CHAPTER 7

STRENGTH AND FAILURE THEORIES

7.1 Introduction

In the past sections focus has been directed upon the functional requirements of beam, plate and shell structural elements as subjected to particular loading environments. The material presented in this section now addresses the broader based objective of having acquired a knowledge of the load analysis methodology, how can one now apply this knowledge to design a structural system to ensure a potentially safe design.

This objective, for composite structures design, is captured in the closed loop stress design profile shown below in Figure 7.1. It should be noted that as with any structural design, a key feature in the design is the selection of an appropriate failure criterion and the inherent iterative nature of the design process proper. In addition, for the case of monolithic materials such as metals it is sufficient to use one observable metric such as the ultimate tensile, compressive or shear stress to describe failure. For composites, however, the structural engineer is faced with the dilemma of selecting a suitable failure criterion based upon a number of observable stress metrics.

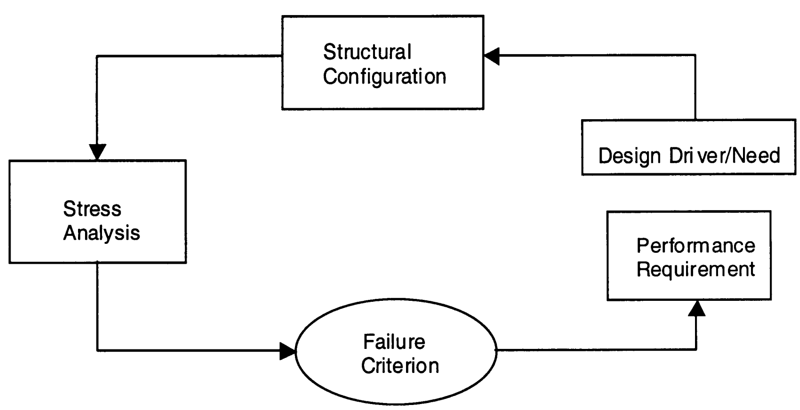


FIGURE 7.1. Closed Loop Stress Design Profile.

Thus one of the most difficult and challenging subjects to which one is exposed to in the mechanics of advanced composite materials involves finding a suitable failure criterion for these systems. This is compounded by our still limited grasp of understanding and predicting adequately the failure of all classes of monolithic materials to which we seek recourse for guidance in establishing descriptive failure criterion for a composite system. To a large degree the development of such criteria must be associated with philosophical notions of what the concept of failure is about. For example, in most instances failure is perceived to be separation of structural components or material parts of components. This of course need not be the case since the function of the material and/or component may be the design driver and thus, for example, excessive wear in an axle joint may produce a kinematic motion no longer representative of a key design feature. In addition, any micromechanical or substructural failure features associated with early on failure initiation such as flaws and voids, surface imperfections, and built-in residual stresses are generally neglected in these design type approaches to failure. These mechanisms would serve as design drivers for initiating damage and/or degradation in the composite. Thus, failure identification can best be classified along a spectrum of disciplines and in the particular case of composites further subjugated to different levels of definition of failure dependent upon the level of material characterization. To this end, Figure 7.2 appears useful for focusing attention on the different levels of failure characterization and discipline linkage necessary for identity with each failure type. We focus attention in this section on laminate failure criteria, which are based upon lamina and micromechanical failure initiators. We do not address in this section the more global problem of failure at joints and attachments.

Failure Characterization of Composites

Failure Characterization	Characterization of Disciplines: Mechanistic		Characterization of Disciplines: Phenomenological (Level of Observation)
Physics of Failure			Molecular
Micromechanics of Failure	Molecular →	Physics	→ Fiber-Matrix
Lamina Failure Criterion	Fiber-Matrix →	Stress Analysis Micromechanics	→ Homogeneous Lamina
Laminate Failure Criterion	Homogeneous Lamina →	Stress Analysis Laminate Plate Theory	→ Homogeneous Laminate
Structural Strength	Homogeneous Laminate →	Structural Analysis	→ Structure

FIGURE 7.2. Classification of composite failure characterization disciplines.

At this point it is worthwhile to identify the important features and mechanisms controlling the microfailure of composite systems, which produce the damage and/or degradation leading to failure in composite laminates. First, it should be emphasized that the matrix and reinforcing fibers, the primary constituents of a composite, have in general widely different strength characteristics. Also, the interface between the fiber and matrix is known to exhibit a response behavior different than that of the bulk matrix in general. The presence of flaws or defects, introduced into the material system during fabrication, may act as stress raisers or failure initiators. Thus, any approach to construction of a micromechanical strength theory should by necessity take into account the influence of such factors. Keeping these thoughts in mind, it appears feasible to characterize failure at the micro level by introducing such local failure modes as

Fiber dominated failure

(Breakage, microbuckling, dewetting)

Bulk matrix dominated failures

(Voids, crazing)

Interface/flaw dominated failures

(Crack propagation, edge delamination)

Once again, such factors while being important failure initiators, are not considered as the primary or causal factors when discussing failure in a global sense. Therefore, as engineers we seek out an observational level of failure to which we can readily relate and feel comfortable with in describing the appropriate failure mechanisms. In this regard we generally attempt to specify lamina failure for an anisotropic unidirectional composite, or alternatively lamina as existing in composite orthotropic laminates.

To begin this discussion, it has been observed that failure of a laminate consisting of a number of plies oriented in different directions relative to the loading configuration occurs gradually. This occurs due to the fact that an individual lamina failure causes a redistribution of stresses within the remaining laminae of the laminate. Although the failure modes can be either fiber dominated, matrix dominated or interface/flaw dominated, these basic mechanisms would appear as more important features to material scientists or material designers. For structural designers, however, the important element to consider is the lamina, made of a chosen fiber-matrix system, which is necessary for strength and failure analysis.

Before describing the failure of isotropic materials, as a precursor to establishing failure theories for composites, it is worthwhile to indicate some relevant and distinctive features concerning the strength characteristics of composite materials. A composite lamina is known to exhibit an anisotropic strength behavior, that is, the strength is directionally dependent and thus its tensile and compressive strengths are found to be widely different. In addition, the orientation of the shear stresses with respect to the fiber direction in the lamina has been observed to have a significant influence on its strength. Finally, another consideration that merits attention for laminate failure, is that the final fracture depends not only on a failure mode or the number of interactive failure modes occurring but also depends on the failure mode which dominates the failure process. These key features of the strength and failure of composites make the subject matter both complex and challenging.

Thus, while we are able to describe the failure of isotropic materials by an allowable stress field associated with the ultimate tensile, compressive and/or shear strength of the material, the corresponding anisotropic (orthotropic) material requires knowledge of at least five principle stresses. These are the longitudinal tensile and compressive stresses, the transverse tensile and compressive stresses, and the corresponding shear stress. In order to attack a failure problem in composites, however, recourse to monolithic material or metals failure criterion are relied upon to serve as role models for both modification and suitability in predicting failure of laminates. It is therefore worthwhile to review some of the classical failure criteria associated with homogeneous isotropic materials to serve as a baseline reference to development of appropriate criteria for the case of failure of the homogeneous anisotropic or orthotropic single-ply lamina. Generally speaking, these combined loading failure (strength) theories can be categorized into three basic types of criteria as follows:

- Stress Dominated
- Strain Dominated
- Interactive

Geometrically speaking, the above criteria can be interpreted in terms of so-called failure envelopes as noted in the accompanying Table 7.1. Further, each of the failure criteria referred to in Table 7.1 can be redefined in either stress or strain space according to specific tabular schemes. We first examine, however, some general comments as related to classical developments of established failure criteria for metals and then proceed to examine these criteria further in the case of composites.

