This work forms part of a programme of fundamental research undertaken by the Board of the British Rubber Producers' Research Association. The work was carried out at the National Bureau of Standards, Washington, D.C., U.S.A., while the author was a guest-worker there.

References

Lamb, H. 1924 *Hydrodynamics*, 5th ed. Camb. Univ. Press.
Love, A. E. H. 1927 *The mathematical theory of elasticity*, 4th ed. Camb. Univ. Press.
Reiner, M. 1945 *Amer. J. Math.* **67**, 350–362.
Weissenberg, K. 1947 *Nature*, **159**, 310.

A theory of the yielding and plastic flow of anisotropic metals

By R. Hill

(*Communicated by E. Orowan, F.R.S.—Received* 13 *November* 1947)

A theory is suggested which describes, on a macroscopic scale, the yielding and plastic flow of an anisotropic metal. The type of anisotropy considered is that resulting from preferred orientation. A yield criterion is postulated on general grounds which is similar in form to the Huber-Mises criterion for isotropic metals, but which contains six parameters specifying the state of anisotropy. By using von Mises' concept (1928) of a plastic potential, associated relations are then found between the stress and strain-increment tensors. The theory is applied to experiments of Körber & Hoff (1928) on the necking under uniaxial tension of thin strips cut from rolled sheet. It is shown, in full agreement with experimental data, that there are generally two, equally possible, necking directions whose orientation depends on the angle between the strip axis and the rolling direction. As a second example, pure torsion of a thin-walled cylinder is analyzed. With increasing twist anisotropy is developed. In accordance with recent observations by Swift (1947), the theory predicts changes in length of the cylinder. The theory is also applied to determine the earing positions in cups deep-drawn from rolled sheet.

1. Introduction

The macroscopic yielding and plastic straining of isotropic ductile metals has been described by theories which may be regarded as essentially satisfactory. The well-known Huber-Mises criterion for determining the elastic limit under combined stresses of metals which deform homogeneously, for example, copper, aluminium, nickel, has been confirmed by many experiments over the last twenty years. There are one or two exceptions, notably annealed mild steel which softens after yielding. The specimen is traversed by Lüders' bands and the deformation is non-uniform. In such circumstances the maximum shear-stress criterion of Coulomb or Tresca is found to be a better approximation than that of Huber-Mises. This is hardly surprising since each Lüders' band is a region where the deformation is a simple shear. For metals obeying the Huber-Mises law of yielding, the relation between the plastic strain-increment and the applied combined stress is given by the Lévy-Mises equations. Strain-hardening can also be taken into account in a relatively simple way which is fairly satisfactory for biaxial stress systems and limited strain ranges.

With increasing strain, however, a preferred orientation of crystal planes and directions gradually develops, and the individual crystals become elongated to form a characteristic fibrous texture in the direction of the most severe tensile strain. In this way an originally isotropic metal becomes anisotropic in respect of many physical properties. It is well known that the fibre texture produced in the technological forming processes, rolling, drawing, and extrusion, is sometimes the cause of undesirable properties in the final product. The most familiar example is perhaps that of the 'earing' of cups deep-drawn from rolled sheet. After severe cold-rolling the yield stress may be as much as 10 % higher in directions across the fibre than it is along the fibre (Cook 1937). Ductility shows even more pronounced directional properties. Such anisotropy can only be removed with difficulty by careful heat treatment. More often the metallurgist must be content with a compromise in which a final preferred orientation of allowable proportions is retained. Preferred orientation is not the only cause of anisotropic plastic properties: laminar inclusions and cavities occasionally produce similar effects. Residual or internal stresses are another cause. The present theory is, however, probably valid only when the anisotropy is due to preferred orientation.

Anisotropy, then, is not to be considered a phenomenon of rather rare occurrence. It is difficult to avoid in metal working and is invariably developed by any severe strain. Whenever it is present the theories of plastic flow for isotropic metals are only valid to a first approximation. This approximation is good enough for many purposes, but there are also many phenomena for which these theories fail to account. The object of the present paper is to formulate a theory capable of describing the macroscopic behaviour of anisotropic metals. Such a theory must be constructed on *a priori* reasoning since no suitable data are as yet available to check individual assumptions. It will be shown finally, however, that the theory is, as a whole, in close agreement with certain experimental observations.

2. The plastic potential

In 1928, von Mises, in a brilliant paper on the plastic distortion of crystals, introduced the fruitful concept of a plastic potential. A detailed account has been given by Geiringer (1937). This concept will now be explained as it is basic in constructing the new theory.

Suppose the criterion of yielding under combined stresses is

$$f(\sigma_{ij}) = \text{constant}, \tag{1}$$

where f is regarded as a function of the components of the stress tensor. The function f will also involve certain parameters characterizing the current state of the material. Von Mises suggested that the relation between the stress tensor σ_{ij} and the natural strain-increment tensor $d\epsilon_{ij}$ which should be used in conjunction with this yield criterion is simply

$$d\epsilon_{ij} = \frac{\partial f}{\partial \sigma_{ij}} d\lambda. \tag{2}$$

Here $d\lambda$ is a positive scalar factor of proportionality. When carrying out the partial differentiation it is important to remember that σ_{ij}, σ_{ji} $(i \neq j)$ must be counted as different for the purposes of tensor analysis. If the metal is isotropic f will not depend on the choice of axes, and equation (2) is then a true tensor relation. When the metal is anisotropic the form of f depends, as we shall see, on the choice of axes of reference. The function f is called the plastic potential.

