Integration-by-parts

$$\int_a^b w \frac{dv}{dx}\,dx = \int_a^b w\,dv = -\int_a^b v\,dw + \Big[wv\Big]_a^b$$

$$= -\int_a^b v \frac{dw}{dx}\,dx + w(b)v(b) - w(a)v(a)$$

We can establish this by:

$$\frac{d}{dx}(wv) = \frac{dw}{dx}v + w\frac{dv}{dx} \Rightarrow \boxed{w\frac{dv}{dx} = \frac{d}{dx}(wv) - \frac{dw}{dx}v}$$

$$\int_a^b w\frac{dw}{dx}\,dx = \int_a^b \left[\frac{d}{dx}(wv) - \frac{dw}{dx}v\right]dx = \int_a^b \frac{d}{dx}(wv)\,dx - \int_a^b \frac{dw}{dx}v\,dw$$

$$= \Big[wv\Big]_a^b - \int_a^b \frac{dw}{dx}v\,dx$$

Ritz or Rayliegh-Ritz Method

utilized the "weak-form" of diff. eqn.

Step 1 Same as ω-r

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x) \quad \text{for} \quad 0 < x < L$$

subject to

$$u(0) = u_0 \qquad \left(a\frac{du}{dx}\right)_{x=L} = Q_L$$

$$\int_0^L \left[-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] - f(x)\right]\delta u \, dx = 0$$

Step 2 Intergrate-by-Parts

$$0 = \int_0^L \left\{ a(x)\frac{d(\delta u)}{dx}\frac{du}{dx} - \delta u\, f(x)\right\}dx - \left[\delta u \; a(x)\frac{du}{dx}\right]_0^L$$

Step 3 Impose B.C.s

$$0 = \int_0^L \left\{ a(x) \frac{d}{dx}(\delta u) \frac{du}{dx} - \delta u\, f(x) \right\} dx - \left[\delta u\, a(x) \frac{du}{dx} \right]_{x=L}$$

$$a(x) \frac{du}{dx}\bigg|_{x=L} = Q_L$$

$$0 = \int_0^L \left\{ a(x) \frac{d}{dx}(\delta u) \frac{du}{dx} - \delta u\, f(x) \right\} dx - \delta u(L)\, Q_L$$

Weak form $\rightarrow$ varational form

Now consider

$$\int_a^b w \frac{d^3u}{dx^2}\, dx = \int_a^b w \frac{d}{dx}\left(\frac{du}{dx}\right) dx = \int_a^b w \frac{dv}{dx}\, dx$$

where $v \equiv \frac{du}{dx}$

$$\int_a^b w \frac{d^2u}{dx^2}\, dx = -\int_a^b v \frac{dw}{dx}\, dx + w(b)v(b) - w(a)v(a)$$

$$= -\int_a^b \frac{du}{dx}\frac{dw}{dx}\, dx + w(b)\frac{du}{dx}\bigg|_b - w(a)\frac{\partial u}{\partial x}\bigg|_a$$

$$\underline{-\int_a^b \frac{dw}{dx}\frac{du}{dx}\, dx} = \int_a^b w \frac{d^2u}{dx^2}\, dx + w(a)\frac{du}{dx}\bigg|_a - w(b)\frac{du}{dx}\bigg|_b$$

$$\boxed{\Pi(u) = \int_0^L \overbrace{\frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2}^{F} dx - P\,u(L)}$$

minimize, subject $\boxed{u(0) = 0}$

$$\delta\Pi(u) = \delta\left[\int_0^L \underbrace{\frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2} dx - P\,u(L)\right] = \boxed{\int_0^L EA\,\frac{du}{dx}\,\frac{d(\delta u)}{dx}\,dx - P\,\delta u(L)}$$

$$\underbrace{\frac{EA}{2}\,\frac{du}{dx}\,\frac{du}{dx}}$$

$$\frac{1}{2}\,B(u,u) = B(u,v)$$

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right) = 0$$

$$B(u,v) = EA\,\frac{du}{dx}\,\frac{dv}{dx}$$

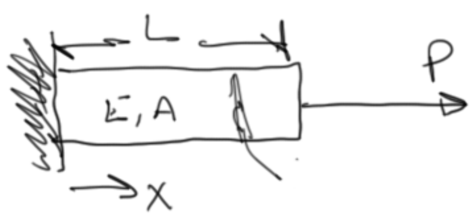
$$B(v,u) = EA\,\frac{dv}{dx}\,\frac{du}{dx}$$

$$\delta\Pi(u) = \int_0^L \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right)\right]\delta u\,dx + \delta u\left[EA\frac{du}{dx}\right]_0^L - P\,\delta u(L)$$

$$\boxed{= \int_0^L \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right)\right]\delta u\,dx + \delta u(L)\left[EA\frac{du}{dx} - P\right]_{x=L} = 0}$$

$$-\left[EA\frac{du}{dx}\right]_{x=0} \cancel{\delta u(0)}^{\,0} = 0$$

$$\boxed{\begin{aligned} &-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0 \\ &EA\frac{du}{dx}\Big|_{x=L} = P \quad \text{natural} \\ &u(0) = 0 \quad \text{essential} \end{aligned}}$$



natural → Neumann   essential b.cs → Dirchelet

Natural + Essential B.C.'s

$$I(u) = \int_a^b F(x, u(x), u'(x))\,dx - Q_a u(a) - Q_b u(b)$$

The necessary condition for I to attain a stationary value yields

$$0 = \int_a^b \delta u \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right] dx + \left(\frac{\partial F}{\partial u'}\,\delta u\right)_a^b - Q_a\,\delta u(a) - Q_b\,\delta u(b)$$

$$= \int_a^b \delta u \underbrace{\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]}_{\text{satisfied by D.E.}} dx + \underbrace{\left(\left.\frac{\partial F}{\partial u'}\right|_b - Q_b\right)\delta u(b) - \left[\left.\frac{\partial F}{\partial u'}\right|_a + Q_a\right]\delta u(a)}_{\text{must} = 0}$$