TABLE 7.1. Geometric Interpretation of Failure

One Stress Metric – Point Failure Envelope
Two Stress Metric – Two Dimensional Failure Envelope
N Dimensional Stress Metric – N Dimensional Failure Envelope

7.2 Failure of Monolithic Isotropic Materials

The ensuing brief discussion in which failure as defined by the occurrence of yielding or fracture is introduced in an attempt to establish a linkage to failure criteria used for anisotropic materials. This enables the reader to focus attention on the use of inductive approaches, based upon well founded contributions, for establishing innovative methodologies in new disciplines. To this end, one of the earliest failure criteria was suggested by Rankine (1858) [1] who proposed a theory for the yielding of homogeneous isotropic materials having unequal tensile and compressive yield strength values.

This theory known as the Maximum Principle Stress Theory simply states that when any state of stress exists in a structural element that exceeds the yield strength of the material in simple tension or compression then failure occurs. Stated mathematically in terms of principal stresses for the case of unequal tensile and compressive yield strengths in the general three dimensional case this theory can be given as:

$$\begin{aligned} &\sigma_{11} \le \sigma_{yp}^T \qquad \sigma_{11} \le \sigma_{yp}^C \\ &\sigma_{22} \le \sigma_{yp}^T \qquad \sigma_{22} \le \sigma_{yp}^C \\ &\sigma_{33} \le \sigma_{yp}^T \qquad \sigma_{33} \le \sigma_{yp}^C \end{aligned} \tag{7.1}$$

For two-dimensional stresses, $\sigma_{33} = 0$ then,

$$\begin{aligned} &\sigma_{11} \le \sigma_{yp}^T \qquad \sigma_{11} \le \sigma_{yp}^C \\ &\sigma_{22} \le \sigma_{yp}^T \qquad \sigma_{22} \le \sigma_{yp}^C \end{aligned} \tag{7.2}$$

If the material has the same yield point in tension and compression then,

$$\begin{aligned} \sigma_{11} &= \pm\sigma_{yp} \\ \sigma_{22} &= \pm\sigma_{yp} \end{aligned} \tag{7.3}$$

The Maximum Strain Criterion states that failure occurs when at any point in a structural element the maximum strain at that point reaches the yield value equal to that occurring in a simple uniaxial tension or compression test. This result can be stated mathematically by equating the principal strains using Hooke's Law to the uniaxial strain at yield. This results in the following set of equations in terms of principal stresses:

$$\begin{aligned} \sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33} &= \sigma_{yp}^T, \sigma_{yp}^C \\ \sigma_{22} - \nu\sigma_{33} - \nu\sigma_{11} &= \sigma_{yp}^T, \sigma_{yp}^C \\ \sigma_{33} - \nu\sigma_{11} - \nu\sigma_{22} &= \sigma_{yp}^T, \sigma_{yp}^C \end{aligned} \tag{7.4}$$

For the case of plane stress, $\sigma_{33} = 0$, the above equations simplify to

$$\begin{gathered} \sigma_{11} - \nu\sigma_{22} = \sigma_{yp}^T, \sigma_{yp}^C, \qquad \sigma_{22} - \nu\sigma_{11} = \sigma_{yp}^T, \sigma_{yp}^C \\ -\nu\left(\sigma_{11} + \sigma_{22}\right) = \sigma_{yp}^T, \sigma_{yp}^C \end{gathered} \tag{7.5}$$

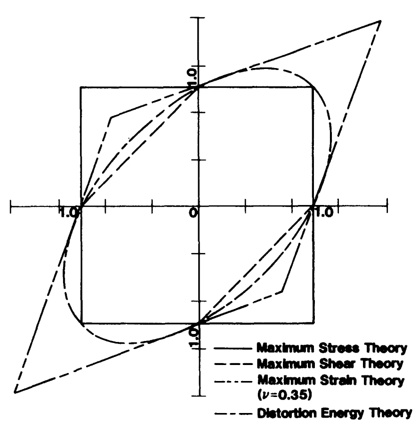
Two other important failure criteria for homogeneous isotropic materials are the Maximum Shear Stress Theory often referred to as Tresca's Theory [2] and the Distortion Energy Criterion often referred to as Von Mises Criterion [3].

The first theory, that is the so-called Maximum Shear Stress Theory, suggests that yielding will occur in a material when the maximum shear stress reaches a critical value, the maximum shear stress at yielding in a uniaxial stress state. Expressed quantitatively in terms of principal stresses this result can be stated as:

$$\begin{aligned} \sigma_{11} - \sigma_{22} &= \pm\sigma_{yp} \\ \sigma_{22} - \sigma_{33} &= \pm\sigma_{yp} \\ \sigma_{33} - \sigma_{11} &= \pm\sigma_{yp} \end{aligned} \tag{7.6}$$

For the special case of plane stress, when $\sigma_{33} = 0$, the expression simplifies to read:

$$\begin{aligned} \sigma_{11} - \sigma_{22} &= \pm\sigma_{yp} \\ \sigma_{22} &= \pm\sigma_{yp} \\ \sigma_{11} &= \pm\sigma_{yp} \end{aligned} \tag{7.7}$$



COMPARISON OF FAILURE THEORIES

FIGURE 7.3. Comparison of failure theories.

Corresponding to Tresca's yield criterion a second popular yield criterion for homogeneous isotropic materials was attributed to von Mises (1913). This theory known as the distortion energy criterion also assumes that the principal influence on material yield is the maximum shearing stress. In this theory, the elastic energy of a material can be considered to consist of a dilatational and distortional component. It is recognized that the dilatational part of the total work is dependent upon a hydrostatic stress state, which does not produce yielding at normal pressures in homogeneous isotropic materials. Thus, the remaining or distortional energy can be expressed in terms of principal stresses and written as:

$$\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11}\sigma_{22} - \sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11} = \sigma_{yp}^2 \tag{7.8}$$

A careful examination of this criterion with the previous Maximum Shearing Stress Theory indicates that at most the two theories deviate by no more than fifteen percent.

These four failure criterion depicted in Figure 7.3, while not exhaustive of all of the types proposed for metals, are representative models of the types often used and as such have led to the anisotropic failure criterion highlighted in the following section.

7.3 Anisotropic Strength and Failure Theories

Attention is now focused on the development of useful macroscopic yield criteria as extended from metals behavior to the case of composite (anisotropic) materials. A more comprehensive review of this subject and the individual theories can be found in Ref. [4]. In the following paragraphs, those theories, which have been most often cited in the composites literature, are reviewed in terms of their origin and salient features for the reader. This review serves as an important interface between the classical theories previously described and as a means for comparison for various composite systems analytically and experimentally.

As mentioned earlier in the text, a fundamental requirement in the design of structural systems from structural elements such as beams, plates and shells is a knowledge of the strength and/or failure of such elements subjected to complex loading states. The strength and failure of these structural elements can be associated with the yield strength of the material or correspondingly its ultimate strength. For brittle materials the ultimate or failure strength appears to be a more suitable base line reference value to determine failure while for ductile type materials the yield strength appears a more useful criterion. In the case of anisotropic materials and in particular high performance composites, the fiber reinforcements are generally elastic up to failure while the matrices can be either of the brittle or ductile type. The latter are mainly associated with metal matrix composites (MMC's), while the majority of advanced composite systems use thermosetting (epoxy) matrices for their binders. Thus, all subsequent reference to failure in this chapter as related to high performance composites will be based upon reference to the ultimate strength of the material. This should, of course, be altered if ductile matrices or fiber types other than brittle elastic are used. One other unique feature of anisotropic material failure is that modes of failure are important considerations in the design process. This is due to the fact that uniqueness in yield or failure is associated with the direction of testing.

