Homework 11

Graded

Student

Abhishek Soundalgekar

Total Points

60 / 60 pts

Question 1

Q1

Resolved **14** / 14 pts

- ✔ **+ 14 pts** Correct
- ✔ **+ 4 pts** Correctly showing NP
- ✔ **+ 4 pts** Correctly constructing and showing reduction
- **+ 6 pts** Proof in both directions

Regrade Request **Submitted on: May 01**

On page 3 in the last paragraph for proof in both directions, I have given the proof, could you please recheck the answer.

oh, sorry, missed that!

Reviewed on: May 06

Question 2

Q2

15 / 15 pts

✔ **+ 15 pts** Correct

+ 2 pts BIG-HAM -CYCLE ∈ NP using a proper certificate

+ 5 pts Correctly constructs and explains the reduction HAM -CYCLE ≤P BIG-HAM -CYCLE
and discusses how the construction is done in polynomial time.

+ 4 pts Forward Direction

+ 4 pts Backward Direction

+ 0 pts Incorrect

− 8 pts Wrong construction

Question 3

Q3

15 / 15 pts

✔ **+ 15 pts** Full points

+ 4 pts Prove in NP

+ 5 pts Construction

+ 6 pts Proof

+ 0 pts Wrong

Question 4

Q4

16 / 16 pts

✔ **+ 16 pts** Full points

+ 4 pts Showing the problem is in NP

+ 2 pts Correct Reduction construction and explanation for m = V/2.

+ 4 pts Correct Reduction construction and explanation/proof for m < V/2.

+ 4 pts Correct Reduction construction and explanation/proof for m > V/2

+ 0 pts Incorrect/missing

+ 6 pts Proof (Kamiar added this)

+ 6 pts Construction (Kamiar added this)

Question assigned to the following page: 1

CSCI - 570 Analysis of Algorithms
Homework Assignment 11

Name: Abhishek Soundalgekar
Reg No: 2089011000

Question 1
Answer 1

We check if STBD problem is in NP and also NP hard to prove it is NP complete.

Proof for STBD in NP

certificate : A spanning tree T of G.

certifier : Check if the STBD 'T' in G is connected i.e. it covers all vertices of G with $|V|-1$ edges & is acyclic which can be done with BFS/DFS in polynomial time. Check if each vertex in T has a vertex degree which is at most k which is done in $O(|V|)$ time.

Since the checks are done in polynomial time STBD is in NP.

Question assigned to the following page: 1

* Proof that STBD is NP-Hard

HP is NP-complete and we reduce HP to STBD with $K=2$.

Reduction Construction:

HP instance: A graph $G = (V, E)$
STBD instance: The same graph with $K = 2$.

If we can solve STBD for $K=2$ in polynomial time, we could solve hamiltonian problem in polynomial time as well. Thus STBD is at least as hard as hamiltonian problem, making it NP hard for $K=2$.

We can in general say the problem remains NP-complete for any fixed $K \geq 2$ because we can reduce the $K=2$ case to higher K by adding extra degree-forcing gadgets.

Question assigned to the following page: 1

★ Proof of reduction correctness

→ Forward direction
If there exists a HP. in $G = (V, E)$ then each vertex in G is visited exactly once meaning the HP will have vertices with either two edges i.e. one entering edge and one exiting or only one entering or exiting. This means that each vertex has a degree $K \leq 2$ ie K. since Hamiltonian path connects all vertices and is acyclic (since all vertices are visited only once) this is a spanning tree & an STBD with $K \leq 2$.

→ Backward Direction
If G has a spanning tree T with maximum degree 2, ~~it~~ T must be a path (the only connected, acyclic graph with max degree 2). Since T spans all vertices, it is a hamiltonian path.

Therefore STBD problem is NP-Hard as reduction is done in polynomial time:
Hamiltonian Path $\leq_p$ STBD
Since STBD is in NP & NP-Hard, STBD is NP-complete

Question assigned to the following page: 2

Question 2
Answer 2

Certificate: There exists a Hamiltonian cycle C.

Certifier: Check if all vertices in G are visited exactly once and no cycles exist with using BFS/DFS.

Check if sum of edges weights in cycle C is at least half of sum of all edge weights in G, done in O(|V|) time.

Since these checks can be performed in polynomial time w.r.t. input the BHC problem is in NP.

* Proof that BHC is NP-Hard:
Reduction Construction
We will choose Hamiltonian cycle problem which is NP-complete to reduce to BHC.

Given an instance G = (V, E) of hamiltonian cycle convert this to a completed graph

Question assigned to the following page: 2

by adding edges. $G' = (V, E')$

Assign weight 1 to the original edges in E and assign weight 0 to the newly added edges.

Total sum of all edge weights = total of original + fake edges = $m = |E|$

We need to decide & check if a H.C. exists whose sum of weights $\geq m/2$.

