CS570

Analysis of Algorithms

Summer 2015

Exam I

Name: ______________________

Student ID: __________________

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_____Check if DEN Student

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Problem 1	20	
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Problem 6	16	
Total	100	

Instructions:

1. This is a 2-hr exam. Closed book and notes
2. If a description to an algorithm or a proof is required please limit your description or proof to within 150 words, preferably not exceeding the space allotted for that question.
3. No space other than the pages in the exam booklet will be scanned for grading.
4. If you require an additional page for a question, you can use the extra page provided within this booklet. However please indicate clearly that you are continuing the solution on the additional page.

20 pts

Mark the following statements as **TRUE** or **FALSE**. No need to provide any justification.

[TRUE]
$2n^3 + \Theta(n^2) = \Theta(n^3)$

[TRUE]
Gale-Shapley algorithm is a greedy algorithm.

[TRUE]
There is a path from any point to any other point in a connected undirected graph

[TRUE]
All operations in a binomial heap have a worst case cost of O(lg n) except for the construction operation.

[FALSE]
All operations in a Fibonacci heap have a worst case cost of O(lg n) except for the construction operation.

[FALSE]
All operations in a binary heap have a worst case cost of O(lg n) except for the construction operation.

[TRUE]
The shortest path between two points in a graph could change if the weight of each edge is increased by an identical number.

[TRUE]
In class, we showed that choosing requests with earliest finish time will lead to an optimal solution to the interval scheduling problem. An optimal solution can also be achieved by choosing requests with latest start time.

[TRUE]
The relaxation step in Dijkstra's shortest path algorithm will have a lower worst case time complexity if we used a Fibonacci heap as opposed to a binary heap.

[FALSE]
BFS can be used to find the shortest path between two nodes in any graph as long as the edge costs are all positive.

2) 16 pts
Given a directed graph with m edges and n nodes where every edge has weight as either 1 or 2, find the shortest path from a given source vertex 's' to a given destination vertex 't'. Expected time complexity is O(m+n).

We can modify the graph and split all edges of weight 2 into one vertex and two edges of weight 1 each, like edge (u,v) of weight 2 to (u,u') of weight 1 and (u',v) of weight 1. In the modified graph, we can use BFS to find the shortest path. Maximum number of new edges and vertices added is O(E), so complexity of this approach is O(V+E).

3) 16 pts

Arrange the following functions in increasing order of asymptotic complexity. If f(n)=Θ(g(n)) then put f=g. Else, if f(n)=O(g(n)), put f < g.

$n^{\pi}\log_2 n$, $\left(\frac{n}{2}\right)^{\frac{n}{2}}$, $n^{\frac{22}{7}}$, $n^{\pi}\log_{\pi} n$, $\log^3 n$, $nlog(n)$, $1.1^{\sqrt{n}}$. (π = 3.1415…)

$$\theta(\log^3 n) < \theta(n\, logn) < \theta(n^{\pi}\log_2 n) = \theta(n^{\pi}\log_{\pi} n) < \theta\left(n^{\frac{22}{7}}\right) < \theta(1.1^{\sqrt{n}}) < \theta\left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)$$

4) 16 pts

For this problem, we will explore the issue of truthfulness in the Stable Matching Problem and specifically in the Gale-Shapley algorithm. The basic question is: Can a man or a woman end up better off by lying about his or her preferences? More concretely, we suppose each participant has a true preference order. Now consider a woman w. Suppose w prefers man m to m', but both m and m' are low on her list of preferences. Can it be the case that by switching the order of m and m' on her list of preferences and running the algorithm with this false preference list, w will end up with a man m'' that she truly prefers to both m and m'?

Resolve this question by doing one of the following two things:

- Give a proof that, for any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the Gale-Shapley algorithm; or
- Give an example of a set of preference lists for which there is a switch that would improve the partner of a woman who switched preferences.

