Bounded Sets, Moving Centers, and Numerical Dynamics

A Geometric, Dynamical, and Numerical Perspective

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Preface

Boundedness is a foundational notion in the qualitative study of dynamical systems. In phase space, many questions—stability, existence of attractors, long–time behavior, and even numerical reliability—begin with a simple requirement: trajectories should remain confined to a finite region. Although the concept appears elementary, it admits a robust geometric formulation that is independent of coordinate choices.

This note is centered around a basic metric-geometric fact: once a set lies inside a closed ball centered at some point, it also lies inside a closed ball centered at any other point, after enlarging the radius by the distance between the two centers. The proof is a direct consequence of the triangle inequality, yet the principle has wide relevance in analysis and dynamics. In particular, it explains why bounding arguments (Lyapunov estimates, absorbing balls, numerical enclosures) do not depend on the choice of origin in phase space.

The exposition proceeds in three layers. First, the geometric theorem is stated and proved in an axiomatic metric-space setting. Second, the theorem is interpreted in the context of ODE trajectories, emphasizing why finite-time trajectory sets are bounded. Third, the theory is connected to computation by an explicit numerical certificate that can be checked during simulation. Finally, schematic figures (2D and 3D-flavored) summarize the geometry underlying the argument.

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1 Center-Independence of Boundedness

1.1 Closed balls and boundedness

Let (X, d) be a metric space. For a center $a \in X$ and radius $r \geq 0$, the *closed ball* is defined by

$$\overline{B}(a, r) := \{x \in X : d(x, a) \leq r\}.$$

A set $S \subseteq X$ is called *bounded* if there exist some $a \in X$ and some $r \geq 0$ such that $S \subseteq \overline{B}(a, r)$. In other words, boundedness means that S fits inside at least one ball (center unspecified).

1.2 The geometric theorem

Theorem 1.1: Center Does Not Matter

Let (X, d) be a metric space and let $S \subseteq X$. Suppose there exist $x_0 \in X$ and $R \geq 0$ such that

$$S \subseteq \overline{B}(x_0, R).$$

Then for every $y \in X$,

$$S \subseteq \overline{B}\big(y,\, R + d(x_0, y)\big).$$

Proof. Fix any $x \in S$. Since $S \subseteq \overline{B}(x_0, R)$, we have $d(x, x_0) \leq R$. The triangle inequality gives

$$d(x, y) \leq d(x, x_0) + d(x_0, y) \leq R + d(x_0, y).$$

Therefore $x \in \overline{B}\big(y,\, R + d(x_0, y)\big)$. Because $x \in S$ was arbitrary, the same containment holds for all of S. $\square$

Remark 1.1: Geometric Interpretation

If S fits inside a ball centered at x_0 with radius R, then shifting the center to an arbitrary point y requires only enlarging the radius by the distance $d(x_0, y)$. Hence boundedness is independent of the choice of "origin" in phase space.

2 Application to Dynamical Systems

2.1 Finite-time trajectory sets are bounded

Consider an ODE

$$\dot{z} = F(z), \qquad z(t) \in \mathbb{R}^m,$$

where F is locally Lipschitz. For any initial condition $z(0) = z_0$ and any $T > 0$ for which the solution exists on $[0, T]$, the map $t \mapsto z(t)$ is continuous. Define the *finite-time trajectory set*

$$S_T := \{z(t) : t \in [0, T]\} \subseteq \mathbb{R}^m.$$

Since $[0, T]$ is compact and $z(\cdot)$ is continuous, the image S_T is compact. Every compact subset of $\mathbb{R}^m$ is bounded. Therefore S_T is bounded and fits inside some closed ball.

Remark 2.1: Finite-Time Boundedness of Trajectories

For any $T > 0$ on which the solution exists, the trajectory set S_T is bounded. Consequently, if $S_T \subseteq \overline{B}(x_0, R)$ for some x_0 and R, then by Theorem 1.1 the same set fits inside a ball centered at any chosen $y \in \mathbb{R}^m$ with radius enlarged by $\|x_0 - y\|$.

3 Absorbing Ball for the Lorenz System

The Lorenz system is given by

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= x(\rho - z) - y, \\ \dot{z} &= xy - \beta z, \end{aligned}$$

where $\sigma, \rho, \beta > 0$. A key structural feature is that it is *dissipative*: solutions eventually enter and remain in a fixed bounded region of $\mathbb{R}^3$.

