P/NP and reductions

NP - MCQ

Which of the following statements are TRUE? (G, k are problem inputs wherever mentioned)

a) The problem of deciding if there exists a cycle in an undirected graph G is in P.

b) The problem of deciding if there exists a cycle in an undirected graph G is in NP.

c) The problem of deciding if there exists a cycle in an undirected graph G of size at least k is in NP.

d) The problem of deciding if there exists a cycle in an undirected graph G of size at least k is NP-complete.

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Detecting a cycle in an undirected graph can be done with a DFS, thus in P. Since it is in P, it is in NP as well.
Deciding if there is a cycle of size at least k, has efficient certification &
reduces FROM Hamiltonian cycle (G has HC **iff** it has a cycle of size k=n), making it NP-hard.

Long Question

In an undirected graph G = (V, E), we say that 3 vertices form a triangle if each pair is connected by an edge. Given G, and positive integers p, q, the triangle-rich subgraph problem (TRIRICH) asks if there exists a subgraph G' with **at most p** vertices, having **at least q** triangles in it. Show that TRIRICH is NP-complete.
(Hint: To show NP-hardness, use the Independent Set (IS) problem)

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a) TRIRICH is in NP: A subgraph G' is a certificate. The certifier checks if it has at most p nodes. For each triplet of nodes, we can check if they form a triangle, and thus, count the total in (cubic) poly-time and check if it is at least q. This can be done in polynomial time.

Intuition for reduction

- IS: Subset should have no edges
- TRIRICH: Subset has lots of triangles (thus, lots of edges)

Graph complement G^C : If (u,v) is an edge in G, it is not in G^C, and vice-versa.

Sparsity in G $\leftrightarrow$ Density in G^C

k vertices with no edges $\leftrightarrow$ k vertices with all pairs having edges

$\leftrightarrow$ k vertices with all triplets making a triangle

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b) TRIRICH is NP-hard: Reduction from IS

Given the input (G, k), simply create the TRIRICH instance (G^C, k, ${}^kC_3$).
If k = 1, return YES if G has a vertex, otherwise no
If k = 2, return YES if G has at least 1 pair of vertices with no edge, otherwise no
If $k \geq 3$, return TRIRICH(G^C, k, ${}^kC_3$).

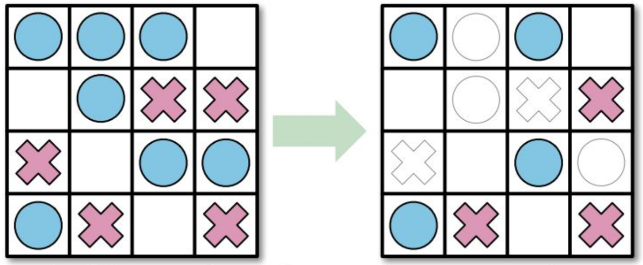
G has an IS of size $\geq$ k iff G^C has a subgraph of size at most k having at least ${}^kC_3$ triangles.

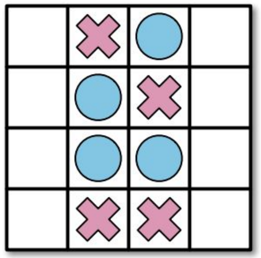
Long Problem

Consider the following solitaire game. The puzzle consists of an n × m grid of squares, where each square may be empty, occupied by a red stone, or occupied by a blue stone. The goal of the puzzle is to remove some of the given stones so that the remaining stones satisfy two conditions:

- every row contains at least one stone, and
- no column contains stones of both colors.

For some initial configurations of stones, reaching this goal is impossible.


A solvable puzzle and one of its many solutions.


An unsolvable puzzle.

