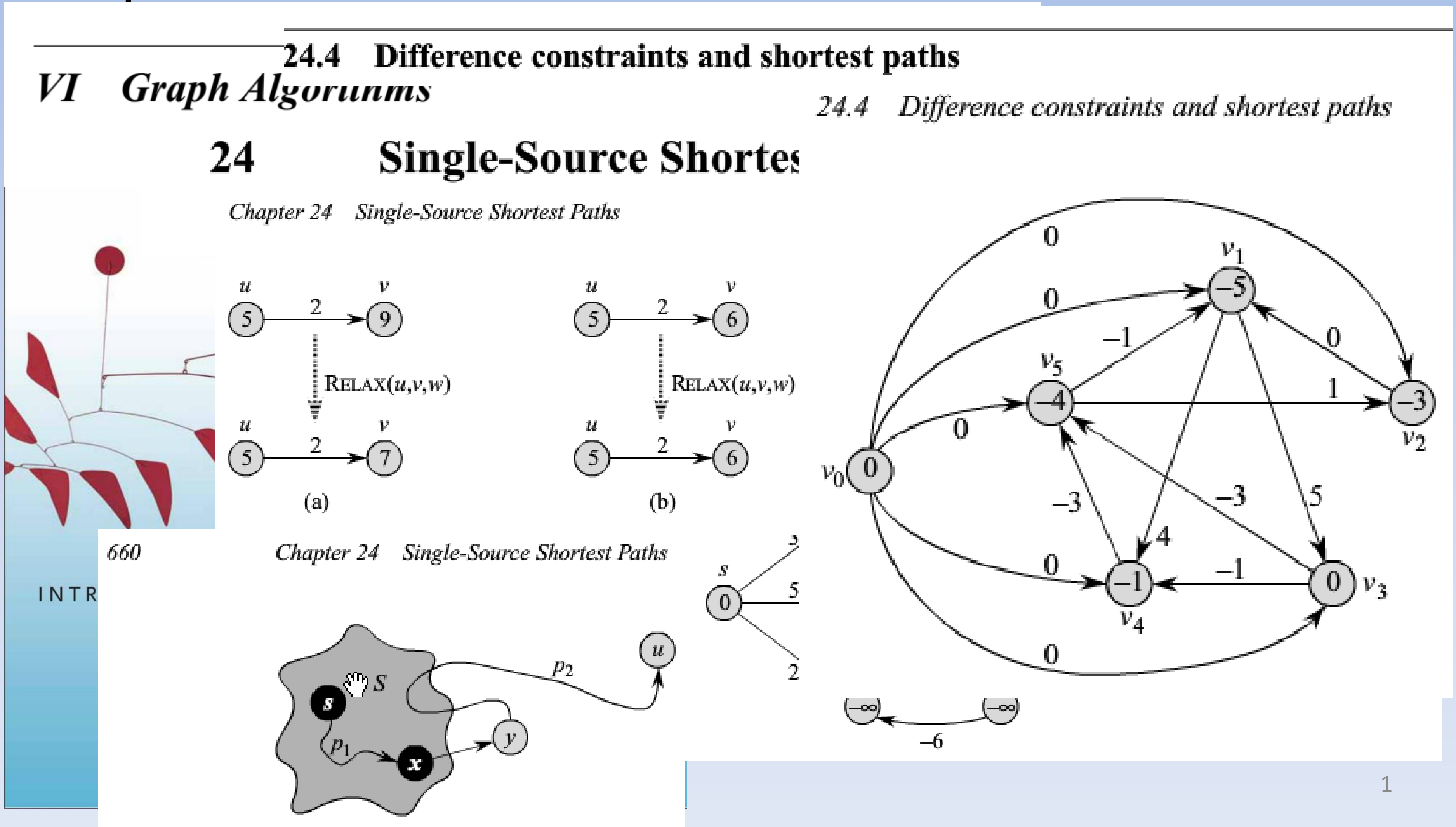
Design and Analysis of Algorithms

Section VI : Graph Algorithms

Chapter 24: Difference Constraints & Shortest Paths



Shortest-Paths Problem

- Given:
  - Weighted, Directed Graph G=(V, E)
  - Weight Function w: E ->
    - Edges -> Real-Valued Weights $\mathbb{R}$
- Weight of path $P=<v_0, v_1, \ldots, v_k>$
  - $w(p) = \sum w(v_{i-1}, v_i)$
- Shortest-Path Weight $\delta(u,v)$ is the minimum weight path w(p) that goes from u to v, otherwise ∞
- The shortest path from u to v is any path p with a weight of $\delta(u,v)$

