MSBA7017 Financial Engineering

1a Bond Analytics

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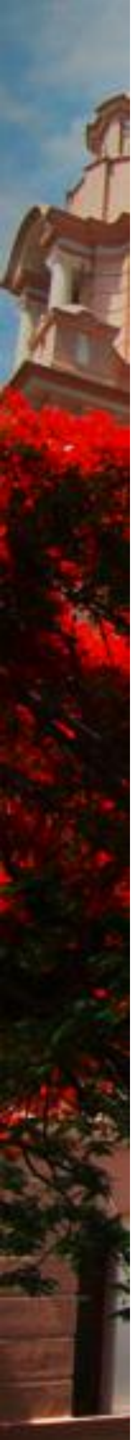
THE UNIVERSITY OF HONG KONG



Agenda

- Basics
- Bond and yield; yield curve
- Interest rate term structure (spot rate curve, forward rate curve).
- Interest Rate Risk (Macaulay Duration, Fisher-Weil Duration, immunization)
- Numerical example:

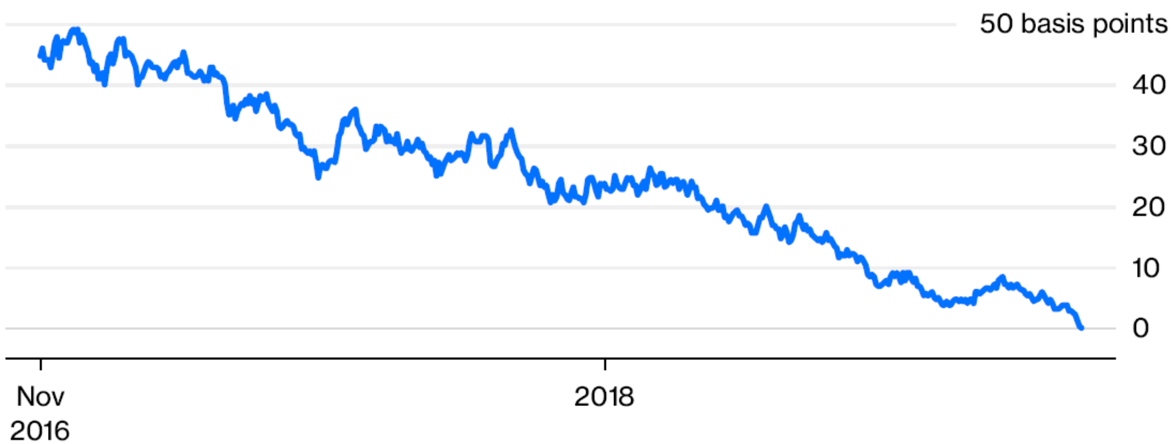
Ex 1 fitting spot rate curve;

Ex 2 Macaulay immunization

A Bloomberg Article

After years of flattening, the yield difference between some Treasury notes falls below zero

Yield spread between 3- and 5-year Treasuries



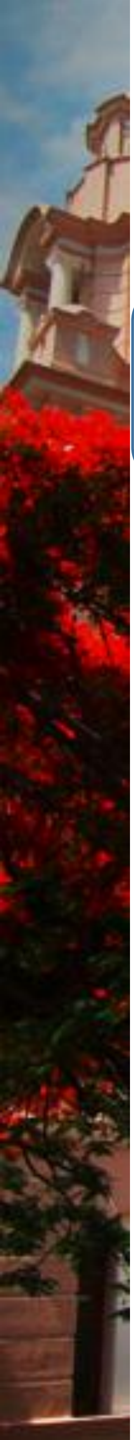
Source: Bloomberg

The move didn’t come out of nowhere. In fact, I wrote a week ago that the spread between short-term Treasury notes was racing toward inversion, and Bloomberg News’s Katherine Greifeld and Emily Barrett noted the failed break below zero on Friday. Still, I wasn’t necessarily expecting this day to come so soon. Rate strategists have long said that being close doesn’t cut it when talking about an inverted yield curve and the well-known economic implications that come with it, namely that the spread between short- and long-term Treasury yields has dropped below zero ahead of each of the past seven recessions.

Basics

On 21 February 2023 3pm (Eastern Time), U.S. Treasury Bill (“T-bill") maturing on 23 February 2023 has price quoted at USD99.9753.

This means: on 21 February 2023 3pm , you pay $99.9753 to the broker and then own this T-bill. If you hold this T-bill until 23 February 2023 , you receive $100, and the T-bill matures (i.e. expires).



Basics

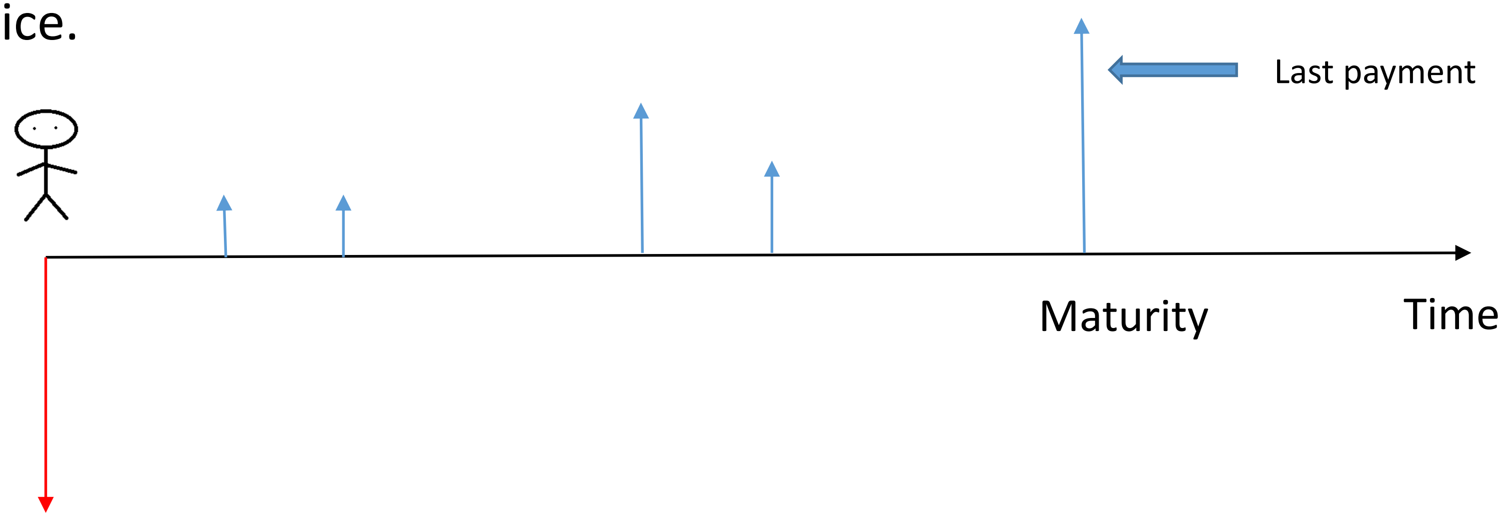
On 21 February 2023 3pm(Eastern Time), U.S. Treasury Note (“T-note") with annual coupon USD1.375 and maturity date 30 September 2023 has price quoted at USD99.2600.

This means: on 21 February 2023 3pm, you pay $99.2600 to own this T-note. If you keep holding this note until the maturity date on 30 September 2023, you will receive:

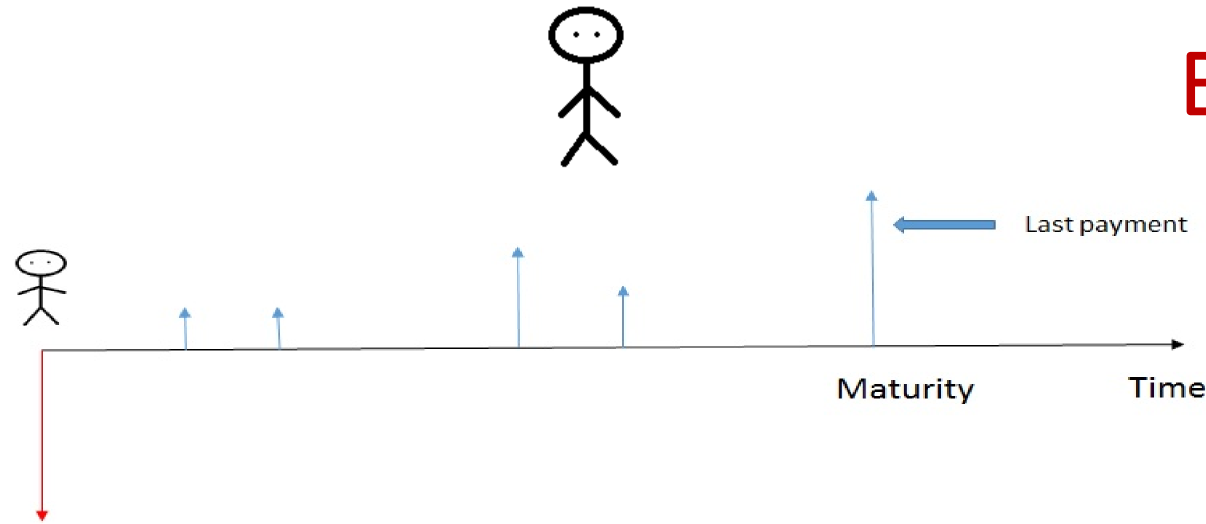
1. Coupon payments on 31 March 2023 and 30 September 2023, each of $0.6875 =1.375/2
2. $100 on 30 September 2023.

