$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}}\left(J_2 - \cancel{\frac{Y^2}{3}}^{\,0}\right) = \frac{\partial J_2}{\partial \sigma_{ij}} = S_{ij}$$

$$= \frac{\partial}{\partial \sigma_{ij}}\left(\frac{1}{2} S_{kl} S_{kl}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial \sigma_{ij}}(S_{kl}) S_{kl} + S_{kl}\frac{\partial}{\partial \sigma_{ij}}(S_{kl})\right)$$

$$= S_{kl}\frac{\partial}{\partial \sigma_{ij}}(S_{kl})$$

$$= S_{kl}\frac{\partial}{\partial \sigma_{ij}}\left(\sigma_{kl} - \frac{1}{3}\sigma_{mm}\delta_{kl}\right)$$

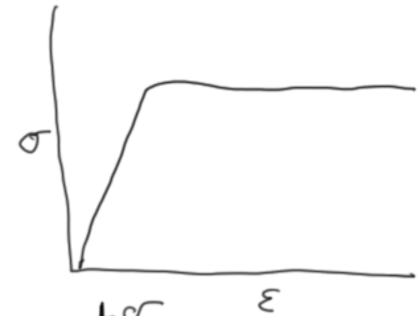
$$= S_{kl}\left(\frac{\partial \sigma_{kl}}{\partial \sigma_{ij}} - \frac{1}{3}\frac{\partial \sigma_{mm}}{\partial \sigma_{ij}}\delta_{kl}\right)$$

$$= S_{kl}\left(\delta_{il}\delta_{kj} - \frac{1}{3}\underbrace{\delta_{im}\delta_{jm}}_{\delta_{ij}}\delta_{kl}\right)$$

$$= S_{ki}\delta_{kj} - \frac{1}{3}S_{kk}\delta_{ij}$$

$$= S_{ji} - \frac{1}{3}\cancel{S_{kk}}^{\,0}\delta_{ij}$$

$$= S_{ji} = S_{ij}$$



$\varepsilon_{11} = 0.20\%$

$$\boxed{\boxed{d\varepsilon^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}}} = d\lambda\, S_{ij}}$$

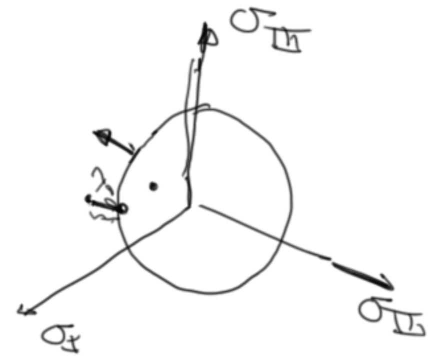
$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}$$

$$\dot{\varepsilon}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}}$$

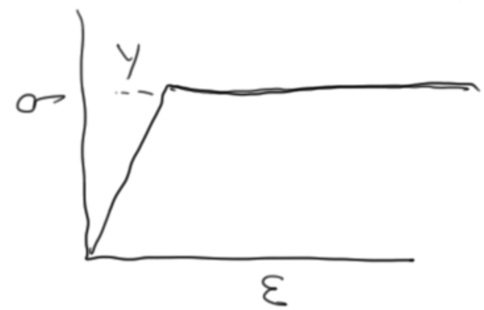
associated flow rule

$$\dot{\varepsilon}^p = \dot{\lambda} \frac{\frac{\partial f}{\partial \sigma_{ij}}}{\left|\frac{\partial f}{\partial \sigma_{ij}}\right|} = \dot{\lambda} \frac{S_{ij}}{|S_{ij}|} \quad Q_{ij}$$

$$|S_{ij}| = \sqrt{S_{ij} S_{ij}}$$



$$\begin{array}{c} \varepsilon_{11}^p \\ d\varepsilon^p = d\lambda \begin{bmatrix} \frac{2Y}{3} & 0 & 0 \\ 0 & -\frac{Y}{3} & 0 \\ 0 & 0 & -\frac{Y}{3} \end{bmatrix} \end{array} \quad -\frac{\varepsilon_{11}^p}{2} \quad -\frac{\varepsilon_{11}^p}{2}$$



