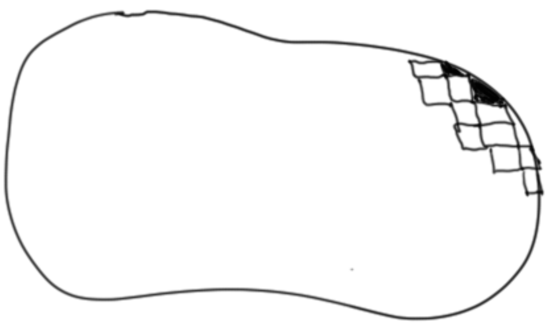
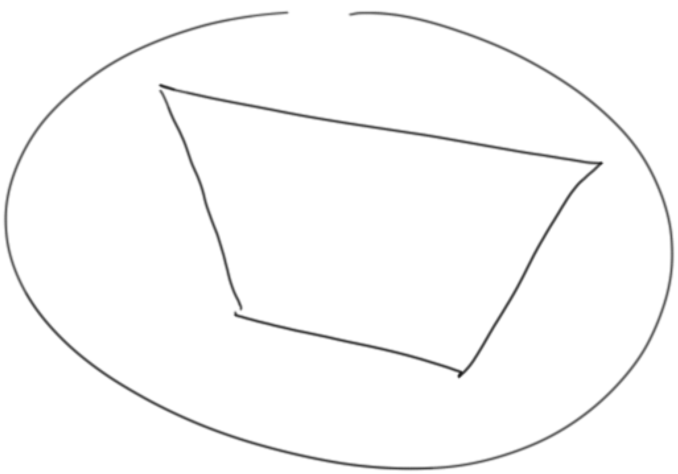
and

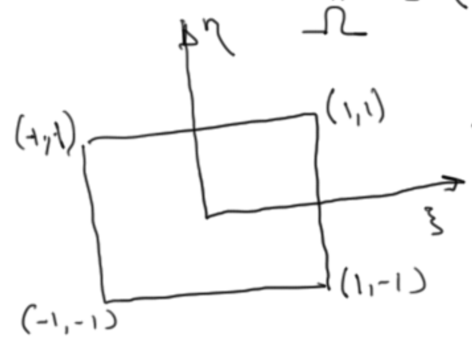
$$K^e = \int_{\Omega^e} B^T \cdot C \cdot B \, d\Omega$$

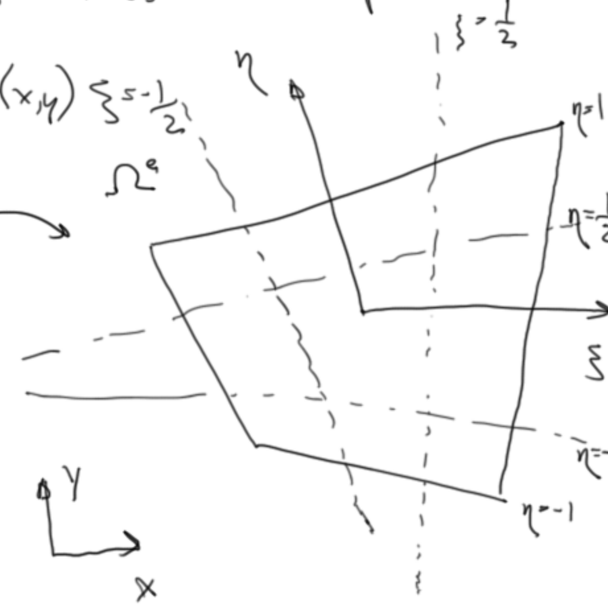




start by defining a "master" or "parent"

$(\xi, \eta) \to (x, y)$





$$\hat{N}_1 = \frac{1}{4}(1-\xi)(1-\eta) \qquad \hat{N}_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$\hat{N}_2 = \frac{1}{4}(1+\xi)(1-\eta) \qquad \hat{N}_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$x = \underline{x_j \hat{N}_j(\xi,\eta)}, \quad y = y_j \hat{N}_j(\xi,\eta)$$

$$K_{ij} = \int_{\Omega} a(x,y) \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + b(x,y)\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + c(x,y) N_i N_j \,]\, dx\,dy$$

$$B^T C B$$

$$C = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$N_i = \hat{N}_i \qquad \frac{\partial N_i}{\partial x} \rightarrow \frac{\partial \hat{N}_i}{\partial \xi} \qquad \frac{\partial N_i}{\partial y} \rightarrow \frac{\partial \hat{N}_i}{\partial \eta}$$

$$\frac{\partial \hat{N}_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \hat{N}_i}{\partial \eta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta}$$

or

$$\begin{Bmatrix} \frac{\partial \hat{N}_i}{\partial \xi} \\ \frac{\partial \hat{N}_i}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$= [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}$$

