Notes 1

Basic Analytics for Fixed Income Securities

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FE readings: this notes; you can skip any section with an asterisk (*) if it does not appear to interest you.

FinTech readings: The following articles from [FB].

1. *Banking and the E-Book Moment.*
2. *Why We're So Excited About FinTech.*
3. *FinTech Impact on Retail Banking — From a Universal Banking Model to Banking Verticalization.*
4. *Embracing the Connected API Economy.*

1 Basics

Fixed-income instruments provide future *cash flows* of pre-specified amounts on pre-specified dates. To receive these cash flows, one pays an upfront amount (i.e. a price).
For example,

- On 21 February 2023 3pm (Eastern Time), U.S. Treasury Bill ("T-bill") maturing on 23 February 2023 has price quoted at USD99.9753.
  This means: on 21 February 2023 3pm , you *pay* \$99.9753 to the broker and then own this T-bill. If you hold this T-bill until 23 February 2023 , you *receive* \$100, and the T-bill matures (i.e. expires).

- On 21 February 2023 3pm(Eastern Time), U.S. Treasury Note ("T-note") with annual coupon USD1.375 and maturity date 30 September 2023 has price quoted at USD99.2600.
  This means: on 21 February 2023 3pm, you *pay* \$99.2600 to own this T-note. If you keep holding this note until the maturity date on 30 September 2023, you will *receive*:

(1) Coupon payments on 31 March 2023 and 30 September 2023, each of \$0.6875 =1.375/2.

(2) \$100 on 30 September 2023.

The examples above, and many others, differ a lot in terms of contract details such as coupon rate and its payment schedule, maturity date, and face value (i.e. the "\$100" in the above two examples). Actual transactions of fixed-income instruments involve even more details like how they are issued on auctions and traded on secondary markets. We will *not* go into these institutional details, but refer the interested readers to [1]. Instead, we will focus on the fundamental feature of fixed-income instruments — paying an upfront price to get a sequence of pre-determined future cash flows — and develop the basic analytics for pricing and hedging. By studying the materials below, you should be able to apply those analytics to a specific fixed-income instrument that is given.

Throughout the following sections, we will make two assumptions on the instrument we study, which we refer to "bond" for short[1]:

- It is *default-free*. In other words, the pre-determined cash flows are guaranteed to be paid in full and on time.
  Note this is the case for most government-backed instruments, such as the U.S. treasury securities.

- There is *no option-related feature* embedded. This means the issuer cannot choose to redeem the bond (typically by making a lump sum payment) to end the instrument before its maturity date.

1.1 Bond Price and Yield

Bond is essentially an IOU ("I-owe-you") instrument: one lends money to the other party today, and receive the payback on some mutually agreed future date(s). In a generic form, this is equivalent to paying an upfront price to exchange for future cash flows.

Then a natural question arises: should the price be just the sum of all future cash flows? The answer is no. A simplest example is the T-bill at the beginning of this section: one pays slightly below \$100 to get an amount of \$100 in two days. This relates to the very

[1]Not to confuse with any specific instrument, such as U.S. Treasury Bond or any specific corporate bond

familiar notion of *interest*: one requires some return if she/he agrees to lend the money. This is because by lending the money today, the lender foregoes potential benefits that can be derived over the time period between now and the payback date. For example, one can spend \$2000 today to buy a high-performance computer. If lending this money today (and expect the payback, say, in a month), she/he has to defer the purchase to a month later and lost the enjoyment of the computer for this month. Alternatively, if this person has a business opportunity to generate additional \$100 out of the \$2000, lending the money forgoes this opportunity.

The price of the bond is exactly the *value* of all the future cash flows it generates, and the value of a future cash flow is determined by certain discounting mechanism. To formalize this, let us refine the setting a little bit. Denote the current time by t, on which the price is paid and the bond is bought. The maturity of the bond is a future date T, on which the last cash flow occurs and the debt is fully cleared. Over the time frame $[t, T]$, there are N cash flows paid on time points $t < t_1 < t_2 < \cdots < t_N = T$, with the cash flow on t_n denoted C_n (i.e. C_1 paid on t_1, C_2 paid on t_2, and so forth). Let the bond price be denoted $B(t, T)$ (meaning "the time t price of the bond which matures on T"). Then, the relationship between the bond price $B(t, T)$ and the sequence of future cash flows $\{C_1, C_2, \cdots, C_N\}$ is captured by a single positive number y, called *yield*:

Formula 1 (Price-Yield Relationship)

$$\begin{aligned} B(t,T) &= \sum_{n=1}^{N} \frac{C_n}{(1+y)^{\frac{t_n-t}{d}}} \\ &= \frac{C_1}{(1+y)^{\frac{t_1-t}{d}}} + \frac{C_2}{(1+y)^{\frac{t_2-t}{d}}} + \cdots + \frac{C_N}{(1+y)^{\frac{t_N-t}{d}}}. \end{aligned} \tag{1}$$

□

Yield is typically interpreted as, to some extent, a measure of return on investment in this bond[1]. The number d is the length of a benchmark interval over which y is prevailing. Here (and below) the "length" of a time interval is denominated in "year". d is decided by market convention, usually in form of a quarter ($d = 0.25$), half a year ($d = 0.5$) or a year ($d = 1$); thus $\frac{t_n-t}{d}$ stands for the number of reference intervals from now (i.e. t) to time t_n, which is not necessarily an integral number (i.e. it can involve fractions).

[1] If you have prior knowledge in financial analysis: yield is just the internal rate of return.

In (1), when y is understood as a return rate, then each $(1+y)^{\frac{t_n-t}{d}}$ stands for the total cash you will receive at time t_n by lending \$1 now (t). Specifically, with a given d, over each time interval of length d, an interest rate y is applied to your current wealth and the interest payment is added to your wealth. Hence your wealth grows by a factor $1+y$ by the end of each interval. There are $\frac{t_n-t}{d}$ intervals from now to t_n, thus \$1 now will grow into $(1+y)^{\frac{t_n-t}{d}}$ at time t_n. In other words, to receive $\$C_n$ on time t_n, all you need is to invest $\frac{C_n}{(1+y)^{\frac{t_n-t}{d}}}$ now, thus this quantity is the current price of $\$C_n$ on the future date t_n[1]. Clearly, (1) indicates that the value of the bond is the sum of prices of its individual cash payments.

Several remarks are in order.

Remark 1 a. For liquid bonds such as treasury securities, $B(t,T)$ is *observed* from market quotes, and the yield y is *implied by* (1). Different bonds have different yields, and you should *not* use yield for one to compute the price of another.

b. The interpretation of yield as investment return makes two implicit assumpitons: (1) the bond is held until maturity (i.e. not sold before that), and (2) each coupon payment is *reinvested* and earns a rate at y. Note that both assumptions are fairly strong (i.e. unlikely to happen).

c. Yield is closely related to prices of the bonds, but not an ingredient for *pricing* (i.e. how to decide price for a sequence of cash flows). To do the pricing, we need *a common interest rate structure* for all bonds in the same class[2], and this is discussed in §2.

d. The number d, which is the length of a reference interval, only relates to market convention. In practice, the price-yield expressions might vary and look slightly different from Formula 1 (usually due to slight different discounting conventions, how to treat non-trading days, etc.), but the underlying logic is the same.

e. In practice yield is reported in *annualized* form. That is, $\frac{1}{d}y$ (recall d is the length of the interval over which y is prevailing). For example, if d in (1) is half year (i.e. $d=0.5$), then the reported yield is $2y$.[3]

[1] If you have prior knowledge in compounding interests, this part exactly talks about that. For an elaborate account on compounding convention in market practice, refer to Chapter 4 of [3].

[2] This typically means the bonds issued by the same institution, for instance, U.S. government.

[3] In practice there are day count conventions (like 360 or 365 for a year); refer to [1] for details.

1.2 Bond Yield Curve

A basic bond analytics is a yield curve: we plot the yield against the time to maturity of the bonds. One example is illustrated below.

