Q1

1.1

The graphs in Figure 1 and Figure 2 are homomorphic.

For the function: f: $V(G_1)$ -> $V(G_2)$

there are f(1) = A, f(2) = B, f(3) = C, f(4) = A, f(5) = A, f(6) = C, f(7) = B, f(8) = C

have edge {1,2} in G_1 and {f(1),f(2)} = {A,B} in G_2 ,

{1,3} in G_1 and {f(1),f(3)} = {A,C} in G_2 ,

{2,3} in G_1 and {f(2),f(3)} = {B,C} in G_2 ,

{2,4} in G_1 and {f(2),f(4)} = {B,A} in G_2 ,

{2,5} in G_1 and {f(2),f(5)} = {B,A} in G_2 ,

{2,8} in G_1 and {f(2),f(8)} = {B,C} in G_2 ,

{3,5} in G_1 and {f(3),f(5)} = {C,A} in G_2 ,

{4,7} in G_1 and {f(4),f(7)} = {A,B} in G_2 ,

{4,8} in G_1 and {f(4),f(8)} = {A,C} in G_2 ,

{5,6} in G_1 and {f(5),f(6)} = {A,C} in G_2 ,

{5,8} in G_1 and {f(5),f(8)} = {A,C} in G_2 ,

{7,8} in G_1 and {f(7),f(8)} = {B,C} in G_2 ,

1.2

The unique subgraphs are:

1234

1236

2354

2358

1256

2456

2568

2478

4578

Q2

2.1

Since there is H^0 and $H^l = \left[h_{v1}^l, h_{v2}^l, h_{v3}^l, h_{v4}^l, h_{v5}^l, h_{v6}^l\right]^T$,we can calculate the $h_{v1}^0, h_{v2}^0, h_{v3}^0, h_{v4}^0, h_{v5}^0, h_{v6}^0$. Then there is :

$$W^1 h_{v1}^0 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.30 \\ -0.60 \\ 0.10 \\ -0.20 \end{pmatrix} = \begin{pmatrix} 0.20 \\ -0.60 \\ -0.70 \\ 0.40 \\ -0.50 \end{pmatrix}$$

$$W^1 h_{v2}^0 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.60 \\ 0.40 \\ 0.40 \\ -0.10 \end{pmatrix} = \begin{pmatrix} 0.90 \\ 0.40 \\ 0.70 \\ 1.00 \\ 0.90 \end{pmatrix}$$

$$W^1 h_{v3}^0 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.20 \\ 0.70 \\ -0.40 \\ 0.50 \end{pmatrix} = \begin{pmatrix} 0.30 \\ 0.70 \\ 0.80 \\ 0.60 \\ 1.40 \end{pmatrix}$$

$$W^1 h_{v4}^0 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -0.40 \\ 0.60 \\ 0.10 \\ 0.80 \end{pmatrix} = \begin{pmatrix} -0.10 \\ 0.60 \\ 1.50 \\ -0.30 \\ 1.00 \end{pmatrix}$$

$$W^1 h_{v5}^0 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.40 \\ 0.90 \\ -0.20 \\ 0.10 \end{pmatrix} = \begin{pmatrix} 0.30 \\ 0.90 \\ 0.80 \\ 0.20 \\ 1.40 \end{pmatrix}$$

$$W^1 h_{v6}^0 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.30 \\ -0.30 \\ 0.90 \\ 0.70 \end{pmatrix} = \begin{pmatrix} 1.90 \\ -0.30 \\ 1.30 \\ 1.20 \\ 0.70 \end{pmatrix}$$

For v1:

