Assignment 2 Solutions

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1 Solutions

1.1 Question 1

Given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} \tag{1}$$

1.1.1 Part A

The spectrum of A is found by finding the eigenvalues of A.

$$\begin{aligned}
A - \lambda I &= 0 && (2)\\
\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} &= 0 && (3)\\
\begin{bmatrix} 1-\lambda & 2 \\ -1 & 2-\lambda \end{bmatrix} &= 0 && (4)\\
((1-\lambda)(2-\lambda)) + 2 &= 0 && (5)\\
\lambda^2 - 3\lambda + 4 &= 0 && (6)
\end{aligned}$$

Now we simply factor to find the eigenvalues.

$$\begin{aligned}
\lambda &= \frac{3 \pm \sqrt{-7}}{2} && (7)\\
\lambda &= \frac{3 \pm \sqrt{7}i}{2} && (8)
\end{aligned}$$

Therefore the spectrum of A,

$$\boxed{\sigma(A) = \frac{3 \pm \sqrt{7}i}{2}} \tag{9}$$

The spectral radius of A is,

$$
\begin{aligned}
p(A) &= \max\{\sigma(A)\} && (10)\\
\text{Therefore, using} & && (11)\\
\lambda &= \frac{3-\sqrt{7}i}{2} && (12)\\
\text{and converting to the reals} & && (13)\\
&= \sqrt{\left(\frac{3\pm\sqrt{7}i}{2}\right)^2} && (14)\\
&= \sqrt{\frac{9}{4}+\frac{7}{4}} && (15)\\
&= \sqrt{\frac{16}{4}} && (16)\\
&= \sqrt{4} && (17)\\
\boxed{p(A)=2} & && (18)
\end{aligned}
$$

1.1.2 Part B

$$
\begin{aligned}
||A||_1 = \max\{2,4\} = \boxed{4} && (19)\\
||A||_{\text{inf}} = \max\{3,3\} = \boxed{3} && (20)\\
||A||_2 = \sqrt{\lambda_{\max}(A'A)} && (21)
\end{aligned}
$$

First we find the spectrum of $A'A$,

$$
\begin{aligned}
A'A &= \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} && (22)\\
&= \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} && (23)
\end{aligned}
$$

The eigenvalues are simply the diagonal entries meaning,

$$
\begin{aligned}
\sigma(A'A) = \{2,8\} & && (24)\\
\text{Therefore, } ||A||_2 &= \sqrt{\max\{2,8\}} && (25)\\
||A||_2 &= \boxed{\sqrt{8}} && (26)
\end{aligned}
$$

1.1.3 Part C

The left singular vectors of A are eigenvectors of AA'; we will denote this U. The right singular vectors of A are eigenvectors of $A'A$; we will dentote this V. The singular values of A are the square roots of the non-zero eigenvalues of $A'A$ and AA'; we will denote this S.

Note,

$$AA' = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}, A'A = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \tag{27}$$

First, we will find the left singular vectors of A. The eigenvalues of AA' are $\lambda = 2, 8$. We will now find eigenvectors for AA'.

Case: $\lambda = 2$,

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{28}$$

$$\text{A solution: } X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{29}$$

Case: $\lambda = 8$,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{30}$$

$$\text{A solution: } X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{31}$$

Therefore,

$$U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{32}$$

Next, we will find the right singular vectors of A. The eigenvalues of $A'A$ are $\lambda = 2, 8$ We will now find eigenvectors for $A'A$.

Case: $\lambda = 2$,

$$\begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{33}$$

$$\text{A solution: } X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{34}$$

Case: $\lambda = 8$,

$$\begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{35}$$

$$\text{A solution: } X = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{36}$$

Therefore,

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{37}$$

The singular values of A are the square roots of the eigenvalues of AA' and $A'A$. Therefore,

$$S = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{8} \end{bmatrix} \tag{38}$$

The singular value decomposition of A is given by $A = USV'$.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{8} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{39}$$

$$A = \begin{bmatrix} 1.4142 & 2.8284 \\ -1.4142 & 2.8282 \end{bmatrix} \tag{40}$$

The matrix above is in fact not A, but it is only off by a scalar. If we divide by 1.4142 we get the original matrix A. Therefore we will encorporate this scalar division into U. Now,

$$U = \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \tag{41}$$

In full,

$$U = \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}, S = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{8} \end{bmatrix}, V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{42}$$

1.2 Question 2

1.2.1 Part A

1.2.2 Part B

1.2.3 Part C

1.2.4 Part D

1.2.5 Part E

1.3 Question 3

1.3.1 Part A

1.3.2 Part B

1.4 Question 4

1.5 Question 5