JACOBIAN MATRIX

Concepts, Geometry, Theory, and Problem Solving

A Geometric and Theoretical Study

A problem-solving handout in Multivariable Calculus

Preface

This handout is designed as a focused and structured study note on the **Jacobian matrix** and its geometry. Rather than presenting a broad treatment of multivariable calculus, we are *strictly concerned* with the Jacobian as a fundamental tool for understanding local linearization, geometric deformation, coordinate transformations, and analytical problem solving. This handout does not attempt to be a comprehensive textbook. Instead, it intentionally focuses on ideas that are directly useful when solving problems. Every concept is introduced with the purpose of being *applied*, and every application is meant to reinforce the underlying geometric intuition.

The presentation is inspired in spirit by the geometric clarity found in *Vector Calculus* by Jerrold E. Marsden, though the treatment here is significantly narrower in scope: our exclusive emphasis is the Jacobian matrix and its role in understanding mappings, transformations, and problem–solving methodologies.

This material is therefore intended to serve as:

- A concise conceptual guide,
- A geometric reference,
- A practical problem–solving handout

for students and practitioners who wish to strengthen their mastery of the Jacobian and its applications.

Jacobian Matrix: Concepts, Geometry, Theory, and Problem Solving

Comprehensive Problem-Solving Strategy

Contents

1 Introduction

In single-variable calculus, the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x_0 is a single number that measures the instantaneous rate of change of f at x_0. In multivariable calculus, for a function

$$F : \mathbb{R}^n \to \mathbb{R}^m,$$

the corresponding object is a *matrix* that encodes the best linear approximation of F near a point. This matrix is called the *Jacobian matrix* of F.

2 Jacobian Matrix: Definition and Basic Properties

2.1 Vector Valued Functions

Let

$$F : \mathbb{R}^n \to \mathbb{R}^m, \qquad F(\mathbf{x}) = F(x_1, \ldots, x_n) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{pmatrix}.$$

Each component f_i is a scalar-valued function $f_i : \mathbb{R}^n \to \mathbb{R}$.

Definition 2.1 (Jacobian Matrix)**.** *Suppose all first-order partial derivatives* $\dfrac{\partial f_i}{\partial x_j}$ *exist at* $\mathbf{x}_0 = (x_1^0, \ldots, x_n^0)$*. The* Jacobian matrix *of* F *at* $\mathbf{x}_0$ *is the* $m \times n$ *matrix*

$$JF(\mathbf{x}_0) := \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \dfrac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \cdots & \dfrac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{pmatrix}.$$

Equivalently, $JF(\mathbf{x}_0)$ has as its i-th row the transpose of the gradient $\nabla f_i(\mathbf{x}_0)^\top$.

Example 2.2. *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be given by*

$$F(x, y) = (xy, x^2 + y).$$

Then

$$JF(x, y) = \begin{pmatrix} \frac{\partial(xy)}{\partial x} & \frac{\partial(xy)}{\partial y} \\ \frac{\partial(x^2+y)}{\partial x} & \frac{\partial(x^2+y)}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ 2x & 1 \end{pmatrix}.$$

2.2 Jacobian as Best Linear Approximation

The Jacobian arises naturally when one formalizes the idea that a differentiable function is "locally linear."

Definition 2.3 (Differentiability via Linear Approximation)**.** *We say F is differentiable at $\mathbf{x}_0$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{\mathbf{h}\to 0} \frac{\|F(\mathbf{x}_0 + \mathbf{h}) - F(\mathbf{x}_0) - L\mathbf{h}\|}{\|\mathbf{h}\|} = 0.$$

In this case, L is called the derivative *of F at $\mathbf{x}_0$ and is represented by the Jacobian matrix $JF(\mathbf{x}_0)$.*

Thus nearby points satisfy

$$F(\mathbf{x}_0 + \mathbf{h}) \approx F(\mathbf{x}_0) + JF(\mathbf{x}_0)\,\mathbf{h},$$

which is the multivariable analogue of the familiar

$$f(x_0 + h) \approx f(x_0) + f'(x_0)h.$$

3 Visualization of a Linear Map and the Jacobian Determinant

The Jacobian matrix can be understood most naturally by asking a simple question:

"If we zoom in very close to a point, what does a nonlinear map look like?"

