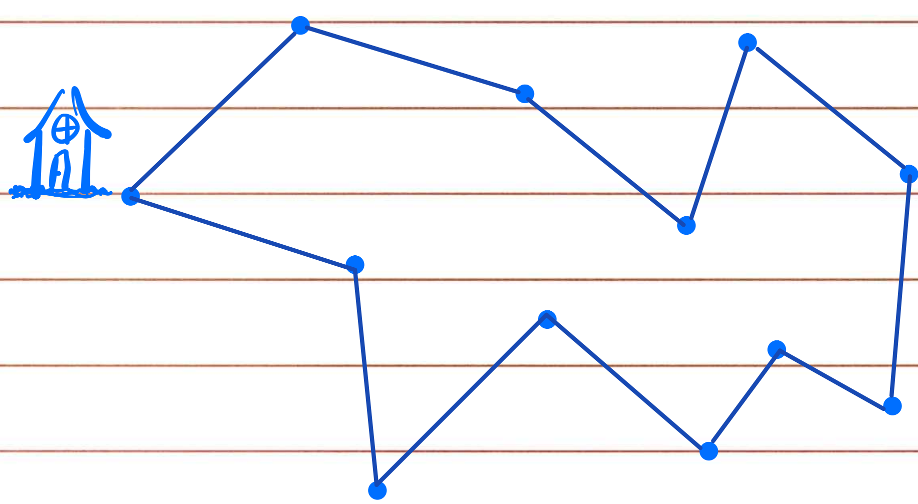
Traveling Salesman Problem (TSP)
&
Hamiltonian Cycle



TSP Problem Statement

Given a set of distances, order $\underline{n}$ cities in a tour $v_{i_1}, v_{i_2}, \ldots v_{i_n}$ with $i_1 = 1$, so it minimizes

$$\sum_{j=1}^{n-1} d(v_{i_j}, v_{i_{j+1}}) + d(v_{i_n}, v_{i_1})$$

TSP has applications in

- Vehicle routing
- Logistics planning
- Cutting / drilling tasks
- ...

Decision version of TSP:

Given a set of distances on n cities and a bound D, is there a tour of length/cost at most D?

Def.: A cycle C in G is a Hamiltonian Cycle if it visits each vertex exactly once.

Problem Statement

Given an undirected graph G, is there a Hamiltonian Cycle in G?

Show that the Hamiltonian Cycle (HC) problem is NP-Complete.

1- We show that the problem is in NP

a) Certificate:
An ordered list of nodes on the Hamiltonian Cycle

b) Certifier:
We will check the following
- All nodes appear on the list
- Nodes only appear once
- Every pair of adjacent nodes in the given order must have an edge between them
- The first and last nodes have an edge between them

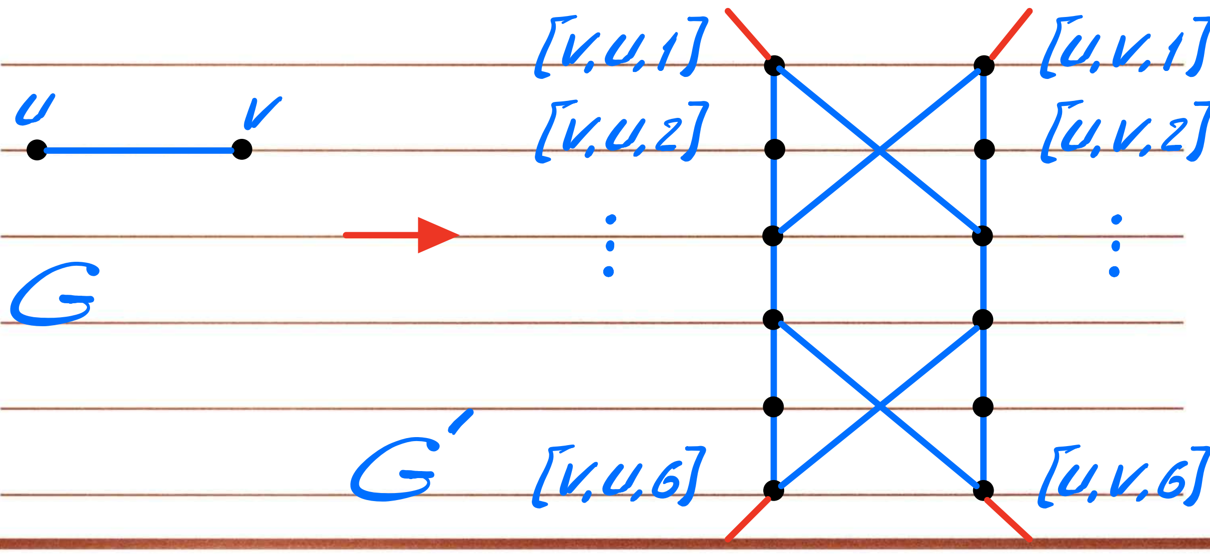
2 - Choose Vertex Cover as the problem known to be NP-Complete