Basics

- Fixed-income instruments provide **future cash flows of pre-specified amounts on pre-specified dates**. They are “IOU” (“I owe you”) instruments.
- To get this sequence of future cash payments, one pays an upfront price.



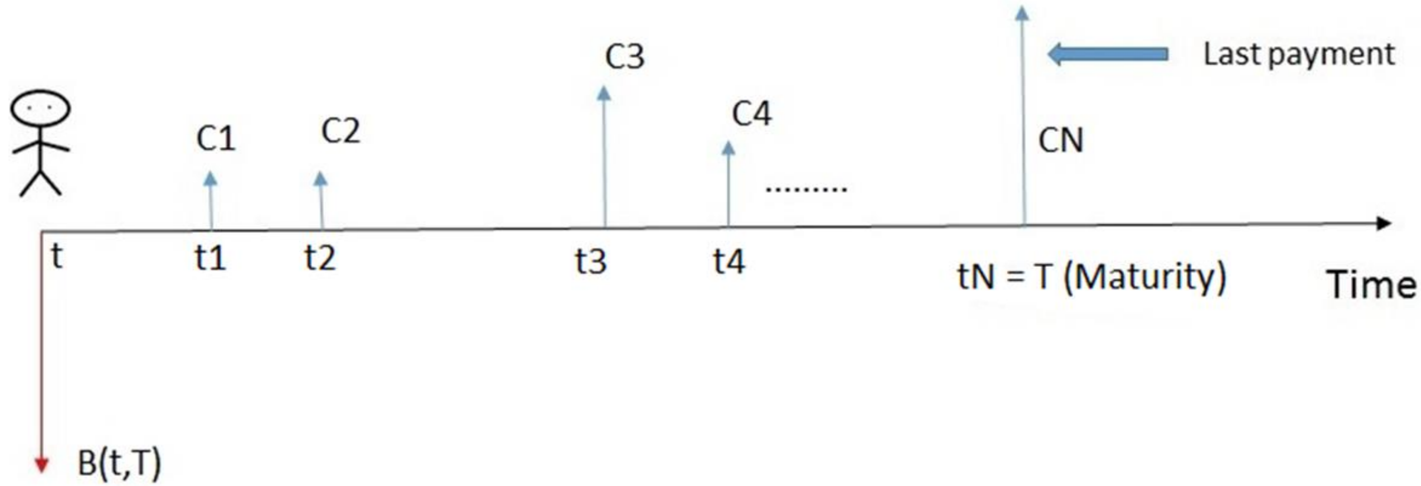
Basics



- The schedule (i.e. when and how much) of future cash payments is determined by the terms of bonds made at the issuance. These are institutional details (not to be discussed in this course).
- The price is determined by market.
- We refer to fixed-income securities in short as bonds.
- Two assumptions throughout this course:

1. The bonds in discussion are default-free.
2. No option-related features are embedded in the bonds.

Bond Price and Yield

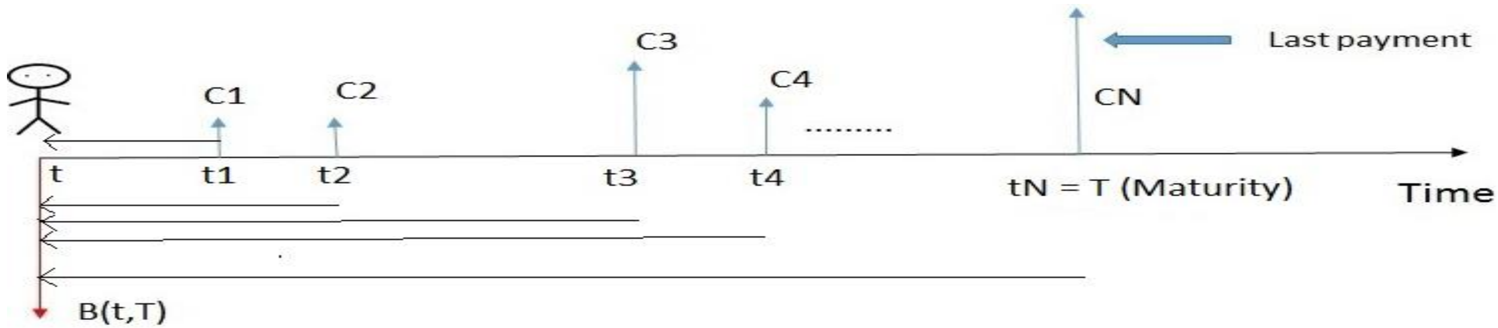


- $B(t,T) = C_1 + C_2 + \cdots + C_N$?

NO. A counter example is the T-Bill example: you pay an amount smaller than \$100 now to receive \$100 on a future date.

- Price is the value expressed in monetary term.
- In general, you demand some reward when lending people money. This is related to interest, also termed time value of money.
- Cash payments occurring on different time points cannot be summed to determine the value.

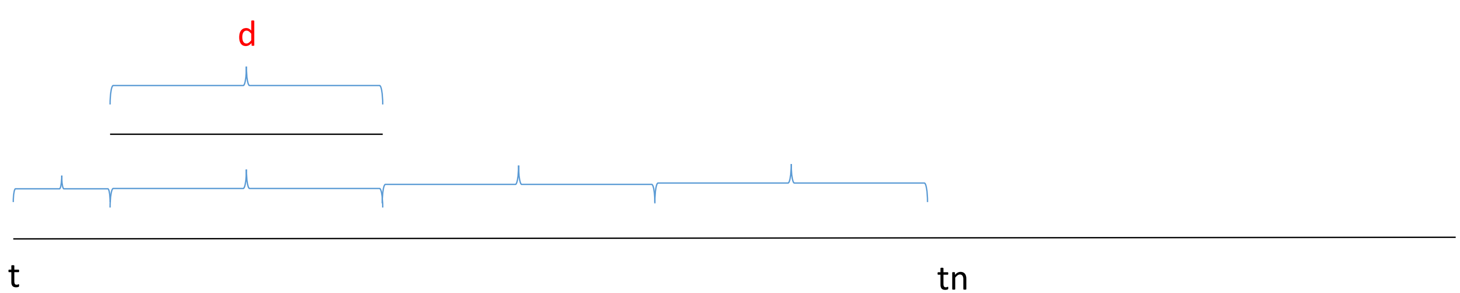
Bond Price and Yield

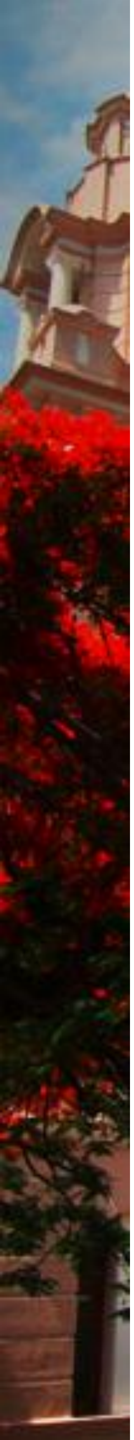


Formula 1 (Bond Price and Yield)

$$B(t,T) = \sum_{n=1}^{N} \frac{C_n}{(1+y)^{\frac{t_n-t}{d}}} = \frac{C_1}{(1+y)^{\frac{t_1-t}{d}}} + \frac{C_2}{(1+y)^{\frac{t_2-t}{d}}} + \cdots + \frac{C_N}{(1+y)^{\frac{t_N-t}{d}}}$$

- y is termed yield. It is quoted with respect to a reference time interval of length d.
- Heuristically it reflects the "interest" prevailing over a time period of length d.
- Then, $\frac{t_n-t}{d}$ is number of the reference time intervals between now (t) and the n-th cash payment date.

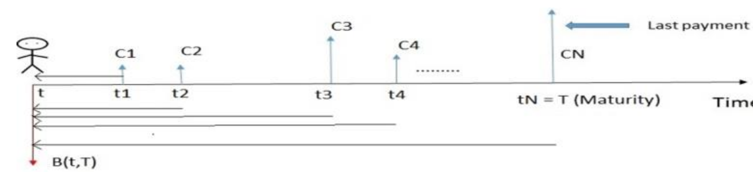




- $B(t,T) = \sum_{n=1}^{N} \frac{C_n}{(1+y)^{\frac{t_n - t}{d}}} = \frac{C_1}{(1+y)^{\frac{t_1 - t}{d}}} + \frac{C_2}{(1+y)^{\frac{t_2 - t}{d}}} + \cdots + \frac{C_N}{(1+y)^{\frac{t_N - t}{d}}}$

Bond Price and Yield

- $\frac{C_n}{(1+y)^{\frac{t_n - t}{d}}}$ discounts the n-th cash payment back to now (t).
- Now, we can add the discounted cash flows, that is, $\frac{C_n}{(1+y)^{\frac{t_n - t}{d}}}$, to reach $B(t,T)$.
- Note only cash payments occurring ahead of now (t) enters the formula.