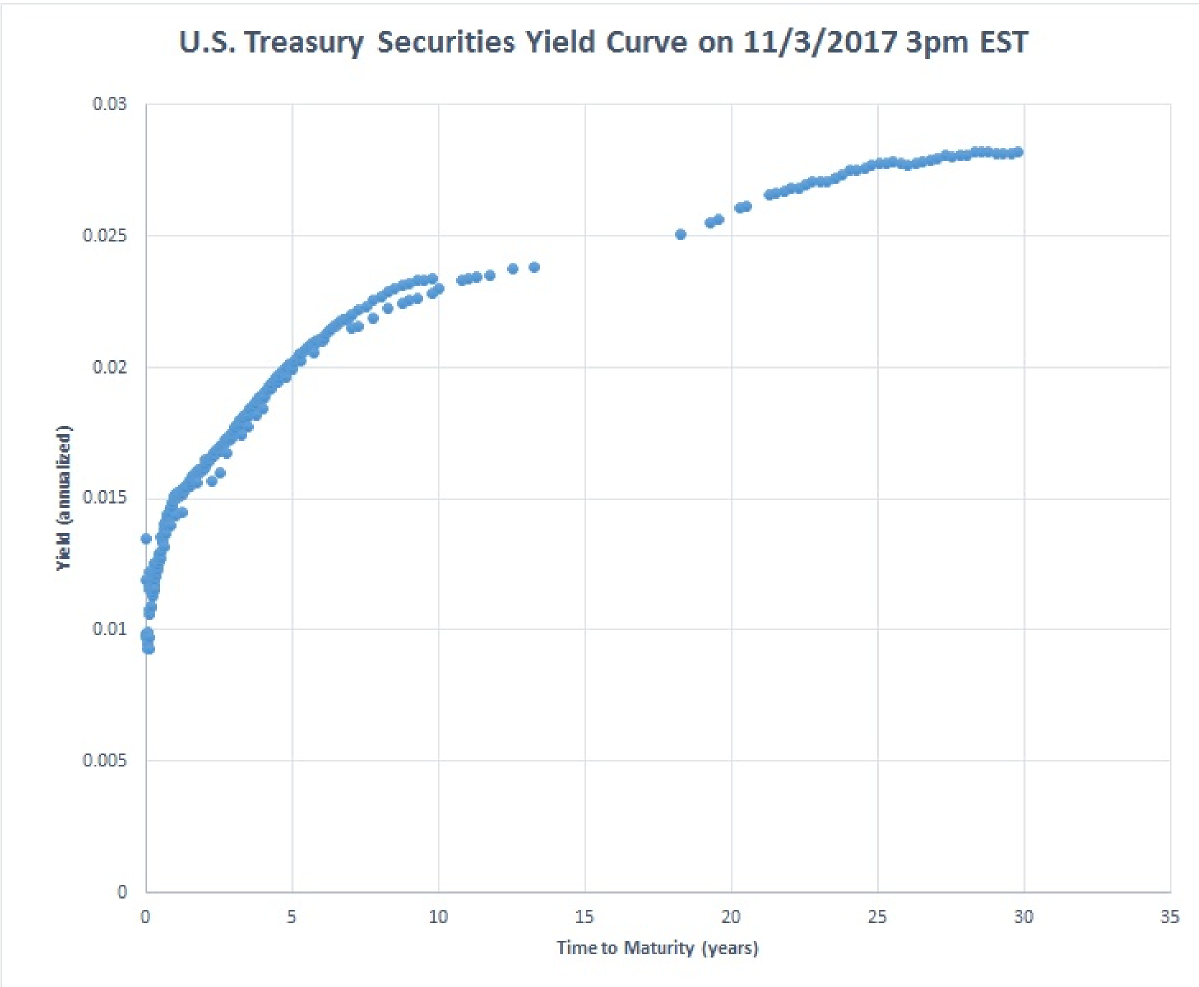

Figure 1: A yield curve.

In Figure 1 above, each point corresponds to a specific U.S. treasury security. Note there can be multiple securities maturing on the same day, hence at one value of time to maturity (x-axis) there can be multiple values for yields (y-axis).

A prominent trend is that the yields increase with time to maturity: bonds with longer maturity into future have higher yields. This pattern is observed for yield curve for most of the time and this typically means that the economy is in a normal state. An *inverted* yield curve flips this pattern and this usually means a warning for the economic state. Below is a graph of treasury yield curves published by U.S. Treasury in February 2007.

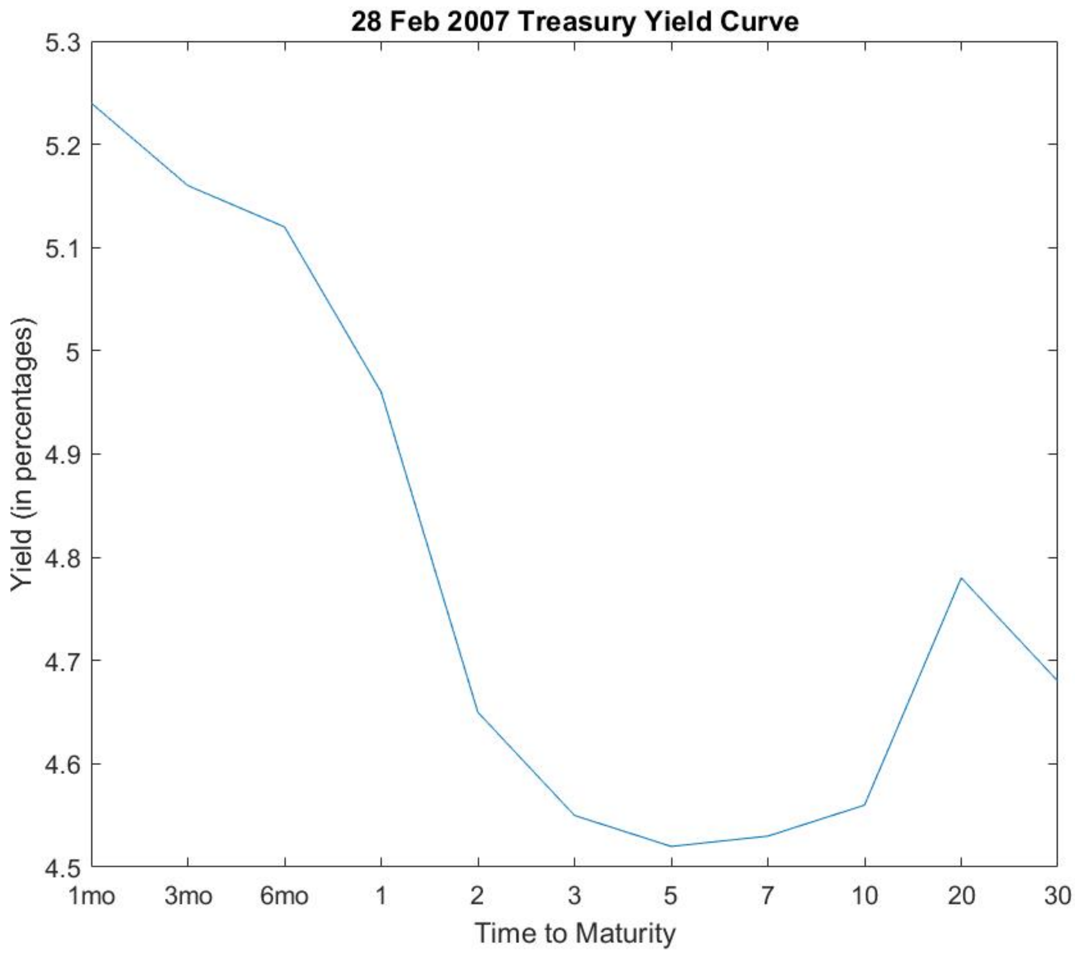

Figure 2: An inverted yield curve, happening in Feb 2007.

Besides what the yield curve can tell us about the economic state, at a micro level it sends an important message to investment decision makers: interests (which are embedded in yields) demanded on money vary with lengths of holding period. In §2 we will give a more rigorous treatment of this point.

2 Term Structure

From last section, we have seen the yield-price relationship. Recall that, yield is a rate implied by the observed bond price and it is a measure (partially) describing the investment return of the bond. Also recall, c of Remark 1 says that it is in general not appropriate to use yield for bond pricing. The reason is that yield is not a bona fide *interest rate*. In other words, yield does not represent a return rate to be earned if one lends money; this argument is also reinforced by b of Remark 1.

We need interest rates for pricing (and hedging). And the notion of interest rate is formalized by *spot rate*. We will see that a defining parameter of spot rate is the length of holding period, i.e. *time to maturity*. A fundamental property of spot rate is: spot rate varies with lengths of time to maturity. In fact this is quite a familiar property of interest rates. For instance, a certificate of deposit (CD) account offers substantially higher yearly interest rate than a checking account does. We have also seen this property in §1.2, as yields partially reflect the interest rates.

The relationship between spot rates and time to maturity is called the *term structure* of spot rates (another word for "time to maturity" is "term"). Once we know, at current time point, the spot rates prevailing various lengths of periods starting from now, we can infer interest rates prevailing a time period between two future dates, *forward rates*. Forward rate varies with how far into future we are looking at, hence like spot rate, it also has a term structure. These two term structures are the foundations for bond analytics in practice.