3 - We show that Vertex Cover $\leq_p$ HC

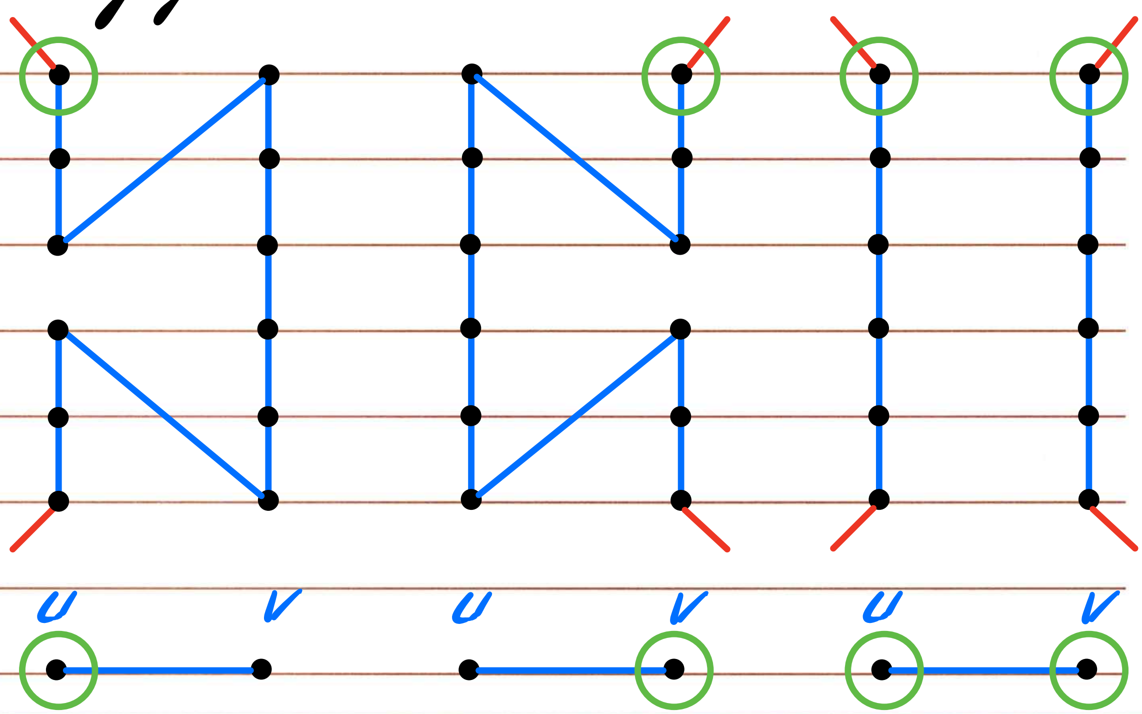
Plan: Given an undirected graph $G = (V, E)$ and an integer $\underline{k}$, we construct $G' = (V', E')$ that has a Hamiltonian Cycle iff G has a vertex cover of size at most $\underline{k}$.

Construction of G'

For each edge (VU) in G, G' will have one gadget W_{VU} with following node labeling:



Some intuition behind the construction of the gadget:



- There are only 3 ways that a HC can go through all the node of one gadget. These three ways correspond to the 3 ways that an edge can be covered in the vertex cover problem.

- The gadget is constructed such that if a HC enters the gadget on one side, it has to leave the gadget on the same side.

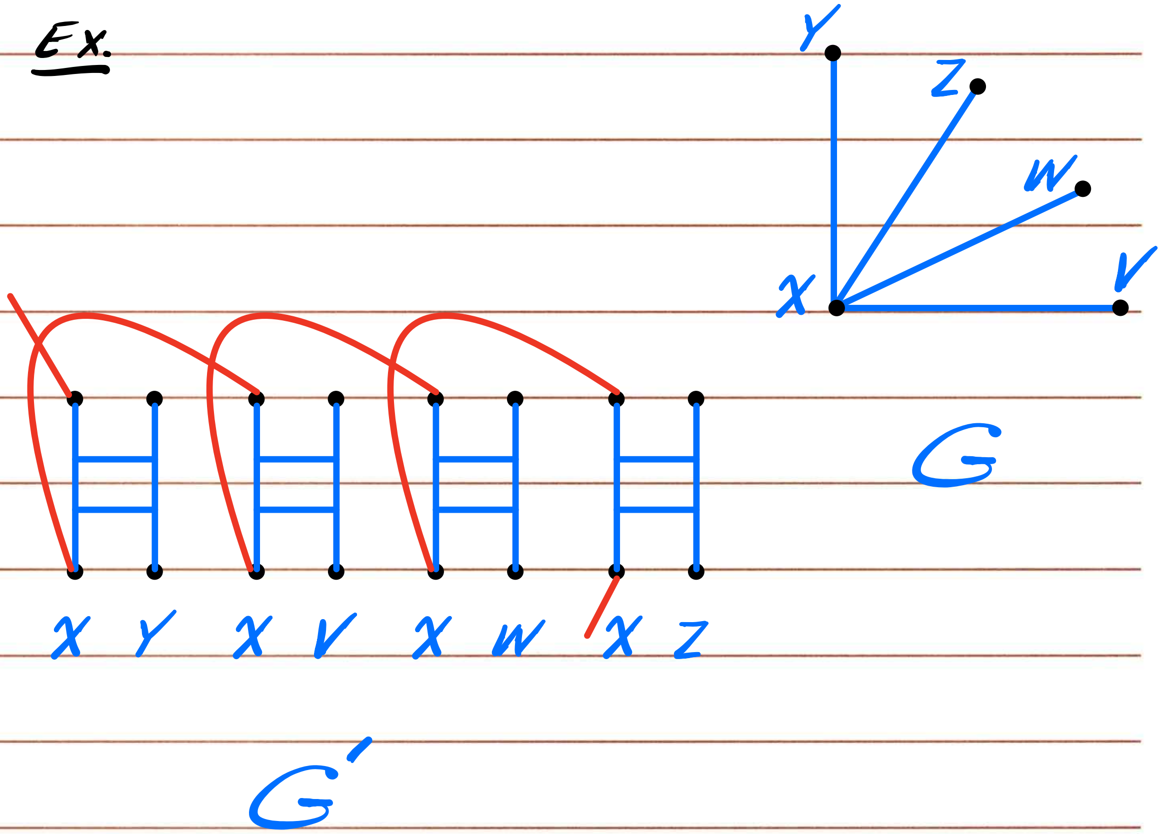
Other vertices in G'

- Selector vertices: There are $\underline{k}$ selector vertices in G', $s_1 \dots s_k$