We now proceed t o examine anisotropic failure criteria in the light of the aforementioned remarks, and as extensions of failure theories based upon the three basic types established for monolithic materials, that is,

- Stress Dominated
- Strain Dominated
- Interactive

The historical development, as presented, reflects a somewhat selective one and the reader is referred to Ref. [4] for a more comprehensive discussion. In addition, the reader is cautioned that the theories, as constructed and referred to, do not include environmental factors such as temperature and humidity, and also do not include initial (residual) stresses that are usually present in composites.

7.3.1 MAXIMUM STRESS THEORY

Because of research in the forest products area, the Maximum Stress Theory was extended to orthotropic materials. It was stated that failure occurred when one or all of the orthotropic stress values exceed their maximum limits as obtained in uniaxial tension, compression, or pure shear stress tests, when the material is tested to failure. This can be stated analytically as:

$$\begin{aligned} \sigma_{11} &= X \\ \sigma_{22} &= Y \\ \sigma_{12} &= S \end{aligned} \tag{7.9}$$

Further extensions of this theory were presented by Stowell and Liu (1961) [5] and Kelly and Davies (1965) [6].

7.3.2 MAXIMUM STRAIN THEORY

The maximum strain theory states that failure occurs when the strain obtained along the principal material axes exceed their limiting values. Waddoups [7] utilized these conditions in assessing the failure strength of orthotropic materials expressing the results analytically as

$$\bar{\varepsilon}_{11} = \frac{\sigma_{11}}{\mathrm{E}_{11}} - \nu_{12}\frac{\sigma_{22}}{\mathrm{E}_{11}}$$

$$\bar{\varepsilon}_{22} = \frac{\sigma_{22}}{\mathrm{E}_{22}} - \frac{\nu_{12}}{\mathrm{E}_{11}} \qquad \bar{\gamma}_{12} = \frac{\sigma_{12}}{\mathrm{G}_{12}} \tag{7.10}$$

$$\bar{\varepsilon}_{33} = -\frac{\nu_{31}}{E_{33}}\sigma_{11} - \frac{\nu_{32}}{E_{33}}\sigma_{22}$$

The three principal strain values can be referred to modes of failure in the corresponding strain directions.

7.3.3 INTERACTIVE FAILURE THEORIES

One of the earliest interactive failure criterion for anisotropic materials was initiated by Hill [8]. This theory is a generalization of the isotropic yield behavior of ductile metals for the case of large strains. That is, as a metal is strained in a certain direction, for example in a rolling process, the material grains tend to become aligned and a self-induced anisotropy occurs. Hill thus formulated an interactive yield criterion for such materials, which can be written in terms of the stress components as:

$$\begin{aligned} &F(\sigma_{22}-\sigma_{33})^2 + G(\sigma_{33}-\sigma_{11})^2 + H(\sigma_{11}-\sigma_{22})^2 \\ &\quad + 2L\sigma_{23}^2 + 2M\sigma_{13}^2 + 2N\sigma_{12}^2 = 1 \end{aligned} \tag{7.11}$$

where the quantities F, G, H, L, M and N reflect the current state of material anisotropy.

For unidirectionally reinforced composites $M = N$, $G = H$, we obtain

$$\begin{aligned} &F(\sigma_{22}-\sigma_{33})^2 + G(\sigma_{33}-\sigma_{11})^2 + H(\sigma_{11}-\sigma_{22})^2 \\ &\quad + 2L\sigma_{23}^2 + 2M\left(\sigma_{31}^2 + \sigma_{12}^2\right) = 1. \end{aligned} \tag{7.12}$$

For a composite lamina or laminates in a plane state of stress $\sigma_{33} = 0$, the above equation reduces to

$$F\sigma_{22}^2 + G\sigma_{11}^2 + H(\sigma_{11}-\sigma_{22})^2 + 2N\sigma_{12}^2 = 1. \tag{7.13}$$

An extension of Hill's criterion to account for unequal tension and compression for anisotropic materials was introduced by Marin [9]. Stated in terms of principal stresses, failure (yielding) is assumed to occur when the following condition is satisfied

$$\begin{aligned} &(\sigma_{11}-a)^2 + (\sigma_{22}-b)^2 + (\sigma_{33}-c)^2 + q[(\sigma_{11}-a)(\sigma_{22}-b) \\ &\quad + (\sigma_{22}-b)(\sigma_{33}-c) + (\sigma_{33}-c)(\sigma_{11}-a)] = \sigma^2 \end{aligned} \tag{7.14}$$

The quantities a, b, c, q and r are experimentally determined parameters.

Marin's anisotropic failure criterion was further extended by Norris (1962) [10] who introduced nine stress components to define failure. These nine properties consist of three tensile, three compressive and three shear strength values. The equations, which must be satisfied for failure to occur, are thus given by:

$$\left(\frac{\sigma_{11}}{X}\right)^2+\left(\frac{\sigma_{22}}{Y}\right)^2-\frac{\sigma_{11}\sigma_{22}}{XY}+\left(\frac{\sigma_{12}}{S}\right)^2=1$$

$$\left(\frac{\sigma_{22}}{Y}\right)^2+\left(\frac{\sigma_{33}}{Z}\right)^2-\frac{\sigma_{22}\sigma_{33}}{YZ}+\left(\frac{\sigma_{23}}{T}\right)^2=1 \qquad (7.15)$$

$$\left(\frac{\sigma_{33}}{Z}\right)^2+\left(\frac{\sigma_{11}}{X}\right)^2-\frac{\sigma_{33}\sigma_{11}}{ZX}+\left(\frac{\sigma_{13}}{R}\right)^2=1$$

where the quantities X, Y and Z are the tensile or compressive yield strengths of the material which R, S and T are the corresponding shear strengths.

For the case of plane stress the above equations reduce to

$$\left(\frac{\sigma_{11}}{X}\right)^2+\left(\frac{\sigma_{22}}{Y}\right)^2-\frac{\sigma_{11}\sigma_{22}}{XY}+\left(\frac{\sigma_{12}}{S}\right)^2=1$$

$$\left(\frac{\sigma_{22}}{Y}\right)^2=1 \qquad \left(\frac{\sigma_{11}}{X}\right)^2=1 \qquad (7.16)$$

The plane stress results by Hill were simplified for the case of fiber reinforced composites by Azzi and Tsai [11] considering the composite to be transversely isotropic. Thus with $Z = Y$, the modified form of Hill's plane stress criterion can be written as

$$\left(\frac{\sigma_{11}}{X}\right)^2+\left(\frac{\sigma_{22}}{Y}\right)^2-\frac{\sigma_{11}\sigma_{22}}{X^2}+\left(\frac{\sigma_{12}}{S}\right)^2=1. \qquad (7.17)$$

The above result is equivalent to that obtained by Norris with the distinction that the latter's data is somewhat more general in accounting for difference in yield in both tension and compression.

