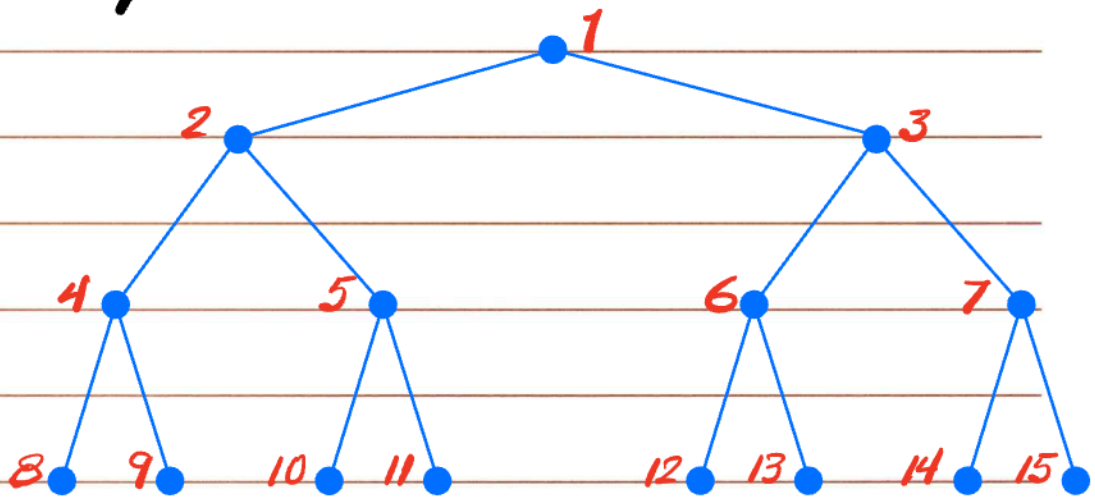
Priority Queues

A priority queue has to perform these two operations fast!

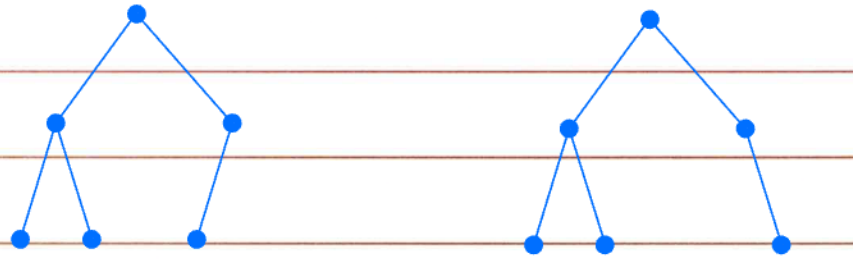
1. Insert an element into the set
2. Find the smallest element in the set

	Insert	Find Min
Array Implementation	O(1)	O(n)
Sorted array "	O(n)	O(1)
Linked list "	O(1)	O(n)
Sorted linked list "	O(n)	O(1)

Def. A binary tree of depth k which has exactly $2^k - 1$ nodes is called a full binary tree.



Def. A binary tree with n nodes and of depth k is complete iff its nodes correspond to the nodes which are numbered 1 to n in the full binary tree of depth k.



Complete

Not Complete

Traversing a complete binary tree stored as an array.

Parent(i) is at $\lfloor i/2 \rfloor$ if $i \neq 1$
if $i = 1$, i is the root

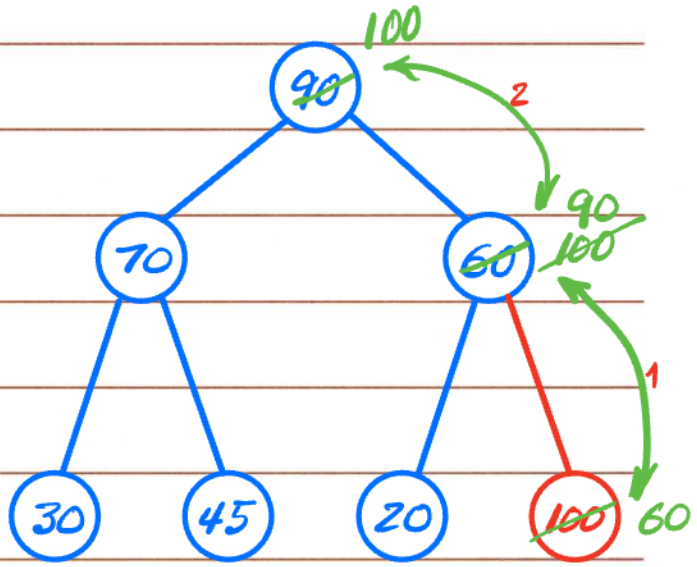
Lchild(i) is at $2i$ if $2i \leq n$
otherwise, it has no left child

