$$E = \frac{1}{2}\left(F^T F - I\right) \qquad \text{recall} \qquad F = I + \nabla u \qquad \|\nabla u\| << I$$

$$= \frac{1}{2}\left(\nabla_X \vec{u} + (\nabla_X \vec{u})^T + (\nabla_X \vec{u})^T (\nabla_X u)\right)$$

$$\to E = \boxed{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_i} + \frac{\partial u_j}{\partial X_j}\right) + \cancel{\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}}} \longrightarrow \boxed{\varepsilon = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right)}$$

LINEAR STRAIN → CAUCHY

$$(ds)^2 - (dS)^2 = d\vec{x}^T d\vec{x} - d\vec{X}^T d\vec{X} = d\vec{x}^T d\vec{x} - (F^{-1} d\vec{x})^T (F^{-1} d\vec{x})$$

$$F^{-1} d\vec{x} = \cancel{F^{-1} F}^{\,I}\, d\vec{X}$$

$$= d\vec{x}^T d\vec{x} - d\vec{x}^T F^{-T} F^{-1} d\vec{x}$$

$$F^{-1} d\vec{x} = d\vec{X}$$

$$= d\vec{x}^T \underbrace{\left(I - F^{-T} F^{-1}\right)}_{2e} d\vec{x}$$

$$F = I + \nabla u$$

$$\to e = \boxed{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \cancel{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}}$$

$$e = \frac{1}{2}\left(I - F^{-T} F^{-1}\right)$$

Eulerian strain → Almansi strain

$$\vec{x} \approx \vec{X} + \cancel{\vec{u}}^{\,0}$$

Seth-Hill Strain

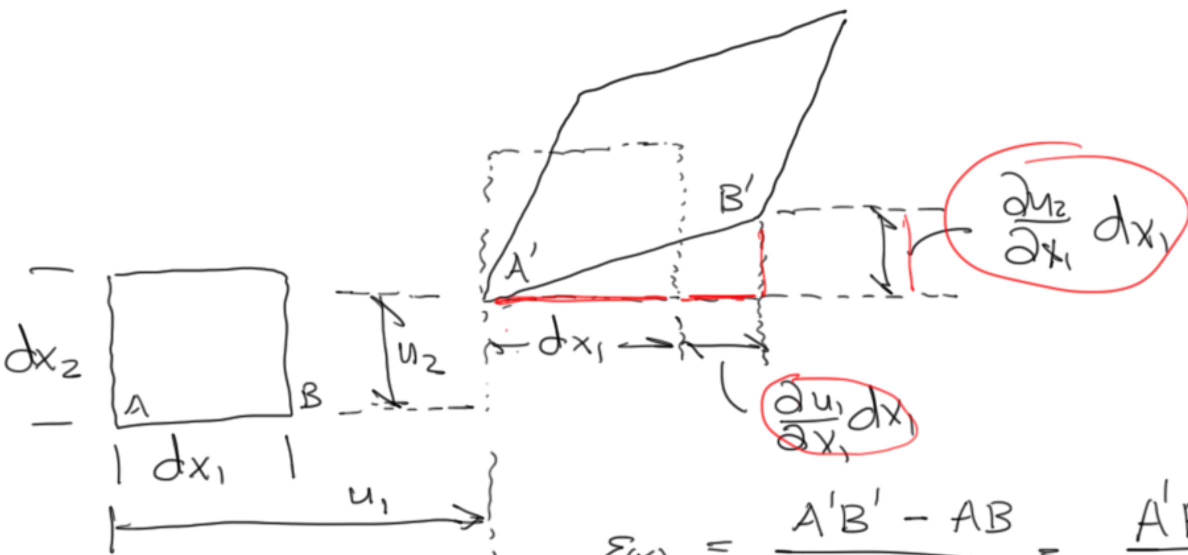
$$E_{(m)} = \frac{1}{2m}\left(C^m - I\right) = \frac{1}{2m}\left(\left(F^T F\right)^m - I\right)$$

$m = 1 \rightarrow$ Green Langrange Strain

$m = -1 \rightarrow$ Eulerian strain

$$E_{(m)} = \underline{\underline{\varepsilon}} + \frac{1}{2}\nabla\vec{u}^T \nabla\vec{u} - (1-m)\,\varepsilon^T\cdot\varepsilon \qquad \varepsilon = \frac{1}{2}\left(\nabla\vec{u} + \nabla\vec{u}^T\right)$$

Geometric interpretation of small strain



$$\varepsilon_{(x_1)} = \frac{A'B' - AB}{AB} = \frac{A'B' - dx}{dx} = \frac{A'B'}{dx} - 1$$

$$(A'B')^2 = \left[\left(dx + \frac{\partial u_1}{\partial x_1} dx_1\right)^2 + \left(\frac{\partial u_2}{\partial x_1} dx_1\right)^2 \right]$$

$$\varepsilon_{(x_1)} = \frac{A'B'}{dx} - 1 \quad \Rightarrow \quad (A'B')^2 = \left[dx\left(\varepsilon_{(x_1)} + 1\right)\right]^2$$