Bellman-Ford

- Solves Single-Source Shortest-Paths Problem in General Case.
  - Weights may be negative.
- Given Graph G=(V,E) with source s and weight function w returns:
  - False: if negative-weight cycle exist
  - True: Otherwise

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| - 1
    for each edge (u, v) ∈ G.E
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

Relaxation
Step 1: Initialize-Single-Source

```
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex v ∈ G.V
    v.d = ∞
    v.π = NIL
s.d = 0
```

Relax (Pseudocode)

```
RELAX(u, v, w)
    if v.d > u.d + w(u, v)
        v.d = u.d + w(u, v)
        v.π = u
```

Bellman-Ford Complexity

```
BELLMAN-FORD(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  for i = 1 to |G.V| - 1
3      for each edge (u, v) ∈ G.E
4          RELAX(u, v, w)
5  for each edge (u, v) ∈ G.E
6      if v.d > u.d + w(u, v)
7          return FALSE
8  return TRUE
```

- Lines 2-4: |V|-1 passes over Edges in E
- O(VE)

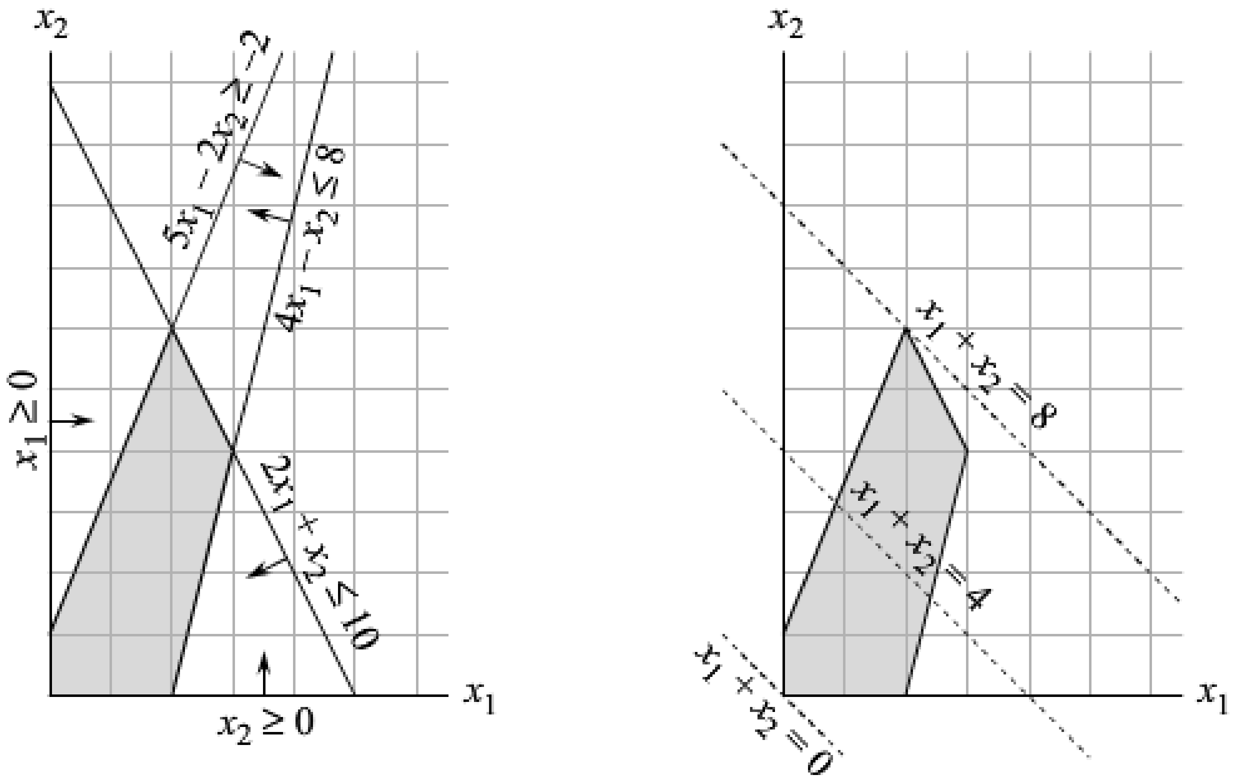
Linear Programming

- Problem-solving model for optimal allocation of scarce resources, among a number of competing activities.

$$
\begin{array}{lrcrcr}
\text{maximize} & 13A & + & 23B & & \\
\text{subject} & 5A & + & 15B & \leq & 480 \\
\text{to the} & 4A & + & 4B & \leq & 160 \\
\text{constraints} & 35A & + & 20B & \leq & 1190 \\
 & A & , & B & \geq & 0
\end{array}
$$

SEE LP PDF

Chapter 29



(a) (b)

Simplified Problem: 24.4 Difference Constraints & Shortest Paths

- Sometimes we don’t really care about the objective function, we just wish to find any FEASIBLE SOLUTION.
  - Any vector x that satisfiex Ax <- b
  - OR determine no solution exists.
- Lets look at Feasibility Problem.