Since $N(v1) \cup \{v1\} = \{v1, v2, v3\}, a_{v1u} = \frac{1}{3}$

$$h_{v1}^1 = \sigma\left(\sum_{N(v1)\cup\{v1\}} a_{v1u} W^1 h_u^0\right)$$

$$= \sigma\left(\frac{1}{3} W^1 h_{v1}^0 + \frac{1}{3} W^1 h_{v2}^0 + \frac{1}{3} W^1 h_{v3}^0\right)$$

$$= \sigma\left(\frac{1}{3}\left(\begin{pmatrix}0.20\\-0.60\\-0.70\\0.40\\-0.50\end{pmatrix}+\begin{pmatrix}0.90\\0.40\\0.70\\1.00\\0.90\end{pmatrix}+\begin{pmatrix}0.30\\0.70\\0.80\\0.60\\1.40\end{pmatrix}\right)\right)$$

$$= \sigma\left(\frac{1}{3}\begin{pmatrix}1.40\\0.50\\0.80\\2.00\\1.80\end{pmatrix}\right)$$

$$= \sigma\left(\begin{pmatrix}0.47\\0.17\\0.27\\0.67\\0.60\end{pmatrix}\right)$$

$$= \begin{pmatrix}0.47\\0.17\\0.27\\0.67\\0.60\end{pmatrix}$$

For v2:

Since $N(v2) \cup \{v2\} = \{v1, v2, v4, v5\}$, $a_{v2u} = \frac{1}{4}$

$$h_{v2}^{1} = \sigma\left(\sum_{N(v2)\cup\{v2\}} a_{v2u} W^1 h_u^0\right)$$

$$= \sigma\left(\frac{1}{4}W^1 h_{v1}^0 + \frac{1}{4}W^1 h_{v2}^0 + \frac{1}{4}W^1 h_{v4}^0 + \frac{1}{4}W^1 h_{v5}^0\right)$$

$$= \sigma\left(\frac{1}{4}\left(\begin{pmatrix}0.20\\-0.60\\-0.70\\0.40\\-0.50\end{pmatrix}+\begin{pmatrix}0.90\\0.40\\0.70\\1.00\\0.90\end{pmatrix}+\begin{pmatrix}-0.10\\0.60\\1.50\\-0.30\\1.00\end{pmatrix}+\begin{pmatrix}0.30\\0.90\\0.80\\0.20\\1.40\end{pmatrix}\right)\right)$$

$$= \sigma\left(\frac{1}{4}\begin{pmatrix}1.30\\1.30\\2.30\\1.30\\2.80\end{pmatrix}\right)$$

$$= \sigma\left(\begin{pmatrix}0.33\\0.33\\0.58\\0.33\\0.70\end{pmatrix}\right)$$

$$= \begin{pmatrix}0.33\\0.33\\0.58\\0.33\\0.70\end{pmatrix}$$

For v3:

Since $N(v3) \cup \{v3\} = \{v1, v3, v5, v6\}$, $a_{v3u} = \frac{1}{4}$

$$h_{v3}^1 = \sigma\left(\sum_{N(v3)\cup\{v3\}} a_{v3u} W^1 h_u^0\right)$$

$$= \sigma\left(\frac{1}{4}W^1 h_{v1}^0 + \frac{1}{4}W^1 h_{v3}^0 + \frac{1}{4}W^1 h_{v5}^0 + \frac{1}{4}W^1 h_{v6}^0\right)$$

$$= \sigma\left(\frac{1}{4}\left(\begin{pmatrix}0.20\\-0.60\\-0.70\\0.40\\-0.50\end{pmatrix} + \begin{pmatrix}0.30\\0.70\\0.80\\0.60\\1.40\end{pmatrix} + \begin{pmatrix}0.30\\0.90\\0.80\\0.20\\1.40\end{pmatrix} + \begin{pmatrix}1.90\\-0.30\\1.30\\1.20\\0.70\end{pmatrix}\right)\right)$$

$$= \sigma\left(\frac{1}{4}\begin{pmatrix}2.70\\0.70\\2.20\\2.40\\3.00\end{pmatrix}\right)$$

$$= \sigma\left(\begin{pmatrix}0.68\\0.18\\0.55\\0.60\\0.75\end{pmatrix}\right)$$

$$= \begin{pmatrix}0.68\\0.18\\0.55\\0.60\\0.75\end{pmatrix}$$

For v4:

