MAT389 Fall 2016, Problem Set 10

Laurent series

10.1 The function $f(z) = (1-z)^{-1}$ is holomorphic everywhere on the complex plane except at the point $z = -1$. Consequently, there are two series expansions about $z = 0$ that converge to it. The first one is its Taylor series expansion about $z = 0$,

$$f(z) = \sum_{n=0}^{\infty} z^n,$$

that converges absolutely in the disk $|z| < 1$. The other one is a Laurent series expansion that converges absolutely for $1 < |z| < +\infty$. Compute the latter.

Hint: we have done this in class several times, but it's good to hammer it in a little bit.

We express $f(z)$ as

$$f(z) = -\frac{1}{z} \cdot \frac{1}{1-1/z}$$

and use the geometric series expansion for the second factor, convergent for $|1/z| < 1$:

$$f(z) = -\frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}.$$

10.2 The function $f(z) = 1/(z-1)(z-2)$ is holomorphic on the disc $|z| < 1$, and on two differential annular domains centered at $z = 0$: , the annulus $1 < |z| < 2$, and the outside of the circle of radius 2 (i.e., $|z| > 2$). In the first of these, we have a Taylor series expansion about 0; in the other two, we have (different!) Laurent series expansion. Compute all of them.

Hint: this is a slight variation of what we did in class. Try a partial fraction decomposition first.

We start by performing a partial fraction decomposition of $f(z)$:

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

In the disc $|z| < 1$, we expand

$$f(z) = -\frac{1}{2}\frac{1}{1-z/2} + \frac{1}{1-z} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right) z^n$$

For $1 < |z| < 2$, we have

$$f(z) = -\frac{1}{2}\frac{1}{1-z/2} - \frac{1}{z}\frac{1}{1-1/z} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}$$

Outside of the circle of radius 2, it is

$$f(z) = \frac{1}{z}\frac{1}{1-2/z} - \frac{1}{z}\frac{1}{1-1/z} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}(2^{n-1}-1)\frac{1}{z^n}$$

10.3 Consider again the function $f(z) = 1/(z-1)(z-2)$.

(i) It is holomorphic on a punctured neighborhood of the point $z = 1$. Compute its Laurent series expansion about $z = 1$, and reason why it converges on $0 < |z-1| < 1$.

(ii) Do the same about the point $z = 2$.

Hint: once again, perform a partial fraction decomposition first.

(i) We can massage the partial fraction decomposition of $f(z)$,

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

to make the combination $z-1$ appear in the first fraction:

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

This first fraction is holomorphic at $z = 1$, and we can use the geometric series to expand it.

$$f(z) = -\sum_{n=0}^{\infty}(z-1)^n - \frac{1}{z-1}$$

This gives us a Laurent series expansion for $f(z)$ that converges around $z = 1$. The part in positive powers of $z-1$ converges for $|z-1| < 1$, since the singularity closest to the expansion point (the *only* singularity, in fact), of $1/(z-2)$ is at unit distance of $z = 1$. The Laurent series above then converges at all points of that disc, except at the center.

(ii) We now want to make $z-2$ appear in the second term of the partial fraction decomposition of $f(z)$. Reasoning as above, we find

$$f(z) = \frac{1}{z-2} - \frac{1}{1+(z-2)} = \frac{1}{z-2} - \sum_{n=0}^{\infty}(-1)^n(z-2)^n$$

The series in positive terms in $z-2$ converges on the disc $|z-2| < 1$; the Laurent series converges on $0 < |z-2| < 1$.

10.4 The following functions are holomorphic on a punctured neighborhood of the given point z_0. Leverage your knowledge of Taylor series to compute their respective Laurent expansions in such a punctured neighborhood.