Theorem 3.1: Existence of an Absorbing Ball for the Lorenz System

For the Lorenz system above, there exist constants $R > 0$ and $t_0 \geq 0$ such that every solution satisfies

$$(x(t), y(t), z(t)) \in \overline{B}(0, R) \subset \mathbb{R}^3 \qquad \text{for all } t \geq t_0.$$

Proof sketch. One constructs a quadratic Lyapunov function of the form

$$V(x, y, z) = x^2 + y^2 + (z - \rho)^2,$$

and shows by direct differentiation along solutions that there exist $c, C > 0$ (depending only on σ, ρ, β) such that

$$\dot{V} \leq -cV + C.$$

Gronwall's inequality implies that $V(t)$ is ultimately bounded uniformly in time, which yields an absorbing ball. $\square$

Remark 3.1: Connection to Center-Independence

Once an absorbing ball $\overline{B}(0, R)$ is known, Theorem 1.1 implies that the absorbing set can be re-centered at any point $y \in \mathbb{R}^3$ by enlarging the radius by $\|y\|$. Thus the boundedness statement is not tied to a special choice of origin.

4 Numerical Verification Algorithm

4.1 Discrete trajectory sets and computable radii

Suppose a numerical method (e.g. Runge–Kutta) produces a discrete trajectory

$$S_N := \{z_0, z_1, \dots, z_N\} \subseteq \mathbb{R}^m,$$

intended to approximate an ODE trajectory. Choose a reference center $x_0 \in \mathbb{R}^m$ and define the computed radius

$$R_N := \max_{0 \le k \le N} \|z_k - x_0\|.$$

Then, by construction, $S_N \subseteq \overline{B}(x_0, R_N)$. Theorem 1.1 predicts that for any other center y,

$$S_N \subseteq \overline{B}\big(y,\, R_N + \|x_0 - y\|\big).$$

This predicted containment can be checked numerically via a worst-case residual.

Algorithm 4.1: Numerical Verification of Center-Independence

Given a numerically computed trajectory $S_N = \{z_0, \dots, z_N\} \subseteq \mathbb{R}^m$:

1. Choose a reference center x_0.
2. Compute $R_N := \max_k \|z_k - x_0\|$.
3. For a candidate center y, compute

$$\Delta(y) := \max_k \Big(\|z_k - y\| - \big(R_N + \|x_0 - y\|\big) \Big).$$

4. Verify that $\Delta(y) \le 0$ (up to numerical precision).

Remark 4.1: Interpretation

The quantity $\Delta(y)$ measures the maximal violation of the containment predicted by the triangle inequality. In exact arithmetic, Theorem 1.1 forces $\Delta(y) \le 0$ for all y. In floating-point arithmetic, one expects $\Delta(y)$ to be nonpositive up to small rounding errors, providing a practical computational certificate.

