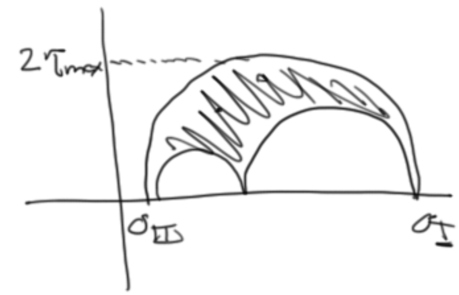
$$f(J_2) = \sqrt{3J_2} - Y = 0$$

Other yield surfaces

Maximum shear stress (Tresca criterion)



$$f = \frac{1}{2}\max\{|\sigma_I - \sigma_{II}|, |\sigma_{II} - \sigma_{III}|, |\sigma_I - \sigma_{III}|\}$$

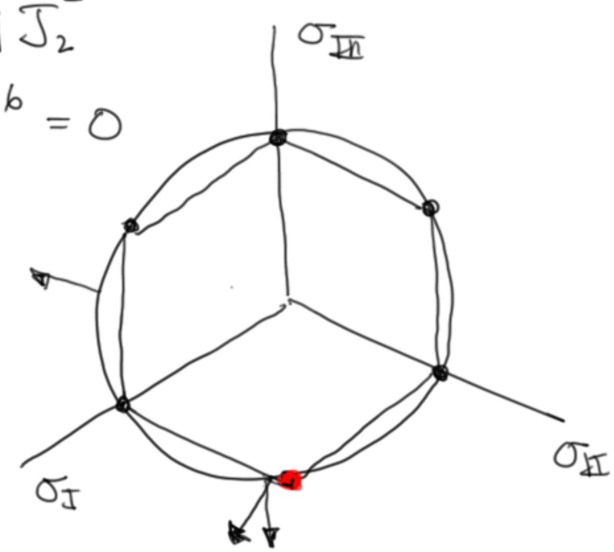
$$= 4J_2^3 - 27J_3^2 - 9J_2^2 + Y^2 - 6J_2Y^4 - Y^6 = 0$$

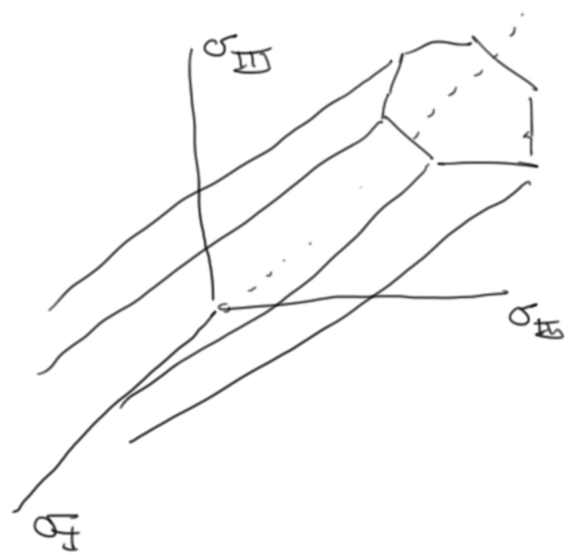
$$J_3 = \det(\sigma) = \sigma_I \sigma_{II} \sigma_{III}$$

$$S_{ij} = \begin{bmatrix} -\frac{2}{3}Y & 0 & 0 \\ 0 & +\frac{1}{3}Y & 0 \\ 0 & 0 & +\frac{1}{3}Y \end{bmatrix}$$

$J_3 = \frac{2}{27}Y^3$ tension

$J_3 = -\frac{2}{27}Y^3$ compression





$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda} S_{ij}$$

Drucker - Prager

$$f(p, J_2) = \sqrt{3 J_2} - \beta p - Y = 0$$

pressure dep. in plastic flow

$$p = -\frac{1}{3}\sigma_{kk}$$

$$\frac{\partial f}{\partial \sigma_{ij}}$$

$$\beta \frac{\partial p}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}}\left\{\frac{1}{-3}\sigma_{kk}\right\} = -\frac{1}{3}\delta_{ik}\delta_{jk} = -\frac{1}{3}\delta_{ij}$$