Since $N(v4) \cup \{v4\} = \{v2, v4, v6\}$, $a_{v4u} = \frac{1}{3}$

$$h_{v4}^1 = \sigma\left(\sum_{N(v4)\cup\{v4\}} a_{v4u} W^1 h_u^0\right)$$

$$= \sigma\left(\frac{1}{3}W^1 h_{v2}^0 + \frac{1}{3}W^1 h_{v4}^0 + \frac{1}{3}W^1 h_{v6}^0\right)$$

$$= \sigma\left(\frac{1}{3}\left(\begin{pmatrix}0.90\\0.40\\0.70\\1.00\\0.90\end{pmatrix} + \begin{pmatrix}-0.10\\0.60\\1.50\\-0.30\\1.00\end{pmatrix} + \begin{pmatrix}1.90\\-0.30\\1.30\\1.20\\0.70\end{pmatrix}\right)\right)$$

$$= \sigma\left(\frac{1}{3}\begin{pmatrix}2.70\\0.70\\3.50\\1.90\\1.70\end{pmatrix}\right)$$

$$= \sigma\left(\begin{pmatrix}0.90\\0.23\\1.17\\0.63\\0.57\end{pmatrix}\right)$$

$$= \begin{pmatrix}0.90\\0.23\\1.17\\0.63\\0.57\end{pmatrix}$$

For v5:

Since $N(v5) \cup \{v5\} = \{v2, v3, v5, v6\}$, $a_{v5u} = \frac{1}{4}$

$$h_{v5}^{1} = \sigma\left(\sum_{N(v5)\cup\{v5\}} a_{v5u} W^{1} h_{u}^{0}\right)$$

$$= \sigma\left(\frac{1}{4}W^{1}h_{v2}^{0} + \frac{1}{4}W^{1}h_{v3}^{0} + \frac{1}{4}W^{1}h_{v5}^{0} + \frac{1}{4}W^{1}h_{v6}^{0}\right)$$

$$= \sigma\left(\frac{1}{4}\left(\begin{pmatrix}0.90\\0.40\\0.70\\1.00\\0.90\end{pmatrix} + \begin{pmatrix}0.30\\0.70\\0.80\\0.60\\1.40\end{pmatrix} + \begin{pmatrix}0.30\\0.90\\0.80\\0.20\\1.40\end{pmatrix} + \begin{pmatrix}1.90\\-0.30\\1.30\\1.20\\0.70\end{pmatrix}\right)\right)$$

$$= \sigma\left(\frac{1}{4}\begin{pmatrix}3.40\\1.70\\3.60\\3.00\\4.40\end{pmatrix}\right)$$

$$= \sigma\left(\begin{pmatrix}0.85\\0.42\\0.90\\0.75\\1.10\end{pmatrix}\right)$$

$$= \begin{pmatrix}0.85\\0.42\\0.90\\0.75\\1.10\end{pmatrix}$$

For v6:

Since $N(v6) \cup \{v6\} = \{v3, v4, v5, v6\}$, $a_{v6u} = \frac{1}{4}$

$$h_{v6}^{1} = \sigma\left(\sum_{N(v6)\cup\{v6\}} a_{v6u} W^{1} h_{u}^{0}\right)$$

$$= \sigma\left(\frac{1}{4}W^{1}h_{v3}^{0} + \frac{1}{4}W^{1}h_{v4}^{0} + \frac{1}{4}W^{1}h_{v5}^{0} + \frac{1}{4}W^{1}h_{v6}^{0}\right)$$

$$= \sigma\left(\frac{1}{4}\left(\begin{pmatrix}0.30\\0.70\\0.80\\0.60\\1.40\end{pmatrix}+\begin{pmatrix}-0.10\\0.60\\1.50\\-0.30\\1.00\end{pmatrix}+\begin{pmatrix}0.30\\0.90\\0.80\\0.20\\1.40\end{pmatrix}+\begin{pmatrix}1.90\\-0.30\\1.30\\1.20\\0.70\end{pmatrix}\right)\right)$$

$$= \sigma\left(\frac{1}{4}\begin{pmatrix}2.40\\1.90\\4.40\\1.70\\4.50\end{pmatrix}\right)$$

$$= \sigma\left(\begin{pmatrix}0.60\\0.48\\1.10\\0.43\\1.13\end{pmatrix}\right)$$

