Homework 4

1. [10 points] Design a data structure that has the following properties (assume n elements in the data structure, and that the data structure properties need to be preserved at the end of each operation):
   - Find median takes O(1) time
   - Insert takes O(log n) time

   Do the following:

   1. Describe how your data structure will work.
   2. Give algorithms that implement the Find-Median() and Insert() functions.

2. [10 points] In a busy grocery store, there are k self-checkout lanes, each with customers waiting in line. The store organizes these lanes so that within each line, customers are arranged in increasing order of their time of arrival—nobody skips ahead, and everyone follows the rules. The store manager wants to merge all k lines into a single checkout queue, ensuring that customers are still ordered by their arrival time. However, since the store is crowded, they need to do this as efficiently as possible. How can the manager merge these k sorted lines into one single sorted line in the fastest possible way, ensuring an optimal time complexity of O(n log k), where n is the total number of customers across all lines? Can you describe an algorithm that achieves this, justify its correctness, and explain why it runs in O(n log k) time?

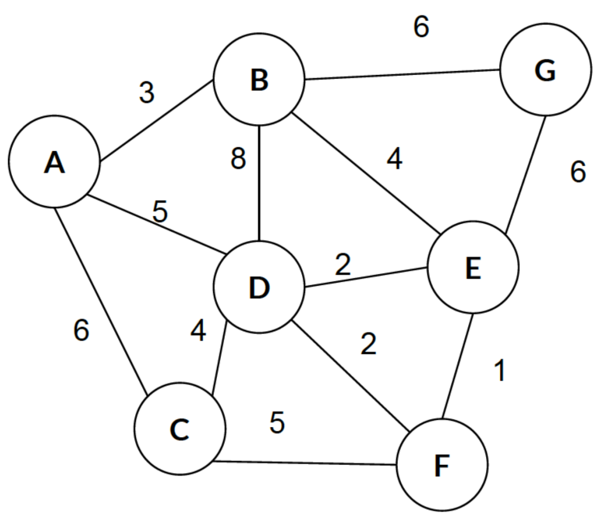
3. [5 points] A town wants to build roads connecting all neighborhoods at the lowest total cost, with each road having a unique construction cost. The mayor needs to know: Can there be more than one way to build this minimum-cost road network, or is the solution always unique? Prove why the answer must be yes or no.

4. [10 points] Suppose you are given a connected undirected graph where every edge weight is either 1 or 2. You want to find an MST.
   - Show that Prim's algorithm will always find an MST in O(n log n) time for this graph.
   - Can we do better than O(n log n)? Justify your answer—if yes, provide an algorithm with its time complexity; if no, give a proof

5. [5 points] Design an algorithm to find the maximum bandwidth path between a given source vertex s and a destination vertex t in a weighted graph G = (V, E), where each edge weight represents bandwidth capacity. The bandwidth of a path is defined as the minimum edge weight along that path (i.e., the bottleneck). Explain how to modify Dijkstra's algorithm to do this.

Ungraded Problems

6. Considering the following graph G in Fig 1:
   a. In graph G, if we use the Kruskals Algorithm to find the MST, what is the third edge added to the solution?
   b. In graph G, if we use the Prims Algorithm to find MST starting at A, what is the second edge added to the solution?
   c. What is the cost of the MST in the graph?



7. Given a connected graph G = (V, E) with positive edge weights. In V , s and t are two nodes for shortest path computation, prove or disprove with explanations:
   a. If all edge weights are unique, then there is a single shortest path between any two nodes in V .
   b. If each edge's weight is increased by k, the shortest path cost between s and t will increase by a multiple of k.
   c. If the weight of some edge e decreases by k, then the shortest path cost between s and t will decrease by at most k.
   d. If each edge's weight is replaced by its square, i.e., w to w2, then the shortest path between s and t will be the same as before but with different costs.