Ex

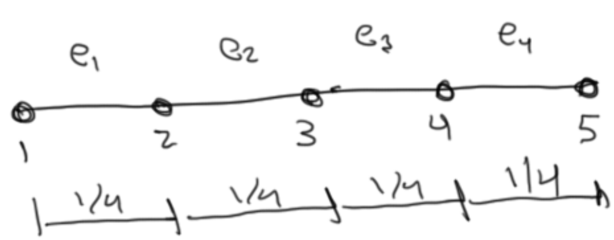
$$a = 1 \quad , \quad c = -1 \quad , \quad f = -x^2$$

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \qquad 0 < x < 1$$

$$u(0) = 0 \qquad u(1) = 0$$

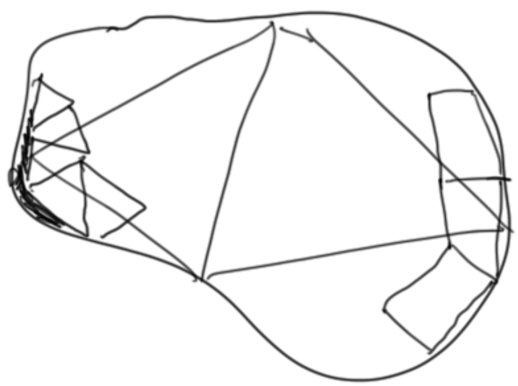
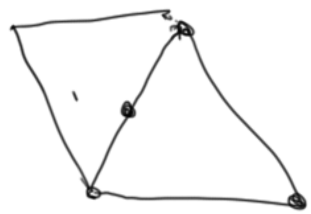
$$K_{ij} = \int_{x_a}^{x_b} \left(\frac{dN_i}{dx} \frac{dN_j}{dx} - N_i N_j \right) dx$$

$$f_i = \int_{x_a}^{x_b} (-x^2) N_j \, dx$$



Problems in 2D with scalar field variables, u

1D  no mesh error

2D  

General rules

1. Element should the governing eqn.
2. #, shape, type → accurate
3. The mesh density should cover areas of high graients

Element type
 Triangles
 Quad
elements
Degree of accuracy - interpolatunt

– no magic

4. Grade away gradually.

$$\underbrace{-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)}_{\nabla\cdot(A\nabla u)} + a_{00}u - f = 0$$

$u = u(x,y)$ a_{ij} are the data with B.C.'S

Models

Heat transfer in 2D

Irrotational flow of an ideal fluid

Groundwater flow through permeable geology

Weak Form

$$0 = \int_{\Omega_e} \delta u \left[-\frac{\partial}{\partial x}(F_1) - \frac{\partial}{\partial y}(F_2) + a_{00}u - f \right] dx\,dy$$

where $F_1 = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}$, $F_2 = a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}$

Integrate - by - parts

$$\frac{\partial}{\partial x}(\delta u\, F_1) = \frac{\partial u}{\partial x} F_1 + \delta u \frac{\partial F_1}{\partial x} \Rightarrow -\delta u \frac{\partial F_1}{\partial x} = \frac{\partial \delta u}{\partial x} F_1 - \frac{\partial}{\partial x}(\delta u\, F_1)$$

$$\frac{\partial}{\partial y}(\delta u\, F_2) = \frac{\partial \delta u}{\partial y} F_2 + \delta u \frac{\partial F_2}{\partial y} \Rightarrow -\delta u \frac{\partial F_2}{\partial y} = \frac{\partial \delta u}{\partial y} F_2 - \frac{\partial}{\partial y}(\delta u\, F_2)$$

Diverge. Theorem

$$\int_\Omega \frac{\partial}{\partial x}(\delta u\, F_1)\, dx\,dy = \oint_\Gamma \delta u\, F_1\, n_x\, dS_\Gamma$$

$$\int_\Omega \frac{\partial}{\partial y}(\delta u\, F_2)\, dx\,dy = \oint_\Gamma \delta u\, F_2\, n_y\, dS_\Gamma$$

where n_x + n_y are components of the unit normal

$$\hat{n} = n_x \hat{e}_x + n_y \hat{e}_y \quad \text{on } \Gamma_e$$

$$0 = \int_\Omega \Big[\frac{\partial \delta u}{\partial x}\left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y}\right) + \frac{\partial \delta u}{\partial y}\left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y}\right) + a_{00}\, \delta u\, u$$

$$- \delta u\, f \Big] - \oint_\Gamma \Big[\underbrace{n_x \left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y}\right) + n_y \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y}\right)}_{= q_n} dS$$

$$0 = \int_\Omega \left[\frac{\partial \delta u}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial \delta u}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}\,\delta u\, u \right.$$

$$\left. - \delta u\, f \right] dx\,dy - \oint_\Gamma \delta u\, q_n\, dS \quad \backslash\backslash\backslash$$

$$B(\delta u, u) = \int_\Omega \left[\frac{\partial \delta u}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial \delta u}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}\,\delta u\, u \right] dx\,dy$$

$$l(\delta u) = \int_\Omega \delta u\, f\, dx\,dy + \oint_\Gamma \delta u\, q_n\, dS$$