Rchild(i) is at $2i+1$ if $2i+1 \leq n$
otherwise, it has no right child

Def. A binary heap is a complete binary tree with the property that the value of the key at each node is at least as large as the key values at its children (Max heap)

Find_Max

Takes O(1)



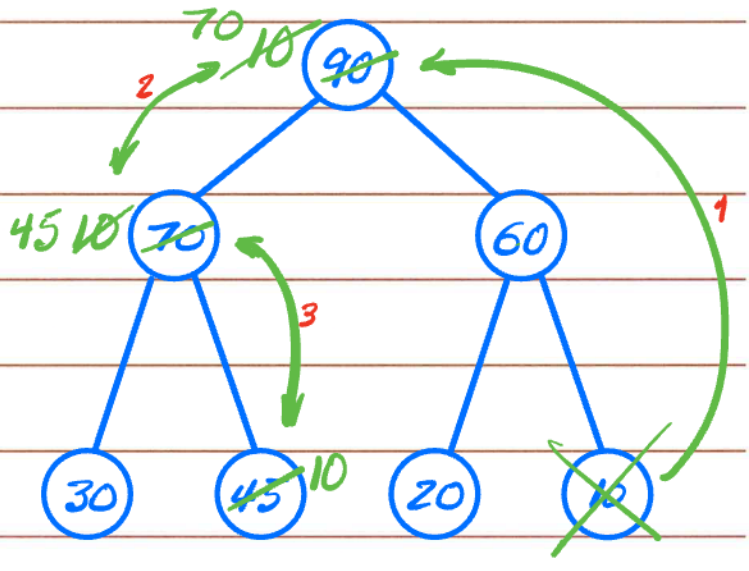
Insert

ex. insert (100)

Takes O(lgn)

Extract_Max

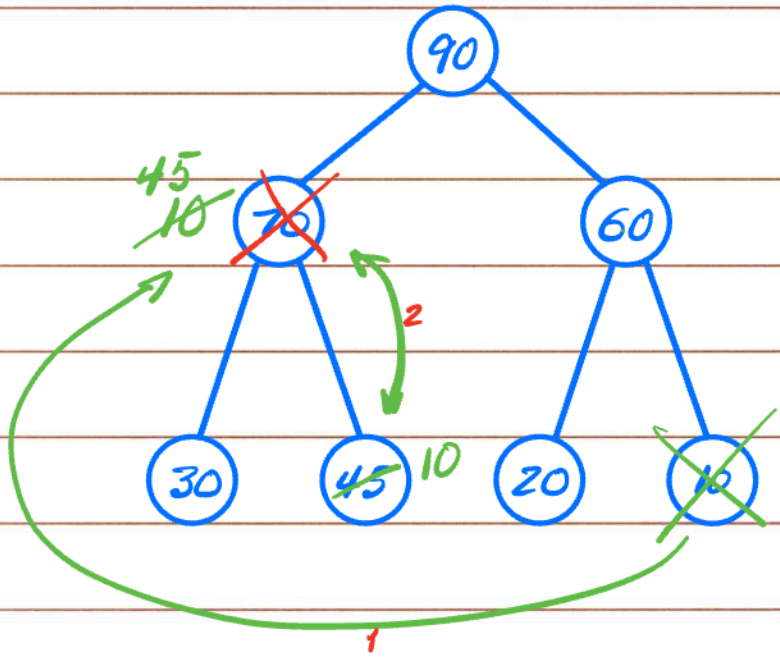
Takes O(lgn)



Delete

ex. Delete (70)