$$= \begin{pmatrix}0.60\\0.48\\1.10\\0.43\\1.13\end{pmatrix}$$

Therefore,

$$H^1 = \left[h_{v1}^1, h_{v2}^1, h_{v3}^1, h_{v4}^1, h_{v5}^1, h_{v6}^1\right]^T$$

$$= \begin{pmatrix}0.47 & 0.33 & 0.68 & 0.90 & 0.85 & 0.60\\0.17 & 0.33 & 0.18 & 0.23 & 0.42 & 0.48\\0.27 & 0.58 & 0.55 & 1.17 & 0.90 & 1.10\\0.67 & 0.33 & 0.60 & 0.63 & 0.75 & 0.43\\0.60 & 0.70 & 0.75 & 0.57 & 1.10 & 1.13\end{pmatrix}^T$$

$$= \begin{pmatrix}0.47 & 0.17 & 0.27 & 0.67 & 0.60\\0.33 & 0.33 & 0.58 & 0.33 & 0.70\\0.68 & 0.18 & 0.55 & 0.60 & 0.75\\0.90 & 0.23 & 1.17 & 0.63 & 0.57\\0.85 & 0.42 & 0.90 & 0.75 & 1.10\\0.60 & 0.48 & 1.10 & 0.43 & 1.13\end{pmatrix}$$

2.2

a. True

b. True

c. False

d. False

Q3
3.1

```
class DistanceAndCount{
public:
	int distance;
	int path_count;
	int visited;

	DistanceAndCount() : distance(INF), path_count(0), visited(-1) {}
	DistanceAndCount(int dist, int count, int pred) : distance(dist),
path_count(count), visited(pred) {}
};


class ShortestPathCountVertex
	: public Vertex<int, DistanceAndCount, DistanceAndCount> {

	void Compute(MessageIterator* msgs) {
		int mindist = IsSource(vertex_id()) ? 0 : INF;
		int path_count = IsSource(vertex_id()) ? 1 : 0;
		int visited= -1;

		for (; !msgs->Done(); msgs->Next()) {
			DistanceAndCount msg = msgs->Value();
			int received_distance = msg.distance;
			int received_count = msg.path_count;
			int visited = msg.visited;

			if (received_distance < mindist) {
				mindist = received_distance;
				path_count = received_count;
			} else if (received_distance == mindist) {
				path_count += received_count;
			}
		}

		if (mindist < GetValue().distance) {
			*MutableValue() = DistanceAndCount(mindist, path_count, visited);
		}

		OutEdgeIterator iter = GetOutEdgeIterator();
		for (; !iter.Done(); iter.Next()) {
			int neighbor = iter.Target();
			if (neighbor != visited) {
```

```
                SendMessageTo(neighbor, DistanceAndCount(mindist + 1,
                                        path_count, vertex_id()));
            }
        }
        VoteToHalt();
    }
};
```

3.2
Iteration 1:
set of active vertices: source node 1
sends a message to its neighbor:
  1 send to 2
  1 send to 4
  1 send to 5
3 messages are sent

Iteration 2:
set of active vertices: node 2,4 and 5
sends a message to its neighbor:
  2 send to 3
  4 send to 3
  5 send to 7
3 messages are sent

Iteration 3:
set of active vertices: node 3 and 7
sends a message to its neighbor:
  3 send to 6
  7 send to 6
2 messages are sent

Iteration 4:
set of active vertices: node 6
sends a message to its neighbor:
  there is no new neighbor
0 message is sent

Iteration 1: 3 messages
Iteration 2: 3 messages
Iteration 3: 2 messages
Iteration 4: 0 messages

3.3
Pseudocode:
```
class MinIntCombiner: public Combiner<DistanceAndPredecessor> {
    virtual void Combine(MessageIterator* msgs) {
        int mindist    = INF;
        int path_count = 0;
        int visited= -1;

        for (; !msgs->Done(); msgs->Next()) {
            DistanceAndPredecessor msg = msgs->Value();
            if (msg.distance < mindist ) {
                mindist    = msg.distance;
                path_count = msg.path_count;
                visited= msg.visited;
            } else if (msg.distance == mindist ) {
                path_count += msg.path_count;
            }
        }

        Output("combined_source", DistanceAndPredecessor(mindist ,
                                  path_count, visited));
    }
};
```

