

Assignment 2 Solutions

Aaron Cahn
University of Wisconsin-Madison
cahn@cs.wisc.edu

February 24, 2015

1 Solutions

1.1 Question 1

Given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} \quad (1)$$

1.1.1 Part A

The spectrum of A is found by finding the eigenvalues of A .

$$A - \lambda I = 0 \quad (2)$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0 \quad (3)$$

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 2-\lambda \end{bmatrix} = 0 \quad (4)$$

$$((1-\lambda)(2-\lambda)) + 2 = 0 \quad (5)$$

$$\lambda^2 - 3\lambda + 4 = 0 \quad (6)$$

Now we simply factor to find the eigenvalues.

$$\lambda = \frac{3 \pm \sqrt{-7}}{2} \quad (7)$$

$$\lambda = \frac{3 \pm \sqrt{7}i}{2} \quad (8)$$

Therefore the spectrum of A ,

$$\boxed{\sigma(A) = \frac{3 \pm \sqrt{7}i}{2}} \quad (9)$$

The spectral radius of A is,

$$p(A) = \max\{\sigma(A)\} \quad (10)$$

Therefore, using (11)

$$\lambda = \frac{3 - \sqrt{7}i}{2} \quad (12)$$

and converting to the reals (13)

$$= \sqrt{\left(\frac{3 \pm \sqrt{7}i}{2}\right)^2} \quad (14)$$

$$= \sqrt{\frac{9}{4} + \frac{7}{4}} \quad (15)$$

$$= \sqrt{\frac{16}{4}} \quad (16)$$

$$= \sqrt{4} \quad (17)$$

$$\boxed{p(A) = 2} \quad (18)$$

Listing 1: Matlab Commands

```
A = [1,2;-1,2];
eig(A)

ans =

    1.5000 + 1.3229 i
    1.5000 - 1.3229 i

abs(ans)

ans =

    2.0000
    2.0000

diary off
```

1.1.2 Part B

$$\|A\|_1 = \max\{2, 4\} = \boxed{4} \quad (19)$$

$$\|A\|_{\inf} = \max\{3, 3\} = \boxed{3} \quad (20)$$

$$\|A\|_2 = \sqrt{\lambda_{\max}(A'A)} \quad (21)$$

First we find the spectrum of $A'A$,

$$A'A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \quad (22)$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \quad (23)$$

The eigenvalues are simply the diagonal entries meaning,

$$\sigma(A'A) = \{2, 8\} \quad (24)$$

$$\text{Therefore, } \|A\|_2 = \sqrt{\max\{2, 8\}} \quad (25)$$

$$\|A\|_2 = \boxed{\sqrt{8}} \quad (26)$$

Listing 2: Matlab Commands

```
norm(A,1)
ans =

    4

norm(A,2)
ans =

    2.8284

norm(A,inf)
ans =

    3
```

1.1.3 Part C

The left singular vectors of A are eigenvectors of AA' ; we will denote this U . The right singular vectors of A are eigenvectors of $A'A$; we will denote this V . The singular values of A are the square roots of the non-zero eigenvalues of $A'A$ and AA' ; we will denote this S .

Note,

$$AA' = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}, A'A = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \quad (27)$$

First, we will find the left singular vectors of A . The eigenvalues of AA' are $\lambda = 2, 8$. We will now find eigenvectors for AA' .

Case: $\lambda = 2$,

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad (28)$$

$$\text{A solution: } X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (29)$$

Case: $\lambda = 8$,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad (30)$$

$$\text{A solution: } X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (31)$$

Therefore,

$$U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad (32)$$

Next, we will find the right singular vectors of A . The eigenvalues of $A'A$ are $\lambda = 2, 8$. We will now find eigenvectors for $A'A$.

Case: $\lambda = 2$,

$$\begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad (33)$$

$$\text{A solution: } X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (34)$$

Case: $\lambda = 8$,

$$\begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad (35)$$

$$\text{A solution: } X = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (36)$$

Therefore,

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (37)$$

The singular values of A are the square roots of the eigenvalues of AA' and $A'A$. Therefore,

$$S = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{8} \end{bmatrix} \quad (38)$$

The singular value decomposition of A is given by $A = USV'$.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{8} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (39)$$

$$A = \begin{bmatrix} 1.4142 & 2.8284 \\ -1.4142 & 2.8282 \end{bmatrix} \quad (40)$$

The matrix above is in fact not A , but it is only off by a scalar. If we divide by 1.4142 we get the original matrix A . Therefore we will incorporate this scalar division into U . Now,

$$U = \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \quad (41)$$

In full,

$$U = \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}, S = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{8} \end{bmatrix}, V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (42)$$