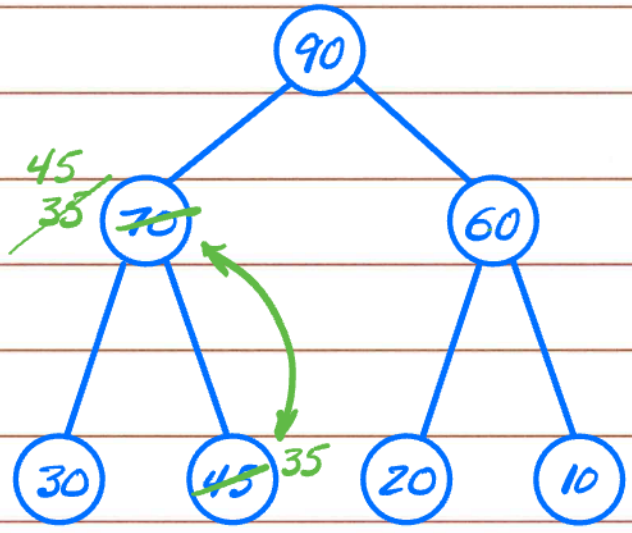
Takes O(lgn)



Decrease-key

ex. Decrease-key (70) to (35)

Takes O(lgn)

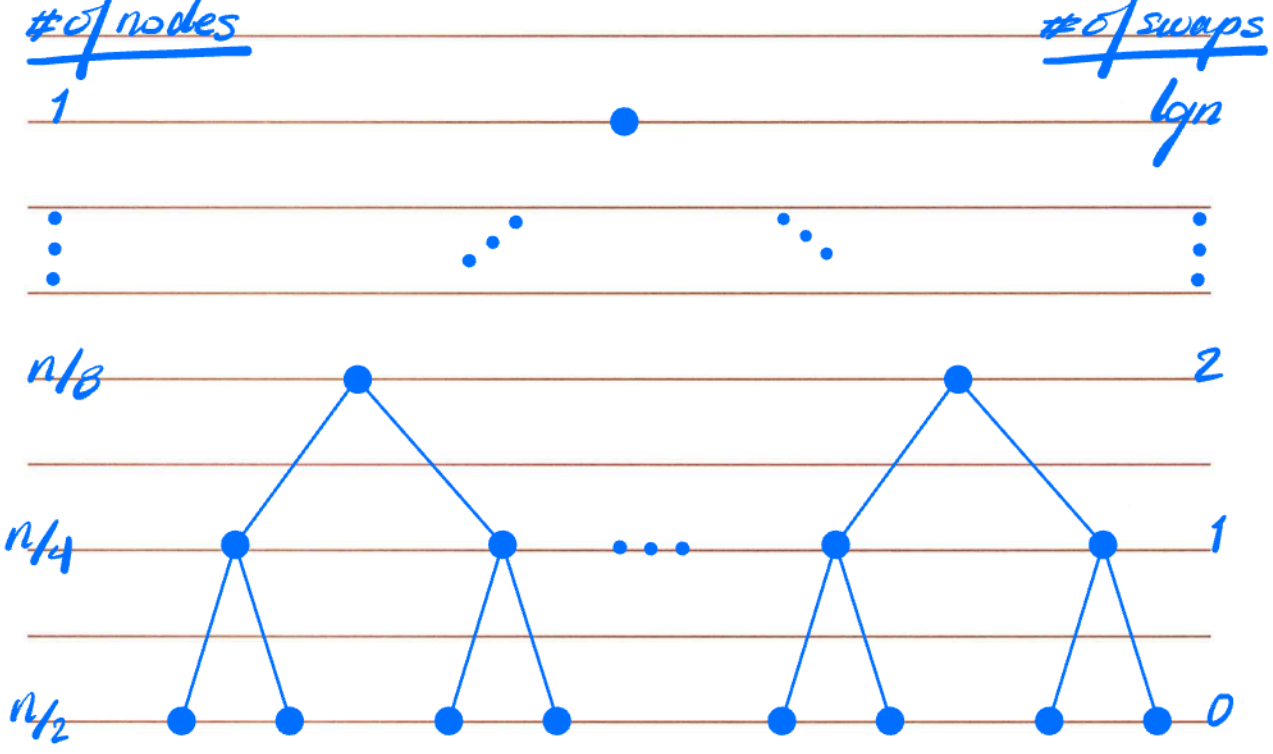


Construction

Can be done in $O(n \lg n)$ time using n insert operations.