Any of following

$\delta u(a) \Rightarrow u(a) = 0$ same for b

1.) $\delta u(a) = 0$, $\delta u(b) = 0$

2.) $\delta u(a) = 0$, $\left.\frac{\partial F}{\partial u'}\right|_b - Q_b = 0$

3.) $-\left.\frac{\partial F}{\partial u'}\right|_a - Q = 0$, $\delta u(b) = 0$

4.) $-\left.\frac{\partial F}{\partial u'}\right|_a - Q_a = 0$

$\left.\frac{\partial F}{\partial u'}\right|_b - Q_b = 0$

Variational Formulations

Classically "variational formulations" refer to constucting a functional or a variational principle that is equivalent to the governing equation.

The modern use refers to a formulation where the governing eqns. are translated into an equivalent weighted-integral statement.

Weighted Integral Statement

$$u \approx u^h = \sum_{j=1}^{n} u_j \phi_j + \sum_{j=1}^{m} c_j \phi_j$$

$u_j \Rightarrow$ "nodes", but we have no "nodes"

$$u \approx u^h = \sum_{j=1}^{m} c_j \phi_j + \phi_0$$

$\nwarrow$ sole purpose is to satisfy the B.C.s

Consider the ODE

$$-\frac{d}{dx}\left[a(x)\,\frac{du}{dx}\right] + c(x)\,u = f(x) \qquad 0 < x < L$$

$$u(0) = u_0 \qquad \left[a\,\frac{\partial u}{\partial x}\right]_{x=L} = Q_0$$

Lets choose

$$\phi_1 = x^2 - 2x \qquad \text{Let } L=1,\ u_0 = 1,\ Q_0 = 0$$

$$\phi_2 = x^3 - 3x \qquad a(x) = x\,,\ c(x) = 1,\ f(x) = 0$$

$$\phi_0 = 1$$

$$-\frac{d}{dx}\left[x\,\frac{du}{dx}\right] + u = 0 \implies -\cancelto{1}{\frac{dx}{dx}}\,\frac{du}{dx} + x\,\frac{d^2u}{dx^2} + u = 0$$

$$u(0) = 1 \quad \& \quad x\,\frac{\partial u}{dx}\bigg|_{x=L} = 0$$

$$u = u^h = c_1\left(x^2 - 2x\right) + c_2\left(x^3 - 3x\right) + 1$$

$$\frac{dy}{dx} = c_1(2x-2) + c_2(3x^2-3)$$

$$\frac{d^2y}{dx} = c_1(2) + c_2(6x)$$

$$-c_1(2x-2) - c_2(3x^2-3) - 2xc_1 - 6c_2x^2 + c_1(x^2-2x) + c_2(x^3-3x) + 1$$

$$= 0$$

$$\left.\begin{aligned}
x^3&: \quad c_2 = 0 \\
x^2&: \quad -3c_2 - 6c_2 + c_1 = -9c_2 + c_1 = 0 \\
x^1&: \quad -2c_1 - 2c_1 - 2c_1 - 3c_2 = -6c_1 - 3c_2 = 0 \\
x^0&: \quad 2c_1 + 3c_2 + 1 = 0
\end{aligned}\right\} \text{ Doesn't work}$$

Go back

$$\delta u\left[-\frac{d}{dx}\left[x\frac{du}{dx}\right]+u\right]=\left[0\right]\delta u$$

$$\int_0^L \underbrace{\delta u}_{w}\underbrace{\left[-\frac{d}{dx}\left[x\frac{du}{dx}\right]+u\right]}_{R}dx=0 \qquad \int_0^L w\,R\,dx=0$$

$$R=c_2x^3+(c_1-9c_2)x^2+(-6c_1-3c_2)x+2c_1+3c_2+1$$

choose $\delta u_1=1$, $\delta u_2=x$

$$\left.\begin{aligned}0&=\int_0^1 1\cdot R\,dx=(1+2c_1+3c_2)+\frac{1}{2}(-6c_1-3c_2)+\frac{1}{3}(c_1-9c_2)+\frac{1}{4}c_2\\0&=\int_0^1 x\cdot R\,dx=(1+2c_1+3c_2)+\frac{1}{3}(-6c_1-3c_2)+\frac{1}{4}(c_1-9c_2)+\frac{1}{5}c_2\end{aligned}\right\}\text{Solve}$$

$$\boxed{c_1=\frac{222}{23} \qquad c_2=-\frac{100}{23}}$$

Depending on the choice of δu_i we arrive at the different weighted residual methods. If we shoose δu_i

$$\begin{cases} \delta u_i = \varphi_i & \Rightarrow \text{ Galerkin Method} \\ \delta u_i \neq \varphi_i & \Rightarrow \text{ Petrov-Galerkin Method} \\ \delta u_i = \dfrac{d}{dx}\left(a(x)\dfrac{d\varphi_i}{\partial x}\right) & \Rightarrow \text{ Least-squares method} \\ \delta u_i = \Delta(x - x_i) & \Rightarrow \text{ Collocation method} \end{cases}$$

where Δ is Dirac Delta function

$$\Delta = 0 \quad x \neq x_i$$
$$\Delta = 1 \quad x = x_i$$

Only L-S method results in symm. coeff. matrix

$$\begin{bmatrix} 7/3 & -5/4 \\ -3/4 & -31/20 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \leftarrow \text{not symm.}$$