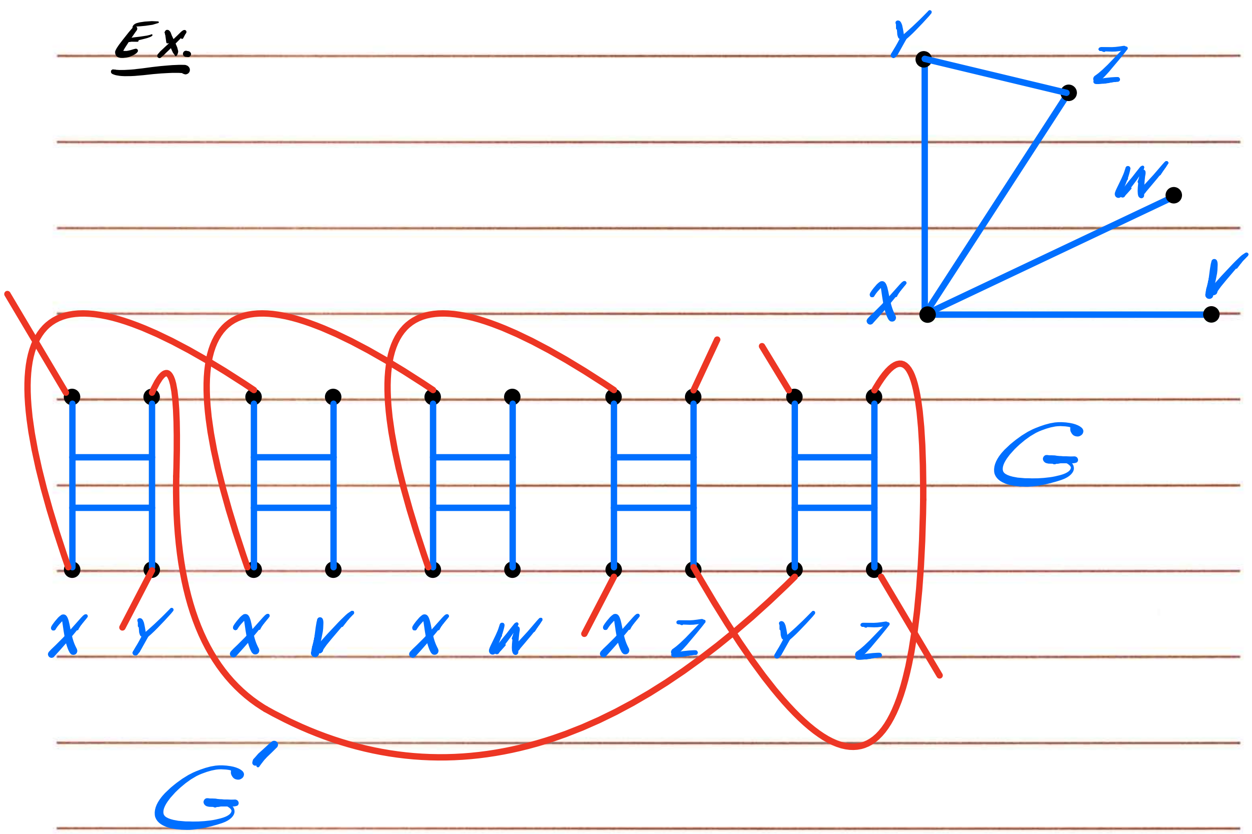
Other edges in G'

1 - For each vertex $u \in V$, we add edges to join pairs of gadgets in order to form a path going through all the gadgets corresponding to edges incident on node u in G.

Ex.

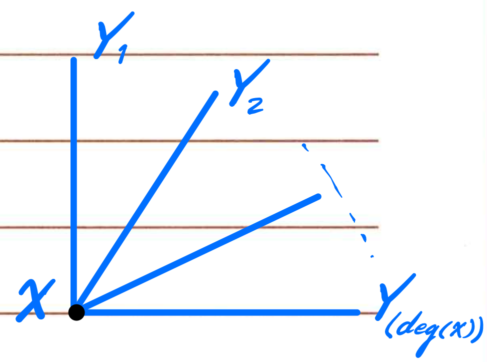


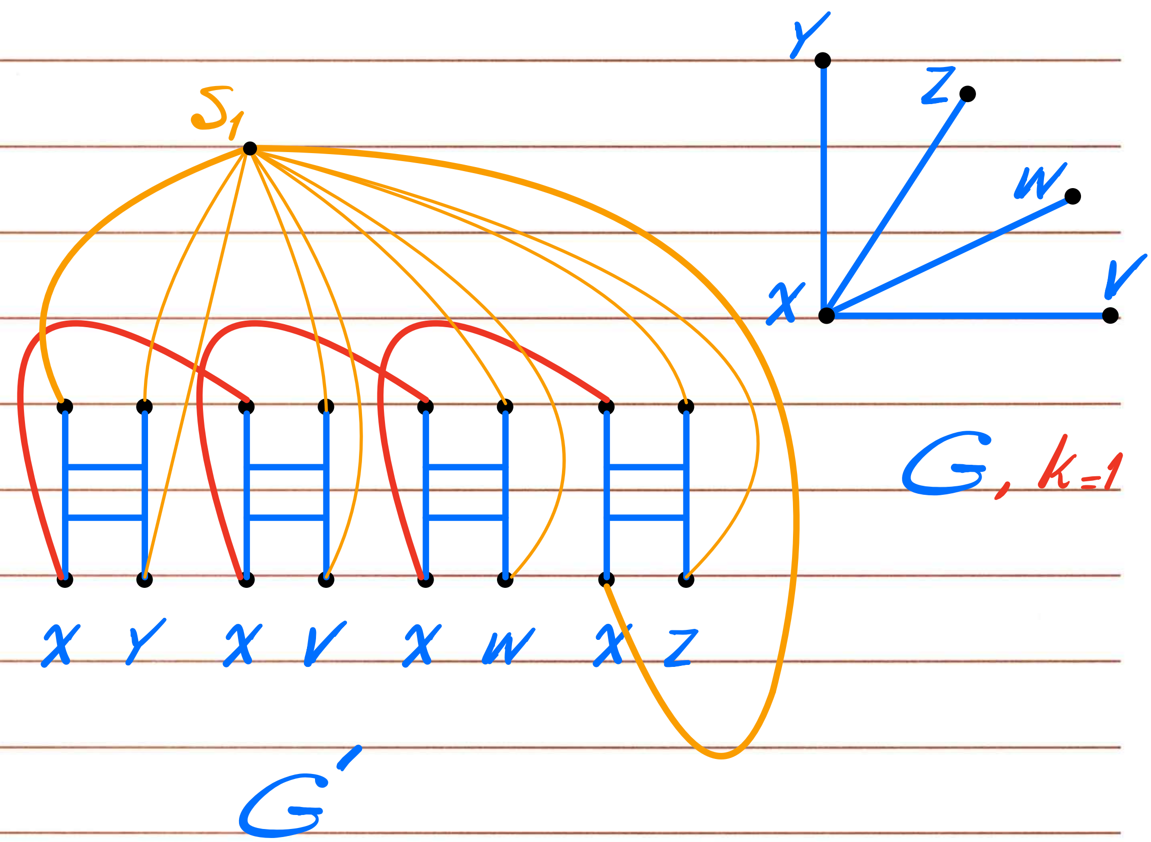
Ex.

