Homework 11

CS570 Spring 2025

Due: Apr 25, 2025

1. (**14 points**) Consider the problem Spanning Tree with Bounded Degree (STBD): Given an undirected graph $G = (V, E)$ and integer $k > 0$, determine whether there is a spanning tree where the maximum vertex degree *in the tree* is not greater than k, i.e, each vertex has at most k neighbors within the tree, but may have other neighbors in G overall. Prove that STBD is NP-complete.

2. (**15 points**) Given an undirected graph with positive edge weights, the BIG HAM CYCLE (BHC) problem is to decide if it contains a Hamiltonian cycle C such that the sum of weights of edges in C is at least half of the total sum of weights of edges in the graph. Show that BHC is NP-complete.

3. (**15 points**) In the Bipartite Directed Hamiltonian Cycle (BDHC) problem, we are given a directed graph $G = (V, E)$ that is bipartite, i.e., V can be partitioned as $L \cup R$ s.t. each edge goes from 'L to R' or 'R to L', and we are asked whether there is a simple cycle which visits every vertex exactly once. It is known that the Directed Hamiltonian Cycle is an NP-complete problem. Prove that BDHC is NP-Complete as well.

4. (**16 points**) Suppose we have a variation of the 3-SAT problem called Min-3-SAT, where the literals are never negated. Of course, in this case it is possible to satisfy all clauses by simply setting all variables to true. But, we are additionally given a number k, and are asked to determine whether we can satisfy all clauses while setting at most k variable to be true. Prove that Min-3-SAT is NP-Complete.

Ungraded Problems

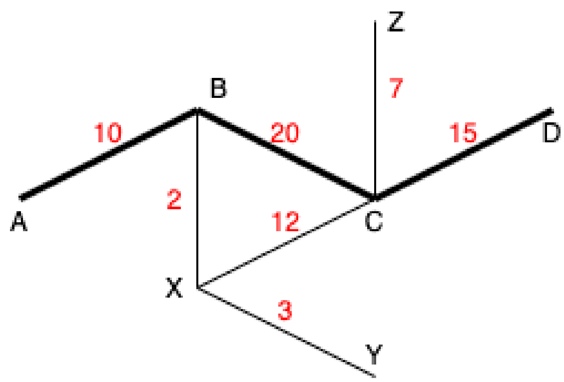
5. The government wants to build a multi-lane highway across the country. The plan is to choose a route and rebuild the roads along this route. We model this problem with a simple weighted undirected graph G with the nodes denoting the cities and the edges capturing the existing road network. The weight of an edge denotes the length of the road connecting the corresponding two cities (assumed positive).

   Let d_{uv} denote the shortest path distance between nodes u and v.

   Let $d(v, P)$ denote the shortest path distance from a node v to the *closest* node on a path P (i.e. $\min_{u \in P} d_{uv}$).

   Finally, we define the *aggregate remoteness* of P as $r(P) = \sum_{v \in V} d(v, P)$.

   In the example shown in the figure below, path $P = ABCD$ is a potential highway. Then, $d(A, P) = d(B, P) = d(C, P) = d(D, P) = 0$, while $d(X, P) = d_{XB} = 2$, $d(Y, P) = d_{YB} = 5$, and, $d(Z, P) = d_{ZC} = 7$. The remoteness of path ABCD is $0 + 0 + 0 + 0 + 2 + 5 + 7 = 14$.



   The government wants a highway with the minimum aggregate remoteness, so that all the cities are somewhat close to the highway. Formally, we state the problem as follows, "Given a graph G, and a non-negative number k, does there exist a path P in G having remoteness $r(P)$ at most k"? Show that this problem REMOTE-PATH is NP-complete.

6. For any positive k, the graph k-coloring problem is stated as follows: Determine if the vertices of a given graph G can be colored using k colors such that no two adjacent vertices have the same color. The graph 3-coloring problem is known to be NP-complete. Prove that the 5-coloring problem is NP-complete.

7. You are given a directed graph $G = (V, E)$ with weights on its edges $e \in E$. The weights can be negative or positive. The Zero-Weight-Cycle Problem is to decide if there is a simple cycle in G so that the sum of the edge weights on this cycle is exactly 0. Prove that this problem is NP-complete. Hint: For NP-hardness, reduce from the Subset sum problem that can be stated as "Given the set of positive numbers $S = \{w_1, ..., w_n\}$ and target sum $W > 0$, is there is a subset of S that adds up to exactly W ?".

8. The Directed Disjoint Paths Problem is defined as follows. We are given a directed graph G and k pairs of nodes $(s_1, t_1), (s_2, t_2), ..., (s_k, t_k)$. The problem is to decide whether there exist node-disjoint paths $P_1, P_2, ..., P_k$ so that P_i goes from s_i to t_i. Show that Directed Disjoint Paths is NP-complete.