = np.linspace(-3,3,400)
34 plt.plot(xx, 0*xx, "--", linewidth=1) # y=0
35 plt.plot(xx, -xx, "--", linewidth=1) # y=-x
36 plt.axhline(0, linewidth=0.8)
37 plt.axvline(0, linewidth=0.8)
38 plt.xlabel("x"); plt.ylabel("y")
39 plt.title("Phase portrait:  $x' = y$ ,  $y' = -x - y$ ")
40 plt.show()
41
42 # --- Lyapunov V(t) ---
43 z0 = (2.0, 0.0)
44 t, Z = rk4(f, 0.0, z0, h, nsteps)
45 V = Z[:,0]**2 + Z[:,1]**2
46
47 plt.figure()
48 plt.plot(t, V, linewidth=1)
49 plt.xlabel("t"); plt.ylabel("V(t)=x(t)^2+y(t)^2")
50 plt.title("Lyapunov function V(t) (non-increasing since  $dV/dt=-2y^2$ )")
51 plt.show()
52
53 # --- Estimate decay rate from log ||z(t)|| ---
54 norm = np.sqrt(Z[:,0]**2 + Z[:,1]**2)
55 log_norm = np.log(norm + 1e-14)
56
57 plt.figure()
58 plt.plot(t, log_norm, linewidth=1)
59 plt.xlabel("t"); plt.ylabel("log ||z(t)||")
60 plt.title("Log-norm vs time (slope ~ -1/2 after transients)")
61 plt.show()
62
63 mask = (t > 5.0) & (t < 20.0)
64 slope, intercept = np.polyfit(t[mask], log_norm[mask], 1)
65 print("Estimated slope:", slope, " (theory ~ -0.5)")

```

Python Experiment: 3D: simulate $\dot{Z} = BZ$ for the literal system and see possible growth

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 # 3D:  $x' = v$ ,  $v' = y$ ,  $y' = -x - y$ 
5 B = np.array([[0.0, 1.0, 0.0],
6               [0.0, 0.0, 1.0],
7               [-1.0, 0.0, -1.0]])
8
9 def rk4_lin(M, t0, z0, h, nsteps):
10     z0 = np.array(z0, dtype=float)
11     t = np.zeros(nsteps+1)
12     Z = np.zeros((nsteps+1, len(z0)))
13     t[0], Z[0] = t0, z0
14     for k in range(nsteps):
15         zk = Z[k]
16         k1 = M @ zk
17         k2 = M @ (zk + 0.5*h*k1)
18         k3 = M @ (zk + 0.5*h*k2)
19         k4 = M @ (zk + h*k3)
20         Z[k+1] = zk + (h/6)*(k1 + 2*k2 + 2*k3 + k4)
21         t[k+1] = t[k] + h
22     return t, Z
23
24 z0 = (0.4, 0.0, 0.4) # (x(0), v(0), y(0))
25 T, h = 40.0, 0.01
26 nsteps = int(T/h)
27 t, Z = rk4_lin(B, 0.0, z0, h, nsteps)
28
29 x, v, y = Z[:,0], Z[:,1], Z[:,2]
30 norm = np.sqrt(x*x + v*v + y*y)
31
32 plt.figure()
33 plt.plot(t, norm, linewidth=1)
34 plt.xlabel("t"); plt.ylabel("||(x,v,y)(t)||")
35 plt.title("3D system norm (may grow if unstable oscillatory mode
36           dominates)")
37 plt.show()
38
39 # Projection plots
40 plt.figure()
41 plt.plot(x, v, linewidth=1)
42 plt.xlabel("x"); plt.ylabel("v")
43 plt.title("(x,v) projection")
44 plt.show()
45
46 plt.figure()
47 plt.plot(x, y, linewidth=1)
48 plt.xlabel("x"); plt.ylabel("y")
49 plt.title("(x,y) projection")
50 plt.show()
```

```

51 plt.figure()
52 plt.plot(v, y, linewidth=1)
53 plt.xlabel("v"); plt.ylabel("y")
54 plt.title("(v,y) projection")
55 plt.show()

```

6 Conclusion

Conclusion

In this article we solved and interpreted the linear planar system

$$\dot{x} = y, \quad \dot{y} = -x - y,$$

through a unified problem-solving workflow that moves from *structure* to *proof* to *computation*.

(1) Linear-algebraic structure. Writing the dynamics as $\dot{z} = Az$ with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

the eigenvalues $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ immediately identify the origin as a stable focus and explain the spiral geometry of trajectories. The explicit solution confirms exponential decay with rate $e^{-t/2}$.

(2) Lyapunov analysis and the meaning of $\dot{V} \leq 0$. For the natural energy candidate $V(x, y) = x^2 + y^2$, we computed

$$\dot{V} = -2y^2 \leq 0,$$

showing that $V(t)$ is nonincreasing along trajectories. Since \dot{V} is only negative semidefinite, Lyapunov's basic theorem guarantees stability but does not automatically force convergence.

(3) Rigorous asymptotic stability via LaSalle. LaSalle's invariance principle upgrades $\dot{V} \leq 0$ to asymptotic stability by identifying the largest invariant set inside $\{\dot{V} = 0\} = \{y = 0\}$; the flow leaves this set unless $x = 0$, so the invariant set reduces to $(0, 0)$. Hence every trajectory converges to the origin, giving a clean stability proof independent of explicitly solving the ODE.