At a sufficiently small scale, a smooth map

$$F : \mathbb{R}^n \to \mathbb{R}^m$$

behaves *almost* like a linear map. This phenomenon can be stated informally:

$$F(x_0 + h) \approx F(x_0) + JF(x_0)\,h \qquad \text{for very small } h.$$

That is, the Jacobian matrix $JF(x_0)$ is the best linear approximation to F near x_0. This approximation is not merely algebraic—it has genuine geometric content.

Imagine drawing a tiny shape around the point x_0—a small square, circle, ball, or any nice region. When this tiny region is mapped by F, the image is *almost indistinguishable* from the image produced by the linear map

$$L(h) = JF(x_0)\,h.$$

Thus:

- A small *circle* is sent to an *ellipse*.
- A small *square* is sent to a *parallelogram*.
- A small *cube* becomes a *parallelepiped*.
- Small vectors are stretched, rotated, sheared, or flipped according to the linear part.

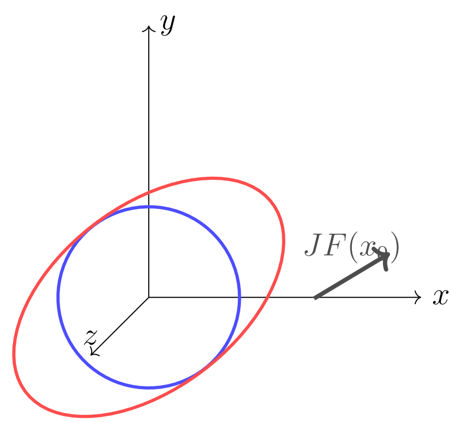
In each case, the shape is transformed exactly as a linear transformation would,because curvature effects become negligible at sufficiently small scales. The most important geometric quantity is the determinant:

$$\det JF(x_0).$$

It gives the factor by which areas (in $\mathbb{R}^2$) or volumes (in $\mathbb{R}^3$) are scaled. For example:

- If $|\det JF(x_0)| > 1$, tiny regions expand.
- If $|\det JF(x_0)| < 1$, tiny regions shrink.
- If $\det JF(x_0) < 0$, the transformation reverses orientation.
- If $\det JF(x_0) = 0$, the map locally collapses dimensions.

So, the Jacobian determinant is nothing more mysterious than the *local area– or volume–scaling factor*.



A tiny circle around a point (blue) is mapped by the Jacobian $JF(x_0)$ to an ellipse (red). This captures the local linear behaviour of the function near x_0.

Zoom in near x_0 so far that F becomes nearly linear. Your tiny shape is effectively being transformed by the matrix $JF(x_0)$. Everything that happens—stretching, rotating, flattening, skewing—comes from this single matrix. The Jacobian is therefore not just a derivative: *it is the entire local geometry of the map, captured at a single point.*

3.1 Visual: Unit Square Under a Linear Map

The determinant of the Jacobian measures local area (or volume) scaling and orientation.The Jacobian determinant of a differentiable map measures the local area–scaling factor. In the special case of a linear transformation $L(x) = Ax$, this area scaling is exactly the area of the parallelogram spanned by the column vectors of A.Thus the determinant is not an abstract algebraic quantity—it is a direct geometric measurement. Visualizing the action of A in 3D clarifies this geometry without crowding or overlapping the image, and highlights the fact that the Jacobian determinant is the area ratio between the transformed and original infinitesimal regions. To understand the geometric meaning of the Jacobian determinant, it is important to examine how a linear map

$$L : \mathbb{R}^2 \to \mathbb{R}^2, \qquad L(x) = Ax, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

acts on simple geometric objects.

