

## Lecture 1, 2

### Function, Limit and Continuity

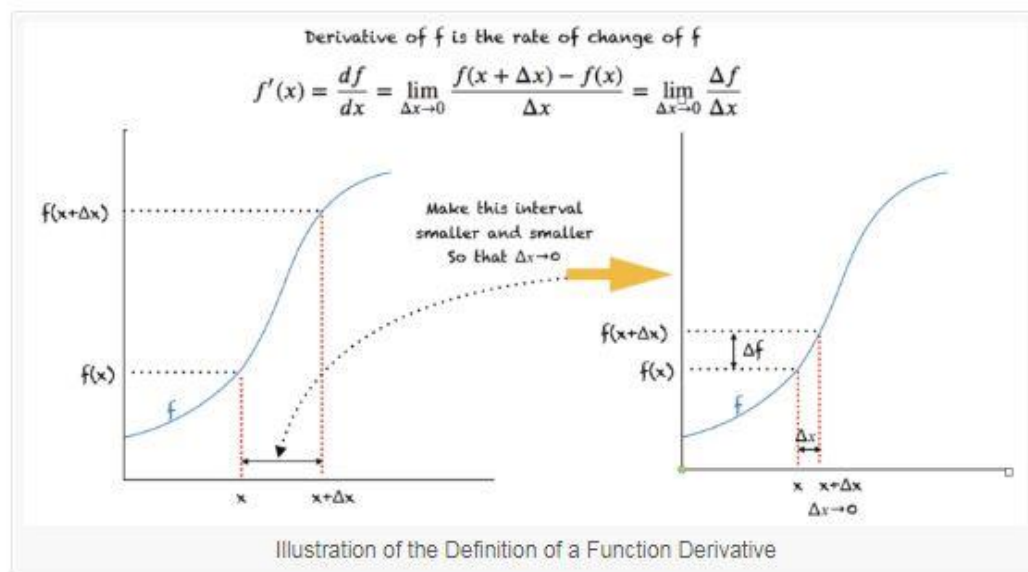
## Lecture 3

### Derivative:

The derivative is **the instantaneous rate of change of a function with respect to one of its variables**. This is equivalent to finding the slope of the tangent line to the function at a Point.

## What is the Derivative of a Function

In very simple words, the derivative of a function  $f(x)$  represents its rate of change and is denoted by either  $f'(x)$  or  $df/dx$ . Let's first look at its definition and a pictorial illustration of the derivative.



In the figure,  $\Delta x$  represents a change in the value of  $x$ . We keep making the interval between  $x$  and  $(x + \Delta x)$  smaller and smaller until it is infinitesimal. Hence, we have the limit ( $\Delta x \rightarrow 0$ ). The numerator  $f(x + \Delta x) - f(x)$  represents the corresponding change in the value of the function  $f$  over the interval  $\Delta x$ . This makes the derivative of a function  $f$  at a point  $x$ , the rate of change of  $f$  at that point.

Find the derivative of the following

i.  $y = f(x) = x^n$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} x^n = nx^{n-1}$$

ii. Example  $y = f(x) = x^5$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} x^5 = 5x^{5-1} = 5x^4$$

iii.  $y = f(x) = c$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} c = 0$$

iv.  $y = f(x) = 4$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} 4 = 0$$

v.  $y = f(x) = cx^n$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} cx^n = c \frac{d}{dx} x^n = cnx^{n-1}$$

vi.  $y = f(x) = 3x^6$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} 3x^6 = 3 \frac{d}{dx} x^6 = 3 \times 6x^{6-1} = 18x^5$$

vii.  $y = f(x) = \log x$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} \log x = \frac{1}{x}$$

viii.  $y = f(x) = e^x$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} e^x = e^x$$

ix.  $y = f(x) = uv$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} uv = u \frac{d}{dx} v + v \frac{d}{dx} u$$

x.  $y = f(x) = x^3 e^x$

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} x^3 e^x = x^3 \frac{d}{dx} e^x + e^x \frac{d}{dx} x^3 = x^3 e^x + 3e^x x^2$$

