Assignment 3 Solutions

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1 Solutions

1.1 Question 1

Listing 1: Matlab Commands

```
function [P] = find_vectors(m, n, s)

    % create the x and y ranges
    x=(1:m);
    y=(1:n);

    % create the meshgrid (x,y) coordinates
    [X,Y]=meshgrid(x,y);

    % create (x,y) pairs in form [j;k]
    A=[X(:)';Y(:)'];

    % solve each s*[p;o]=[j;k] for all [j;k]
    B=s\A;

    % find columns with integer solutions
    cols=all(mod(B,1)==0);

    % return [j;k]'s with integer [p;o]'s
    P=A(:,cols);
end
```

1.1.1 Part A

Listing 2: Matlab Commands

```
s=[2,0;0,2];
find_vectors(4,6,s)

ans =

     2     2     2     4     4     4
     2     4     6     2     4     6
```

1.1.2 Part B

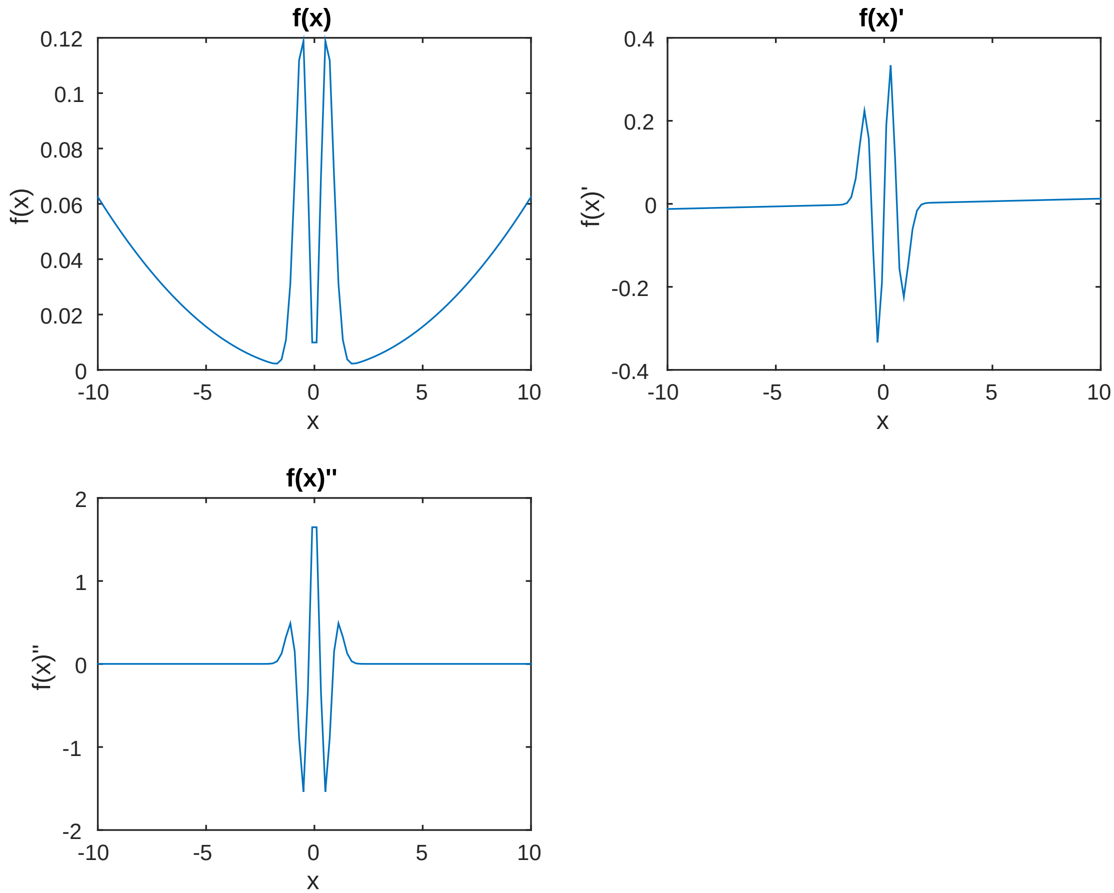
Listing 3: Matlab Commands

```
s=[1,1;1,-1];
find_vectors(5,6,s)

ans =

 1   1   1   2   2   2   3   3   3   4   4   4   5   5   5
 1   3   5   2   4   6   1   3   5   2   4   6   1   3   5
```

1.2 Question 2



Listing 4: Matlab Commands

```
% max/min for f(x)
x(find(yi(x)==max(yi(x))))
ans =

    -0.5051     0.5051

x(find(yi(x)==min(yi(x))))
ans =

    -1.7172     1.7172
```

