We can show

$$d\psi = -s\,dT + \tau_j\,d\lambda_j$$

$$\dot\psi = -s\,\cancel{\dot T}^{\,0} + \tau_j\,\dot\lambda_j$$

$$\dot\psi = \tau_j\,\dot\lambda_j \qquad (\star)$$

$$\rho\dot\psi = \sigma_{ij}\,\dot\varepsilon_{ij}$$

$$= \tau_j\,\dot\lambda_j$$

<u>Energy Eqn.</u>

$$\rho\frac{Du}{Dt} = \sigma_{ij}D_{ij} - \frac{\partial q_j}{\partial x_j} + \rho r$$

$$\psi = u - sT \quad \Rightarrow \quad \boxed{u = \psi + sT}$$

$$(\star\star)$$

$$\rho\dot\psi = \sigma_{ij}D_{ij} + \rho r - \frac{\partial q_j}{\partial x_j} - \rho\dot s - \cancel{\rho\dot T}^{\,0} = \sigma_{ij}D_{ij} \simeq \sigma_{ij}\dot\varepsilon_{ij}$$

C-D

$$\dot s = \frac{1}{T}\left[r - \frac{1}{\rho}\frac{\partial q_j}{\partial x_j}\right]$$

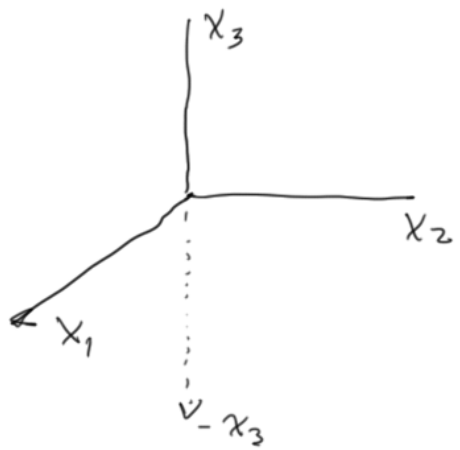
$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \vec{v}\,\cancel{\nabla u}^{\,0}$$

$$u = u(s, \vec{\Omega}, \vec{\nabla})$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{ij}$$

$$\underbrace{\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ & & & C_{2323} & C_{2331} & C_{2312} \\ & \text{symm.} & & & C_{3131} & C_{3112} \\ & & & & & C_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix}}_{\text{triclinic}}$$

Consider that has a plane of symm.



$x_1 - x_2$ plane is a plane of symm.

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\sigma' = R \sigma R^T = \begin{bmatrix} \sigma_{11} & \sigma_{12} & -\sigma_{31} \\ & \sigma_{22} & -\sigma_{23} \\ & & \sigma_{33} \end{bmatrix}$$

similarly for $\varepsilon' = R \varepsilon R^T$

$\varepsilon_{31}' = -\varepsilon_{31}$ & $\varepsilon_{23}' = -\varepsilon_{23}$

rest same

$$\sigma_{11}' = C_{1111}\varepsilon_{11}' + C_{1122}\varepsilon_{22}' + C_{1133}\varepsilon_{33}' + 2C_{1123}\varepsilon_{23}' + 2C_{1131}\varepsilon_{31}' + 2C_{1121}\varepsilon_{12}'$$

$$\sigma_{11} = C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}\varepsilon_{33} - 2C_{1123}\varepsilon_{23} - 2C_{1131}\varepsilon_{31} + 2C_{1121}\varepsilon_{12}'$$

$$\sigma_{11} = C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}\varepsilon_{33} + 2C_{1123}\varepsilon_{23} + 2C_{1131}\varepsilon_{31} + 2C_{1121}\varepsilon_{12}$$

$$0 = -4\, C_{1123}\, \varepsilon_{23} - 4\, C_{1131}\, \varepsilon_{31}$$

$$C_{1123} = C_{1131} = 0$$

Similar arguments

$$C_{2223} = C_{2231} = C_{3323} = C_{3331} = 0$$

$$C = \begin{bmatrix} C_{1111} & C_{1112} & C_{1133} & 0 & 0 & C_{1112} \\ & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ & & C_{3333} & 0 & 0 & C_{3312} \\ & & & C_{2323} & C_{2331} & 0 \\ & \text{symm.} & & & C_{3131} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