Lecture 4, 5

Find the differential coefficient of the following

i)  $y = x^{\tan x} + (\sin x)^{\cos x}$

ii)  $y = x^{\cos^{-1} x} + (\sin x)^{\log x}$

Solution: (i)

$$y = x^{\tan x} + (\sin x)^{\cos x}$$

$$y = u + v$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \dots\dots\dots(1)$$

$$u = x^{\tan x} \text{ and } v = \sin x^{\cos x}$$

$$\log u = \tan x \log x \text{ and } \log v = \cos x \log \sin x$$

$$\frac{1}{u} \frac{du}{dx} = \tan x \frac{1}{x} + \log x \sec^2 x$$

$$\frac{du}{dx} = u \left[ \tan x \frac{1}{x} + \log x \sec^2 x \right]$$

$$\frac{du}{dx} = x^{\tan x} \left[ \tan x \frac{1}{x} + \log x \sec^2 x \right]$$

Again  $\frac{1}{v} \frac{dv}{dx} = \cos x \frac{\cos x}{\sin x} - \sin x \log \sin x$

$$\frac{dv}{dx} = (\sin x)^{\cos x} \left[ \cos x \frac{\cos x}{\sin x} - \sin x \log \sin x \right]$$

From (1), We get

$$\frac{dy}{dx} = x^{\tan x} \left[ (\tan x) \frac{1}{x} + \log x \sec^2 x \right] + (\sin x)^{\cos x} \left[ \cos x \frac{\cos x}{\sin x} - \sin x \log \sin x \right]$$

$$\frac{dy}{dx} = x^{\tan x} \left[ (\tan x) \frac{1}{x} + \log x \sec^2 x \right] + (\sin x)^{\cos x} \left[ \frac{\cos^2 x}{\sin x} - \sin x \log \sin x \right]$$

Find the differential coefficient of the following

i)  $y = \tan^{-1} \frac{\sqrt{1+x^2}-1}{x}$

ii)  $y = \cos^{-1} \frac{1-x^2}{1+x^2}$

Solution:  $x = \tan \theta, \quad \tan^{-1} x = \theta$

Find the differential coefficient of the following

i)  $\tan^{-1} \frac{2x}{1-x^2}$  with respect to  $\sin^{-1} \frac{2x}{1+x^2}$

Solution:  $y = \tan^{-1} \frac{2x}{1-x^2}$  with respect to  $z = \sin^{-1} \frac{2x}{1+x^2}$

We have to find  $\frac{dy}{dz} = \frac{dy}{dx} / \frac{dz}{dx}$

$x = \tan \theta, \quad \tan^{-1} x = \theta$

$$y = \tan^{-1} \frac{2x}{1-x^2}$$

## Lecture 6

If  $y = f(x)$ , the successive derivatives are also denoted by

$$y_1, y_2, y_3, \dots, y_n$$

$$y', y'', y''', \dots, y^{(n)}$$

$$f'(x), f''(x), f'''(x), \dots, f^{(n)}(x)$$

$$Df(x), D^2f(x), D^3f(x), \dots, D^n f(x)$$

$D$  standing for the symbol  $\frac{d}{dx}$

### 1. The $n$ th derivative of some special functions

$$y = x^n$$

$$y_1 = nx^{n-1}$$

$$y_2 = n(n-1)x^{n-2}$$

$$y_3 = n(n-1)(n-2)x^{n-3}$$

$$y_3 = n(n-1)\{n-(3-1)\}x^{n-3}$$

and proceeding in a similar manner

$$y_r = n(n-1)(n-2)\dots\{(n-(r-1))\}x^{n-r}$$

$$y_n = n(n-1)(n-2)\dots\{(n-(n-1))\}x^{n-n}$$

$$y_n = n(n-1)(n-2)\dots 3, 2, 1 = n!$$

### 2. $y = (ax+b)^m$

$$y_1 = ma(ax+b)^{m-1}$$

$$y_2 = a^2m(m-1)(ax+b)^{m-2}$$

$$y_3 = a^3m(m-1)(m-2)(ax+b)^{m-3}$$

$$y_3 = a^3m(m-1)\{m-(3-1)\}(ax+b)^{m-3}$$

and proceeding in a similar manner

$$y_n = a^n m(m-1)(m-2).....\{(m-(n-1))\}(ax+b)^{m-n}$$

$$y_n = a^n \frac{m!}{(m-n)!} (ax+b)^{m-n}$$

**Leibnitz's theorem:** (nth derivative of the product of two functions)

If  $u$  and  $v$  are two functions of  $x$ , then the nth derivative of their product i.e.,

$$(uv)_n = u_n v + {}^n C_1 u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2 + ..... + {}^n C_r u_{n-r} v_r + ..... + uv_n$$

where the suffixes in  $u$  and  $v$  denote the order of differentiations of  $u$  and  $v$  with respect to  $x$