5 Geometry of Moving Centers

5.1 A Schematic Geometric Picture (2D)

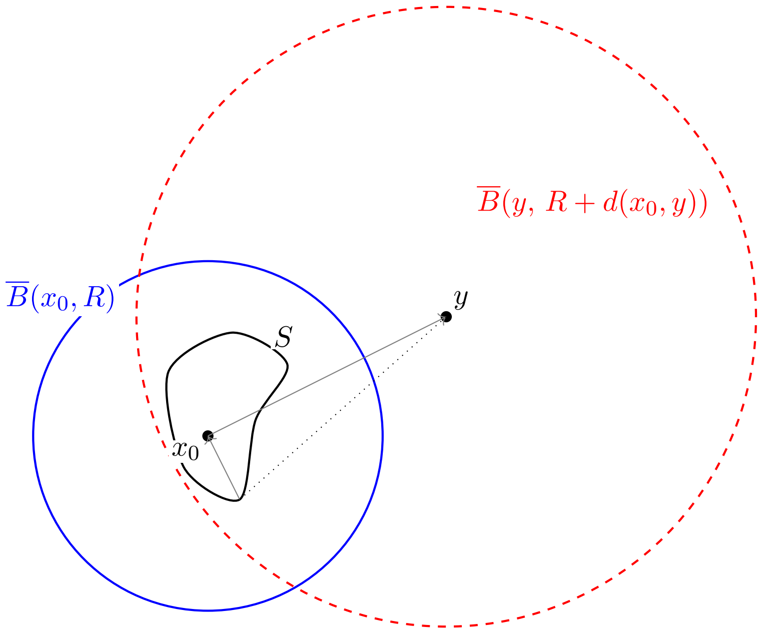


Figure 1: Geometry underlying Theorem 1.1. If $S \subseteq \overline{B}(x_0, R)$ then, for any y, the triangle inequality implies $S \subseteq \overline{B}(y,\ R + d(x_0, y))$.

This section explains the geometric content encoded by Figure 1. Although the drawing is two-dimensional, the underlying statement is valid in any metric space and, in particular, in any Euclidean phase space $\mathbb{R}^m$ used in dynamical systems.

5.2 Theoretical Viewpoint: Why Re-centering Works

Assume (X, d) is a metric space and that a set $S \subseteq X$ is bounded in the sense that there exist a reference center $x_0 \in X$ and a radius $R \geq 0$ such that

$$S \subseteq \overline{B}(x_0, R).$$

Equivalently, for every $x \in S$ one has the uniform bound $d(x, x_0) \leq R$.

Now fix an arbitrary point $y \in X$ (the "new center"). For any $x \in S$, the triangle inequality gives

$$d(x, y) \leq d(x, x_0) + d(x_0, y).$$

Using $d(x, x_0) \leq R$ yields

$$d(x, y) \leq R + d(x_0, y).$$

Since this holds for every $x \in S$, it follows that

$$S \subseteq \overline{B}\big(y,\ R + d(x_0, y)\big).$$

Thus, boundedness is independent of the choice of center: shifting the center from x_0 to y increases the required radius by exactly the center-shift distance $d(x_0, y)$.

5.3 How The Diagram Represent Theory

Figure 1 visualizes the inequality above.

- The closed curve labelled S represents an arbitrary bounded set. In dynamics, one may interpret it as a finite-time trajectory set $S_T = \{z(t) : t \in [0, T]\}$ or a numerically sampled trajectory $S_N = \{z_0, \ldots, z_N\}$.
- The blue circle represents the original enclosure $\overline{B}(x_0, R)$: all points of S lie within distance R of the reference center x_0.
- The red dashed circle represents the re-centered enclosure $\overline{B}(y,\ R + d(x_0, y))$. It is larger because the center is shifted from x_0 to y.
- The arrows encode the triangle inequality geometry for a representative point $x \in S$: one compares the direct distance $d(x, y)$ with the broken path length $d(x, x_0) + d(x_0, y)$.

A useful way to interpret the enlargement is the additive decomposition

$$\text{New radius} = \underbrace{R}_{\text{size of } S \text{ around } x_0} + \underbrace{d(x_0, y)}_{\text{shift of center}}.$$

5.4 Computational and Numerical Viewpoint

In numerical dynamics, the set S is typically represented by discrete samples. Suppose a numerical solver produces a trajectory

$$S_N := \{z_0, z_1, \ldots, z_N\} \subset \mathbb{R}^m.$$

Fix a reference center $x_0 \in \mathbb{R}^m$ (common choices are $x_0 = z_0$, or $x_0 = 0$, or the empirical mean $\bar{z}$). Define the computed enclosure radius

$$R_N := \max_{0 \le k \le N} \|z_k - x_0\|.$$

Then, by construction,

$$S_N \subseteq \overline{B}(x_0, R_N).$$

For any chosen (possibly moving) center $y \in \mathbb{R}^m$, the theoretical result predicts

$$S_N \subseteq \overline{B}\big(y,\ R_N + \|x_0 - y\|\big).$$

To certify this numerically, one evaluates the maximal residual

$$\Delta(y) := \max_{0\le k\le N}\Big(\|z_k - y\| - \big(R_N + \|x_0 - y\|\big)\Big).$$

In exact arithmetic, the triangle inequality forces $\Delta(y) \le 0$ for all y. In floating-point arithmetic, one expects $\Delta(y)$ to be nonpositive up to a small tolerance (e.g. 10^{-12} to 10^{-8}, depending on scaling), making $\Delta(y)$ a practical runtime certificate in simulations and animations.