(4) Consistency and modelling discipline. We also highlighted a common notation pitfall: replacing $\dot{x} = y$ by $\ddot{x} = y$ forces an additional state variable $v = \dot{x}$, enlarging the phase space from \mathbb{R}^2 to \mathbb{R}^3 and changing the qualitative dynamics. This reinforces a key modelling principle: the phase variables must be chosen so that the system is genuinely first-order.

(5) Computation as verification. Finally, RK4 simulations and projection plots provide numerical confirmation of the theoretical predictions, help visualize phase portraits and Lyapunov decay, and offer a reproducible way to estimate decay rates from data.

Takeaway: For linear systems, eigenvalues, Lyapunov functions, invariance principles, and numerical experiments are not separate topics; they are complementary tools for a single goal—*understanding stability and long-time behavior*.

7 Practice Problem Set

Level I

Problem: 1: Linear Classification

Consider the linear system

$$\dot{x} = y, \quad \dot{y} = -x - 2y, \quad z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

- (a) Write the system in matrix form $\dot{z} = Az$ and determine A explicitly.
- (b) Compute the eigenvalues of A .
- (c) Classify the equilibrium point at the origin (node/focus/saddle/center).

Level II

Problem: 2: Lyapunov Stability (Semi-Definite Case)

Consider the nonlinear system

$$\dot{x} = y, \quad \dot{y} = -x - y^3.$$

Let $V(x, y) = x^2 + y^2$.

- (a) Show that $(0, 0)$ is an equilibrium.
- (b) Compute $\dot{V}(x, y)$ along trajectories of the system.
- (c) Determine whether \dot{V} is negative definite, negative semidefinite, or indefinite.
- (d) State what Lyapunov's direct method guarantees about the stability of $(0, 0)$.

Level III

Problem: 3: Asymptotic Stability via LaSalle

For the same system

$$\dot{x} = y, \quad \dot{y} = -x - y^3,$$

with $V(x, y) = x^2 + y^2$:

- (a) Identify the set $E = \{(x, y) : \dot{V}(x, y) = 0\}$.
- (b) Determine the largest invariant subset $M \subset E$.
- (c) Use LaSalle's invariance principle to conclude whether $(0, 0)$ is asymptotically stable.
- (d) Clearly state the compact positively invariant set Ω on which you apply LaSalle.

Level IV

Problem: 4: Consistency and Phase-Space Dimension

Consider the system

$$\ddot{x} = -x - y, \quad \dot{y} = -y.$$

- Explain why this is not a first-order system on \mathbb{R}^2 in variables (x, y) .
- Introduce a new variable to rewrite the system as a first-order system $\dot{Z} = BZ$.
- State the dimension of the resulting phase space.
- Find the matrix B explicitly.
- Explain qualitatively how the eigenvalues of B influence the long-time behavior of solutions.

Level V

Problem: 5: Computational Verification of Stability

Consider the linear system

$$\dot{x} = y, \quad \dot{y} = -x - y.$$

- Design a numerical experiment using a classical fourth-order Runge–Kutta method (RK4) to approximate trajectories.
- Choose at least three distinct initial conditions and predict the qualitative behavior of trajectories in the phase plane.
- Compute and plot the Lyapunov function $V(t) = x(t)^2 + y(t)^2$ along each trajectory.
- Plot $\log \|z(t)\|$ versus t and explain how to estimate the exponential decay rate from this plot.
- Compare your numerical observations with the theoretical stability classification using eigenvalues.

8 Hints-Only Version

Hints for Problem 1

- Collect coefficients: $\dot{x} = 0 \cdot x + 1 \cdot y$, $\dot{y} = (-1) \cdot x + (-2) \cdot y$.
- For 2×2 matrices: $\lambda^2 - (\text{tr}A)\lambda + \det(A) = 0$.
- Complex eigenvalues with negative real part \Rightarrow stable focus.

Hints for Problem 2

- Use the chain rule: $\dot{V} = 2x\dot{x} + 2y\dot{y}$.
- After substitution, factor terms to see the sign of \dot{V} .
- If $\dot{V} \leq 0$ only, Lyapunov gives stability but may not give convergence.

Hints for Problem 3

- First compute $E = \{\dot{V} = 0\}$; it will be a set (often a curve/line), not just a point.
- To find the largest invariant subset, check whether the flow keeps you inside E : compute \dot{y} (or other components) on E .
- Apply LaSalle on a compact sublevel set $\Omega_c = \{V \leq c\}$ and argue it is positively invariant because $\dot{V} \leq 0$.

Hints for Problem 4

- A first-order system needs all derivatives determined by the state. Introduce $v = \dot{x}$.
- Use (x, v, y) as the state: then $\dot{x} = v$, $\dot{v} = \ddot{x}$, and \dot{y} is given.
- The matrix B is found by writing each RHS as a linear combination of x, v, y .

Hints for Problem 5

- In RK4, implement k_1, k_2, k_3, k_4 and update z_{n+1} .
- Exponential decay shows as a straight line in a plot of $\log \|z(t)\|$ vs t .
- The slope should be close to the dominant real part of the eigenvalues (most weakly negative).

9 References

References (suggested)

1. H. K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice Hall, 2002.
2. J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*, Prentice Hall, 1991.
3. S. H. Strogatz, *Nonlinear Dynamics and Chaos*, 2nd ed., Westview Press, 2015.
4. L. Perko, *Differential Equations and Dynamical Systems*, 3rd ed., Springer, 2001.
5. M. W. Hirsch, S. Smale, and R. L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 3rd ed., Academic Press, 2013.
6. R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2012.
7. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955.

End of article.