A generalization of this failure criterion to incorporate the effects of brittle materials was considered by Hoffman [12]. In terms of stress components, failure occurs when the following equation is satisfied:

$$C_1(\sigma_{22}-\sigma_{33})^2+C_2(\sigma_{33}-\sigma_{11})^2+C_3(\sigma_{11}-\sigma_{22})^2+C_4\sigma_{11}+C_5\sigma_{22}$$
$$+C_6\sigma_{33}+C_7\tau_{23}^2+C_8\tau_{13}^2+C_9\sigma_{12}^2=1 \tag{7.18}$$

Quantities $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$ and C_9 are constants determined from material properties tests.

$$C_1=\frac{1}{2}\left[\frac{1}{Y_TY_C}+\frac{1}{Z_TZ_C}-\frac{1}{X_TX_C}\right]$$

$$C_2=\frac{1}{2}\left[\frac{1}{Z_TZ_C}+\frac{1}{X_TX_C}-\frac{1}{Y_TY_C}\right]$$

$$C_3=\frac{1}{2}\left[\frac{1}{X_TX_C}+\frac{1}{Y_TY_C}-\frac{1}{Z_TZ_C}\right]$$

$$C_4=\frac{1}{X_T}-\frac{1}{X_C} \qquad C_7=\frac{1}{T^2} \tag{7.19}$$

$$C_5=\frac{1}{Y_T}-\frac{1}{Y_C} \qquad C_8=\frac{1}{R^2}$$

$$C_6=\frac{1}{Z_T}-\frac{1}{Z_C} \qquad C_9=\frac{1}{S^2}$$

This theory also incorporates unequal tensile and compressive failure strengths as an inherent part of the analytical development. For the case of plane stress this equation reduces to:

$$\frac{\sigma_{11}^2}{X_TX_C}+\frac{\sigma_{22}^2}{Y_TY_C}-\sigma_{11}\sigma_{22}\left[\frac{1}{X_TX_C}+\frac{1}{Y_TY_C}\right]+\frac{X_C-X_T}{X_TX_C}\sigma_{11}$$
$$+\frac{Y_C-Y_T}{Y_TY_C}\sigma_{22}+\frac{\sigma_{12}^2}{S^2}=1 \tag{7.20}$$

A generalization of the Hoffman result to incorporate a more comprehensive definition for failure was later proposed by Tsai and Wu [13]. In this criterion failure is assumed to occur when the following equations are satisfied:

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j = 1 \quad (i, j = 1, 2, \ldots, 6)$$

$$F_{ii}F_{jj} - F_{ij}^2 \geq 0 \quad (i, j = 1, 2, \ldots, 6) \tag{7.21}$$

The quantities F_i and F_{ii} (i = 1, 2, 3) are related to the tensile and compressive yield strengths of the material while six shear tests both positive and negative are required to define the parameters F_4 through F_6 and F_{44} through F_{66}.

$$F_1 = \frac{1}{X_T} - \frac{1}{X_C} \qquad F_{11} = \frac{1}{X_T X_C}$$

$$F_2 = \frac{1}{Y_T} - \frac{1}{Y_C} \qquad F_{22} = \frac{1}{Y_T Y_C}$$

$$F_3 = \frac{1}{Z_T} - \frac{1}{Z_C} \qquad F_{33} = \frac{1}{Z_T Z_C}$$

$$F_4 = \frac{1}{T^+} - \frac{1}{T^-} \qquad F_{44} = \frac{1}{T^+ T^-}$$

$$F_5 = \frac{1}{R^+} - \frac{1}{R^-} \qquad F_{55} = \frac{1}{R^+ R^-}$$

$$F_6 = \frac{1}{S^+} - \frac{1}{S^-} \qquad F_{66} = \frac{1}{S^+ S^-}$$

Tsai and Hahn [14] suggest the following for the constant F_{12}

$$F_{12} = -\frac{1}{2\left(X_T X_C Y_T Y_C\right)^{1/2}}$$

The ultimate test for any of the criterion as discussed above is based upon its correlation with available test data. At the present time an adequate number of tests has not been made to ascertain any deterministic values for comparative purposes. However, a survey of failure criteria, which have been proposed for use in practice, have been featured in

Ref. [15]. The results of this survey indicate that the following failure criteria appear most popular in practical use today:

- Maximum Strain
- Maximum Stress
- Interactive Criteria
  - Tsai-Hill
  - Tsai-Wu
  - Hoffman

A number of these theories will be used in selected examples in the latter part of this chapter. We begin our discussion of failure criteria by first examining failure of single-ply unidirectional fiber composites and progress to multi-ply composites consisting of variable angle ply geometries and orientations with respect to complex stress fields.

7.4 Lamina Strength Theories

We first examine the single ply strength of uniaxial lamina subjected to tensile loads. At the level of analysis being discussed and due to the success of micromechanical principles in predicting the elastic properties of composites, similar principles have been advanced for predicting the overall strength of composites as well. The problem with using this approach is that in applying a deterministic approach to the case where probabilistic events predominate, we generally obtain an invalid result. This is due to the wide variety of fiber and/or fiber bundle dominated strengths exhibited in composite networks. Further, the effects of these load perturbations can result in high stress concentrations in the vicinity of the newly failed fibers or be propagated/dispersed to remote locations in the composite. This results in a failure model, which is both dependent on the relative strength levels in the different regions of the composite as well as involvement of interactive failure modes in a broad sense. Thus in the end, a requirement for well-planned experiments appears to be a necessary building block for understanding and predicting such failure phenomena. As an example, in the case of compressive loads both material and configurational failures need to be addressed. That is both the compressive strength and buckling strengths need to be considered. Local fiber instabilities identified by in-phase and out-of-phase buckling (1), and fiber eccentricities (2) also need to be considered. The corresponding transverse tension and compression loadings are generally dominated by the so-called matrix mode of failure. In the case of compressive transverse loads, there appears to be several microscopic shear failure modes active, these being in the plane of and perpendicular to the filaments. Longitudinal and transverse shear stresses in general also result in matrix dominated failure modes although there is some theoretical evidence to suggest an increase in the in-plane shear stress.

Since most structural components are subjected to complex loading states the question of what happens to structural elements under combined loads is an important question. In the preceding qualitative discussion of strength investigation for single-ply laminae it was mentioned that material characterization was possible by obtaining data through relatively simple uniaxial and shear tests. For the case of composites, a systematic experimental program to investigate failure for all possible complex stress states becomes prohibitive and recourse to other predictive methodologies appears necessary.

In order to establish guidelines for strength and failure of laminae, which are the building blocks of laminates, it is necessary to consider a number of fundamental features related to composite material behavior. To this end, it is important to point out that while we recognize the inherent microlevel mechanisms associated with failure in laminae we only consider failure to occur at the macrolevel. This observation is predicated to a large extent by the current state of the art in failure analysis which precludes an ability to quantitatively relate in any interactive fashion events occurring at the micro and macro level of analysis. Thus, for design purposes the engineer must resort to consideration of strength degradation as observed for laminae and relate these data subsequently to laminates.

Further, since we deal with laminae as our fundamental building units in composites we characterize their properties through means of either analytical methodologies (micromechanics) or experimental (empirical) data. The data as shown in Figures 7.4 and 7.5 serve as a means for obtaining the necessary uniaxial stress and strain allowables in tension, compression and shear. Since most laminae can be considered as either transversely isotropic or specially orthotropic, we generally need only four independent elastic compliances or stiffnesses and five strengths to define the material system.

Again, as mentioned previously a single lamina is considered to be in a plane state of stress and principal strengths are used as comparative strength criteria.

- Maximum Stress
- Maximum Strain
- Interactive Laws

Each of these theory types will be used in testing for the failure of laminae subjected to several states of stress.

