Linear Algebra Basics

Matrix-Vector multiplication

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$c_1 = a_{11}b_1 + a_{12}b_2 + a_{13}b_3$$

$$c_2 = a_{21}b_1 + a_{22}b_2 + a_{23}b_3$$

$$c_3 = a_{31}b_1 + a_{32}b_2 + a_{33}b_3$$

In words: c_i is the dot product of the i^{th} row of $\mathbf{a}$ with $\vec{b}$...

Matrix-Matrix multiplication

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$

$$\vdots$$

In words: c_{ij} is the dot product of the i^{th} row of $\mathbf{a}$ with the j^{th} column of $\mathbf{b}$

Examples

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 7 \end{Bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 5 & 13 \end{bmatrix}$$

The determinant of a 2 x 2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(\mathbf{A}) = a \cdot d - b \cdot c$$

The determinant of a 3 x 3 matrix

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det(\mathbf{A}) = a \cdot (e \cdot i - f \cdot h) - b \cdot (d \cdot i - f \cdot g) + c \cdot (d \cdot h - e \cdot g)$$

Matrix row operations

Used in solving matrix equations, i.e.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$\mathbf{A}\vec{x} = \vec{b}$$

- Swaping rows doesn't change solution
- Adding rows together doesn't change solution
- Multiplying row by a scalar doesn't change solution

Example

Solve for $\vec{x}$

$$\begin{bmatrix} 2 & 3 & -2 \\ 0 & 0 & 3 \\ 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -6 \\ 3 \end{Bmatrix}$$

Eigenvalue problem

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

$$\mathbf{A}\vec{v} - \lambda\vec{v} = 0$$

$$(\mathbf{A} - \lambda\mathbf{I})\vec{v} = 0$$

A non-trivial solution for $\vec{v}$ exists, if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

The λ's are called the **eigenvalues**. Examples to follow in the context of stress.

Vector transformation

$$\vec{v}' = \mathbf{Q}\vec{v}$$

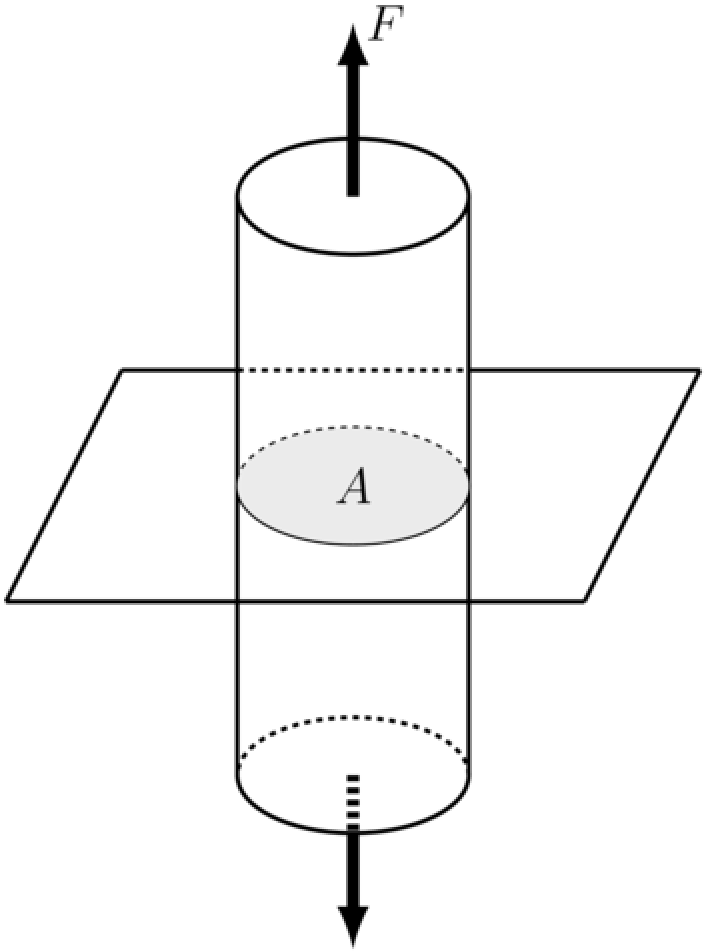
$$\mathbf{Q} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Matrix transformation

$$\mathbf{S}' = \mathbf{Q}^{-1}\mathbf{S}\mathbf{Q}$$

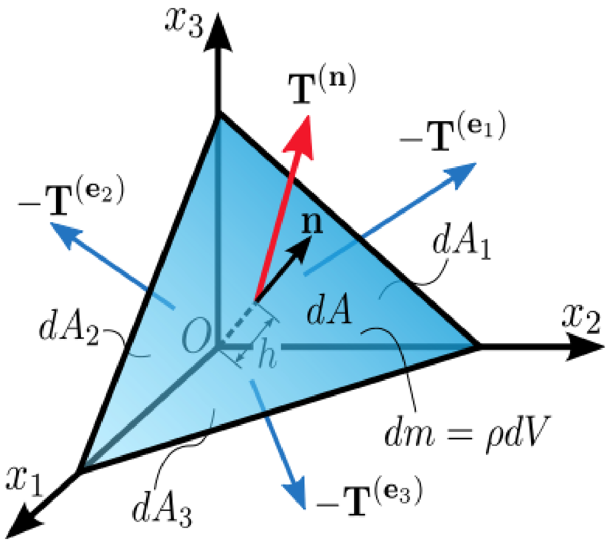
If $\mathbf{Q}$ is chosen such that its columns are eigenvectors of $\mathbf{S}$, then $\mathbf{S}'$ will be *diagonal* with its entries cooresponding to the eigenvalues $\mathbf{S}$ (and $\mathbf{S}'$).

Stress



$$\sigma = \frac{F}{A}$$

Cauchy tetrahedron



"Cauchy tetrahedron" by Sanpaz - Own work.
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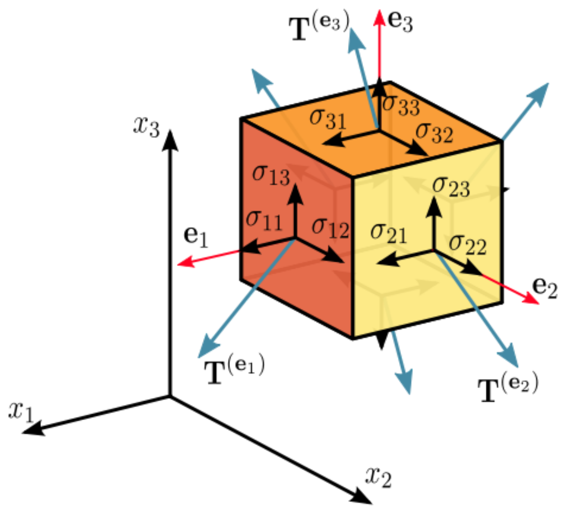
Stress tensor

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Visual definition



- First subscripted index refers to the index of the unit vector that is normal to the face.
- Second subscripted index refers to the component of traction vector.