Let us examine what validity von Mises' idea has in relation to the known plastic properties of single crystals and quasi-isotropic polycrystals. The Huber-Mises criterion of yielding for the latter is

$$2f(\sigma_{ij}) \equiv (\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2) = 2Y^2, \tag{3}$$

where Y is the yield stress of a bar in uniaxial tension. Regarding the nine stress components as distinct independent variables, we find

$$\frac{\partial f}{\partial \sigma_{xx}} = (2\sigma_{xx}-\sigma_{yy}-\sigma_{zz}), \quad \frac{\partial f}{\partial \sigma_{yz}} = \frac{\partial f}{\partial \sigma_{zy}} = 3\sigma_{yz}, \quad \text{etc.}$$

Now the Lévy-Mises stress-strain relations are

$$d\epsilon_{xx} = d\lambda(2\sigma_{xx}-\sigma_{yy}-\sigma_{zz}), \quad d\epsilon_{yz} = 3d\lambda\sigma_{yz}, \quad \text{etc.}, \tag{4}$$

where $d\epsilon_{xx}$, $d\epsilon_{yz}$, etc. are components of the natural strain-increment tensor. Hence equations (4) can be written in the form of equation (2).

Consider next a metal single-crystal which is generally accepted as yielding when the shear stress, acting over a certain crystal plane in a certain specific direction, reaches a limiting value. If axes are chosen so that the shear stress is σ_{yz}, then the criterion of yielding is $f(\sigma_{ij}) \equiv \sigma_{yz}$ = constant. From equation (2) we find that the strain-increment tensor has only two non-vanishing components viz. $d\epsilon_{yz}$ and $d\epsilon_{zy}$, which are of course equal. This represents a shear in the yz plane in the direction of the shear stress, which is what is actually observed.

In annealed mild steel yielding begins when the maximum shear stress reaches a limiting value. Proceeding as with the single crystal, equation (2) implies that the ensuing strain is a shear in the direction of the maximum shear stress. This is just what happens within a Lüders' band. Prestrained mild steel, however, is like most ductile metals in following both the Huber-Mises and Lévy-Mises laws (Taylor & Quinney 1931).

The success of the plastic potential in these examples gives confidence in applying it to anisotropic polycrystalline metals. Von Mises himself was concerned with single crystals. However, the idea seems of wider validity, for in spite of its apparent artificiality it can be given a physical interpretation. This, too, is due to von Mises. Consider the work $\sigma_{ij} d\epsilon_{ij}$ done on a small element in deforming it through a given plastic strain-increment $d\epsilon_{ij}$. Suppose we arbitrarily vary σ_{ij}, subject only to the restriction imposed by the yield criterion $f(\sigma_{ij}) = \text{constant}$. Then for given $d\epsilon_{ij}$ the work done has a stationary value when σ_{ij} assumes its actual value given by equation (2). Stationary values of $\sigma_{ij} d\epsilon_{ij}$, subject to $f(\sigma_{ij}) = \text{constant}$, are given, in the usual way, by

$$\frac{\partial}{\partial \sigma_{ij}}(\sigma_{ij} d\epsilon_{ij} - f d\lambda) = 0,$$

where $d\lambda$ is an undetermined constant multiplier. This leads immediately to equation (2) and the statement is proved. In the special case when f is the Huber-Mises expression the work done has a maximum value (Hill, Lee & Tupper 1947), but this may not be true in general. It is true for the yield criterion proposed in § 3, except possibly when one of F, G, H is negative.

3. A yield criterion for anisotropic metals

It is assumed that the anisotropy has three mutually orthogonal planes of symmetry at every point. The three planes of symmetry meet in three orthogonal directions which may be called the principal axes of anisotropy. These axes can, and frequently will, vary in direction at any moment from point to point of the bulk metal. For example, if anisotropy is developed through the severe uniform expansion of a hollow cylinder by internal pressure, the principal axes of anisotropy can be expected to lie in the radial, circumferential and longitudinal directions. An example of uniformly directional anisotropy would be that in a bar cut from the central part of a cold-rolled sheet; the principal axes would lie in the direction of rolling, transversely in the plane of the sheet, and normal to this plane. Again, the principal axes in a given element can vary relatively to the element itself as deformation continues, for example, in simple torsion of a hollow cylinder.

Let us fix our attention on a given element in a certain state of anisotropy and choose the principal axes of anisotropy as Cartesian axes of reference. By analogy with the Huber-Mises yield criterion for isotropic metal it is natural to select some homogeneous quadratic in the stresses to represent the plastic potential. This would imply that the yield stress in tension for any direction is the same as that in compression, i.e. the absence of a Bauschinger effect. Now cold work does generally give rise to a Bauschinger effect, just as it produces a preferred orientation and fibre texture. The former can be removed by mild annealing, but the latter remains. Hence it is perfectly possible, both in principle and in practice, to have an anisotropic metal with no Bauschinger effect. The Bauschinger effect will be ignored in the present analysis; it will not be difficult to take account of it when experiment shows such a refinement in the theory to be desirable. We can, then, safely choose a homogeneous quadratic for the plastic potential. Furthermore, in view of the symmetry assumption, terms in which any one shear stress occurs linearly must be rejected. If it be also assumed that the superposition of hydrostatic pressure does not influence yielding, the plastic potential or yield criterion takes the form

$$2f \equiv F(\sigma_{yy}-\sigma_{zz})^2+G(\sigma_{zz}-\sigma_{xx})^2+H(\sigma_{xx}-\sigma_{yy})^2+2L\sigma_{yz}^2+2M\sigma_{zx}^2+2N\sigma_{xy}^2 = 1. \quad (5)$$

F, G, H, L, M, N are constants characteristic of the current state of anisotropy. The expression for f only has this form when the principal axes of anisotropy are chosen as axes of reference. If other axes of reference are taken, the form changes in a way that can be found by transforming the stress components.

If X, Y, Z are the tensile yield stresses in the principal anisotropic directions, it is easily seen that

$$\left.\begin{aligned} \frac{1}{X^2} &= G+H, \quad 2F = \frac{1}{Y^2}+\frac{1}{Z^2}-\frac{1}{X^2},\\ \frac{1}{Y^2} &= H+F, \quad 2G = \frac{1}{Z^2}+\frac{1}{X^2}-\frac{1}{Y^2},\\ \frac{1}{Z^2} &= F+G, \quad 2H = \frac{1}{X^2}+\frac{1}{Y^2}-\frac{1}{Z^2}.\end{aligned}\right\} \quad (6)$$

It is clear that only one of F, G, H can be negative, a situation which, however, is unlikely to occur in practice, since it implies rather large differences between the yield stresses. Also $F > G$ if and only if $X > Y$, etc. These inequalities will be used later when comparing theory and experiment. If R, S, T are the yield stresses in shear with respect to the principal axes of anisotropy, then

$$2L = \frac{1}{R^2}, \quad 2M = \frac{1}{S^2}, \quad 2N = \frac{1}{T^2}. \quad (7)$$

L, M, N are thus essentially positive.