(i) $f(z) = z^{-2}e^{iz}$ about $z_0 = 0$,

(ii) $f(z) = \cos z^{-2}$ about $z_0 = 0$
(iii) $f(z) = (z-\pi)^{-1} \sin z$ about $z_0 = \pi$ (**Hint:** start by expressing $\sin z$ in terms of $\sin(z-\pi)$),

(i) $f(z)$ is holomorphic away from the origin, and so its Laurent series converges on $\mathbb{C}^\times$:

$$f(z) = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n}{n!} z^{n-2}$$

(ii) Once again, $f(z)$ is holomorphic on $\mathbb{C}^\times$, and it has the following Laurent expansion, convergent on $\mathbb{C}^\times$:

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\, z^{4n}}$$

(iii) Writing $\sin z = -\sin(z - \pi)$ an using the Taylor expansion of sine, we have

$$f(z) = -\frac{1}{z-\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z-\pi)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (z-\pi)^{n-1}$$

convergent everywhere except at the point $z = \pi$ itself.

Isolated singularities

10.5 For each of the cases below, determine whether the (unique) isolated singular point is a removable singularity, a pole or an essential singularity.

(i) $f(z) = ze^{1/z}$,
(ii) $f(z) = z^2/(1+z)$,
(iii) $f(z) = (\sin z)/z$,
(iv) $f(z) = (2-z)^{-3}$.

(i) The singularity of $f(z) = ze^{1/z}$ is located at $z = 0$ and its Laurent series around that point is

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n! z^{n-1}}.$$

There are infinitely many terms in negative powers of z, implying that $z = 0$ is an essential singularity of f.

(ii) The point $z = -1$ is an isolated singularity of $f(z) = z^2/(1+z)$. We have

$$f(z) = \frac{z^2}{1+z} = \frac{1}{z+1} + \frac{z^2-1}{z+1} = \frac{1}{z+1} - 2 + (z+1)$$

There is only term in negative powers of $z+1$—that is, $z = -1$ is a pole (of order 1).

(iii) The singularity of $f(z) = \sin z/z$ is located at $z = 0$. The Laurent series expansion about that point is

$$f(z) = \frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

There are no terms in negative powers of z, so $z = 0$ is a removable singularity of f.

(iv) The function $f(z) = (2-z)^{-3}$ is singular at $z = 2$. It is almost already expressed as a Laurent series: we only need to pull out a sign:

$$f(z) = \frac{-1}{(z-2)^3}.$$

It is now clear that f has a pole of order 3 at $z = 2$.

Residues

10.6 Calculate the following residues:

(i) $\operatorname*{Res}_{z=0} \dfrac{1}{z+z^2}$, (ii) $\operatorname*{Res}_{z=1} \dfrac{e^{2z}}{(z-1)^2}$,

(iii) $\operatorname*{Res}_{z=0} \dfrac{1-\cosh z}{z^3}$, (iv) $\operatorname*{Res}_{z=0} \dfrac{1-e^{2z}}{z^4}$,

(v) $\operatorname*{Res}_{z=0} z\cos\dfrac{1}{z}$, (vi) $\operatorname*{Res}_{z=0} \dfrac{\sinh z}{z^4(1-z^2)}$,

(vii) $\operatorname*{Res}_{z=1} \dfrac{\sinh z}{z^4(1-z^2)}$, (viii) $\operatorname*{Res}_{z=1} \dfrac{z^2+2}{z-1}$,

(ix) $\operatorname*{Res}_{z=-1/2} \left(\dfrac{z}{2z+1}\right)^3$, (x) $\operatorname*{Res}_{z=i\pi} \dfrac{e^z}{z^2+\pi^2}$.

We have two ways of calculating the residue $\operatorname{Res}_{z=z_0} f(z)$ of a function f at an isolated singular point $z = z_0$. The first one is to simply write down the Laurent series expansion of $f(z)$ in a punctured neighborhood of $z = z_0$ and take the coefficient of the term $(z-z_0)^{-1}$.

