Description of a first order multi-plane model. The different planes are supposed to be independant and interact with the incoming plane wave by assuming that this latter was not modified by the upstream planes.

Notations:

- $A \times B$: matrix multiplication
- $A.B$: element wise product
- $\frac{A}{B}$: element wise division
- e^{A}: element wise exponentiation
- $A^{\star}$: hermitian transposition
- $\overline{A}$: complex conjugate
- $\mathcal{R}\left(A\right)$: element wisereal part of a complex matrix
- P_{z_p}: nb_{plane} propagators on distances equal to z_p
- θ: variable with nb_{plane} complex planes θ_p describing the complex optical length of the different planes
- pad: padding operators
- U_{BP}: background wavefront

1 Coherent interferences

Direct model:

$$\begin{aligned} U_{tot}\left(\theta\right) &= U_{BP} + \sum_{p=1}^{nb_{plane}} e^{-ikz_p}\left(pad^{-1}\left(P_{z_p} \circledast pad\left(e^{\frac{2i\pi}{\lambda}\theta_p} - 1\right)\right)\right) \\ &= U_{BP} + \sum_{p=1}^{nb_{plane}} O_p \times \left(e^{\frac{2i\pi}{\lambda}\theta_p} - 1\right) \end{aligned}$$

with $O_p = e^{-ikz_p} pad^{-1} \circ P_{z_p} \circledast pad$ a linear operator.

Application of the hermitian transposition of the Jacobian at a given point θ for a given plane p:

$$J_p^{\star}\left(\theta\right) = -\frac{2i\pi}{\lambda} e^{-\frac{2i\pi}{\lambda}\theta_p}.O_p^{\star}$$

2 Incoherent interferences

Note on the interference of two incoherent waves U_1 and U_2:

$$\begin{aligned}|U_{tot}|^2 &= (U_1 + U_2)\,\overline{(U_1 + U_2)} \\ &= |U_1|^2 + |U_2|^2 + 2\mathcal{R}\left(\overline{U_1}U_2\right)\end{aligned}$$

$2\mathcal{R}\left(\overline{U_1}U_2\right)$ is the interference term. For coherent waves, its temporal average is not null. But it disappears with temporal integration for incoherent waves.

Let's now assume nb_{plane} incoherent waves U_p but coherent with a background wave U_{BP}. Let's determine the resulting intensity:

$$\begin{aligned}|U_{tot}|^2 &= \left(U_{BP} + \sum_{p=1}^{nb_{plane}} U_p\right)\overline{\left(U_{BP} + \sum_{p=1}^{nb_{plane}} U_p\right)} \\ &= |U_{BP}|^2 + \sum_{p=1}^{nb_{plane}} \left(|U_p|^2 + 2\mathcal{R}\left(\overline{U_{BP}}U_p\right)\right) + \sum_{p\neq p'} \mathcal{R}\left(\overline{U_p}U_{p'}\right)\end{aligned}$$

According to the previous remark, $\forall\,(p,p')$, it comes $\mathcal{R}\left(\overline{U_p}U_{p'}\right) = 0$. Then :

$$\begin{aligned}|U_{tot}|^2 &= (1 - nb_{plane})\,|U_{BP}|^2 + \sum_{p=1}^{nb_{plane}} \left(|U_p|^2 + 2\mathcal{R}\left(\overline{U_{BP}}U_p\right) + |U_{BP}|^2\right) \\ &= (1 - nb_{plane})\,|U_{BP}|^2 + \sum_{p=1}^{nb_{plane}} |U_p + U_{BP}|^2\end{aligned}$$

The direct model for incoherent waves is then:

$$\begin{aligned}|U_{tot}(\theta)|^2 &= (1 - nb_{plane})\,|U_{BP}|^2 + \\ &cdots \\ &\qquad \sum_{p=1}^{nb_{plane}} \left|U_{BP} + e^{-ikz_p}\left(pad^{-1}\left(P_{z_p} \circledast pad\left(e^{\frac{2i\pi}{\lambda}\theta_p} - 1\right)\right)\right)\right|^2 \\ &= (1 - nb_{plane})\,|U_{BP}|^2 + \sum_{p=1}^{nb_{plane}} \left|U_{BP} + O_p \times \left(e^{\frac{2i\pi}{\lambda}\theta_p} - 1\right)\right|^2\end{aligned}$$

with $O_p = e^{-ikz_p} pad^{-1} \circ P_{z_p} \circledast pad$ a linear operator.

Application of the hermitian transposition of the Jacobian at a given point θ for a given plane p computed in y:

$$J_p^{\star}(\theta) = -\frac{2i\pi}{\lambda} \frac{\overline{U_{BP} + O_p \times \left(e^{\frac{2i\pi}{\lambda}\theta_p} - 1\right)}}{\left|U_{BP} + O_p \times \left(e^{\frac{2i\pi}{\lambda}\theta_p} - 1\right)\right|} . e^{-\frac{2i\pi}{\lambda}\theta_p} . O_p^{\star}$$

Important note: this approach can correctly deal with the spatial coherence length which can be different for each plane.

3 Cost function

It is possible to take into account a global spatial coherence length by convolving the simulated intensity $|U_{tot}|^2$ by a filter F_{coh} corresponding to a diaphragm of the source seen by the sample and the sensor. This is a strong approximation as it is supposed to be equal for all the planes because it is applied after their interference. In the following, F_{coh} being a linear operator, it is seen as its matrix shape.

