$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0 \qquad \text{for} \quad a<x<b$$

Subject to von Neumann B.C.'s

$$\left(a\frac{\partial u}{\partial x}\right)_{x=a}=Q_a \qquad \& \qquad \left(a\frac{\partial u}{\partial x}\right)\Big|_{x=b}=Q_b$$

<u>Weak form</u>

$$\int_a^b\left[a\frac{d\delta u}{dx}\frac{du}{dx}+c\,\delta u\,u-\delta u\,f\right]dx-\delta u(a)Q_a-\delta u(b)Q_b$$

$$B(\delta u,u)=\int_a^b\left[a\frac{d\delta u}{dx}\frac{du}{dx}+c\,\delta u\,u\right]dx$$

$$l(\delta u)=\int_a^b\delta u\,f\,dx+\delta u(a)Q_a+\delta u(b)Q_b$$

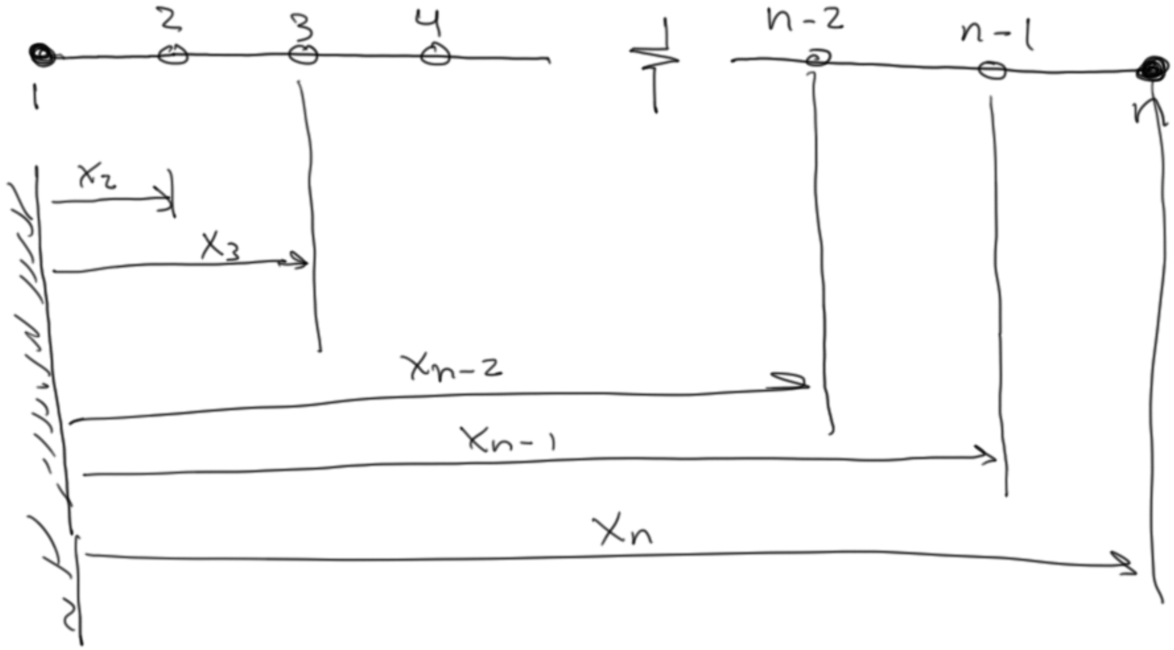
$$B(\delta u,u)=l(\delta u)$$

$u\approx u^h=\sum_{j=1}^{n}N_j u_j$ $\quad$ N_j of degree $n-1$ & u_j are unknowns

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - f = 0 \qquad 0 < x < L$$

$$u(0) = u_0 \quad + \quad \left(a\frac{du}{dx}\right)_{x=L} = Q_0$$

	u	a	c	f	Q
Heat Transfer	Temp, T	Thermal conductivity kA	conv.	heat gen.	Heat, Q
Flow	p	resistance	0	0	point source
Elasticity	Disp	stiffness AE	0	Axial dist. force	Point load



$$(x_1, x_2), \; (x_2, x_3), \; \ldots, \; (x_{n-1}, x_n)$$

$$0 = \sum_{i=1}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} \left[a \frac{d\delta u}{dx} \frac{du}{dx} + c\, \delta u\, u - \delta u\, f \right] dx - \left[\delta u(x)\, a \frac{du}{dx} \right]_{x_i}^{x_{i+1}} \right\}$$

$$= \int_{x_1}^{x_n} \left[a \frac{d\delta u}{dx} \frac{du}{dx} + c\, \delta u\, u - \delta u\, f \right] dx - \delta u(x_1) \left[-a \frac{du}{dx} \right]_{x_1} - \delta u(x_2) \left[a \frac{du}{dx} \right]_{x_2}$$

$$- \delta u(x_2) \left[-a \frac{du}{dx} \right]_{x_2} - \delta u(x_3) \left[a \frac{du}{dx} \right]_{x_3} - \cdots - \delta u(x_{n-1}) \left[a \frac{du}{dx} \right]_{x_{n-1}}$$