Because a linear map is determined entirely by its action on the basis vectors

$$e_1 = (1, 0), \qquad e_2 = (0, 1),$$

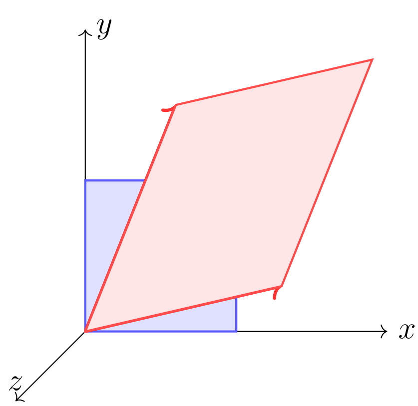
the image of any region can be described using the vectors

$$Ae_1 = (a, c), \qquad Ae_2 = (b, d).$$

The unit square spanned by $\{e_1, e_2\}$ is transformed into the parallelogram spanned by $\{Ae_1, Ae_2\}$. The area of this parallelogram is exactly

$$|\det A| .$$

To present this geometry cleanly, we embed $\mathbb{R}^2$ into $\mathbb{R}^3$. The third dimension is not used by the transformation, but it allows a perspective view that avoids visual overlap and improves readability.



3D perspective view of a linear transformation A: the unit square (blue) is mapped to a parallelogram (red). The area scaling equals $|\det A|$.

Examples. The determinant reveals how different linear transformations affect area:

- **Pure scaling:**
$$A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \quad \Rightarrow \quad \det A = s^2.$$
The map enlarges all areas by a factor of s^2.

- **Shear transformation:**
$$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \det A = 1.$$
A shear slants shapes but preserves area.

- **Rotation matrix:**
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \Rightarrow \quad \det A = 1.$$
Rotations preserve both shape and area.

- **Reflection:**
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \det A = -1.$$
The map reverses orientation but preserves area.

- **General area expansion:**
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \quad \Rightarrow \quad \det A = 2 \cdot 3 - 1 \cdot 1 = 5.$$
The image parallelogram has 5 times the area of the unit square.

3.2 Jacobian Determinant as Local Area (or volume) Scaling

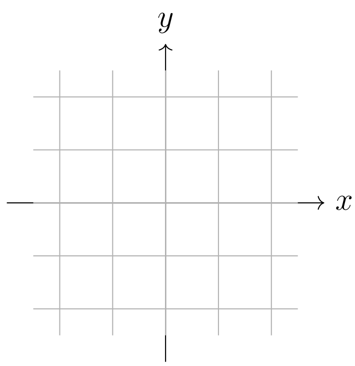
Proposition 3.1. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be differentiable at $\mathbf{x}_0$, and consider a very small region R around $\mathbf{x}_0$ with area ΔA. Then the image region $F(R)$ has area approximately*

$$\Delta A' \approx |\det JF(\mathbf{x}_0)| \, \Delta A.$$

The same idea extends to n dimensions: the absolute value of $\det JF(\mathbf{x}_0)$ is the local volume scaling factor.

3.3 Visual: Grid Deformation by a Nonlinear Map

Suppose $F(x, y) = (u, v)$ deforms the plane. Locally, one can see how coordinate grid lines are stretched and rotated.



Original (x, y)-grid

Deformed (u, v)-grid

The Jacobian at a point describes the *local* linear behaviour of this nonlinear deformation.

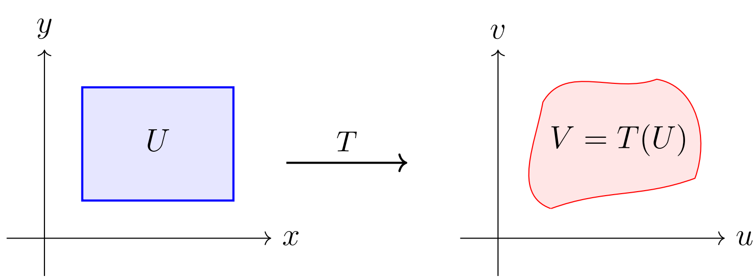
4 Change of Variables and Jacobian Determinant

4.1 Change of Variables Formula (2D)

Let $T : U \subset \mathbb{R}^2 \to V \subset \mathbb{R}^2$ be a C^1 bijection with C^1 inverse and nonzero Jacobian determinant everywhere in U. Then for a suitable integrable function g,

$$\int_V g(u, v)\, du\, dv = \int_U g(T(x, y))\ |\det JT(x, y)|\ dx\, dy.$$

4.2 Visual: Mapping a Rectangle to a Curved Region



The *Jacobian determinant* adjusts area when integrating over V via coordinates on U.

