$$0 = \int_0^L \left\{ a(x) \frac{d}{dx}(\delta u) \frac{du}{dx} - \delta u \, f(x) \right\} dx - \delta u(L) \, Q_L$$

Bilinear Form

The weak form will contain 2 types of expressions, those involving δu + u and those involving only δu

Group:

$$B(\delta u, u) = \int_0^L a(x) \frac{d}{dx}(\delta u) \frac{du}{dx} \, dx$$

$$l(\delta u) = \int_0^L \delta u \, f(x) \, dx + \delta u(L) \, Q_L$$

We can writer (the problem is stated as, find u:

$B(\delta u, u) = l(\delta u) \rightarrow$ this is known as the "variational problem"

The functional $B(\delta u, u)$ is said to be <u>bilinear</u>

$$B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v)$$

$$B(u, \alpha v_1 + \beta v_2) = \alpha B(u, v_1) + \beta B(u, v_2)$$

and $\quad B(u, v) = B(v, u)$

If $B(\cdot,\cdot)$ is bilinear & symm. and $\ell(\cdot)$ is linear

we have

$$B(\delta u, u) = \frac{1}{2}\delta B(u,u) \quad , \quad \ell(\delta u) = \delta \ell(u)$$

$$B(\delta u, u) = \int_0^L a \frac{d}{dx}(\delta u)\frac{du}{dx}\,dx = \delta \int_0^L \frac{a}{2}\left(\frac{du}{dx}\right)^2 dx$$

$$= \frac{1}{2}\delta \int_0^L a \frac{du}{dx}\frac{du}{dx} = \frac{1}{2}\delta B(u,u)$$

Rewrite

$$B(\delta u, u) = l(\delta u) \quad \Rightarrow \quad B(\delta u, u) - l(\delta u) = 0$$

$$\frac{1}{2}\delta B(u,u) - \delta l(u) \equiv \delta I(u) = 0$$

So we can restate the variational problem as a minimization (stationary value)

$$\underline{I(u) = \frac{1}{2} B(u,u) - l(u)}$$

<u>Ritz Method</u>

Use the "weak form". Has the advantage that the approximating functions (ϕ_i's) only need to satisfy the essential B.C.'s, since the natural conditions are included. We seek an approximate solution of the form

$$u \approx u^h = \sum_{j=1}^{n} c_j \phi_j(x)$$

$$I(u) = \frac{1}{2} B(u,u) - l(u)$$

$$u \to u^h$$

$$\frac{\partial I}{\partial c_j} = 0$$

Example

$$-\frac{\partial^2 u}{\partial x^2} + u + x^2 = 0 \quad \text{for} \quad 0 < x < 1$$

with $u(0) = 0$, $u(1) = 0$

$$\underbrace{\int_0^1 \left\{ \frac{d}{dx}(\delta u)\frac{du}{dx} - \delta u \, u \right\} dx}_{B(\delta u, u)} + \underbrace{\int_0^1 \delta u \, x^2 \, dx}_{l(\delta u)} = 0$$

$$I(u) = \frac{1}{2} B(u,u) - l(u) = \frac{1}{2}\int_0^1 \left[\left(\frac{\partial u}{\partial x}\right)^2 - u^2 + 2x^2 u\right] dx$$

Now $u^h = c_1 x(1-x) + c_2 x^3(1-x) + c_3 x^3(1-x)$

$$\frac{\partial I}{\partial c_1} = 0 \qquad \frac{\partial I}{\partial c_2} = 0 \qquad \frac{\partial I}{\partial c_3} = 0$$

Interpolation Functions?

Again we

$$u \approx u_h = c_j \phi_j + \phi_0$$

ϕ_0 satisfies the essential B.C.'s

otherwise the ϕ_j have to satisfy the following:

1.) - ϕ_j must be selected such that $B(\phi_i, \phi_j)$ is defined and nonzero

i.e. the must have the proper continuouty requirement

$$\frac{\partial^2 u}{\partial x^2} \qquad u^h = x$$

- ϕ_j must satisfy the homogenuous form of the specified B.C.'s i.e. $u(0) = u_0 \rightarrow \phi_j$ must satisfy $u(0) = 0$

2.) The set of $\{\phi_j\}$ must be linearly independent

$$c_1 = x(1-x) \qquad c_2 = x^3(1-x) \qquad c_3 = \cancel{2x^3(1-x)}$$

3.) The set $\{\phi_j\}$'s must be <u>complete</u>

$\{x, x^2, x^3, x^4\} \to$ complete

$\{x, y, xy, x^2y, xy^2, x^2y^2\} \to$ complete

$\{x^3, x^5, x^{25}\} \to$ not complete

$\{x, x^2, xy^3\} \to$ not complete

4.) ϕ_o must be the lowest order function that satisfies the essential B.C.'s

<u>Almost polynomials</u> ///