- Everything else being equal, B(t,T) decreases with y, and the vice versa.
- (Not req.) Everything else being equal, B(t,T) is *convex* in y.
- <u>Summary:</u> yield provides a way to discount the cash payments, such that the discounted payments amount to the observed bond price.

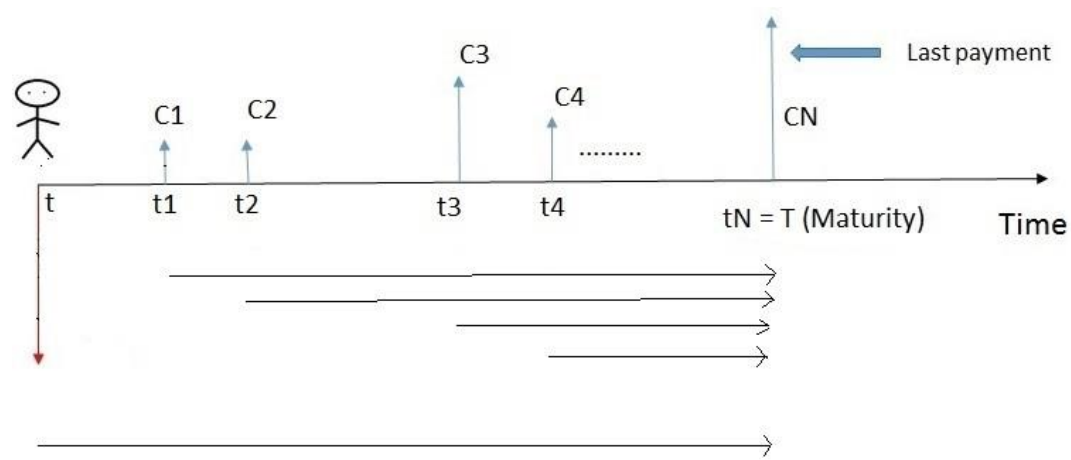
Bond Price and Yield

Remarks

a) Price determines yield. That is: first, price is observed, which is determined by market force; then, yield is implied by the formula. Thus, yield is a representation, or a summarizing measure, of the bond price, not used for pricing.

b) Yield partially reflects (or, summarizes) interest rate.

$$B(t,T)(1+y)^{\frac{t_N-t}{d}} = \sum_{n=1}^{N} C_n(1+y)^{\frac{t_N-t_n}{d}}$$



Bond Price and Yield

(Remarks ct'd)

b) (ct'd) This assumes two things: first, the bond is held until maturity; second, each cash payment is reinvested at rate y (the yield) after being received. Both assumptions are unlikely to happen.

c) In light of a) and b), again, yield is not used for pricing, because it is not a bona-fide interest rate. To price bonds, we need a common interest rate structure.

d) The yield, y, is always quoted with a reference interval length d. But this is not important, just a convention. Changing values of d changes the value of y to make the price-yield formula hold.

e) Commonly the yield is reported in annualized form. E.g. if d = 0.5, then 2y is reported.

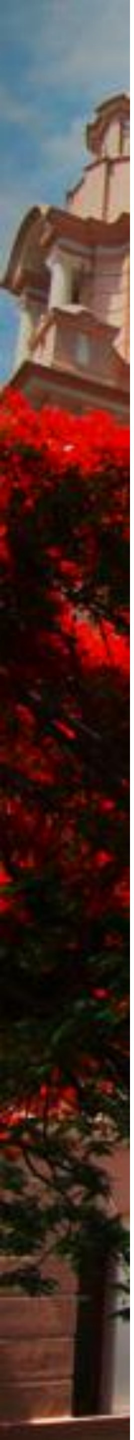
f) (complement to d) above). Although d is not important, when you see a yield quoted with a bond price, or simply a quoted yield, you should still pay attention to which d is used (U.S. treasury uses half year).

Bond Price and Yield

- Yield summarizes a bond in one number, thus convenient for comparison (e.g. high yield vs low yield). But in general it is not a measure of comparative advantage.
- A class of convenient risk analytics is bases on yields (to be discussed later).
- Yield is easy to compute, and shape of yield curve (next slide) is a useful indicator.

- **Question**: Formula 1 is used to compute the yield. But usually the computed yield might deviate from the actual yield (due to human mistake, limit of numerical precision, etc.). Thus, when you plug the computed yield back to Formula 1, the LHS (i.e. the bond price) looks different from the quoted price.

Which is more sensitive to the yield error, long term bond or short term bond? (And why?)



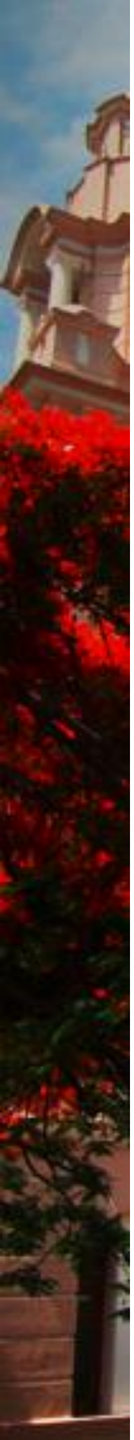
Yield Curve

- x-axis is time to maturity (or maturity); y-axis is the yield.

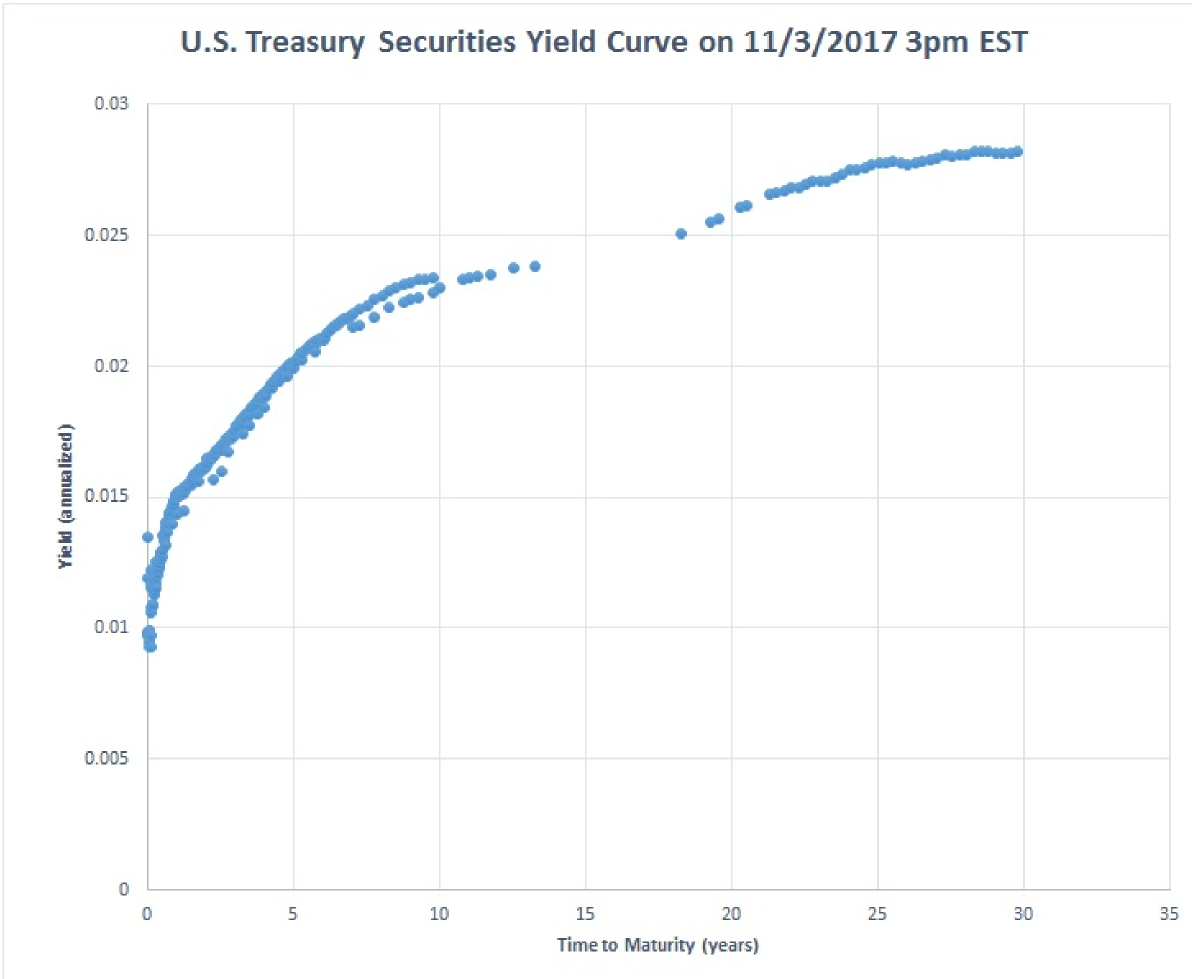
- Different points on the curve reflect bonds with different maturities (e.g. 10- year T-note of coupon rate 1.5% versus 30-year T-bond of coupon rate 2.0%).

