1 2D Gaussian pattern

Formulation

Simulation of a gaussian pattern according to a list of parameters $\left\{x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp, a, o\right\}$ where $x_c$and y_c are the coordinate of the center of the gaussian pattern, θ its orientation (in degree), $\sigma_\parallel$ its elongation along the parralel axis given by θ, $\sigma_\perp$ its elongation along the orthogonal axis given by θ, a its amplitude and o its offset:

$$
\begin{aligned}
G\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp, a, o\right) &= a.\exp\left\{-\frac{1}{2}\left[\left[\frac{(x-x_c)\cos\left(\frac{\pi}{180}\theta\right)+(y-y_c)\sin\left(\frac{\pi}{180}\theta\right)}{\sigma_\parallel}\right]^2\right.\right.\\
&\quad\left.\left.+\left[\frac{-(x-x_c)\sin\left(\frac{\pi}{180}\theta\right)+(y-y_c)\cos\left(\frac{\pi}{180}\theta\right)}{\sigma_\perp}\right]^2\right]\right\}+o\\
&= a.\exp\left\{-\frac{1}{2}\left[\left(\frac{\cos^2\left(\frac{\pi}{180}\theta\right)}{\sigma_\parallel^2}+\frac{\sin^2\left(\frac{\pi}{180}\theta\right)}{\sigma_\perp^2}\right)(x-x_c)^2\right.\right.\\
&\quad+\left(\frac{\sin^2\left(\frac{\pi}{180}\theta\right)}{\sigma_\parallel^2}+\frac{\cos^2\left(\frac{\pi}{180}\theta\right)}{\sigma_\perp^2}\right)(y-y_c)^2\\
&\quad\left.\left.+2(x-x_c)(y-y_c)\cos\left(\frac{\pi}{180}\theta\right)\sin\left(\frac{\pi}{180}\theta\right)\left(\frac{1}{\sigma_\parallel^2}-\frac{1}{\sigma_\perp^2}\right)\right]\right\}+o\\
&= a.\exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_\perp^2}+\left(\frac{1}{\sigma_\parallel^2}-\frac{1}{\sigma_\perp^2}\right)\cos^2\left(\frac{\pi}{180}\theta\right)\right)(x-x_c)^2\right.\right.\\
&\quad+\left(\frac{1}{\sigma_\perp^2}+\left(\frac{1}{\sigma_\parallel^2}-\frac{1}{\sigma_\perp^2}\right)\sin^2\left(\frac{\pi}{180}\theta\right)\right)(y-y_c)^2\\
&\quad\left.\left.+(x-x_c)(y-y_c)\sin\left(\frac{\pi}{90}\theta\right)\left(\frac{1}{\sigma_\parallel^2}-\frac{1}{\sigma_\perp^2}\right)\right]\right\}+o\\
&= a.E\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right)+o
\end{aligned}
$$

with

$$E\left(x_c, y_c, \theta, \sigma_{\|}, \sigma_{\perp}\right) = \exp\left\{-\frac{1}{2}\left[\left[\frac{(x-x_c)\cos\left(\frac{\pi}{180}\theta\right)+(y-y_c)\sin\left(\frac{\pi}{180}\theta\right)}{\sigma_{\|}}\right]^2\right.\right.$$
$$\left.\left.+\left[\frac{-(x-x_c)\sin\left(\frac{\pi}{180}\theta\right)+(y-y_c)\cos\left(\frac{\pi}{180}\theta\right)}{\sigma_{\perp}}\right]^2\right]\right\}$$

Partial derivatives

$$\frac{\partial G}{\partial x_c} = \left\{\left(\frac{1}{\sigma_{\perp}^2}+\left(\frac{1}{\sigma_{\|}^2}-\frac{1}{\sigma_{\perp}^2}\right)\cos^2\left(\frac{\pi}{180}\theta\right)\right)(x-x_c)\right.$$
$$\left.+\frac{1}{2}(y-y_c)\sin\left(\frac{\pi}{90}\theta\right)\left(\frac{1}{\sigma_{\|}^2}-\frac{1}{\sigma_{\perp}^2}\right)\right\} a.E\left(x_c, y_c, \theta, \sigma_{\|}, \sigma_{\perp}\right)$$