Let  $y = uv$ . By actual differentiation, we have  $y_1 = u_1 v + uv_1$

$$y_2 = u_2 v + 2u_1 v_1 + uv_2 = u_2 v + {}^2 C_1 u_1 v_1 + uv_2$$

$$y_3 = u_3 v + 3u_2 v_1 + 3u_1 v_2 + uv_3 = u_3 v + {}^3 C_1 u_2 v_1 + {}^3 C_2 u_1 v_2 + uv_3$$

The theorem is thus seen to be true when  $n=2$  and  $n=3$ .

Let us assume therefore that

$$y_n = u_n v + {}^n C_1 u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2 + ..... + {}^n C_r u_{n-r} v_r + ..... + uv_n$$

Where  $n$  has any particular value.

Differentiating,

$$y_{n+1} = u_{n+1} v + ({}^n C_1 + 1) u_n v_1 + ({}^n C_2 + {}^n C_1) u_{n-1} v_2 + ..... + ({}^n C_r + {}^n C_{r-1}) u_{n-r+1} v_r + ..... + uv_{n+1}$$

Since  $({}^n C_r + {}^n C_{r-1}) = {}^{n+1} C_r$  and  $({}^n C_1 + 1) = {}^{n+1} C_1$

$$y_{n+1} = u_{n+1} v + {}^{n+1} C_1 u_n v_1 + {}^{n+1} C_2 u_{n-1} v_2 + ..... + {}^{n+1} C_r u_{n-r+1} v_r + ..... + uv_{n+1}$$

Thus, if the theorem holds for  $n$  differentiations, it also holds for  $n+1$ . But it was proved to hold for 2 and 3 differentiations. Hence it holds for four, and so on, and thus the theorem is true for every positive integral value of  $n$ .

Example: If  $y = e^{\tan^{-1} x}$  then (i)  $(1+x^2)y_2 + (2x-1)y_1 = 0$

$$(ii) (1+x^2)y_{n+2} + \{2(n+1)x-1\}y_{n+1} + n(n+1)y_n = 0$$

Solution:

$$y = e^{\tan^{-1} x}$$

$$\log y = \tan^{-1} x$$

$$\frac{1}{y} y_1 = \frac{1}{1+x^2}$$

$$(1+x^2)y_1 = y$$

$$(1+x^2)y_2 + 2xy_1 = y_1$$

$$(1+x^2)y_2 + (2x-1)y_1 = 0$$

$$y_2(1+x^2) + y_1(2x-1) = 0$$

By leibnitz's theorem

$$y_{n+2}(1+x^2) + {}^n c_1 y_{n+1}(2x) + {}^n c_2 y_n 2 + y_{n+1}(2x-1) + {}^n c_1 y_n 2 = 0$$

$$(1+x^2)y_{n+2} + n y_{n+1}(2x) + \frac{n(n-1)}{2} y_n 2 + (2x-1)y_{n+1} + n y_n 2 = 0$$

$$(1+x^2)y_{n+2} + (2xn + 2x-1)y_{n+1} + (n^2 - n + 2n)y_n = 0$$

$$(1+x^2)y_{n+2} + \{2(n+1)x-1\}y_{n+1} + n(n+1)y_n = 0$$

Example: If  $y = \tan^{-1} x$  then

$$(i) (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

Example: If  $y = \sin(m \sin^{-1} x)$  then

$$(i) (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$$

CHAPTER VII  
MAXIMA AND MINIMA  
( Functions of a Single Variable )