- Bonds on the same yield curve should belong to the same class (e.g. U.S. government bonds; or investment grade corporate bonds, etc.)

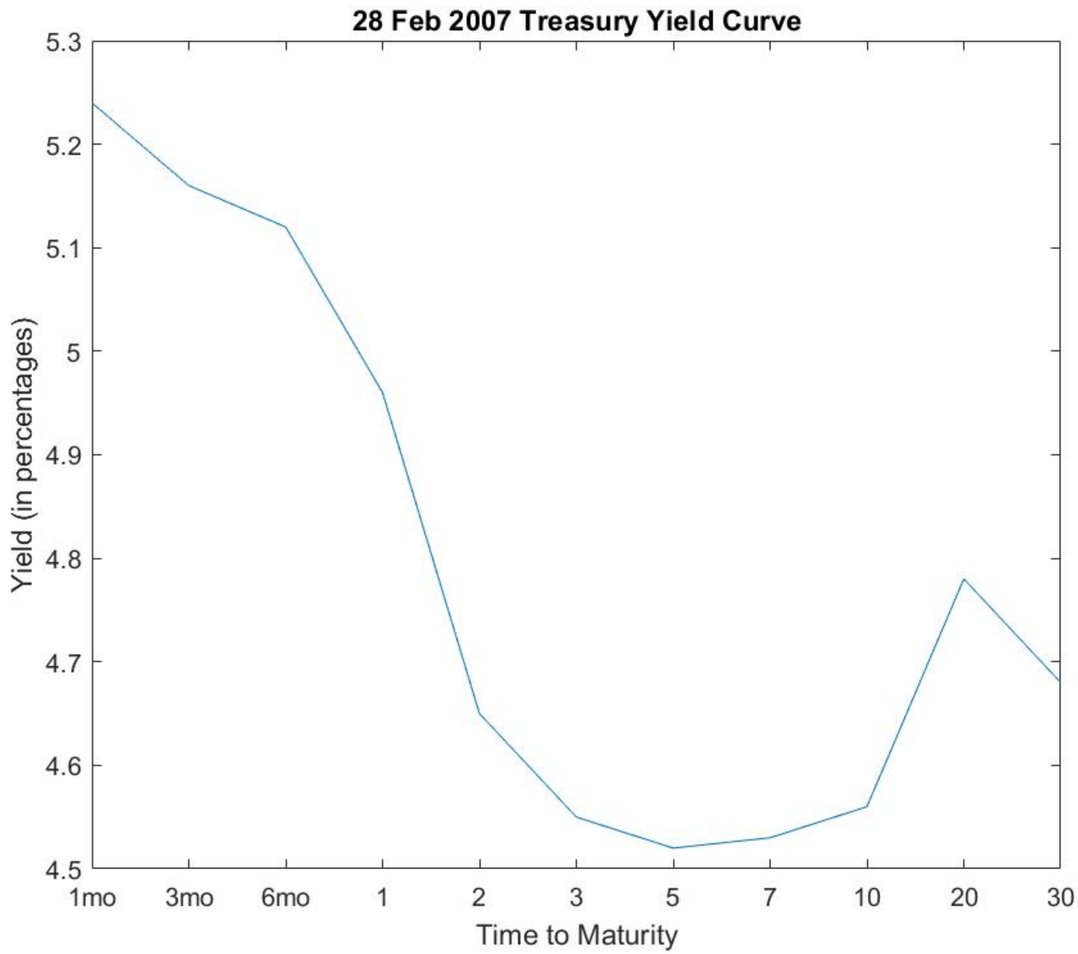
- In normal situation, the yield curve is “increasing and concave”.

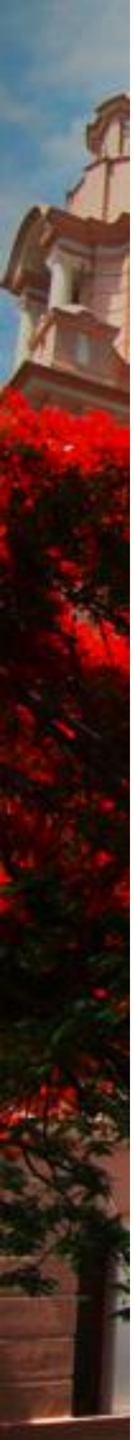


Yield Curve


U.S. Treasury Securities Yield Curve on 11/3/2017 3pm EST
0.03
0.025
0.02
0.015
0.01
0.005
0
Yield (annualized)
0
5
10
15
20
25
30
35
Time to Maturity (years)

Yield Curve


28 Feb 2007 Treasury Yield Curve
Yield (in percentages)
5.3
5.2
5.1
5
4.9
4.8
4.7
4.6
4.5
1mo
3mo
6mo
1
2
3
5
7
10
20
30
Time to Maturity

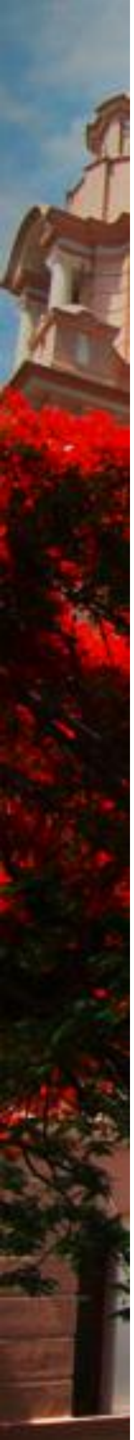


- Now you should be able to read the kind of article put in the beginning.

Spot Rate

- We looked at relationship between bond price and yields.

- Yield is derived from observed bond price, useful in comparison and risk analytics (to be discussed later) and being indicative of economic/market condition, but not used for pricing: a yield is tied up to a specific bond, and never enters the price formula of any other bond.



Spot Rate

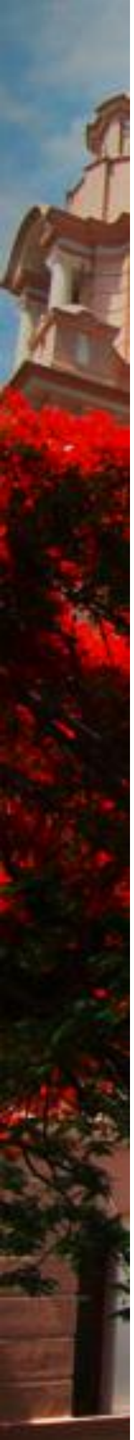
- For bond pricing, we need a common structure for interest rates.
- This starts from an important property of interest rate: money lent for different lengths of time demands different amount of interest.
- That is, interest rates vary with time to maturity, and another word for time to maturity is "term".

Spot Rate

- Your checking account pays extremely low interest rate; CD (certificate of deposit) pays relatively higher.

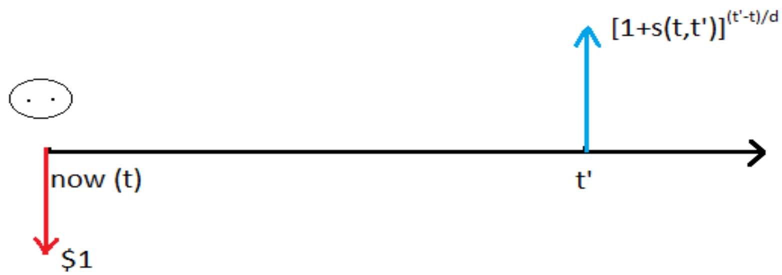
- Long-term bonds tend to have higher yields (hence “cheaper”) than the short-term bonds in normal situation.

- You take a loan and spread the payment into a long period, you bear higher interest rate.



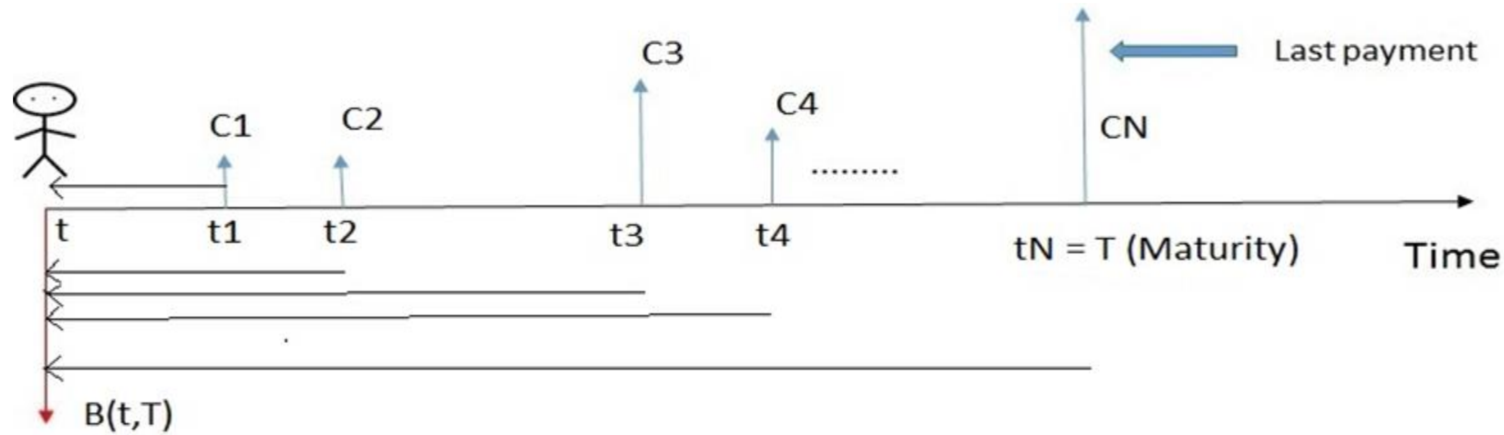
Spot Rate

- Money lent now (t) and to be paid (with interest) on a future date t’ is called a spot loan.
- The interest rate prevailing from t to t’ is called spot rate. Spot rate is denoted by $s(t, t')$, and it is agreed on t (the loan date).



