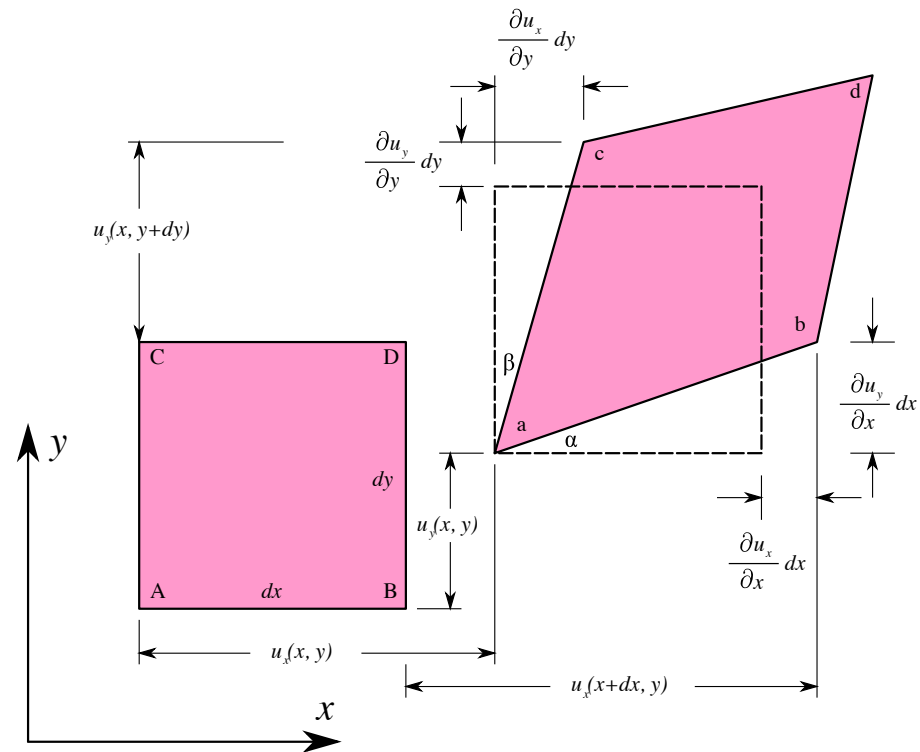


Constitutive modeling

Kinematics of strain



"2D geometric strain" by Sanpaz. Licensed under Public Domain via Wikimedia Commons.

Strain tensor

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{for } i = 1, 2, 3 \quad j = 1, 2, 3$$

Volumetric strain

$$\epsilon_{vol} = \text{tr}(\boldsymbol{\epsilon}) = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$$

