Homework 1

Due January 24 at 11:59 PM

1. Consider the G-S algorithm for n men and n women. What is the maximum number of times a man may be rejected as a function of n? Give an example where this happens. (5pts)

2. Determine whether the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. (5pts)

   For all $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every man is matched with their most preferred woman.

3. Solve Kleinberg and Tardos, Chapter 1, Exercise 3. (5pt)

4. Solve Kleinberg and Tardos, Chapter 1, Exercise 4. (15pts)

5. Consider a stable marriage problem where the set of men is given by $M = m_1, m_2, ..., m_N$ and the set of women is $W = w_1, w_2, ..., w_N$. Consider their preference lists to have the following properties:

   $$\forall w_i \in W : w_i \text{ prefers } m_i \text{ over } m_j \qquad \forall j > i$$

   $$\forall m_i \in M : m_i \text{ prefers } w_i \text{ over } w_j \qquad \forall j > i$$

   Prove that a unique stable matching exists for this problem. Note: the $\forall$ symbol means "for all". (10pts)

6. State True/False: An instance of the stable marriage problem has a unique stable matching if and only if the version of the Gale-Shapely algorithm where the male proposes and the version where the female proposes both yield the exact same matching. (10pts)

Ungraded Problems

7. Solve Kleinberg and Tardos, Chapter 1, Exercise 2.

8. Determine whether the following statement is true or false. If it is true, give an example. If it is false, give a short explanation.

   For some $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm, every woman is matched with their most preferred man, even though that man does not prefer that woman the most.

9. Four students, a, b, c, and d, are rooming in a dormitory. Each student ranks the others in strict order of preference. A *roommate matching* is defined as a partition of the students into two groups of two roommates each. A roommate matching is *stable* if no two students who are not roommates prefer each other over their roommate. Does a stable roommate matching always exist? If yes, give a proof. Otherwise, give an example of roommate preferences where no stable roommate matching exists.

10. Solve Kleinberg and Tardos, Chapter 1, Exercise 8.