Representation

- System of Difference Constraints:
  - Each row of linear-programming matrix A contains one 1 and one -1.
  - All other entries of A are 0
- Constraints Ax <= b are a set of m different constraints involving n unknowns.
- Each constraints is a simple linear inequality of the form:
  - ❑ $x_j - x_i <= b_k$
  - ❑ where 1 <= i,j <= n, i <> j, and 1 <= k <= m

Example

$$
\begin{aligned}
x_1 - x_2 &\leq 0\,, &(24.3)\\
x_1 - x_5 &\leq -1\,, &(24.4)\\
x_2 - x_5 &\leq 1\,, &(24.5)\\
x_3 - x_1 &\leq 5\,, &(24.6)\\
x_4 - x_1 &\leq 4\,, &(24.7)\\
x_4 - x_3 &\leq -1\,, &(24.8)\\
x_5 - x_3 &\leq -3\,, &(24.9)\\
x_5 - x_4 &\leq -3\,. &(24.10)
\end{aligned}
$$

Example

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \leq \begin{pmatrix} 0 \\ -1 \\ 1 \\ 5 \\ 4 \\ -1 \\ -3 \\ -3 \end{pmatrix}.$$

Example Solution

- Solution 1
  - X = (-5, -3, 0, -1, -4)
- Solution 2
  - X’ = (0, 2, 5, 4, 1)

- Solution are related!

Lemma 24.8

- Let $x = (x_1, x_2, \ldots, x_n)$ be a solution to a system $Ax \le b$ of difference constraints
- Let d be any constant
- $X + d = (x_1+d, x_2+d, \ldots, x_n+d)$ is a solution

Lemma 24.8 w/ Proof

- Let $x = (x_1, x_2, \ldots, x_n)$ be a solution to a system $Ax \le b$ of difference constraints
- Let d be any constant
- $X + d = (x_1+d, x_2+d, \ldots, x_n+d)$ is a solution

- For each xi and xj, we have:
  - $(x_j+d)-(x_i+d)=x_j-x_i$
- Thus is x satisfies, the x+d satisfies.

Constraint Graph

- We can interpret our Difference Constraints as a Graph, G = (V,E)
- Each vertex v_i corresponds to one of the n unknowns
- Each edge corresponds to one of the m inequalities.