If $z = z_0$ is a pole of order m of f, we can also write $f(z) = \phi(z)/(z-z_0)^m$ with $\phi(z)$ holomorphic and nonzero at $z = z_0$. The residue is then given by

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\Big[(z-z_0)^m f(z)\Big]\Bigg|_{z=z_0} = \frac{\phi^{(m-1)(z_0)}}{(m-1)!}$$

If you can't quite remember this formula (I tend to forget it), think in terms of the Taylor series of $\phi(z)$ about z_0: we want the term with $(z-z_0)^{-1}$ in the Laurent series of $f(z)$, and we have a factor of $(z-z_0)^m$ in the denominator.

(i) The easiest approach here is to calculate the whole Laurent series. We have

$$\frac{1}{z+z^2} = \frac{1}{z}\,\frac{1}{z+1} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^n = \frac{1}{z} - 1 + O(z)$$

The residue is the coefficient of z^{-1}, which is 1.

(ii) Since e^{2z} is an entire function and it does not vanish at $z = 1$, $e^{2z}/(z-1)^2$ has a double pole at $z = 1$. Writing $\phi(z) = e^{2z}$, we have

$$\operatorname*{Res}_{z=1} \frac{e^{2z}}{(z-1)^2} = \frac{\phi'(1)}{1!} = 2e^2$$

(iii) With the knowledge of the Taylor series expansion of $\cosh z$ about $z = 0$, we calculate

$$\frac{1-\cosh z}{z^3} = \frac{1-\sum_{n=0}^{\infty} z^{2n}/(2n)!}{z^3} = -\sum_{n=1}^{\infty} \frac{z^{2n-3}}{(2n)!} = -\frac{1}{2z} + O(z)$$

so the residue is $-1/2$.

(iv) Again we take the Taylor series expansion of the numerator to get

$$\frac{1-e^{2z}}{z^4} = \frac{1-\sum_{n=0}^{\infty} 2^n z^n/n!}{z^4} = -\sum_{n=1}^{\infty} \frac{2^n z^{n-4}}{n!}$$

The term of degree -1 corresponds to $n = 3$, and its coefficient is $-2^3/3! = -4/3$.

(v) The function given has an essential singularity at $z = 0$, as can easily be seen by using the Taylor series expansion of cosine:

$$z\cos\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{1-2n}}{(2n)!}$$

The residue is the coefficient of the term $n = 1$ in the above sum; that is, $-1/2$.

(vi) In this case we need to take two Taylor series expansions:

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^4}\left(\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\right)\left(\sum_{m=0}^{\infty} z^{2m}\right)$$

There are two terms in the above product that results in a term with z^{-1}: $n = 1$, $m = 0$ (with coefficient $1/6$), and $n = 0$, $m = 1$ (with coefficient 1). Hence the residue is $7/6$.

(vii) At $z = 1$ the function given has a simple pole, with $\phi(z) = -\sinh z/z^4(1+z)$. We can calculate the residue by evaluating $\phi(1) = -\sinh(1)/2$.

(viii) It is clear that $z = 1$ is a simple pole, so we just need to evaluate the numerator at that point to get a residue of $(z^2+2)\big|_{z=1} = 3$.

(ix) It is apparent that we have a pole of order 3, with $\phi(z) = z^3/8$. Hence

$$\operatorname*{Res}_{z=-1/2}\left(\frac{z}{2z+1}\right)^3 = \frac{1}{2!}\frac{d^2}{dz^2}z^3\bigg|_{z=-1/2} = -\frac{3}{16}$$

(x) The function given has a simple pole at $z = i\pi$, with $\phi(z) = e^z/(z+i\pi)$, so

$$\operatorname*{Res}_{z=i\pi}\frac{e^z}{z^2+\pi^2} = \frac{e^z}{z+i\pi}\bigg|_{z=i\pi} = -\frac{1}{2\pi i} = \frac{i}{2\pi}$$