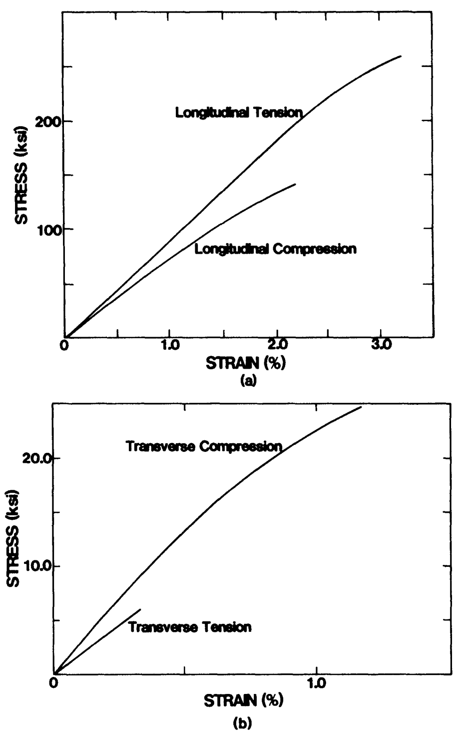
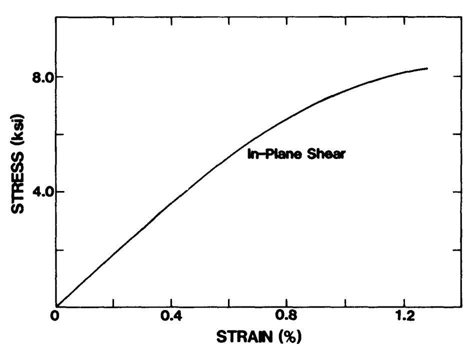


Figure 7.4.



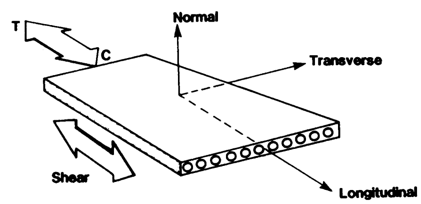
(c)

FIGURE 7.5. Typical stress-strain curves for S-glass fibers in an epoxy matrix.

(a) Unidirectional laminate with $E_{11} = 8.8 \times 10^6$ psi for longitudinal tension and $E_{11} = 7.2 \times 10^6$ psi for longitudinal compression.

(b) Unidirectional laminate with $E_{22} = 2.5 \times 10^6$ psi for transverse compression $E_{22} = 1.9 \times 10^6$ psi for transverse tension.

(c) In-plane shear with $G_{12} = 0.8 \times 10^6$ psi.



SINGLE LAMINA

FIGURE 7.6.

Example 1

We begin by examining a single stress applied to an angle ply lamina material made of S-glass/epoxy, with stress, $\sigma_x = 500\,\text{psi}$, making an angle, $\theta = 60^\circ$, with respect to the principal fiber directions. This loading is shown in Figure 7.7 along with the design properties corresponding to this material system as indicated in Table 7.2.

We now examine each of the failure laws in turn and in terms of the loading shown.

Maximum Stress

This theory states that failure will occur if any stress along the principal directions of the lamina exceeds the specified allowables. Analytically the following set of inequalities must be satisfied to ensure a fail safe design.

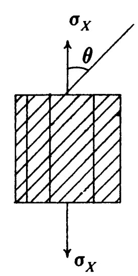


FIGURE 7.7.

TABLE 7.2. S-glass/epoxy.

$E_L = E_x = 8.42 \times 10^6$ psi
$E_T = E_Y = 2.00 \times 10^6$ psi
$G_{LT} = 0.77 \times 10^6$ psi
$\nu_{LT} = 0.293$
$\nu_{TL} = 0.067$
$\sigma_L^T = 280{,}000$ psi
$\sigma_L^C = 136{,}000$ psi
$\sigma_T^T = 4{,}000$ psi
$\sigma_T^C = 20{,}000$ psi
$\sigma_{LT}^S = 6{,}000$ psi
$V_f = 0.67$

Tension		Compression
$\sigma_L < \sigma_{LU}^t$		$\sigma_L < \sigma_{LU}^c$
$\sigma_T < \sigma_{TU}^t$		$\sigma_T < \sigma_{TU}^c$
	$\sigma_{LT} < \sigma_{LTU}$	

For the present example and in order to test the failure theory, it is first necessary to transform the given stress σ_x along the direction of the principal material direction. Using the transformation equations we can thus write in a general form the following results:

$$\sigma_L = \sigma_X \cos^2\theta = 125 \text{ psi}$$

$$\sigma_T = \sigma_X \sin^2\theta = 375 \text{ psi}$$

$$\sigma_{LT} = \sigma_X \sin\theta \cos\theta = 216.5 \text{ psi}$$

Substituting the given values for σ_X and θ we can test the computed stresses in the material directions against the allowable stresses obtaining,

$$\sigma_L = 125 < \sigma_{LU} = 136{,}000 \text{ psi}$$

$$\sigma_T = 375 < \sigma_{TU} = 4{,}000 \text{ psi}$$

$$\sigma_{LT} = 216.5 < \sigma_{LTU} = 6{,}000 \text{ psi}$$

The results obtained can also be cast in terms of non-dimensional stress values and plotted as shown in Figure 7.8.

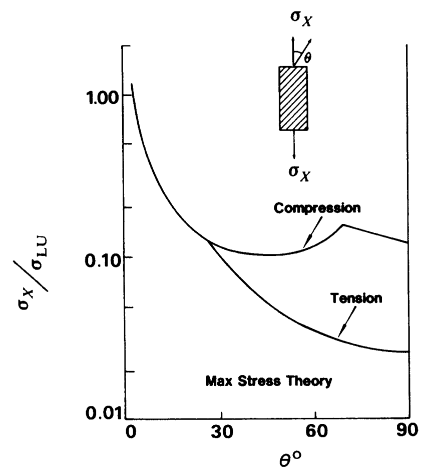


FIGURE 7.8. Analysis of an orthotropic lamina.

Maximum Strain

The maximum strain theory states that failure occurs when the strain along the material axes exceeds the allowable strain in the same principal direction as obtained in the same mode of testing. That is,

Tension		Compression
$\varepsilon_L < \varepsilon_{LU}^t$		$\varepsilon_L < \varepsilon_{LU}^c$
$\varepsilon_T < \varepsilon_{TU}^t$		$\varepsilon_T < \varepsilon_{TU}^c$
	$\gamma_{LT} < \gamma_{LTU}$	

Once again, for the orthotropic lamina subjected to a load σ_X, we make use of the transformed stresses as found previously in the principal material coordinate directions. In order to calculate the corresponding strains in the principal directions (material coordinates) we use Hooke's Law for orthotropic materials. Therefore, we can write:

$$\varepsilon_L = \frac{1}{E_L}\left[\cos^2\theta - \nu_{LT}\sin^2\theta\right]\sigma_X$$

$$\varepsilon_T = \frac{1}{\mathrm{E}_T}\left[\sin^2\theta - \nu_{TL}\cos^2\theta\right]\sigma_X$$

$$\gamma_{LT} = \frac{1}{\mathrm{G}_{LT}}\left[\sin\theta\cos\theta\right]\sigma_X$$

Substituting the given values of $\sigma_X, \theta, \nu_{LT}, \mathrm{E}_L, \mathrm{E}_T, \text{and}\ \mathrm{G}_{LT}$ from Table 7.2 we determine $\varepsilon_L, \varepsilon_T$ and γ_{LT}. These values are computed and tested against the allowable strain values which are given below,

$$\varepsilon_L < \varepsilon_{LU} \qquad \varepsilon_L = 0.000001 < \varepsilon_{LU} = 0.033$$

$$\varepsilon_T < \varepsilon_{TU} \qquad \varepsilon_T = 0.000183 < \varepsilon_{TU} = 0.002$$

$$\gamma_{LT} < \gamma_{LTU} \qquad \gamma_{LT} = 0.000281 < \gamma_{LTU} = 0.0078$$

The Maximum Strain Theory can also be graphically depicted and compared with the Maximum Stress theory for the example cited. The differences in the theoretical predictions are small, this due to the fact that the material is considered to behave as linearly elastic to failure and consequently such a plot is not shown.