$$f = \sqrt{3J_2} - Y = 0$$

$$f < 0$$

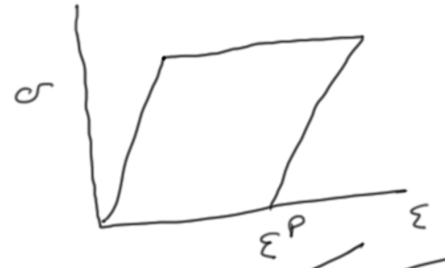
$$f = 0$$

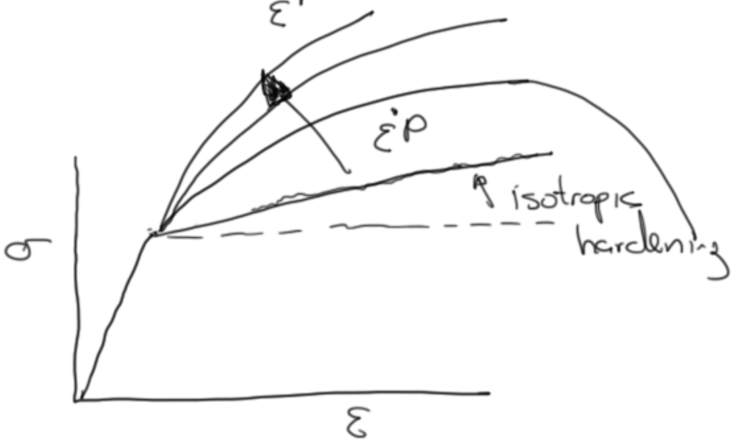
$$\varepsilon_{11} = \underbrace{\frac{Y}{E}}_{\varepsilon_{11}^e} + \varepsilon_{11}^p \Rightarrow \varepsilon_{11}^p = \varepsilon_{11} - \frac{Y}{E}$$

0.20%

$$\varepsilon_{22} = \frac{-\nu}{E} Y - \frac{1}{2}(\varepsilon_{11}^p) = \underbrace{\frac{-\nu}{E} Y - \frac{1}{2}\left(\varepsilon_{11} - \frac{Y}{E}\right)}$$

ε_{33} likewise





equivalent plastic strain, ε^p

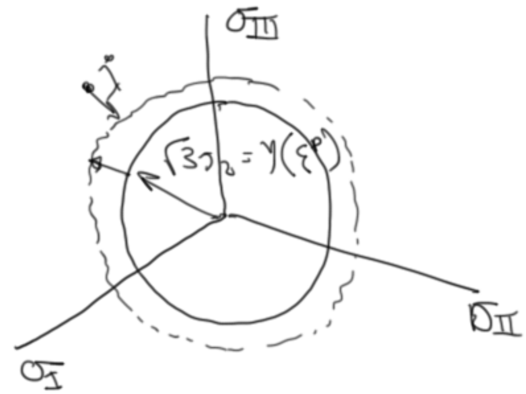
$$\dot{\varepsilon}^p = \sqrt{\frac{2}{3}\,\dot{\varepsilon}^p_{ij}\,\dot{\varepsilon}^p_{ij}} = \sqrt{\frac{2}{3}\,\dot{\lambda}^2\,Q_{ij}Q_{ij}} = \sqrt{\frac{2}{3}}\,\dot{\lambda}$$

$$\varepsilon^p = \int_0^{t_f} \dot{\varepsilon}^p\,dt$$

$$\boxed{\dot{\varepsilon}^p_{ij} = \dot{\lambda}\,Q_{ij}} = \dot{\lambda}\,\frac{\frac{\partial f}{\partial \sigma_{ij}}}{\left|\frac{\partial f}{\partial \sigma_{ij}}\right|} = \dot{\lambda}\,\frac{S_{ij}}{|S_{ij}|}$$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$



$$\dot{e}_{ij} = \dot{e}^e_{ij} + \dot{e}^p_{ij}$$

$$\dot{e}_{ij} - \dot{e}^e_{ij} - \dot{e}^p_{ij} = 0$$

$$\dot{e}_{ij} = \dot{\lambda}\,\frac{S_{ij}}{|S_{ij}|} \quad Q_{ij}$$

$$\dot{e}_{ij}Q_{ij} - \dot{e}^e_{ij}Q_{ij} - \dot{e}^p_{ij}Q_{ij} = 0$$

$$\sigma_{ij} = 2\mu e_{ij} + \underbrace{K \tfrac{1}{3} \varepsilon_{kk} \delta_{ij}}$$

$$S_{ij} = 2\mu e_{ij}$$

$$\dot{e}^{e}_{ij} = \frac{\dot{S}_{ij}}{2\mu}$$

$$0 = \dot{e}_{ij} Q_{ij} - \dot{e}^{e}_{ij} Q_{ij} - \dot{e}^{p}_{ij} Q_{ij}$$

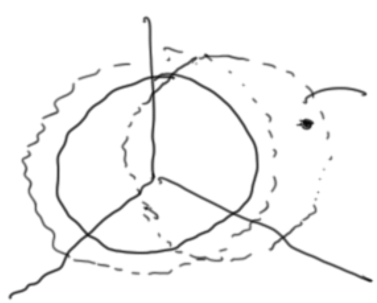
$$0 = \dot{e}_{ij} Q_{ij} - \frac{\dot{S}_{ij}}{2\mu} Q_{ij} - \dot{\lambda} \underbrace{Q_{ij} Q_{ij}}_{=1}$$

$$\boxed{0 = \dot{e}_{ij} Q_{ij} - \overbrace{\frac{\dot{S}_{ij}}{2\mu} Q_{ij}}^{\dot{S}} - \dot{\lambda}}$$

$$S = \sqrt{S_{ij} S_{ij}}$$

<u>Ex.</u> Isotropic Hardening

$$\boxed{\dot{e}_{ij} Q_{ij} - \frac{\dot{S}}{2\mu} - \dot{\lambda} = 0} \Rightarrow \bigstar$$



