CSCI 570 Fall 2025

Homework 9

Due: April 11

Note: In the following questions we assume that the Ford-Fulkerson algorithm is provided. You do not need to rewrite the Ford-Fulkerson algorithm.

Question 1

Given a weighted, directed graph $G = (V, E)$, determine whether there exist k paths such that each path starts at a unique vertex in the set

$$A = \{a_1, a_2, \ldots, a_k\} \subset V,$$

and ends at a unique vertex in the set

$$B = \{b_1, b_2, \ldots, b_k\} \subset V,$$

with no two paths sharing any vertices or edges. (Note that $B \cap A = \emptyset$.)

Task: Design an algorithm to solve this problem and prove its correctness.

(Hint: The proof will have two directions. 1)Forward Direction: If there exist k paths, then [statement]. 2) Backward Direction: If [statement] then there exist k paths.)

Question 2

We have n traders, each possessing W_i Swiss Francs (for $i = 1, \ldots, n$). They wish to convert their Francs into m different currencies $c_1, \ldots, c_m$. However:

- Each currency c_j has a maximum conversion limit B_j, meaning the bank can convert at most B_j Francs into currency c_j (for $j = 1, \ldots, m$).

- Each trader i can convert at most $S_{i,j}$ Francs into currency c_j.

The question is: *Is it possible to convert all of the traders' Francs, i.e., $\sum_{i=1}^{n} W_i$, subject to the given constraints?*

Design an algorithm to decide feasibility by formulating and solving a **maximum flow** problem. Clearly describe the construction of the flow network: specify all vertices, edges, and capacities, and explain how each constraint in the original problem is modeled within this network.

Question 3

We have m disjoint time intervals. There are n guards, and each guard is *available* for some subset of these intervals (i.e., the guard can only be assigned to intervals in which they are available). We need to assign guards so that:

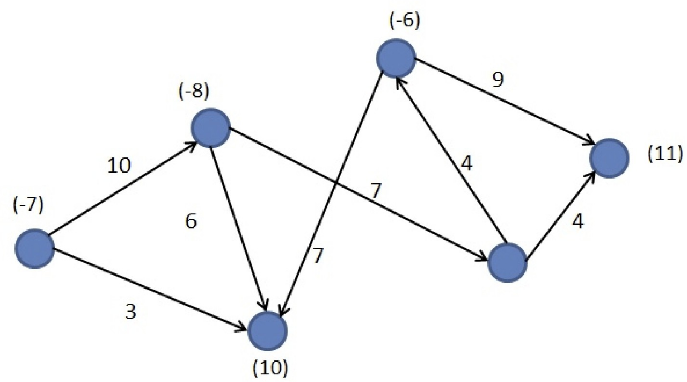
- Each interval is covered by at least one and at most two guards.
- Each guard is deployed to *at most* Q intervals.
- Each guard is deployed to *at least* L intervals.

The goal is to decide whether there exists a valid assignment of guards to intervals satisfying all these conditions.

Task: Design an algorithm to decide feasibility by formulating and solving a maximum flow problem.
Clearly describe the construction of the flow graph: specify all vertices, edges, capacities, and constraints, and explain how each constraint in the original problem is modeled within this network. Reduce the problem to a standard maximum flow problem without lower bounds by introducing new nodes where necessary.

Question 4

Graph G is an instance of a circulation problem with demands. The edge weights indicate capacities, and the node weights (shown in parentheses) indicate demands. A negative demand indicates that a node acts as a source.



- Transform this graph into an instance of max-flow problem. *Just draw the the new graph.*
- Now, assume that each edge of G has a constraint of lower bound of 1 unit, i.e., one unit must flow along all edges. Find the new instance of max-flow problem that includes the lower bound constraint. *Just draw the the new graph.*

Ungraded 1

Kleinberg and Tardos, Chapter 7, Exercise 9

Ungraded 2

Counter Espionage Academy instructors have designed the following problem to see how well trainees can detect SPY's in an $n \times n$ grid of letters S, P, and Y. Trainees are instructed to detect as many disjoint copies of the word SPY as possible in the given grid. To form the word SPY in the grid they can start at any S, move to a neighboring P, then move to a neighboring Y. (They can move north, east, south or west to get to a neighbor.) The following figure shows one such problem on the left, along with two possible optimal solutions with three SPY's each on the right.

Give an efficient network flow-based algorithm to find the largest number of SPY's.

Note: We are only looking for the largest **number** of SPYs not the actual location of the words. No proof is necessary.

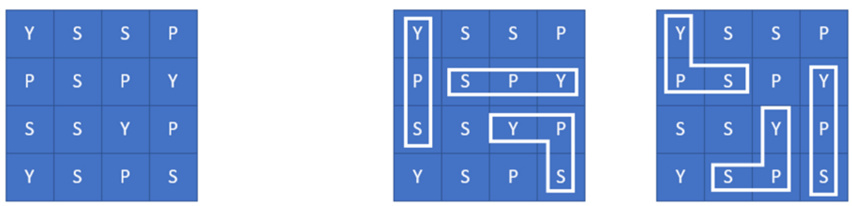


Figure 3: SPY detection problem and two optimal solutions