Assignment 1 Solutions

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1 Solutions

1.1 Question 1

1.1.1 Question 1.1

1.1.2 Part A

1. double column 1 (postmultiply):

$$X\{1\} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

2. halve row 3 (premultiply):

$$X\{2\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

3. add row 3 to row 1 (premultiply):

$$X\{3\} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

4. interchange columns 1 and 4 (postmultiply):

$$X\{4\} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

5. subtract row 2 from each of the other rows (premultiply):

$$X\{5\} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{5}$$

6. replace column 4 by column 3 (postmultiply):

$$X\{6\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

7. delete column 1 (postmultiply):

$$X\{7\} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

The result as a product of eight matrices is as follows:

$$\boxed{M = X\{5\} * X\{3\} * X\{2\} * B * X\{1\} * X\{4\} * X\{6\} * X\{7\}} \tag{8}$$

1.1.3 Part B

(8) can be rewritten as a product of three matrices (same B) as follows:

$$A = X\{5\} * X\{3\} * X\{2\} \tag{9}$$

$$A = \begin{pmatrix} 1 & -1 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0.5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{10}$$

$$C = X\{1\} * X\{4\} * X\{6\} * X\{7\} \tag{11}$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{12}$$

This results in:

$$\boxed{M = A * B * C} \tag{13}$$

1.1.4 Matlab Code for Parts A & B

Listing 1: Matlab Commands

```
% Part A
X={};
X{1}=eye(4);
X{1}(1,1)=2;
X{2}=eye(4);
X{2}(3,3)=.5;
X{3}=eye(4);
X{3}(1,3)=1;
X{4}=eye(4);
X{4}(1,1)=0;
X{4}(4,1)=1;
X{4}(4,4)=0;
X{4}(1,4)=1;
X{5}=eye(4);
X{5}(1,2)=-1;
X{5}(3:4,2)=-1;
X{6}=eye(4);
X{6}(4,4)=0;
X{6}(3,4)=1;
X{7}=eye(4);
X{7}=X{7}(:,2:end);

B=rand(4);
ANS1=X{5}*X{3}*X{2}*B*X{1}*X{4}*X{6}*X{7};

% Part B
A=X{5}*X{3}*X{2};
C=X{1}*X{4}*X{6}*X{7};
ANS2=A*B*C;

% Check equivalence
isequal(ANS1,ANS2)
```

1.1.5 Question 2.3

Let $A \in \mathbb{C}^{m\times m}$ be hermitian. By definition $A = A^*$.

1.1.6 Part A

Prove that all eigenvalues of A are real. Assume λ is an eigenvalue of A and v is eigenvector associated with λ. The proof is as follows:

$$\lambda \langle v,v\rangle = \langle \lambda v,v\rangle = \langle Av,v\rangle = \langle v,A^*v\rangle = \langle v,Av\rangle = \langle v,\lambda v\rangle = \bar{\lambda}\langle v,v\rangle \tag{14}$$

$$\lambda \langle v,v\rangle = \bar{\lambda}\langle v,v\rangle \tag{15}$$

Positive-definiteness says,

$$\langle x,x\rangle \geq 0 \tag{16}$$

$$\langle x,x\rangle = 0 \Rightarrow x = 0 \tag{17}$$

By definition eigenvectors cannot be the zero vector. Therefore,

$$\langle v,v\rangle \neq 0 \tag{18}$$

This means we can divide both sides of (15) by $\langle v,v\rangle$. This leaves,

$$\lambda = \bar{\lambda} \tag{19}$$

By definition, for a real scalar z, $\bar{z} = z$. Therefore, λ is real valued and since we choose λ to be an arbitrary eigenvalue of A, all eigenvalues of A are real.

1.1.7 Part B

Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal. Note, a pair of vectors w_1 and w_2 are orthogonal if $\langle w_1, w_2 \rangle = 0$. We will therefore prove orthogonality of x and y by proving the equivalent statement: $\langle x, y \rangle = 0$. The proof is as follows:

Assume λ is the eigenvalue associated with x, μ is the eigenvalue associated with y, and $\lambda \neq \mu$. Then,

$$\lambda \langle x,y \rangle = \langle \lambda x, y \rangle = \langle Ax, y \rangle = \langle x, A^* y \rangle = \langle x, Ay \rangle = \langle x, \mu y \rangle = \bar{\mu} \langle x, y \rangle \tag{20}$$

From the previous proof we know that $\bar{\mu} = \mu$ Therefore we are left with the following equation,

$$\lambda \langle x,y \rangle = \mu \langle x,y \rangle \tag{21}$$

We can then reorder terms,

$$\lambda \langle x,y \rangle - \mu \langle x,y \rangle = 0 \tag{22}$$

This reduces to,

$$(\lambda - \mu) \langle x,y \rangle = 0 \tag{23}$$

Allowing $(\lambda - \mu) = 0$ would imply $\lambda = \mu$. However, this contradicts our assumption that λ and μ are distinct; $\lambda \neq \mu$. Therefore, $\langle x, y \rangle = 0$. This proves the equivalent statement that eigenvectors x and y are orthogonal.

1.2 Question 2

1.2.1 Part A

1.2.2 Part B

1.3 Question 3