4.3 Polar Coordinates as a Classical Example

The mapping from polar coordinates (r, θ) to Cartesian (x, y) is

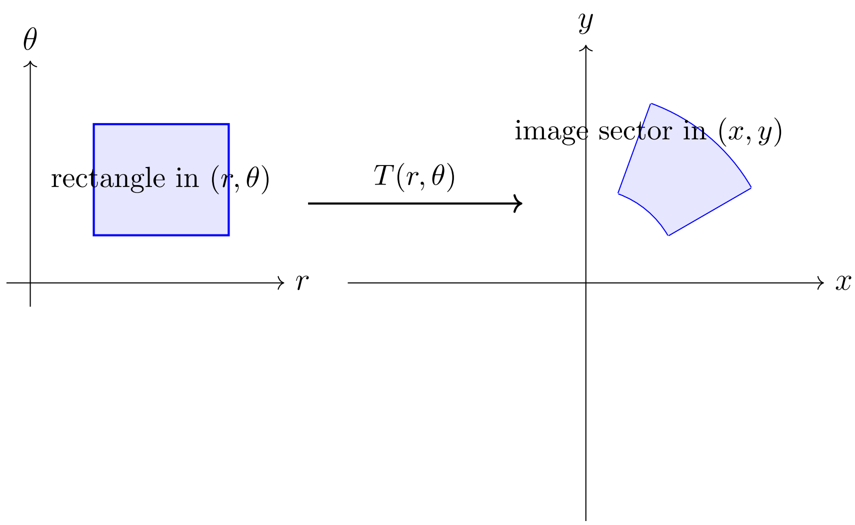
$$T(r, \theta) = (x, y) = (r \cos \theta, r \sin \theta).$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$
$$= r(\cos^2 \theta + \sin^2 \theta) = r.$$

Thus the area element transforms as

$$dx\, dy = r\, dr\, d\theta.$$

4.4 Visual: Rectangle in (r, θ) Becomes a Sector in (x, y)



The area in (x, y) equals the area in (r, θ) multiplied by the Jacobian r.

5 Theoretical Principles: Chain Rule, Inverse Map, Implicit Maps

5.1 Matrix Chain Rule

Theorem 5.1 (Chain Rule in Matrix Form)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^k$ be differentiable. Then the composition $H = G \circ F : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable and*

$$JH(\mathbf{x}) = J(G \circ F)(\mathbf{x}) = JG(F(\mathbf{x})) \cdot JF(\mathbf{x}).$$

5.2 Inverse Function Theorem (Local Invertibility)

Theorem 5.2 (Inverse Function Theorem)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 in a neighborhood of $\mathbf{x}_0$ and suppose*

$$\det JF(\mathbf{x}_0) \neq 0.$$

Then there exist neighborhoods U of $\mathbf{x}_0$ and V of $F(\mathbf{x}_0)$ such that $F : U \to V$ is a diffeomorphism, and

$$J(F^{-1})(F(\mathbf{x}_0)) = (JF(\mathbf{x}_0))^{-1}.$$

5.3 Implicit Function Theorem (Solving Constraints)

Theorem 5.3 (Implicit Function Theorem, Informal Version)**.** *Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be C^1, and write variables as $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. Suppose $F(x_0, y_0) = 0$ and the $m \times m$ Jacobian*

$$\frac{\partial F}{\partial y}(x_0, y_0)$$

is invertible. Then, near x_0, one can solve the equation $F(x, y) = 0$ uniquely for y as a function $y = \varphi(x)$, and φ is C^1.