Interactive Theory

The third theory is of the interactive type, and states that failure occurs under some combined (multiplicative and/or additive) set of stresses. For the current examples, we select one of the interactive criteria for use specifically the Azzi/Tsai type. This failure criterion can be stated analytically for the case of plane stress as:

$$\left(\frac{\sigma_L}{\sigma_{LU}}\right)^2 - \left(\frac{\sigma_L}{\sigma_{LU}}\right)\left(\frac{\sigma_T}{\sigma_{LU}}\right) + \left(\frac{\sigma_T}{\sigma_{TU}}\right)^2 + \left(\frac{\sigma_{LT}}{\sigma_{LTU}}\right)^2 < 1$$

In the present example for an applied stress σ_X, the above equation can be written as:

$$\left(\frac{\cos^2\theta}{\sigma_{LU}}\right)^2 - \left(\frac{\cos\theta\sin\theta}{\sigma_{LU}}\right)^2 + \left(\frac{\sin^2\theta}{\sigma_{TU}}\right)^2 + \left(\frac{\sin\theta\cos\theta}{\sigma_{LTU}}\right)^2 < \frac{1}{\sigma_X^2}$$

Using the numbers given in Table 7.2 this equation is written as:

$$\left(\frac{0.25}{280{,}000}\right)^2-\left(\frac{0.433}{280{,}000}\right)^2+\left(\frac{0.75}{4{,}000}\right)^2+\left(\frac{0.433}{6{,}000}\right)^2<\frac{1}{(500)^2}$$

As was done in the case of the Maximum Stress Theory, a similar graphical plot of the off axis composite strength can be presented. This is shown in Figure 7.9 for comparison with the Maximum Stress (Strain) Theory. A comparison of these theories leads to the fact that the Maximum Stress Theory as does the Maximum Strain predicts somewhat higher strength values. The major discrepancy between the theories occurs in the angle at which a transition from one failure mode to an alternative mode occurs (compare curves).

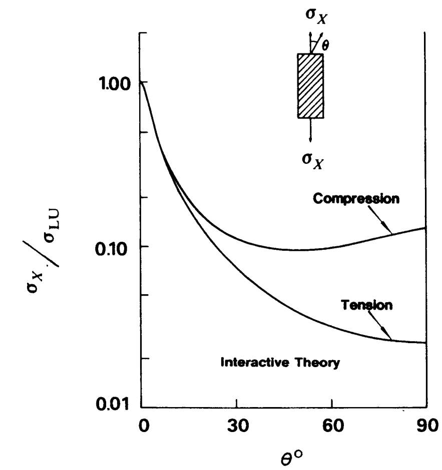


FIGURE 7.9.

Example 2

As a second example we consider the lamina composite shown in Figure 7.10 considering the material properties as given in Table 7.2.

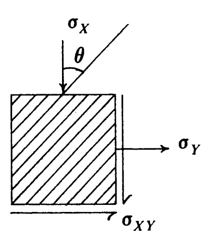


FIGURE 7.10.

Maximum Stress Theory

As stated previously, this theory predicts failure when any one of the ultimate (yield) strength values exceeds the corresponding allowable stresses in a principal axes direction. Thus, in order to ensure a safe design once again the following inequalities must be satisfied in both tension and compression.

Tension		Compression
$\sigma_L \leq \sigma^t_{LU}$		$\sigma_L \leq \sigma^c_{LU}$
$\sigma_T \leq \sigma^t_{TU}$		$\sigma_T \leq \sigma^c_{TU}$
	$\sigma_{LT} \leq \sigma_{LTU}$	

In the example given, the stresses as shown must be translated from the geometric axes to the principal material directions. This can be done through equilibrium considerations on the element or use of the transformation equations presented previously. Thus, we can use

$$\sigma_{11} = \sigma_X \cos^2\theta + \sigma_Y \sin^2\theta + 2\sigma_{XY} \cos\theta \sin\theta$$

$$\sigma_{22} = \sigma_X \sin^2\theta + \sigma_Y \cos^2\theta - 2\sigma_{XY} \cos\theta \sin\theta$$

$$\sigma_{12} = -\sigma_X \sin\theta \cos\theta + \sigma_Y \sin\theta \cos\theta + \sigma_{XY}\left(\cos^2\theta - \sin^2\theta\right)$$

or the transformation equations

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \\ T \\ \\ \end{bmatrix} \begin{bmatrix} \sigma_X \\ \sigma_Y \\ \sigma_{XY} \end{bmatrix}$$

Using the latter and substituting $\theta = 60^\circ$ into the transformation matrix we obtain

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 & \sqrt{3/2} \\ 3/4 & 1/4 & -\sqrt{3/2} \\ -\sqrt{3/4} & \sqrt{3/4} & -1/2 \end{bmatrix} \begin{bmatrix} -500 \\ 1000 \\ -200 \end{bmatrix} = \begin{bmatrix} 451.8 \\ 48.2 \\ 749.5 \end{bmatrix} \text{ psi}$$

Comparing the stresses calculated in the right-hand column vector with the allowable values indicates that the lamina is fail safe, that is

$$\sigma_L < \sigma'_{LU}$$

$$\sigma_T < \sigma'_{TU}$$

$$\sigma_{LT} < \sigma'_{LTU}$$

Importance of Shear Stresses

Some comment on the role of the shear stress in determining the strength of lamina needs to be addressed. For homogeneous isotropic materials the direction of the shear stress can be either positive of negative and is of little consequence in determining the strength of such materials. These simplistic arguments do not carry over to orthotropic lamina and composites. Consider the lamina consisting of (–45°) oriented fibers as shown in Figure 7.11, and loaded by means of positive and negative shear stresses respectively, the signs determined from generally accepted strength of materials conventions. It can be seen that each of these laminae leads to important differences in loads along the principal material coordinate axes. In particular, for the case of positive shear tensile stresses are developed in the transverse direction and compressive stresses in the fiber direction. For the case of negative shear the reverse loading situation occurs.

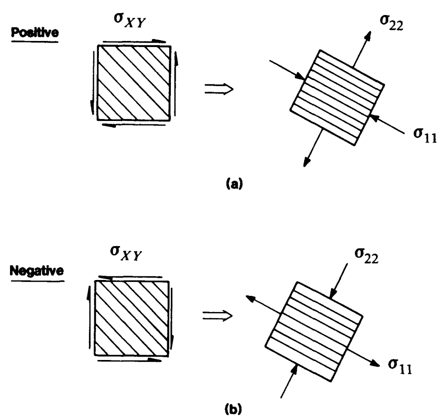


FIGURE 7.11.

Since shear strength of a lamina is controlled by the transverse strength of the composite, we can see that for the case of positive shear and negative shear fiber orientation we are led to lower predictions of apparent strength than when the reverse case prevails. Thus, the off-axis shear of laminae must be carefully examined in the light of the applied direction of the shear stress.

