Homework 10

CS570 Spring 2025

Due: Apr 18, 2025

1. (**15 points**)

   Suppose you move to a new city. Due to busy work schedule, you have to eat out very often. There are m different food items that you are fond of. You seek recommendations from friends and online, and you get a list of n restaurants, each offering one or more (or none) of your m favorite foods. You realize this list is too big to try them all, so you want to remove as many from the list as possible while making sure that each of your favorite foods is offered in at least one of the remaining restaurants. Thus, the problem can be stated as deciding if at least k restaurants can be removed from the given list so that each of your favorite foods is offered in at least one of the remaining restaurants. Prove that the problem of *shortening restaurant list* (SRL) is

   a) in NP.

   b) NP-Hard using a poly-time reduction from Vertex Cover.

2. (**16 points**)

   Given a graph $G = (V, E)$ and a positive integer r, the dominating set problem (DSP) is to decide if there is a subset $B \subseteq V$ of size r such that every vertex that is not in B, has an adjacent vertex in B. Prove that the vertex cover (VC) problem is polynomial time reducible to DSP. For simplicity, you may assume that VC remains NP-complete when restricted to input graphs with no isolated vertices (i.e., each vertex has at least one neighbor), and reduce from this version of VC if required.

   Useful terminology: Similar to the context of Vertex cover where an edge is *covered* by an endpoint vertex being included in the VC set, here, we say a vertex u is *dominated* if u itself or one of its neighbors is included in the dominating set (DS); thus, the DS solution is a set of vertices that collectively *dominates* all the vertices in the graph (similar to how a VC solution collectively *covers* all the edges).

3. (**15 points**)

Suppose you are a movie director. For your next movie, you are considering n actors that you like. You want the audience to witness all the new combinations of acting talent working together - i.e., you want to pick a set of actors from the n candidates, such that no two have worked together in a movie before. For this, you make a combined list of all the movies in which any of these actors have acted, giving you a total of m movies (thus, each of the movies features one or more of these n actors). You want to find at least k actors to cast for your movie such that no two have acted in the same movie before. Your input consists of n actors, m movies, along with the information of which actors have acted in which movies, and the number k. Prove that this problem of *Novel Combination Casting* (NCC) is.

a) in NP.

b) NP-hard (by choosing a suitable problem studied in class to show a poly-time reduction).

4. (**14 points**) Given an undirected graph $G = (V, E)$, and integer m, the *CLIQUE* problem is to decide if there is a subset $A \subseteq V$ of vertices (called as a *clique*) of size at least m such that for every pair of vertices $u, v \in A$ with $u \neq v$, (u, v) is an edge in E.

   The *HALF-CLIQUE* problem asks if G has a clique of size at least $\frac{|V|}{2}$ - Note that it does not take integer m as input unlike the *CLIQUE* problem. Show that *HALF-CLIQUE* is

   a) in NP.

   b) NP-hard by reducing from *CLIQUE*, which is known to be NP-complete.

Ungraded problems

5. Consider the vertex cover problem that restricts the input graphs to be those where all vertices have even degree. Call this problem VC-EDG. Show that VC-EDG is NP-complete. (15pts)

   Hint: To show NP-hardness, reduce from VC, i.e., think of a construction where given the input of VC, i.e., a graph G and a number K, you can construct G', K', (where G' has even degrees for all its vertices), such that G has a vertex cover of size K if and only if G' has a vertex cover of size K'.

6. Consider the ***partial*** satisfiability problem, denoted as 3-Sat(α) defined with a fixed parameter α where $0 \leq \alpha \leq 1$. As input, we are given a collection of k clauses, each of which contains exactly three literals (i.e. the same input as the 3-SAT problem from lecture). The goal is to determine whether there is an assignment of true/false values to the literals such that at least αk clauses will be true. Note that for $\alpha = 1$, we require all k clauses true, thus 3-Sat(1) is exactly the regular 3-SAT problem.

   Prove that 3-Sat($\frac{15}{16}$)is NP-complete. (20 points)

   Hint: If x, y, and z are variables, note that there are eight possible clauses containing them: $(x \vee y \vee z), (!x \vee y \vee z), (x \vee !y \vee z), (x \vee y \vee !z), (!x \vee !y \vee z), (!x \vee y \vee !z), (x \vee !y \vee !z), (!x \vee !y \vee !z)$ Think about how many of these are true for a given assignment of x, y, and z.