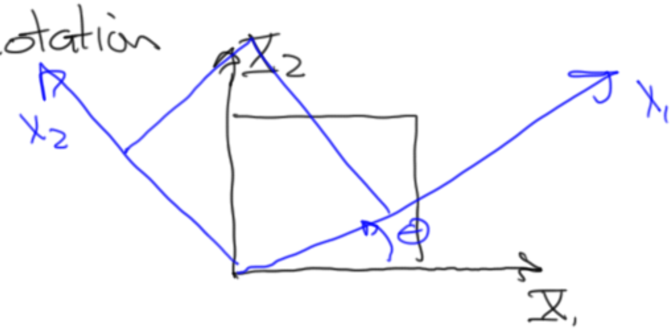
$$\cancelto{0}{\varepsilon^2_{(x_1)}} + \cancel{2}\,\varepsilon_{(x_1)} + \cancel{1} = \cancel{1} + \cancel{2}\,\frac{\partial u_1}{\partial x_1} + \cancelto{0}{\left(\frac{\partial u_1}{\partial x_1}\right)^2} + \cancelto{0}{\left(\frac{\partial u_2}{\partial x_1}\right)^2}$$

$$\varepsilon_{(x_1)} = \frac{\partial u_1}{\partial x_1} \qquad \text{likewise for} \quad \varepsilon_{(x_2)} = \frac{\partial u_2}{\partial x_2}$$

Def. through Linearization Green-Lagrange strain was

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \cancel{\frac{1}{2}}\Bigg(\underbrace{\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}}_{\cancel{2}\,\frac{\partial u_1}{\partial x_1}}\Bigg)$$

Rotation



$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

$$d\vec{x} = R\,d\vec{X} \qquad d\vec{x} = F\,d\vec{X} \qquad F = R$$

$F = R$

$$E = \frac{1}{2}\left(F^T F - I\right) = \frac{1}{2}\left(R^T R - I\right) = \frac{1}{2}\left(\overset{I}{\cancel{R^{-1} R}} - I\right)$$

0

Unity $R^T = R^{-1}$

$$\varepsilon = \frac{1}{2}\left(\nabla\vec{u} + \nabla\vec{u}^T\right)$$

$$\frac{\partial \vec{x}_i}{\partial X} = \frac{\partial \vec{X}}{\partial X} + \frac{\partial \vec{u}}{\partial X}$$

$$F = I + \nabla\vec{u} \Rightarrow \nabla\vec{u} = F - I$$

$$\varepsilon = \frac{1}{2}\left(F - I + (F - I)^T\right)$$

$$= \frac{1}{2}\left(F - I + F^T - I\right)$$

$$= \frac{1}{2}\left(F + F^T\right) - I$$

$R = F$

$$= \underbrace{\frac{1}{2}\left(R + R^T\right) - I}_{\neq 0}$$

only time $R = I$

Strain-rate

$$\frac{d}{dt}\left(d\vec{x}\right) = d\left(\frac{d\vec{x}}{dt}\right) = d\vec{v}$$

$\vec{v}$

$$\vec{v} = \vec{v}(x_1, x_2, x_3, t)$$

$$d\vec{v} = \frac{\partial v_i}{\partial x_j} d\vec{x}$$

$\downarrow$

$L \rightarrow$ velocity gradient

$$= L\, d\vec{x} \qquad d\vec{x} = F\, d\vec{X}$$

$$= L F\, d\vec{X} \quad \rightarrow \frac{\partial v_k}{\partial X_j} \qquad F = \frac{\partial x_i}{\partial X_j} \rightarrow F^{-1} = \frac{\partial X_j}{\partial x_i}$$

$$\frac{\partial X_k}{\partial x_j}\left(\frac{dF}{dt}\right) = \frac{d}{dt}\left(\frac{\partial x_i}{\partial X_j}\right) = \left[\frac{\partial}{\cancel{\partial X_j}}\left(\frac{\partial x_i}{dt}\right)\right]\frac{\cancel{\partial X_k}}{\partial x_j} = \frac{\partial v_i}{\partial x_j} = L$$

v

$F^{-1}\dot{F}$

$$L = F^{-1}\dot{F}$$

$$\frac{d}{dt}\left[(ds)^2-(dS')^2\right] = \frac{d}{dt}\left((ds)^2\right) - \frac{d}{dt}\left(\cancel{(dS)^2}^{\,0}\right) \qquad \frac{d}{dt}(ds^2) = 2ds\frac{d}{dt}ds$$

$$= \frac{d}{dt}(ds^2) = \frac{d}{dt}\left(d\vec{x}^T d\vec{x}\right)$$

$$= d\left(\frac{d\vec{x}}{dt}\right)^{\vec{v}} d\vec{x} + d\vec{x}^T d\left(\cancel{\frac{d\vec{x}}{dt}}^{\,\vec{v}}\right)$$

$$= d\vec{v}^T d\vec{x} + d\vec{x}^T d\vec{v} \qquad d\vec{v} = L\,d\vec{x}$$

$$= (L\,d\vec{x})^T d\vec{x} + d\vec{x}^T L\,d\vec{x}$$

$$= d\vec{x}^T L^T d\vec{x} + d\vec{x}^T L\,d\vec{x}$$

$$= d\vec{x}^T \underbrace{\left(L^T + L\right)}_{2D} d\vec{x}$$

$$D = \frac{1}{2}\left(L^T + L\right) \Rightarrow \text{rate-of-deformation tensor}$$

$$D^T = \frac{1}{2}\left(L^{T^T} + L^T\right) = \frac{1}{2}\left(L^T + L\right) = D = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$\vec{x} = \vec{X} + \cancel{\vec{u}}^{\,0}$$

$$\dot{\varepsilon} = \frac{1}{2}\left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i}\right)$$