From Izhikevich ABC model to the I-V formulation

## ABC → I/V Translation Try 3.

Existing model:

$$\frac{1}{T}\dot{v} = Av^2 + Bv + C - u + I \qquad (\mathrm{I})$$
$$\frac{1}{T}\dot{u} = a'(b'v - u) \qquad (\mathrm{II})$$

$v > v_{peak} \Rightarrow v = c$, $u = u + d$

With: $A = 0.032$, $B = 4$, $C = 113.147$, $T = 0.4$, $a' = 0.015$, $b' = 0.191$, $c = -64$, $d = 0.05$, $V_{peak} = 30$

Translate to I/V formulation like:

$$Cap\,\dot{v} = k(v - v_r)(v - v_t) - u' + I \qquad (\mathrm{III})$$
$$\dot{u}' = a(b(v - v_r) - u') \qquad (\mathrm{IV})$$

$v > v_{peak} \Rightarrow v = c$, $u' = u' + d$

noting that we have allowed for a change in variable for u such that $u' = u + u_0$. Rewrite (III):

$$Cap\,\dot{v} = k(v - v_r)(v - v_t) - u - u_0 + I$$
$$\Rightarrow Cap\,\dot{v} = k(v^2 - vv_t - vv_r + v_r v_t) - u - u_0 + I$$
$$\Rightarrow Cap\,\dot{v} = kv^2 - k(v_t + v_r)v + kv_r v_t - u_0 - u + I \qquad (\mathrm{V})$$

Compare (V) with (I):

$$Cap\,\dot{v} = kv^2 + (-k(v_t + v_r))v + (kv_r v_t - u_0) - u + I$$
$$\frac{1}{T}\dot{v} = Av^2 + Bv + C - u + I$$

and see that $\frac{1}{T} = Cap$; $A = k$; $B = -k(v_t + v_r)$; $C = (kv_r v_t - u_0)$.

Now Rewrite (IV) interms of u:

$$\dot{u}' = a(b(v - v_r) - u') \Rightarrow \dot{u} = a(b(v - v_r) - u - u_0) \qquad (\mathrm{VI})$$

Consider the nullcline for u both from VI, above & from (II)

$$\dot{u} = \phi = a(b(v - v_r) - u - u_0) \Rightarrow u = b(v - v_r) - u_0$$
$$\dot{u} = Ta'(b'v - u) = 0 \Rightarrow u = b'v$$

Slope of u nullcline to be identical so $b = b'$.

Set these equal: $bv = b(v - v_r) - u_0$

$$bv = bv - bv_r - u_0$$

$$0 = -bv_r - u_0$$

$$\Rightarrow \boxed{u_0' = -bv_r} \quad (\text{VII})$$

This offset ensures that whereas in the I/V formulation, the linear nullcline crosses the v axis at $v_r$; in the ABC formulation, it crosses at $v = \phi$.

Now solve for $k$, $v_r$, $v_t$ in terms of A, B, C & b:

| | | |
|---|---|---|
| $A = k$ | $B = -k(v_t + v_r)$ | $C = kv_r v_t + bv_r$ |
| | $\Rightarrow B = -kv_t - kv_r$ | $= (kv_t + b)v_r$ |
| | $\Rightarrow kv_t = -kv_r - B$ | $\Rightarrow v_r = \dfrac{C}{kv_t + b}$ |
| | $\Rightarrow v_t = -v_r - \dfrac{B}{k}$ | |

$$v_r = \frac{C}{k\left(-v_r - \frac{B}{k}\right) + b}$$

$$\Rightarrow C = k v_r\left(-v_r - \frac{B}{k}\right) + bv_r$$

$$\Rightarrow C = -kv_r^2 - Bv_r + bv_r$$

$$\Rightarrow \phi = -kv_r^2 - (B-b)v_r - C$$

$$\Rightarrow kv_r^2 + \underbrace{(B-b)}_{\text{"B"}}v_r + C = \phi$$

Now, $k = A$, so:

$$Av_r^2 + (B-b)v_r + C = \phi$$

for which we can find roots;

$$v_r = \frac{-(B-b) \pm \sqrt{(B-b)^2 - 4AC}}{2A}$$

(one root is $v_r$, the other is $v_t$)

$$v_r = \frac{-(4 - 0.191) \pm \sqrt{(4-0.191)^2 - 4\,0.032 . 113 . 147}}{2 . 0.032}$$

$$= \frac{-3.809 \pm \sqrt{3.809^2 - 14.482816}}{0.064}$$

$$V_r = \frac{-3.809 \pm \sqrt{14.508481 - 14.482816}}{0.064}$$

$$= \frac{-3.809 \pm 0.025665}{0.064}$$

$$= \frac{-3.783335}{0.064} \quad \text{or} \quad \frac{-3.834665}{0.064}$$

$$V_r = -59.1146. \quad \text{or} \quad -59.9166. \qquad \text{VIII}$$

So we now have:

$$\boxed{\begin{aligned} Cap\,\dot{v} &= k(v-V_r)(v-V_t) - u + bV_r + I \\ \dot{u} &= a(b(v-V_r) - u + bV_r) \end{aligned}}$$

Where $Cap = \frac{1}{T}$, $k = A$, $V_r, V_t$ = roots of above – VIII, $b = b'$, $a = Ta'$

Nullclines are: $u = k(v-V_r)(v-V_t) + bV_r + I$

$u = bv.$