Can we do better?

Bottom up construction



T = Maximum number of swaps needed

$T = n/4 * 1 + n/8 * 2 + n/16 * 3 + \dots$

$T/2 = n/8 * 1 + n/16 * 2 + n/32 * 3 + \dots$

$T - T/2 = \underbrace{n/4 * 1 + n/8 * 1 + n/16 * 1 + \dots}_{n/2}$

$T/2 = n/2$

$T = n$

Bottom up construction takes $O(n)$

Q: What is the best run time to merge two binary heaps of size n?

A: $O(n)$ using linear time construction

Finding the top k elements in an array.

Input: An unsorted array of length n.

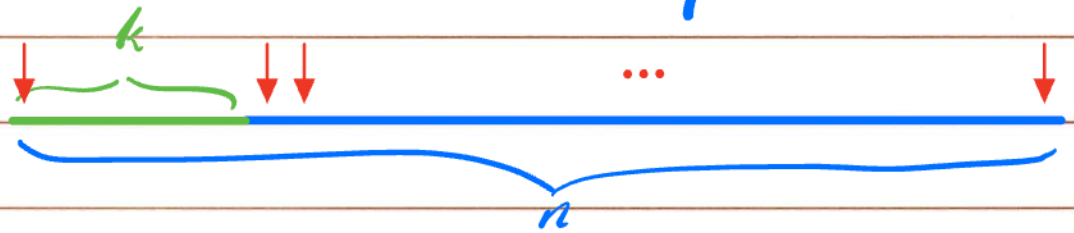
Output: Top k values in the array ($k < n$)

Constraints:

- You cannot use any additional memory
- Your algorithm should run in time $O(n \lg k)$

Solution:

- Construct a Minheap of size k.



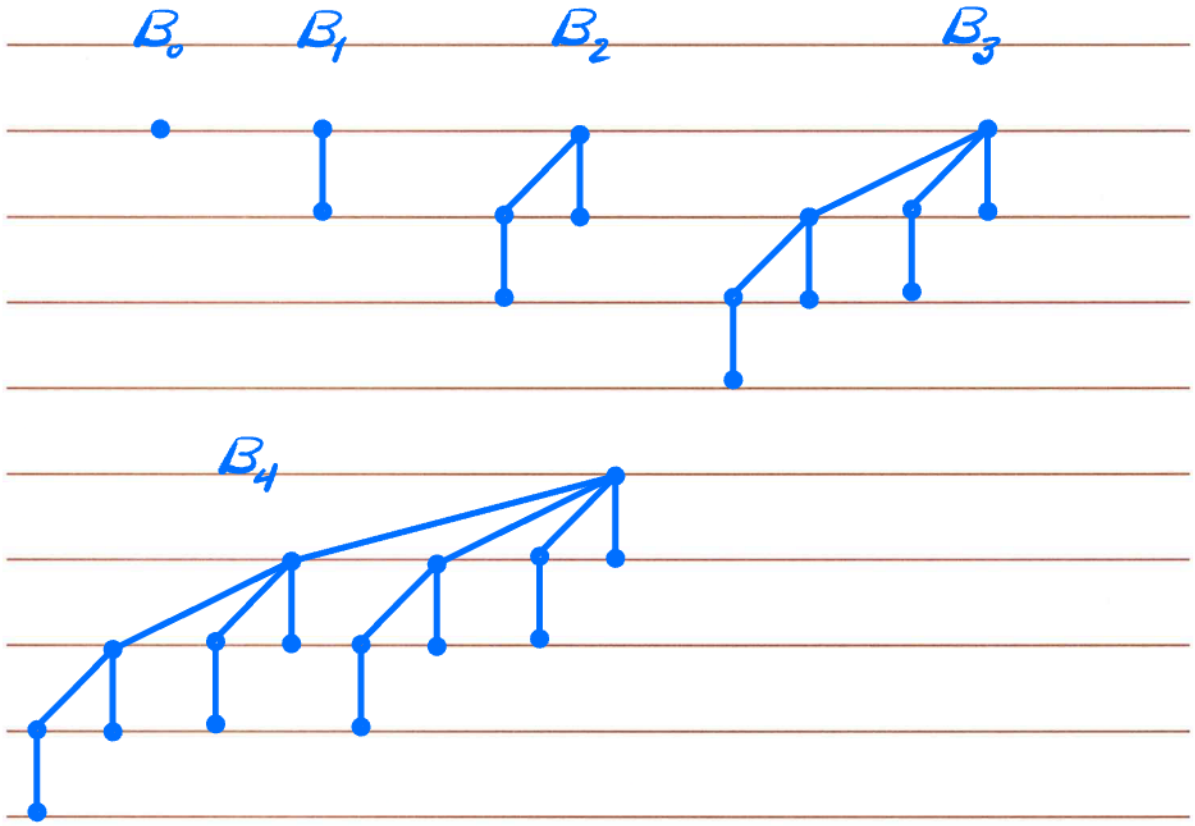
- Move through the rest of the $(n-k)$ elements. If we encounter an element larger than the element at the top of our minheap, swap them. This involves extract-min and insert ($O(\lg k)$)