$$- \delta u(x_n) \left[a \frac{du}{dx} \right]_{x_n}$$

$$0 = \int_{x_1}^{x_n} \left[a \frac{d\delta u}{dx} \frac{du}{dx} + c\, \delta u\, u - \delta u\, f \right] dx - \delta u(x_1) Q_1 - \delta u(x_2) Q_2$$

$$- \cdots - \delta u(x_{n-1}) Q_{n-1} - \delta u(x_n) Q_n$$

where

$$\left.\begin{aligned} Q_1 &= \left[-a \frac{du}{dx} \right]_{x_1} \\ Q_2 &= \left[\left(a \frac{du}{dx} \right)_{x_2^-} - \left(a \frac{du}{dx} \right)_{x_2^+} \right] \\ &\vdots \\ Q_{n-1} &= \left[\left(a \frac{du}{dx} \right)_{x_{n-1}^-} - \left(a \frac{du}{dx} \right)_{x_{n-1}^+} \right] \end{aligned}\right\} = 0$$

$$Q_n = \left(a \frac{du}{dx} \right)_n$$

$$0 = \underbrace{\int_{x_a}^{x_b} \left(a \frac{d\delta u}{dx} \frac{du}{dx} + c\, \delta u\, u \right) dx}_{B(\delta u, u)} - \underbrace{\int_{x_a}^{x_b} \delta u\, f\, dx - \sum_{j=1}^{n} \delta u(x_j) Q_j}_{l(\delta u)}$$

$$=$$

Let $u = N_j u_j$ & $\delta u_i = N_i$

For $i=1$

$$0 = \int_{x_a}^{x_b} \left[a \frac{dN_1}{dx} \left(\frac{d}{dx}(N_j u_j) \right) + c N_1 \left(N_j u_j \right) \right] dx - \int_{x_a}^{x_b} N_1 f \, dx - \sum_{j=1}^{n} N_1(x_j) Q_j$$

$\vdots$

For $i=n$

$$0 = \int_{x_a}^{x_b} \left[a \frac{\partial N_n}{dx} \frac{dN_j}{dx} u_j + c N_n N_j u_j \right] dx - \int_{x_a}^{x_b} N_n f \, dx - \sum_{j=1}^{n} N_n(x_j) Q_j$$

$\therefore$ $i^{\underline{th}}$ equation we have

$$0 = \underbrace{\left[\int_{x_a}^{x_b} a \frac{dN_i}{dx} \frac{dN_j}{dx} + c N_i N_j \, dx \right]}_{} u_j - \underbrace{\int_{x_a}^{x_b} N_i f \, dx}_{f_i} - Q_i$$

$$N_i(x_j) = \begin{cases} 1 & i=1 \\ 0 & \text{otherwise} \end{cases}$$

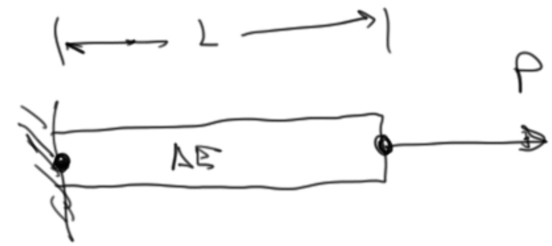
$$\underbrace{B(N_i, N_j)} u_j \underbrace{- f_i - Q_i}_{F_i} = 0$$

$$K_{ij} u_j = F_i \qquad K \vec{u} = \vec{F}$$

$$K\vec{u} = \vec{F}$$

coefficient or "stiffness matrix" or "transmissibility"

$$\vec{u} = K^{-1}F$$



$u(0) = 0$

<u>Example</u>

Let $(x_a, x_b) = (0, L)$

$c = 0$

$a = EA$ find K_{ij}

$$K_{ij} = \int_0^L \left(EA \frac{dN_i}{dx} \frac{dN_j}{dx} \right) dx = \begin{bmatrix} \cancelto{1}{\frac{AE}{L}} & -\cancelto{0}{\frac{AE}{L}} \\ -\cancelto{0}{\frac{AE}{L}} & \frac{AE}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