Prove that it is NP-hard to determine, given an initial configuration of red and blue stones, whether this puzzle can be solved. (Hint: reduce from SAT/3SAT

We show that this puzzle is NP-hard by describing a reduction from 3SAT (can choose SAT as well, works the same way).
Let Φ be a 3CNF boolean formula with n variables and m clauses. We transform this formula into a puzzle configuration in polynomial time as follows. The size of the board is m × n (rows × cols). The stones are placed as follows, for all cells (i,j):

- If the literal x_j appears in the ith clause of Φ, we place a blue stone at (i, j)
- If the negated variable $\sim x_j$ appears in the ith clause of Φ, we place a red stone at (i, j).
- Otherwise, we leave cell (i, j) blank

This reduction clearly requires only O(mn) time, thus, polynomial time.

For example:

We have Φ as follows:

$(a \lor b \lor c) \land (a \lor \sim b \lor d) \land (c \lor b \lor \sim d) \land (\sim a \lor c \lor d)$

a appears at clause 1 (= row 1) and it's the first var so it goes to column 1 as Blue.

~a appears at clause 4 (= row 4) and it's the first var so it goes to column 1 as Red.

D appears at clause 2 (= row 2) and it's the fourth var so it goes to column 4 as Blue.

B	B	B	
B	R		B
	B	B	R
R		B	B

We claim that this puzzle has a solution if and only if Φ is satisfiable. This claim immediately implies that solving the puzzle is NP-hard. We prove our claim as follows:
First, suppose Φ is satisfiable; consider an arbitrary satisfying assignment. For each index j, remove stones from column j according to the value assigned to xj :
– If xj = True, remove all red stones from column j (keep blue if any).
– If xj = False, remove all blue stones from column j (keep red if any).

In other words, remove precisely the stones that correspond to False literals. Because every variable is either True/False, each column now contains stones of only one color (if any). On the other hand, each clause of Φ must contain at least one True literal, and thus each row still contains at least one stone. We conclude that the puzzle is solvable.

On the other hand, suppose the puzzle is solvable; consider an arbitrary solution. For each index j, assign a value to xj depending on the color of stones left in column j (Each column has at most one color since we have a valid puzzle solution):

– If column j contains blue stones, set xj = True.

– If column j contains red stones, set xj = False.

– If column j is empty, set xj arbitrarily.

In other words, assign values to the variables so that the literals corresponding to the stones in puzzle solution are all True. Each row still has at least one stone, so each clause of Φ contains at least one True literal, so this assignment makes Φ = True. We conclude that Φ is satisfiable.

Linear Programming

Manufacturing Problem

A company manufactures four items A, B, C, and D on two machines X and Y. The time (in minutes) to manufacture one unit of each item on the two machines is shown on the right. Assume all four (on any machine) can be produced in non-integer quantities.

The profit per unit for A, B, C, and D is \$10, \$12, \$17, and \$8 respectively.

The floor space taken up by each unit of A, B, C, and D is 0.1, 0.15, 0.5, and 0.05 sq. meters respectively. Total floor space available is 50 sq. meters.

Customer demands require that at least twice as many units of B should be produced as C.

Machine X is out of action (maintenance/breakdown) for 5% of the time, and machine Y for 7% of the time.

Assuming a working week 35 hours long, formulate a linear program to decide how many products of each to manufacture per week and on which machines, so as to maximize profit. You don't need to solve it.

Item	Time taken	
	Machine X	Machine Y
A	10	29
B	12	19
C	13	33
D	8	23

Solution

a) Variables: Let x_A, x_B, x_C, and x_D be the units of A, B, C, D manufactured by machine X.