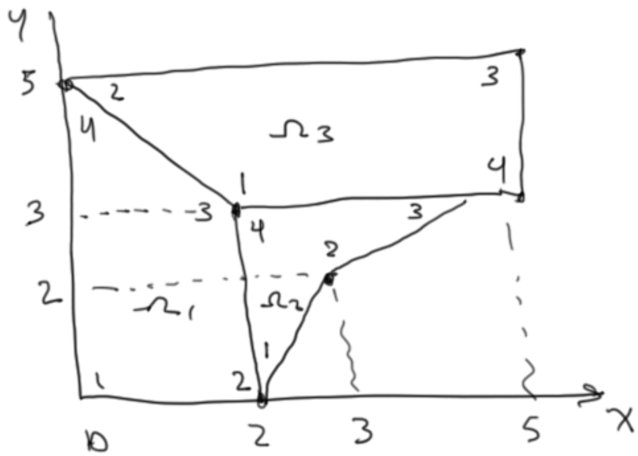
$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \hat{N}_i}{\partial \xi} \\ \dfrac{\partial \hat{N}_i}{\partial \eta} \end{Bmatrix}$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}(x_j \hat{N}_j) = x_j \frac{\partial \hat{N}_j}{\partial \xi} \qquad \frac{\partial y}{\partial \xi} = y_j \frac{\partial \hat{N}_j}{\partial \xi}$$

$$\frac{\partial x}{\partial \eta} = x_j \frac{\partial \hat{N}_j}{\partial \eta} \qquad \frac{\partial y}{\partial \eta} = y_j \frac{\partial \hat{N}_j}{\partial \eta}$$

$$[J] = \begin{bmatrix} x_i \dfrac{\partial \hat{N}}{\partial \xi} & y_i \dfrac{\partial \hat{N}_i}{\partial \xi} \\ x_i \dfrac{\partial \hat{N}_i}{\partial \eta} & y_i \dfrac{\partial \hat{N}_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \hat{N}_1}{\partial \xi} & \dfrac{\partial \hat{N}_2}{\partial \xi} & \dfrac{\partial \hat{N}_3}{\partial \xi} & \cdots & \dfrac{\partial \hat{N}_n}{\partial \xi} \\ \dfrac{\partial \hat{N}_1}{\partial \eta} & \dfrac{\partial \hat{N}_2}{\partial \eta} & & \cdots & \dfrac{\partial \hat{N}_n}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \\ x_n & y_n \end{bmatrix}$$

$$J = \det([J]) = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} > 0$$



For Ω_1

$$[J] = \begin{bmatrix} 1 & -\frac{1}{2}(1+\eta) \\ 0 & 2 - \frac{1}{2}\xi \end{bmatrix} \Rightarrow \det([J]) = \frac{1}{2}(4-\xi) > 0$$

$$-1 < \xi < 1$$ always invertable

For Ω_2

$$[J] = \begin{bmatrix} 1 + \frac{1}{2}\eta & \frac{1}{2}(1-\eta) \\ \frac{1}{2}(1+\xi) & 1 - \frac{1}{2}\xi \end{bmatrix} \Rightarrow \det([J]) = \frac{3}{4}(1+\eta-\xi)$$

not invertable

$$\xi = 1+\eta \Rightarrow J = 0$$

For Ω_3

$$[J] = \begin{bmatrix} -\frac{1}{2}(1-\eta) & 1 \\ 2+\frac{1}{2}\xi & 0 \end{bmatrix} \Rightarrow \det([J]) = -\left(2+\frac{1}{2}\xi\right) < 0$$

invertable

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial \hat{N}_i}{\partial \xi} \\ \frac{\partial \hat{N}_i}{\partial \eta} \end{Bmatrix} = [J]^{*} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

$$\hat{a} = a(\xi,\eta) \qquad \hat{b} = b(\xi,\eta) \qquad \hat{c} = c(\xi,\eta)$$