If $a_{21} = a_{12}$

$$I(u) = \frac{1}{2} B(u,u) - l(u)$$

$$I(u) = \frac{1}{2}\int_\Omega \left[a_{11}\left(\frac{\partial u}{\partial x}\right)^2 + 2a_{12}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + a_{22}\left(\frac{\partial u}{\partial y}\right)^2 + a_{00}\, u^2 \right] dx\,dy$$

$$+ \int_\Omega u\, f\, dx\,dy + \oint_\Gamma u\, q_n\, dS$$

$$B(\delta\vec{u}, \vec{u}) = l(\delta\vec{u})$$

Let

$$C = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{00} \end{bmatrix}, \quad D = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ 1 \end{bmatrix}$$

$$B(\delta\vec{u}, \vec{u}) = \int_\Omega \begin{bmatrix} \frac{\partial \delta u}{\partial x} \\ \frac{\partial \delta u}{\partial y} \\ \delta u \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{00} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ u \end{bmatrix} dx\,dy$$

$$l(\delta u) = \int_\Omega \{\delta u\}^T \{f\}\, dx\,dy + \oint_\Gamma \{\delta u\}^T \{q_n\}\, dS$$

$$\left.\begin{aligned} B(\delta\vec{u}, \vec{u}) &= \int_\Omega (D\,\delta u)^T C\, D u\, dx\,dy \\ l(\delta\vec{u}) &= \int_\Omega \delta\vec{u}^T \hat{f}\, dx\,dy + \int_\Gamma \delta\vec{u}^T \vec{q}_n\, dS \end{aligned}\right\} - \underline{\text{Weak Form}}$$

FE model in 2D

$$u(x,y) \approx u_h(x,y) = \sum_{j=1}^{n} N_j u_j$$

$$0 = \int_\Omega \Big[\frac{\partial \delta u}{\partial x}\Big(a_{11}\frac{\partial N_j}{\partial x}u_j + a_{12}\frac{\partial N_j}{\partial y}u_j\Big) + \frac{\partial \delta u}{\partial y}\Big(a_{21}\frac{\partial N_j}{\partial x}u_j + a_{22}\frac{\partial N_j}{\partial y}u_j\Big)$$

$$+ a_{00}\,\delta u\, N_j u_j - \delta u\, f \Big] dx\,dy - \oint_\Gamma \delta u\, q_n\, dS$$

or

$$0 = \int_\Omega (D\delta u)^T C D(N^T u)\, dx\,dy - \int_\Omega \delta \vec{u}^T \vec{f}\, dx\,dy - \oint_\Gamma \delta\vec{u}\, \vec{q}_n\, dS$$

Let $\delta u_i = N_i$

$$0 = \sum_{i}^{n} \overbrace{\Big\{ \int_\Omega \Big[\frac{\partial N_i}{\partial x}\Big(a_{11}\frac{\partial N_j}{\partial x} + a_{21}\frac{\partial N_j}{\partial y}\Big) + \frac{\partial N_i}{\partial y}\Big(a_{21}\frac{\partial N_j}{\partial x} + a_{22}\frac{\partial N_j}{\partial y}\Big) + a_{00} N_i N_j \Big] dx\,dy \Big\}}^{K_{ij}} u_j$$

$$- \int_\Omega f N_i\, dx\,dy - \oint_\Gamma N_i q_n\, dS \Rightarrow K_{ij} u_j = f_i + Q_i$$

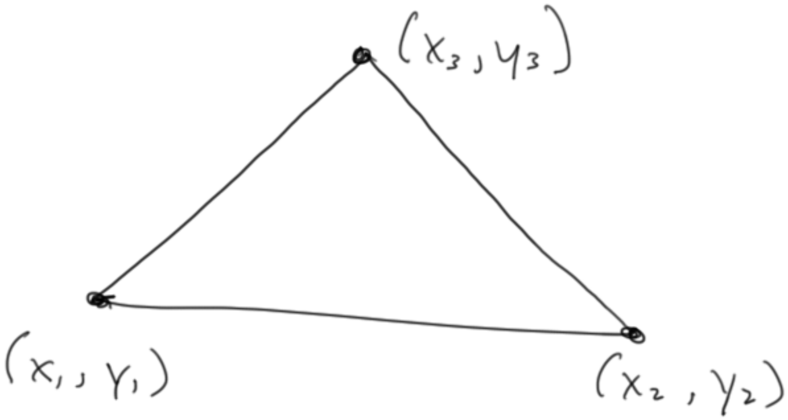
$$K\bar{u} = \vec{F} = \vec{f} + \vec{Q}$$

