1 Fourier transform of a haxagon

From [Smith_1973] “Diffraction patterns of simple apertures” the Fourier transform of a hexagonal pattern to a list of parameters $\{x_c, y_c, \theta, \sigma, a, o\}$ where $x_c$and y_c are the coordinate of the center of the gaussian pattern, θ its orientation (in degree), σ its size is:

$$H\left(x_c, y_c, \theta, \sigma\right) = \frac{2}{v_+ + v_-}\left[\frac{\sin u_-}{u_-}\sin v_+ + \frac{\sin u_+}{u_+}\sin v_-\right]$$

with:

$$\begin{cases} u_\pm = \frac{1}{3}\left(X \pm \sqrt{3}Y\right) \\ v_\pm = X \pm \frac{Y}{\sqrt{3}} \\ X = \frac{2\pi}{\sigma}\left[\left(x - x_c\right)\cos\theta + \left(y - y_c\right)\sin\theta\right] \\ Y = \frac{2\pi}{\sigma}\left[-\left(x - x_c\right)\sin\theta + \left(y - y_c\right)\cos\theta\right] \end{cases}$$

The point spread function for an amplitude a and offset o is given by:

$$PSF\left(x_c, y_c, \theta, \sigma, a, o\right) = a\frac{2}{\sqrt{3}\sigma^2}H^2\left(x_c, y_c, \theta, \sigma\right) + o$$

2 Partial derivatives

Main derivatives

$$H\left(u_-, u_+, v_-, v_+\right) = \frac{2}{v_+ + v_-}\left[\frac{\sin u_-}{u_-}\sin v_+ + \frac{\sin u_+}{u_+}\sin v_-\right]$$

$$\frac{\partial H}{\partial u_-} = \frac{2}{v_+ + v_-}\frac{\sin v_+}{u_-^2}\left[u_-\cos u_- - \sin u_-\right]$$

$$\frac{\partial H}{\partial u_+} = \frac{2}{v_+ + v_-}\frac{\sin v_-}{u_+^2}\left[u_+\cos u_+ - \sin u_+\right]$$

$$\begin{aligned}\frac{\partial H}{\partial v_-} &= -\frac{2}{\left(v_+ + v_-\right)^2}\left[\frac{\sin u_-}{u_-}\sin v_+ + \frac{\sin u_+}{u_+}\sin v_-\right] + \frac{2}{v_+ + v_-}\frac{\sin u_+}{u_+}\cos v_- \\ &= \frac{1}{v_+ + v_-}\left[2\frac{\sin u_+}{u_+}\cos v_- - H\left(u_-, u_+, v_-, v_+\right)\right]\end{aligned}$$

$$
\begin{aligned}
\frac{\partial H}{\partial v_+} &= -\frac{2}{\left(v_+ + v_-\right)^2}\left[\frac{\sin u_-}{u_-}\sin v_+ + \frac{\sin u_+}{u_+}\sin v_-\right] + \frac{2}{v_+ + v_-}\frac{\sin u_-}{u_-}\cos v_+ \\
&= \frac{1}{v_+ + v_-}\left[2\frac{\sin u_-}{u_-}\cos v_+ - H\left(u_-, u_+, v_-, v_+\right)\right]
\end{aligned}
$$

Parameters derivatives

$$\frac{\partial H}{\partial p} = \frac{\partial u_-}{\partial p}\frac{\partial H}{\partial u_-} + \frac{\partial u_+}{\partial p}\frac{\partial H}{\partial u_+} + \frac{\partial v_-}{\partial p}\frac{\partial H}{\partial v_-} + \frac{\partial v_+}{\partial p}\frac{\partial H}{\partial v_+}$$

$$\frac{\partial u_\pm}{\partial x_c} = \frac{2\pi}{3\sigma}\left(-\cos\theta \pm \sqrt{3}\sin\theta\right)$$

$$\frac{\partial v_\pm}{\partial x_c} = \frac{2\pi}{\sigma}\left(-\cos\theta \pm \frac{1}{\sqrt{3}}\sin\theta\right)$$

$$\frac{\partial u_\pm}{\partial y_c} = \frac{2\pi}{3\sigma}\left(-\sin\theta \pm -\sqrt{3}\cos\theta\right)$$

$$\frac{\partial v_\pm}{\partial y_c} = \frac{2\pi}{\sigma}\left(-\sin\theta \pm -\frac{1}{\sqrt{3}}\cos\theta\right)$$

$$\frac{\partial u_\pm}{\partial \theta} = \frac{1}{3}\frac{\pi}{180}\left(Y \pm -\sqrt{3}X\right)$$

$$\frac{\partial v_\pm}{\partial \theta} = \frac{\pi}{180}\left(Y \pm \frac{-X}{\sqrt{3}}\right)$$

$$\frac{\partial u_\pm}{\partial \sigma} = -\frac{1}{\sigma}u_\pm$$

$$\frac{\partial v_\pm}{\partial \sigma} = -\frac{1}{\sigma}v_\pm$$

PSF derivatives

$$\frac{\partial PSF}{\partial x_c} = a\frac{4}{\sqrt{3}\sigma^2}H\frac{\partial H}{\partial x_c}$$

$$\frac{\partial PSF}{\partial y_c} = a\frac{4}{\sqrt{3}\sigma^2}H\frac{\partial H}{\partial y_c}$$

$$\frac{\partial PSF}{\partial \theta} = a\frac{4}{\sqrt{3}\sigma^2}H\frac{\partial H}{\partial \theta}$$

$$\frac{\partial PSF}{\partial \sigma} = a\frac{4}{\sqrt{3}\sigma^2}H\left(\frac{\partial H}{\partial \sigma} - \frac{1}{\sigma}H\right)$$

$$\frac{\partial PSF}{\partial a} = \frac{2}{\sqrt{3}\sigma^2}H^2$$

$$\frac{\partial PSF}{\partial o} = \mathbf{1}$$