Material constants for isotropic materials

Young's modulus

$$E = \frac{S_{11}}{\epsilon_{vol}} = \frac{S_{11}}{\epsilon_{11}}$$

Bulk modulus

$$K = \frac{S_{11} + S_{22} + S_{33}}{3\epsilon_{vol}}$$

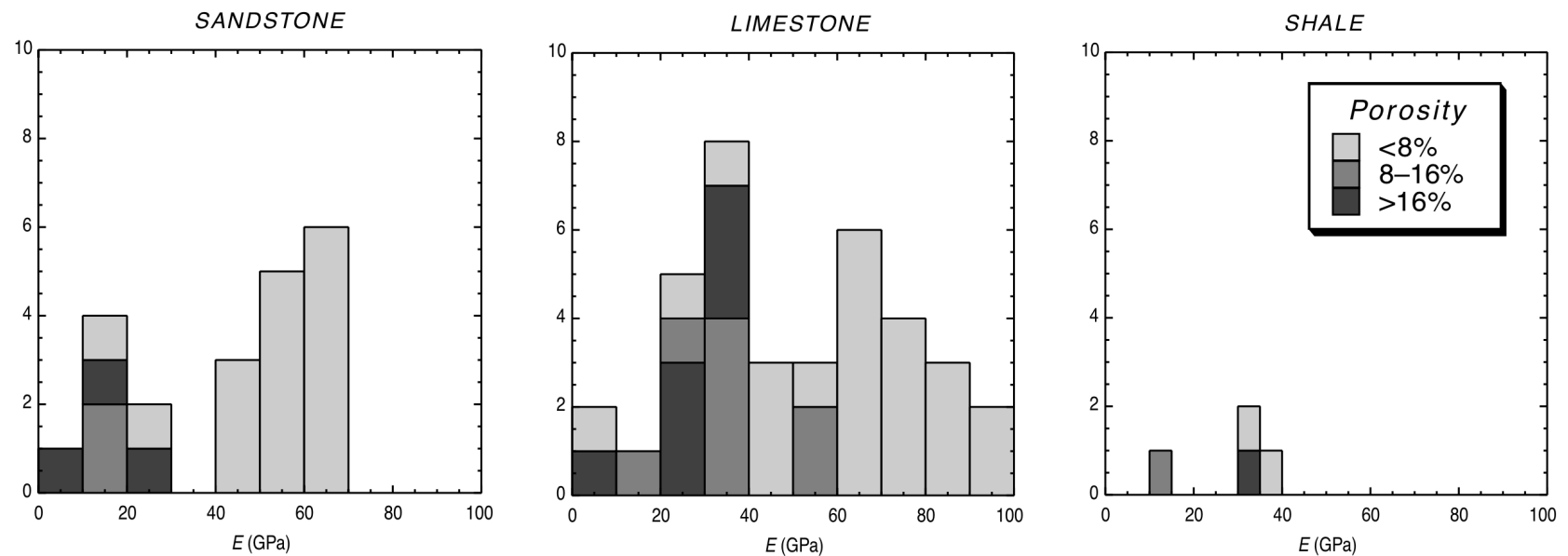
Shear Modulus

$$G = \frac{1}{2} \frac{S_{13}}{\epsilon_{13}}$$

Poisson's ratio

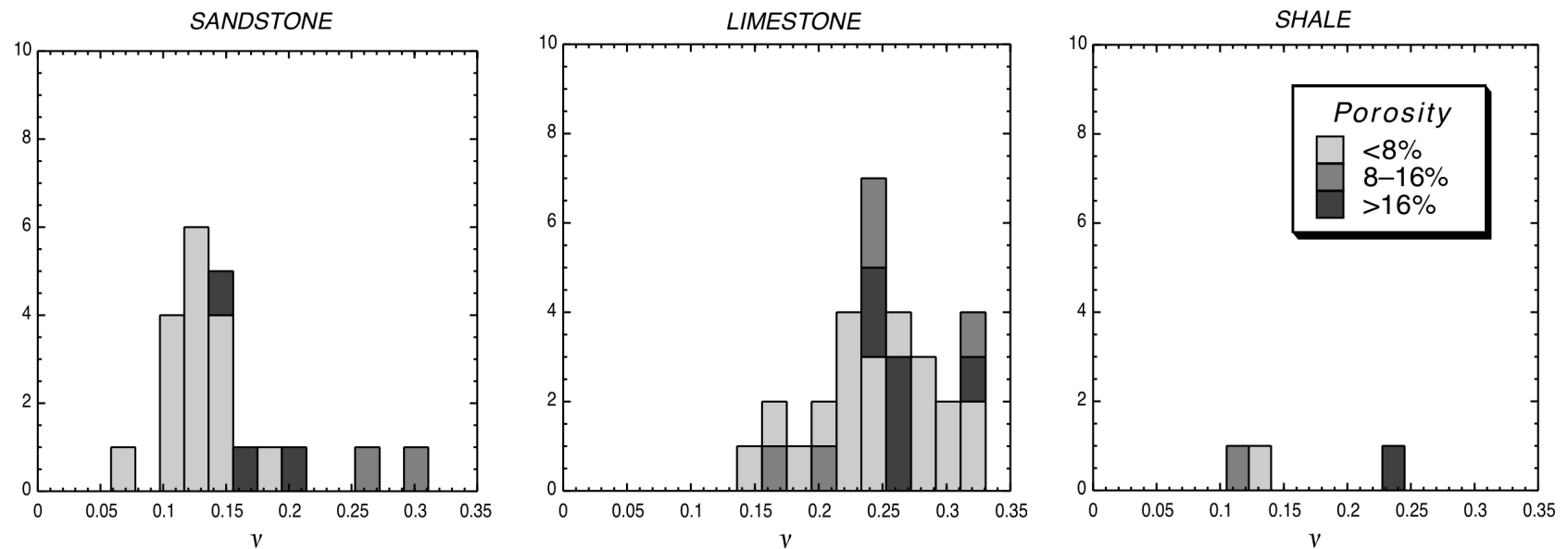
$$\nu = \frac{\epsilon_{33}}{\epsilon_{11}}$$