2.1 Spot Rate and Term Structure

Spot rate is the interest rate charged on money lent for a period of time starting now. This rate and the payback date are agreed on the current time point by both parties (the borrower and the lender), and thus it represents an interest rate to be truly earned from now to a designated future date. Usually we call such practice (i.e. lend money now and receive payback on a specified future date) a *spot loan* (and this is why the associated interest rate is called spot rate).

Here we formalize the notation. Let the current time be denoted t, and for any future time $t' > t$, the associated spot rate (corresponding to a reference length of interval, d) is denoted $s(t, t')$ (meaning "the rate prevailing from now (t) to t'"). This means: one lends \$1 now and then gets back $\$(1 + s(t,t'))^{\frac{t'-t}{d}}$ on time t'[1]. In other words, to receive \$1 on t', one needs to pay out-of-pocket cost of $\$\frac{1}{(1+s(t,t'))^{\frac{t'-t}{d}}}$ now, which is also the current price of the spot loan that pays back \$1 on t'. Of course the payment can be numbers other than

[1]This term comes from the reasoning based on investment return, the same as we did for the bond yield.

\$1, and in such case the price is just $\frac{1}{(1+s(t,t'))^{\frac{t'-t}{d}}}$ times the payment. Let us explicitly write the relationship below:

$$\text{price of the spot loan} \times \left[1+s(t,t')\right]^{\frac{t'-t}{d}} = \text{payment of the spot loan.} \tag{2}$$

t', the price and the payment are all agreed now (t) by the lender and the borrower, hence so is $s(t,t')$. The curve of $s(t,t')$ versus t' is called the *term structure*, which is observed at current time t.

With the collection of spot rates, price formula of the same bond in §1.1 follows immediately. Recall this bond offers cash payments on future time points $t_1 < t_2 < \cdots < t_N$, and the payment on t_n is C_n. This is essentially synthesized by N cash payments, with each occurring on a specific date and thus associating with the corresponding spot rate. Therefore, buying the bond means simultaneously entering N spot loans, and for the loan paying \$$C_n$ on t_n, one needs to pay $\frac{C_n}{(1+s(t,t_n))^{\frac{t_n-t}{d}}}$ now. This immediately gives the pricing formula of the bond (recall, the bond price is denoted $B(t,T)$):

Formula 2 (Bond Pricing by Spot Rates)

$$\begin{aligned} B(t,T) &= \sum_{n=1}^{N} \frac{C_n}{(1+s(t,t_n))^{\frac{t_n-t}{d}}} \\ &= \frac{C_1}{(1+s(t,t_1))^{\frac{t_1-t}{d}}} + \frac{C_2}{(1+s(t,t_2))^{\frac{t_2-t}{d}}} + \cdots + \frac{C_N}{(1+s(t,t_N))^{\frac{t_N-t}{d}}}. \end{aligned} \tag{3}$$

□

Both Formula 2 and Formula 1 give price expressions of the same bond, hence equating them we get:

$$\sum_{n=1}^{N} \frac{C_n}{(1+y)^{\frac{t_n-t}{d}}} = \sum_{n=1}^{N} \frac{C_n}{(1+s(t,t_n))^{\frac{t_n-t}{d}}}. \tag{4}$$

On the left hand side, the same yield y applies to all cash payments, whereas on the right hand side, each cash payment is associated with a different spot rate, $s(t,t_n)$. Clearly, this indicates that yield is a *summary* of all spot rates over the relevant time period.

Once the spot rates $s(t,t')$ are obtained for all relevant $t' \geq t$ (i.e. we have a spot rate curve), we can invoke the pricing formula to price bonds with known payment structures. The discussion on how to build the spot curve from bond prices data are collected in §2.1.1

below.

Getting rid of d Before we discuss in §2.1.1 on how to obtain $s(t, t')$ from actual data of bond quotes, let us look into the number d which appears repeatedly in the previous content. By the end of this discussion, we shall see that, when coming to building spot rate curve and the pricing using this obtained curve, which d to use in this context does not really matter.

Recall, the spot rate $s(t, t')$ for some $t' > t$ is defined through the interest charged for a spot loan given now (t) and ends on t'. The spot rate, based on a reference interval length d, connects the price and payment of the spot loan via (2), which is copied below:

$$\text{price of the spot loan} \times \left[1 + d \times \frac{s(t,t')}{d}\right]^{\frac{t'-t}{d}} = \text{payment of the spot loan.} \tag{5}$$

Note here the spot rate $s(t, t')$ is deliberately written in its equivalent form $d \times \frac{s(t,t')}{d}$ to explicitly involve its annualized version $\frac{s(t,t')}{d}$. (5) is immediately rearranged into:

$$\left[1 + d \times \frac{s(t,t')}{d}\right]^{\frac{t'-t}{d}} = \frac{\text{payment of the spot loan}}{\text{price of the spot loan}}. \tag{6}$$

The key is that, the right hand side is decided by the market and does not change with choice of d: we always pay the market price to receive the designated spot loan payment. In Formula 3 and equation (4), we choose d to be the same as that of the bond's yield, but changing it to other values does not alter the right hand side of (6). If we choose to use another value for the reference interval, say d_1, then the associated spot rate $s_1(t, t')$ satisfies:

$$\left[1 + d_1 \times \frac{s_1(t,t')}{d_1}\right]^{\frac{t'-t}{d_1}} = \frac{\text{payment of the spot loan}}{\text{price of the spot loan}} = \left[1 + d \times \frac{s(t,t')}{d}\right]^{\frac{t'-t}{d}}. \tag{7}$$

Of course, different values of d give us different values of $s(t, t')$, but they are all connected by (7), and the value of the right hand side of Formula 2 stays the same. To this point, we should be clear that d only relates to the quoting convention and it only affects how we *represent* the same spot rate.

Reporting spot rate term structure together with its associated d is cumbersome. Also, different people might use different d, resulting the differences in reported values and causing unnecessary ambiguity. Therefore, practitioners assume that d *is very small*, and this generates a particularly elegant formula for the corresponding *annualized* spot rate $s(t, t')$:

$$\Big[1 + d \times s(t, t')\Big]^{\frac{t'-t}{d}} = e^{s(t,t')(t'-t)}. \tag{8}$$

Recall, e is the exponential constant and $e \approx 2.71828$.[1] We call the $s(t, t')$ with extremely small d the *continuous compounding rate.*

Using (8), Formula 2 reduces to:

Formula 3 (Bond Pricing by Continuous Compounding Spot Rates) Let $s(t, t')$ be the continuous compounding spot rates; recall they are already in annualized form.

$$\begin{aligned} B(t, T) &= \sum_{n=1}^{N} C_n e^{-s(t,t_n)(t_n-t)} \\ &= C_1 e^{-s(t,t_1)(t_1-t)} + C_2 e^{-s(t,t_2)(t_2-t)} + \cdots + C_1 e^{-s(t,t_N)(t_N-t)}. \end{aligned} \tag{9}$$

□

In §2.1.1, we will use the continuous compounding spot rates and Formula 3 to build the spot rate curve from data.

Here are some remarks for this section.

Remark 2 a. In practice, spot rates are fitted from market price quotes of liquid bonds. Then the obtained rates can be used to price bonds which have missing or unreliable price quotes. For instance, market quotes for very illiquid bonds might not reflect executable prices, so the spot rates can be used to provide more reliable prices. More generally, the spot curve can be used to price any given sequence of cash flows that may arise in more complicated fixed-income instruments (of the same class), or cash flows from private transactions.

b. If we choose to represent spot rates based on the reference interval d that is the same as that of the bond yield, then the yield of bonds with a single payment (called "zero-coupon bonds"; an example is a T-bill) coincides with the spot rate associated with its maturity date.This should be obvious via (4). And some practitioners use the terms

[1] The equation (8) uses $\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x$.

"spot rate curve" and "yield curve" interchangeably, and this is not to be confused with the bond yield curve in §1.2.

c. Continue from Item b, when the spot rate term structure is *flat*, that is, $s(t, t')$ takes the same value for all $t' \geq t$, this value coincides with the bond yield y. i.e. $s(t, t') = y$ for all $t' \geq t$. This is also obvious from (4).

d. Formula 3 is just a special case of Formula 2 by taking the d on the right hand side of equation (9) to be extremely small. With a given spot rate curve of continuous compounding spot rates, the same curve represented by any other reference interval length d can be obtained.

2.1.1 Build Spot Rate Curve from Data

Throughout the spot rate curve building, we use continuous compounding spot rates, hence the associated bond pricing formula is Formula 3.

Now suppose we are standing at time t as usual, and we have bond price data $B(t, T_m)$ (recall, T_m stands for the bond maturity) for a sequence of T_m; $m = 1, \cdots, M$. Usually we also have the yield data associated with each bond, denoted y_m.