- On the other hand, \$1 paid on t’ is worth \$$\frac{1}{[1+s(t,t')]^{(t'-t)/d}}$ now.
- We can also say: the price of a spot loan paying \$1 on t is the number above.

Spot Rate



$$B(t,T) = \sum_{n=1}^{N} \frac{C_n}{(1+y)^{\frac{t_n-t}{d}}} = \frac{C_1}{(1+y)^{\frac{t_1-t}{d}}} + \frac{C_2}{(1+y)^{\frac{t_2-t}{d}}} + \cdots + \frac{C_N}{(1+y)^{\frac{t_N-t}{d}}}$$

1. Yield is a single rate, y, and it "serves as" the interest rate. It is the same for all terms.

2. A bond is just a collection of several spot loans. So the bond price is just the sum of prices of all the spot loans.

Bond Price and Spot Rates

Formula 2 (Bond Price and Spot Rates)

$$B(t,T)=\sum_{n=1}^{N}\frac{C_n}{(1+s(t,t_n))^{\frac{t_n-t}{d}}}=\frac{C_1}{(1+s(t,t_1))^{\frac{t_1-t}{d}}}+\frac{C_2}{(1+s(t,t_2))^{\frac{t_2-t}{d}}}+\cdots+\frac{C_N}{(1+s(t,t_N))^{\frac{t_N-t}{d}}}$$

- The LHS of the yield formula is bond price, and the same here.
- The structure of the RHS is also the same.
- We just change y into $s(t,t_n)$.
- Each $\frac{C_n}{(1+s(t,t_n))^{\frac{t_n-t}{d}}}$ is the price of the spot loan paying C_n on t_n.

Spot Rates

- $s(t,t')$ is in general not directly observable. It is an underlying interest rate model.
- It is up to us to define the reference interval length, d.
- Trick: make d very small, that is, $d \approx 0$. The associate $s(t,t')$ is called continuous compounding spot rate.
- This gives a much cleaner formula.

Formula 3 (Bond Price and Continuous Compounding Spot Rates)

$$B(t,T) = \sum_{n=1}^{N} C_n e^{-s(t,t_n)(t_n-t)} = C_1 e^{-s(t,t_1)(t_1-t)} + C_2 e^{-s(t,t_2)(t_2-t)} + \cdots + C_N e^{-s(t,t_N)(t_N-t)}$$

- Recall, e is the exponential constant, $e = 2.718281 \ldots$
- Formula 3 and Formula 2 are *not* conflicting --- the s(t,t') in these two formulae are different. I used the same symbols just to prevent more notation.

Spot Rate and Yield

Equating Formula 1 and Formula 3:

$$\sum_{n=1}^{N} \frac{C_n}{(1+y)^{\frac{t_n - t}{d}}} = B(t,T) = \sum_{n=1}^{N} C_n e^{-s(t,t_n)(t_n - t)}$$

- The yield, y, relates to a collection of spot rates.
- It reflects (or summarizes) the interest rates.
- Equating Formula 2 with Formula 1 draws the same insight.

Spot Rate Curve

- x-axis: $t' - t$; y-axis: $s(t,t')$.
- We have a spot rate curve on time t.
- The shape of spot rate curve is called the Spot Rate Term Structure.
- Spot rate curve is built from data.
- Once spot rate curve is built, it is used for pricing:

1. bonds with quoted prices. You can compare the quoted price with the price predicted by the spot rate curve.
2. bonds with missing prices.
3. interest rate derivatives.
4. cash flows from private transactions.
5. other cash flows.

In any case, spot rate curve for one class of bonds can only be used to value assets associated with that class (for example, do not use treasury spot rate curve to price junk bonds).

Exercise: Spot Rate Curve

- A spot rate curve is fitted as follows (current time is t):

$$s(t, t') = 0.05 + 0.005(t' - t), \quad t' \geq t.$$

- There are two bonds, with payment schedule summarized in the following table:

	t+0.25	t+0.5	t+0.75
Bond 1	\$1		
Bond 2			\$2

- Question: the two bonds are customized to meet the client's cash flow needs, so no quoted prices for them yet. What should be the price for each?

Building Spot Rate Curve

- **Step 1**: Pre-impose a functional form on s(t, t'). This usually involves a set of parameters, denoted Θ.

- **Step 2**: for bond m, compute the model-implied price, $\hat{B}_m$, by invoking Formula 2 or 3. Let B_m denote the observed price. Note $\hat{B}_m$ depends on Θ.

- **Step 3**: solve the MSE problem:

$$\min_{\Theta} \sum_{m=1}^{M} \left(\hat{B}_m - B_m \right)^2$$

The MSE problem is mostly likely not solvable by mathematical derivation, but there is a set of out-of-shelf optimization software and packages. We will use software (Excel solver, or Python packages).

Building Spot Rate Curve

- Functional forms for Step 1 commonly applied are polynomial-based. Denote $\tau = t' - t$.

- 3rd-order polynomial: $s(t, t') = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3$.

  Then, $\Theta = \{a_0, a_1, a_2, a_3\}$, which are determined by solving the MSE in Step 3.

- 4th-order polynomial: $s(t, t') = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4$.

  Then, $\Theta = \{a_0, a_1, a_2, a_3, a_4\}$.
- Caveat: polynomials of high orders behave crazily out of the data range.
- Low-order polynomial has too simple a shape to possibly describe the underlying interest rate structure.
- Cubic splines is a remedy. We will investigate cubic splines in numerical example.

Building Spot Rate Curve

Remark

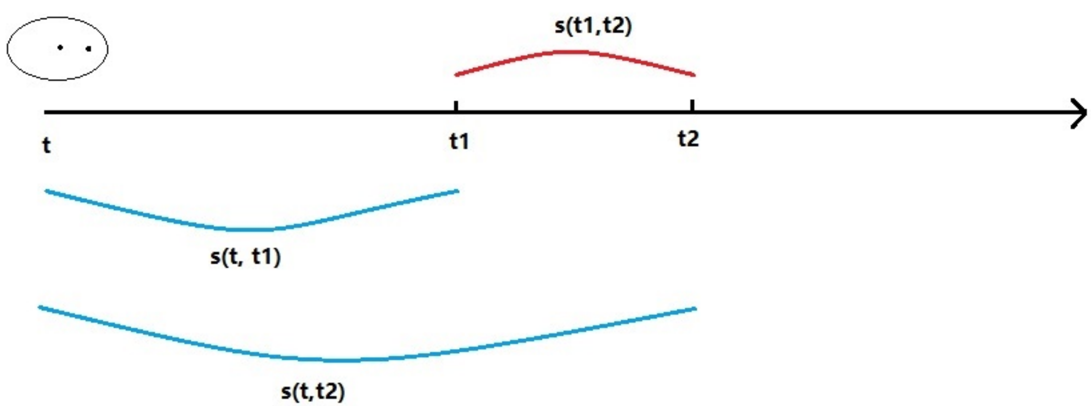
Building spot rate curve by price matching (match the model-implied price to data) is natural. But it might under-weigh the yield errors for the short-term bonds.

Practitioners may want to match the yield: minimize the error between observed yield and model-implied yield. Formula 1 is used to convert the model-implied bond price into yield (also implied by model).

Forward Rate

- Spot rate prevails over [t ,t’]. That is, prevailing from now (t) to a future date t’.
- Two spot rates, s(t, t1) and s(t, t2), lock interest rate prevailing [t1, t2].

- This rate is locked now (t), to be earned over the future time window [t1, t2]. So we denote it by $f(t, t_1, t_2)$, called forward rate.



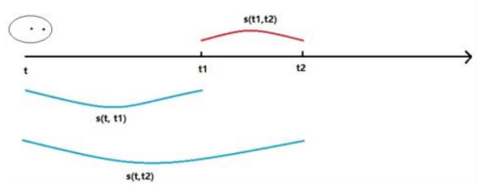
Forward Rate

- The forward rate is implied by spot rates (and the vice versa). Hence the reference interval length d appears also in forward rate, and it is the same as that for spot rates.
- We got rid of d by introducing the continuous compounding spot rate. This also gives associated continuous compounding forward rate.