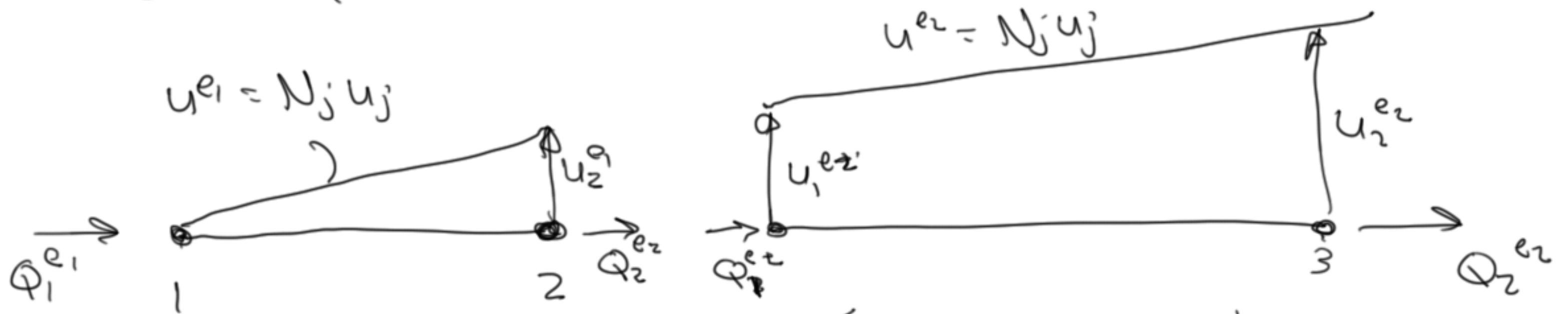
$u_1 = 0$

$$\frac{AE}{L} u_2 = P \Rightarrow u_2 = \frac{PL}{AE}$$

2n unknowns

$$(u_1, u_2, u_3 \dots, u_n) + (Q_1, Q_2, \dots, Q_n)$$

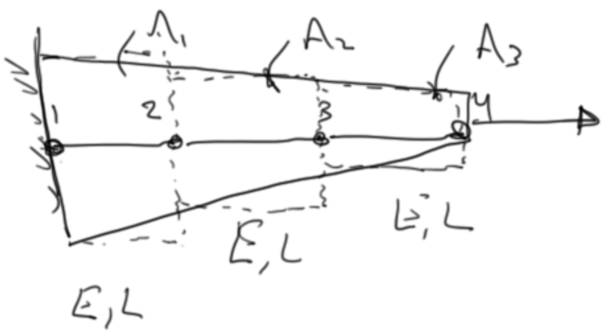
Continuity in u_i's + "balance" Q_i's



$$Q_2^{e_1} + Q_1^{e_2} = \begin{cases} 0 & \text{if no external source} \\ Q_I & \text{if external source of mag. } Q_I \end{cases}$$

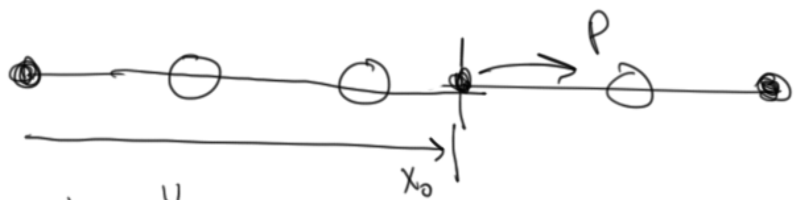
$$K^{e_1} = \begin{bmatrix} K_{11}^1 & K_{12}^1 \\ K_{21}^1 & K_{22}^1 \end{bmatrix} + K^{e_2} = \begin{bmatrix} K_{11}^2 & K_{12}^2 \\ K_{12}^2 & K_{22} \end{bmatrix}$$

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 \\ K_{21}^1 & \boxed{K_{22}^1 + K_{11}^2} & K_{12}^2 \\ 0 & K_{21}^2 & K_{22}^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q^2 \end{Bmatrix}$$



$$
\begin{bmatrix}
\overset{1}{\cancel{\dfrac{A_1 E}{L}}} & \overset{0}{\cancel{-\dfrac{A_1 E}{L}}} & 0 & 0 \\[2ex]
\overset{0}{\cancel{-\dfrac{A_1 E}{L}}} & \dfrac{(A_1 + A_2)E}{L} & -\dfrac{A_2 E}{L} & 0 \\[2ex]
0 & -\dfrac{A_2 E}{L} & \dfrac{(A_2 + A_3)E}{L} & -\dfrac{A_3 E}{L} \\[2ex]
0 & 0 & -\dfrac{A_3 E}{L} & \dfrac{A_3 E}{L}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ P \end{Bmatrix}
$$

What happens if Q_j is not at a node

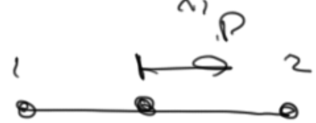


"function"

$$f(x) = P\,\Delta(x - x_0)$$

$$\int_{-\infty}^{\infty} F(x)\,\Delta(x - x_0)\,dx = F(x_0)$$

$$f_i^e = \int_{x_1}^{x_n} f(x) N_i(x)\,dx = \int_{x_1}^{x_n} P\,\Delta(x - x_0) N_i(x)\,dx = P N_i(x_0)$$



$x_0 = L/2$

$$f_1 = P\left(1 - \frac{x_0}{L}\right), \quad f_2 = P\left(\frac{x_0}{L}\right) \Rightarrow f_1 + f_2 = P$$

$$f_1 = f_2 = \frac{P}{2}$$

$$P_1 = A\sigma_1 = AE\varepsilon_1 = AE\left[\frac{u_1 - u_2}{L}\right] = \frac{AE}{L}u_1 - \frac{AE}{L}u_2$$

$$P_2 = A\sigma_2 = AE\varepsilon_2 = AE\left[\frac{u_2 - u_1}{L}\right] = -\frac{AE}{L}u_1 + \frac{AE}{L}u_2$$

$$\begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$