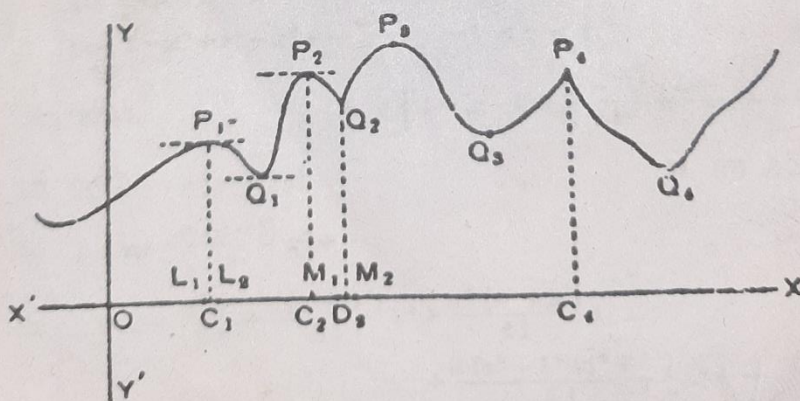
7.1. By the maximum value of a function  $f(x)$  in Calculus we do not necessarily mean the absolutely greatest value attainable by the function. A function  $f(x)$  is said to be maximum for a value  $c$  of  $x$ , provided  $f(c)$  is greater than every other value assumed by  $f(x)$  in the immediate neighbourhood of  $x=c$ . Similarly, a minimum value of  $f(x)$  is defined to be the value which is less than other values in the immediate neighbourhood. A formal definition is as follows :

A function  $f(x)$  is said to have a maximum value for  $x=c$ , provided we can get a positive quantity  $\delta$  such that for all values of  $x$  in the interval  $c-\delta < x < c+\delta$ , ( $x \neq c$ )  $f(c) > f(x)$  ;

i.e., if  $f(c+h) - f(c) < 0$ , for  $|h|$  sufficiently small.

Similarly, the function  $f(x)$  has a minimum value for  $x=d$ , provided we can get an interval  $d-\delta' < x < d+\delta'$  within which  $f(d) < f(x)$  ( $x \neq d$ ) ;

i.e., if  $f(d+h) - f(d) > 0$ , for  $|h|$  sufficiently small.



Thus, in the above figure which represents graphically the function  $f(x)$  (a continuous function here), the function



has a maximum value at  $P_1$ , as also at  $P_2, P_3, P_4$ , etc. and has minimum values at  $Q_1, Q_2, Q_3, Q_4$ , etc. At  $P_1$ , for instance, corresponding to  $x = OC_1$  ( $= c_1$  say), the value of the function, namely, the ordinate  $P_1C_1$  is not necessarily bigger than the value  $Q_2D_2$  at  $x = OD_2$ , but we can get a range, say  $L_1C_1L_2$  in the neighbourhood of  $C_1$  on either side of it, (i.e., we can find a  $\delta = L_1C_1 = C_1L_2$  say) such that for every value of  $x$  within  $L_1C_1L_2$  (except at  $C_1$ ), the value of the function (represented by the corresponding ordinate) is less than  $P_1C_1$  (the value at  $C_1$ ). Hence, by definition, the function is maximum at  $x = OC_1$ . Similarly, we can find out an interval  $M_1D_2M_2$  ( $M_1D_2 = D_2M_2 = \delta'$  say) in the neighbourhood of  $D_2$  within which for every other value of  $x$  the function is greater than that at  $D_2$ . Hence, the function at  $D_2$  (represented by  $Q_2D_2$ ) is a minimum.

From the figure the following features regarding maxima and minima of a continuous function will be apparent :

(i) that the function may have several maxima and minima in an interval ; (ii) that a maximum value of the function at some point may be less than a minimum value of it at another point ( $C_1P_1 < D_2Q_2$ ) ; (iii) maximum and minimum values of the function occur alternately, i.e., between any two consecutive maximum values there is a minimum value, and *vice versa*.

## 7.2. A necessary condition for maximum and minimum.

If  $f(x)$  be a maximum, or a minimum at  $x = c$ , and if  $f'(c)$  exists, then  $f'(c) = 0$ .

By definition,  $f(x)$  is a maximum at  $x = c$ , provided we can find a positive number  $\delta$ , such that

$$f(c+h) - f(c) < 0 \text{ whenever } -\delta < h < \delta, (h \neq 0).$$

$$\therefore \frac{f(c+h) - f(c)}{h} < 0 \text{ if } h \text{ be positive and sufficiently}$$

small, and  $> 0$  if  $h$  be negative and numerically sufficiently small.

Thus,  $\lim_{h \rightarrow 0+} \frac{f(c+h) - f(c)}{h} < 0$ , [See Ex. 6, § 2'11]

and similarly,  $\lim_{h \rightarrow 0-} \frac{f(c+h) - f(c)}{h} > 0$ .

Now, if  $f'(c)$  exists, the above two limits, which represent the right-hand and left-hand derivatives respectively of  $f(x)$  at  $x=c$ , must be equal. Hence, the only common value of the limit is zero. Thus,  $f'(c) = 0$ .