Let y_A, y_B, y_C, and y_D be the units of A, B, C, D manufactured by machine Y.

b) Objective Function:

Maximize $10(x_A + y_A) + 12(x_B + y_B) + 17(x_C + y_C) + 8(x_D + y_D)$

Solution

c) Constraints:

a. Floor Space

$0.1(x_A + y_A) + 0.15(x_B + y_B) + 0.5(x_C + y_C) + 0.05(x_D + y_D) \leq 50$

b. Customer requirement

$2(x_C + y_C) = x_B + y_B$

c. Time constraint

$10\ x_A + 12\ x_B + 13\ x_C + 8\ x_D \leq 0.95 \times 35 \times 60$

$29\ y_A + 19\ y_B + 33\ y_C + 23\ y_D \leq 0.93 \times 35 \times 60$

d. Non-negative units

$x_A, x_B, x_C, x_D, y_A, y_B, y_C, y_D \geq 0$

Question 2 – MAX TFSAT

A variation of the satisfiability problem is the MAX True-False SAT, or MAX TFSAT for short. Given n Boolean variables $x_1, \dots, x_n$, and m clauses $c_1, \dots, c_m$, each of which involves two of the variables. We are guaranteed that each clause is of the form $x_i \land \overline{x_j}$. Formulate an integer linear program to maximize the number of satisfied clauses.

Solution 2 – MAX TFSAT (1/2)

Objective

- Maximize the number of satisfied clauses

Constraints

- Each clause is of the form $x_i \land \overline{x_j}$
  - $x_i \land \overline{x_j}$ is satisfied if and only if x_i is set to TRUE and x_j is set to FALSE

Variables

- y_i: Binary variable, 1 if x_i is set to TRUE
- z_k: Binary variable, 1 if c_k is satisfied

Solution 2 – MAX TFSAT (2/2)

Maximize $z_1 + \cdots + z_m$

subject to $z_k \leq y_i$, for $c_k = x_i \wedge \overline{x_j}$ (k = 1, …, m)

$z_k \leq 1 - y_j$, for $c_k = x_i \wedge \overline{x_j}$ (k = 1, …, m)

$y_i \in \{0, 1\}$, for $i = 1, \dots, n$.

$z_k \in \{0,1\}$, for k = 1, …, m

Approximation Algorithms

- An independent set (IS) in a graph is a collection of vertices that are mutually non-adjacent. For general independent set problem, no constant factor approximation algorithm exists unless P=NP.
- IS problem for a bounded degree graph:
    - For graphs with Δ as the maximum degree of any vertex

Notation: $N(v)$ is the set of neighbors of v.