To describe fully the state of anisotropy in an element the six independent yield stresses, X, Y, Z, R, S, T need to be given. They must be considered as dependent on the strain-history of the material and so varying with strain. For present purposes it is not necessary to specify the state of anisotropy more closely than this. No attempt will be made to examine, for example, how the yield stresses change with increasing distortion, or how they can be quantitatively related to the degree of preferred orientation.

If there is rotational symmetry of the anisotropy in an element about the z axis, then

$$F = G, \quad N = G+2H. \quad (8)$$

This may be verified by requiring the form of f to remain invariant for arbitrary x, y axes of reference. For complete spherical symmetry, i.e. isotropy, $F = G = H$ and $L = M = N = 3F$. The expression (5) is then identical with (3) when $2F$ is put equal to $1/Y^2$.

It might be asked why f is not taken as the expression for the distortional energy due to shear alone in an anisotropic elastic medium, by analogy with Hencky's interpretation of the Huber-Mises yield criterion for isotropic materials. There are two objections to this procedure. A hydrostatic pressure generally produces shear strain in an anisotropic elastic material. Hence we cannot take the expression for shear-strain energy as the plastic potential without implying that a superposed hydrostatic pressure influences yielding. Again, the anisotropic shear-strain energy is not independent of elastic constants as is the isotropic shear-strain energy, apart from a multiplying factor. The elastic constants will therefore appear in the expression (2) for the plastic strain-increment. This is of course not physically plausible.

4. Relations between stress and strain-increment for anisotropic metals

By applying the equation (2) to the yield criterion (5) the following expressions are obtained, relating the components of the strain-increment tensor to the stress components.

$$\left.\begin{aligned} d\epsilon_{xx} &= d\lambda[H(\sigma_{xx}-\sigma_{yy})+G(\sigma_{xx}-\sigma_{zz})], & d\epsilon_{yz} &= d\lambda\, L\sigma_{yz}, \\ d\epsilon_{yy} &= d\lambda[F(\sigma_{yy}-\sigma_{zz})+H(\sigma_{yy}-\sigma_{xx})], & d\epsilon_{zx} &= d\lambda\, M\sigma_{zx}, \\ d\epsilon_{zz} &= d\lambda[G(\sigma_{zz}-\sigma_{xx})+F(\sigma_{zz}-\sigma_{yy})], & d\epsilon_{xy} &= d\lambda\, N\sigma_{xy}. \end{aligned}\right\} \tag{9}$$

It will be noticed that $d\epsilon_{xx}+d\epsilon_{yy}+d\epsilon_{zz} = 0$ identically for all stress systems. This of course is essential in view of the known fact that the volume change associated with plastic strain is negligible. The zero volume change is directly connected with the assumption that yielding is independent of hydrostatic pressure. Either is a consequence of the other for any yield criterion whatever. This can most readily be seen by considering the stationary work interpretation of the plastic potential method. For a superposed hydrostatic pressure does not change the work done in a given strain when the volume change is zero. Thus if we accept, as experimental facts, the zero plastic volume change and the independence of the yield criterion with respect to hydrostatic pressure, then the method of the plastic potential is not only consistent with both but provides a link between them.

When using the above equations it must be remembered that they only have this form when the stresses and strain-increments are referred to the principal axes of anisotropy. Further, the principal axes of stress and strain-increment coincide when the principal stress directions are along the principal axes of anisotropy, but not in general otherwise.

Consider now some simple stress systems for a material with anisotropy uniformly distributed in magnitude and direction. In pure tension X parallel to the principal x axis of anisotropy the incremental strains at a certain stage are

$$d\epsilon_{xx} : d\epsilon_{yy} : d\epsilon_{zz} = (G+H) : -H : -G.$$

The specimen contracts in the transverse directions unless one of G and H is negative, which seems hardly likely to occur. The contraction in the y direction is greater than that in the z direction if $H > G$, i.e. if $Z > Y$. An experimental determination of the ratio $d\epsilon_{yy}/d\epsilon_{zz}$ would give the value of H/G. Similarly from tension tests in the y, z directions values of G/F, F/H can be found. Apart from the check on the theory in view of the identity $\frac{H}{G}\times\frac{G}{F}\times\frac{F}{H} = 1$, this would allow an alternative way of determining the ratios X/Y, Y/Z of the yield stresses. If the yielding is not particularly sharp, direct measurement of X, Y, Z would be unreliable. Similar experiments can, in principle, be devised to find the shear-stress ratios R/S, R/T.

In plane strain, where motion is prevented in the z principal direction of anisotropy, we find by putting $d\epsilon_{zz} = 0$ in equation (9) that

$$\sigma_{zz} = \frac{G\sigma_{xx}+F\sigma_{yy}}{G+F},$$

and thus from equation (5) that

$$\left(\frac{FG+GH+HF}{F+G}\right)(\sigma_{xx}-\sigma_{yy})^2+2N\sigma_{xy}^2 = 1. \tag{10}$$

It is easy to show that the tensile yield stresses are equal in any pair of orthogonal directions in the plane, although σ_{zz} is different in the two cases. It can also be shown that the dependence of the tensile yield stress on orientation is such that maxima or minima occur in directions along the anisotropic axes, or in directions making 45° with the axes. If

$$\frac{2(FG+GH+HF)}{N(F+G)}<1,$$

the yield stress has a minimum in the 45° directions and a maximum in the x and y directions; for the opposite inequality the reverse is true. When the equality holds, the yield stress does not vary with orientation. In view of the identities

$$\frac{2(FG+GH+HF)}{N(F+G)} = \frac{F+2H}{N}+\frac{F(G-F)}{N(F+G)} = \frac{G+2H}{N}+\frac{G(F-G)}{N(F+G)},$$

it is clear that when

$$N>F+2H,\quad G+2H,\quad \text{then}\quad \frac{2(FG+GH+HF)}{N(F+G)}<1;$$

and when $$N<F+2H,\quad G+2H,\quad \text{then}\quad \frac{2(FG+GH+HF)}{N(F+G)}>1.$$

The physical significance of these inequalities will be further exemplified in § 5.