Mohr - Coulomb

$$\sigma_I - \sigma_{III} + (\sigma_I + \sigma_{III})\sin(\phi) = Y\cos(\phi)$$

angle of repose
angle of internal friction

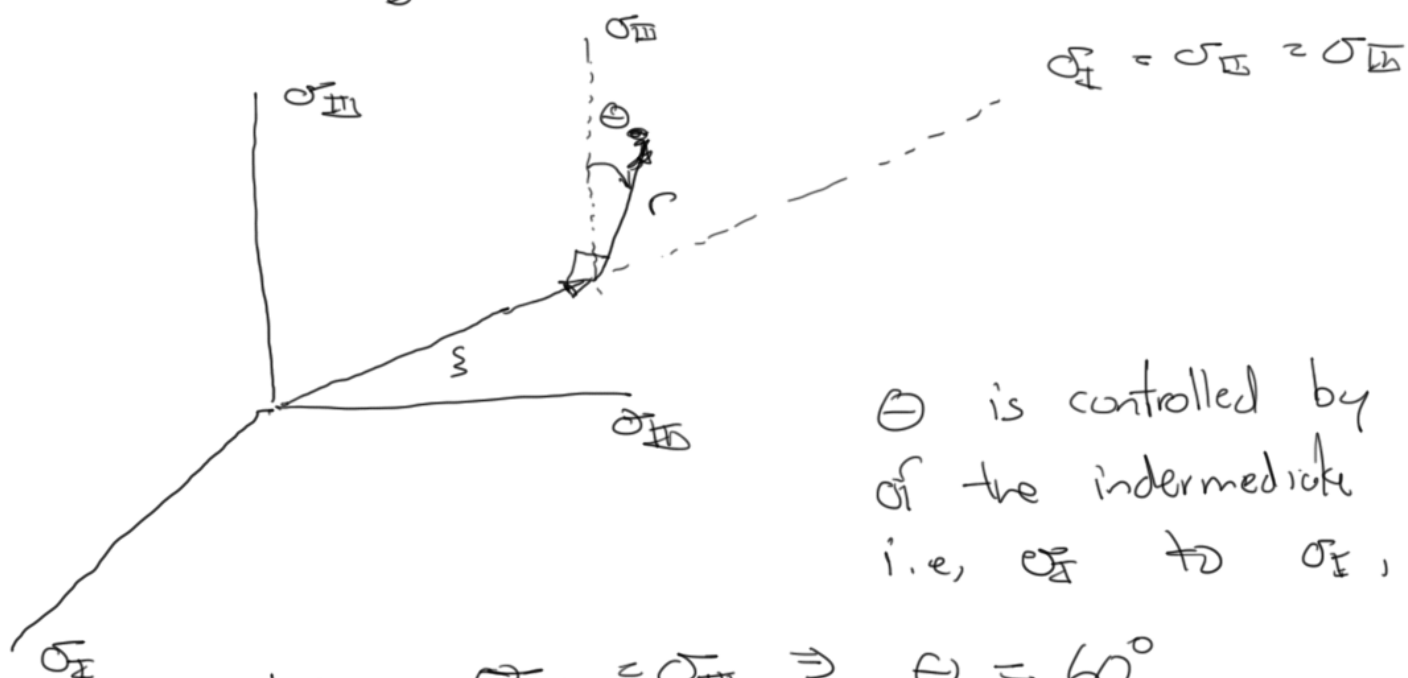
In terms of invariats

$$\frac{1}{3} I_1 \sin\phi + \sqrt{J_2}\left\{\cos\theta - \frac{1}{\sqrt{3}}\sin\theta \sin\phi\right\} = \frac{Y}{2}\cos\phi$$

Lode angle

$$\Theta = \frac{1}{3} \sin^{-1}\left(\frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}} \right)$$

$$-\frac{\pi}{6} \leq \Theta \leq \frac{\pi}{6}$$



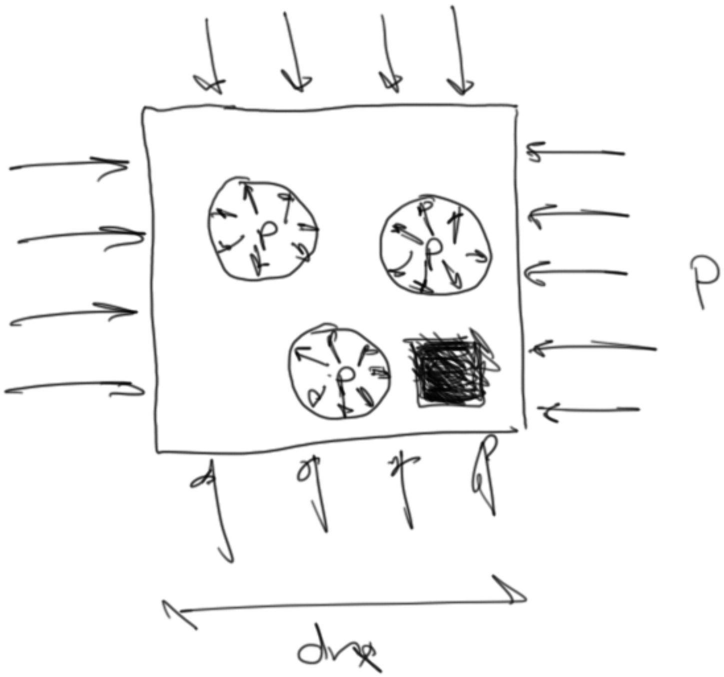
Θ is controlled by the relationship of the indermediate principple stress, i.e, σ_{II} to σ_I, σ_{III}