Exactly similar is the proof when  $f(c)$  is a minimum.

Note. In case  $f'(c)$  does not exist,  $f(c)$  may be a maximum or a minimum, as is apparent from the figure for points  $Q_1$  and  $P_1$ . At the former point  $f(x)$  is a minimum, and at the latter it is a maximum.  $f'(x)$  is however not zero at these points, for  $f'(x)$  does not exist at all at these points.

### 7.3. Determination of Maxima and Minima.

(A) If  $c$  be a point in the interval in which the function  $f(x)$  is defined, and if  $f'(c) = 0$ , and  $f''(c) \neq 0$ , then  $f(c)$  is (i) a maximum if  $f''(c)$  is negative and (ii) a minimum if  $f''(c)$  is positive.

Proof: Suppose  $f'(c) = 0$ , and  $f''(c)$  exists, and  $\neq 0$ .

By the Mean Value Theorem\*,

$$\begin{aligned} f(c+h) - f(c) &= hf'(c+\theta h), \quad 0 < \theta < 1, \\ &= \theta h^2 \frac{f'(c+\theta h) - f'(c)}{\theta h}. \end{aligned}$$

Since  $0 < \theta < 1$ ,  $\theta h \rightarrow 0$  as  $h \rightarrow 0$ , and writing  $\theta h = k$ , the coefficient of  $\theta h^2$  on the right side  $\rightarrow \lim_{k \rightarrow 0} \frac{f'(c+k) - f'(c)}{k} = f''(c)$ . Accordingly, since  $\theta h^2$  is positive,  $f(c+h) - f(c)$  has the same sign as that of  $f''(c)$  when  $|h|$  is sufficiently small.

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\* Since  $f''(c)$  exists,  $f'(x)$  also exists in the neighbourhood of  $c$ .



$\therefore$  if  $f''(c)$  is positive,  $f(c+h)-f(c)$  is positive, whether  $h$  is positive or negative, provided  $|h|$  is small. Hence  $f(c)$  is a minimum, by definition.

Similarly, if  $f''(c)$  is negative,  $f(c+h)-f(c)$  is negative, whether  $h$  is positive or negative, when  $|h|$  is small, and so  $f(c)$  is a maximum.

(B) Let  $c$  be an interior point of the interval of definition of the function  $f(x)$ , and let

$$f'(c) = f''(c) = \dots = f^{n-1}(c) = 0, \text{ and } f^n(c) \neq 0;$$

then (i) if  $n$  be even,  $f(c)$  is a maximum or a minimum according as  $f^n(c)$  is negative or positive,

and (ii) if  $n$  be odd,  $f(c)$  is neither a minimum, nor a maximum.

*Proof:* By the Mean Value Theorem of Higher order, here

$$\begin{aligned} f(c+h) - f(c) &= \frac{h^{n-1}}{(n-1)!} f^{n-1}(c+\theta h), \quad 0 < \theta < 1 \\ &= \frac{\theta h^n}{(n-1)!} \frac{f^{n-1}(c+\theta h) - f^{n-1}(c)}{\theta h}. \end{aligned}$$

Since  $0 < \theta < 1$ , as  $h \rightarrow 0$ ,  $\theta h \rightarrow 0$  and the coefficient of  $\theta h^n/(n-1)!$ , on the right side  $\rightarrow f^n(c)$ .

Now, suppose  $n$  is even; then,  $\theta h^n/(n-1)!$  is positive.

$\therefore f(c+h)-f(c)$  has the same sign as of  $f^n(c)$ , whether  $h$  is positive or negative, provided  $|h|$  is sufficiently small. Hence, if  $f^n(c)$  be positive,  $f(c+h)-f(c)$  is positive for either sign of  $h$ , when  $|h|$  is small, and so  $f(c)$  is a minimum. Similarly, if  $f^n(c)$  is negative,  $f(c)$  is a maximum.

Next suppose  $n$  is odd; then  $\theta h^n/(n-1)!$  is positive or negative according as  $h$  is positive or negative. Hence,  $f(c+h)-f(c)$  changes in sign with the change of  $h$  whatever the sign of  $f^n(c)$  may be, and so  $f(c)$  cannot be either a maximum or a minimum at  $x=c$ .