Listing 3: Matlab Commands

```
[U,S,V] = svd(A)
```

U =

```
    0.7071    -0.7071  
    0.7071     0.7071
```

S =

```
    2.8284     0  
         0    1.4142
```

V =

```
     0    -1  
     1     0
```

1.2 Question 2

1.2.1 Part A

The example symmetric matrix is,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 2 \\ 3 & 7 & 2 & 9 \\ 4 & 2 & 9 & 4 \end{bmatrix} \quad (43)$$

The eigenvalues and eigenvectors for A are thus,

$$\lambda = \begin{bmatrix} 17.4770 & 0 & 0 & 0 \\ 0 & -7.5267 & 0 & 0 \\ 0 & 0 & 3.0000 & 0 \\ 0 & 0 & 0 & -0.9502 \end{bmatrix} \quad (44)$$

$$V = \begin{bmatrix} 0.3043 & 0.0760 & 0.2673 & 0.9112 \\ 0.4769 & -0.3446 & -0.8018 & 0.1047 \\ 0.6022 & 0.7534 & -0.0000 & -0.2640 \\ 0.5633 & -0.5549 & 0.5345 & -0.2986 \end{bmatrix} \quad (45)$$

And the eigenvalues and eigenvectors for $A'A$ are thus,

$$\lambda = \begin{bmatrix} 305.4452 & 0 & 0 & 0 \\ 0 & 56.6519 & 0 & 0 \\ 0 & 0 & 9.0000 & 0 \\ 0 & 0 & 0 & 0.9030 \end{bmatrix} \quad (46)$$

$$V = \begin{bmatrix} 0.3043 & -0.0760 & 0.2673 & -0.9112 \\ 0.4769 & 0.3446 & -0.8018 & -0.1047 \\ 0.6022 & -0.7534 & 0.0000 & 0.2640 \\ 0.5633 & 0.5549 & 0.5345 & 0.2986 \end{bmatrix} \quad (47)$$

Therefore, it is clear from this example that the eigenvectors for A and $A'A$ are the same and the eigenvalues of $A'A$ are the eigenvalues of A squared. The conjecture is thus, (λ, v) is an eigenpair of $A'A$ if and only if $(\sqrt{\lambda}, v)$ is an eigenpair of A ; (for symmetric A).

Listing 4: Matlab Commands

```
A = [1,2,3,4;2,5,7,2;3,7,2,9;4,2,9,4];  
[P1,D1] = eigs(A)
```

P1 =

0.3043	0.0760	0.2673	0.9112
0.4769	-0.3446	-0.8018	0.1047
0.6022	0.7534	-0.0000	-0.2640
0.5633	-0.5549	0.5345	-0.2986

D1 =

17.4770	0	0	0
0	-7.5267	0	0
0	0	3.0000	0
0	0	0	-0.9502

```
[P2,D2] = eigs(A'*A)
```

P2 =

0.3043	-0.0760	0.2673	-0.9112
0.4769	0.3446	-0.8018	-0.1047
0.6022	-0.7534	0.0000	0.2640
0.5633	0.5549	0.5345	0.2986

D2 =

305.4452	0	0	0
0	56.6519	0	0
0	0	9.0000	0
0	0	0	0.9030

```
diary off
```

1.2.2 Part B

Since A is a symmetric matrix we know that A can be decomposed to $A = PDP^{-1}$. Where the columns of P are the eigenvectors of A and D is a diagonal matrix where the diagonal entries are the eigenvalues of A . The proof is as follows,

$$A = PDP^{-1} \quad (48)$$

$$\text{Squaring both sides,} \quad (49)$$

$$A^2 = PDP^{-1}PDP^{-1} \quad (50)$$

$$= PD(P^{-1}P)P^{-1} \quad (51)$$

$$= PDDP^{-1} \quad (52)$$

$$= PD^2P^{-1} \quad (53)$$

We know that $A^2 = A'A$. The columns of P are the eigenvectors of $A'A$ and D^2 is a diagonal matrix where the diagonal entries are the eigenvalues of $A'A$. Therefore, the eigenvalues of $A'A$ are the eigenvalues of A squared. This concludes the proof.

1.2.3 Part C

Theorem: The 2-norm of a symmetric matrix A is as follows,

$$\|A\|_2 = p(A) \quad (54)$$

$$\text{Where, } p(A) := \max\{|\lambda| : \lambda \in \sigma(A)\} \quad (55)$$

We must check the theorem against,

$$B = \begin{bmatrix} -92 & 144 \\ 144 & -8 \end{bmatrix} \quad (56)$$

The eigenvalues of B are -200 and 100 . Therefore, $\sigma(B) = \{-200, 100\}$ and $p(B) = 200$. Using Matlab we find $\|B\|_2 = 200$. Therefore, the theorem holds for B .

