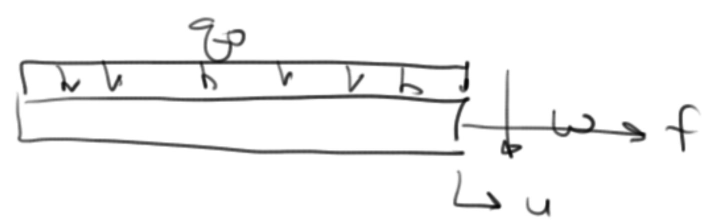
Nonlinear Problems

- Development of FE is exactly same
- Solution techniques

Large deflection E-B beam

$$-\frac{d}{dx}\left\{ EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\} - f = 0$$

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right) - \frac{d}{dx}\left\{EA\,\frac{dw}{dx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\} - q = 0$$



$$\frac{dw}{dx}\frac{du}{dx} \approx 0$$

$$\left(\frac{dw}{dx}\right)^2 \approx 0$$

$x_a, \; x_b$

$$0 = \int_{x_a}^{x_b} \left\{ EA \frac{d\delta u}{dx} \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right] - \delta u f \right\} dx$$
$$- Q_1 \delta u(x_a) - Q_4 \delta u(x_b)$$

$$0 = \int_{x_a}^{x_b} \left\{ EI \frac{d^2\delta w}{dx^2} \frac{d^2 w}{dx^2} + EA \frac{d\delta w}{dx} \frac{dw}{dx} \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right] - \right.$$
$$\left. \delta w \, q \right\} dx - Q_2 \, \delta w(x_a) - Q_3 \left(-\frac{d\delta w}{dx} \right)\Big|_{x_a} - Q_5 \, \delta w(x_b)$$
$$- Q_6 \left(-\frac{d\delta w}{dx} \right)\Big|_{x_b}$$

$$Q_1 = - \left\{ EA \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right] \right\}\Big|_{x_a}$$

$$Q_2 = \left\{ \frac{d}{dx}\left(EI \frac{d^2 w}{dx^2} \right) - EI \frac{dw}{dx} \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right] \right\}\Big|_{x_a}$$

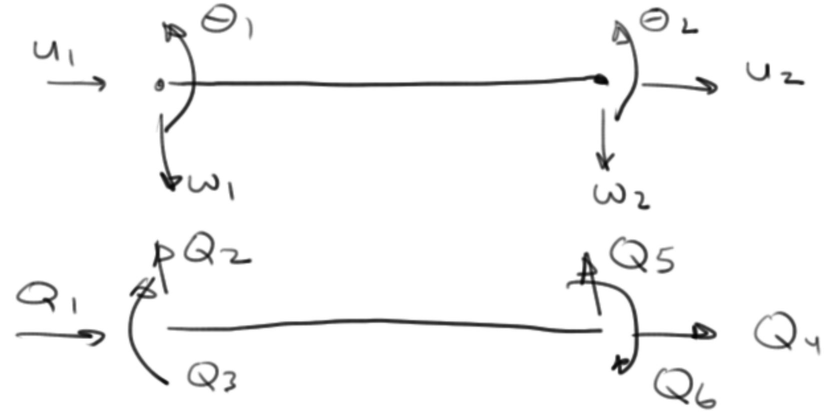
$$Q_3 = \left(EI \frac{d^2 w}{dx^2} \right)\Big|_{x_a} \qquad Q_4 = \left\{ EA \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right] \right\}\Big|_{x_b}$$

$$Q_5 = -\left\{ \frac{d}{dx}\left(EI \frac{d^2 w}{dx^2} \right) - EA \frac{dw}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx} \right)^2 \right] \right\}\Bigg|_{x_b}$$

$$Q_6 = -\left\{ EI \frac{d^2 w}{dx^2} \right\}\Bigg|_{x_b}$$

$$u = \Sigma u_j N_j \qquad \delta u = N_i$$

$$w = w_j M_j \qquad \delta w = M_i$$



$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{Bmatrix} \vec{u} \\ \vec{w} \end{Bmatrix} = \begin{Bmatrix} F^1 \\ F^2 \end{Bmatrix}$$

$$K^{11}_{ij} = \int_{x_a}^{x_b} EA \frac{dN_i}{dx} \frac{dN_j}{dx}\, dx \qquad K^{12}_{ij} = \int_{x_b}^{x_b} \frac{1}{2} EA \frac{dw}{dx} \frac{dN_i}{dx} \frac{dM_j}{dx}\, dx$$

$$K^{22}_{ij} = \int_{x_a}^{x_b} EI \frac{d^2 M_i}{dx^2} \frac{d^2 M_j}{dx^2}\, dx + \int_{x_a}^{x_b} \frac{1}{2} EA \left(\frac{dw}{dx}\right)^2 \frac{dM_i}{dx} \frac{dM_j}{dx}\, dx$$

$$K_{ij}^{21} = \int_{x_a}^{x_b} EA \frac{dw}{dx} \frac{\partial N_i}{\partial x} \frac{\partial m_j}{\partial x} \, dx$$

$$F_i^1 = \int_{x_a}^{x_b} N_i f \, dx + Q_{3i-2}$$

$$F_i^2 = \int_{x_a}^{x_b} M_i \, q \, dx + Q_{i+I} \quad \begin{Bmatrix} I = 1 & \forall & i = 1,2 \\ I = 2 & \forall & i = 3,4 \end{Bmatrix}$$

$$K(U)\, U = F$$

Use iteration

$$K(U^n)\, U^{n+1} = F$$

$$U^{n+1} = \left[K(U^n)\right]^{-1} F$$

$$\frac{|U^{n+1} - U^n|}{|U^{n+1}|} < \epsilon \quad \text{say } 10^{-6}$$

Newton-Raphson

$$R \equiv K(U)\,U - F = 0$$

$$R = R^n + \left(\frac{\partial R}{\partial U}\right)_n \left(U^{n+1} - U^n\right) + \frac{1}{2!}\left(\frac{\partial^2 R}{\partial U^2}\right)_n \left(U^{n+1} - U^n\right) + \dots$$