Maximum Strain Theory

This theory states that a fail safe design will exist when the following inequalities are satisfied:

Tension		Compression
$\varepsilon_{11} \le \varepsilon_{11}^{T}$		$\varepsilon_{11} \le \varepsilon_{11}^{C}$
$\varepsilon_{22} \le \varepsilon_{22}^{T}$		$\varepsilon_{22} \le \varepsilon_{22}^{C}$
	$\varepsilon_{12} \le \varepsilon_{12}$	

The Maximum Strain Theory is directly analogous to the Maximum Stress Theory in terms of correspondence in that stresses are simply replaced by allowable strain values in the fail safe procedure. This is predicated upon the assumption that the material remains elastic up to failure and thus one obtains the allowable strain from:

$$\varepsilon_{LU} = \frac{\sigma_{LU}}{E_L} \qquad \varepsilon'_{LU} = \frac{\sigma_{LU}}{E_L}$$

$$\varepsilon_{TU} = \frac{\sigma_{TU}}{E_T} \qquad \gamma_{LTU} = \frac{\sigma_{LTU}}{G_{LT}} \qquad \varepsilon'_{TU} = \frac{\sigma'_{TU}}{E'_T}$$

The strains in the *L*, *T* directions can be calculated from the Hooke's law relations using the calculated stress values for σ_L, σ_T and σ_{LT} as input values, that is:

$$\varepsilon_L = \frac{\sigma_L}{E_L} - \nu_{TL}\frac{\sigma_T}{E_T} = 0.000052$$

$$\varepsilon_T = \frac{\sigma_T}{E_T} - \nu_{LT}\frac{\sigma_L}{E_L} = 0.0000084$$

$$\gamma_{LT} = \frac{\sigma_{LT}}{G_{LT}} = 0.00097$$

Comparing the above values with the allowable strain we see that:

$$\varepsilon_L < \varepsilon_{LU}$$

$$\varepsilon_T < \varepsilon_{TU}$$

and,

$$\gamma_{LT} < \gamma_{LTU}$$

Interactive Theory

Two key features of the Maximum Stress and Maximum Strain failure theories should be noted. These are

- Interaction between strengths is not accounted for
- Failure is dictated by a governing inequality

The first of the above features is sometimes considered an inadequacy of these types of failure theorems and thus one attempts to define involved composite failure theories based upon homogeneous isotropic metal based systems. Due to the large number of such theories available, as discussed previously, only one of these theories will be presented and used in the present problem. The specific failure theory selected is based upon an extension of Hill's generalization of the von Mises or Distortional Energy failure theory which was further extended by Azzi and Tsai (1965). In functional form this equation is of the type,

$$\left(\frac{\sigma_L}{\sigma_{LU}}\right)^2 - \left(\frac{\sigma_L}{\sigma_{LU}}\right)\left(\frac{\sigma_T}{\sigma_{LU}}\right) + \left(\frac{\sigma_T}{\sigma_{TU}}\right)^2 + \left(\frac{\sigma_{LT}}{\sigma_{LTU}}\right)^2 < 1$$

For the present example using the calculated results for $\sigma_L, \sigma_T, \sigma_{LT}$ and the data included in Table 7.2, we can write,

$$\left(\frac{3{,}160}{280{,}000}\right)^2 + \left(\frac{340}{4{,}000}\right)^2 - \frac{(3{,}160)(340)}{(280{,}000)^2} + \left(\frac{5{,}240}{6{,}000}\right)^2 < 1$$

The failure criteria described in the preceding paragraphs can be summarized in terms of the three types mentioned earlier in the chapter. That is stress dominated, strain dominated and so-called stress interactive types. A graphical interpretation of these yield criteria for the case of equal tensile and compressive yield points can be graphically displayed, while an analytical description couched in terms of either strain or stress space follows in the accompanying tables.

TABLE 7.3. Failure Criteria
(Theories with and without Independent Failure Modes)
Maximum Stress Criterion

Stress Space	Strain Space
$\sigma_{LU}^{1} \le \sigma_1 \le \sigma_{LU}$	$\varepsilon_1 = -\frac{C_{12}}{C_{11}}\varepsilon_2 - \frac{C_{16}}{C_{11}}\varepsilon_6 + \frac{\sigma_{LU}}{C_{11}}$
$\sigma_{TU}^{1} \le \sigma_2 \le \sigma_{TU}$	$\varepsilon_1 = -\frac{C_{12}}{C_{11}}\varepsilon_2 - \frac{C_{16}}{C_{11}}\varepsilon_6 - \frac{\sigma'_{LU}}{C_{11}}$
$\sigma_{LTU}^{1} \le \sigma_6 \le \sigma_{LTU}$	$\varepsilon_3 = -\frac{C_{12}}{C_{22}}\varepsilon_1 - \frac{C_{26}}{C_{22}}\varepsilon_6 + \frac{\sigma_{TU}}{C_{22}}$
	$\varepsilon_4 = -\frac{C_{12}}{C_{22}}\varepsilon_1 - \frac{C_{26}}{C_{22}}\varepsilon_6 - \frac{\sigma'_{TU}}{C_{22}}$
	$\varepsilon_6 = -\frac{C_{16}}{C_{66}}\varepsilon_1 - \frac{C_{26}}{C_{66}}\varepsilon_2 + \frac{\sigma_{LTU}}{C_{66}}$
	$\varepsilon_6 = -\frac{C_{16}}{C_{66}}\varepsilon_1 - \frac{C_{26}}{C_{66}}\varepsilon_2 - \frac{\sigma_{LTU}}{C_{66}}$

TABLE 7.4. Maximum Strain Criterion

Strain Space	Stress Space
$\varepsilon_{LU} = S_{11}\sigma_{LU} \ge \varepsilon_1$	$\sigma_1 = -\frac{S_{12}}{S_{11}}\sigma_2 - \frac{S_{16}}{S_{11}}\sigma_6 + \sigma_{LU}$
$\varepsilon'_{LU} = S_{11}\sigma'_{LU} \ge -\varepsilon_1$	$\sigma_1 = -\frac{S_{12}}{S_{11}}\sigma_2 - \frac{S_{16}}{S_{11}}\sigma_6 - \sigma'_{LU}$
$\varepsilon_{TU} = S_{22}\sigma_{TU} \ge \varepsilon_2$	$\sigma_2 = -\frac{S_{12}}{S_{22}}\sigma_1 - \frac{S_{26}}{S_{22}}\sigma_6 + \sigma_{TU}$
$\varepsilon'_{TU} = S_{22}\sigma'_{TU} \ge -\varepsilon_2$	$\sigma_2 = -\frac{S_{12}}{S_{22}}\sigma_1 - \frac{S_{26}}{S_{22}}\sigma_6 - \sigma'_{TU}$
$\varepsilon_{LTU} = S_{66}\sigma_{LTU} \ge \varepsilon_6$	$\sigma_6 = -\frac{S_{16}}{S_{66}}\sigma_1 - \frac{S_{26}}{S_{66}}\sigma_2 + \sigma_{LTU}$
$\varepsilon'_{LTU} = S_{66}\sigma'_{LTU} \ge -\varepsilon_6$	$\sigma_6 = -\frac{S_{16}}{S_{66}}\sigma_1 - \frac{S_{26}}{S_{66}}\sigma_2 - \sigma'_{LTU}$

TABLE 7.5. Interactive Failure Criterion

Stress Space	Strain Space
$f(\tau_i) = F_i\sigma_i + F_{ij}\sigma_i\sigma_j$	$g(\varepsilon_i) = G_i\varepsilon_i + G_{ij}\varepsilon_i\varepsilon_j$
$+ F_{ijk}\sigma_i\sigma_j\sigma_k + \ldots$	$+ G_{ijk}\varepsilon_i\varepsilon_j\varepsilon_k + \ldots$
F's and G's are material constants	

7.5 Laminate Strength Analysis

We now turn our attention from single lamina strength analysis to the failure behavior of multiple laminae or laminates. The objective of this analysis is to determine the strength behavior of each lamina in the laminate assuming that a plane state of stress exists for each lamina independent of its orientation and position within the laminate. The application of a suitable failure criterion must be adopted, the stress on each lamina calculated and transformed to the material axes of the lamina, and a fail safe measure of each lamina determined. Several points must therefore be addressed in terms of general laminate strength analysis. We begin by examining several approaches to laminate strength analysis, which have been advanced.

