

# Assignment 3 Solutions

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# 1 Solutions

## 1.1 Question 1

Listing 1: Matlab Commands

```
function [P] = find_vectors(m, n, s)

    % create the x and y ranges
    x=(1:m);
    y=(1:n);

    % create the meshgrid (x,y) coordinates
    [X,Y]=meshgrid(x,y);

    % create (x,y) pairs in form [j;k]
    A=[X(:)' ;Y(:)'];

    % solve each  $s*[p;o]=[j;k]$  for all  $[j;k]$ 
    B=s\A;

    % find columns with integer solutions
    cols=all(mod(B,1)==0);

    % return  $[j;k]$ 's with integer  $[p;o]$ 's
    P=A(:,cols);

end
```

### 1.1.1 Part A

#### Listing 2: Matlab Commands

```
s=[2,0;0,2];  
find_vectors(4,6,s)
```

**ans =**

2	2	2	4	4	4
2	4	6	2	4	6

### 1.1.2 Part B

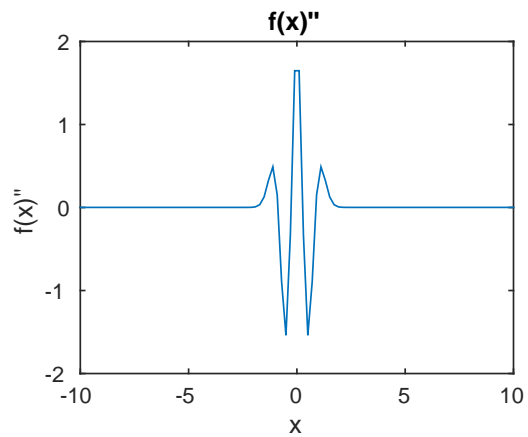
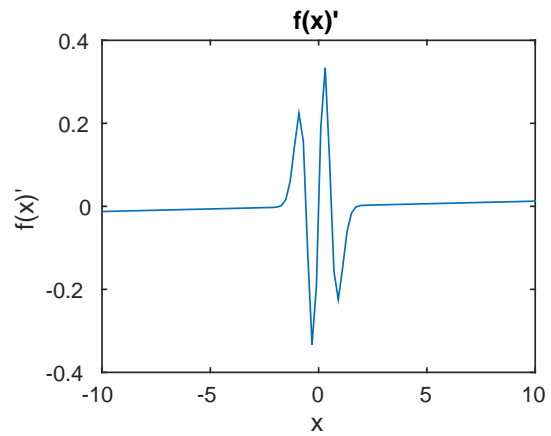
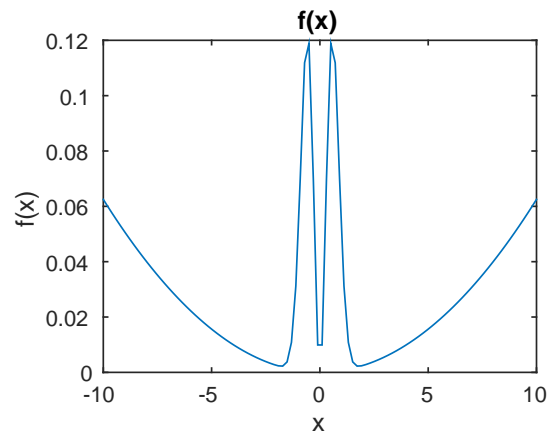
#### Listing 3: Matlab Commands

```
s=[1,1;1,-1];  
find_vectors(5,6,s)
```

**ans =**

1	1	1	2	2	2	3	3	3	4	4	4	5	5	5
1	3	5	2	4	6	1	3	5	2	4	6	1	3	5

## 1.2 Question 2



### Listing 4: Matlab Commands

```
% max/min for f(x)
x( find( yi(x)==max( yi(x) ) ) )
ans =

    -0.5051    0.5051

x( find( yi(x)==min( yi(x) ) ) )
ans =

    -1.7172    1.7172
```