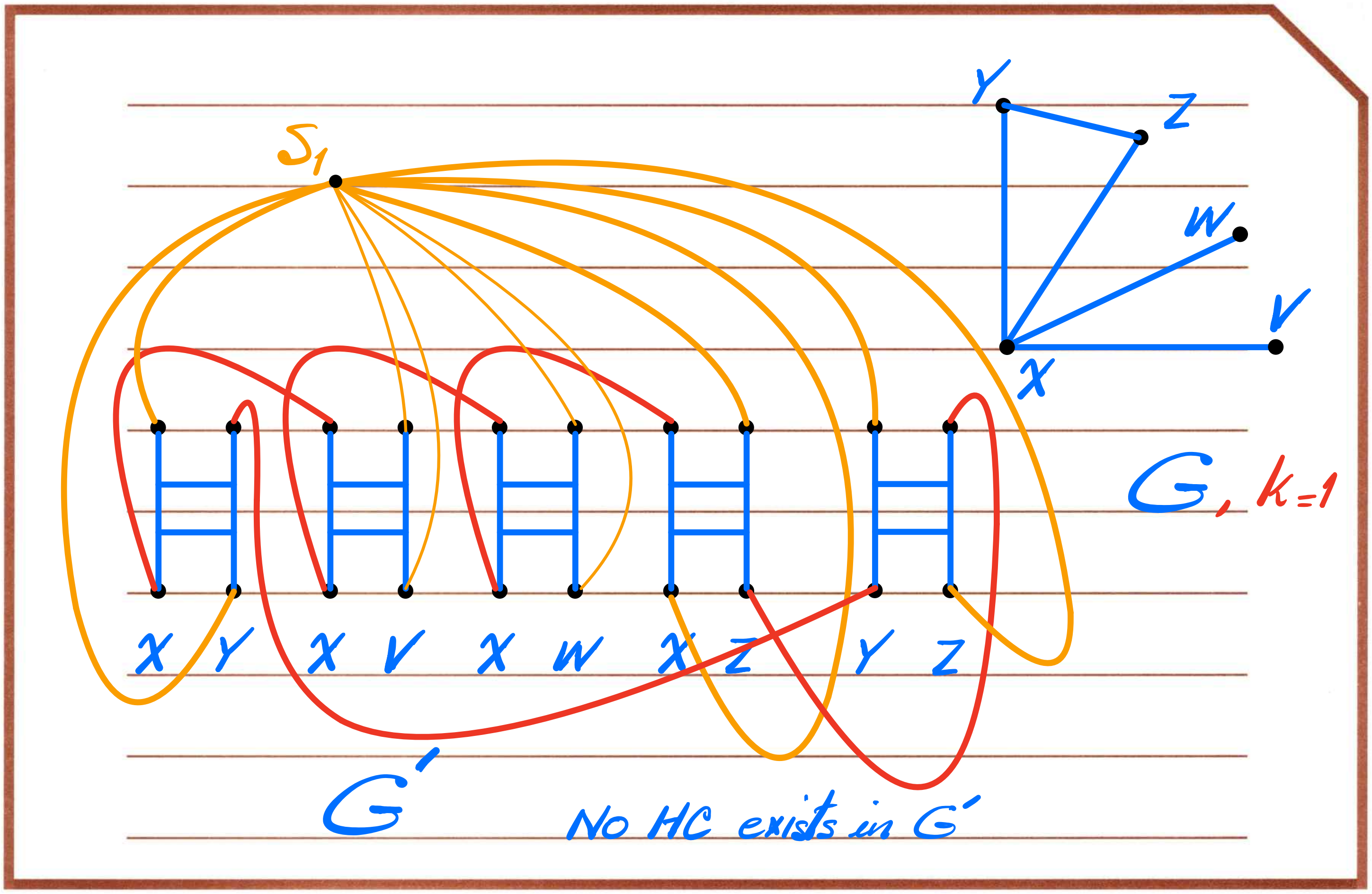


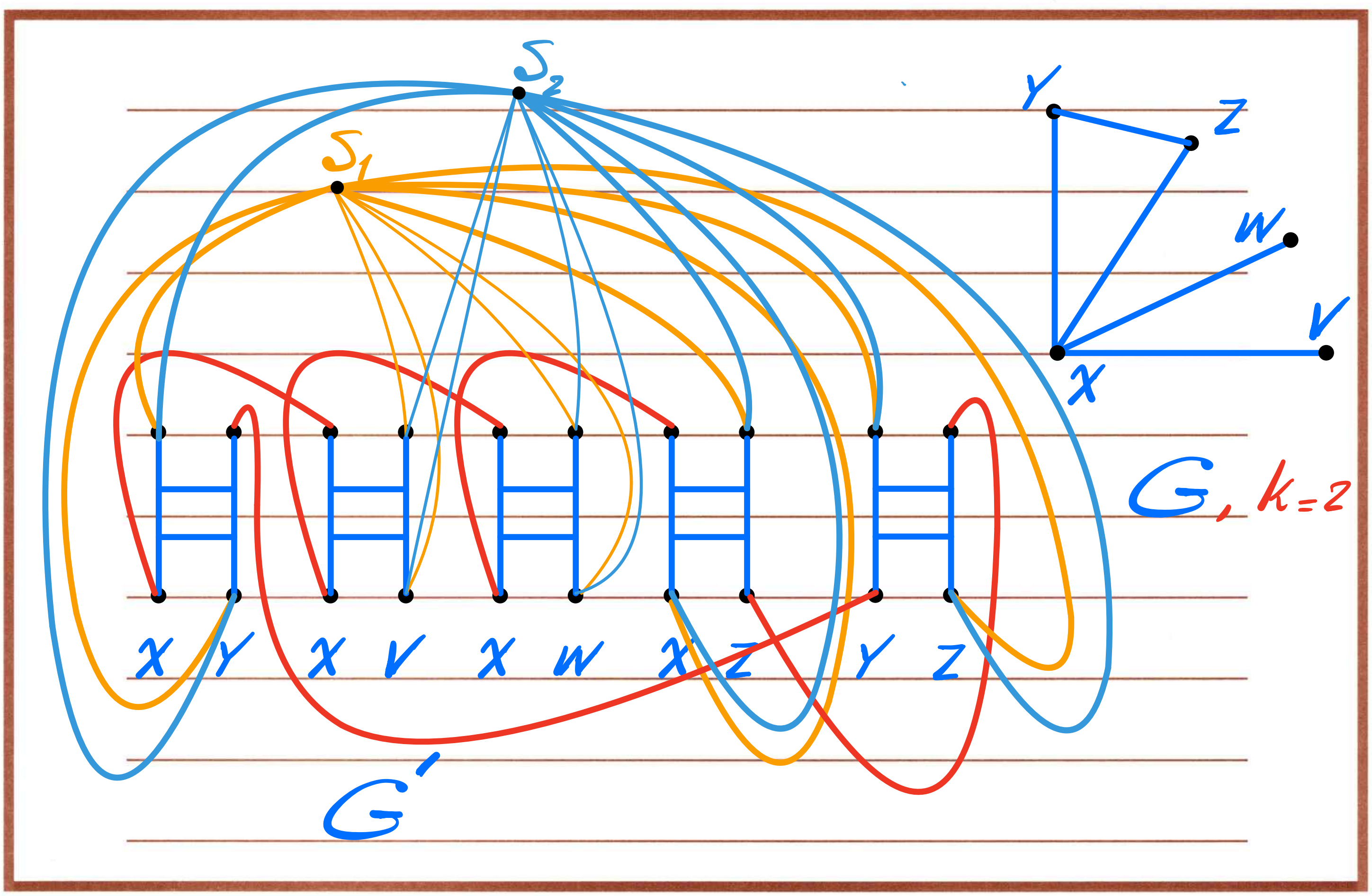
Other edges in G'

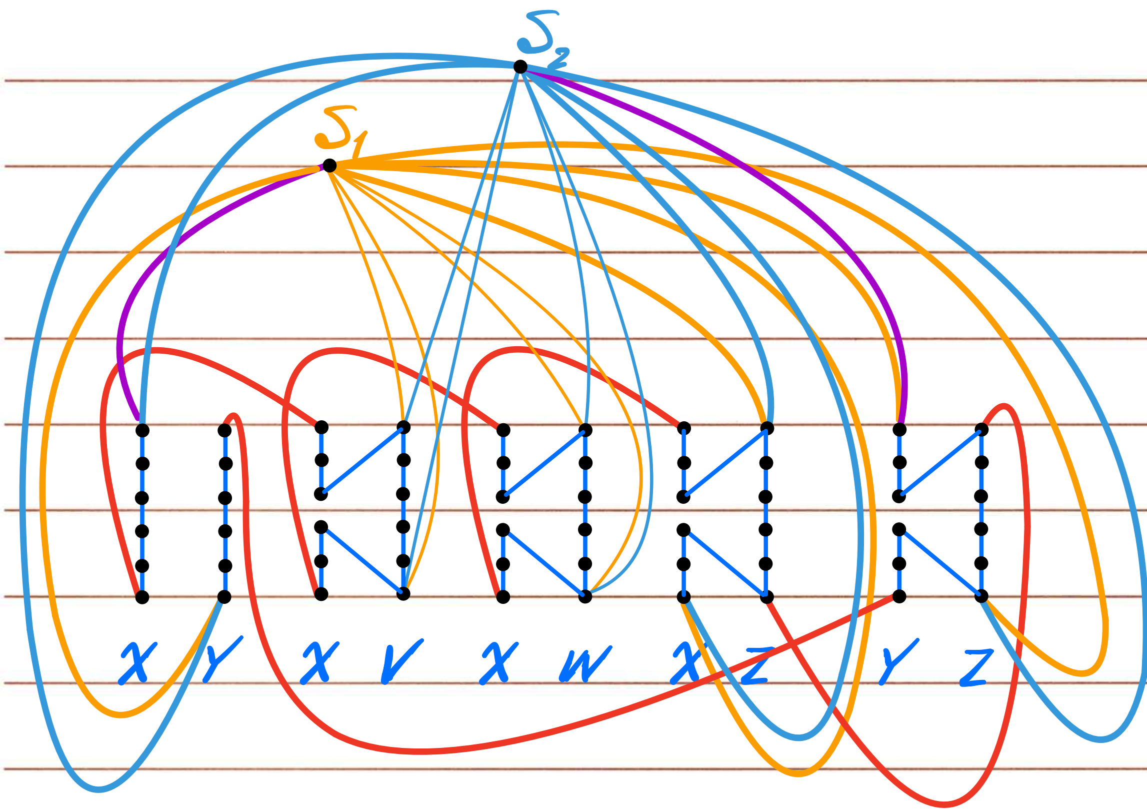
2 _ Final set of edges in G' join the first vertex $[X, Y_1, 1]$ and the last vertex $[X, Y_{(deg(X))}, 6]$ of each of these paths to each of the selector vertices.