10.7 Use the Residue Theorem to evaluate the following integrals:

(i) $\displaystyle\oint_{C_1(0)} \frac{e^{-z}}{z^2}\,dz$,

(ii) $\displaystyle\oint_{C_1(0)} z^2 e^{1/z}\,dz$,

(iii) $\displaystyle\oint_{C_3(0)} \frac{z+1}{z^2-2z}\,dz$,

(iv) $\displaystyle\oint_{C_4(0)} \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$,

(v) $\displaystyle\oint_{C_2(2)} \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$,

(vi) $\displaystyle\oint_{C_2(0)} \frac{dz}{z^3(z+4)}$,

(vii) $\displaystyle\oint_{C_3(-2)} \frac{dz}{z^3(z+4)}$, (viii) $\displaystyle\oint_{C_2(0)} \tan z\, dz$,

(ix) $\displaystyle\oint_{C_2(0)} \frac{dz}{\sinh 2z}$, (x) $\displaystyle\oint_{C_2(0)} \frac{\cosh \pi z}{z(z^2+1)}\, dz$.

(i) The integrand has a double pole at the origin, with $\phi(z) = e^{-z}$ (in the notation of the last problem). Hence

$$\oint_{C_1(0)} \frac{e^{-z}}{z^2}\, dz = 2\pi i \operatorname*{Res}_{z=0} \frac{e^{-z}}{z^2} = 2\pi i \left.\frac{de^{-z}}{dz}\right|_{z=0} = -2\pi i$$

(ii) The integrand has an essential singularity at $z = 0$, and its Laurent series expansion in a punctured neighborhood of the origin is

$$z^2 e^{1/z} = z^2 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^{n-2}}$$

Its residue at $z = 0$ is then $1/6$ and

$$\oint_{C_1(0)} z^2 e^{1/z}\, dz = 2\pi i\, \frac{1}{6} = \frac{\pi i}{3}$$

(iii) The integrand has two simple poles inside the contour of integration:

$$\begin{aligned}\oint_{C_3(0)} \frac{z+1}{z^2-2z}\, dz &= 2\pi i \left(\operatorname*{Res}_{z=0} \frac{z+1}{z(z-2)} + \operatorname*{Res}_{z=2} \frac{z+1}{z(z-2)}\right)\\ &= 2\pi i \left(\left.\frac{z+1}{z-2}\right|_{z=0} + \left.\frac{z+1}{z}\right|_{z=2}\right)\\ &= 2\pi i \left(-\frac{1}{2} + \frac{3}{2}\right) = 4\pi i\end{aligned}$$

(iv) The integrand has three simple poles inside the contour of integration:

$$\begin{aligned}\oint_{C_4(0)} \frac{3z^3+2}{(z-1)(z^2+9)}\, dz &= 2\pi i \left(\operatorname*{Res}_{z=1} + \operatorname*{Res}_{z=3i} + \operatorname*{Res}_{z=-3i}\right) \frac{3z^3+2}{(z-1)(z^2+9)}\\ &= 2\pi i \left(\left.\frac{3z^3+2}{z^2+9}\right|_{z=1} + \left.\frac{3z^3+2}{(z-1)(z+3i)}\right|_{z=3i} + \left.\frac{3z^3+2}{(z-1)(z-3i)}\right|_{z=-3i}\right)\\ &= 2\pi i \left(\frac{1}{2} + \frac{5}{4} + \frac{49}{12i} + \frac{5}{4} - \frac{49}{12i}\right) = 6\pi i\end{aligned}$$

(v) This is the same function as before, but the only pole inside $C_2(2)$ is that at $z = 1$, so

$$\oint_{C_2(2)} \frac{3z^3+2}{(z-1)(z^2+9)}\, dz = 2\pi i \operatorname*{Res}_{z=1} \frac{3z^3+2}{(z-1)(z^2+9)} = 2\pi i\, \frac{1}{2} = \pi i$$