#### Listing 5: Matlab Commands

```
x=linspace(-10,10);

y=sym('x^2*exp(-3*x^2)+(x/40)^2')
y =
x^2*exp(-3*x^2) + x^2/1600

dy=diff(y)
dy =
x/800 + 2*x*exp(-3*x^2) - 6*x^3*exp(-3*x^2)

ddy=diff(dy)
ddy =
2*exp(-3*x^2) - 30*x^2*exp(-3*x^2) + 36*x^4*exp(-3*x^2) + 1/800

yi=inline(y);
dyi=inline(dy);
ddyi=inline(ddy);

subplot

% Plot f(x)
subplot(2,2,1);
plot(x,yi(x));
title('f(x)');
xlabel('x');
ylabel('f(x)');

% Plot f(x)'
subplot(2,2,2);
plot(x,dyi(x));
title('f(x)''');
xlabel('x');
ylabel('f(x)''');

% Plot f(x)''
subplot(2,2,3);
plot(x,ddyi(x));
title('f(x)''''');
xlabel('x');
ylabel('f(x)''''');
```

## 1.3 Question 3

### 1.3.1 Part C

The claim made from assignment 2 question 3 part B was as follows:

$$\|A^{-1}\|_2 = \max_{s_A \in s(A)} \left\{ \frac{1}{s_a} \right\} \quad (1)$$

We must find a formula for the condition number of  $A$ ,  $c_2(A)$ , in terms of  $\sigma(A'A)$ . Note, we know  $\|A\|_2 = \max\{s(A)\}$  where  $s(A)$  are the singular values of  $A$ . The formula for the condition number of  $A$  is as follows:

$$c_2(A) = \|A\|_2 \|A^{-1}\|_2 \quad (2)$$

Therefore, to define  $c_2(A)$  in terms of the  $\sigma(A'A)$  we simply substitute our formulas for  $\|A\|_2$  and  $\|A^{-1}\|_2$  into (2). This yeilds,

$$c_2(A) = \max\{s(A)\} \max_{s_A \in s(A)} \left\{ \frac{1}{s_a} \right\} \quad (3)$$

### 1.3.2 Part D

The claim made from assignment 2 question 2 part E was as follows:

$$s(A) = \{|\lambda| : \lambda \in \sigma(A), \lambda \neq 0\} \quad (4)$$

Stated, this says "The singular values of a matrix  $A$  are the absolute values of the non-zero eigenvalues of  $A$ , where  $A$  is symmetric".

The formula for the condition number of a symmetric matrix  $A$  is the same as the formula stated in part C. The only change in definition is that of the singular values of  $A$ . Thefore the formula, using the definition of  $s(A)$  from 4, is,

$$c_2(A) = \max\{s(A)\} \max_{s_A \in s(A)} \left\{ \frac{1}{s_a} \right\} \quad (5)$$

### 1.3.3 Part E

#### Listing 6: Matlab Commands

```
% Check for general matrix A
A = [1,2;-1,2];
[P,D] = eig(A'*A);
singA = sqrt(diag(D));
a_norm = max(singA);
inva_norm = max(1./singA);
a_norm*inva_norm

ans =

    2

cond(A,2)

ans =

    2.0000

% Check for a symmetric matrix A
A = [-92,144;144,-8];
[P,D] = eig(A);
singA = abs(diag(D));
a_norm = max(singA);
inva_norm = max(1./singA);
a_norm*inva_norm

ans =

    2.0000

cond(A,2)

ans =

    2.0000
```

## 1.4 Question 4

### 1.4.1 Part A

Listing 7: Matlab Commands

```
function [A] = q4_partA(T,k)
    [exp,base]=meshgrid(0:k,T);
    A=base.^exp;
end
```