S1
Y
Z
W
V
X
X Y X V X W X Z Y Z
G, k=1
G′
No HC exists in G′


S2
S1
Y
Z
W
V
X
X Y X V X W X Z Y Z
G, k=2
G′



Ham. Cycle in G'

Edge between S_1 and [X, Y, 1] indicates that S_1 has selected node X

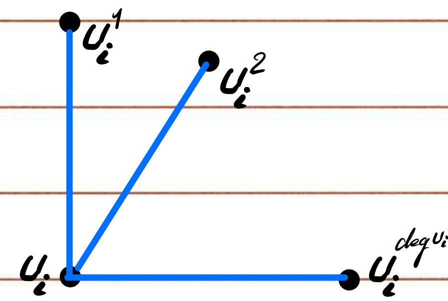
Edge between S_2 and [Y, Z, 1] indicates that S_2 has selected node Y

Therefore the vertex cover set identified by this HC is the set {X, Y}

Proof: A) Suppose that $G=(V,E)$ has a vertex cover of size $\underline{k}$. Let the vertex cover set be

$$S=\{u_1, u_2, \quad u_k\}$$

We will identify the neighbors of u_i as shown here:



We can form a Ham. Cycle in G' by following the nodes in G' in this order:

start at s_1 and go to

$[u_1, u_1^1, 1] \quad \ldots \quad [u_1, u_1^1, 6]$

$[u_1, u_1^2, 1] \quad \ldots \quad [u_1, u_1^2, 6]$

$[u_1, u_1^{\deg u_1}, 1] \quad \ldots \quad [u_1, u_1^{\deg u_1}, 6]$

Then go to S_2 and follow the nodes

$[u_2, u_2^1, 1]$ ... $[u_2, u_2^1, 6]$

$[u_2, u_2^2, 1]$ ... $[u_2, u_2^2, 6]$

$[u_2, u_2^{\deg u_1}, 1]$ ... $[u_2, u_2^{\deg u_1}, 6]$

Then go to S_3

⋮

⋮

$[u_k, u_k^1, 1]$ ... $[u_k, u_k^1, 6]$

$[u_k, u_k^2, 1]$ ... $[u_k, u_k^2, 6]$

$[u_k, u_k^{\deg u_1}, 1]$ ... $[u_k, u_k^{\deg u_1}, 6]$

Finally return back to S_1 to complete the Ham. Cycle.

B) Suppose G' has a Ham. Cycle, then the set

$$S = \{ v_i \in V : (s_j, [v_i, v_i^1, 1]) \in C$$

$$\text{for some } 1 \leq j \leq k \}$$

will be a vertex cover set in G.
Since segments of the HC between s_j and s_{j+1} go through all gadgets corresponding to edges that are

incident on v_i in G (indicating that node v_i covers all edges incident on it in G)
And because the Ham. Cycle goes through all gadgets in G', then all corresponding edges will be covered by the nodes in the set S.

Prove that TSP is NP-Complete

1- Show that $TSP \in NP$

a. Certificate:
A tour of cost at most D

b. Certifier:
- All checks we did for HC, +
- Check that cost of tour $\leq D$

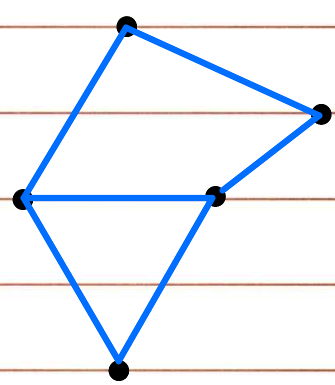
2- Choose an NP-Complete problem:
Hamiltonian Cycle

3- Prove that $HC \leq_p TSP$

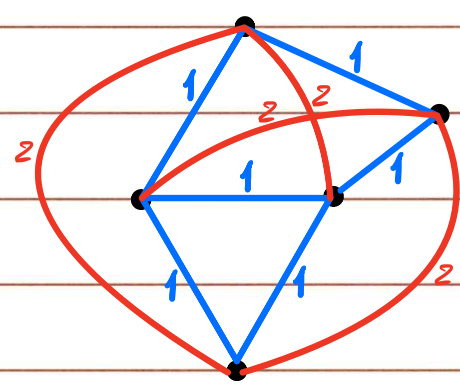
Plan: Given an instance of the HC problem on graph $G = (V, E)$, we will construct G' such that G has a HC iff G' has a tour of cost $\leq |V|$.

Construction of G':

- G' has the same set of nodes as in G.
- G' is a fully connected graph
- Edges in G' that are also in G have a cost of 1.
- Other edges in G' have a cost of 2.



