# Assignment 3 Solutions

Aaron Cahn
University of Wisconsin-Madison
cahn@cs.wisc.edu

March 9, 2015

# 1 Solutions

## 1.1 Question 1

Listing 1: Matlab Commands

```
function [P] = find_vectors(m, n, s)

    % create the x and y ranges
    x=(1:m);
    y=(1:n);

    % create the meshgrid (x,y) coordinates
    [X,Y]=meshgrid(x,y);

    % create (x,y) pairs in form [j;k]
    A=[X(:)';Y(:)'];

    % solve each s*[p;o]=[j;k] for all [j;k]
    B=s\A;

    % find columns with integer solutions
    cols=all(mod(B,1)==0);

    % return [j;k]'s with integer [p;o]'s
    P=A(:,cols);
end
```

### 1.1.1 Part A

Listing 2: Matlab Commands

```
s=[2,0;0,2];
find_vectors(4,6,s)

ans =

     2     2     2     4     4     4
     2     4     6     2     4     6
```

### 1.1.2 Part B

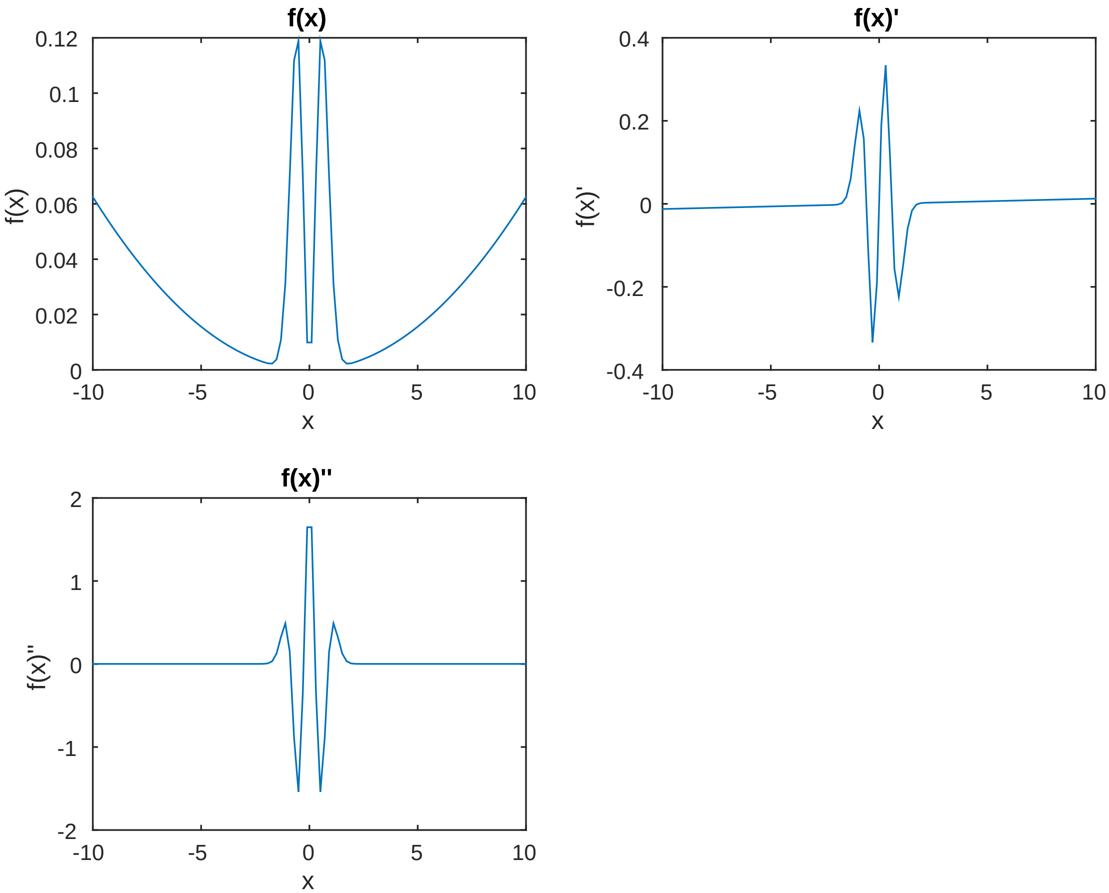
Listing 3: Matlab Commands

```
s=[1,1;1,-1];
find_vectors(5,6,s)

ans =

  1   1   1   2   2   2   3   3   3   4   4   4   5   5   5
  1   3   5   2   4   6   1   3   5   2   4   6   1   3   5
```