The high-level purpose is to find a formula for $s(t, t')$ such that the model-predicted quantities are close to the observed quantities. Here we summarize the general procedure into three steps.

Step 1. Pre-impose a *parameterized* formula for $s(t, t')$, and let the parameter(s) be denoted θ. Note θ can stand for one or several parameters. So, let us denote the imposed spot rate model by $s(t, t'; \theta)$ to stress its dependence on θ. The ultimate goal is to find θ from data.

In practice *polynomial-based* models are usually used. For example, a 4-th order polynomial:

$$s(t, t') = a_0 + a_1(t' - t) + a_2(t' - t)^2 + a_3(t' - t)^3 + a_4(t' - t)^4. \tag{10}$$

For this case, the parameters to be found are a_0, a_1, a_3, a_4 (hence θ stands for the collection of these four parameters).

Step 2. Compute the model-implied quantities, based on the model in Step 1, for each of the bond and record its difference from the observed value. (i.e. record the error).

An *error measure* is then defined based on the collection of individual errors. Note the error measure depends on the θ which enters the spot rate model.

In practice, people match bond prices or bond yields. If we choose to *match price*, then we use Formula 3 with the spot rate model set up in Step 1 to compute the model-implied bond price for each bond, denoted $\hat{B}_m(\theta)$ for m-th bond. Obviously, the associated error is just $\hat{B}_m(\theta) - B(t, T_m)$. If we choose to *match yield*, then we still compute $\hat{B}_m$ first, then use Formula 1 to infer the model-implied yield by replacing $B(t,T)$ with $\hat{B}_m$ in (1) and solving for y. Denote the inferred yield by $\hat{y}_m(\theta)$, then apparently the error is $\hat{y}_m(\theta) - y_m$. Here, be careful that when solving for $\hat{y}_m(\theta)$ via (1), the d on the right hand side of (1) must be the same as that used in the quoted prices data.

Usually we define the error measure to be the sum of squares of individual errors. Formally, for bond price matching, the error measure is

$$\sum_{m=1}^{M} [\hat{B}_m(\theta) - B(t, T_m)]^2 \tag{11}$$

For yield matching, the measure is:

$$\sum_{m=1}^{M} [\hat{y}_m(\theta) - y_m]^2 \tag{12}$$

Step 3. Use *optimization methods* to find the θ that minimizes the error measure computed in Step 2. Plug this optimal θ back to the formula for $s(t, t')$ in Step 1, and the resulted expression is the spot rate curve fitted from data.

Remark 3 a. Price matching is a natural approach. But many practitioners prefer to match the yields because price matching under-weighs the yield error of short-term bonds whose prices are less sensitive to yield changes.

b. We can rely on many off-the-shelf optimization software to implement the finding-the-θ task of Step 3. But to do this, we still need to write codes to implement Step 1 and 2. In §4, we will show how to develop these analytics in Excel with help of VBA coding.

2.2 Forward Rate and Term Structure

Spot rates give interest rates to be earned over periods of various lengths, all *starting now*. In fact we can use spot rates now to *lock* interest rates to be earned *over a future period*. This is quite intuitive to understand: $s(t, t_1)$ and $s(t, t_2)$ (with $t_2 > t_1$) are both locked now and these two spot rates must *imply* a rate prevailing from time t_1 to t_2. We call this implied rate a *forward rate*, and below we formalize the definition and give its expression based on spot rates.

A forward rate, denoted $f(t, t_1, t_2)$ (with $t < t_1 < t_2$), is the rate agreed upon now (t) and is to be prevailing over the time period from t_1 to t_2. In other words, both parties (lender and borrower) set up $f(t, t_1, t_2)$ now. Then, the lender gives money to the borrower at a later date t_1 and the borrower pays back at t_2, with the interest applied to this loan being $f(t, t_1, t_2)$ (which is agreed now (t)). The expression of $f(t, t_1, t_2)$ is given in the following formula.

Formula 4 (Forward Rate)
For $t < t_1 < t_2$, the forward rate on time t for period $[t_1, t_2]$, denoted $f(t, t_1, t_2)$, is determined by:

$$1 + f(t, t_1, t_2) = \frac{\big[1 + s(t, t_2)\big]^{\frac{t_2 - t}{t_2 - t_1}}}{\big[1 + s(t, t_1)\big]^{\frac{t_1 - t}{t_2 - t_1}}}. \tag{13}$$

Recall, $s(t, t_1)$ and $s(t, t_2)$ are spot rates now (t), and the reference interval length is d. $\square$

To understand this formula, rearrange the terms of (13) and obtain an equivalent equation:

$$[1 + s(t, t_1)]^{\frac{t_1 - t}{d}} [1 + f(t, t_1, t_2)]^{\frac{t_2 - t_1}{d}} = [1 + s(t, t_2)]^{\frac{t_2 - t}{d}}. \tag{14}$$

The left hand side of the equation relates to this strategy: lend \$1 from t to t_1 and on t_1 *roll* this loan to t_2. The total payment at t_2 is exactly the left hand side: \$1 grows to $[1+s(t, t_1)]^{\frac{t_1 - t}{d}}$ on t_1, enjoying the spot rate $s(t, t_1)$ prevailing $[t, t_1]$; then the money further grows by a factor $[1 + f(t, t_1, t_2)]^{\frac{t_2 - t_1}{d}}$ on t_2, enjoying the forward rate $f(t, t_1, t_2)$ prevailing $[t_1, t_2]$. The right hand side relates to this strategy: lend \$1 from t to t_2. So, the expression is just the money received on t_2, reflecting the spot rate $s(t, t_2)$. Clearly, (14) says that the two strategies are equivalent: both comprise of out-of-pocket investment of \$1 on t_1 and the same amount of money received on t_2. This essentially says: investment in a longer loan is the same as investment in a shorter loan then rolling the proceeds at the forward rate to match the maturity of the longer one.

While the above interpretation looks intuitive to interpret $f(t, t_1, t_2)$, it might still appear unclear on how to actually *realize* this forward rate? In other words, how can we make sure that something can be done on time t which guarantees the rate to be actually made real later from t_1 to t_2, so as $f(t, t_1, t_2)$ is a bona fide interest rate? The answer is this strategy: on time t, sell the spot loan with maturity t_1 and use the proceeds to buy the spot loan with maturity t_2. The details are summarized in the table below.

Time	Action	Net Cash Flow
t(now)	Sell one unit of $B_0(t, t_1)$ and buy $\frac{B_0(t,t_1)}{B_0(t,t_2)}$ units of $B_0(t, t_2)$	0
t_1 (short bond matures)	pay \$1 and the short bond is cleared	-1
t_2 (long bond matures)	receive $\$\frac{B_0(t,t_1)}{B_0(t,t_2)}$ and the long bond is cleared	$\frac{B_0(t,t_1)}{B_0(t,t_2)}$

Table 1: Using bonds of different maturities to lock future cash flows. $B_0(t, t_1)$ is a spot loan paying \$1 at time t_1. $B_0(t, t_2)$ is a spot loan paying \$1 at time t_2. $t < t_1 < t_2$, and $B_0(t, t_1)$ is referred as "short bond" and $B_1(t, t_2)$ is referred as "long bond".

Note "sell a loan (or a bond)" means borrowing the money, and "buy a loan (or a bond)" means lending the money. Here one unit of the loan promises \$1 payment on maturity[1], and the interest rate charged is reflected in the bond price which is below \$1. In the table above, sell a unit of the short bond means: we receive $B_0(t, t_1)$ on t and hence have the obligation to pay (to the lender) \$1 on t_1. And the proceeds, $B_0(t, t_1)$, is used up to buy the longer bond, and the number of units we can buy is $\frac{B_0(t,t_1)}{B_0(t,t_2)}$ (i.e. available funds divided by the price). Hence this buying of the longer bond entitles us to received $\frac{B_0(t,t_1)}{B_0(t,t_2)}$ on t_2. Both actions are conducted now (t), but the cash flows created are on later dates: \$1 out-of-pocket payment on t_1 (to pay the lender of the short bond) and $\frac{B_0(t,t_1)}{B_0(t,t_2)}$ to be received on t_2 (payment by the borrower of the short bond). Both cash flows are definite, since the actions creating them are done now. The created cash flows indicate, at the current time point, we agree to pay \$1 later on t_1 and receive $\frac{B_0(t,t_1)}{B_0(t,t_2)}$ on t_2, and this locks the interest rate to be prevailing from t_1 to t_2. Working out the details using Formula 2 will lead to Formula 4.

[1] In bond terminology, this is said as "the face value is \$1", or the "par value is \$1", or "the principal is \$1".