(vi) Of the poles of the integrand, only that at $z = 0$ is inside the contour of integration. It is a pole of order 3, and so we have

$$\oint_{C_2(0)} \frac{dz}{z^3(z+4)} = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^3(z+4)} = 2\pi i\, \frac{1}{2!} \left.\frac{d^2(z+4)^{-1}}{dz^2}\right|_{z=0} = 2\pi i\, \frac{1}{64} = \frac{\pi i}{32}$$

(vii) Here we have both poles inside the contour of integration, so

$$\begin{aligned}\oint_{C_3(-2)} \frac{dz}{z^3(z+4)} &= 2\pi i \left(\operatorname*{Res}_{z=0} \frac{1}{z^3(z+4)} + \operatorname*{Res}_{z=-4} \frac{1}{z^3(z+4)} \right) \\ &= 2\pi i \left(\frac{1}{64} - \frac{1}{64} \right) = 0\end{aligned}$$

(viii) Realizing $\tan z$ as the quotient of $\sin z$ by $\cos z$, it is clear that it has two isolated singularities inside the contour of integration: $z = \pm\pi/2$. From the Taylor series expansions of $\sin z$ and $\cos z$ we can see that these are poles or order 1 and calculate the residues there:

$$\tan z = \frac{\sin z}{\cos z} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z-\pi/2)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (z-\pi/2)^{2n+1}} = \frac{1}{z-\pi/2} \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z-\pi/2)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (z-\pi/2)^{2n}}$$

$$\Rightarrow \operatorname*{Res}_{z=\pi/2} \tan z = \left. \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (z-\pi/2)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (z-\pi/2)^{2n}} \right|_{z=\pi/2} = -1$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} (z+\pi/2)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!} (z+\pi/2)^{2n+1}} = \frac{1}{z+\pi/2} \frac{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} (z+\pi/2)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!} (z+\pi/2)^{2n}}$$

$$\Rightarrow \operatorname*{Res}_{z=-\pi/2} \tan z = \left. \frac{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} (z+\pi/2)^{2n}}{\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!} (z+\pi/2)^{2n}} \right|_{z=-\pi/2} = -1$$

Hence,

$$\oint_{C_2(0)} \tan z \, dz = 2\pi i \left(\operatorname*{Res}_{z=\pi/2} \tan z + \operatorname*{Res}_{z=-\pi/2} \tan z \right) = -4\pi i$$

(ix) We know that $\sinh 2z$ has zeroes at $z = k\pi i/2$ for $k \in \mathbb{Z}$. Moreover, they all have multiplicity one, and hence $1/\sinh 2z$ has simple poles at those points. Since the function is periodic with period πi, the residues at those points with k even are all the same. We only need to calculate it at, e.g., $z = 0$. For that, we write

$$\frac{1}{\sinh 2z} = \frac{1}{\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} z^{2n+1}} = \frac{1}{z} \frac{1}{\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} z^{2n}}$$

so that

$$\operatorname*{Res}_{z=0} \frac{1}{\sinh 2z} = \left. \frac{1}{\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} z^{2n}} \right|_{z=0} = \frac{1}{2}$$

Similarly, the periodicity of the function implies that the residues at the poles $z = k\pi i/2$ with k odd all coincide. We do the calculation for $k = 1$:

$$\frac{1}{\sinh 2z} = \frac{1}{\sum_{n=0}^{\infty} \frac{-2^{2n+1}}{(2n+1)!} (z-\pi i/2)^{2n+1}} = \frac{1}{z-\pi i/2} \frac{-1}{\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} (z-\pi i/2)^{2n}}$$

$$\operatorname*{Res}_{z=\pi i/2} \frac{1}{\sinh 2z} = \left. \frac{-1}{\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} (z-\pi i/2)^{2n}} \right|_{z=\pi i/2} = -\frac{1}{2}$$