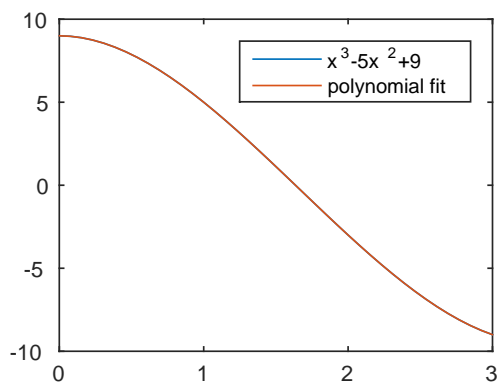
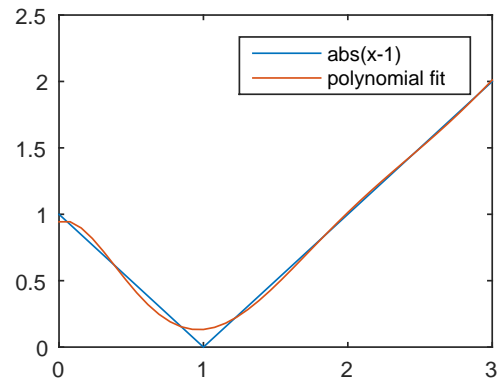
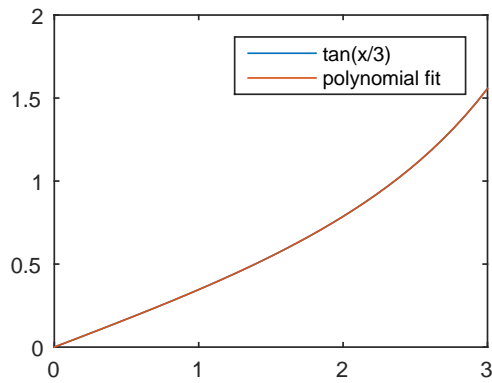
### 1.4.2 Part B

Listing 8: Matlab Commands

```
function [p] = q4_partB(f,a,b,m,k)
    T=linspace(a,b,m);
    p = polyfit(T,f(T),k);
end
```



### 1.4.3 Part C



From these plots we can see that a higher order polynomial is quite good at fitting a lower order polynomial on a small range. This is apparent from the plots of  $\tan(\frac{x}{3})$  and  $x^3 - 5x^2 + 9$ . However, the fit is less tight for  $|x - 1|$ , a function that has a point where it is not differentiable (e.g., in general, non-smooth functions).

### Listing 9: Matlab Commands

```
% Variables used throughout
a=0; b=3; m=40; k=6;
T=linspace(a,b,m);
A=q4_partA(T,k);
subplot;

% Running on function tan(x/3)
f = inline(sym('tan(x/3)'));
p=q4_partB(f,a,b,m,k);
p=p(end:-1:1);

% Plotting the function and the fit
subplot(2,2,1);
plot(T,f(T));
hold on
plot(T,A*p');
legend('tan(x/3)', 'polynomial fit');

% Running on function abs(x-1)
f = inline(sym('abs(x-1)'));
p=q4_partB(f,a,b,m,k);
p=p(end:-1:1);

% Plotting the function and the fit
subplot(2,2,2);
plot(T,f(T));
hold on
plot(T,A*p');
legend('abs(x-1)', 'polynomial fit');

% Running on function x^3-5x^2+9
f = inline(sym('x^3-5*x^2+9'));
p=q4_partB(f,a,b,m,k);
p=p(end:-1:1);

% Plotting the function and the fit
subplot(2,2,3);
plot(T,f(T));
hold on
plot(T,A*p');
legend('x^3-5x^2+9', 'polynomial fit');
```

## 1.5 Question 5

### 1.5.1 Part A

We must prove that a matrix  $A$  with a real Cholesky factorization is symmetric positive definite. Therefore we must prove two things. The first is that  $A$  is positive definite. The second is that  $A$  is symmetric. To this end we know  $A = C'C$ .

Therefore we can prove positive definiteness simply as follows,

$$(Av, v) \geq 0 \quad (6)$$

$$(C'Cv, v) \geq 0 \quad (7)$$

$$(Cv, Cv) \geq 0 \quad (8)$$

This concludes the proof for positive definiteness.

Next, we will prove that  $A$  is symmetric. Note, a matrix  $A$  is symmetric if  $A = A'$ . The proof is a simple substitution as follows,

$$A = A' \quad (9)$$

$$C'C = (C'C)'\quad (10)$$

$$C'C = C'C \quad (11)$$

This concludes the proof that  $A$  is symmetric. The proof that  $A$  is symmetric positive definite is complete.

### 1.5.2 Part B

### 1.5.3 Part C

### 1.5.4 Part D

## 1.6 Question 6