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Greedy-IS(G):
    $S \leftarrow$ Null
    While G is not empty:
        Choose v s.t. $d(v) = \min [d(w): w \in V(G)]$
        $S \leftarrow S \cup \{v\}$,    $K \leftarrow \{v\} \cup N(v)$,    $G \leftarrow G - K$
    Output S
```

Show that Greedy-IS yields $(1/\Delta)$-approximation.

Intuition to Proof

For each vertex added to the solution, at most Δ others are removed

Where Δ = maximum degree in the bounded degree graph,

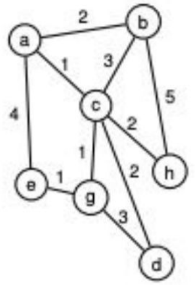
Suppose t iterations in total for Greedy-IS.

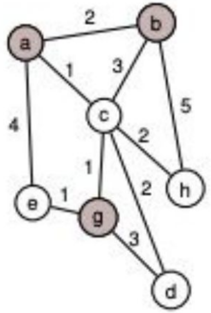
Proving for value of ρ

- If K_i be the set in the i-th iteration in Greedy-IS, then size of any K_i is:
- $|K_i| \leq \Delta + 1$ as K_i only includes some v and N(v) which can be at most Δ
- Additionally, $|K_i \cap OPT| \leq \Delta$
  - This is because OPT either includes v and none of the members in N(v), OR OPT includes some members from N(v) (at most Δ) but definitely not v.
- So $|OPT| \leq \sum_{i=1}^{t} \Delta = \Delta t$
- t is the minimum possible size of S, i.e., $|S| \geq t$
- $|S| \geq |OPT|/\Delta$

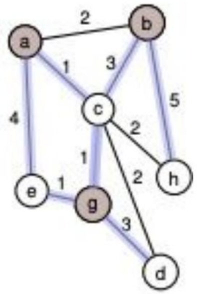
Maximum Cut Problem

For a graph G = (V, E) with edge weights c_e, the maximum cut problem is to find a cut S ⊆ V such that the weight of edges across the vertices (S, V\S) is maximized.





$S = \{a, b, g\}$



$S = \{a, b, g\}$

$w(S) = 18$

Greedy Algorithm

These greedy moves give us a simple optimization procedure:

1. Begin with an arbitrary cut (e.g. S = ∅).

2. If for a node v the total weight of edges from v to nodes in its current side of the partition exceeds the total weight of edges from v to nodes on the other side of the partition, then we move v to the other side of the partition.

$$\sum_{e \in \delta(v) \cap (S \times S)} w(e) > \sum_{e \in \delta(v) \cap (S \times (V \setminus S))} w(e)$$

Why is this a ½-approximation?

To analyze the approximation ratio, observe that at optimality:

$$\sum_{e \in \delta(v) \cap (S \times S)} w(e) \leq \sum_{e \in \delta(v) \cap (S \times (V \setminus S))} w(e) \qquad \forall v \in S$$

and similarly for $v \in V \setminus S$. We can find an equivalent form by adding the weight of cut edges to both sides:

$$\sum_{e \in \delta(v) \cap (S \times S)} w(e) + \sum_{e \in \delta(v) \cap (S \times (V \setminus S))} w(e) \leq \sum_{e \in \delta(v) \cap (S \times (V \setminus S))} w(e) + \sum_{e \in \delta(v) \cap (S \times (V \setminus S))} w(e)$$

Why is this a ½-approximation?

Simplifying:

$$\sum_{e \in \delta(v)} w(e) \leq 2 \sum_{e \in \delta(v) \cap C} w(e)$$

If we sum over all vertices:

$$\sum_{v} \sum_{e \in \delta(v) \cap C} w(e) \geq \frac{1}{2} \sum_{v} \sum_{e \in \delta(v)} w(e)$$

The left hand side is exactly twice the value of the cut, while the right hand side (sum of degree cuts) counts every edge twice.

$$2w(C) \geq \frac{1}{2}\left(2 \cdot \sum_{e} w(e)\right)$$

Since OPT uses a subset of edges:

$$2w(C) \geq \sum w(e) \geq \text{OPT}$$

Approximation algorithm: MVC

Recall the Minimum Vertex Cover (MVC) problem. While finding vertex cover on arbitrary graphs is a hard problem, having certain special structures on edges can make the problem easier - for instance, vertex cover can be found in poly-time for trees. Using similar ideas, Lily designs an algorithm for computing an (approximate) MVC given input G = (V,E):

1) Identify subgraphs G_1 and G_2 which partition the edges in G, i.e., they have edges E_1 and E_2 such that every edge in E is either in E_1 or E_2. This is done in a way that E_1 and E_2 have some useful properties (specifics not relevant for the problem).

2) Compute V_1 & V_2 which are 1.5-approx VC for G_1 and G_2 resp.

3) Return $V_1 \cup V_2$

Note that the solution returned is indeed a vertex cover of G. Show that this is a c-approx. Algorithm for vertex cover for a constant c > 1. Find the best possible value of c such that this is true.

Approximation algorithm: MVC (Solution)

Suppose the min. Vertex covers of G_1 and G_2 are S_1 and S_2 are resp. Suppose $|S_1| \geq |S_2|$.

Now, S, the MVC of G, must be at least as big than both S_1 and S_2, (Since otherwise, S will be provide a smaller vertex cover for G_1 or G_2). Thus, $|S| \geq |S_1|$.

The returned solution has a size $|V_1 \cup V_2| \leq |V_1| + |V_2| \leq 1.5|S_1| + 1.5|S_2| \leq 3|S_1|$

Altogether, we get $|V_1 \cup V_2| \leq 3|S|$.
Thus, this is a 3-approx. algorithm