The relations between the components of the strain-increment and stress tensors in a state of plane strain are

$$\left.\begin{aligned} d\epsilon_{xx}+d\epsilon_{yy} &= 0, \\ \text{and}\quad \frac{d\epsilon_{xx}-d\epsilon_{yy}}{2d\epsilon_{xy}} &= \frac{2(FG+GH+HF)}{N(F+G)}\frac{(\sigma_{xx}-\sigma_{yy})}{2\sigma_{xy}}. \end{aligned}\right\} \tag{11}$$

Hence if ψ is the angle between a principal stress direction and the x axis, and ψ' is the angle between a principal strain-increment direction and the x axis, then

$$\frac{\tan 2\psi}{\tan 2\psi'} = \frac{2(FG+GH+HF)}{N(F+G)}. \tag{12}$$

Hence $\psi = \psi'$ if, and only if, $\psi = 0, \pm\frac{1}{4}\pi, \pm\frac{1}{2}\pi$. Thus the principal axes of stress and strain-increment coincide in the directions in which the tensile yield stress has maximum or minimum values. The set of equations (10) and (11) together with the equilibrium equations, considered in terms of the five dependent variables (three stress components and two incremental displacement components) is hyperbolic. There are only two distinct characteristics: these are the orthogonal directions of maximum shear strain-increment or of zero extension. These, as has been remarked, are not generally the directions of maximum shear stress. The four differential relations holding along the characteristics are found to be analogous to the well-known Hencky relations for an isotropic metal. It is scarcely worth while to include an account of these in the present paper while the theory has not yet been sufficiently tested.

For a thin sheet perpendicular to the Z direction and in a state of plane stress

$$(G+H)\sigma_{xx}^2 - 2H\sigma_{xx}\sigma_{yy} + (H+F)\sigma_{yy}^2 + 2N\sigma_{xy}^2 = 1. \tag{13}$$

The maxima and minima of the yield stress in uniaxial tension occur along the anisotropic axes and in a direction θ with respect to the x axis, where

$$\tan^2\theta = \frac{N-G-2H}{N-F-2H}. \tag{14}$$

If $N > F+2H,\ G+2H$, the yield stress has maximum (unequal) values in the x, y directions and minimum (equal) values in the θ directions.* If $N < F+2H,\ G+2H$, the converse is true. If N is intermediate to $F+2H$ and $G+2H$ there is no real θ satisfying equation (14). The yield stress then has a maximum in the x direction and a minimum in the y direction if $F > G$, and vice versa. The relations between the components of the strain-increment and stress tensors in a state of plane stress are

$$\left.\begin{aligned} \frac{d\epsilon_{xx}}{d\epsilon_{yy}} &= \frac{(G+H)\sigma_{xx} - H\sigma_{yy}}{(F+H)\sigma_{yy} - H\sigma_{xx}}, \\ \frac{d\epsilon_{xx} - d\epsilon_{yy}}{2d\epsilon_{xy}} &= \frac{(G+2H)\sigma_{xx} - (F+2H)\sigma_{yy}}{2N\sigma_{xy}}. \end{aligned}\right\} \tag{15}$$

The principal axes of stress and strain-increment coincide when the principal axes of stress are along the anisotropic axes, and also for special states of stress such that

$$\frac{\sigma_{yy}}{\sigma_{xx}} = \frac{N-G-2H}{N-F-2H}.$$

If the state of stress is a uniaxial tension σ directed at an angle θ with the x axis then

$$\sigma_{xx} = \sigma\cos^2\theta, \quad \sigma_{yy} = \sigma\sin^2\theta, \quad \sigma_{xy} = \sigma\sin\theta\cos\theta.$$

The principal axes of stress and strain-increment then coincide when θ has the value given by equation (14). θ is real only when $(N-F-2H)$ and $(N-G-2H)$ have the

* Compare values for 0·1 % proof stress given by Cook (*loc. cit.*, tables I and II).

same sign. The directions for which the principal axes of stress and strain-increment coincide in uniaxial tension are thus those for which the yield stress has maximum and minimum values.

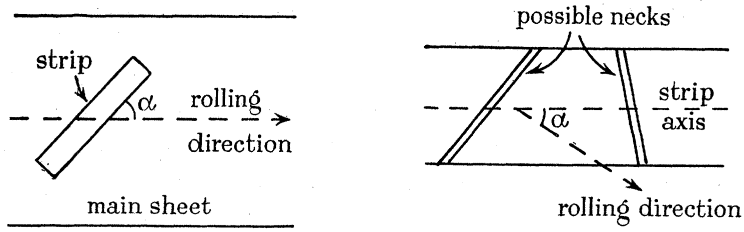
The characteristics for a state of plane stress are in directions satisfying

$$[(G+H)\sigma_{xx}-H\sigma_{yy}]\,dx^2+2N\sigma_{xy}\,dx\,dy+[(F+H)\sigma_{yy}-H\sigma_{xx}]\,dy^2=0. \tag{16}$$

The characteristics are in the directions of zero extension in the plane of the sheet. They are not generally perpendicular because of the $d\epsilon_{zz}$ strain, nor are they always real. The plane stress problem is thus sometimes hyperbolic and sometimes elliptic, just as in the case of an isotropic metal. The coincidence of the characteristics with the directions of zero extension in both plane strain and plane stress is closely connected with the plastic potential. In these special cases the characteristics can be found for the Cauchy problem in the stresses and the incremental displacements separately, since the full set of equations breaks down into distinct halves. The two pairs of characteristic directions corresponding to the two halves can be shown to be the same if, and only if, the stress-strain equations and the yield criterion are connected on the plastic potential basis.