Yes, it's possible. Consider the following instance with n=3 men and women. Woman w3 will lie in this instance; the first six columns are true preference lists, and the final one is w3's false, but stated, preference list. For the sake of this example, let's assume that G-S breaks ties by using the lowest-numbered unmatched man to ask; similar examples exist for other tiebreakers.

m1	m2	m3	w1	w2	w3	w3'
w3	w1	w3	m1	m1	m2	m2
w1	w3	w1	m2	m2	m1	m3
w2	w2	w2	m3	m3	m3	m1

Initially, with the listed tie-breaker, Gale-Shapley will produce the pairs (m1,w3) (m2, w1) and (m3,w2). However, if w3's false preference list is used (and the other five remain truthful), we are left with (m1,w1), (m2,w3), and (m3,w2) -- leaving w3 with her truly first choice.

5) 16 pts

The diameter of a graph is defined as the distance between the pair of vertices which have the maximum shortest distance among all the pairs of vertices. Prove or disprove that there exists an undirected graph with diameter $2^{2015} + 1$ such that it has a BFS tree of depth 2^{2014} 2^{2014}. (Note: BFS tree with only one node has depth of 0)

Suppose such a graph exists. Let r be the root of the BFS tree. Let u and v be the vertices that are furthest apart in the graph.

Diameter = dist(u, v) $\leq$ dist(u, r) + dist(v, r)

Since, BFS tree provides the shortest paths from the root r, none of these distances can exceed the depth.

$$\text{dist}(u, r) + \text{dist}(v, r) \leq 2^{2014} + 2^{2014}$$
$$\leq 2^{2015}$$

leading to

$$2^{2015} + 1 \leq 2^{2015}$$

which is absurd. Hence, such a graph cannot exist.

6) 16 pts

There are *n* types of valuable items with type 1, type 2, …, type *n* valued at $v_1, v_2, \ldots, v_n$ respectively. You are allowed to pick k_1 items of one type, k_2 items of another, and so on. How should you match the sets $\{v_1, v_2, \ldots, v_n\}$ and $\{k_1, k_2, \ldots, k_n\}$, so that the net value is maximum. In other words, find a permutation Π that maximizes $\sum_i v_i k_{\prod(i)}$. Also, show that your strategy is optimal.

The strategy is to pick most number of best valued item. Sort the two lists in increasing order. Let the sorted sequence corresponding to $\{v_1, v_2, \ldots, v_n\}$ be $(V_1, V_2, \ldots, V_n)$ and that corresponding to $\{k_1, k_2, \ldots, k_n\}$ be $(K_1, K_2, \ldots, K_n)$. Then one should pick K_1 items of value V_1, K_2 of $V_2, \ldots, K_n$ of V_n.

The proof is based on proving that no other permutation on $\{K_1, K_2, \ldots, K_n\}$ can result in a higher value. We prove it by induction on n. When the list is of size 1 (n=1), the optimality of the strategy is trivial. For n = 2, we can prove the claim as follows - since $(K_1 - K_2)$ and $(V_1 - V_2)$

$(V_2 - V_1) \times (K_2 - K_1) \geq 0$

=) $V_1 K_1 + V_2 K_2 \geq V_1 K_2 + V_2 K_1$.

Suppose the strategy is optimal for n = j items in the sequences. Assume that the optimal strategy is

$$\sum_{i=1}^{j+1} V_i K_{\sigma(i)}$$

where σ is a permutation over {1, 2, . . . , j + 1}. Let σ (p) = j + 1. Then using the result from n = 2 with increasing sequences (Vp, Vp+1) and $(K_{\sigma(p)+1}, K_{\sigma(p)})$

$V_p K_{\sigma(p)} + V_{p+1} K_{\sigma(p)+1} \leq Vp K_{\sigma(p)+1} + V_{p+1} K_{\sigma(p)}$

$V_1 K_{\sigma(1)} + \ldots + (V_p K_{\sigma(p)} + V_{p+1} K_{\sigma(p)+1}) + \ldots V_{j+1} K_{\sigma(j+1)}$

$\leq V_1 K_{\sigma(1)} + \ldots + (V_p K_{\sigma(p)+1} + V_{p+1} K_{\sigma(p)}) + \ldots + V_{j+1} K_{\sigma(j+1)}$

which means swapping $\sigma(p)$ with $\sigma(p+1)$ in the permutation leads to a higher
total. Proceeding thus we end up with p to be the rightmost element, i.e., $\sigma(j+1)$ = j + 1.

Additional Space

Additional Space