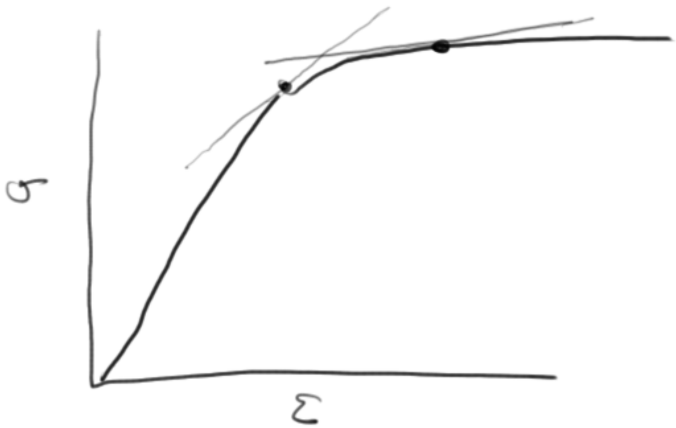
$$0 \approx R^n + K_T^n \,\Delta U + \mathcal{O}(\Delta U^2)$$

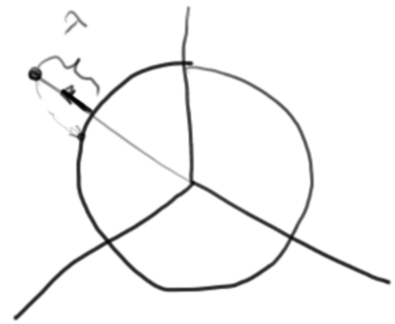
K_T is the tangent stiffness matrix

$K_T = \frac{\partial R}{\partial U}$ evalutated at $U = U^n$

$$\Delta U = -(K_T)^{-1} R^n = \left(K_T(U^n)\right)^{-1}\left(F - K(U^n)\,U^n\right)$$

$$U^{n+1} = U^n + \Delta U$$





From 9/25/2014

$$\dot{e}_{ij} Q_{ij} + \frac{\dot{S}}{2\mu} - \dot{\lambda} = 0 \qquad S = |S_{ij}| = \sqrt{S_{ij}S_{ij}} \qquad Q_{ij} = \frac{S_{ij}}{|S_{ij}|}$$

$$\frac{\Delta e_{ij}}{\cancel{\Delta t}} Q_{ij} + \frac{\Delta S}{2\mu \cancel{\Delta t}} - \frac{\Delta \lambda}{\cancel{\Delta t}} = 0 \qquad \dot{e}_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3}\dot{\varepsilon}_{kk}\delta_{ij}$$

$$\Delta e_{ij} Q_{ij} + \frac{\Delta S}{2\mu} - \Delta\lambda = 0 \Rightarrow \Delta e_{ij} Q_{ij} + \frac{S_{n+1} - S_n}{2\mu} - \Delta\lambda = 0$$

$S_n \rightarrow$ known

$S_{n+1} \rightarrow \sqrt{\frac{2}{3}}\, Y$

$$\sigma = \begin{bmatrix} Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \therefore \underline{\underline{S}} = \begin{bmatrix} \frac{2}{3}Y & & \\ & -\frac{1}{3}Y & \\ & & -\frac{1}{3}Y \end{bmatrix}$$

$$Y = Y(\varepsilon^p, \dot{\varepsilon}^p) = Y(\varepsilon^p, \sqrt{\tfrac{2}{3}}\,\dot{\lambda})$$

$$Y_{n+1} = Y\left(\varepsilon^p_n, \sqrt{\tfrac{2}{3}}\,\frac{\Delta\lambda}{\Delta t}\right) \quad (1)$$

$$\varepsilon^p_{n+1} = \varepsilon^p_n + \dot{\varepsilon}^p \Delta t$$

$$\varepsilon^p_n + \sqrt{\tfrac{2}{3}}\Delta\lambda$$

$$= Y\left(\varepsilon^p_{n+1}, \sqrt{\tfrac{2}{3}}\,\frac{\Delta\lambda}{\Delta t}\right)$$

$$= Y\left(\varepsilon^p_n + \sqrt{\tfrac{2}{3}}\Delta\lambda, \sqrt{\tfrac{2}{3}}\Delta\lambda/\Delta t\right) \quad (2)$$

$$\Delta e_{ij} Q_{ij} - \Delta\lambda - \frac{1}{2\mu}\left[\sqrt{\tfrac{2}{3}}\, Y\left(\varepsilon^p_n, \sqrt{\tfrac{2}{3}}\,\frac{\Delta\lambda}{\Delta t}\right) - S_n\right] = 0$$

$$\overset{\circ}{\sigma} = C\,\dot{\varepsilon} \qquad \Delta\varepsilon$$

$$\underline{\sigma_{n+1}} - \underline{\sigma_n} = C(\varepsilon_{n+1} - \varepsilon_n)$$

$$\underline{\underline{S}}_{n+1} = \begin{cases} \underline{\underline{S}}^{tr} & f^{tr} < 0 \\ \sqrt{\tfrac{2}{3}}\, Y_{n+1} Q^{tr} & f^{n+1} = 0 \end{cases}$$

$$\sigma_{n+1} = \underline{\underline{S}}_{n+1} + p_{n+1} I, \quad p_{n+1} = p_n + K\dot{\varepsilon}_{kk}\Delta t$$