Homework 10

● Graded

Student

Abhishek Soundalgekar

Total Points

60 / 60 pts

Question 1

Q1

Resolved **15** / 15 pts

✔ **+ 15 pts** Correct

+ 4 pts Correctly showing NP

+ 5 pts Correctly constructing and showing reduction

+ 6 pts Proof in both directions

+ 0 pts Incorrect

Regrade Request **Submitted on: Apr 23**

Hi
I have Correctly constructed and showed reduction and also proved in both directions. Can you please recheck it once.
Thank you so much.

Thats odd, sorry about that.
Added the points!
Also please improve your handwriting

Reviewed on: Apr 24

Question 2

Q2

16 / 16 pts

✔ **– 0 pts** Correct

– 6 pts Incorrect construction of DSP instance from Vertex-Cover

– 5 pts Incorrectly explains/proves forward direction

– 5 pts Incorrectly explains/proves the backward direction

Question 3

Q3 — **15** / 15 pts

✔ **+ 15 pts** Correct

+ 4 pts Correctly shows in NP

+ 5 pts Correctly constructs and explains the reduction.

+ 6 pts Proof in both directions.

+ 0 pts Wrong ans

Question 4

Q4

14 / 14 pts

✔ **+ 14 pts** Full points

+ 4 pts Showing the problem is in NP

+ 2 pts Correct Reduction construction and explanation for $m = V/2$.

+ 4 pts Correct Reduction construction and explanation/proof for $m < V/2$.

+ 4 pts Correct Reduction construction and explanation/proof for $m > V/2$

+ 0 pts Incorrect/missing

Question assigned to the following page: 1

CSCI-570 Homework Assignment 10

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Question 1

Answer 1

(a) Prove the SRL problem is in NP:

Given:

m food items, n restaurants and an integer k, every food item is offered by one remaining restaurant.

Certificate :

A certificate will be a subset of n-k or fewer restaurants that will remain after removing atleast k restaurants out of n.

Certifier :

checks in polynomial time whether:

i) Number of restaurants in the proposed subset is $\leq n-k$.

ii) Union of food items offered in all these n-k restaurants covers all 'm' favourite food items.

Here (i) & (ii) can both be done in polynomial time i.e. $O(mn)$ and $O(m)$. Therefore, SRL is in NP.

Question assigned to the following page: 1

[b.] Prove that SRL problem is NP-Hard :

We will be using vertex cover and reducing to SRL to prove that it is NP-Hard. Construct SRL instance from vertex cover where G is a graph such that $G = (V, E)$, $|V| = n$ restaurant and $|E| = m$ food items. Edges correspond to food items and vertices are restaurants. Let $k = n - l$ (l is an integer) restaurants such that removing k restaurants can still cover all edges i.e. food items.

Proof of correctness of reduction :

(i) Suppose we have vertex cover $C \subseteq V$ of size l. Then removing $n-l$ restaurants i.e. $n-l$ nodes not in C will still give us a solution that covers all food items i.e. all m edges in E such that each food item is covered by at least one restaurant.

(ii) If we remove $k = n-l$ restaurants and are able to cover all m food items i.e. 'm' edges then the remaining l restaurants (nodes) must include at least one restaurant which offers each food item. This constitutes a vertex cover of size l.
Therefore Vertex cover $\leq_p$ SRL.
Therefore, SRL is NP-Hard. Since SRL is both in NP & NP-Hard, SRL is NP-complete.

Question assigned to the following page: 2

Question 2
Answer 2

Given
A VC instance $G = ((V, E), k)$ we can construct in polynomial time a DSP instance $G' = ((V', E'), r)$ such that G has a vertex cover of size $\leq k$ iff G' has a dominating set of size $\leq r = k$.