where

$$K = \int_\Omega B^T C B \, dx\,dy \quad , \quad \vec{f} = \int_\Omega N^T \vec{f} \, dy\,dx \quad Q = \oint_\Gamma N^T \hat{q}_n \, dS$$

$$B = DN^T = \begin{bmatrix} N_{1,x} & N_{2,x} & \cdots & N_{n,x} \\ N_{1,y} & N_{2,y} & \cdots & N_{n,y} \\ N_1 & N_2 & \cdots & N_n \end{bmatrix}$$

Constrant Strain (CST) 3-nodes

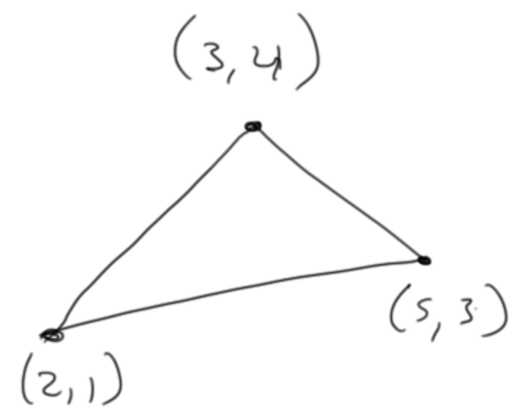


$$u_n = C_1 + C_2\, x + C_3\, y$$

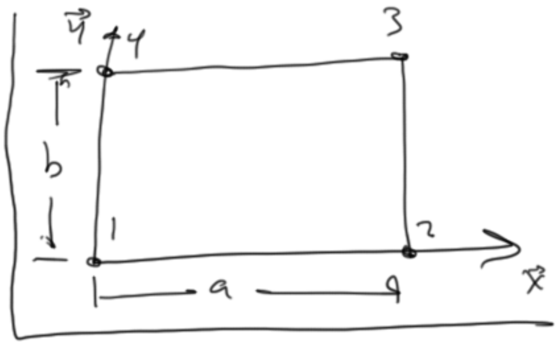
$$X = \begin{bmatrix} 1 & x & y \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}_{A} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix}$$

$$N = X\, A^{-1}$$



Linear Rectangle Element (Quad 4)



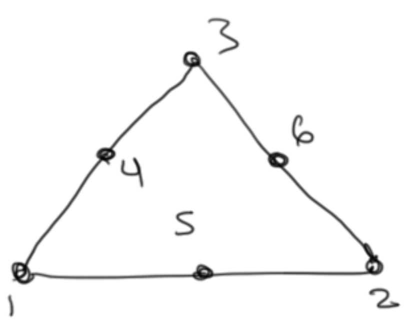
$$u_h(x,y) = C_1 + C_2 X + C_3 Y + C_4 XY$$

$$N_1 = \left(1 - \frac{\bar{x}}{a}\right)\left(1 - \frac{\bar{y}}{b}\right) \qquad N_2 = \frac{\vec{x}}{a}\left(1 - \frac{\bar{y}}{b}\right)$$

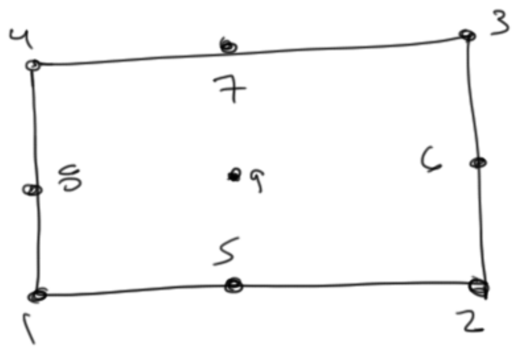
$$N_3 = \frac{\bar{x}}{a}\,\frac{\vec{y}}{b} \qquad N_4 = \left(1 - \frac{\bar{x}}{b}\right)\frac{\bar{y}}{b}$$

Quadritic Triangle

$$u^h(x,y) = C_1 + C_2 X + C_3 Y + C_4 XY + C_5 X^2 + C_6 Y^2$$



(Quad 8)



$$u_h(x,y) = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2$$
$$+ c_7 xy^2 + c_8 x^2 y + c_9 x^2 y^2$$