G



G'

Proof Template:

A- HC in G $\rightarrow$ tour of cost $|V|$ in G'

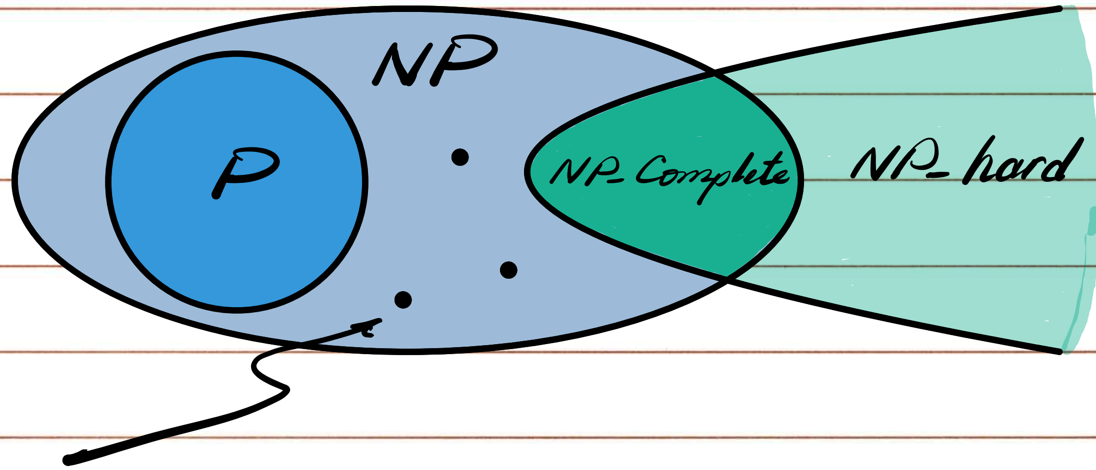
B- Tour of cost $|V|$ in G' $\rightarrow$ HC in G

The set of known NP-Complete problems we can choose from in our proof of NP-Completeness:

- 3-SAT
- Independent Set
- Vertex cover
- Set cover
- Set packing
- Hamiltonian Cycle and Ham. Path
- TSP

- We can also use the decision versions of
  - 0-1 Knapsack
  - Subset sum

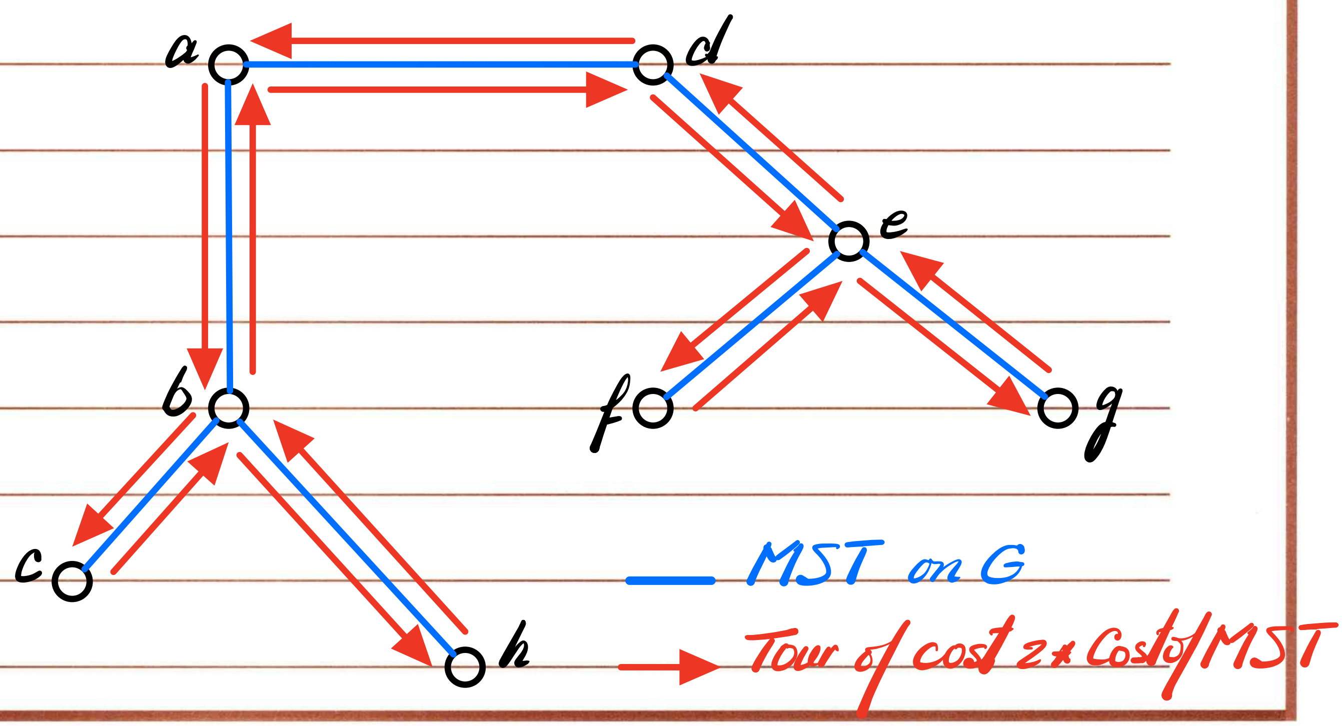
Since we are already familiar with these problems, although a proof of their NP-Completeness has not been presented in lecture.