$$\frac{\partial G}{\partial y_c} = \left\{\left(\frac{1}{\sigma_{\perp}^2}+\left(\frac{1}{\sigma_{\|}^2}-\frac{1}{\sigma_{\perp}^2}\right)\sin^2\left(\frac{\pi}{180}\theta\right)\right)(y-y_c)\right.$$
$$\left.+\frac{1}{2}(x-x_c)\sin\left(\frac{\pi}{90}\theta\right)\left(\frac{1}{\sigma_{\|}^2}-\frac{1}{\sigma_{\perp}^2}\right)\right\} a.E\left(x_c, y_c, \theta, \sigma_{\|}, \sigma_{\perp}\right)$$

$$\frac{\partial G}{\partial \theta} = \left\{\frac{\pi}{180}\left(\frac{1}{\sigma_{\|}^2}-\frac{1}{\sigma_{\perp}^2}\right)\cos\left(\frac{\pi}{180}\theta\right)\sin\left(\frac{\pi}{180}\theta\right)(x-x_c)^2\right.$$
$$-\frac{\pi}{180}\left(\frac{1}{\sigma_{\|}^2}-\frac{1}{\sigma_{\perp}^2}\right)\cos\left(\frac{\pi}{180}\theta\right)\sin\left(\frac{\pi}{180}\theta\right)(y-y_c)^2$$
$$\left.-\frac{\pi}{180}(x-x_c)(y-y_c)\cos\left(\frac{\pi}{90}\theta\right)\left(\frac{1}{\sigma_{\|}^2}-\frac{1}{\sigma_{\perp}^2}\right)\right\} a.E\left(x_c, y_c, \theta, \sigma_{\|}, \sigma_{\perp}\right)$$
$$= \frac{\pi}{360}\left(\frac{1}{\sigma_{\|}^2}-\frac{1}{\sigma_{\perp}^2}\right)\left\{\left[(x-x_c)^2-(y-y_c)^2\right]\sin\left(\frac{\pi}{90}\theta\right)\right.$$
$$\left.-2(x-x_c)(y-y_c)\cos\left(\frac{\pi}{90}\theta\right)\right\} a.E\left(x_c, y_c, \theta, \sigma_{\|}, \sigma_{\perp}\right)$$

$$\frac{\partial G}{\partial \sigma_{\|}} = \frac{\left[(x-x_c)\cos\left(\frac{\pi}{180}\theta\right)+(y-y_c)\sin\left(\frac{\pi}{180}\theta\right)\right]^2}{\sigma_{\|}^3} a.E\left(x_c, y_c, \theta, \sigma_{\|}, \sigma_{\perp}\right)$$

$$\frac{\partial G}{\partial \sigma_\perp} = \frac{\left[-\left(x - x_c\right)\sin\left(\frac{\pi}{180}\theta\right) + \left(y - y_c\right)\cos\left(\frac{\pi}{180}\theta\right)\right]^2}{\sigma_\perp^3} a.E\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right)$$

$$\frac{\partial G}{\partial a} = E\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right)$$