5. The necking of strip under tension

A check on the present theory can be obtained by analyzing the experimental data of Körber & Hoff (1928). These workers carried out tensile tests on strips cut from thin rolled sheet. Now it is well known that a sufficiently thin strip of isotropic metal, pulled in tension, develops a neck making an angle of about 55° with the direction of pull. Körber & Hoff, however, observed that for a strip cut at a definite angle to the rolling direction either of two distinct necking angles occurred apparently at random. Suppose a strip is cut with its length making an angle α with the rolling direction in the original sheet (figure 1*a*). In figure 1*b* the strip is shown with the two equally likely directions of necking.



a *b*

Figure 1. Positions of the two possible necks in a flat strip cut from rolled sheet and pulled in tension.

Körber & Hoff plotted graphs giving the two necking angles as functions of α. Figure 2 indicates qualitatively the general trend of their results; in detail the graphs depended on the material used and reduction in rolling.

The ordinate scale represents the acute angle between the line of the neck and the strip axis; the abscissa measures α, which of course ranges from 0 to $\frac{1}{2}\pi$. It is convenient to distinguish the two possible necking directions according as they tend to lie with the fibre (i.e. the rolling direction) or across it. The full curve in the figure corresponds to necking across the fibre; the dashed curve corresponds to necking with the fibre. It will be seen that there are three values of α for which the two possible necks make the same angle with the strip axis, thus giving the graph a figure of eight appearance. In the main sheet the principal axes of anisotropy are along and transverse to the rolling directions, and normal to the sheet. Hence the equality of the necking angles when α is either 0 or $\frac{1}{2}\pi$ is what would be expected from symmetry considerations since the axes of anisotropy are then parallel to the sides of the strip. The reason for the coincidence of the angles for a certain intermediate α will become clear when we analyze the experiments theoretically.

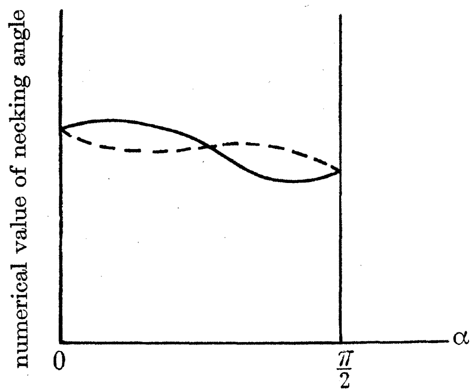


FIGURE 2. Qualitative representation of Körber & Hoff's measured values of the necking angle as a function of the angle α between strip axis and rolling direction. —— across fibre; --- with fibre.

To begin with, consider a flat strip of isotropic metal strained in tension. Necking will begin after some preliminary extension at a point where there is a slight non-uniformity in the strip, either geometrical or structural. In view of the property of characteristics as directions along which small disturbances propagate, it is natural to suppose that the line of the neck will coincide with one or other of the two characteristics through the origin of the disturbance. If this origin lies not on the edge, but in the middle of the bar, a **V**-shaped neck is sometimes observed, with its branches coinciding with parts of both characteristics. We can immediately find the directions of the characteristics from equation (16) for a state of plane stress. Take the x axis in the direction of pulling and set $F = G = H = \frac{1}{3}N$ in view of the assumed isotropy. The directions of the characteristics are found to make an angle $\tan^{-1}\sqrt{2}$, or approximately 55°, with the strip axis. It has already been mentioned that the incremental extension in the characteristic directions is zero. The extension along the line of the neck is thus zero, and the neck forms by thinning in the thickness of the strip.

Orowan, in unpublished work on the pulling of isotropic strip, takes as a starting-point the assumption that necking will begin along a direction of zero extension; the final result for the necking angle is therefore the same. Körber & Siebel (1928) have also obtained this value by an entirely different argument which is, however, unconvincing.

When the strip is anisotropic and the anisotropy is uniformly distributed, let the x principal axis of anisotropy be taken along the fibre or rolling direction (figure 3). The x axis makes an acute angle α with the strip axis or pulling direction. The axis of z is as usual normal to the plane of the strip (i.e. to the plane of the paper in figure 3).

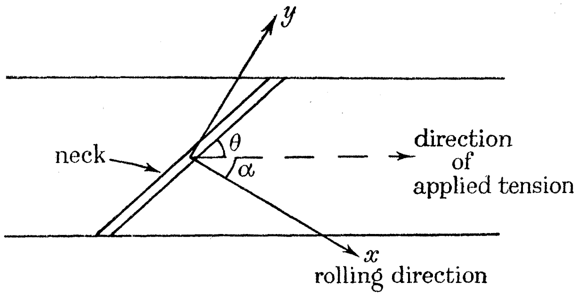


FIGURE 3. Co-ordinate system for analysis of the necking of flat strips in tension.

θ, measured from the axis in a sense away from the rolling direction denotes the inclination of a possible necking direction. It is the numerical value of θ that is plotted in figure 2. Except for material that has received very little cold work by rolling, the extension prior to necking is small compared with the rolling strains, and so the state of anisotropy in the strip at the beginning of necking will be effectively the same as in the rolled sheet. Let F, G, H, L, M, N, as in § 2, denote the anisotropic parameters specifying this state. If σ is the tensile stress in the strip when necking begins, the stress components referred to the x, y axes are

$$\sigma_{xx} = \sigma \cos^2 \alpha, \quad \sigma_{yy} = \sigma \sin^2 \alpha, \quad \sigma_{xy} = \sigma \sin \alpha \cos \alpha.$$