Intensity cost function for a given θ and intensity data I_d:

$$D(\theta) = \left\| F_{coh} \times |U_{tot}(\theta)|^2 - I_d \right\|_W^2$$

Getting inspered by the appendix D of my PhD, let's now find the first order development of $D(\theta + \delta\theta)$ according to $\delta\theta$. Assuming that $\delta\theta$ is real by decomposing the complex values on their real and imaginary part, it comes:

$$
\begin{aligned}
D\left(\theta+\delta\theta\right) &= \left\|F_{coh}\times\left|U_{tot}\left(\theta+\delta\theta\right)\right|^{2}-I_{d}\right\|_{W}^{2}\\
&= \left\|F_{coh}\times\left|U_{tot}\left(\theta\right)+J\left(\theta\right)\times\delta\theta\right|^{2}-I_{d}\right\|_{W}^{2}\\
&= \Big\|F_{coh}\times\Big(\left|U_{tot}\left(\theta\right)\right|^{2}+\left|J\left(\theta\right)\times\delta\theta\right|^{2}+\cdots\\
&\qquad 2\mathcal{R}\left(\overline{U_{tot}}\left(\theta\right).\left(J\left(\theta\right)\times\delta\theta\right)\right)\Big)-I_{d}\Big\|_{W}^{2}\\
&= \left\|F_{coh}\times\left|U_{tot}\left(\theta\right)\right|^{2}-I_{d}\right\|_{W}^{2}+\cdots\\
&\qquad 2\Big\langle F_{coh}\times\left|U_{tot}\left(\theta\right)\right|^{2}-I_{d},F_{coh}\times\cdots\\
&\qquad 2\mathcal{R}\left(\overline{U_{tot}}\left(\theta\right).\left(J\left(\theta\right)\times\delta\theta\right)\right)\Big\rangle_{W,\mathbb{R}}+o\left(\left\|\delta\theta\right\|\right)\\
&= D\left(\theta\right)+4\mathcal{R}\Big\langle F_{coh}\times\left|U_{tot}\left(\theta\right)\right|^{2}-I_{d},F_{coh}\times\cdots\\
&\qquad \overline{U_{tot}}\left(\theta\right).\left(J\left(\theta\right)\times\delta\theta\right)\Big\rangle_{W,\mathbb{C}}+o\left(\left\|\delta\theta\right\|\right)\\
&= D\left(\theta\right)+4\mathcal{R}\Big\langle W.\left(F_{coh}\times\left|U_{tot}\left(\theta\right)\right|^{2}-I_{d}\right),F_{coh}\times\cdots\\
&\qquad \overline{U_{tot}}\left(\theta\right).\left(J\left(\theta\right)\times\delta\theta\right)\Big\rangle_{\mathbb{C}}+o\left(\left\|\delta\theta\right\|\right)\\
&= D\left(\theta\right)+4\mathcal{R}\Big\langle U_{tot}\left(\theta\right).F_{coh}^{\star}\times W.\left(F_{coh}\times\left|U_{tot}\left(\theta\right)\right|^{2}-I_{d}\right),\cdots\\
&\qquad J\left(\theta\right)\times\delta\theta\Big\rangle_{\mathbb{C}}+o\left(\left\|\delta\theta\right\|\right)\\
&= D\left(\theta\right)+4\mathcal{R}\left\langle J^{\star}\left(\theta\right)\times\left[U_{tot}\left(\theta\right).F_{coh}^{\star}\times W.\cdots\right.\right.\\
&\qquad \left.\left.\left(F_{coh}\times\left|U_{tot}\left(\theta\right)\right|^{2}-I_{d}\right)\right],\delta\theta\right\rangle_{\mathbb{C}}+o\left(\left\|\delta\theta\right\|\right)\\
&= D\left(\theta\right)+\left\langle 4\mathcal{R}\left[J^{\star}\left(\theta\right)\times\left[U_{tot}\left(\theta\right).F_{coh}^{\star}\times W.\cdots\right.\right.\right.\\
&\qquad \left.\left.\left.\left(F_{coh}\times\left|U_{tot}\left(\theta\right)\right|^{2}-I_{d}\right)\right]\right],\delta\theta\right\rangle_{\mathbb{R}}+o\left(\left\|\delta\theta\right\|\right)
\end{aligned}
$$

The gradient is then:

$$\nabla D\left(\theta\right)=4\mathcal{R}\left[J^{\star}\left(\theta\right)\times\left[U_{tot}\left(\theta\right).F_{coh}^{\star}\times W.\left(F_{coh}\times\left|U_{tot}\left(\theta\right)\right|^{2}-I_{d}\right)\right]\right]$$

Note here that it assumes that θ is decomposed on its real and imaginary part. Using the formalism presented in the chapter 3, section 3.3 for my PhD, it is possible to directly express the gradient in terms of complex values:

$$\nabla D\left(\theta\right)=4J^{\star}\left(\theta\right)\times\left[U_{tot}\left(\theta\right).F_{coh}^{\star}\times W.\left(F_{coh}\times\left|U_{tot}\left(\theta\right)\right|^{2}-I_{d}\right)\right]$$