$$\frac{\partial G}{\partial o} = \mathbf{1}$$

2 Normalized 2D Gaussian pattern

Formulation

$$
\begin{aligned}
G\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp, a, o\right) &= \frac{a}{2\pi\sigma_\parallel\sigma_\perp}.\exp\left\{-\frac{1}{2}\left[\left[\frac{(x-x_c)\cos\left(\frac{\pi}{180}\theta\right)+(y-y_c)\sin\left(\frac{\pi}{180}\theta\right)}{\sigma_\parallel}\right]^2\right.\right.\\
&\quad\left.\left.+\left[\frac{-(x-x_c)\sin\left(\frac{\pi}{180}\theta\right)+(y-y_c)\cos\left(\frac{\pi}{180}\theta\right)}{\sigma_\perp}\right]^2\right]\right\}+o\\
&= \frac{a}{2\pi\sigma_\parallel\sigma_\perp}.\exp\left\{-\frac{1}{2}\left[\left(\frac{\cos^2\left(\frac{\pi}{180}\theta\right)}{\sigma_\parallel^2}+\frac{\sin^2\left(\frac{\pi}{180}\theta\right)}{\sigma_\perp^2}\right)(x-x_c)^2\right.\right.\\
&\quad+\left(\frac{\sin^2\left(\frac{\pi}{180}\theta\right)}{\sigma_\parallel^2}+\frac{\cos^2\left(\frac{\pi}{180}\theta\right)}{\sigma_\perp^2}\right)(y-y_c)^2\\
&\quad\left.\left.+2\left(x-x_c\right)\left(y-y_c\right)\cos\left(\frac{\pi}{180}\theta\right)\sin\left(\frac{\pi}{180}\theta\right)\left(\frac{1}{\sigma_\parallel^2}-\frac{1}{\sigma_\perp^2}\right)\right]\right\}+o\\
&= \frac{a}{2\pi\sigma_\parallel\sigma_\perp}.\exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_\perp^2}+\left(\frac{1}{\sigma_\parallel^2}-\frac{1}{\sigma_\perp^2}\right)\cos^2\left(\frac{\pi}{180}\theta\right)\right)(x-x_c)^2\right.\right.\\
&\quad+\left(\frac{1}{\sigma_\perp^2}+\left(\frac{1}{\sigma_\parallel^2}-\frac{1}{\sigma_\perp^2}\right)\sin^2\left(\frac{\pi}{180}\theta\right)\right)(y-y_c)^2\\
&\quad\left.\left.+\left(x-x_c\right)\left(y-y_c\right)\sin\left(\frac{\pi}{90}\theta\right)\left(\frac{1}{\sigma_\parallel^2}-\frac{1}{\sigma_\perp^2}\right)\right]\right\}+o\\
&= \frac{a}{2\pi\sigma_\parallel\sigma_\perp}.E\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right)+o\\
&= a.\tilde{E}\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right)+o
\end{aligned}
$$

with

$$
\begin{aligned}
E\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right) = \ &\exp\left\{-\frac{1}{2}\left[\left[\frac{(x-x_c)\cos\left(\frac{\pi}{180}\theta\right)+(y-y_c)\sin\left(\frac{\pi}{180}\theta\right)}{\sigma_\parallel}\right]^2\right.\right.\\
&\left.\left.+\left[\frac{-(x-x_c)\sin\left(\frac{\pi}{180}\theta\right)+(y-y_c)\cos\left(\frac{\pi}{180}\theta\right)}{\sigma_\perp}\right]^2\right]\right\}
\end{aligned}
$$

and

$$\tilde{E}\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right) = \frac{1}{2\pi\sigma_\parallel\sigma_\perp}.E\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right)$$

Partial derivatives

$$\begin{aligned}
\frac{\partial G}{\partial x_c} = & \left\{\left(\frac{1}{\sigma_\perp^2} + \left(\frac{1}{\sigma_\parallel^2} - \frac{1}{\sigma_\perp^2}\right)\cos^2\left(\frac{\pi}{180}\theta\right)\right)(x - x_c)\right. \\
& \left. + \frac{1}{2}(y - y_c)\sin\left(\frac{\pi}{90}\theta\right)\left(\frac{1}{\sigma_\parallel^2} - \frac{1}{\sigma_\perp^2}\right)\right\} a.\tilde{E}\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right) \\
\frac{\partial G}{\partial y_c} = & \left\{\left(\frac{1}{\sigma_\perp^2} + \left(\frac{1}{\sigma_\parallel^2} - \frac{1}{\sigma_\perp^2}\right)\sin^2\left(\frac{\pi}{180}\theta\right)\right)(y - y_c)\right. \\
& \left. + \frac{1}{2}(x - x_c)\sin\left(\frac{\pi}{90}\theta\right)\left(\frac{1}{\sigma_\parallel^2} - \frac{1}{\sigma_\perp^2}\right)\right\} a.\tilde{E}\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right) \\
= & \frac{\pi}{360}\left(\frac{1}{\sigma_\parallel^2} - \frac{1}{\sigma_\perp^2}\right)\left\{\left[(x - x_c)^2 - (y - y_c)^2\right]\sin\left(\frac{\pi}{90}\theta\right)\right. \\
& \left. - 2(x - x_c)(y - y_c)\cos\left(\frac{\pi}{90}\theta\right)\right\} a.\tilde{E}\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right)
\end{aligned}$$