Inserting these in equation (16) with $dy/dx = \tan(\theta+\alpha)$ we obtain after some reduction:

$$a \tan^2 \theta + b \tan \theta - c = 0, \tag{17}$$

where

$$a = H + \sin^2 \alpha \cos^2 \alpha [(N-F-2H)+(N-G-2H)],$$

$$b = 2 \sin \alpha \cos \alpha [(N-F-2H) \sin^2 \alpha - (N-G-2H) \cos^2 \alpha],$$

$$c = F \sin^2 \alpha + G \cos^2 \alpha + H + \sin^2 \alpha \cos^2 \alpha [(N-F-2H)+(N-G-2H)].$$

By using (5) c can be shown to equal $1/\sigma^2$ and is therefore essentially positive for all values of α. a is positive except when $2N < F+G$. Since $N = 3F = 3G$ for an isotropic metal, the possibility of a being negative is most unlikely and can be disregarded. There are therefore in general two distinct necks with different inclinations corresponding to the roots of this quadratic in $\tan\theta$. The roots are numerically equal, but opposite in sign when $b = 0$, which is true for $\alpha = 0$, $\frac{1}{2}\pi$, and for an intermediate value $\bar{\alpha}$ given by

$$\tan^2\bar{\alpha} = \frac{N-G-2H}{N-F-2H}. \tag{18}$$

For these three values of α the two possible necks are symmetrically situated with respect to the strip axis. Since the neck is a direction of zero extension the principal axes of strain-increment are along and perpendicular to the axis for these three α-values, and so coincide with the principal stress axes. The theory thus predicts the figure of eight pattern of figure 2. For $\bar{\alpha}$ to be real, $(N-G-2H)$ and $(N-F-2H)$ must have the same sign and this condition must presumably have been satisfied for the materials used by Körber & Hoff. Further, if $(N-F-2H)$ and $(N-G-2H)$ are positive, the coefficient b is negative for $\alpha < \bar{\alpha}$ and positive for $\alpha > \bar{\alpha}$. Hence the sum $(-b/a)$ of the roots is positive for $\alpha < \bar{\alpha}$ and negative for $\alpha > \bar{\alpha}$. This means that the angle for necking across the fibre is numerically larger than that for necking with the fibre when $\alpha < \bar{\alpha}$, and vice-versa when $\alpha > \bar{\alpha}$. This agrees with the relative positions of the full and dashed curves in figure 2. If $(N-F-2H)$ and $(N-G-2H)$ are negative the curves are interchanged. We must therefore presume further that the anisotropy of the materials used by Körber & Hoff is such that

$$N > F+2H, \quad N > G+2H.$$

That there is nothing intrinsically impossible in these inequalities may be realized by remembering that $N = F+2H = G+2H$ when there is rotational symmetry of the anisotropy about the z axis.

The final feature of figure 2 which must be verified theoretically is that the general trend of Körber & Hoff's data was such that the figure of eight sloped downward from $\alpha = 0$ to $\alpha = \frac{1}{2}\pi$. Suppose θ_1 is the value of θ when $\alpha = 0$, and θ_2 the value when $\alpha = \frac{1}{2}\pi$. Then

$$\tan^2\theta_1 = \frac{G}{H}+1, \quad \tan^2\theta_2 = \frac{F}{H}+1.$$

By using the equations (6) it can be shown that the following possibilities arise, depending on the relative magnitude of the yield stresses X, Y, Z:

(i) If $Z > X$, Y then $\theta_2 \lesseqgtr \theta_1$, according as $X \lesseqgtr Y$. Both θ_1 and θ_2 are less than 55°.

(ii) If $Y > Z > X$ then $\theta_1 > 55° > \theta_2$.

(iii) If $X > Z > Y$ then $\theta_1 < 55° < \theta_2$.

(iv) If X, $Y > Z$, then, if H is positive, $\theta_2 \lesseqgtr \theta_1$ according as $X \lesseqgtr Y$. Both θ_1, θ_2 are greater than 55°. If H is negative the θ's are imaginary and expansion occurs in the transverse direction; according to the theory necking should be inhibited.

It is found for many rolled metals that after severe reductions the yield stress in the rolling direction is less than that in the transverse direction. In particular this was observed by Körber & Hoff for the metals used by them. In the present notation x is the rolling directionand so $Y > X$. The magnitude of Z was not measured. Now Körber & Hoff found $\theta_1 > \theta_2$, and this, according to the above analysis, is required by $Y > X$. Their values of θ_1, θ_2 were distributed haphazardly about 55° and so the relative magnitude of Z must have varied similarly.

The theory is thus in agreement with the experimental data at all points where a test is possible. Closer quantitative agreement could not be checked without a knowledge of the absolute magnitudes of the anisotropic parameters.

6. The effect of anisotropy developed in a torsion test

Consider a torsion test performed on an initially isotropic, thin-walled, cylinder. Let us fix attention on a small element of the wall. In figure 4*a* the plane of the paper is to be regarded as tangential to the cylinder wall; AA' is the direction of shear and BB' is the axial direction.

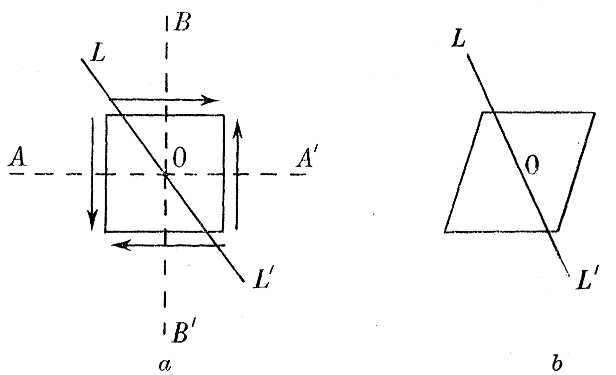


Figure 4. Deformation in pure torsion.