5.5 Link to Computational Implementations

For extended numerical experiments (phase portraits, moving-center enclosures, Plotly animations, and live evaluation of $\Delta(y)$), the implementation is available at Git-Hub

5.6 Numerical Illustration: Moving Centers Along a Trajectory

Let $z(t) \in \mathbb{R}^3$ denote a numerically computed trajectory of the Lorenz system. Fix a reference center $x_0 \in \mathbb{R}^3$ and define the sampled trajectory set

$$S_N := \{z(t_k) : k = 0, 1, \dots, N\}.$$

By construction,

$$R_N := \max_{0\le k\le N} \|z(t_k) - x_0\| \quad \text{satisfies} \quad S_N \subseteq \overline{B}(x_0, R_N).$$

Moving the center along the trajectory. Define a moving center by $y(t_k) := z(t_k)$. For each k, Theorem 1.1 predicts

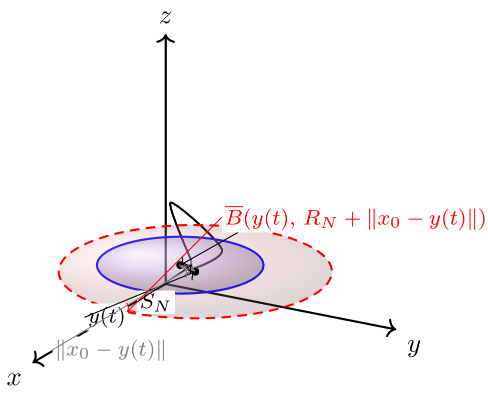
$$S_N \subseteq \overline{B}\big(y(t_k),\, R_N + \|x_0 - y(t_k)\|\big).$$

This inclusion is verified numerically through

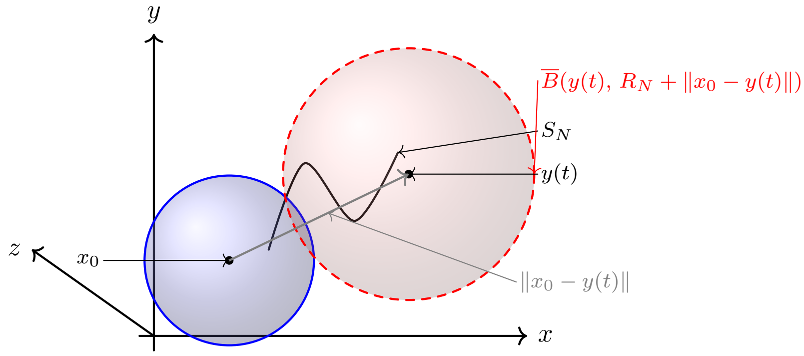
$$\Delta\big(y(t_k)\big) := \max_{0\le j\le N} \left(\|z(t_j) - y(t_k)\| - \big(R_N + \|x_0 - y(t_k)\|\big)\right).$$

Remark 5.1: Numerical Certificate of Containment

The inequality $\Delta(y(t_k)) \le 0$ is a numerical realization of the triangle inequality. Its satisfaction for all k provides a runtime certificate that the entire sampled trajectory set remains inside the translated ball centered at $y(t_k)$.



(a) 3D Schematic



(b) Projection Schematic

Figure 2: Moving-center geometry. If a bounded trajectory set lies inside $\overline{B}(x_0, R_N)$, then it also lies inside $\overline{B}(y(t), R_N + \|x_0 - y(t)\|)$ for any (even moving) center $y(t)$.

6 Conclusion

The "center does not matter" principle is an immediate consequence of the triangle inequality, yet it provides a powerful geometric viewpoint for boundedness in phase space. In finite-time ODE dynamics, trajectory sets are bounded by compactness, while in dissipative systems (such as the Lorenz equations) one can often prove the existence of absorbing balls that confine trajectories for all sufficiently large times. Numerically, discrete trajectory sets admit computable enclosing radius and a verifiable residual $\Delta(y)$ that serves as a runtime certificate of the containment predicted by the theorem. Together, these perspectives unify metric geometry, dynamical systems, and computation into a single coherent framework.

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