Hence to determine maxima and minima of  $f(x)$ , we proceed with the following working rule:

## 7.5. Illustrative Examples.

Ex. 1. Find for what values of  $x$ , the following expression is maximum and minimum respectively :

$$2x^3 - 21x^2 + 36x - 20.$$

Find also the maximum and minimum values of the expression.

[ C. P. 1936 ]

Let  $f(x) = 2x^3 - 21x^2 + 36x - 20.$

$\therefore f(x) = 6x^2 - 42x + 36$ , which exists for all values of  $x$ .

Now, when  $f(x)$  is a maximum or a minimum,  $f'(x) = 0$ .

$\therefore$  we should have  $6x^2 - 42x + 36 = 0$ , i.e.,  $x^2 - 7x + 6 = 0$ ,

or,  $(x-1)(x-6) = 0$ ;  $\therefore x = 1$  or  $6$ .

Again,  $f''(x) = 12x - 42 = 6(2x - 7).$

Now, when  $x = 1$ ,  $f'(x) = -30$  which is negative,

when  $x = 6$ ,  $f'(x) = 30$ , which is positive.

Hence, the given expression is maximum for  $x = 1$ , and minimum for  $x = 6$ .

The maximum and minimum values of the given expression are respectively  $f(1)$ , i.e.,  $-3$ , and  $f(6)$ , i.e.,  $-128$ .

Ex. 2. Investigate for what values of  $x$ ,

$$f(x) = 5x^5 - 18x^3 + 15x^2 - 10$$

is a maximum or minimum.

Here,  $f'(x) = 30(x^4 - 3x^2 + 2x^2).$

Putting  $f'(x) = 0$ , we have  $x^4(x^2 - 3x + 2) = 0$ .

i.e.,  $x^4(x-1)(x-2) = 0$ , whence,  $x = 0, 1$  or  $2$ .

Again,  $f''(x) = 30(5x^4 - 12x^2 + 6x^2).$

When  $x = 1$ ,  $f''(x)$  is negative, and hence  $f(x)$  is a maximum for  $x = 1$ .

When  $x = 2$ ,  $f''(x)$  is positive, and hence  $f(x)$  is a minimum for  $x = 2$ .

When  $x = 0$ ,  $f''(x) = 0$ ; so the test fails, and we have to examine higher order derivatives.

$$f'''(x) = 120(5x^3 - 9x^2 + 3x).$$

$$f^{(4)}(x) = 360(5x^2 - 6x + 1).$$

$$\therefore f'''(0) = 0.$$

$$\therefore f^{(4)}(0) \text{ is positive.}$$

Since even order derivative is positive for  $x=0$ ,

$\therefore$  for  $x=0$ ,  $f(x)$  is a minimum.

Ex. 3. Show that  $f(x) = x^3 - 6x^2 + 24x + 4$  has neither a maximum nor a minimum.

Here,  $f'(x) = 3(x^2 - 4x + 8) = 3\{(x-2)^2 + 4\}$

which is always positive and can never be zero.

$\therefore f(x)$  has neither a maximum nor a minimum.

Ex. 4. Examine  $f(x) = x^3 - 9x^2 + 24x - 12$  for maximum or minimum values.

Here,  $f'(x) = 3(x^2 - 6x + 8) = 3(x-2)(x-4)$ .

Putting  $f'(x) = 0$ , we find  $x = 2$  or  $4$ .

Now,  $f'(2-h) = 3(-h)(-2-h) = +$ ,

and  $f'(2+h) = 3(h)(h-2) = -$ , since,  $h$  is positive and small.

$\therefore$  by § 7.3, Note 1, for  $x=2$ ,  $f(x)$  has a maximum value, and this is  $f(2) = 8$ .

Again,  $f'(4-h) = 3(2-h)(-h) = -$ , since  $h$  is positive and small,  
 $f'(4+h) = 3(2+h)(h) = +$ .

$\therefore$  by § 7.3, Note 1, for  $x=4$ ,  $f(x)$  has a minimum value, and this is  $f(4) = 4$ .

Note. In this case we could have easily applied rule of Art. 7.3.

Ex. 5. Find the maxima and minima of

$$1 + 2 \sin x + 3 \cos^2 x. \quad (0 \leq x \leq \frac{1}{2}\pi).$$

Let  $f(x) = 1 + 2 \sin x + 3 \cos^2 x$ .