Typical Young's modulus values



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Typical Poissons' modulus values



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Generalized Hooke's law

$$\vec{\sigma} = \mathbf{C} \vec{\varepsilon}$$

For isotropic materials

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}$$

$$\mathbf{S} = K\varepsilon_{vol}\mathbf{I} + 2G\left(\boldsymbol{\varepsilon} - \frac{1}{3}\varepsilon_{vol}\mathbf{I}\right), \quad G = \frac{E}{2(1+\nu)} \Rightarrow \text{shear modulus}$$

$$\mathbf{S} = \lambda\varepsilon_{vol}\mathbf{I} + 2G\boldsymbol{\varepsilon}, \quad \lambda = K - \frac{2}{3}G \Rightarrow \text{Lamé's constant}$$

Relationships between constants

	$K =$	$E =$	$\lambda =$	$G =$	$\nu =$	$M =$
(K, E)	K	E	$\frac{3K(3K-E)}{9K-E}$	$\frac{3KE}{9K-E}$	$\frac{3K-E}{6K}$	$\frac{3K(3K+E)}{9K-E}$
(K, λ)	K	$\frac{9K(K-\lambda)}{3K-\lambda}$	λ	$\frac{3(K-\lambda)}{2}$	$\frac{\lambda}{3K-\lambda}$	$3K - 2\lambda$
(K, G)	K	$\frac{9KG}{3K+G}$	$K - \frac{2G}{3}$	G	$\frac{3K-2G}{2(3K+G)}$	$K + \frac{4G}{3}$
(K, ν)	K	$3K(1 - 2\nu)$	$\frac{3K\nu}{1+\nu}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	ν	$\frac{3K(1-\nu)}{1+\nu}$
(K, M)	K	$\frac{9K(M-K)}{3K+M}$	$\frac{3K-M}{2}$	$\frac{3(M-K)}{4}$	$\frac{3K-M}{3K+M}$	M
(E, λ)	$\frac{E+3\lambda+R}{6}$	E	λ	$\frac{E-3\lambda+R}{4}$	$\frac{2\lambda}{E+\lambda+R}$	$\frac{E-\lambda+R}{2}$
(E, G)	$\frac{EG}{3(3G-E)}$	E	$\frac{G(E-2G)}{3G-E}$	G	$\frac{E}{2G} - 1$	$\frac{G(4G-E)}{3G-E}$
(E, ν)	$\frac{E}{3(1-2\nu)}$	E	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	ν	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$

Relationships between constants (contd.)

	$K =$	$E =$	$\lambda =$	$G =$	$\nu =$	$M =$
(E, M)	$\frac{3M-E+S}{6}$	E	$\frac{M-E+S}{4}$	$\frac{3M+E-S}{8}$	$\frac{E-M+S}{4M}$	M
(λ, G)	$\lambda + \frac{2G}{3}$	$\frac{G(3\lambda+2G)}{\lambda+G}$	λ	G	$\frac{\lambda}{2(\lambda+G)}$	$\lambda + 2G$
(λ, ν)	$\frac{\lambda(1+\nu)}{3\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	λ	$\frac{\lambda(1-2\nu)}{2\nu}$	ν	$\frac{\lambda(1-\nu)}{\nu}$
(λ, M)	$\frac{M+2\lambda}{3}$	$\frac{(M-\lambda)(M+2\lambda)}{M+\lambda}$	λ	$\frac{M-\lambda}{2}$	$\frac{\lambda}{M+\lambda}$	M
(G, ν)	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$2G(1 + \nu)$	$\frac{2G\nu}{1-2\nu}$	G	ν	$\frac{2G(1-\nu)}{1-2\nu}$
(G, M)	$M - \frac{4G}{3}$	$\frac{G(3M-4G)}{M-G}$	$M - 2G$	G	$\frac{M-2G}{2M-2G}$	M
(ν, M)	$\frac{M(1+\nu)}{3(1-\nu)}$	$\frac{M(1+\nu)(1-2\nu)}{1-\nu}$	$\frac{M\nu}{1-\nu}$	$\frac{M(1-2\nu)}{2(1-\nu)}$	ν	M

Siesmic wave velocity

$$V_p = \sqrt{\frac{M}{\rho}}, \quad V_s = \sqrt{\frac{G}{\rho}}$$