Listing 5: Matlab Commands

```
B = [-92,144;144,-8];
```

```
eigs(B)

ans =

    -200.0000
     100.0000

abs(ans)

ans =

    200.0000
    100.0000

norm(B,2)

ans =

    200

diary off
```

1.2.4 Part D

This theorem does not hold for the non-symmetric matrix,

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad (57)$$

We find $\sigma(C) = \{5.3723, -0.3723\}$ and $p(C) = 5.3723$. Using Matlab we find $\|C\|_2 = 5.4650$. Therefore, the theorem does not hold for C .

Listing 6: Matlab Commands

```
C = [1,2;3,4];
eigs(C)

ans =
```

```

    5.3723
   -0.3723

abs(ans)

ans =

    5.3723
    0.3723

norm(C,2)

ans =

    5.4650

diary off

```

1.2.5 Part E

The singular values of a symmetric matrix A are the absolute values of the non-zero eigenvalues of A .

1.3 Question 3

1.3.1 Part A

We want to prove that (λ, v) is an eigenpair of A if and only if $(\frac{1}{\lambda}, v)$ is an eigenpair of A^{-1} . The proof is as follows,

$$Av = \lambda v \quad (58)$$

$$A^{-1}Av = A^{-1}\lambda v \quad (59)$$

$$(A^{-1}A)v = A^{-1}\lambda v \quad (60)$$

$$Iv = A^{-1}\lambda v \quad (61)$$

$$v = A^{-1}\lambda v \quad (62)$$

$$\frac{1}{\lambda}v = A^{-1}v \quad (63)$$

The 'only if' portion follows by symmetry.

1.3.2 Part B

The 2-norm of A^{-1} should be $1 / (\text{the smallest singular value of } A)$. We will denote the singular values of A as $s(A)$. Stated formally,

$$\|A^{-1}\|_2 = \max_{s_A \in s(A)} \left\{ \frac{1}{s_A} \right\} \quad (64)$$

This is true because we know from (a) that (λ, v) is an eigenpair for A if and only if $(\frac{1}{\lambda}, v)$ is an eigenpair of A^{-1} . Therefore, it must be the case that (λ, v) is an eigenpair for $A'A$ if and only if $(\frac{1}{\lambda}, v)$ is an eigenpair of $(A^{-1})'A^{-1}$ because the 2-norm is preserved. Since the λ 's of $A'A$ are the singular values of A it follows that $\frac{1}{\lambda}$'s of $A'A$ are the singular values of A^{-1} . Therefore, since $\|A\|_2 = \max\{s(A)\}$,

$$\|A^{-1}\|_2 = \max_{s_A \in s(A)} \left\{ \frac{1}{s_A} \right\} \quad (65)$$

1.4 Question 4

QR factor the matrix Z where matrix Z is,

$$Z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \\ 4 & 2 & 3 \\ 4 & 2 & 2 \end{bmatrix} \quad (66)$$

This will require three iterations.

Iteration 1:

$$x = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 4 \\ 4 \end{bmatrix}, y = \begin{bmatrix} 9.8995 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w = \begin{bmatrix} -0.6704 \\ 0.3013 \\ 0.5273 \\ 0.3013 \\ 0.3013 \end{bmatrix} \quad (67)$$

$$H = \begin{bmatrix} 0.1010 & 0.4041 & 0.7071 & 0.4041 & 0.4041 \\ 0.4041 & 0.8184 & -0.3178 & -0.1816 & -0.1816 \\ 0.7071 & -0.3178 & 0.4438 & -0.3178 & -0.3178 \\ 0.4041 & -0.1816 & -0.3178 & 0.8184 & -0.1816 \\ 0.4041 & -0.1816 & -0.3178 & -0.1816 & 0.8184 \end{bmatrix} \quad (68)$$

$$Q = \begin{bmatrix} 0.1010 & 0.4041 & 0.7071 & 0.4041 & 0.4041 \\ 0.4041 & 0.8184 & -0.3178 & -0.1816 & -0.1816 \\ 0.7071 & -0.3178 & 0.4438 & -0.3178 & -0.3178 \\ 0.4041 & -0.1816 & -0.3178 & 0.8184 & -0.1816 \\ 0.4041 & -0.1816 & -0.3178 & -0.1816 & 0.8184 \end{bmatrix} \quad (69)$$

$$R = \begin{bmatrix} 9.8995 & 9.4954 & 9.6975 \\ -0.0000 & 1.6311 & 2.9897 \\ -0.0000 & 2.1044 & 1.7320 \\ -0.0000 & -1.3689 & -0.0103 \\ -0.0000 & -1.3689 & -1.0103 \end{bmatrix} \quad (70)$$