Listing 5: Matlab Commands

```
x=linspace(-10,10);

y=sym('x^2*exp(-3*x^2)+(x/40)^2')
y =
x^2*exp(-3*x^2) + x^2/1600

dy=diff(y)
dy =
x/800 + 2*x*exp(-3*x^2) - 6*x^3*exp(-3*x^2)

ddy=diff(dy)
ddy =
2*exp(-3*x^2) - 30*x^2*exp(-3*x^2) + 36*x^4*exp(-3*x^2) + 1/800

yi=inline(y);
dyi=inline(dy);
ddyi=inline(ddy);

subplot

% Plot f(x)
subplot(2,2,1);
plot(x,yi(x));
title('f(x)');
xlabel('x');
ylabel('f(x)');

% Plot f(x)'
subplot(2,2,2);
plot(x,dyi(x));
title('f(x)''');
xlabel('x');
ylabel('f(x)''');

% Plot f(x)''
subplot(2,2,3);
plot(x,ddyi(x));
title('f(x)''''');
xlabel('x');
ylabel('f(x)''''');
```

1.3 Question 3

1.3.1 Part C

The claim made from assignment 2 question 3 part B was as follows:

$$||A^{-1}||_2 = \max_{s_A \in s(A)} \{\frac{1}{s_a}\} \tag{1}$$

We must find a formula for the condition number of A, $c_2(A)$, in terms of $\sigma(A'A)$. Note, we know $||A||_2 = \max\{s(A)\}$ where $s(A)$ are the singular values of A. The formula for the condition number of A is as follows:

$$c_2(A) = ||A||_2||A^{-1}||_2 \tag{2}$$

Therefore, to define $c_2(A)$ in terms of the $\sigma(A'A)$ we simply substitute our formulas for $||A||_2$ and $||A^{-1}||_2$ into (2). This yeilds,

$$c_2(A) = \max\{s(A)\} \max_{s_A \in s(A)} \{\frac{1}{s_a}\} \tag{3}$$

1.3.2 Part D

The claim made from assignment 2 question 2 part E was as follows:

$$s(A) = \{|\lambda| : \lambda \in \sigma(A), \lambda \neq 0\} \tag{4}$$

Stated, this says "The singular values of a matrix A are the absolute values of the non-zero eigenvalues of A, where A is symmetric".

The formula for the condition number of a symmetric matrix A is the same as the formula stated in part C. The only change in definition is that of the singular values of A. Thefore the formula, using the definition of $s(A)$ from 4, is,

$$c_2(A) \quad = \quad \max\{s(A)\} \max_{s_A \in s(A)} \{\frac{1}{s_a}\} \tag{5}$$

1.3.3 Part E

Listing 6: Matlab Commands

```
% Check for general matrix A
A = [1,2;-1,2];
[P,D] = eig(A'*A);
singA = sqrt(diag(D));
a_norm = max(singA);
inva_norm = max(1./singA);
a_norm*inva_norm

ans =

     2

cond(A,2)

ans =

    2.0000

% Check for a symmetric matrix A
A = [-92,144;144,-8];
[P,D] = eig(A);
singA = abs(diag(D));
a_norm = max(singA);
inva_norm = max(1./singA);
a_norm*inva_norm

ans =

    2.0000

cond(A,2)

ans =

    2.0000
```

1.4 Question 4

1.4.1 Part A

Listing 7: Matlab Commands

```
function [A] = q4_partA(T,k)
    [exp,base]=meshgrid(0:k,T);
    A=base.^exp;
end
```

1.4.2 Part B

Listing 8: Matlab Commands

```
function [p] = q4_partB(f,a,b,m,k)
    T=linspace(a,b,m);
    p = polyfit(T,f(T),k);
end
```

1.4.3 Part C

1.5 Question 5

1.5.1 Part A

We must prove that a matrix A with a real Cholesky factorization is symmetric positive definite. Therefore we must prove two things. The first is that A is positive definite. The second is that A is symmetric. To this end we know $A = C'C$.

Therefore we can prove positive definiteness simply as follows,

$$
\begin{aligned}
(Av, v) &\geq 0 \qquad (6)\\
(C'Cv, v) &\geq 0 \qquad (7)\\
(Cv, Cv) &\geq 0 \qquad (8)
\end{aligned}
$$

This concludes the proof for positive definiteness.

Next, we will prove that A is symmetric. Note, a matrix A is symmetric if $A = A'$. The proof is a simple substitution as follows,

$$
\begin{aligned}
A &= A' \qquad (9)\\
C'C &= (C'C)' \qquad (10)\\
C'C &= C'C \qquad (11)
\end{aligned}
$$

This concludes the proof that A is symmetric. The proof that A is symmetric positive definite is complete.

1.5.2 Part B

1.5.3 Part C

1.5.4 Part D

1.6 Question 6