Overall complexity $= O(k) + O((n-k)\lg k) = O(n \lg k)$

Def. A binomial tree B_k is an ordered tree defined recursively

- Binomial tree B_0 consists of one node

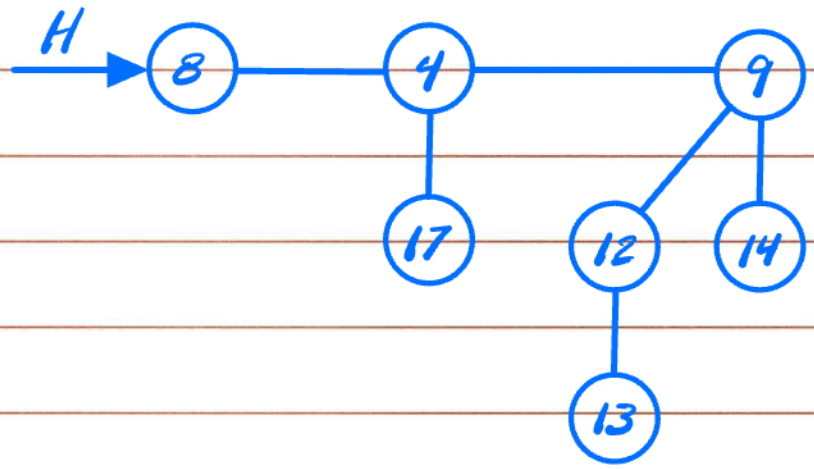
- Binomial tree B_k consists of 2 binomial trees B_{k-1} that are linked together such that root of one is the leftmost child of the root of the other.



Def. A binomial heap H is a set of binomial trees that satisfies the following properties:

1. Each binomial tree in H obeys the min-heap property.

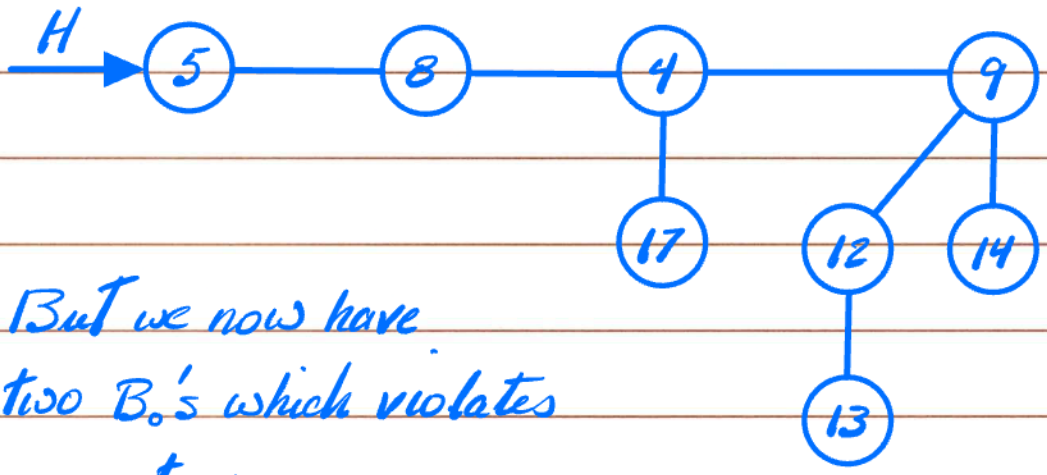
2. For any non-negative integer k, there is at most one binomial tree in H whose root has degree k.



