Homework 12

CSCI570 Spring 2025

Due: May 4, 2025

1. (**15 points**)

   A small coffee shop sells three types of coffee-based drinks. The first drink uses 10g coffee, 12g sugar, and 8 fl oz of milk. The second uses 16g coffee, 6g sugar, 3 fl oz milk. The third type uses 8g coffee, 15g sugar, 10 fl oz milk. The profit made on the three types of drinks is \$2, \$2.8, and \$1.5, respectively. The cafe makes 200 drinks of the first type, 120 of the second type, and 180 of the third type every day (with no ingredients leftover and there is enough demand that they're completely sold). On a certain day, there is a shortage of coffee by 600 g, and sugar by 400 g, but 2 gallons extra milk is available (1 gallon = 128 fl oz). How should the production of the three types of drinks be adjusted (increased or decreased as necessary) **to minimize the decrease in profit**? Formulate this problem as a linear programming problem (no need to solve the LP).

   (a) Define your variables. (Describe what they represent in English)

   (b) What is the objective function?

   (c) What are the constraints in your LP?

2. (**12 points**) A clique in a graph $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that each pair of vertices in C is adjacent, i.e., $\forall u, v \in C, (u, v) \in E$. We want to find the largest clique (i.e., a clique with the maximum number of vertices) in a given graph. Write an integer linear program that will find the largest clique.

   (a) Define your variables. (Describe what they represent in English)

   (b) What is the objective function?

   (c) What are the constraints in your LP?

3. (**16 points**) In today's fast paced world, online grocery shopping is becoming increasingly common as customers can enjoy the comfortable option of home-delivered groceries. Suppose there are n stores in your area to choose from. Suppose you want to buy m items which are all divisible goods - i.e., you could buy them in quantities expressed in real numbers. Suppose store j sells item i for a price p_{ij} per unit. Your consumption needs you want to buy at least a_i units of item i in total, but due to the limitations of storage and preservation, you want to buy at most b_i units of it. To afford the transportation costs of delivering, the stores do not accept orders that are too small - store j requires the total order cost to be at least S_j (all $S_j's$ are positive). Finally, you want to place the order with any particular store as a single transaction on your credit card which has a per-transaction cost limit of W.

   Grocery stores commonly sell some items for cheap to attract customers, and some items expensive to reap high profits. As a smart customer, you are planning to strategically spread your orders across various stores to minimize your expenditure - i.e. you want to decide how much of each item to order from each of the stores. Also, you are planning to place an order with all n stores not to leave any store unexplored. Formulate this problem as a Linear Program (LP).

   (a) Define your variables. (Describe what they represent in English)

   (b) What is the objective function?

   (c) What are the constraints in your LP?

4. (**17 points**) Given a graph G and a partition (A, B), let $E(A, B)$ denote the set of edges with one endpoint in A and one endpoint in B. The Max Equal Cut problem is defined as follows
Instance Graph $G(V, E), V = 1, 2, ..., 2n$.
Question Find a partition of V into two n-vertex sets A and B, maximizing the size of $E(A, B)$.

Provide a $\frac{1}{2}$-approximation algorithm for solving the Max Equal Cut problem.
(Hint: Consider an algorithm that iteratively builds the partition (A, B) in a *greedy* manner while ensuring A, B have equal size at any point.)

Ungraded problems

For all of the LP problems, follow the structure:

(a) Define your variables. (Describe what they represent in English)

(b) What is the objective function?

(c) What are the constraints in your LP?

5. Write down the problem of finding a Min-s-t-Cut of a directed network with source s and sink t as an Integer Linear Program and explain your program.

6. The edge-coloring problem is to color the edges of a graph with the fewest number of colors in such a way any two edges that share a vertex have different colors. Suppose you are given the algorithm ALG that colors a graph with at most $m/n + d$ colors when the graph has m edges, n vertices, and a maximum vertex-degree of d (We do not need to know how the algorithm works). Prove that this algorithm is a $\frac{3}{2}$-approximation algorithm for the edge-coloring problem. You may assume that the input graph is not empty (i.e., has at least one edge).

7. 720 students have pre-enrolled for the "Analysis of Algorithms" class in Fall. Each student must attend one of the 16 discussion sections, and each discussion section i has capacity for D_i students. The **aggregate** happiness level of students assigned to a discussion section i is proportionate to $\alpha_i(D_i - S_i)$, where α_i is a known parameter reflecting how well the air-conditioning system works for the room used for section i (the higher the better), and S_i is the actual number of students assigned to that section. We want to find out how many students to assign to each section in order to maximize total student happiness. Express the problem as an integer linear program problem.

8. Mary is a primary school teacher. She decides to distribute candy to the students in her class to congratulate them on their exam grades. There are n students sitting in a line. The j^{th} student in the line has grades g_j. Mary has m different type of candy, and N_i pieces available for each candy type i. A single piece of candy i costs c_i.

   Mary wants to distribute the candy satisfying the following criteria:

   (a) Each student receives at least 3 pieces of candy, and each should be of different kind.

   (b) For any students s and s' that are sitting next to each other in the line:

      i. If they have equal grades, they should get equal pieces of candy, if not, the one with higher grades should get more pieces.

      ii. They should not both get the candy of any same kind.

   Formulate a linear program to model this problem with an objective to minimize the overall money spent by Mary.

9. A set of n space stations need your help in building a radar system to track spaceships traveling between them. The i^{th} space station is located in 3D space at coordinates (x_i, y_i, z_i). The space stations never move. Each space station i will have a radar with power r_i, where r_i is to be determined. You want to figure how powerful to make each space station's radar transmitter, so that whenever any spaceship travels in a straight line from one station to another, it will always be in radar range of either the first space station (its origin) or the second space station (its destination). A radar with power r is capable of tracking space ships anywhere in the sphere with radius r centered at itself. Thus, a space ship is within radar range through its strip from space station i to space station j if every point along the line from (x_i, y_i, z_i) to (x_j, y_j, z_j) falls within either the sphere of radius r_i centered at (x_i, y_i, z_i) or the sphere of radius r_j centered at (x_j, y_j, z_j). The cost of each radar transmitter is proportional to its power, and you want to minimize the total cost of all of the radar transmitters. You are given all of the $(x_1, y_1, z_1), ..., (x_n, y_n, z_n)$ values, and your job is to choose values for $r_1, ..., r_n$. Express this problem as a linear program.

   (a) Describe your variables for the linear program (3 pts).

   (b) Write out the objective function (5 pts).

   (c) Describe the set of constraints for LP. You need to specify the number of constraints needed and describe what each constraint represents (8 pts).