Iteration 2:

$$x = \begin{bmatrix} 9.4954 \\ 1.6311 \\ 2.1044 \\ -1.3689 \\ -1.3689 \end{bmatrix}, y = \begin{bmatrix} 9.4954 \\ 3.2919 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 0 \\ -0.5023 \\ 0.6364 \\ -0.4140 \\ -0.4140 \end{bmatrix} \quad (71)$$

$$H = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0.4955 & 0.6393 & -0.4158 & -0.4158 \\ 0 & 0.6393 & 0.1900 & 0.5269 & 0.5269 \\ 0 & -0.4158 & 0.5269 & 0.6572 & -0.3428 \\ 0 & -0.4158 & 0.5269 & -0.3428 & 0.6572 \end{bmatrix} \quad (72)$$

$$Q = \begin{bmatrix} 0.1010 & 0.3162 & 0.8185 & 0.3316 & 0.3316 \\ 0.4041 & 0.3534 & 0.2714 & -0.5649 & -0.5649 \\ 0.7071 & 0.3906 & -0.4537 & 0.2661 & 0.2661 \\ 0.4041 & -0.5580 & 0.1590 & 0.5082 & -0.4918 \\ 0.4041 & -0.5580 & 0.1590 & -0.4918 & 0.5082 \end{bmatrix} \quad (73)$$

$$R = \begin{bmatrix} 9.8995 & 9.4954 & 9.6975 \\ -0.0000 & 3.2919 & 3.0129 \\ -0.0000 & -0.0000 & 1.7026 \\ -0.0000 & 0.0000 & 0.0089 \\ -0.0000 & 0.0000 & -0.9911 \end{bmatrix} \quad (74)$$

Iteration 3:

$$x = \begin{bmatrix} 9.6975 \\ 3.0129 \\ 1.7026 \\ 0.0089 \\ -0.9911 \end{bmatrix}, y = \begin{bmatrix} 9.6975 \\ 3.0129 \\ 1.9701 \\ 0 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 0 \\ -0.2606 \\ 0.0086 \\ -0.9654 \end{bmatrix} \quad (75)$$

$$H = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0.8642 & 0.0045 & -0.5031 \\ 0 & 0 & 0.0045 & 0.9999 & 0.0167 \\ 0 & 0 & -0.5031 & 0.0167 & -0.8641 \end{bmatrix} \quad (76)$$

$$Q = \begin{bmatrix} 0.1010 & 0.3162 & 0.5420 & 0.3408 & -0.6928 \\ 0.4041 & 0.3534 & 0.5162 & -0.5730 & 0.3422 \\ 0.7071 & 0.3906 & -0.5248 & 0.2684 & 0.0028 \\ 0.4041 & -0.5580 & 0.3871 & 0.5006 & 0.3534 \\ 0.4041 & -0.5580 & -0.1204 & -0.4825 & -0.5273 \end{bmatrix} \quad (77)$$

$$R = \begin{bmatrix} 9.8995 & 9.4954 & 9.6975 \\ -0.0000 & 3.2919 & 3.0129 \\ -0.0000 & -0.0000 & 1.9701 \\ -0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -0.0000 & 0.0000 \end{bmatrix} \quad (78)$$

Therefore the QR factorization of Z is

$$Q = \begin{bmatrix} 0.1010 & 0.3162 & 0.5420 & 0.3408 & -0.6928 \\ 0.4041 & 0.3534 & 0.5162 & -0.5730 & 0.3422 \\ 0.7071 & 0.3906 & -0.5248 & 0.2684 & 0.0028 \\ 0.4041 & -0.5580 & 0.3871 & 0.5006 & 0.3534 \\ 0.4041 & -0.5580 & -0.1204 & -0.4825 & -0.5273 \end{bmatrix} \quad (79)$$

$$R = \begin{bmatrix} 9.8995 & 9.4954 & 9.6975 \\ -0.0000 & 3.2919 & 3.0129 \\ -0.0000 & -0.0000 & 1.9701 \\ -0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -0.0000 & 0.0000 \end{bmatrix} \quad (80)$$

Listing 7: Matlab Commands

```
Z=[1,2,3;4,5,6;7,8,7;4,2,3;4,2,2];  
R=Z;
```

```
% Iteration 1
```

```
x=R(:,1);  
a=norm(x(1:5),2);  
y=[x(1:0)' a zeros(1,4)]';  
w=(x-y)/norm(x-y,2);  
H=eye(5)-2*w*w';  
Q=eye(5)*H;  
R=H*R;
```

```
% Iteration 2
```

```
x=R(:,2);  
a=norm(x(2:5),2);  
y=[x(1:1)' a zeros(1,3)]';  
w=(x-y)/norm(x-y,2);  
H=eye(5)-2*w*w';  
Q=Q*H;  
R=H*R;
```

```
% Iteration 3
```

```
x=R(:,3);  
a=norm(x(3:5),2);  
y=[x(1:2)' a zeros(1,2)]';  
w=(x-y)/norm(x-y,2);  
H=eye(5)-2*w*w';  
Q=Q*H;  
R=H*R;
```

1.5 Question 5