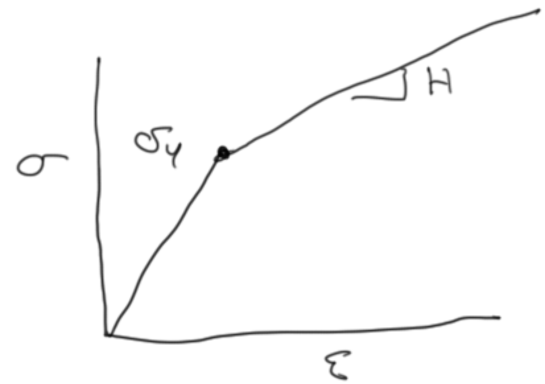
Translation

→ Kinematic Hardening

$$f = \sqrt{3J_2} - Y(\varepsilon^p) = 0$$

$$= \sqrt{3J_2} - \sigma_y - H\varepsilon^p$$

$$\sqrt{3J_2} \to \sqrt{\tfrac{3}{2}}\, s$$



$f < 0$

$f = 0$

Kuhn-Tucker Constrait equations

$$f \leq 0 \,, \quad \dot{\lambda} \geq 0 \,, \quad \dot{\lambda} f = 0$$

$$\boxed{\dot{f} = 0}$$

$$\sqrt{\tfrac{3}{2}}\, \dot{s} - H\dot{\varepsilon}^p = 0 \qquad \dot{\varepsilon}^p = \sqrt{\tfrac{2}{3} e^p_{ij} e^p_{ij}} = \sqrt{\tfrac{2}{3}}\,\dot{\lambda}$$

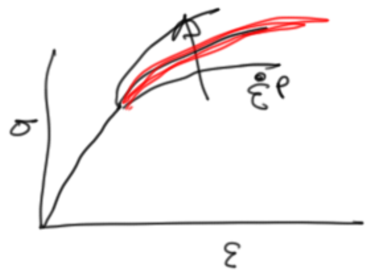
$$\sqrt{\tfrac{3}{2}}\, \dot{s} - H\sqrt{\tfrac{2}{3}}\,\dot{\lambda} = 0$$

$$\dot{s} = \tfrac{2}{3} H \dot{\lambda} \to \star$$

$$\dot{\lambda} = e_{ij} Q_{ij} \left(\frac{H}{3\mu} + 1\right)^{-1}$$

$$\dot{\sigma} = \begin{cases} \left(K + \dfrac{2\mu^3}{H+3\mu}\right)\dot{\varepsilon}_{kk}\,\delta_{ij} + \left(2\mu - \dfrac{6\mu^2}{H+3\mu}\right)\dot{\varepsilon}_{ij} & f = 0 \\ K\,\dot{\varepsilon}_{kk}\,\delta_{ij} + 2\mu\,\dot{\varepsilon}_{ij} & f < 0 \end{cases}$$

<u>Viscoplasticity</u>



$$f = \sqrt{\frac{3}{2}}\, s - \sigma_y \left(1 + \beta\,\dot{\varepsilon}^p\right)^N$$

$$0 = \sqrt{\frac{3}{2}}\,\dot{s} - N\sigma_y\beta\,\ddot{\varepsilon}^p\left(1+\beta\,\dot{\varepsilon}^p\right)^{N-1}$$

$$= \sqrt{\frac{3}{2}}\,\dot{s} - \sqrt{\frac{2}{3}}\,N\sigma_y\beta\,\ddot{\lambda}\left(1+\beta\sqrt{\frac{2}{3}}\,\dot{\lambda}\right)^{N-1}$$

$$\dot{s} = \frac{2}{3}N\sigma_y\beta\,\ddot{\lambda}\left(1+\beta\sqrt{\frac{2}{3}}\,\dot{\lambda}\right)^{N-1} \Rightarrow \bigstar$$

$$0 = \dot{e}_{ij}Q_{ij} - \frac{1}{3\mu}N\sigma_y\,\ddot{\lambda}\left(1+\beta\sqrt{\frac{2}{3}}\,\dot{\lambda}\right)^{N-1} - \dot{\lambda}$$