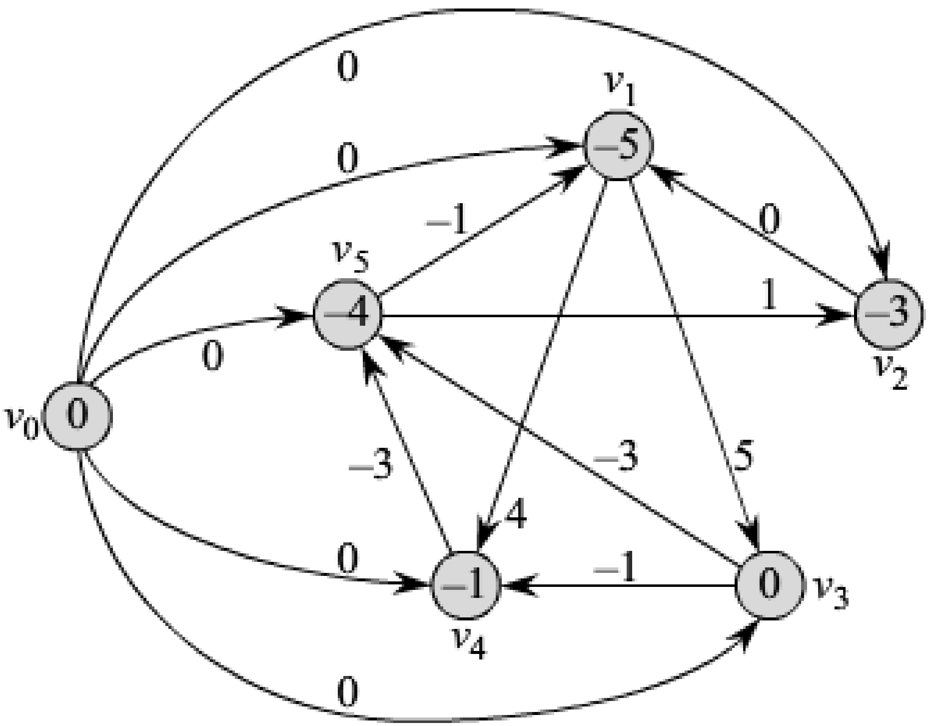
Constraint Graph

- We can interpret our Difference Constraints as a Graph, G = (V,E)
- Each vertex v_i corresponds to one of the n unknowns
- Each edge corresponds to one of the m inequalities.
  - V={v_0,v_1, v_2, …, v_n}
  - E={(v_i,v_j): xj-xi<=bk is a constraint} UNION
  - {(v_0,v_1), (v_0,v_2), … (v_0, v_n)}

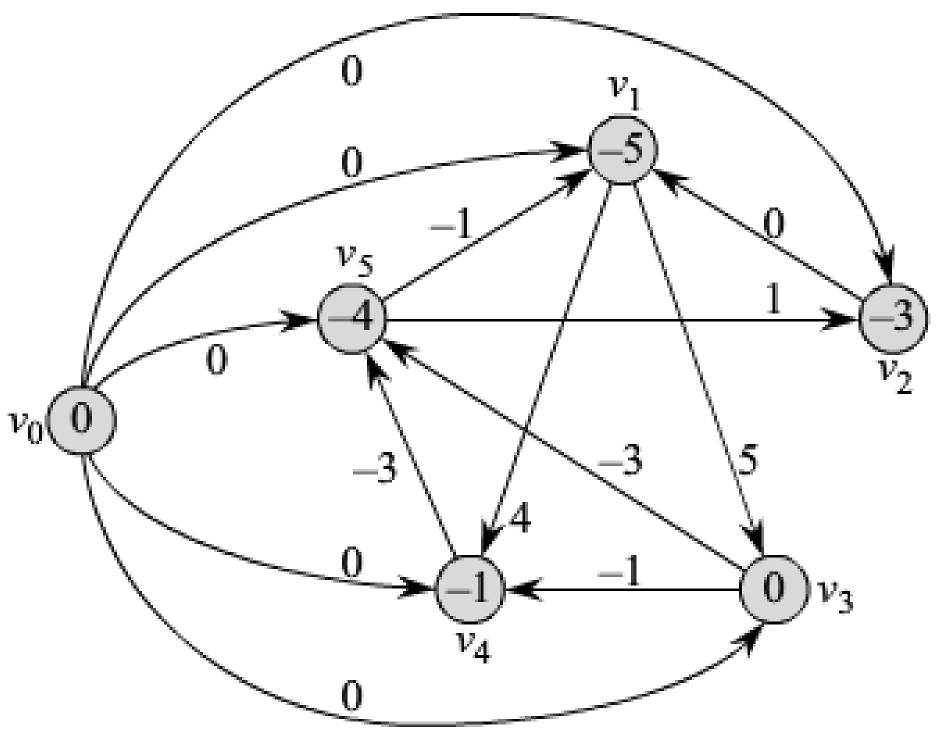
V_0

- V_0 is included with an edge to each vertex to insure all vertices are reachable.
- $\{(v_0,v_1), (v_0,v_2), \dots (v_0, v_n)\}$