When $\sigma_{II} = \sigma_{III} \Rightarrow \Theta = 60^\circ$

$\sigma_{II} = \sigma_I \Rightarrow \Theta = 0^\circ$

$$\sigma_{ij} = S_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} = S_{ij} - p\,\delta_{ij}$$

$$p = -\frac{1}{3}\sigma_{kk}$$



$$\underline{\sigma_{ij} = -p\,\delta_{ij}}$$

$$\sigma_{ij} = \lambda\,\varepsilon_{kk}\delta_{ij} + 2\mu\,\varepsilon_{ij}$$

$$\lambda = \left(K - \frac{2}{3}\mu\right)$$

$$\sigma_{ij} = \left(K - \frac{2}{3}\mu\right)\varepsilon_{kk}\delta_{ij} + 2\mu\,\varepsilon_{ij}$$

$$= K\,\varepsilon_{kk}\delta_{ij} - 2\mu\left(\frac{1}{3}\varepsilon_{kk}\delta_{ij}\right) + 2\mu\,\varepsilon_{ij}$$

$$= K\,\varepsilon_{kk}\delta_{ij} + 2\mu\left(\underbrace{\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}}_{\cancelto{0}{e_{ij}}}\right)$$

$$\sigma_{ij} = K\,\varepsilon_{kk}\delta_{ij}$$

$$-p\,\cancel{\delta_{ij}} = K\,\varepsilon_{kk}\cancel{\delta_{ij}} \Rightarrow \varepsilon_{kk} = -\frac{p}{K_s}$$

$$\varepsilon_{ij} = \cancel{e_{ij}}^{0} + \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

$$= \frac{-p}{3K_s}\delta_{ij} \leftarrow$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \Longrightarrow \varepsilon_{kl} = D_{klij}\,\sigma_{ij}$$

where $D = C^{-1}$ or $C_{ijkl}\,D_{mnop} = \underbrace{\delta_{im}\,\delta_{jn}\,\delta_{ko}}\,\delta_{lp}$

4^{th} identity tensor

so for arbitrary σ + p

$$\varepsilon_{kl} = \underbrace{D_{klij}\left(\sigma_{ij} + p\,\delta_{ij}\right)} - \frac{1}{3K_s}\,p\,\delta_{kl}$$

effective stress

$$\sigma' = \sigma_{ij} + \alpha p\,\delta_{ij} = C_{ijkl}\,\varepsilon_{kl} = \underbrace{C_{ijkl}\left[D_{klij}\right.}_{\delta_{ik}\delta_{jl}\delta_{ki}\delta_{lj}\sigma_{ij}}\left.\left(\sigma_{ij} + p\,\delta_{ij}\right) - \frac{1}{3K_s}\,p\,\delta_{kl}\right]$$

$$\cancel{\sigma_{ij}} + \alpha \cancel{p}\,\delta_{ij}\cancel{\delta_{ij}}^{3} = \cancel{\sigma_{ij}} + \cancel{p}\,\delta_{ij}\cancel{\delta_{ij}}^{3} - \frac{C_{ijkl}}{3K_s}\,p\,\delta_{kl}\delta_{ij}$$

$$\frac{\cancel{3}^{1}}{\cancel{3}}\alpha = \frac{\cancel{3}^{1}}{\cancel{3}} + \frac{\delta_{ij}\,C_{ijkl}\,\delta_{kl}}{3\cdot\cancel{3}\,K_s}$$

$$\alpha = 1 - \frac{\delta_{ij} C_{ijkl} \delta_{kl}}{9 K_s}$$

$$\frac{\delta_{ij} C_{ijkl} \delta_{kl}}{9} = \frac{9\lambda + 6\mu}{9} = K_T$$

$$\alpha = 1 - \frac{K_T}{K_s}$$

Biot's coef.

$$K_s \gg K_T \qquad \alpha \to 1$$

rocks & concrete $\quad \alpha \approx \frac{2}{3}$

$$\sigma' = \sigma_{ij} + \alpha p \, \delta_{ij}$$