The integral is now easily calculated:

$$\begin{aligned}\oint_{C_2(0)} \frac{dz}{\sinh 2z} &= 2\pi i \left(\operatorname*{Res}_{z=-\pi i/2} \frac{1}{\sinh 2z} + \operatorname*{Res}_{z=0} \frac{1}{\sinh 2z} + \operatorname*{Res}_{z=\pi i/2} \frac{1}{\sinh 2z} \right) \\ &= 2\pi i \left(-\frac{1}{2} + \frac{1}{2} - \frac{1}{2} \right) = -\pi i\end{aligned}$$

(x) The integrand has three simple poles, with residues

$$\operatorname*{Res}_{z=0} \frac{\cosh \pi z}{z(z^2+1)} = \left. \frac{\cosh \pi z}{z^2+1} \right|_{z=0} = 1$$

$$\operatorname*{Res}_{z=\pm i} \frac{\cosh \pi z}{z(z^2+1)} = \left. \frac{\cosh \pi z}{z(z \pm i)} \right|_{z=\pm i} = \frac{1}{2}$$

The integral is

$$\begin{aligned}\oint_{C_2(0)} \frac{\cosh \pi z}{z(z^2+1)}\, dz &= 2\pi i \left(\operatorname*{Res}_{z=0} \frac{\cosh \pi z}{z(z^2+1)} + \operatorname*{Res}_{z=i} \frac{\cosh \pi z}{z(z^2+1)} + \operatorname*{Res}_{z=-i} \frac{\cosh \pi z}{z(z^2+1)} \right) \\ &= 2\pi i \left(1 + \frac{1}{2} + \frac{1}{2} \right) = 4\pi i\end{aligned}$$

Trigonometric integrals

10.8 Use residues to establish the following integration formulas:

(i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$

(ii) $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2}\,\pi$

(iii) $\displaystyle\int_0^{2\pi} \frac{\cos\theta\, d\theta}{5-3\cos\theta} = \frac{\pi}{6}$

(iv) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5-4\cos 2\theta} = \frac{3\pi}{8}$

(v) $\displaystyle\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{a\pi}{(a^2-1)^{3/2}}$ where $a > 1$

(vi) $\displaystyle\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\pi \ (n \in \mathbb{Z}_{>0})$

Note: beware the limits of integration in the last two cases!

The basic strategy for calculating integrals of this kind is to use the formula

$$\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta = \oint_{C_1(0)} F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i} \right) \frac{dz}{iz}$$

that we gave in class, followed by an application of the Residue theorem.

(i) This is the example we did in class. We follow our nose:

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \oint_{C_1(0)} \frac{1}{5-2i(z-z^{-1})} \frac{dz}{iz} = \oint_{C_1(0)} \frac{dz}{2z^2+5iz-2}$$

The denominator of this last integrand factors as $2z^2 + 5iz - 2 = 2(z+2i)(z+i/2)$; of its two zeroes, only the second lies inside the contour of integration, and so

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi i \operatorname*{Res}_{z=-i/2} \frac{1}{2(z+2i)(z+i/2)} = 2\pi i \left.\frac{1}{2(z+2i)}\right|_{z=-i/2} = \frac{2\pi}{3}$$

(ii) Although the limits of integration are slightly different, the result is the same due to the periodicity of $\sin\theta$:

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta} = \oint_{C_1(0)} \frac{1}{1-(z-z^{-1})^2/4}\,\frac{dz}{iz} = \oint_{C_1(0)} \frac{4iz\,dz}{z^4-6z^2+1}$$

Note that

$$z^4 - 6z^2 + 1 = \left[z^2 - (3+2\sqrt{2})\right]\left[z^2 - (3-2\sqrt{2})\right]$$