monoclinic $\rightarrow$ 13 independent constants

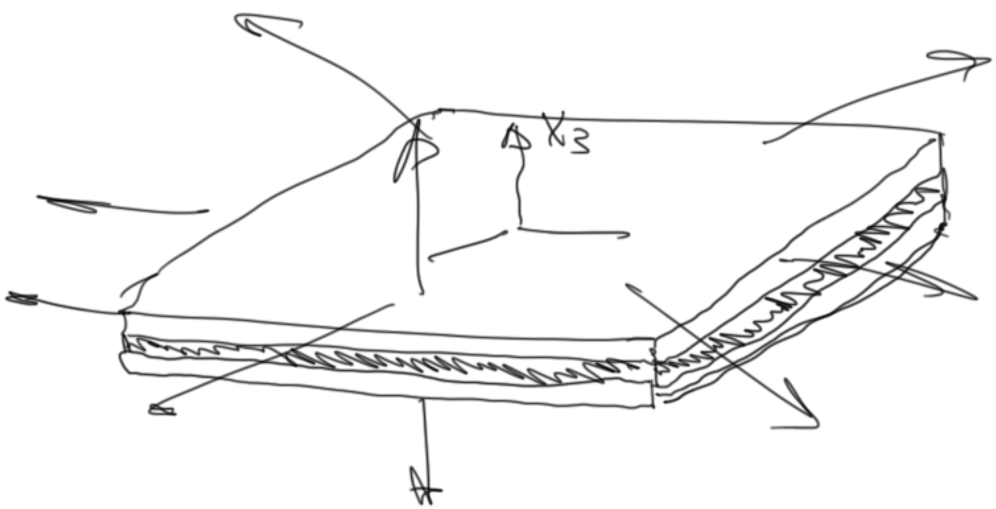
If 3 orthogonal planes of symm.

$$C_{1122} = C_{2223} = C_{2231} = C_{2212} = C_{3312} = 0$$

9 independent constants $\rightarrow$ orthotropic material

If there exist an axes about which a material has indentical prop. then $\underline{\underline{5}}$ independent constants

Transversely isotropic



For a material in which every plane is a plane of symm,

Isotropic

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix}$$

$$\lambda = \frac{2G}{1-2\nu} = \frac{G(E-2G)}{3G-E} = K - \frac{2}{3}G = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$= \frac{3K\nu}{1+\nu} = \frac{3K(3K-E)}{9K-E}$$

$$\mu = G = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(K-\lambda) = \frac{E}{2(1-\nu)} = \frac{3K(1-2\nu)}{2(1+\nu)} = \frac{3KE}{9K-E}$$

$\lambda \to$ Lamé's constant, $K \to$ Bulk modulus, $E \to$ Young's modulus, $\nu \to$ Poisson's ratio, $\mu = G \to$ shear mod.

For isotropic materials

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ij}\delta_{jl} + \delta_{il}\delta_{jk})$$

Sub. in to Gen. Hooke's Law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ij}\delta_{jl} + \delta_{il}\delta_{jk})]\varepsilon_{kl}$$

$$\delta_{ik}\delta_{jk}\varepsilon_{kl} = \varepsilon_{jl} = \varepsilon_{ji}$$

$$\boxed{\sigma_{ij} = \lambda \delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}}$$

let $i=j$

$$\sigma_{ii} = (3\lambda + 2\mu)\varepsilon_{ii} \Rightarrow \varepsilon_{kk} = \frac{\sigma_{kk}}{(3\lambda+2\mu)}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$\boxed{\varepsilon_{ij} = \frac{-\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}}$$

$$\begin{cases} \varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right] \\ \varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right] \\ \varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right] \\ \varepsilon_{23} = \frac{1}{2\mu}\sigma_{23} \\ \varepsilon_{31} = \frac{1}{2\mu}\sigma_{23} \\ \varepsilon_{12} = \frac{1}{2\mu}\sigma_{12} \end{cases}$$

$$\leftarrow \quad \varepsilon = \frac{\sigma}{E} \Rightarrow \underline{\sigma = E\varepsilon}$$

<u>Plane stress</u>

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

$$\underbrace{\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix}}_{\vec{\varepsilon}} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & (1+\nu) \end{bmatrix} \underbrace{\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix}}_{\vec{\sigma}} \Rightarrow \begin{Bmatrix} \vec{\sigma} \end{Bmatrix} = \frac{E}{(1-\nu)^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}\begin{Bmatrix} \vec{\varepsilon} \end{Bmatrix}$$

Plane strain

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix}$$

$$\left\{ \vec{\varepsilon} \right\} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \vec{\sigma} \right\}$$