After twisting, the initially square element shears over into a parallelogram (figure 4*b*). An arbitrary direction LOL' fixed in the element rotates into a new position and simultaneously suffers a resultant extension or compression. For a given total shear it is clear that there will be some initial direction which undergoes most compression, and also one which undergoes most extension. A fibre structure will thus develop, and the principal axes of anisotropy in the plane of the wall at any moment can be expected to coincide approximately with the final positions of the directions of greatest extension and compression. These can be determined by a simple construction (figure 5). TT' is the direction of shear; ON, which can be taken of unit length, is the perpendicular on TT' from the centre O of an element. For a given total shear-strain γ there are two directions which undergo no resultant strain. One of these is clearly parallel to OT. If $DD' = \gamma$ and D, D' are reflexions in ON, then OD is the initial position of the other and OD' is its final position. This direction undergoes compression and then extension. Now let OC, OE be the internal and external bisectors of the angle between OT and OD. It is easy to prove that OC is the initial position of the direction of greatest compression and OE is that of the direction of greatest extension. These directions are mutually orthogonal. The final position of OC is OC' where C' is the reflexion of E in ON: the final position of OE is OE' where E' is the reflexion of C in ON. OE', OC' are of course the bisectors of the angle between OT' and OD'. Further, $CC' = DD' = EE' = \gamma$. We see therefore that, as the twist increases, the principal axes of anisotropy OE', OC' rotate from positions making $\frac{1}{4}\pi$ with ON, TT' towards the limiting positions ON, OT', and further that they vary relatively to the element itself. The degree of the anisotropy will also vary with progressive twisting, though not necessarily monotonically because of a possible complex Bauschinger effect due to alternate compression and extension.

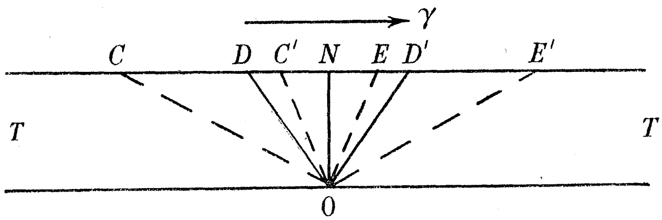


FIGURE 5. Construction for directions of greatest extension and compression in pure torsion.

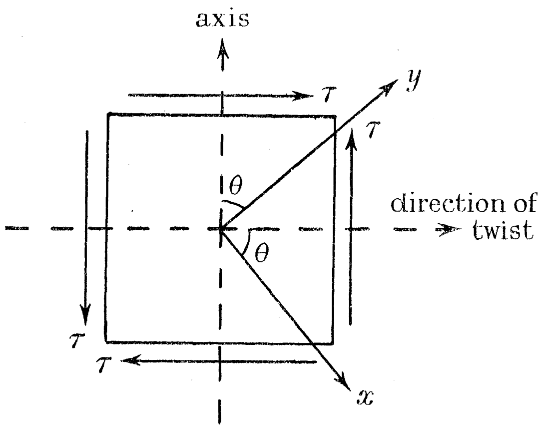


FIGURE 6. Co-ordinate system for analysis of the torsion of an anisotropic tube.

Suppose that at some stage during the torsion the x, y axes of anisotropy make an angle θ with the direction of twist and the cylinder axis (figure 6). As has been shown θ increases steadily from $\frac{1}{4}\pi$ towards $\frac{1}{2}\pi$. The x, y axes correspond respectively to $C'O$ and OE' in figure 5. Oy thus marks the fibre direction. If τ is the shear stress,

$$\sigma_{xx} = -\sigma_{yy} = -\tau \sin 2\theta, \quad \sigma_{xy} = \tau \cos 2\theta.$$

All other stress components are zero. The value of τ for which flow will continue is

$$\tau = [(F+G+4H)\sin^2 2\theta + 2N\cos^2 2\theta]^{-\frac{1}{2}}. \tag{19}$$

The anisotropic parameters continually change in magnitude during the twisting because of work-hardening and varying anisotropy. The strain-increments are

$$\left.\begin{aligned} d\epsilon_{xx} &= -d\lambda(G+2H)\tau\sin 2\theta, \\ d\epsilon_{yy} &= \quad d\lambda(F+2H)\tau\sin 2\theta, \\ d\epsilon_{zz} &= \quad d\lambda(G-F)\tau\sin 2\theta, \\ d\epsilon_{xy} &= \quad d\lambda\, N.\tau\cos 2\theta. \end{aligned}\right\} \tag{20}$$

There is thus thinning or thickening of the cylinder wall. The strain-increment $d\epsilon_A$ in the axial direction is obtained by tensor transformation of the elements in (20):

$$d\epsilon_A = d\lambda.\tau\sin 2\theta[(N-G-2H)\sin^2\theta-(N-F-2H)\cos^2\theta]. \tag{21}$$

With an ideally isotropic cylinder both the wall thinning and axial extension are of course zero.

Just after twisting is begun, when θ is a little larger than $\frac{1}{4}\pi$, the axial strain-increment has the sign of $(F-G)$ or of $(X-Y)$. In § 5 it was seen that in heavily rolled metals the yield stress along the fibre is usually less than that across the fibre. For small strains, when the anisotropy is still small, there seems to be no general rule. The cylinder may therefore shorten or lengthen initially according to the metal used. If for larger angles of twist we are entitled to assume as in § 5 that $N > G+2H$, then $d\epsilon_A$ is finally positive. The most that can be said at present, then, is that shortening or lengthening of a torsion specimen, or even reversal of axial strain, are all *a priori* possible on the basis of the theory. Such changes in length have been observed by Swift (1947).

7. The earing of deep-drawn cups

When a cup is deep drawn from a flat circular blank cut from rolled sheet, it is often found that the height of the rim above the base is not uniform, as would be expected in a symmetrical operation on an isotropic blank. Instead it is observed that 'ears' (to use the technical terminology) form in positions symmetrically situated with respect to the direction of rolling in the original sheet. Generally four ears occur, either at the ends of the two diameters making 45° with the rolling direction, or at the ends of the diameters making 0 and 90° with the rolling direction. The positions and height of the ears which are observed in any given instance depend among other things on the particular metal and on the prior mechanical and heat treatment (see, for example, Wilson & Brick 1945). Both types of earing can be produced in the same metal by suitably varying the treatment before drawing, e.g. in copper and steel (Barrett 1943). With brass, six ears have also been observed in the 0 and 60° positions (Cook 1937). In a few instances other earing positions have been reported (e.g. Cook & Richards 1943). It is recognized that the presence of earing is due to anisotropy in the rolled sheet, and several inconclusive attempts have been made to correlate the observed behaviour with the crystal texture and the mechanical properties of the material. The present approach is somewhat different. The anisotropy is specified by the six parameters of the theory, whose values are related in some complicated way to the previous treatment of the material. The earing positions can, in principle, be calculated in terms of these parameters and the stresses and strains characterizing the deep-drawing process.