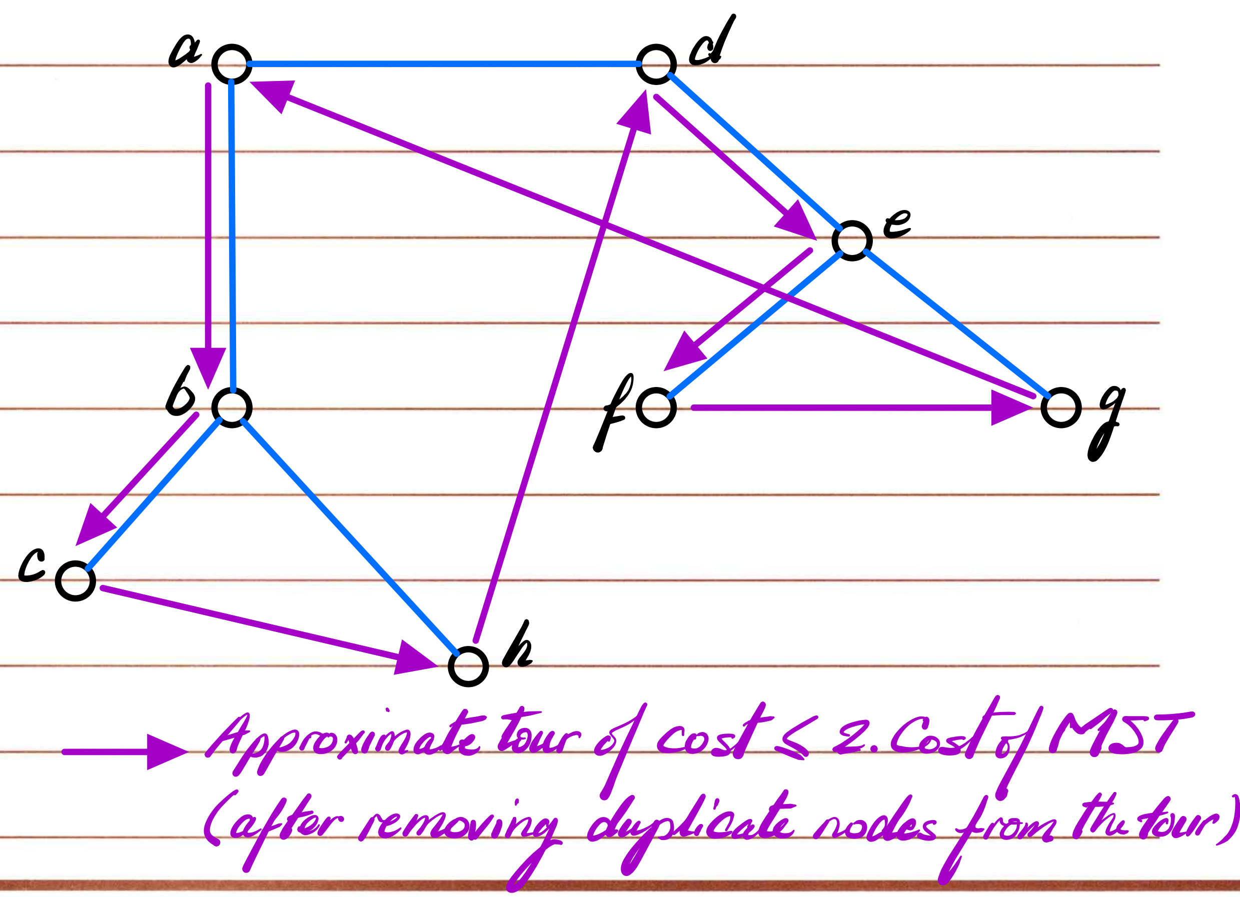


We know of only a handful of problems in NP that are neither proven to be NP-Complete, nor do we have a polynomial time solution for.

These problem are called NP-intermediate.
examples of such problems:
- Graph Isomorphism
- Integer factoring

TSP with Triangle Inequalities





Claim:
The cost of our approximate slolution to TSP is within a factor of 2 of the cost of the optimal tour.

Proof:
- Cost of our initial tour = 2 * Cost of MST
- Since triangle inequalities hold in G, after removing duplicate nodes from our initial tour, we have:

Cost of our approx. tour $\leq$ 2 * Cost of MST
- Since Cost of the opt. tour $>$ Cost of MST, then

Cost of our approx. tour $<$ 2 * Cost of opt. tour

Our solution is called a 2-approximation since it guarantees to come to within a factor of 2 of the optimal solution.

General TSP

Theorem: If $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial time approximation algorithm with approximation ratio ρ for the general TSP.

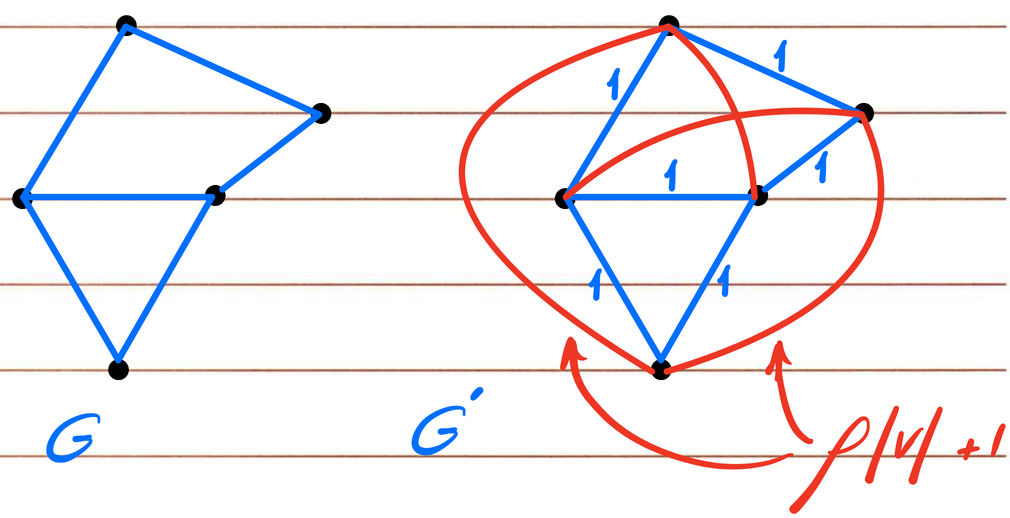
Plan for the proof of the theorem:

We will assume that such an approximation algorithm exists. We will then use it to solve the HC problem.

Proof:

Given an instance of the HC problem on graph $G=(V,E)$, we will construct G' as follows

- G' has the same set of nodes as in G.
- G' is a fully connected graph
- Edges in G' that are also in G have a cost of 1.
- Other edges in G' have a cost of $\rho|V|+1$



If we have a HC in G, there will be a tour of cost $|V|$ in G'

If we have a tour of cost $\leq \rho|V|$ in G', there will be a HC in G.

So we can now run the approximation alg. on G'. Since it guarantees to find a tour of cost no more than a factor of ρ from the optimal solution, if G has a HC, it must return a tour of cost no more than $\rho|V|$. And as mentioned above, we can use this tour to find a HC in G.