The zeroes of the first factor must have modulus greater than one and lie outside the unit circle, since $3+2\sqrt{2} > 1$. Those of the second factor are a bit tricky to find; we make the ansatz that they are of the form $a+b\sqrt{2}$, with $a, b \in \mathbb{Q}$. Then,

$$(a+b\sqrt{2})^2 = (a^2+2b^2) + 2ab\sqrt{2} = 3 - 2\sqrt{2} \iff \begin{cases} a^2+2b^2 = 3 \\ 2ab = -2 \end{cases} \iff a = -b = \pm 1.$$

That is, they are $1-\sqrt{2}$ and $-1+\sqrt{2}$. We can now compute residues:

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} &= 2\pi i \left[\operatorname*{Res}_{z=1-\sqrt{2}} \frac{4iz}{z^4-6z^2+1} + \operatorname*{Res}_{z=-1+\sqrt{2}} \frac{4iz}{z^4-6z^2+1}\right] \\
&= 2\pi i \left[\left.\frac{4iz}{(z^2-3-2\sqrt{2})(z+1-\sqrt{2})}\right|_{z=1-\sqrt{2}} + \left.\frac{4iz}{(z^2-3-2\sqrt{2})(z-1+\sqrt{2})}\right|_{z=-1+\sqrt{2}}\right] \\
&= 2\pi i \left(-\frac{i}{2\sqrt{2}} - \frac{i}{2\sqrt{2}}\right) = \sqrt{2}\pi
\end{aligned}$$

(iii) This was a question in a previous year's Final Exam! Our basic strategy yields

$$\int_0^{2\pi} \frac{\cos\theta\,d\theta}{5-3\cos\theta} = \oint_{C_1(0)} \frac{\frac{1}{2}(z+z^{-1})}{5-\frac{3}{2}(z+z^{-1})}\,\frac{dz}{iz} = i\oint_{C_1(0)} \frac{z^2+1}{z(3z^2-10z+3)}\,dz$$

The quadratic polynomial in the denominator factors as $3z^2 - 10z + 3 = 3(z-1/3)(z-3)$. The singularity of the integrand at $z=3$ is outside the unit circle, and so it does not contribute to the integral. The other two singular points, $z=0$ and $z=1/3$, are simple poles. Thus,

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos\theta\,d\theta}{5-3\cos\theta} &= 2\pi i \left(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=1/3}\right) \frac{i(z^2+1)}{3z(z-1/3)(z-3)} \\
&= 2\pi i\,\frac{i}{3}\left[\left.\frac{z^2+1}{(z-1/3)(z-3)}\right|_{z=0} + \left.\frac{z^2+1}{z(z-3)}\right|_{z=1/3}\right] \\
&= -\frac{2\pi}{3}\left[-\frac{5}{4}+1\right] = \frac{\pi}{6}
\end{aligned}$$

(iv) Our original formula does not apply quite so directly here, but the modification we need to make is easy: substitute $\cos m\theta$ by $(z^m + z^{-m})/2$. Then,

$$\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{5 - 4\cos 2\theta} = \oint_{C_1(0)} \frac{(z^3 + z^{-3})^2/4}{5 - 2(z^2 + z^{-2})} \frac{dz}{iz} = \oint_{C_1(0)} \frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} \, dz$$

The singularities of this last integrand lying inside the unit circle are located at 0 and $\pm 1/\sqrt{2}$. The residues at the last two points are computed in a straightforward manner:

$$\operatorname*{Res}_{z=1/\sqrt{2}} \frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} = \left. \frac{i(z^6+1)^2}{8z^5(z^2-2)(z+1/\sqrt{2})} \right|_{z=1/\sqrt{2}} = -\frac{27}{64}\, i$$

$$\operatorname*{Res}_{z=-1/\sqrt{2}} \frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} = \left. \frac{i(z^6+1)^2}{8z^5(z^2-2)(z-1/\sqrt{2})} \right|_{z=-1/\sqrt{2}} = -\frac{27}{64}\, i$$