Formula 5 (Continuous Compounding Forward Rate)

$$f(t,t_1,t_2) = \frac{s(t,t_2)(t_2-t) - s(t,t_1)(t_1-t)}{t_2-t_1}$$

- The version with reference interval being d is in Formula 4 of [N].



Forward Rate

- Let $B_0(t, t_1)$ be the price of a bond paying \$1 on t_1, call it "short bond". Likewise define for $B_0(t, t_2)$, and call it "long bond".
- Now (t), issue one unit of short bond, and use the proceeds to buy the long bond: you can buy $\frac{B_0(t,t_1)}{B_0(t,t_2)}$ units of the long bond.
- Your net cash flow now is 0. But these two transactions create a guaranteed cash outflow \$1 on t_1, and cash inflow $\frac{B_0(t,t_1)}{B_0(t,t_2)}$ on t_2.
- That is, now(t), you are sure that you will lend \$1 on t_1, and later on t_2 you receive $\frac{B_0(t,t_1)}{B_0(t,t_2)}$.
- In continuous compounding form, this means:

$$e^{f(t,t_1,t_2)\times(t_2-t_1)} = \frac{B_0(t, t_1)}{B_0(t, t_2)}$$

- Plugging Formula 3 to re-express $\frac{B_0(t,t_1)}{B_0(t,t_2)}$ gives Formula 5.

$$B(t,T) = \sum_{n=1}^{N} C_n e^{-s(t,t_n)(t_n-t)} = C_1 e^{-s(t,t_1)(t_1-t)} + C_2 e^{-s(t,t_2)(t_2-t)} + \cdots + C_N e^{-s(t,t_N)(t_N-t)}$$

Forward Rate Curve

- In practice, people fix $t_2 - t_1$, i.e. the length of the future interval at some number τ.
- Then, forward rate is written as $f(t, t_1, t_1 + \tau)$.
- Plot $f(t, t_1, t_1 + \tau)$ against t_1 gives the forward rate curve. This is the term structure for forward rates.

- Questions: Let the spot rate (hence also forward rate) be continuous compounding.

1. Given $s(t, t_1)$ and $f(t, t_1, t_2)$, can we determine $s(t, t_2)$?
2. Given $s(t, t_1)$, $f(t, t_1, t_2)$, $f(t, t_2, t_3)$, how to determine $s(t, t_3)$?
3. Given $f(t, t_1, t_2)$ and $f(t, t_2, t_3)$, how to determine $f(t, t_1, t_3)$?
4. What's the pattern here?

Exercise: Forward Rate Agreement (FRA)

- All rates are continuous compounding.
- You agree with a counter party to exchange cash flows on a future time period [t1, t2]. The agreement is made now(t).
- On t1, the spot rate prevailing [t1,t2] will be observed, denote s(t1,t2).
- The agreement is implemented on time t2: you pay the counter-party $\$e^{s(t_1,t_2)(t_2-t_1)}$.
- In return, the counter party pays you (also on time t_2) $\$e^{f(t_2-t_1)}$. The value f is determined and agreed between you and the counter party.
- Assumption: bonds of any payment structure are freely traded w/o transaction costs; investors can issue any bonds.
- Question: what is the value of f which makes entering either positions of this agreement of price 0?

FRA (ct'd)

- Answer: the forward rate $f(t, t_1, t_2)$.
- Elaboration: let $B_0(t, t_1)$ denote a bond paying \$1 on t_1; likewise $B_0(t, t_2)$ is a bond paying off \$1 on t_2. On t, enter the FRA (as the party paying the $\$e^{s(t_1,t_2)(t_2-t_1)}$); meanwhile, buy $B_0(t, t_1)$, and issue $\frac{B_0(t,t_1)}{B_0(t,t_2)}$ units of $B_0(t, t_2)$. This creates 0 cash flow on t. Note by this construction, you are entitled to receive \$1 on t_1 and obliged to pay $\$\frac{B_0(t,t_1)}{B_0(t,t_2)}$ on t_2.
- On t_1, the spot rate $s(t_1, t_2)$ is observed. Also, you receive \$1 (from the investment in $B_0(t, t_1)$) and use this money immediately to buy a bond that pays off \$1 on t_2 (denote this bond by $B_0(t_1, t_2)$). Note you can buy $\frac{1}{B_0(t_1,t_2)}$ units of it. Note $B_0(t_1, t_2) = \$e^{-s(t_1,t_2)(t_2-t_1)}$.
- On t_2, you receive $\$\frac{1}{B_0(t_1,t_2)}$; pass this money to the other party of the FRA, meanwhile receive $\$e^{f(t_2-t_1)}$. Recall you are also obliged to pay $\$\frac{B_0(t,t_1)}{B_0(t,t_2)}$.

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FRA (ct'd)

- In summary, the net effect of the above gives you a cash flow of \$$\left(e^{f(t_2-t_1)} - \frac{B_0(t,t_1)}{B_0(t,t_2)}\right)$.
- The only additional cash flow is the price of entering FRA on t.
- For this price to be 0, we need $e^{f(t_2-t_1)} - \frac{B_0(t,t_1)}{B_0(t,t_2)} = 0$.

- Now you know yield curve, spot rate curve and forward rate curve.
- Don’t confuse among them.

Interest Rate Risk

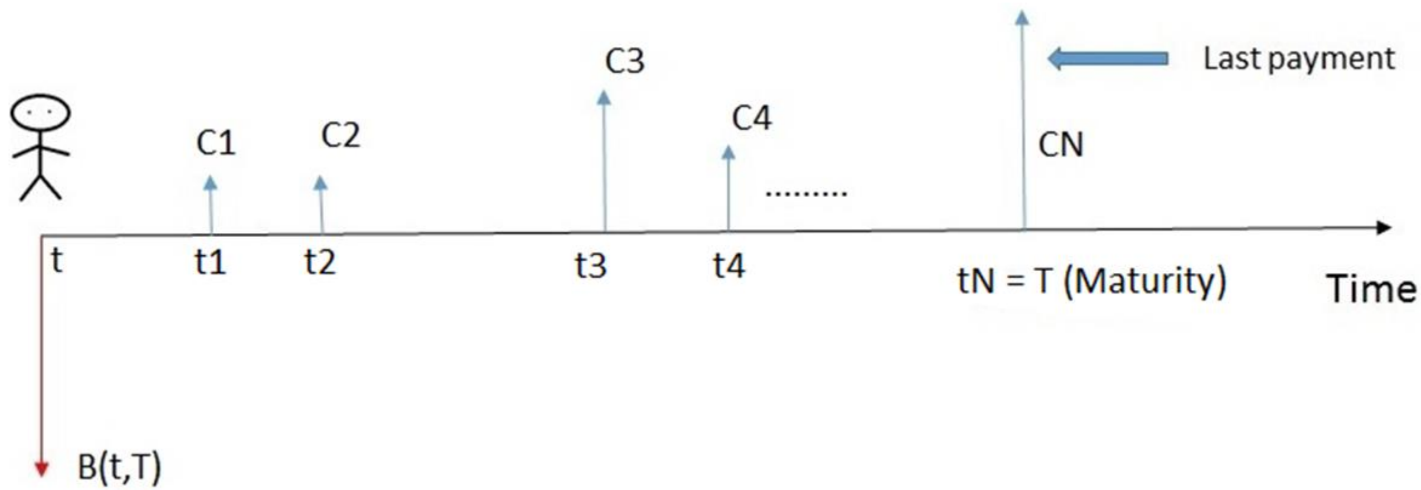
- Bonds can be invested to reserve funds for future cash outlay:

1. Children's university education.
2. Buy houses for children.
3. Retirement plan.
4. Insurance companies facing regular claims.
5. etc.

- At time t, we know there is a cash outlay of amount $C on time T.
- We cannot find a bond whose payment will happen exactly on T.
- If the cash payment happens before T, we will have to put the money somewhere. This creates reinvestment risk.
- If the cash payment happens after T, then we have to sell the bond to raise cash for the obligation. Then price is uncertain, because interest at that time (which determines bond price) is uncertain.
- Another perspective: when interest rate is down, $C becomes more "expensive"; otherwise, it is "cheaper".
- They all relate to interest risk: interest rate is volatile, and the bond prices also go up and down.

Macaulay Duration

- Macaulay duration is a yield-based bond's interest risk measure.
- Recall, yield is understood to partly reflect or summarize interest rates (again, it is NOT an interest rate).
- Macaulay duration quantifies (to some extent) how bond price changes when the yield moves a little bit.



Macaulay Duration

Formula 6 (Macaulay Duration)

Write $B(t,T)$ as $B(y)$. Then, the associated Macaulay duration is defined as:

$$D = \sum_{n=1}^{N} \left[\frac{C_n}{(1+y)^{\frac{t_n - t}{d}} B(y)} \right] (t_n - t)$$

$D_M = D/(1+y)$ is called modified duration. If the yield y changes to $y + d\delta$ for a small δ (due to a small shift in the annualized yield), the corresponding change in bond prices is:

$$B(y + d\delta) - B(y) \approx -D_M B(y)\delta = -\frac{D}{1+y} B(y)\delta.$$

- The change in yield (from y to $y + d\delta$) is assumed to be instantaneous (i.e. happening in an instant), and after the change we are still standing on time t.
- Note the number, d, is the same d as that for y.
- Read [N] for derivation of Formula 6.

