Lecture 26: Linear Programming

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Recall that in a Linear program, you have some variables, an objective w.r.t. variables, and constraints on those variables.

Objective:

$$\max \text{ or } \min \sum_{i=1}^{n} c_i x_i$$

Constraints: s.t.

$$\sum_i a_i^{(j)} x_i \leq b^{(j)} \text{ for } j \in \{1, \ldots, k\}$$

The goal is to find the maximum value to the objective subject to the constraints. The reason why this is efficiently solvable is because the region turns out to be convex. As we discussed last time, if the region is not convex, it may become hard to solve.

Notice that the objective could be represented as a vector dot product $\max cx$ where c are the coefficients to the objective and X is the input vector. Similarly, the constraints could also be represented by m vector dot products; this can be turned into a matrix vector multiplication $Ax \leq b$. This is formalized as follows.

Objective:

$$\max \text{ or } \min c^t x$$

Constraints: s.t.

$$Ax \leq b$$

where c, x are vectors containing $c_1, \ldots, c_n$ and $x_1, \ldots, x_n$ respectively and A is a matrix containing $a_i^{(j)}$.

1 Duality

For each linear program, there is exactly one "dual" linear program. The motivation behind duality is for us to provide a verifiable certificate that proves the optimality of the solution of the linear program. Given some linear program, the "primal", $\max c^T x$, s.t. $Ax \leq b, x \geq 0$. We find the "dual" linear program by using the equation $\min b^T y$, s.t. $A^t y \geq c, y \leq 0$. [1]

Theorem (Strong Duality): The optimal value for the primal linear program is equal to the optimal value for the dual linear program. In other words $(\exists x, Ax \leq b) \implies \min b^T y = \max c^T x$.

This is a very powerful theorem whose effects can be felt in mathematics, economics, and game theory. You might notice the word "strong" in name of the theorem; this is done

[1] For an example of duality worked out, refer to the actual lecture notes.

to differentiate it from “weak” duality which only says that the primal is less than the dual. Though, it is not strong enough to show that the primal equals the dual. [2]

Recall max-flow and min-cut; we showed that the maximum flow is the minimum cut. To remind you of the proof, essentially assume the maximum flow exceeded the minimum cut, this creates a contradiction because there could not be more flow across that cut. Both of these can be represented as linear programs, and more importantly they’re duals of each other, meaning the optimal maximum flow equals the minimum cut.

2 Takeaways

1. Linear programs are powerful and can be solved in polynomial time.

2. Dual linear programs gives a proof of optimality (by strong duality).

We also covered some randomized algorithms, but we’re going to move this discussion to the Thursday notes, so the notes are more concise.

[2] It turns out that many LP solvers, in practice, solve both the primal and dual LPs simultaneously to check for correctness. Since both the primal and the dual have the same optimum, it is sufficient to use either to check the optimality of the solution for the primal LP.