6 Worked Examples: From Basic to Intermediate

6.1 Example: A Nontrivial Jacobian in $\mathbb{R}^2$

Let

$$F(x, y) = (u, v) = (x^2 - y^2, 2xy).$$

This is the complex squaring map $(x + iy)^2 = (x^2 - y^2) + i(2xy)$ written in real coordinates.

Compute:

$$JF(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}.$$

The Jacobian determinant is

$$\det JF(x, y) = (2x)(2x) - (-2y)(2y) = 4x^2 + 4y^2 = 4(x^2 + y^2).$$

This vanishes only at the origin, reflecting that the complex squaring map is 2-to-1 except at 0.

6.2 Example: 3D Transformation and Volume Scaling

Consider

$$T(r, \theta, z) = (x, y, z) = (r\cos\theta, r\sin\theta, z),$$

which is the cylindrical coordinate map.

$$JT(r, \theta, z) = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The determinant is

$$\det JT(r, \theta, z) = r,$$

so

$$dx\, dy\, dz = r\, dr\, d\theta\, dz.$$

7 Advanced Problems with Detailed Solutions

In this section we present more challenging problems involving Jacobians, with full solutions that emphasize both the algebra and the geometric interpretation.

7.1 Problem 1: Nonlinear Change of Variables with Curved Boundary

Problem. Consider the transformation

$$u = x^2 - y^2, \qquad v = 2xy.$$

(a) Compute the Jacobian determinant $\dfrac{\partial(u, v)}{\partial(x, y)}$.

(b) The unit disk in (x, y)-coordinates,

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\},$$

is mapped by $(x, y) \mapsto (u, v)$ onto a region D' in the (u, v)-plane. Describe D'.

(c) Evaluate

$$I = \iint_D (x^2 + y^2)\, dx\, dy$$

using the change of variables via (u, v) and your Jacobian.

7.1.1 Solution

(a) Jacobian Determinant. We already computed in a previous example:

$$JF(x,y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix},$$

so

$$\frac{\partial(u,v)}{\partial(x,y)} = \det JF(x,y) = 4(x^2+y^2).$$

(b) Description of the Image Region. Observe that

$$u^2+v^2 = (x^2-y^2)^2 + (2xy)^2 = (x^2+y^2)^2.$$

Therefore, if $(x,y) \in D$, then $x^2+y^2 \le 1$ and hence

$$u^2+v^2 = (x^2+y^2)^2 \le 1.$$

So the image D' is contained in the unit disk

$$D' \subseteq \{(u,v) : u^2+v^2 \le 1\}.$$

Conversely, given a point (u,v) with $u^2+v^2 \le 1$, we have $r := \sqrt{x^2+y^2} = \sqrt[4]{u^2+v^2} \le 1$, and there are (generically) points (x,y) with $x^2+y^2 = r^2$ that map to (u,v) because this transformation is essentially squaring in polar form. Thus the image D' is exactly the unit disk as a set, but the mapping is *two-to-one* except at the origin.

For purposes of the integral, we note we must handle the multiplicity carefully.

(c) Evaluating the Integral. We want

$$I = \iint_D (x^2+y^2)\,dx\,dy.$$

A simpler route is to use polar coordinates directly, but here we illustrate the Jacobian method carefully.

In polar coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, so $x^2+y^2 = r^2$, $dx\,dy = r\,dr\,d\theta$, and D is $0 \le r \le 1$, $0 \le \theta \le 2\pi$. Then

$$I = \int_0^{2\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\pi}{2}.$$

Now, to see how the (u,v) transformation fits, notice that $u^2+v^2 = (x^2+y^2)^2$, so let

$$\rho = \sqrt{u^2+v^2} = (x^2+y^2) = r^2.$$

One can think of the mapping $(r,\theta) \mapsto (\rho,\varphi)$ with $\rho = r^2$, $\varphi = 2\theta$ (this is the polar form of the squaring map). Then

$$d\rho = 2r\,dr, \qquad d\varphi = 2\,d\theta,$$

and one must keep track of the factor 4 from (r,θ) to (ρ,φ), plus the fact the map is 2-to-1. Ultimately, the result is consistent with the value $\pi/2$. This illustrates how multiple layers of Jacobians can be combined, but for clarity the polar computation above is the neatest.