## 1.2 Question 2




Listing 4: Matlab Commands

```
% max/min for f(x)
x(find(yi(x)==max(yi(x))))
ans =

   -0.5051    0.5051

x(find(yi(x)==min(yi(x))))
ans =

   -1.7172    1.7172
```

Listing 5: Matlab Commands

```
x=linspace(-10,10);

y=sym('x^2*exp(-3*x^2)+(x/40)^2')
y =
x^2*exp(-3*x^2) + x^2/1600

dy=diff(y)
dy =
x/800 + 2*x*exp(-3*x^2) - 6*x^3*exp(-3*x^2)

ddy=diff(dy)
ddy =
2*exp(-3*x^2) - 30*x^2*exp(-3*x^2) + 36*x^4*exp(-3*x^2) + 1/800

yi=inline(y);
dyi=inline(dy);
ddyi=inline(ddy);

subplot

% Plot f(x)
subplot(2,2,1);
plot(x,yi(x));
title('f(x)');
xlabel('x');
ylabel('f(x)');

% Plot f(x)'
subplot(2,2,2);
plot(x,dyi(x));
title('f(x)''');
xlabel('x');
ylabel('f(x)''');

% Plot f(x)''
subplot(2,2,3);
plot(x,ddyi(x));
title('f(x)''''');
xlabel('x');
ylabel('f(x)''''');
```

## 1.3 Question 3

### 1.3.1 Part C

The claim made from assignment 2 question 3 part B was as follows:

$$||A^{-1}||_2 = \max_{s_A \in s(A)} \{\frac{1}{s_a}\} \tag{1}$$

We must find a formula for the condition number of $A$, $c_2(A)$, in terms of $\sigma(A'A)$. Note, we know $||A||_2 = \max\{s(A)\}$ where $s(A)$ are the singular values of $A$. The formula for the condition number of $A$ is as follows:

$$c_2(A) = ||A||_2||A^{-1}||_2 \tag{2}$$

Therefore, to define $c_2(A)$ in terms of the $\sigma(A'A)$ we simply substitute our formulas for $||A||_2$ and $||A^{-1}||_2$ into (2). This yeilds,

$$c_2(A) = \max\{s(A)\} \max_{s_A \in s(A)} \{\frac{1}{s_a}\} \tag{3}$$

### 1.3.2 Part D

The claim made from assignment 2 question 2 part E was as follows:

$$s(A) = \{|\lambda| : \lambda \in \sigma(A), \lambda \neq 0\} \tag{4}$$

Stated, this says "The singular values of a matrix $A$ are the absolute values of the non-zero eigenvalues of $A$, where $A$ is symmetric".

The formula for the condition number of a symmetric matrix $A$ is the same as the formula stated in part C. The only change in definition is that of the singular values of $A$. Thefore the formula, using the definition of $s(A)$ from 4, is,

$$c_2(A) \quad = \quad \max\{s(A)\} \max_{s_A \in s(A)} \{\frac{1}{s_a}\} \tag{5}$$

### 1.3.3 Part E

Listing 6: Matlab Commands

```
% Check for general matrix A
A = [1,2;-1,2];
[P,D] = eig(A'*A);
singA = sqrt(diag(D));
a_norm = max(singA);
inva_norm = max(1./singA);
a_norm*inva_norm

ans =

     2

cond(A,2)

ans =

    2.0000

% Check for a symmetric matrix A
A = [-92,144;144,-8];
[P,D] = eig(A);
singA = abs(diag(D));
a_norm = max(singA);
inva_norm = max(1./singA);
a_norm*inva_norm

ans =

    2.0000

cond(A,2)

ans =

    2.0000
```

## 1.4 Question 4

### 1.4.1 Part A

Listing 7: Matlab Commands

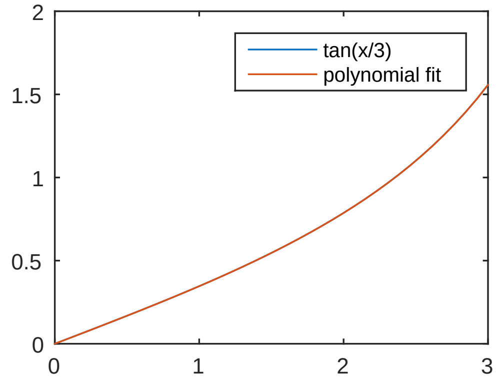
```
function [A] = q4_partA(T,k)
    [exp, base]=meshgrid(0:k,T);
    A=base.^exp;
end
```