Then  $f'(x) = 2 \cos x - 6 \cos x \sin x$ .

$\therefore f'(x) = 0$  when  $2 \cos x(1 - 3 \sin x) = 0$ , i.e., when  $\cos x = 0$ , and also when  $\sin x = \frac{1}{3}$ .

$$f''(x) = -2 \sin x - 6(\cos^2 x - \sin^2 x).$$

When  $\cos x = 0$ ,  $x = \frac{1}{2}\pi$ .  $\therefore \sin x = 1$ .  $\therefore f''(x) = -2 + 6 = 4$  (positive).

$\therefore$  for  $\cos x = 0$ ,  $f(x)$  is a minimum, and the minimum value is 3.

When  $\sin x = \frac{1}{3}$ ,

$$f''(x) = -2 \sin x - 6(1 - 2 \sin^2 x) = -\frac{2}{3} - 6(1 - \frac{1}{9}) \text{ (negative).}$$



Therefore, for  $\sin x = \frac{1}{2}$ ,  $f(x)$  is a maximum and the maximum value is  $1 + 2 \cdot \frac{1}{2} + 3 \cdot (1 - \frac{1}{2}) = 4\frac{1}{2}$ .

Ex. 6. Examine whether  $x^{\frac{1}{2}}$  possesses a maximum or a minimum, and determine the same. [C. P. 1941, '43]

Let  $y = x^{\frac{1}{2}}$ .  $\therefore \log y = \frac{1}{2} \log x$ .

$$\therefore \frac{1}{y} \frac{dy}{dx} = \frac{1}{x^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}} \log x = \frac{1}{x^{\frac{1}{2}}} (1 - \log x). \quad \dots (1)$$

$\therefore$  when  $\frac{dy}{dx} = 0$ ,  $1 - \log x = 0$ .  $\therefore \log x = 1 = \log e$ .  $\therefore x = e$ .

Again, differentiating (1) with respect to  $x$ ,

$$-\frac{1}{y^2} \left( \frac{dy}{dx} \right)^2 + \frac{1}{y} \frac{d^2y}{dx^2} = x^{\frac{1}{2}} \cdot (-1/x) - (1 - \log x) \frac{2x}{x^{\frac{3}{2}}} = \frac{-3 + 2 \log x}{x^{\frac{3}{2}}}.$$

$\therefore$  when  $x = e$ ,  $\frac{d^2y}{dx^2} = e^{\frac{1}{2}} \cdot \frac{-3 + 2}{e^{\frac{3}{2}}} = -\frac{e^{\frac{1}{2}}}{e^{\frac{3}{2}}}$ , which is negative.

$$\left( \therefore \text{for } x = e, \frac{dy}{dx} = 0. \right)$$

$\therefore$  for  $x = e$ , the function is a maximum, and the maximum value is  $e^{\frac{1}{2}}$ .

Ex. 7. Find the maximum and minimum values of  $u$  where

$$u = \frac{4}{x} + \frac{36}{y} \text{ and } x + y = 2.$$

Eliminating  $y$  between the two given relations

$$u = \frac{4}{x} + \frac{36}{2-x} \quad \therefore \frac{du}{dx} = -\frac{4}{x^2} + \frac{36}{(2-x)^2} = \frac{16(2x^2 + x - 1)}{x^2(2-x)^2}.$$

$$\frac{du}{dx} = 0, \text{ gives } x = \frac{1}{2} \text{ or } -1. \quad \text{Also, } \frac{d^2u}{dx^2} = \frac{8}{x^3} + \frac{72}{(2-x)^3}.$$

When  $x = \frac{1}{2}$ ,  $\frac{d^2u}{dx^2} = \frac{8}{(\frac{1}{2})^3} + \frac{72}{(\frac{3}{2})^3}$ , which is positive.

$\therefore$  for  $x = \frac{1}{2}$ ,  $u$  is a minimum.

$$\therefore \text{minimum value of } u = \frac{4}{\frac{1}{2}} + \frac{36}{2 - \frac{1}{2}} = 32.$$

When  $x = -1$ ,  $\frac{d^2u}{dx^2} = -8 + \frac{72}{27}$ , which is negative.