7.2 Problem 2: A Nontrivial Change of Variables in a Double Integral

Problem. Let R be the region in the xy-plane bounded by the curves

$$y = x, \quad y = 2x, \quad xy = 1, \quad xy = 2.$$

Use the change of variables

$$u = xy, \qquad v = \frac{y}{x},$$

to evaluate

$$I = \iint_R \frac{1}{x^2 + y^2}\,dx\,dy.$$

Solution

Step 1: Describe R in (u,v)-coordinates. We have

$$u = xy, \qquad v = \frac{y}{x}.$$

From $y = x$ we get $v = 1$. From $y = 2x$ we get $v = 2$. From $xy = 1$ we get $u = 1$. From $xy = 2$ we get $u = 2$.

Thus the region R becomes the rectangle

$$R' = \{(u,v) : 1 \le u \le 2,\ 1 \le v \le 2\}.$$

Step 2: Compute the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$. We have

$$u = xy, \qquad v = \frac{y}{x}.$$

Compute partial derivatives:

$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x,$$
$$\frac{\partial v}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{x}.$$

So

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = y \cdot \frac{1}{x} - x \cdot \left(-\frac{y}{x^2}\right) = \frac{y}{x} + \frac{xy}{x^2} = \frac{y}{x} + \frac{y}{x} = 2\frac{y}{x} = 2v.$$

Thus

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}.$$

Step 3: Rewrite the integrand in terms of (u,v). We need x and y in terms of u and v. From $u = xy$ and $v = y/x$, we get:

$$y = vx, \qquad u = x \cdot vx = vx^2 \Rightarrow x^2 = \frac{u}{v}, \quad x = \sqrt{\frac{u}{v}} \text{ (take } x > 0 \text{ in this region).}$$

Then

$$y = vx = v\sqrt{\frac{u}{v}} = \sqrt{uv}.$$

Compute

$$x^2 + y^2 = \frac{u}{v} + uv = \frac{u}{v}(1+v^2).$$

Thus

$$\frac{1}{x^2+y^2} = \frac{1}{\frac{u}{v}(1+v^2)} = \frac{v}{u(1+v^2)}.$$

Step 4: Transform the integral. We have

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{2v}\,du\,dv.$$

Therefore

$$I = \iint_R \frac{1}{x^2+y^2}\,dx\,dy = \iint_{R'} \frac{v}{u(1+v^2)} \cdot \frac{1}{2v}\,du\,dv = \iint_{R'} \frac{1}{2u(1+v^2)}\,du\,dv.$$

The region R' is the rectangle $1 \le u \le 2,\ 1 \le v \le 2$, so

$$I = \int_{v=1}^{2} \int_{u=1}^{2} \frac{1}{2u(1+v^2)}\,du\,dv.$$

Integrate with respect to u first:

$$\int_{u=1}^{2} \frac{1}{2u(1+v^2)}\,du = \frac{1}{2(1+v^2)} \int_1^2 \frac{1}{u}\,du = \frac{1}{2(1+v^2)} \ln 2.$$

So

$$I = \int_{v=1}^{2} \frac{\ln 2}{2(1+v^2)}\,dv = \frac{\ln 2}{2} \int_1^2 \frac{1}{1+v^2}\,dv = \frac{\ln 2}{2}\left[\arctan v\right]_1^2.$$

Thus

$$I = \frac{\ln 2}{2}\,(\arctan 2 - \arctan 1) = \frac{\ln 2}{2}\left(\arctan 2 - \frac{\pi}{4}\right).$$

This is a compact exact value, obtained via a clean Jacobian-based change of variables.

Problem 3: Jacobian and Local Invertibility

Problem. Consider the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = (u, v) = (x^3 - y, x + y^3).$$

(a) Compute the Jacobian matrix $JF(x, y)$ and its determinant.

(b) Show that F is locally invertible at $(0, 0)$.

(c) Compute the Jacobian of the local inverse map F^{-1} at $(u, v) = (0, 0)$.