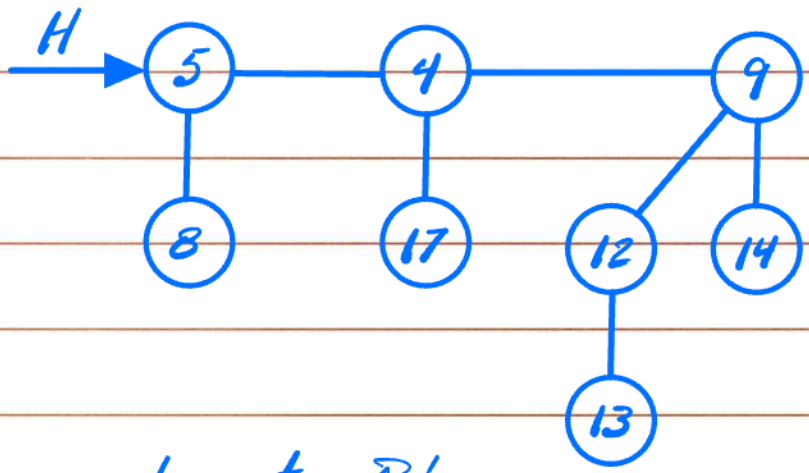
Find-min takes $O(\lg n)$

Insert

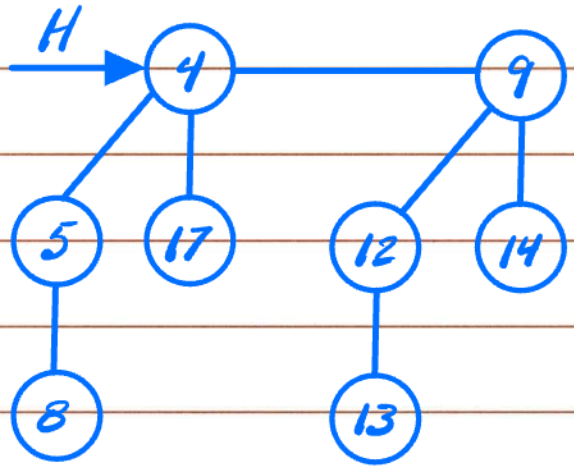
ex. insert 



But we now have two B_0's which violates property #2.

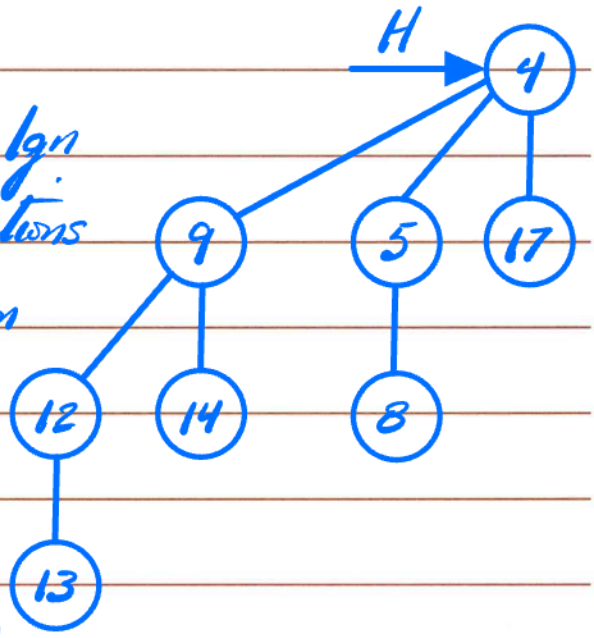


Now we have two B_1's...



Now we have two B_2's...