$\therefore$  for  $x = -1$ ,  $u$  is a maximum.

$$\therefore \text{maximum value of } u = \frac{4}{-1} + \frac{36}{2+1} = 8.$$

## Lecture 10

Define homogeneous function

### Euler's theorem ( FROM BOOK)

If  $f(x, y)$  be a homogeneous function of  $x$  and  $y$  of degree  $n$ , then  $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf(x, y)$

Proof: Since  $f(x, y)$  is a homogeneous function of  $x$  and  $y$  of degree  $n$

Let

$$f(x, y) = x^n \phi\left(\frac{y}{x}\right)$$

$$= x^n \phi(v)$$

$$\frac{\partial f}{\partial x} = nx^{n-1} \phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right) \frac{-y}{x^2} \dots\dots\dots i$$

$$\frac{\partial f}{\partial y} = x^n \phi'\left(\frac{y}{x}\right) \frac{1}{x} \dots\dots\dots ii$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nx^n \phi\left(\frac{y}{x}\right) = nf(x, y)$$

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Ex: 1

## Lecture 11,12

### Tangent and Normal

- i) The tangent to the curve  $y = f(x)$  at  $(x, y)$  (not parallel to  $y$  axis) is

$$Y - y = \frac{dy}{dx}(X - x)$$

- ii) When the equation of the curve is  $f(x, y) = 0$ , since  $\frac{dy}{dx} = -\frac{f_x}{f_y}$ ,  $f_y \neq 0$

The equation of the tangent to the curve at  $(x, y)$  is

$$(X - x)f_x + (Y - y)f_y = 0$$

- iii) When the equation of the curve is  $x = \phi(t)$ ,  $y = \varphi(t)$ , since  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{\varphi'(t)}{\phi'(t)}$ ,  $\phi'(t) \neq 0$

The equation of the tangent to the curve at the point ' $t$ ' is

$$Y - \varphi(t) = \frac{\varphi'(t)}{\phi'(t)}(X - \phi(t))$$



**Application of Differentiation**  
**Calculus: Equations of Tangent and normal to a C...**  
**Tangents and Normals to Curves**

Watch Later

(1) Find the gradient of the curve  $y = 2 - 3x - 2x^2$  at the point T(x, y) where  $x = -3$ . Hence, find the equation of the tangent to the curve at the point T.

$$\frac{dy}{dx} = -3 - 4x \quad \left| \quad m = \frac{dy}{dx} \right|_{x=-3} = -3 - 4(-3) = 9$$

$$y = 2 - 3(-3) - 2(-3)^2 = 2 + 9 - 18 = -7$$

Equation of tangent at  $(-3, -7)$

$$m = \frac{y - y_1}{x - x_1}$$

$$9 = \frac{y - (-7)}{x - (-3)}$$

$$y + 7 = 9x + 27$$

### Lecture 13

#### Tangent and Normal

- i) The Normal to the curve  $y = f(x)$  at  $(x, y)$  (not parallel to the co-ordinate axis) is

$$\frac{dy}{dx}(Y - y) + (X - x) = 0$$

- ii) When the equation of the curve is  $f(x, y) = 0$

The equation of the Normal to the curve at  $(x, y)$  is

$$\frac{X - x}{f_x} = \frac{Y - y}{f_y}$$

iii) When the equation of the curve is  $x = \phi(t)$ ,  $y = \varphi(t)$ , since  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\varphi'(t)}{\phi'(t)}$ ,  $\phi'(t) \neq 0$

The equation of the Normal to the curve at the point ' $t$ ' is

$$Y - \varphi(t) = -\frac{\phi'(t)}{\varphi'(t)}(X - \phi(t))$$

Find the equation of the tangent at any point  $(x, y)$  to the curve  $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$

The equation of the curve  $f(x, y) = \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} - 1 = 0$

The equation of the tangent is  $(X - x)f_x + (Y - y)f_y = 0$

$$(X - x)\frac{2}{3}x^{-\frac{1}{3}}/a^{\frac{2}{3}} + (Y - y)\frac{2}{3}y^{-\frac{1}{3}}/b^{\frac{2}{3}} = 0$$

$$Xx^{-\frac{1}{3}}/a^{\frac{2}{3}} + Yy^{-\frac{1}{3}}/b^{\frac{2}{3}} = (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}}$$

$$Xx^{-\frac{1}{3}}/a^{\frac{2}{3}} + Yy^{-\frac{1}{3}}/b^{\frac{2}{3}} = 1$$

**Find angle between two curves  $f(x, y) = 0$  and  $g(x, y) = 0$**