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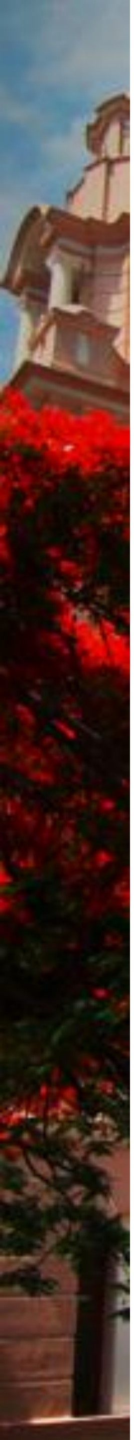

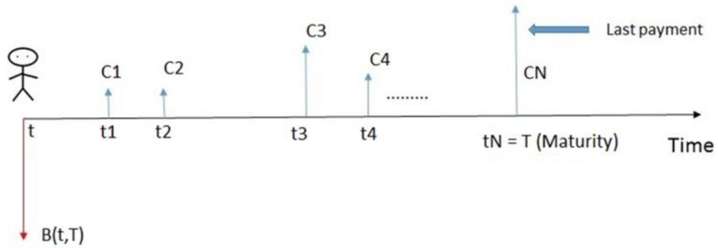

Macaulay Duration

$$D = \sum_{n=1}^{N} \left[\frac{C_n}{(1+y)^{\frac{t_n - t}{d}} B(y)} \right] (t_n - t)$$

- $\dfrac{C_n}{(1+y)^{\frac{t_n-t}{d}} B(y)} = \dfrac{C_n/(1+y)^{\frac{t_n-t}{d}}}{B(y)}$
- $C_n/(1+y)^{\frac{t_n-t}{d}}$ is the "value" of the n-th cash payment, if we "treat" y as the interest rate (again, y is not interest rate…).
- All $C_n/(1+y)^{\frac{t_n-t}{d}}$ add up to $B(y)$.
- Hence, the term in [ ] is the fraction of value of n-th payment in the total bond price.
- $t_n - t$ is the time (from now) to the n-th cash payment.
- Hence, Macaulay duration is a weighted average of all payment times.
- Question: What does N = 1 imply?

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Macaulay Duration of A Bond Portfolio

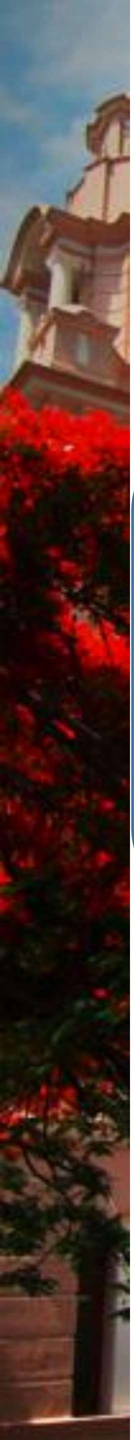
Formula 7 (Macaulay Duration for A Bond Portfolio)

We have multiple bonds, indexed by i= 1,2,…,m. Suppose they have a common yield, y. Let B_i be the bond price of i-th bond, and D_i be its Macaulay duration. Let $B = B_1 + B_2 + \cdots B_m$.

Then, the Macaulay duration of this bond portfolio, denoted D, is:

$$D = \sum_{i=1}^{m} \frac{B_i}{B} D_i.$$

- Macaulay duration of a bond portfolio is a weighted average of the Macaulay durations of individual bonds in this portfolio.
- The weight is the proportion of that bond price of the sum of bond prices.
- The assumption that the bonds have the same yield is quite strong. In practice it can be replaced by the average yield.



Formula 3 (Bond Price and Continuous Compounding Spot Rates)

$$B(t,T) = \sum_{n=1}^{N} C_n e^{-s(t,t_n)(t_n-t)} = C_1 e^{-s(t,t_1)(t_1-t)} + C_2 e^{-s(t,t_2)(t_2-t)} + \cdots + C_N e^{-s(t,t_N)(t_N-t)}$$

Fisher-Weil Duration

- Fisher-Weil duration is based on continuous compounding spot rate curve (Formula 3).
- More reasonable reflection of interest rate risk.

Formula 8a (Fisher-Weil Duration)

Let $s(t,t')$ be the spot rate *curve* for all $t' \geq t$. Let this spot rate curve be denoted s. Write $B(t,T)$ as $B(s)$.

Then, the Fisher-Weil duration of a bond is defined as:

$$D_{FW} = \sum_{n=1}^{N} \left[\frac{C_n e^{-s(t,t_n)(t_n-t)}}{B(s)} \right] (t_n - t)$$

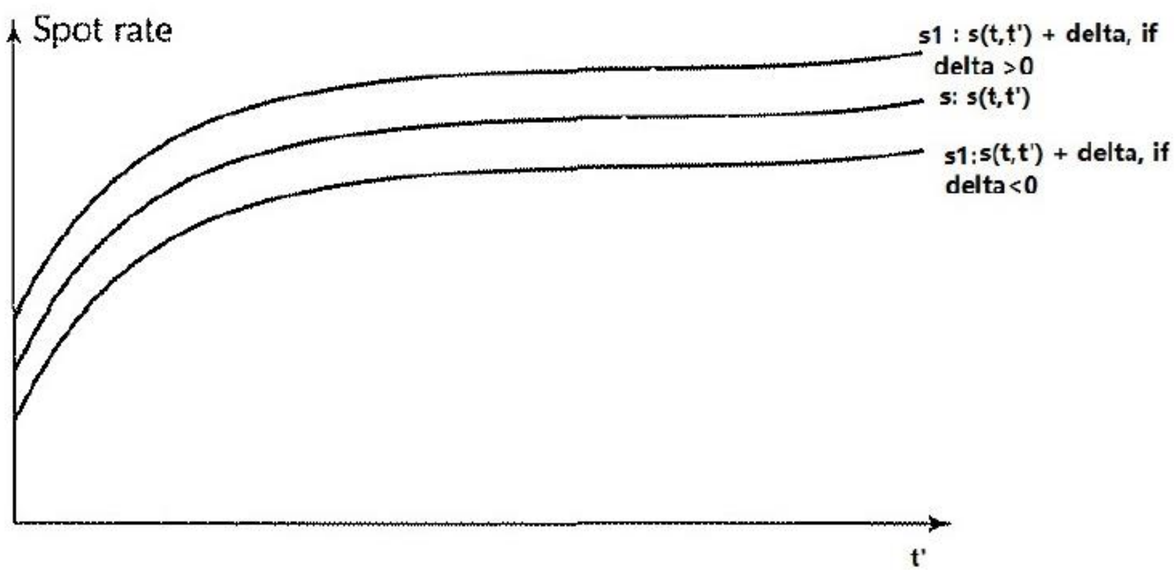
- Similar interpretation as Macaulay duration. Even better, $C_n e^{-s(t,t_n)(t_n-t)}$ is indeed the value of of n-th payment.
- When dealing with multiple bonds, FW-duration does not require them to have the same yield (since yield does not appear). See Formula 9 of [N].

Formula 3 (Bond Price and Continuous Compounding Spot Rates)

$$B(t,T) = \sum_{n=1}^{N} C_n e^{-s(t,t_n)(t_n-t)} = C_1 e^{-s(t,t_1)(t_1-t)} + C_2 e^{-s(t,t_2)(t_2-t)} + \cdots + C_N e^{-s(t,t_N)(t_N-t)}$$

Fisher-Weil Duration

- To compute the bond price change, we first have to specify how the original spot rate curve changes.
- There are many spot rates on the same curve.
- Compromise: parallel-shift.
- $s_1(t,t') = s(t,t') + \delta$, for all $t' \geq t$.



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Fisher-Weil Duration

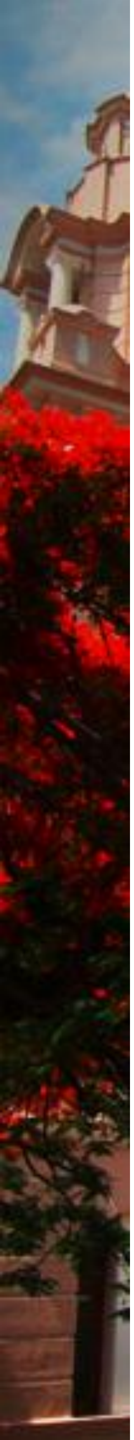
Formula 8b (Fisher-Weil Duration and Bond Price Sensitivity)

Continue from Formula 8a. Let the spot rate curve experiences an instantaneous parallel shift from s to s_1. That is, $s_1(t,t') = s(t,t') + \delta$, for all $t' \geq t$.

Then,

$$B(s_1) - B(s) \approx -D_{FW} B(s)\delta$$

- Note there is no modified duration needed.
- FW-duration for bond portfolio is the same as that for Macaulay duration; see Formula 9.