Iteration 1:
set of active vertices: source node 1
sends a message to its neighbor:
    1 send to 2
    1 send to 4
    1 send to 5
3 messages are sent

Iteration 2:
set of active vertices: node 2,4 and 5
sends a message to its neighbor:
    2 send to 3
    4 send to 3
    5 send to 7
messages send to 3 are combined
2 messages are sent

Iteration 3:
set of active vertices: node 3 and 7
sends a message to its neighbor:
    3 send to 6
    7 send to 6
messages send to 6 are combined
1 messages are sent

Iteration 4:
set of active vertices: node 6
sends a message to its neighbor:
    there is no new neighbor
0 message is sent

Iteration 1: 3 messages
Iteration 2: 2 messages
Iteration 3: 1 messages
Iteration 4: 0 messages

Q4

4.1

Betweenness Centrality:

For node C:

B-C-H (has other shortest path, B-A-H, B-F-H)

B-C-I

B-C-I-J (has other shortest path, B-F-G-J)

$c_C = 1/3 + 1 + 1/2 = 1.53$

For node H:

A-H-C (has other shortest path, A-B-C)

A-H-F (has other shortest path, A-B-F, A-D-F)

A-H-G

A-H-I

A-H-G-J, A-H-I-J

C-H-F-D (has other shortest path, C-B-F-D, C-B-A-D)

C-H-A-E (has other shortest path, C-B-A-E)

C-H-F (has other shortest path, C-B-F)

C-H-G (has other shortest path, C-I-G)

D-F-H-I, D-A-H-I (has other shortest path, D-F-G-I)

E-A-H-F (has other shortest path, E-A-B-F, E-A-D-F)

E-A-H-G

E-A-H-I

E-A-H-G-J, E-A-H-I-J

F-H-I (has other shortest path, F-G-I)

$c_H = 1/2 + 1/3 + 1 + 1 + 1 + 1/3 + 1/2 + 1/2 + 1/2 + 2/3 + 1/3 + 1 + 1 + 1 + 1/2 = 10.17$

Closeness Centrality:

For node C:

C-B-A, C-H-A: 2

C-B: 1

C-H-F-D, C-B-F-D, C-B-A-D: 3

C-B-A-E, C-H-A-E:3

C-B-F, C-H-F: 2

C-H-G, C-I-G: 2

C-H: 1

C-I: 1

C-I-J: 2

$c_C = 1/(2+1+3+3+2+2+1+1+2) = 1/17 = 0.06$

For node H:

H-A: 1

H-A-B, H-C-B: 2

H-C: 1

H-A-D, H-F-D: 2

H-A-E: 2
H-F: 1
H-G: 1
H-I: 1
H-G-J, H-I-J: 2
c_H = 1/(1+2+1+2+2+1+1+1+2) = 1/ 13 = 0.08

Therefore, for Betweenness Centrality c_C = 1.53 and c_H = 10.17, for Closeness Centrality c_C = 0.06 and c_H = 0.08

4.2
For node A:
0: none
1: B-A-D, B-A-E, B-A-H, D-A-E, D-A-H, E-A-H
2: A-B-C-I, A-B-F-G, A-D-F-G, A-H-G-J, A-H-I-J
3: B-A-H-G, B-A-H-I, D-A-B-C, D-A-H-C, D-A-H-G, D-A-H-I, E-A-B-C, E-A-B-F, E-A-D-F, E-A-H-C, E-A-H-F, E-A-H-G, E-A-H-I
4: B-A-D,E, B-A-D,H, B-A-E,H, D-A-E,H
GDV(A) = [0,6,5,13,4]
For node G:
0: F-G-H, H-G-I, I-G-J
1: F-G-J
2: G-F-B-A, G-F-B-C, G-F-D-A, G-H-A-B, G-H-A-D, G-H-A-E, G-H-C-B, G-I-C-B
3: F-G-I-C, I-G-F-B, I-G-F-D, J-G-F-B, J-G-F-D, J-G-H-A, J-G-H-C
4: H-G-F,J, I-G,F,J
GDV(G) = [3,1,8,7,2]