Example Constraint Graph



Example Constraint Graph



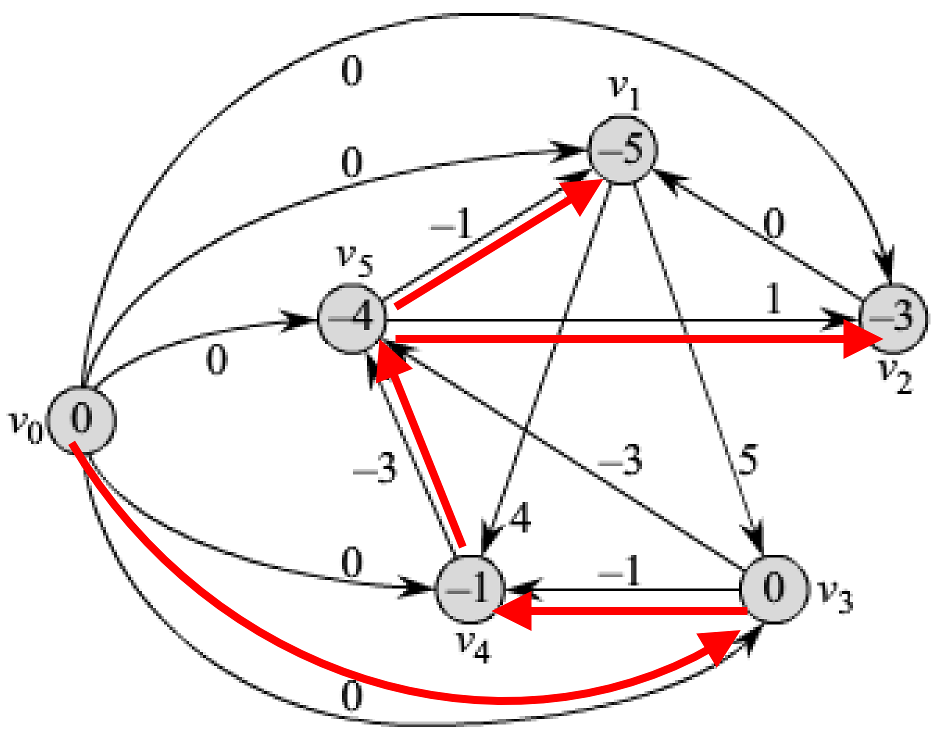
$$
\begin{aligned}
x_1 - x_2 &\le 0\,, \\
x_1 - x_5 &\le -1\,, \\
x_2 - x_5 &\le 1\,, \\
x_3 - x_1 &\le 5\,, \\
x_4 - x_1 &\le 4\,, \\
x_4 - x_3 &\le -1\,, \\
x_5 - x_3 &\le -3\,, \\
x_5 - x_4 &\le -3\,.
\end{aligned}
$$

Theorem 24.9

- Given a system $Ax \leq b$ of difference constraints, let $G=(V,E)$ be the corresponding constraint graph. If G contains no negative-weight cycles, a feasible solutions is:
  - ❑ $x = (\delta(v_0,v_1), \delta(v_0,v_2), \delta(v_0,v_3), \ldots , \delta(v_0,v_n))$

Example Constraint Graph

24.4 Difference constraints and shortest paths 667



$$
\begin{aligned}
x_1 - x_2 &\le 0 , \\
x_1 - x_5 &\le -1 , \\
x_2 - x_5 &\le 1 , \\
x_3 - x_1 &\le 5 , \\
x_4 - x_1 &\le 4 , \\
x_4 - x_3 &\le -1 , \\
x_5 - x_3 &\le -3 , \\
x_5 - x_4 &\le -3 .
\end{aligned}
$$

Theorem 24.9 w/Proof

- Consider any edge (v_i,v_j) in E
- By Triangle Inequality:
  - $\delta(v_0,v_j) \le \delta(v_0,v_i) + w(v_i,v_j)$
- OR: $\delta(v_0,v_j) - \delta(v_0,v_i) \le w(v_i,v_j)$
- Let $x_i = \delta(v_0,v_i)$ and $x_j = \delta(v_0,v_j)$
- THEN: x_i and x_j satisfies $x_j - x_i \le w(v_i,v_j)$

Theorem 24.9 w/Proof (b)

- If constraint graph contains a negative weight cycle, then it has no feasible solution.

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- If constraint graph contains a negative weight cycle, then it has no feasible solution.

➢ Let negative weight cycle be:
  - ➢ $c = (v_1, v_2, \ldots, v_k)$
  - ➢ It cannot contain v_0 since it has no entering edge.

Theorem 24.9 w/Proof (b)

- Let negative weight cycle be:
  - c = (v_1, v_2, … , v_k)
- Cycle corresponds to difference equations:

$$
\begin{aligned}
x_2 - x_1 &\leq w(v_1, v_2)\,, \\
x_3 - x_2 &\leq w(v_2, v_3)\,, \\
&\vdots \\
x_{k-1} - x_{k-2} &\leq w(v_{k-2}, v_{k-1})\,, \\
x_k - x_{k-1} &\leq w(v_{k-1}, v_k)\,.
\end{aligned}
$$

- Summing the left and right hand side must also be true.
- Given $x_1 = x_k$ Left side sums to 0 yielding:
  - $0 \leq w(c)$
  - $w(c) < 0$: CONTRADICTION

Bellman-Ford

- Bellman-Ford to find $\delta(v_0,v_i)$
- Shortest Paths from v_0 with negative weights and no negative-weight cycles.