Solution

(a) Jacobian matrix and determinant. Compute partial derivatives:

$$\frac{\partial u}{\partial x} = 3x^2, \quad \frac{\partial u}{\partial y} = -1,$$

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 3y^2.$$

Thus

$$JF(x, y) = \begin{pmatrix} 3x^2 & -1 \\ 1 & 3y^2 \end{pmatrix},$$

and

$$\det JF(x, y) = (3x^2)(3y^2) - (-1) \cdot 1 = 9x^2y^2 + 1.$$

In particular,

$$\det JF(0, 0) = 1 \neq 0.$$

(b) Local invertibility at $(0, 0)$. Since F is C^1 (indeed, smooth) and $\det JF(0, 0) \neq 0$, the Inverse Function Theorem applies. Therefore, there exist neighborhoods U of $(0, 0)$ and V of $F(0, 0) = (0, 0)$ such that

$$F : U \to V$$

is a C^1 bijection with a C^1 inverse.

(c) Jacobian of the inverse at $(0,0)$. The Inverse Function Theorem also tells us

$$J(F^{-1})(0,0) = (JF(0,0))^{-1}.$$

Compute

$$JF(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The inverse of this matrix is

$$(JF(0,0))^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which is the transpose of $JF(0,0)$ with a minus sign adjusted as expected for a 90° rotation. Hence

$$J(F^{-1})(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Geometrically, near the origin, F behaves (to first order) like a linear map that is a 90° rotation, and the inverse behaves like the opposite rotation.

7.3 Problem 4: Jacobian in Probability Density Transformation

Problem. Let (X,Y) be a continuous random vector with joint density

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}, \qquad x, y \in \mathbb{R},$$

i.e. a standard bivariate normal with independent components. Define the transformation

$$R = \sqrt{X^2+Y^2}, \qquad \Theta = \arctan\left(\frac{Y}{X}\right),$$

interpreted suitably so that $\Theta \in (-\pi, \pi]$. Use the Jacobian to find the joint density of (R,Θ) and then marginal densities $f_R(r)$ and $f_\Theta(\theta)$.

Solution

Step 1: Transformation and its Jacobian. We already know the inverse mapping:

$$X = R\cos\Theta, \qquad Y = R\sin\Theta.$$

The Jacobian determinant of (X,Y) with respect to (R,Θ) is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

Thus

$$dx\,dy = r\,dr\,d\theta.$$

Step 2: Joint density of (R, Θ). We use the formula

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = f_{X,Y}(r\cos\theta, r\sin\theta)\cdot r.$$

Plug in $f_{X,Y}$:

$$f_{X,Y}(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}e^{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}} = \frac{1}{2\pi}e^{-\frac{r^2}{2}}.$$

Hence

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi}e^{-\frac{r^2}{2}}\cdot r,$$

for $r \geq 0$, $\theta \in (-\pi, \pi]$.

Step 3: Marginal density of R. Integrate over θ:

$$f_R(r) = \int_{-\pi}^{\pi} f_{R,\Theta}(r,\theta)\,d\theta = \int_{-\pi}^{\pi}\frac{1}{2\pi}e^{-\frac{r^2}{2}}r\,d\theta = re^{-\frac{r^2}{2}}\int_{-\pi}^{\pi}\frac{1}{2\pi}\,d\theta = re^{-\frac{r^2}{2}}.$$

So

$$f_R(r) = re^{-\frac{r^2}{2}}, \qquad r \geq 0.$$

Step 4: Marginal density of Θ. Integrate over r:

$$f_\Theta(\theta) = \int_0^\infty f_{R,\Theta}(r,\theta)\,dr = \int_0^\infty \frac{1}{2\pi}e^{-\frac{r^2}{2}}r\,dr.$$

Make the substitution $s = \frac{r^2}{2}$, so $ds = r\,dr$. Then

$$f_\Theta(\theta) = \frac{1}{2\pi}\int_0^\infty e^{-s}\,ds = \frac{1}{2\pi}.$$

Thus

$$f_\Theta(\theta) = \frac{1}{2\pi}, \qquad \theta \in (-\pi, \pi],$$

which means Θ is uniform, and independent of R. The Jacobian calculation is central to this derivation.