Recall from §2.1, spot rate curve can be represented based on different reference interval lengths and we prefer to use $d \approx 0$ which induces the continuous compounding spot rates. For this case, Formula 4 reduces to a very clean form, which deserves a separate formula.

Formula 5 (Continuous Compounding Forward Rate)
For $t < t_1 < t_2$, the forward rate with continuous compounding on time t for period $[t_1, t_2]$, denoted $f(t, t_1, t_2)$, is determined by:

$$f(t, t_1, t_2) = \frac{s(t, t_2)(t_2 - t) - s(t, t_1)(t_1 - t)}{t_2 - t_1}. \tag{15}$$

Here $s(t, t')$ is the spot rate curve represented in continuous compounding. □

One spot rate curve leads to many forward rates, and they cannot be plotted on the same 2-D curve since both t_1 and t_2 can vary. In practice, people fix $t_2 - t_1$ at some number τ and plot $f(t, t_1, t_1 + \tau)$ as the *forward rate curve*, which is the term structure for forwards rates. Like spot rate curve, the forward rate curve is the basic analytics for pricing and hedging of various interest rate products.

3 Interest Rate Risk

Bonds are heavily used to reserve funds for future obligations or expenditures. For example:

- Insurance companies project claims to arrive on regular basis.
- A family wants to save for the college education cost for children.

Suppose at the current time t, we know there is an obligation occurring on $T > t$ that requires a cash payment of amount $\$C$. If there is any bond in the market that matures exactly on T, then the strategy is simple: we just buy that bond today (in an amount such that the total payments amount to $\$C$), hold it until maturity, and use the bond payment to settle the obligation. However, in reality, this does not work: bonds with maturity coinciding exactly with T typically may not exist. Then, we might want to make a compromise by purchasing bonds with a maturity T' such that T' is relatively close to T. Then two kinds of uncertainty arises. If $T' < T$, then we have to invest the proceeds from the bond payment somewhere, and this introduces reinvestment risk because currently we do not quite know what interest rate will be over that period of time.[1] If $T' > T$, then we will sell the bond on T and use

[1]Of course one can keep the cash and hold it until T, but this is throwing away potential interests.

the proceeds to settle the obligation, but we are not sure if the bond price at that time is sufficient to cover the obligation amount \$C.

The difficulties described above come from the volatility of interest rates, and in fact *interest rate risk* is the fundamental risk associated with fixed-income securities. Concretely, in Formula 1 and Formula 2, the bond payments C_n are fixed, but the yield or spot rates can change as time goes by. As the rates change, the bond price also changes.

This section gives two simple but heavily used interest risk measures that describe how bond price changes with interest rates. They provide basic analytics for interest risk mitigation, termed *immunization* in this context.

3.1 Macaulay Duration

Macaulay duration measures how bond price changes as its yield changes. More concretely, it measures the change of $B(t,T)$ in Formula 1 when y changes (by a small amount). We start with a little bit of derivation and then summarize the results into a formula.

Recall the bond setting: we are now standing at time t, and the bond will pay cash $\$C_n$ on time t_n, with $t < t_1 < t_2 < \cdots < t_n$, and the yield for a given reference time interval length d is denoted y. The bond price $B(t,T)$ is connected with y by Formula 1. Now, suppose over the next instant, there is a small change in the (annualized) yield, hence y becomes $y + d\delta$ (δ is small).[1] The time change is so small that we ignore it and still consider ourselves to be on time t, and the bond price changes to reflect the change in yield. Here t and T are fixed, and we write the bond prices by $B(y)$ (before yield change) and $B(y + d\delta)$ (after yield change) to stress the dependence of price on yield. Formally,

$$
\begin{aligned}
B(y+d\delta) - B(y) &= \sum_{n=1}^{N} \frac{C_n}{(1+y+d\delta)^{\frac{t_n-t}{d}}} - \sum_{n=1}^{N} \frac{C_n}{(1+y)^{\frac{t_n-t}{d}}} \\
&= \sum_{n=1}^{N} \left[\frac{C_n}{(1+y+d\delta)^{\frac{t_n-t}{d}}} - \frac{C_n}{(1+y)^{\frac{t_n-t}{d}}} \right] \\
&\approx -\left\{ \left(\frac{1}{1+y} \right) \left[\sum_{n=1}^{N} \left(\frac{C_n}{(1+y)^{\frac{t_n-t}{d}} B(y)} \right) (t_n - t) \right] B(y) \right\} \delta. \qquad (16)
\end{aligned}
$$

[1]The convention when dealing with Macaulay duration is to assume there is a small change δ in the annualized yield, so the corresponding change of y (recall it is the yield prevailing the reference interval which is of length d expressed in years) is $d\delta$.

The first line directly plugs Formula 1 to express $B(y+d\delta)$ and $B(y)$ respectively, and the second line combines the two sums by pairing up the summands. The third line uses first order Taylor series expansion[1] treating y as the argument to approximate $\frac{C_n}{(1+y+\delta)^{\frac{t_n-t}{d}}} - \frac{C_n}{(1+y)^{\frac{t_n-t}{d}}}$, and the terms are rearranged deliberately to reach the final expression.

The last line in (16) looks messy due to the term involving the sum, but let's write it into the following format:

$$B(y+d\delta) - B(y) \approx -\left\{\left[\frac{\text{the messy term involving the sum}}{1+y}\right] B(y)\right\}\delta. \tag{17}$$

Now we can see the change of bond price (i.e. $B(y+d\delta) - B(y)$) is controlled by three quantities in (17): (i) The term inside the bracket "[ ]"; (ii) $B(y)$, the bond price before the yield change; (iii) δ, the yield change. All three are positive numbers, and the minus sign reflects the price-yield relationship: yield goes up and price goes down, and the vice versa. (ii) and (iii) are easy to understand: the bond price change should depend on the yield change δ, and this change is expressed in proportion to the original bond price, $B(y)$. Therefore, what actually controls the magnitude of price change is (i), and in particular, "the messy term involving the sum". This term is called Macaulay duration, and we formalize it in the following formula.

Formula 6 (Macaulay Duration and Bond Price Sensitivity to Yield)
The Macaulay duration corresponding to Formula 1 is denoted D and defined as

$$D = \sum_{n=1}^{N}\left[\frac{C_n}{(1+y)^{\frac{t_n-t}{d}} B(y)}\right](t_n - t). \tag{18}$$

(Recall $B(y)$ is the bond price $B(t,T)$ in Formula 1.)

In addition, the associated *modified duration* is denoted by D_M and defined as

$$D_M = \frac{D}{1+y}. \tag{19}$$

[1]If you are unfamiliar or uncomfortable with this concept, just take a leap of faith to accept the resulted expression. It is just a mathematical technique to express how the function value changes when the argument changes by a little bit.

These two quantities govern the bond price change due to a small annual yield shift δ by:

$$B(y + d\delta) - B(y) \approx -D_M(B(y)\delta) = -\frac{D}{1+y}(B(y)\delta). \tag{20}$$

$\square$

Note δ is small, hence the resulted bond price change reflects the "sensitivity" of the bond price to its yield. To this point, we can see that so long we have computed D_M and D, we can immediately have an estimate of the bond price change due to a given small shift in yield.

We are ready to look into the expression of D in (18). It is a sum of terms, each summand is a product of $\frac{C_n}{(1+y)^{\frac{t_n-t}{d}} B(y)}$ and $t_n - t$. Formula 1 indicates that $\frac{C_n}{(1+y)^{\frac{t_n-t}{d}} B(y)}$ is the *proportion* of cash flow C_n's value in the bond price. The term $t_n - t$ is length of time from now(t) to t_n (on which C_n is paid). Therefore, Macaulay duration is interpreted as a *weighted average* of the maturities associated with each payment, and each weight is the proportion of that cash flow's value in the bond price. In other words, Macaulay duration is a measure of the *bond's life*, and this is why it is called a "duration".

Now, suppose we have several bonds, i.e., we have a *portfolio* of bonds. Also suppose the bonds in this portfolio have a *common yield* value. We can use Formula 6 to compute the Macaulay duration for each bond. If this yield shifts a bit, we can use these durations to compute for a change in the total value of these bonds, i.e. a change in the portfolio value. It turns out that based on this reasoning, we can define a Macaulay duration for this portfolio and use it to quickly compute its value change, just as we did for a single bond.