Fisher-Weil Duration and Macaulay Duration

Difference

- Macaulay duration is based on yield: it quantifies change in a bond price due to change of its yield. Whereas FW duration is based on spot rate curve: it quantifies change in a bond price due to a parallel shift of spot rate curve.
- When computing for bond portfolio's duration, Macaulay duration requires their yield to be the same; FW-duration does not.

Similarity

- Both quantify bond price changes due to a changes in interest (yield partially reflects/summarizes interest rates).
- Both have the form of weighted average of payment times.

Immunization

- We are on time t, there is a cash outlay (expenditure, obligation,..etc.) to happen on a future time T of amount $C.
- Want to buy bonds now to help with this outlay.
- Now we have two bonds, bond 1 and bond 2, with *durations* (either Macaulay or FW) D_1 and D_2; and bond prices B_1 and B_2.
- To decide: how many bond 1 to buy, and how many bond 2 to buy?

Immunization

- We want to use bond 1 and bond 2 to hedge for the cash outlay, \$C.
- First, we want the value of the bond portfolio coincide with the outlay value, \$C. Because ultimately we will liquidate the bond portfolio to cover the outlay.
- Second, we want the duration of the bond portfolio to coincide with that of the outlay.
- In this lecture we only do Macaulay duration immunization. The one for FW-duration is in [N], left for self reading (and also homework).

Immunization (Macaulay)

Formula 10 (Immunization Based on Macaulay Duration)

Current time is t, a cash outlay of amount $C happens on T > t. There are two bonds, bond 1 and bond 2, with prices B_1 and B_2, Macaulay duration D_1 and D_2. They have a common yield y, associated with a reference interval length d.

The immunization portfolio using bond 1, 2 and immunizes against the cash outlay involves buying x_1 units of bond 1, and x_2 units of bond 2. x_1 and x_2 are decided by:

$$x_1 B_1 + x_2 B_2 = \frac{C}{(1+y)^{\frac{T-t}{d}}} \ldots .(value\ match)$$

$$\left[\frac{x_1 B_1}{C/(1+y)^{\frac{T-t}{d}}}\right] D_1 + \left[\frac{x_2 B_2}{C/(1+y)^{\frac{T-t}{d}}}\right] D_2 = T - t \ldots \ldots .(risk\ match)$$

Immunization (Macaulay)

$$\left[\frac{x_1 B_1}{C/(1+y)^{\frac{T-t}{d}}}\right] D_1 + \left[\frac{x_2 B_2}{C/(1+y)^{\frac{T-t}{d}}}\right] D_2 = T - t$$

- LHS: note by the first equation, LHS is equivalent to:

$$\left[\frac{x_1 B_1}{x_1 B_1 + x_2 B_2}\right] D_1 + \left[\frac{x_2 B_2}{x_1 B_1 + x_2 B_2}\right] D_2 = T - t$$

- By Formula 7, the LHS is the portfolio Macaulay duration.
- RHS: the Macaulay duration of the cash outlay.
- Hence this equation matches the Macaulay duration of the immunization portfolio to the cash outlay.

Numerical Examples

- Data: (asked) prices and yields for U.S. treasury coupon bonds as of 17 February 2023 3pm (EST).
- Time to maturity range from 2 weeks to 5 years.
- Data source: Wall Street Journal Data Center.
- U.S. treasury securities pay semiannual interest payments, called "coupons". Coupon rates vary across different bonds.
- At the maturity date, the principle (also called "face value", or "par value") is paid.

- Example 1: build a spot rate curve
- Example 2: immunizing $1 Million future obligation.

- VBA and Python codes are available on Moodle.

Ex1: Building Spot Rate Curve

- Let us investigate the cubic splines, one of primary curve fitting techniques.
- Recall, the first step of building spot rate curve is to impose a parameterized functional form for $s(t, t')$
- $\tau = t' - t$
- Cubic splines essentially use 3rd order polynomials as the basic functional form:

$$a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3$$

- But for different ranges of the argument (for our case, it is τ), the coefficients should be different.
- The point on which the coefficients change is called knot.

Ex1: Building Spot Rate Curve

Cubic splines with one *knot* on $\tau = k_1$:

$$\begin{aligned} s(t,t') &= a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3, \quad \tau \le k_1 \\ s(t,t') &= b_0 + b_1(\tau - k_1) + b_2(\tau - k_1)^2 + b_3(\tau - k_1)^3, \quad \tau \ge k_1. \end{aligned}$$

We require, on $\tau = k_1$:

- The function should not "jump": (left function value = right function value)

$$b_0 = a_0 + a_1 k_1 + a_2 k_1^2 + a_3 k_1^3$$

- There should be no "kink": (left derivative = right derivative)

$$b_1 = a_1 + 2a_2 k_1 + 3a_3 k_1^2$$

- It should be smooth: (left 2nd order derivative = right 2nd order derivative)

$$2b_2 = 2a_2 + 6a_3 k_1$$

- Not these three conditions reduce the # of variables to 5.

Ex1: Building Spot Rate Curve

- We consider three models for the spot rate curve

1. 4-th order polynomial (spotRateCurve_p4)

$$s(t,t') = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4$$

2. Cubic spline with one knot (on k_1)

(spotRateCurve_p3SplineKnot1)

$$\begin{aligned} s(t,t') &= a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3, \quad \tau \le k_1 \\ s(t,t') &= b_0 + b_1(\tau - k_1) + b_2(\tau - k_1)^2 + b_3(\tau - k_1)^3, \quad \tau \ge k_1. \end{aligned}$$

Ex1: Building Spot Rate Curve

3. Cubic splines with four knots on k_1, k_2, k_3, k_4 (spotRateCurve_p3SplineKnot4)

$$
\begin{aligned}
s(t,t') &= a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3, \quad \tau \le k_1 \\
s(t,t') &= b_0 + b_1(\tau - k_1) + b_2(\tau - k_1)^2 + b_3(\tau - k_1)^3, k_1 \le \tau \le k_2 \\
s(t,t') &= c_0 + c_1(\tau - k_2) + c_2(\tau - k_2)^2 + c_3(\tau - k_2)^3, \quad k_2 \le \tau \le k_3 \\
s(t,t') &= d_0 + d_1(\tau - k_3) + d_2(\tau - k_3)^2 + d_3(\tau - k_3)^3, \quad k_3 \le \tau \le k_4 \\
s(t,t') &= e_0 + e_1(\tau - k_4) + e_2(\tau - k_4)^2 + e_3(\tau - k_4)^3, \quad \tau \ge k_4.
\end{aligned}
$$

Ex1: Building Spot Rate Curve

- Helper function: **getBondCashFlows(…)**

  --Retrieve the bond payment schedule for each bond quote.

  --only works for U.S. treasuries (since it calls **CoupNcd()** which only works for U.S. treasuries).

- For each spot rate model.

  -- a function that computes the spot rate for a given set of coefficients (a_0, a_1, a_2**….**)

  spotRate(…), spotRateRate1(…), etc.

  -- a function that computes the bond price based on the spot rate model

  getBondPrice(), getBondPrice1(…), etc.

Ex1: Building Spot Rate Curve

For each price quote,

- We first retrieve its cash payment schedule
- Then, for the given spot rate model, compute the bond price according to Formula 2.
- For this exercise, we fix the location of knots (which are selected by heuristic argument), and then the only parameters we can (and should) tune are the polynomial coefficients.
- We find the best coefficients via minimizing the price errors by using Excel solver.
- Data → Analyze → Solver.
- Do remember to choose “Multistart” in Options of the Solver. (The problem is mostly likely nonconvex).

Ex1: Building Spot Rate Curve

- We can repeat the same procedure by minimizing the yield error (recommended).
- This involves one more step: compute the model-implied yield based on the mode-implied price.
- Use the worksheet function **YIELD(…).** Note it assumes U.S. treasury securities.
- For a customized sequence of cash payments, use **IRR(…).**

Ex1: Building Spot Rate Curve

- 4th-order polynomial: polynomial pattern in error plot.
- Cubic splines with knots placed on 0.25, 0.6, 1.5, 3 performs best.
- There is prominent curvature in yield within [0,1.5] with a sharp turn at around 0.6.
- Short-term (less than one year) rates are likely to behave differently than longer term rates.

Model	Average Price Error	Max Price Error	Average Yield Error	Max Yield Error
4th-order polynomial	0.7849	2.1259	0.0019	0.0146
Cubic spline; knot: 1	0.7010	1.9695	0.0010	0.0076
Cubic spline; knots: 1,2,3,4	0.6640	2.0419	0.0010	0.0079
Cubic spline; knots: 0.25, 0.6, 1.5, 3	0.6971	2.0093	0.0009	0.0052

Ex2: Immunization

- Use two bonds to hedge for $1Million to be paid out on 17 Feb 2026.
- Now(t): 17 Feb 2023 3pm (EST).
- Question: how to choose the two bonds?
- At least one of them should have duration greater than 3.
- Yields should be close to each other.
- Maturity should be close to the obligation date (not too close to now, not too far into future).
- Trial and error: if the resulted positions are too extreme, or become negative (negative number means shorting; to be covered in next topic), or with other undesirable characteristics, try other bonds.

Ex2: Immunization

- macaulayImmunization.xlsm
- Very straightforward: just write the Formula 10 into a function.