- Insert requires at most lgn binomial tree merge operations
- Each tree merge operation takes O(1)
- So the worst case complexity of the insert operation in a binomial heap is O(lgn)



Amortized Cost Analysis

Ex. 1 Starting from an empty stack

```
for i = 1 to n
    Push or Pop
endfor
```

Push and Pop take $O(1)$ each

Worst case runtime complexity = $O(n)$

Ex. 2 Starting from an empty stack

```
for i = 1 to n
    Push or Pop or Multipop
endfor
```

Push and Pop take $O(1)$ each (worst case)

Multipop takes $O(n)$ (worst case)

Worst case runtime complexity = $O(n^2)$

But is $O(n^2)$ a tight upper bound?

Aggregate Analysis

- Specify the set of operations involved.

- Show that a sequence of n operations (for all n) takes worst case time $T(n)$ total.

- In the worst case, the amortized cost (average cost) per operation will be $T(n)/n$

Back to Ex. 2

Observation 1 : Multipop takes $O(n)$ time if there are n elements pushed on the stack.

I.e. the total number of pops cannot be greater than the total number of pushes.

Observation 2 : We can only do $O(n)$ pushes

So the sequence of n Push, Pop or Multipop operations will take $O(n)$ time in the worst case

$\therefore$ Amortized cost of each operation $= \frac{O(n)}{n} = O(1)$

Ex. 2 Starting from an empty stack

for $i = 1$ to n

Push or Pop or Multipop

endfor

Push takes $O(1)$ amortized

Pop " $O(1)$ "

Multipop " $O(1)$ "

Worst case runtime complexity = $\Theta(n)$

Accounting Method

- Assign different charges to different operations.

- If the charge for an operation exceeds its actual cost, the excess is stored as credit.

- The credit can later help pay for operations whose actual cost is higher than their amortized cost.

- Total credit at any time =

$$\underbrace{\text{Total amortized cost} - \text{Total actual cost}}_{\text{total charges}}$$

- Total credit can never be negative.

Ex. 2 Starting from an empty stack

```
for i = 1 to n
        Push or Pop or Multipop
endfor
```

try #1: assign a charge of 1 to each operation

Op.	Charge	Actual Cost	Total Credit
Push	1	1	0
Push	1	1	0
Multipop	1	2	-1

try #1: assign charges as follows:

Push 2 → O(1)

Pop 0 → O(1)

Multipop 0 → O(1)

Op.	Charge	Actual Cost	Total Credit
Push	2	1	1
Push	2	1	2
Push	2	1	3
Multipop	0	3	0

Ex. 2 Starting from an empty stack

for $i = 1$ to n

Push or Pop or Multipop

endfor

Push takes $O(1)$ amortized

Pop " $O(1)$ "

Multipop " $O(1)$ "

Worst case runtime complexity = $\Theta(n)$

Fibonacci Heaps

- Fibonacci heaps are loosely based on binomial heaps.

- A Fibonacci heap is a collection of min-heap trees similar to binomial heaps. However, trees in a Fibonacci heap are not constrained to be binomial trees.

- Unlike binomial heaps, trees in Fibonacci heaps are not ordered.

See Fibonacci heap animation at:

www.cs.usfca.edu/~galles/JavascriptVisual/FibonacciHeap.html

Worst case costs

Amortized costs

	Binary Heap	Binomial Heap	Fibonacci Heap
Find_Min	$O(1)$	$O(\lg n)$	$O(1)$
Insert	$O(\lg n)$	$O(\lg n)$	$O(1)$
Extract_Min	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$
Delete	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$
Decrease-key	$O(\lg n)$	$O(\lg n)$	$O(1)$
Merge	$O(n)$	$O(\lg n)$	$O(1)$
Construct	$O(n)$	$O(n)$	$O(n)$

3. The values 1, 2, 3, . . . , 63 are all inserted (in any order) into an initially empty min-heap. What is the smallest number that could be a leaf node?

5. Suppose you have two min-heaps, A and B, with a total of *n* elements between them. You want to discover if A and B have a key in common. Give a solution to this problem that takes time O(n log n) and explain why it is correct. Give a brief explanation for why your algorithm has the required running time. For this problem, do not use the fact that heaps are implemented as arrays; treat them as abstract data types.