Exercises on Laurent Series

Problem Set

Exercise 1. *Expand*

$$f(z) = \frac{2z+1}{z^2(1-z)}$$

as a Laurent series about $z_0 = 0$ for

(a) $|z| < 1$,

(b) $|z| > 1$.

In each case, identify the principal part and the analytic part.

Exercise 2. *Find the Laurent series of*

$$f(z) = \frac{1}{z^2(z-3)}$$

about $z_0 = 0$ in the region $0 < |z| < 3$.

Exercise 3. *Compute the residue of*

$$f(z) = \frac{e^z}{z-1}$$

at $z = 1$ using its Laurent expansion.

Exercise 4. *Show that the Laurent expansion of*

$$\frac{\sin z}{z^3}$$

about $z = 0$ contains infinitely many negative powers. Classify the singularity at $z = 0$.

Exercise 5. *Expand*

$$f(z) = \frac{1}{z^2+4}$$

about $z_0 = 2i$, and determine the annulus of convergence.

Exercise 6. *Explain why a single function may have* different *Laurent expansions about the same center z_0. Provide an explicit example to illustrate this behaviour.*

Exercise 7. *Let*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

be a Laurent series. Prove that the annulus of convergence is uniquely determined by the distances from z_0 to the nearest singularities of f.

Exercise 8. *Suppose f is analytic in $0 < |z| < 1$. Show that the principal part of its Laurent expansion around $z = 0$ has infinitely many negative-power terms if and only if $z = 0$ is an essential singularity.*

Exercise 9. *Evaluate the contour integral*

$$\oint_{|z|=2} \frac{\cos z}{z(z-1)}\,dz$$

using residues obtained from Laurent expansions.

Exercise 10. *Determine the Laurent series of*

$$f(z) = \frac{1}{z^2 - 1}$$

about $z_0 = 0$ in the region $1 < |z| < \infty$. Use your expansion to evaluate

$$\oint_{|z|=R} f(z)\,dz$$

for sufficiently large R.