If G has Hamiltonian cycle containing only real edges then sum $= |V| = n$ edges

If fake edges are used then sum drops below $|V|$.

G has a hamiltonian cycle iff BHC is found i.e. sum $\geq m/2$.

(i) Forward Proof: If G has a hamiltonian cycle then it uses edges from E only which have weight 1 such that sum $= |V|$. All vertices in G are visited i.e. n. Since $m \geq n$ is connected graph, $n \geq m/2$ which is weight of the cycle.

Question assigned to the following page: 2

Therefore, BHC is found with weight $\geq \frac{m}{2}$.

(ii) Backward Proof: If BHC with sum of weights $\geq m/2$ is found then there exists a hamiltonian cycle in G'. This means there must be at least $m/2$ real edges used since they can only contribute weight 1 to total. But for a hamiltonian cycle we must have $n = |V|$ edges in $G' = (V, E')$ which means all the edges in cycle should be of weight 1 such that sum $= n \geq m/2$.

This means that the BHC is the H.C. is original graph G.

Since the reduction is done in $O(|V|^2)$ time to construct G' i.e. complete graph and proved the BHC is NP-Hard.

Hamiltonian cycle $\leq_P$ BHC.

Therefore, BHC is NP-complete since it is in NP and NP-Hard.

Question assigned to the following page: 3

Question 3
Answer 3

BDHC is in NP:

Certificate: candidate Hamiltonian cycle in the directed bipartite graph.

certifier: check whether it visits every vertex exactly once.
Verify that all edges in the cycle are valid and follow the directed edges.
These checks can be done in polynomial time
$\therefore$ BDHC $\in$ NP.

* Prove that BDHC is NP-hard
Reduction Construction:
Construct $G' = (V', E')$ from $G = (V, E)$ by connecting all vertices to V_{in} and V_{out} with a directed edge between V_{in} and V_{out} in that direction.
We will use DHC which is a known NP-complete.

The V_{in} & V_{out} vertices can be divided into L and R such that $V' = L \cup R$

Question assigned to the following page: 3

$L = \{ V_{1in}, V_{2in}, \text{---} V_{n\,in} \}$

$R = \{ V_{1out}, V_{2out} \text{---} V_{n\,out} \}$

this means there are edges between L and R making G' a bipartite graph as every V_i out has an edge to V_j in if there existed an edge in original G.

(i) F.P. : If there is a directed H.C in G then it corresponds to a path of size 2n in G' since there is as directed edge between V_i in and V_i out. Therefore, this corresponds to a Bipartite directed H.C in G' as it covers all 2n vertices visited only once.

(ii) B.P : If there exists BDHC in G' then there exists a path covering all 2n vertices in G' will go back and forth between L and R but if the Vin and Vout are removed then we will be left with n edges which corresponds to the hamiltonian cycle in G.

Question assigned to the following page: 3

Since, we can make the reduction in polynomial time $O(n)$ the BDHC is in NP hard.

Directed Hamiltonian Cycle $\leq_p$ BDHC

Therefore, since BDHC is in NP and NP hard it is NP-complete

Question assigned to the following page: 4

Question 4
Answer 4

* Proof that Min-3-SAT is in NP

Certificate : We have a 3-SAT assignment

Certifier : We check how many variables are assigned true ($\leq K$)
check that every clause has at least one true literal.

Since both are polynomial checks so Min-3-SAT is in NP.

* Proof that Min-3-SAT is NP-Hard.

We choose Vertex cover which is NP-complete to complete our reduction

* Reduction construction :
consider a graph $G = (V, E)$ such that there is $S \subseteq V$ which is the vertex cover.

consider x_i as a variable for each vertex $v_i \in V$.

Question assigned to the following page: 4

~~Consider~~ For each edge, (V_i, V_j) creates a clause.

$(x_i \vee x_j \vee x_j)$ (with repeating literals to stay 3-literal)

The clause is ~~iff~~ satisfied iff x_i or x_j is assigned true. Thus the resulting assignment has $|V|$ variables and $|E|$ clauses of 3 literals.

(i) F.P. : ~~If~~ Selecting $\leq K$ vertices in vertex cover such that every edge is covered i.e, every clause is satisfied implies that we are satisfying the Min-3-SAT problem clauses.

(ii) B.P. : Satisfying all clauses by setting $\leq K$ variables true means that we have all clauses or edges covered in a vertex cover of size $\leq K$ vertices

Question assigned to the following page: 4

Therefore, a vertex cover of size $\le k$ is possible iff an assignment with $\le k$ variables true that satisfies all clauses. This takes polynomial $O(|E|)$ time.

$\therefore$ Vertex cover $\le_p$ Min-3-SAT is NP-hard.
Since Min-3-SAT is in NP and NP-hard thus Min-3-SAT is in NP-complete.