Formula 7 (Macaulay Duration for A Bond Portfolio)
Suppose a bond portfolio comprises of m bonds, indexed by $i = 1, \cdots, m$. Suppose these bonds have a *common yield*, y. The i-th bond has price $B_i(y)$ and Macaulay duration D_i. Let $B(y)$ denote the total value of this portfolio, and clearly $B(y) = B_1(y) + \cdots + B_m(y) = \sum_{i=1}^{m} B_i(y)$. The Macaulay duration of this bond portfolio, denoted D, has the following expression:

$$\begin{aligned} D &= \sum_{i=1}^{m} \frac{B_i(y)}{B(y)} D_i \\ &= \frac{B_1(y)}{B(y)} D_1 + \frac{B_2(y)}{B(y)} D_2 + \cdots + \frac{B_m(y)}{B(y)} D_m. \end{aligned} \tag{21}$$

The associated modified duration of this portfolio is:

$$D_M = \frac{D}{1+y}. \tag{22}$$

Then, the portfolio value change due to a small annual yield shift δ is:

$$B(y + d\delta) - B(y) \approx -D_M[B(y)\delta] = -\frac{D}{1+y}[B(y)\delta]. \tag{23}$$

□

Remark 4 a. Recall, Item a and c of Remark 1 say that yield is implied by the bond price and it should not be used for pricing. Likewise, Macaulay duration is *not* used for purpose of pricing or evaluation — it is a risk analytics computed based on the observed bond prices (and associated yields). In practice it is convenient to express views on the yield shifts instead of directly on price shifts, as the bond yields summarize the spot rates (see (4)) and they tend to move together.

b. For bonds with only a single payment (recall, they are called "zero-coupon bonds"), Macaulay duration coincides with the time to the bond's maturity; that is, $D = T - t$.

c. When computing for portfolio Macaulay duration, we have to select bonds with a common yield to construct this portfolio in first place. This is not always possible in practice: yields vary as maturities vary (see §1.2). To compromise, approximation can be made by taking average of the bond yields and treat this average as a common yield.

d. The number, d, appears in expression of Macaulay duration. This number is the same as the d used in computation of the bond's yield; which d to use is decided by market convention.

3.2 Fisher-Weil Duration

This section introduces another widely used measure of interest rate sensitivity, called the *Fisher-Weil duration.* Compared to the Macaulay duration which is yield-based, the Fisher-Weil duration is computed based on spot rate curve, hence involves the interest rate term structure.

The principle is similar to that of the Macaulay duration: we want to measure how the bond price changes when the input interest rate changes; and here the input is the spot rate curve. Specifically, we will represent the spot curve based on *continuous compounding spot rates* and then use Formula 3 to compute the change in bond price resulted from a spot curve change. Again, suppose the current time is t. Let s denote the original spot curve $s(t,t')$ for $t' \geq t$; note s is just a symbol to denote the whole curve. Then, suppose over the next instant, s experiences a *parallel shift* of distance δ and the curve becomes s_1. That is, s_1 stands for the spot curve on which each rate corresponding to t' has value $s(t,t') + \delta$ (i.e. $s_1(t,t') = s(t,t') + \delta$ for each $t' \geq t$). Note this shift is supposed to be instantaneous: we are still standing on time point t, and both $s(t,t')$ and $s_1(t,t')$ give (different) values to spot rates prevailing from now (t) to t'. Figure 3 gives an illustration of the parallel shift of a spot curve.

Let $B(s)$ and $B(s_1)$ denote the bond prices before and after the spot rate shift. Then, we use Formula 3 to derive:

$$\begin{aligned}
B(s_1) - B(s) &= \sum_{n=1}^{N} C_n e^{-(t_n - t)s_1(t,t_n)} - \sum_{n=1}^{N} C_n e^{-(t_n-t)s(t,t_n)} \\
&= \sum_{n=1}^{N} \left[C_n \left(e^{-(t_n-t)(s(t,t_n)+\delta)} - e^{-(t_n-t)s(t,t_n)} \right) \right] \\
&\approx -\left\{ \left[\sum_{n=1}^{N} \frac{C_n e^{-s(t,t_n)(t_n-t)}}{B(s)} (t_n - t) \right] B(s) \right\} \delta. \qquad (24)
\end{aligned}$$

This derivation is the same in nature as that for Macaulay duration (see (16)): we take the difference of bond prices, invoke relevant formula, and apply first order Taylor expansion to approximate the difference that is involved in the sum. To this point, you should observe that it is the term inside the bracket on the last line that controls the change in bond price, which is exactly the Fisher-Weil duration. We formalize this into Formula 8.

Formula 8 (Fisher-Weil Duration and Bond Price Sensitivity to Parallel Shift) Recall, s and s_1 stand for, respectively, the spot rate curve at time t before and after a parallel shift of distance δ. That is, s stands for the original curve $s(t,t')$ and s_1 stands for the shifted curve $s_1(t,t')$ with $s_1(t,t') = s(t,t') + \delta$. Also recall, $B(s)$ and $B(s_1)$ are the bond prices evaluated at s and s_1 respectively, using Formula 3.

The Fisher-Weil duration is denoted D_{FW} and defined as

$$D_{FW} = \sum_{n=1}^{N} \frac{C_n e^{-s(t,t_n)(t_n-t)}}{B(s)}(t_n - t). \tag{25}$$

The Fisher-Weil duration governs the bond price change due to the parallel shift of spot rate curve by:

$$B(s_1) - B(s) \approx -D_{FW}[B(s)\delta]. \tag{26}$$

□

The Fisher-Weil duration for a bond portfolio is defined exactly the same way as that for the Macaulay duration. For completeness we still give a formula here.

Formula 9 (Fisher-Weil Duration for A Bond Portfolio)
Suppose a bond portfolio comprises of m bonds, indexed by $i = 1, \cdots, m$. Let s and s_1 denote the spot rate curve before and after a parallel shift of δ (as described in Formula 8).

Before the curve shift, the i-th bond has price $B_i(s)$ and Fisher-Weil duration D_{FW_i}. Let $B(s)$ denote the total value of this portfolio before curve shift, and clearly $B(s) = B_1(s) + \cdots + B_m(s) = \sum_{i=1}^{m} B_i(s)$. The same notations apply to quantities after the curve shift by changing the argument s to s_1.

The Fisher-Weil duration of this bond portfolio, denoted D_{FW}, has the following expression:

$$\begin{aligned} D_{FW} &= \sum_{i=1}^{m} \frac{B_i(s)}{B(s)} D_{FW_i} \\ &= \frac{B_1(s)}{B(s)} D_{FW_1} + \frac{B_2(s)}{B(s)} D_{FW_2} + \cdots + \frac{B_m(s)}{B(s)} D_{FW_m}. \end{aligned} \tag{27}$$

Then, the portfolio value change due to the parallel shift of spot rate curve from s to s_1 is:

$$B(s_1) - B(s) \approx -D_{FW}[B(s)\delta]. \tag{28}$$

□

Remark 5 a. The key assumption of Fisher-Weil duration is the parallel shift of the spot rate curve.

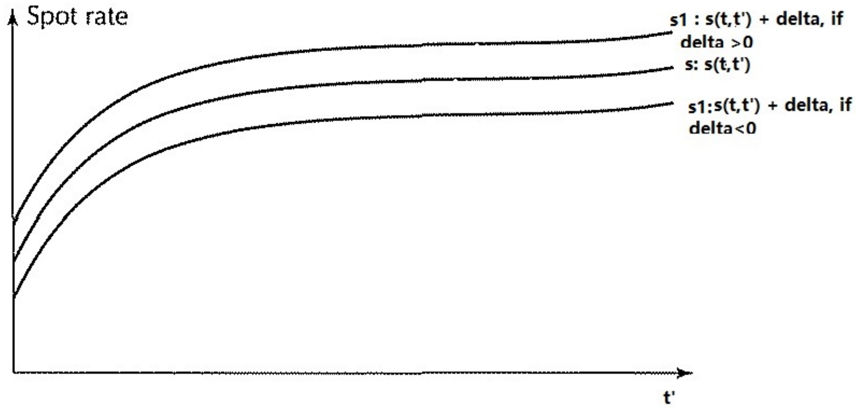

Figure 3: Illustration of parallel shift of spot rate curve.

b. In parallel with Item b of Remark 4, the Fisher-Weil duration of a zero-coupon bond with maturity date T is $T - t$ (the current time is t).

c. The Fisher-Weil duration can be interpreted as "bond's life", the same as what we did for the Macaulay duration. This is left as an exercise.

d. Fisher-Weil duration can also be defined based on Formula 2. That is, we can choose to represent the spot rate curve based on some reference interval length d, and follow the same steps (as those in (24)) to compute for an expression of the resulted duration. Thus, Macaulay duration is usually considered as a special case of Fisher-Weil duration: when the spot rate curve is flat, the spot rate value coincides with the yield (if we choose the reference interval to be the same as that for the yield). See Item c of Remark 2.

3.3 Immunization

Now we are equipped to construct an *immunization* portfolio, thereby answers the question posed at the beginning of this section: how to use bonds to hedge against interest rate change so as to prepare for a future cash obligation.