$$K_{ij} = \int_{\Omega} \Big[\hat{a}\left(J^{*}_{11}\frac{\partial \hat{N}_i}{\partial \xi} + J^{*}_{11}\frac{\partial \hat{N}_i}{\partial \eta}\right)\left(J^{*}_{11}\frac{\partial \hat{N}_j}{\partial \xi} + J^{*}_{12}\frac{\partial \hat{N}_j}{\partial \eta}\right)$$
$$+ \hat{b}\left(J^{*}_{21}\frac{\partial \hat{N}_i}{\partial \xi} + J^{*}_{22}\frac{\partial \hat{N}_i}{\partial \eta}\right)\left(J^{*}_{21}\frac{\partial \hat{N}_j}{\partial \xi} + J^{*}_{22}\frac{\partial \hat{N}_j}{\partial \eta}\right)$$
$$+ \hat{c}\,\hat{N}_i \hat{N}_j \Big] \det([J])\, d\xi\, d\eta \equiv \int_{\Omega} \overbrace{F(\xi,\eta)}^{\text{use Gauss integration}}\, d\xi\, d\eta$$

Gauss Integration

$$\int_{-1}^{1} f(t)\,dt = \omega_1 f(t_1) + \omega_2 f(t_2) \qquad (A)$$

Let $f(t) = t^3$

$$\int_{-1}^{1} t^3\,dt = \left[\frac{1}{4} t^4\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0 = \omega_1 t_1^3 + \omega_2 t_2^3 \qquad (1)$$

Let $f(t) = t^2$

$$\int_{-1}^{1} t^3\,dt = \left[\frac{1}{3} t^3\right]_{-1}^{1} = \frac{1}{3} - \frac{1}{3} = 0 = \omega_1 t_1^2 + \omega_2 t_2^2 \qquad (2)$$

Let $f(t) = t$

$$\int_{-1}^{1} t\,dt = \left[\frac{1}{2} t^2\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0 = \omega_1 t_1 + \omega_1 t_2 \qquad (3)$$

Let $f(t) = 1$

$$\int_{-1}^{1} 1\,dt = [t]_{-1}^{1} = 2 = \omega_1 + \omega_2 \qquad (4)$$

Solve (1) - (4) for $\omega_1, \omega_2, t_1, t_2$

$w_1 = w_2 = 1$

$t_1 = -t_2 = \sqrt{\frac{1}{3}}$

$$\int_{-1}^{1} f(t)\,dt = w_1 f(t_1) + w_2 f(t_2)$$

$$= f\left(-\sqrt{\tfrac{1}{3}}\right) + f\left(\sqrt{\tfrac{1}{3}}\right)$$

Extend

$$\int_{-1}^{1} f(t)\,dt = \sum_{i=1}^{n} w_i f(t_i) \quad \text{for } n \text{ points}$$

Exact for $f(t)$ that are polynomials of degree $2n-1$ or less

Points	Value of t	Weights (w_i)	Valid up to degree
1	0	2	1
2	$-0.5773 = -\sqrt{\frac{1}{3}}$	1	3
	$0.5773 = \sqrt{\frac{1}{3}}$	1	
3	-0.77459	0.5555	5
	0.0	0.8888	
	0.77459	0.5555	

Check the web for larger tables

$$f(t) = 100t^5 - 43t^4 + 75t^3 - t^2 + 5t + 10$$

$$\int_{-1}^{1} f(t)$$

t_i	$f(t_i)$	w_i	$w_i f(t_i)$
-0.5773	-18.8466	1	-18.8466
0.5773	28.6244	1	28.6244
			9.7777

t_i	$f(t_i)$	w_i	$w_i f(t_i)$
-0.77459	-72.6954	0.5555	-40.3864
0	10	0.8888	8.888
0.77458	60.5359	0.5555	33.6308
			2.1333

Exact is 2.1333

For area integrals

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt = \int_{-1}^{1} \underbrace{\sum_{j=1}^{n} w_j f(s,t_j)}\,ds = \sum_{i=1}^{m} w_i \left(\sum_{j=1}^{n} w_j f(s_i,t_j) \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} w_i w_j f(s_i, t_j)$$

Example

$f(t) = 100 s^3 t^2 + 5st + 5$

2×2 quadrature

s_i	t_j	$f(s_i, t_j)$	$w_i w_j$	$w_i w_j f(s_i, t_j)$
$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}$	0.251664	1	~~0.251664~~
$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	-3.08167	1	-3.08167
$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}$	13.0817	1	13.0817
$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	-0.251664	1	~~-0.251664~~
			Exact	10