In the broadest context of laminate strength interrogation, one is generally knowledgeable of either,

- Knowing the loads/Testing the design
- Knowing the design/Testing for allowable loads

For the first case indicated, so-called knowledge of the loads may be available, however, generally this is more in the realm of the designer's province to determine by experience rather than by rigorous deterministic methods. In the second case, many times the design is fixed and knowledge of what loads the structural elements can safely withstand is the challenge. Each of these issues will be examined through examples in this section. An awareness of the semi-global failure analysis of a laminate raises the following issues, that is,

- First Ply Failure
- Behavior after First Ply Failure

First ply failure occurs when the given or calculated loads exceed the ply specified failure criterion. Should this test be met then it is still possible for the system to carry additional loads after first ply failure and several approaches to accounting for ply failure can be introduced. Among these are,

- Total Ply Discount
- Ply Failure Distribution

The first approach, that of total ply discount, implies that the ply still remains within the system as a volume entity but that it no longer carries any load. This represents a conservative approach to failure analysis. The second approach implies that some knowledge of the failure mechanisms of the ply are known. For example, if matrix failure in a particular ply is known to occur, then the transverse properties of that ply can be omitted. Alternatively, if fiber failure occurs within a given ply, then that ply can be treated as having zero stiffness for subsequent analysis.

In addition to examining the effects of ply failure within the laminate, it is sometimes equally advantageous to establish failure envelopes based upon various combinations of a given set of applied loads. The most widely used loading for these latter schemes appear to be that of membrane type loads. Both interrogation of ply failure and failure interaction envelopes will be discussing in the examples to follow. Before discussing these examples, however, it is appropriate to review that pertinent laminate equations, including generalizations to incorporate both temperature and humidity effects, required for analysis of composite failure. These equations will be useful to the reader in examining more complex failure situations where residual stresses or environmental conditions may be important as well as in establishing interactive failure envelopes. Considering a plane state of stress and referring to an arbitrary axes x and y with respect to the principal coordinate material directions, we can write the following stress-strain equations for the kth lamina of a stacking sequence comprising N lamina,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x - \alpha_x \Delta T - \beta_x \Delta m \\ \varepsilon_y - \alpha_y \Delta T - \beta_y \Delta m \\ 2\left(\varepsilon_{xy} - \alpha_{xy} \Delta T - \beta_{xy} \Delta m\right) \end{bmatrix}_k$$

The strains $\varepsilon_x^{(k)}, \varepsilon_y^{(k)}$ and $\varepsilon_{xy}^{(k)}$, can be rewritten in terms of the displacement components u, v and w resulting in the following equation,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k = \begin{bmatrix} \\ \overline{Q}_{ij} \\ \\ \end{bmatrix}_k \begin{bmatrix} \varepsilon_{x_0} + z\kappa_x - \alpha_x \Delta T - \beta_x \Delta m \\ \varepsilon_{y_0} + z\kappa_y - \alpha_y \Delta T - \beta_y \Delta m \\ 2\left(\varepsilon_{xy_0} + z\kappa_{xy} - \alpha_{xy} \Delta T - \beta_{xy} \Delta m\right) \end{bmatrix}_k$$

The stresses for the kth lamina can be redefined in terms of the membrane stress resultants N_x, N_y, N_{xy} and bending stress couples M_x, M_y, M_{xy}. Integrating over the thickness of the lamina as shown in Equation (2.47), we obtain Equation (2.49). In matrix notation this can be written as,

$$[N] = [A]\left[\varepsilon_0\right] + [B][\kappa] - [N]^T - [N]^m$$

A similar set of equations can be found for the bending stress couples, that is,

$$[M] = [B]\left[\varepsilon_0\right] + [D][\kappa] - [M]^T - [M]^m$$

transposing $[N]^T, [N]^m, [M]^T$ and $[M]^m$ from the right hand side to the left hand side of the stress resultant equations we can write:

$$[\overline{N}] = [N] + [N]^T + [N]^m = [A][\varepsilon_0] + [B][\kappa]$$

$$[\overline{M}] = [M] + [M]^T + [M]^m = [B][\varepsilon_0] + [D][\kappa]$$

The barred quantities are often referred to in the literature as the total force and moment resultants.

In some cases, it is advantageous to have the mid-plane strains and curvatures defined in terms of the stress resultants. Thus, inverting the above equations gives:

$$\begin{bmatrix} \varepsilon_0 \\ \hline \kappa \end{bmatrix} = \begin{bmatrix} a - b\alpha^{-1}b^T & b\alpha^{-1} \\ -d^{-1} & d^{-1} \end{bmatrix} \begin{bmatrix} \overline{N} \\ \hline \overline{M} \end{bmatrix}$$

Thus for the case of plane stress anisotropic laminates, in the absence of transverse shear stress, the principal failure criteria can be interrogated as follows,

Maximum Stress Theory

(1) Establish the stresses acting on the system
(2) If the stresses do not coincide with the principal material direction they should be transformed according to the transformation law $[\sigma_L] = [T][\sigma_x]$
(3) The stresses $[\sigma_L]$ should then be compared with the allowable stresses on a ply by ply basis to establish whether first ply failure has occurred.
(4) The process of estimating whether laminate failure has occurred by introducing a selected post first ply failure criterion can then be evaluated.

Maximum Strain Theory

(1) Establish the stresses acting on the system
(2) Calculate the strains corresponding to the applied stresses
(3) If the calculated strains do not coincide with the principal material directions they should be transformed according to the transformation law
(4) The strains $[\varepsilon_L]$ should then be compared with the allowable strains on a ply by ply basis in order to establish whether first ply failure has occurred.
(5) The process of estimating whether laminate failure has occurred by introducing a selected post first ply failure criterion can then be evaluated.

Interactive Failure Criteria

(1) Establish the stresses acting on the system
(2) If the stresses do not coincide with the principal material directions they should be rotated using the transformation $[\varepsilon_L] = [T][\varepsilon_x]$

(3) The stress $[\sigma_L]$ should then be inserted into the selected interactive failure criterion, that is, Tsai-Hill, Tsai-Wu, Hoffman and the criterion tested to ensure existence of a fail safe design on a ply by ply basis.

(4) If a failed ply has been detected, an appropriate first ply failure assessment can be made and the composite system interrogated for laminate failure.

7.6 References

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7.7 Problems

7.1. Consider the following laminate of equal ply thickness $0.0052''$

$$\begin{vmatrix} \theta = 30^\circ \\ \theta = 0^\circ \\ \theta = -30^\circ \end{vmatrix}$$

assume that the maximum strain theory governs failure, with the following inequalities given

$$|\varepsilon_1| \le 0.004$$

$$|\varepsilon_2| \le 0.003$$

$$|\gamma_{12}| \le 0.010$$

Determine whether failure will occur at $z = 0$ for the following loadings.

$$N_x = 100 \text{ lb./in.} \qquad M_x = 5 \text{ in.} - \text{lb./in.}$$

The properties of an unidirectional layer are given by:

$$\begin{aligned} E_{11} &= 29.2 \times 10^6 \text{ psi} \\ E_{22} &= 2.4 \times 10^6 \text{ psi} \\ \nu_{12} &= 0.223 \\ \nu_{21} &= 0.018 \\ G_{12} &= 0.83 \times 10^6 \text{ psi} \end{aligned}$$

7.2. Consider the pure shear loading of a composite material at any angle θ with respect to the principal material directions. Use the Tsai-Hill criterion to find failure of the composite for this loading.