Let us formalize the setting. Suppose currently we are on time t, and we are obliged to pay a cash of amount $\$C$ on some future time point $T > t$. We have two bonds which we call "bond 1" and "bond 2". Then, immunization means to decide on how many units to buy bond 1 and bond 2, respectively. And the decision criteria consists of:

1. *Value match.* The value of the constructed portfolio must equal that of the obligation;

2. *Risk match.* The risk measure of the constructed portfolio must equal that of the obligation.

For Item 2, the risk match criterion, the risk measure is chosen by the decision maker. We give exact formulation of immunization based on the two measures we studied in the two formulae below.

Formula 10 (Immunization Based on Macaulay Duration)
The current time is t. Let the common yield and the current prices of bond 1 and 2 be denoted y with a reference period length d, B_1 and B_2. Let the Macaulay duration of the two bonds be denoted D_1 and D_2. The obligation occurs at a future time $T > t$ of amount $\$C$.
Then, the immunization involves *buying* x_1 *units of bond 1* and x_2 *units of bond 2* at current time. Specifically, x_1 and x_2 satisfy the following equations.

$$x_1 B_1 + x_2 B_2 = \frac{C}{(1+y)^{\frac{T-t}{d}}} \tag{29}$$

$$\left[\frac{x_1 B_1}{\frac{C}{(1+y)^{\frac{T-t}{d}}}}\right] D_1 + \left[\frac{x_2 B_2}{\frac{C}{(1+y)^{\frac{T-t}{d}}}}\right] D_2 = T - t. \tag{30}$$

□

Note (29) matches the value of the portfolio (left hand side) to that of the obligation (the right hand side). In particular, the value of the obligation is calculated based on the bond's yield and this is a compromise made in this method (as in general yields should not be used for pricing). (32) matches the Macaulay duration of the portfolio (left hand side) to that of the obligation (the right hand side). In particular, for left hand side we use Formula 7. For the right hand side, note the obligation's Macaulay duration is $T - t$ as it is a one-shot cash flow thereby treated as a zero-coupon bond (see Item b of Remark 4).

The immunization based on Macaulay duration usually involves some compromise (see Item c of Remark 4). Immunization can be implemented based on Fisher-Weil duration which is usually considered to be more robust than the one based on Macaulay duration (but this requires building a spot rate curve first).

Formula 11 (Immunization Based on Fisher-Weil Duration)
The current time is t, and there are two bonds with current prices denoted B_1 and B_2. Suppose we've already have a spot rate curve $s(t, t')$. Then, the Fisher-Weil duration of the two bonds are denoted D_{FW_1} and D_{FW_2}. The obligation occurs at a future time $T > t$ of amount $\$C$.
The immunization involves *buying* x_1 *units of bond 1* and x_2 *units of bond 2* at current time. Specifically, x_1 and x_2 satisfy the following equations.

$$x_1 B_1 + x_2 B_2 = C e^{-s(t,T)(T-t)} \tag{31}$$

$$\left[\frac{x_1 B_1}{C e^{-s(t,T)(T-t)}}\right] D_{FW_1} + \left[\frac{x_2 B_2}{C e^{-s(t,T)(T-t)}}\right] D_{FW_2} = T - t. \tag{32}$$

□

The interpretation of the two equations above are similar to that for (29) and (32), just replacing the Macaulay duration by the Fisher-Weil duration. Here a point to note is that the right hand side of (31) is now considered to be a well-defined valuation of the obligation, since we are using the spot rate (as opposed to the yield used in (29)).

Remark 6

a. The logic of immunization discussed in this section only provides partial risk mitigation due to the strong assumptions involved. Nevertheless, they are very practical (as opposed to more involved models; see §5.3) and widely used.

b. Immunization portfolio constructed should be updated from time to time, since the yield and spot rate curve can (and will) change.

4 Numerical Examples in Excel/VBA

4.1 Example 1: Building A Spot Rate Curve

For this example, we have prices and yields for U.S. treasury coupon bonds as of 17 February 2023 3pm (EST), with time to maturity spanning from two weeks to around five years. The task is to fit for a 5-year spot rate curve on this particular time stamp. The prices have been adjusted for accrued interests [1] and the bonds are selected to be suitable for fitting [2].

We will work on the following problem instances. For lighter notation, let's denote

$$\tau = t' - t.$$

1. 4-th order polynomial for $s(t,t')$ as below:

$$s(t,t') = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 \tag{33}$$

2. *Cubic splines* with one *knot* on $\tau = k_1$:

$$\begin{aligned}
s(t,t') &= a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3, \quad \tau \leq k_1 \\
s(t,t') &= b_0 + b_1(\tau - k_1) + b_2(\tau - k_1)^2 + b_3(\tau - k_1)^3, \quad \tau \geq k_1.
\end{aligned} \tag{34}$$

3. Cubic splines with four knots on $\tau = k_1, k_2, k_3, k_4$.

$$\begin{aligned}
s(t,t') &= a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3, \quad \tau \leq k_1 \\
s(t,t') &= b_0 + b_1(\tau - k_1) + b_2(\tau - k_1)^2 + b_3(\tau - k_1)^3, k_1 \leq \tau \leq k_2 \\
s(t,t') &= c_0 + c_1(\tau - k_2) + c_2(\tau - k_2)^2 + c_3(\tau - k_2)^3, \quad k_2 \leq \tau \leq k_3 \\
s(t,t') &= d_0 + d_1(\tau - k_3) + d_2(\tau - k_3)^2 + d_3(\tau - k_3)^3, \quad k_3 \leq \tau \leq k_4 \\
s(t,t') &= e_0 + e_1(\tau - k_4) + e_2(\tau - k_4)^2 + e_3(\tau - k_4)^3, \quad \tau \geq k_4.
\end{aligned} \tag{35}$$

It may appear that for the two cubic spline models there are a lot of parameters: 8 in (34) and 20 in (35), but in fact the number is reduced due to the constraints on the knot(s): continuity (*no jump*), first-order derivative continuity (*no kink*) and second-order derivative continuity (*smoothness*)[3]. The constraints for (34) are outlined below:

1. *no jump on* k_1*:* $b_0 = a_0 + a_1 k_1 + a_2 k_1^2 + a_3 k_1^3$.

2. *no kink on* k_1*:* $b_1 = a_1 + 2a_2 k_1 + 3a_3 k_1^2$.

[1]This relates to some quoting convention to accommodate accounting issues. Refer to [3].

[2]In practice this includes excluding certain bonds with special characteristics such as extremely high/low liquidity, call provisions, etc. Refer to [3].

[3]Rigorously, it is "smoothness to second order".

3. *smoothness on* k_1*:* $2b_2 = 2a_2 + 6a_3k_1$.

We can see essentially there are only 5 parameters in (34) to be found: a_0, a_1, a_2, a_3 and b_3. The rest, b_0, b_1 and b_2 are determined by the equations above. The same procedure can be done for (35), and for this case the number of variables is reduced from 20 to 8; this is left as an exercise.

The codes involve the following functions:

1. A helper function that extracts the payment schedules for each quote of the U.S. treasury securities.
2. For each spot rate model, a function that takes in a set of parameters and outputs a spot rate. (33), (34) or (35) (or any other model you are fitting) is coded here.
3. A function that takes in a payment schedule and a spot rate model, and outputs a bond price. Note Formula 3 is coded in this function.

For a specific spot rate model, inputs to the program comprise of: (1) an observed quote ; (2) a set of spot rate model parameters. Output is a model-implied bond price, based on which a model-implied yield is readily computed. For the input part, (1) is data which cannot be changed and we vary values in (2) to minimize the error measure in (11) or (12). This step can be done by many off-the-shelf softwares, and here we use Excel Solver.

We will look into the spreadsheet modeling with VBA codes as well as the solver in lecture (or tutorial). Below is a summary of fitting results for various spot rate models.

Model	Average Price Error	Max Price Error	Average Yield Error	Max Yield Error
4th-order polynomial	0.7849	2.1259	0.0019	0.0146
Cubic spline; knot: 1	0.7010	1.9695	0.0010	0.0076
Cubic spline; knots: 1,2,3,4	0.6640	2.0419	0.0010	0.0079
Cubic spline; knots: 0.25, 0.6, 1.5, 3	0.6971	2.0093	0.0009	0.0052

Table 2: Summary for fitting (33), (34) and (35). Sum of squared yield error, (12), is minimized.

4.2 Example 2: Immunization (Macaulay Duration)

The data set is the same as those in the previous example. Here we work out a simple example on using two coupon bonds to hedge for a future obligation: 1 million USD to be paid on 17 Feb 2026.