Reduction :
Let the input to the VC problem be graph $G = (V, E)$ with no disconnected vertices, and integer k.
$G' = (V', E')$ for the DSP instance can be constructed as :
V' : start with all original vertices V from G for each edge $e = (u, v) \in E$ introduce a new vertex $\rightarrow$ - w_e. $V' = V \cup \{ w_e : e \in E \}$

E' : For each new vertex w_e, connect it to both endpoints of the corresponding edge :
$E' = E \cup \{ (w_e, u), (w_e, v)$ for each $e = (u, v) \in E \}$

this means all the original edges remain and every new vertex w_e is connected to the endpoints of its corresponding edge. Let the target size for dominating set be $r = k$.

Question assigned to the following page: 2

Proof of correctness :
G has a vertex cover of size at most k iff G' has a dominating set of size $\leq k$.
Suppose $C \subseteq V$ is a vertex cover of size $\leq k$ in G:
For every $(u,v) \in E$, either $u \in C$ or $v \in C$.
Then for each added vertex w_e in G', at least one of its neighbours (either u or v) is in C. Since the new vertices w_e are connected only to u & v, and at least one of those is in C, it follows that every w_e is dominated. Also since all vertices in V are either in C or connected via edges in G and G has no isolated vertices all $v \in V$ are also dominated. So C is a dominating set in G', and $|C| \leq k = r$, $\therefore G$ has a vertex cover means G' has a dominating set.

Suppose $B \subseteq V'$ is a dominating set in G' with $|B| \leq k$, New vertex cover of size at most k for G is :
$w_e \in B$, then we can replace it with one of its neighbours (either u or v) since adding u or v also dominates w_e & may dominate additional vertices in V'.

Question assigned to the following page: 2

By replacing each such w_e with one of its neighbours we get a dominating ~~additional~~ ~~vertices~~ set $B' \subseteq V$ of size $\leq k$.

If we claim B' is a vertex cover of G:

(i.) For each edge $e = (u, v)$ there is a w_e in G'.
(ii.) Since B' dominates all w_e and w_e is only adjacent to u and v it means at least u or v is in B'.
(iii.) Thus, each edge is covered by at least one end point in B'. Therefore, G' has a dominating set implies that G has a vertex cover. Since we have $O(|E|)$ time reduction from vertex cover to dominating set preserving solution size. Therefore $VC \leq_P DSP$ means DSP is NP-Hard.

Question assigned to the following page: 3

Question 3
Answer 3

We have n actors, m movies and a target number k. We need to choose at least k actors such that no two selected actors have ever acted together in the same movie.

(a) NCC is in NP:
Certificate: A set of k or more actors
Certifier: checks for the following:
(i.) Build list of actors appeared in that movies.
(ii) For the given actor subset check every pair in the set and verify that no pair has appeared in a movie.
(iii) This takes $O(k^2 m)$ time to check all k^2 actor pairs across m movies.
Since this is polynomial time NCC is in NP.

(b.) NCC is NP-Hard:
To reduce we will choose the Independent set problem which is NP-complete.
Given a graph $G = (V, E)$ construct NCC as follows:
(i.) Each vertex $v \in V$ is an actor, $|V| = n$
(ii.) Each edge, $e \in E$ is a movie of actors u and v such that $e = (u, v)$, $|E| = m$.

Question assigned to the following page: 3

Therefore, actor u appears in movie e if u is an endpoint of e. We claim that independent set of size at least k exists iff there is a subset of at least k actors in NCC such that no two have acted together in any movie.

If independent set exists then let $S \subseteq V$, $|S| \geq k$, and no two vertices in S are adjacent. then in the NCC instance, no two corresponding actors have acted together in any movie. So this is a valid NCC solution since actors are represented as nodes and they do not have movie edge in between.

If NCC solution exists then if we have k actors who have never acted together in any movie. then corresponding nodes in graph cannot be conneted by an edge since they don't have a movie together. this constitutes all vertices in an independent set in G. Therefore, NCC is NP-hard. Since NCC is in NP and in NP-hard, NCC is NP-complete.

Independent set $\leq_P$ NCC.

Question assigned to the following page: 4

Question 4
Answer 4

(a.) Proof that Half-clique is in NP:
To prove this, we must show that given certificate:
Certificate: A subset of vertices $A \subseteq V$, $|A| \geq n/2$.