At $z = 0$, the easiest approach consists of expanding each factor—other than the z^5, of course—in a Taylor series about the origin:

$$\frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} = \frac{i}{8z^5}(1+2z^6+z^{12}) \left(1 + \frac{z^2}{2} + \frac{z^4}{4} + O(z^6)\right) \left(1 + 2z^2 + 4z^4 + O(z^6)\right)$$

Since we need to identify the coefficient of the z^{-1} term, we look for the terms in the product of these three series that have degree four. There are three of them, which yield the following terms in z^{-1}:

$$\frac{i}{8z^5} \cdot 1 \cdot \frac{z^4}{4} \cdot 1, \qquad \frac{i}{8z^5} \cdot 1 \cdot \frac{z^2}{2} \cdot 2z^2, \qquad \frac{i}{8z^5} \cdot 1 \cdot 1 \cdot 4z^4.$$

Hence,

$$\operatorname*{Res}_{z=0} \frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} = \frac{21}{32}\, i$$

and

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{5 - 4\cos 2\theta} &= 2\pi i \left(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=1/\sqrt{2}} + \operatorname*{Res}_{z=-1/\sqrt{2}} \right) \frac{i(z^6+1)^2}{4z^5(z^2-2)(2z^2-1)} \\
&= 2\pi i \left(\frac{21}{32}\, i - \frac{27}{64}\, i - \frac{27}{64}\, i \right) = \frac{3\pi}{8}
\end{aligned}$$

(v) Following our basic strategy, we obtain

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} = \oint_{C_1(0)} \frac{1}{[a + (z + z^{-1})/2]^2} \frac{dz}{iz} = \oint_{C_1(0)} \frac{-4iz}{(z^2 + 2az + 1)^2} \, dz$$

We have

$$z^2 + 2az + 1 = 0 \iff z = -a \pm \sqrt{a^2 - 1}$$

The condition $a > 1$ ensures that this square root is a real number. For the negative sign, we obtain a point outside the unit circle, while the positive sign lands us inside of it—and gives rise to a double pole. Hence,

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} &= 2\pi i \operatorname*{Res}_{z=-a+\sqrt{a^2-1}} \frac{-4iz}{(z^2+2az+1)^2} \\ &= 2\pi i \left.\frac{d}{dz}\frac{-4iz}{(z+a+\sqrt{a^2-1})^2}\right|_{z=-a+\sqrt{a^2-1}} \\ &= \frac{2a\pi}{(a^2-1)^{3/2}}\end{aligned}$$

and

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{a\pi}{(a^2-1)^{3/2}}$$

(vi) Once again, we just apply our basic strategy:

$$\int_0^{2\pi} \sin^{2n}\theta\, d\theta = \oint_{C_1(0)} \frac{(z-z^{-1})^{2n}}{(2i)^{2n}}\frac{dz}{iz} = \frac{(-1)^n}{2^{2n}i}\oint_{C_1(0)} \frac{(z^2-1)^{2n}}{z^{2n+1}}\, dz$$

By the binomial theorem,

$$\frac{(z^2-1)^{2n}}{z^{2n+1}} = \frac{1}{z^{2n+1}}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}z^{2k}$$

The z^{-1} term occurs when $k = n$ in the above sum, and it has coefficient

$$\operatorname*{Res}_{z=0} \frac{(z^2-1)^{2n}}{z^{2n+1}} = (-1)^n\binom{2n}{n} = (-1)^n\frac{(2n)!}{(n!)^2}$$

Consequently,

$$\int_0^{2\pi} \sin^{2n}\theta\, d\theta = \frac{(-1)^n}{2^{2n}i}\cdot 2\pi i \operatorname*{Res}_{z=0}\frac{(z^2-1)^{2n}}{z^{2n+1}} = \frac{(2n)!}{2^{2n-1}(n!)^2}\,\pi$$

and

$$\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta\, d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\,\pi$$

(Just for fun, you could try expressing the fraction on the right hand side of this result in a way similar to that in Problem 8.5.)