Stress

Recall: stress tensor

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

Due to **conservation of angular momentum**: $S_{12} = S_{21}, S_{13} = S_{31}$k and $S_{32} = S_{23}$.

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

Principle stresses and directions

$$\mathbf{S}' = \mathbf{Q}^{-1}\mathbf{S}\mathbf{Q}$$

$$\mathbf{S}' = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$$

with $S_1 > S_2 > S_3$ where the S_i's are the eigenvalues of $\mathbf{S}$

$$\mathbf{Q} = [\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]$$

where $\vec{v}_1$ is the eigenvector corresponding to S_1, $\vec{v}_2$ is the eigenvector corresponding to S_2, and $\vec{v}_3$ is the eigenvector corresponding to S_3.

Example

Determine the principle stresses and directions given:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$S_1 = 4, \quad S_2 = 2, \quad S_3 = 1$$

$$\mathbf{Q} = [\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3] = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Conservation of linear momentum

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{12}}{\partial x_2} + \frac{\partial S_{13}}{\partial x_3} + \rho b_1$$

$$\rho \frac{\partial^2 u_2}{\partial t^2} = \frac{\partial S_{12}}{\partial x_1} + \frac{\partial S_{22}}{\partial x_2} + \frac{\partial S_{23}}{\partial x_3} + \rho b_2$$

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial S_{13}}{\partial x_1} + \frac{\partial S_{23}}{\partial x_2} + \frac{\partial S_{33}}{\partial x_3} + \rho b_3$$