$$\frac{\partial G}{\partial \sigma_\parallel} = \left(\frac{\left[(x - x_c)\cos\left(\frac{\pi}{180}\theta\right) + (y - y_c)\sin\left(\frac{\pi}{180}\theta\right)\right]^2}{\sigma_\parallel^3} - \frac{1}{\sigma_\parallel}\right) a.\tilde{E}\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right)$$

$$\frac{\partial G}{\partial \sigma_\perp} = \left(\frac{\left[-(x - x_c)\sin\left(\frac{\pi}{180}\theta\right) + (y - y_c)\cos\left(\frac{\pi}{180}\theta\right)\right]^2}{\sigma_\perp^3} - \frac{1}{\sigma_\perp}\right) a.\tilde{E}\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right)$$

$$\frac{\partial G}{\partial a} = \tilde{E}\left(x_c, y_c, \theta, \sigma_\parallel, \sigma_\perp\right)$$

$$\frac{\partial G}{\partial o} = \mathbf{1}$$

3 Symmetric case

Formulation

$$\begin{aligned} G\left(x_c, y_c, \sigma, a, o\right) &= a.\exp\left(-\frac{1}{2}\frac{\left(x-x_c\right)^2+\left(y-y_c\right)^2}{\sigma^2}\right)+o \\ &= a.E\left(x_c, y_c, \sigma\right)+o \end{aligned}$$

with

$$E\left(x_c, y_c, \sigma\right) = \exp\left(-\frac{1}{2}\frac{\left(x-x_c\right)^2+\left(y-y_c\right)^2}{\sigma^2}\right)$$

Partial derivatives

$$\frac{\partial G}{\partial x_c} = \frac{x-x_c}{\sigma^2}.a.E\left(x_c, y_c, \sigma\right)$$

$$\frac{\partial G}{\partial y_c} = \frac{y-y_c}{\sigma^2}.a.E\left(x_c, y_c, \sigma\right)$$

$$\frac{\partial G}{\partial \sigma} = \frac{\left(x-x_c\right)^2+\left(y-y_c\right)^2}{\sigma^3}.a.E\left(x_c, y_c, \sigma\right)$$

$$\frac{\partial G}{\partial a} = E\left(x_c, y_c, \sigma\right)$$

$$\frac{\partial G}{\partial o} = \mathbf{1}$$

4 Normalized symmetric case

Formulation

$$\begin{aligned} G\left(x_c, y_c, \sigma, a, o\right) &= \frac{a}{2\pi\sigma^2}.\exp\left(-\frac{1}{2}\frac{\left(x-x_c\right)^2+\left(y-y_c\right)^2}{\sigma^2}\right)+o \\ &= \frac{a}{2\pi\sigma^2}.E\left(x_c, y_c, \sigma\right)+o \\ &= a.\tilde{E}\left(x_c, y_c, \sigma\right)+o \end{aligned}$$

with

$$E\left(x_c, y_c, \sigma\right) = \exp\left(-\frac{1}{2}\frac{\left(x - x_c\right)^2 + \left(y - y_c\right)^2}{\sigma^2}\right)$$

and

$$\tilde{E}\left(x_c, y_c, \sigma\right) = \frac{1}{2\pi\sigma^2}.E\left(x_c, y_c, \sigma\right)$$

Partial derivatives

$$\frac{\partial G}{\partial x_c} = \frac{x - x_c}{\sigma^2}.a.\tilde{E}\left(x_c, y_c, \sigma\right)$$

$$\frac{\partial G}{\partial y_c} = \frac{y - y_c}{\sigma^2}.a.\tilde{E}\left(x_c, y_c, \sigma\right)$$

$$\frac{\partial G}{\partial \sigma} = \left(\frac{\left(x - x_c\right)^2 + \left(y - y_c\right)^2}{\sigma^3} - \frac{2}{\sigma}\right).a.\tilde{E}\left(x_c, y_c, \sigma\right)$$

$$\frac{\partial G}{\partial a} = \tilde{E}\left(x_c, y_c, \sigma\right)$$

$$\frac{\partial G}{\partial o} = \mathbf{1}$$