The crucial quantity to be computed are the Macaulay durations. Since we are using the U.S. treasury securities, we can use a built-in Excel worksheet function. The rest amounts to reading the data into the VBA function, formulating the equation system in Formula 10 and solving it. We will look into the codes in the lecture/tutorial.

The complication here is stated in Item c. of Remark 4: we cannot find two bonds with a common yield. We make compromise by using the average of bond yields as the common yield.

Below is a result summary. The "long" and "short" refer to bonds' time to maturities.

	Time to maturity	Yield	Units to buy
Bond 1 (short)	31 January 2026	0.0431	5147.4584
Bond 2 (long)	31 March 2026	0.0425	4688.5322

Table 3: Immunization example.

5 Further Reading and Pointer to References*

5.1 Explanations of Term Structure and Expectation Hypotheses

The spot rate curves (as well as the yield curves) are *never flat*. Under normal situations, it increases rapidly for short maturities, then slopes more gradually as for the longer end. They can also take other shapes as the market condition changes (e.g. the inverted curve during crisis). The question we want to ask is: why is there a term structure, that is, why spot loans of different maturities bear different interest rates? There are various explanations for this.

The first explanation is called *expectation hypothesis*: most people in the market believe spot rates will rise in the future (for example, due to inflation); in addition, the future spot rate is reflected by the forward rate without bias. The combining effect is an increasing spot rate curve. To see this, note $s(t, T+1)(T+1-t) = s(t,T)(T-t) + f(t,T,T+1)$.

Under the expectation hypothesis, $f(t,T,T+1)$ is the unbiased estimator of $s(T,T+1)$ (note at time t, $s(T,T+1)$ is unknown); that is, $f(t,T,T+1) = E_t[s(T,T+1)]$, where E_t stands for the expectation made at time t. Since $f(t,T,T+1)$ is the expected future spot rate over $[T,T+1]$ and people believe this rate will be larger than $s(t,T)$, we must have: $f(t,T,T+1) \geq s(t,T)$. Then we conclude $s(t,T) \leq s(t,T+1)$, explaining an upward sloping spot rate curve.

The second explanation says investors prefer bonds with short-term maturities over those with longer terms, pushing the prices for the former up and thus sending the spot rates down. One reason for this preference is that investors want their funds back shortly rather than being tied up (with the long term loans). Another reason is that many investors want their money back soon; so, if they hold long term bonds, they have to *sell* them soon. But the long term bonds are more sensitive to interest rate fluctuations, hence their prices are more volatile than the short term ones. Due to this reason, investors are reluctant to hold long term bonds, sending the prices down and the spot rates up.

The third explanation says people buying short term bonds and long term bonds are just different groups of people, so they demand different levels of rates. This argument is called market segmentation.

The three arguments above try to explain why there is a term structure for interest rates; refer to §4.4 of [L] for details.

Another related theory (also called expectation hypothesis; hence expectation hypotheses (plural) is the common name to be refereed) is called "present value form of expectation hypotheses". It says bond prices evolve according to risk-free interest rate. That is, $E_t[B(t+1,T)] = B(t,T)[1+s(t,t+1)]$. Note this means investors are risk-neutral — they do not demand a premium over risk-free rates $s(t,t+1)$. This is in general not true, since investors are risk-averse. But a modification of this hypothesis is the correct bond pricing formula (in expectation form; *not* Formula 2 or Formula 3); refer to Chapter 5 of [2].

5.2 Interest Rate Derivatives

There are many financial instruments that are designed based on bonds and/or interest rates. For example, *bond options* give the holders the rights to buy/sell the bonds at certain pre-determined prices (hence options are sold at prices). *Forward* and *Futures* on bonds are agreements to deliver the bonds at a future time point. *Swaps* are agreements between two parties to exchange streams of cash flows: one arm is based on floating rate (i.e. certain interest rates (like LIBOR)), and the cash flows on the other arm are of pre-specified amount. *Caps* and *floors* entitle the holders to receive payments when interest rates go out of certain boundaries. There are other types of interest rate derivatives.

Pricing interest rate derivatives typically need the spot rate curve and/or forward rate curve as inputs (more specifically, as "state variables").

5.3 Advanced Models and Pointer to References

So far this notes does not concern bond pricing in any modeling framework. Although we can fit the curves and use them to price various bonds *on a specific time stamp*, we do not know *how* the prices will evolve since we do not rely on any specific model describing such evolution. This is restrictive because *hedging* requires a description of how the curves (and bond prices) evolve; as well, the forecasts of portfolio values require the same. Furthermore, interest rate derivative pricing (and hedging) require the same.

By looking at historical evolution of interest rates (possibly via looking at the curves fitted in the past), we may come up with models that describe how the interest rate moves in *physical world*. But when coming to *pricing* interest rate derivative, the evolution in physical world is not relevant; we need the evolution in *risk-neutral world* to derive *arbitrage-free prices.*

There are two main classes of models, both are under continuous time stochastic setting. The first class describes how *short rates* move. Short rate, denoted by r_t, is the continuous compounding interest rate prevailing $[t, t + dt]$, where dt is infinitesimal. Based on short rates, spot rates and forward rates can be expressed. In a nutshell, the bond prices are determined by:

$$B(t, T) = E^M[e^{-\int_t^T r_s ds} | \mathcal{F}_t];$$

$B(t, T)$ is the time-t price of a bond paying \$1 on time T. $\mathcal{F}_t$ is the symbol describing all information up to time t[1]. The E^M means the expectation is taken under *risk-neutral probability measure*, under which prices of all tradable fixed income securities are *martingales.* Below is a collection of short rate models for r_t. All W_t^M mean a Brownian motion under the risk-neutral measure M. First, *Vasicek Model* models r_t by an Ornstein-Uhlenbeck process

[1] If you come from a maths background: $\mathcal{F}_t$ is one element of a filtration.

which is mean-reverting:

$$dr_t = (\theta - \kappa \cdot r_t)dt + \sigma dW_t^M;$$

Then, *Cox-Ingersoll-Ross (CIR) Model* preserves mean-reverting feature while maintaining non-negativity:

$$dr_t = (\theta - \kappa \cdot r_t)dt + \sigma\sqrt{r_t}dW_t^M;$$

Next, *Ho-Lee Model* uses a drifted Brownian motion with time-dependent drift parameter:

$$dr_t = \theta_t dt + \sigma dW_t^M;$$

And, *Hull-White Model* extends Vasicek Model above to allow time-dependent long-term mean and volatility parameter:

$$dr_t = (\theta_t - \kappa \cdot r_t)dt + \sigma_t dW_t^M.$$

There are many others. Note for all the models here, the short rate process is modeled under risk-neutral measure M, and this is for pricing purpose. If we want to do *joint pricing and hedging*, we also need to model their movements in physical world, and this involves modeling of *market price of risk* (which can be a parameter, and can also be a stochastic process).

The second class is called *Heath-Jarrow-Morton (HJM) Framework*, and instead of looking at short rates, it models the *entire forward rate curve*, $f(t,u)$ $(t \leq u)$. Here $f(t,u)$ means the continuous compounding forward rate observed at time t that prevails $[u, u+du]$, where du is infinitesimal. Note $f(t,t)$ is then the short rate on time t. Under this framework, the bond price (for a bond paying \$1 on T) is expressed as:

$$B(t,T) = e^{-\int_t^T f(t,u)du}.$$

Note there is no expectation operator, since the forward curve $f(t,u)$ is observed (i.e. known) on time t. Then, HJM framework says, no matter what kind of model you want to give to $f(t,u)$, it must satisfy that under the risk-neutral measure M, the drift and volatility of $f(t,u)$ conform to a restriction equation (the details of the equation are omitted here since they are mathematically involved). This condition eliminates all arbitrage opportunities. Again, this is for pricing purpose; for joint hedging, description of $f(t,u)$'s movement in physical measure is needed. Note the HJM framework only specifies a condition, not specific models. This allows quite a lot of modeling flexibility (and this is why it is called "framework").

Remark 7 The content in this section is quite brief and only serves as an outline. If you want to learn more about fixed-income securities focusing on practical issues and/or practical methods, use [3] and [1]. But note they can tend to focus more on US markets. If you want to learn the modeling theory in a mathematical manner and focus on the principal/logic, start from [2].

References

[1] FABOZZI, F.J., *The Handbook of Fixed Income Securities.* (2006), 7ed, McGraw Hill.

[2] JARROW, R.A., *Modelling Fixed Income Securities and Interest Rate Options.* (2002), 2ed, Stanford Economics and Finance.

[3] TUCKMAN, B., *Fixed Income Securities: Tools for Today's Markets.* (2002), 2ed, John Wiley & Sons.