Earing begins while the blank is being drawn towards the shoulder of the die, and it is observed that the final positions of the ears coincide approximately with their initial positions (Wilson & Brick 1945). It is probably sufficient, therefore, to analyze the stress and strain distribution immediately after drawing begins, when the rim has just started to move towards the die aperture. A complete analysis of the whole process which allows adequately for the gradual formation of the ears is, in any event, hardly possible. In the initial stages of drawing, the action of the blank-holder must be considered in any calculation of the stresses. There are two main ways in which a blank-holder may be used: it may be held in a fixed position, or it may be pressed against the sheet by a constant load. Possibly a blank-holder may not even be used at all. It seems that the role of the blank-holder in connexion with earing has not been considered; the subsequent analysis indicates its importance. During the process the material eventually forming the wall of the cup is drawn towards the die aperture, undergoing circumferential compression and radial extension. A thickening of the sheet may also occur during this stage, depending on the constraint exerted by the blank-holder. Two extreme situations will be considered in the analysis. In the first the blank-holder is supposed fixed in a position such that the space between the holder and the die is equal to the original sheet thickness. In the second it is supposed either that no holder is used, or that the space between the holder and die is so much wider than the sheet thickness that negligible normal force is exerted in the early stages. If friction can be neglected, the first case corresponds to a state of plane strain, and the second to a state of plane stress. The case when a fixed load is applied to the blank-holder will lie somewhere between these extremes.

It is assumed that the ears begin to form at those points on the rim where the radial direction is one of the principal axes of the strain-increment. This seems reasonable in view of the approximate symmetry of the deformation about the tip of an ear. For consistency it must also be assumed that the hollows between the ears begin to form at points having the same property. On the rim the circumferential stress is the only non-zero stress component in the plane of the sheet. Hence the positions of the ears and hollows correspond to the points at which the principal axes of stress and strain-increment coincide.

Let axes of reference be chosen to coincide with the principal directions of anisotropy in the sheet. The z axis is taken normal to the blank, and the x axis along the direction of rolling. If the blank is drawn under conditions of plane strain, it follows from the discussion in § 4 that the ears and hollows can only form in the 0, 45 and 90° positions. These are also the directions for which the uniaxial yield stress in plane strain has maximum and minimum values. From elementary considerations it is clear either that four ears form in the 0 and 90° positions with hollows in the 45°

positions, or that the reverse situation occurs. This will depend on the state of anisotropy. If the drawing occurs under conditions of plane stress then the discussion in § 4 shows that the ears and hollows can form only in the 0 and 90° positions and in positions making an angle ϕ with the rolling direction, where

$$\tan^2\phi = \frac{N-F-2H}{N-G-2H}. \tag{22}$$

It must be remembered in deriving this equation that the state of stress on the rim is a tangential compression, so that $\phi = 90° - \theta$, where θ is given by equation (14). According to § 4 the tangents at the points where the ears and hollows form are in the directions for which the yield stress in uniaxial tension has stationary values. Elementary considerations show, just as before, that there are four ears in the 0 and 90° positions, or in the ϕ positions. Notice that $\phi = 45°$ when $F = G$ (i.e. $X = Y$), irrespective of the values of N and H. If, however, N is intermediate to $F+2H$ and $G+2H$, ϕ is not real and only two ears form either in the 0 or the 90° positions.

It does not seem possible to distinguish between the possible arrangements of the ears in terms of the relative magnitudes of the anisotropic parameters, without a solution for the displacements on the rim. Such a solution has not yet been found. Failing this it is tempting to surmise that the ears and hollows form respectively at points where the tangents to the rim are in the directions of the minimum and maximum values of the uniaxial yield stresses. This could be tested by experiment. It would be essential to perform plane strain or plane stress tensile tests, according to the conditions of drawing. Summing up the above analysis, the theory indicates that when the blank is drawn under conditions of plane strain, four ears will form in the 0 and 90° positions or in the 45° positions. When ears are found in other positions, this may be due either to friction or to the nature of the constraint applied by the blank-holder.

The author is indebted to Dr E. Orowan, F.R.S., for suggesting the undertaking of this research. The content of the paper owes much to many stimulating discussions with Dr Orowan during the course of the work.

The author is also indebted to the Chief Scientific Officer, Ministry of Supply, for permission to publish this paper.

References

Barrett, C. S. 1943 *The structure of metals*, p. 443. McGraw-Hill.
Cook, M. 1937 *J. Inst. Met.* **60**, 159.
Cook, M. & Richards, T. Ll. 1943 *J. Inst. Met.* **69**, 201.
Cook, M. & Richards, T. Ll. 1947 *J. Inst. Met.* Paper 1061, 541.
Geiringer, H. 1937 *Mémor. Sci. Math.* **86**.
Hill, R., Lee, E. H. & Tupper, S. J. 1947 *Proc. Roy. Soc.* A, **191**, 278.
Körber, F. & Hoff, H. 1928 *Mitt. K.-Wilh.-Inst. Eisenforsch.* **10**, 175.
Körber, F. & Siebel, E. 1928 *Mitt. K.-Wilh.-Inst. Eisenforsch.* **10**, 189.
Swift, H. W. 1947 *Engineering*, **163**, 253.
Taylor, G. I. & Quinney, H. 1931 *Phil. Trans.* A, **230**, 323.
von Mises, R. 1928 *Z. angew. Math. Mech.* **8**, 161.
Wilson, F. H. & Brick, R. M. 1945 *Metals Technol.* **12**.