8 Practice Problems

8.1 Problem 1 (Jacobian Determinants)

Compute the Jacobian determinant of the transformation

$$u = x^2 - y, \qquad v = x + y^2.$$

Evaluate $\det\dfrac{\partial(u,v)}{\partial(x,y)}$ at $(1, 2)$.

8.2 Problem 2 (Inverse Mapping)

Let

$$F(x, y) = (u, v) = (e^x \cos y,\ e^x \sin y).$$

(a) Compute the Jacobian matrix $JF(x, y)$.

(b) Show that $\det JF(x, y) \neq 0$ everywhere and conclude that F is locally invertible.

8.3 Problem 3 (Eigen–Directions of JF)

For the map

$$F(x, y) = (3x + y,\ x + 3y),$$

(a) Compute JF.

(b) Find its eigenvalues and eigenvectors.

(c) Interpret the geometric action of F on small circles near the origin.

8.4 Problem 4 (Change of Variables in Integration)

For the transformation

$$x = r \cos\theta, \qquad y = r \sin\theta,$$

evaluate

$$\iint_D (x^2 + y^2)\, dx\, dy$$

where D is the unit disk, using the Jacobian determinant.

8.5 Problem 5 (Curved Boundary Transformation)

Let

$$u = xy, \qquad v = x/y, \qquad x > 0,\ y > 0.$$

(a) Compute the Jacobian determinant $\dfrac{\partial(u, v)}{\partial(x, y)}$.

(b) Transform the region bounded by $xy = 1$, $xy = 4$, $y = x$, and $y = 2x$ into (u, v)–coordinates and sketch the resulting rectangle.

8.6 Problem 6 (Jacobian in Probability)

Let (X, Y) be uniformly distributed on the unit disk. Define

$$R = \sqrt{X^2 + Y^2}, \qquad \Theta = \arctan(Y/X).$$

(a) Compute the joint density of (R, Θ) using the Jacobian of the transformation.

(b) Find the marginal densities $f_R(r)$ and $f_\Theta(\theta)$.

8.7 Problem 7 (3D Jacobian)

Let the transformation from cylindrical to Cartesian coordinates be

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

(a) Compute the 3×3 Jacobian matrix.

(b) Show that the determinant equals r.

8.8 Problem 8 (Local Linearization)

Let the vector field be

$$\mathbf{V}(x, y) = \begin{pmatrix} y \\ -2x \end{pmatrix}.$$

(a) Compute $J\mathbf{V}(x, y)$.

(b) Classify the critical point at $(0, 0)$ using eigenvalues.

(c) Sketch the linearized flow near the origin.

8.9 Problem 9 (Nonlinear Transformation)

For the mapping

$$F(x, y) = (u, v) = (x^2 + y^2, \arctan(y/x)),$$

(a) Compute the Jacobian.

(b) Show that $\det JF(x, y) = 2$.

(c) Explain geometrically why the factor 2 is expected.

8.10 Problem 10 (Implicit Surfaces)

Consider the constraint

$$F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0.$$

(a) Compute the gradient ∇F.

(b) Show that $\frac{\partial F}{\partial z} \neq 0$ for all points on the unit sphere.

(c) Conclude via the Implicit Function Theorem that z is a differentiable function of (x, y) on the sphere (except at the poles).

9 Conclusion

The Jacobian matrix provides a unified framework for:

- Local linearization of multivariable maps,
- Measuring local stretching, rotation, and orientation,
- Performing changes of variables in integrals,
- Analyzing local invertibility via the Inverse Function Theorem,
- Transforming probability densities under smooth mappings.

Geometric interpretations (via grid deformation, area/volume scaling, and coordinate transformations like polar and cylindrical coordinates) make the Jacobian much more intuitive.

At the same time, the advanced problems show how the Jacobian underlies powerful techniques in analysis, PDEs, and probability. Mastery of these ideas is foundational for further work in differential geometry, dynamical systems, and advanced analysis.

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