Certifier: (i.) check that $|A| \geq n/2$
(ii.) For each pair $(u,v) \in A$, verify $(u,v) \in E$. There are at most $(n^2) = O(n^2)$ checks.
Since this is in polynomial time, Half-clique is in NP.

(b.) Half-clique is NP Hard:

We choose NP-Hard problem CLIQUE to reduce to HALF-CLIQUE.
Let $G = (V, E)$, $|V| = n$ and integer m.
We construct a graph $G' = (V', E')$ such that:
G' has a half-clique (i.e., clique of size $|V|/2$) if and only if G has a clique of size at least m.
Lets assume: If $m \geq n/2$, then this is already a half-clique instance. But if $m < n/2$ we need to pad the graph such that size-m clique in G becomes a half-clique in G'.

Question assigned to the following page: 4

(i) Let $G = (V, E)$, $|V| = n$, target $m < n/2$.
(ii) Add $2m - n$ isolated vertices to G, to form G'.
Let S be the set of added vertices, $|S| = 2m - n$.
Let $V' = V \cup S$, $E' = E$ (no new edges added - S are isolated)

Now $|V'| = n + (2m - n) = 2m$ so half of $|V'| = m$.
We now prove:
G has clique of size $\geq m$ iff G' has half-clique (clique of size $\geq m$)

If G has a clique $C \subseteq V$; $|C| \geq m$
(i) then C is also a clique in G'.
(ii) since $|V'| = 2m$, this is a half-clique
If G' has a half-clique, C', $|C'| \geq m$
(i) since the added vertices S are isolated, they cannot be in any clique of size > 1.
(ii) so all vertices in the clique must be from V.
(iii) Hence, G has unique clique of size at least m.

Thus, solving half-clique on G' solves clique on G.
Half-clique is in NP. Half-clique is NP-hard.
Therefore, Half-clique is NP-complete.

Question assigned to the following page: 4

(b) To prove HALF-CLIQUE is NP-hard by reducing from the CLIQUE problem:

Suppose we are given an instance of the CLIQUE problem, which consists of a graph G = (V, E) and an integer m. We now define a corresponding instance of the HALF-CLIQUE problem from this.

We examine the following two cases:

Case 1: m≥|V|/2m

Add 2m-n isolated vertices to G' such that the total number of vertices in G' becomes 2m.

- In this construction, a clique of size m in the original graph G corresponds to a half-clique of size |V|/2 in G', since the additional vertices are isolated and thus not connected to any other vertex.

This satisfies the HALF-CLIQUE condition.

Case 2: m<|V|/

Add a new set U of 2m−|V| vertices to the graph.

- For every vertex in U, connect it to every vertex in the original graph G.
- The new graph G' has |V|+(2m−|V|)=2m|V| + (2m - |V|) = 2m vertices.

Proof A:
If G has a clique A of size m, then the set A∪U is a clique in G' of size |A| + |U| = m+n-2m
This is exactly half the number of vertices in G'. Thus, the HALF-CLIQUE instance is satisfied.

Proof B:
Suppose G' has a half-clique A' of size |V|/2=|V| - m then A' <= vertices in G'. So the half clique will contain wo parts:
1) x vertices from newly added set U.
2) y vertices from original graph.

Since A' is a half clique we know that x+y>=n-m (|v|=n)…..1

We know that x<= n-2m (U contains n-2m vertices newly added)
Multiplying both sides by -1 we get
-x>=2m-n ……….2

Adding 1 and 2 we have
y>= (n-m) + (2m-n)
y>=m

Question assigned to the following page: 4

So, A’ contains at least m original vertices and since A’ itself is a clique, the original vertices and since A’ itself is a clique, the original vertices are mutually adjacent in the original graph G. Hence G has a clique of size m.

Therefore, we conclude that the original graph G has a clique of size m if and only if the new graph G' has a half-clique of size |V|/2

This completes the reduction, proving that CLIQUE ≤p HALF-CLIQUE, and hence, HALF-CLIQUE is NP-hard.