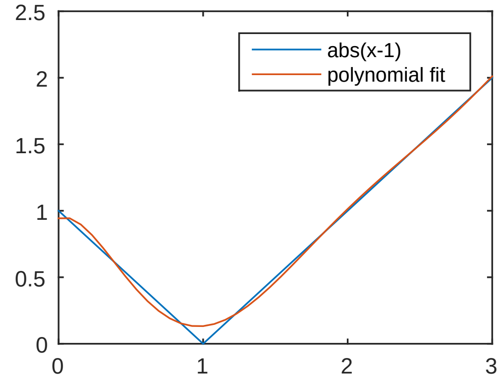
### 1.4.2 Part B

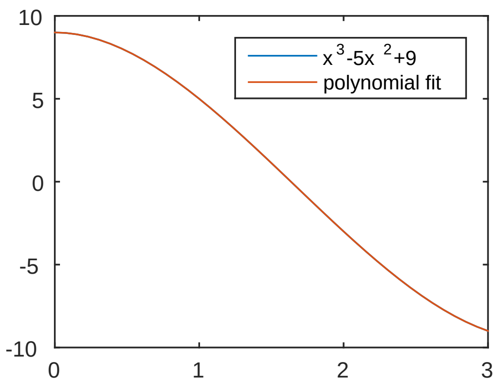
Listing 8: Matlab Commands

```
function [p] = q4_partB(f,a,b,m,k)
    T=linspace(a,b,m);
    p = polyfit(T,f(T),k);
end
```

### 1.4.3 Part C




From these plots we can see that a higher order polynomial is quite good at fitting a lower order polynomial on a small range. This is apparent from the plots of $\tan(\frac{x}{3})$ and $x^3 - 5x^2 + 9$. However, the fit is less tight for $|x-1|$, a function that has a point where it is not differentiable (e.g., in general, non-smooth functions).

Note, the error norm is $||p(T) - f(T)||_2$.

Listing 9: Matlab Commands

```
% Error norm for tan(x/3)
norm((A*p')-f(T)',2)
    5.5146e-04
% Error norm for abs(x-1)
norm((A*p')-f(T)',2)
     0.2366
% Error norm for x^3-5x^2+9
norm((A*p')-f(T)',2)
    1.6582e-14
```

Listing 10: Matlab Commands

```
% Variables used throughout
a=0; b=3; m=40; k=6;
T=linspace(a,b,m);
A=q4_partA(T,k);
subplot;

% Running on function tan(x/3)
f = inline(sym('tan(x/3)'));
p=q4_partB(f,a,b,m,k);
p=p(end:-1:1);

% Plotting the function and the fit
subplot(2,2,1);
plot(T,f(T));
hold on
plot(T,A*p');
legend('tan(x/3)', 'polynomial fit');

% Running on function abs(x-1)
f = inline(sym('abs(x-1)'));
p=q4_partB(f,a,b,m,k);
p=p(end:-1:1);

% Plotting the function and the fit
subplot(2,2,2);
plot(T,f(T));
hold on
plot(T,A*p');
legend('abs(x-1)', 'polynomial fit');

% Running on function x^3-5x^2+9
f = inline(sym('x^3-5*x^2+9'));
p=q4_partB(f,a,b,m,k);
p=p(end:-1:1);

% Plotting the function and the fit
subplot(2,2,3);
plot(T,f(T));
hold on
plot(T,A*p');
legend('x^3-5x^2+9', 'polynomial fit');
```

## 1.5 Question 5

### 1.5.1 Part A

We must prove that a matrix $A$ with a real Cholesky factorization is symmetric positive definite. Therefore we must prove two things. The first is that $A$ is positive definite. The second is that $A$ is symmetric. To this end we know $A = C'C$.

Therefore we can prove positive definiteness simply as follows,

$$\begin{aligned} (Av, v) &\geq 0 && (6) \\ (C'Cv, v) &\geq 0 && (7) \\ (Cv, Cv) &\geq 0 && (8) \end{aligned}$$

This concludes the proof for positive definiteness.

Next, we will prove that $A$ is symmetric. Note, a matrix $A$ is symmetric if $A = A'$. The proof is a simple substitution as follows,

$$\begin{aligned} A &= A' && (9) \\ C'C &= (C'C)' && (10) \\ C'C &= C'C && (11) \end{aligned}$$

This concludes the proof that $A$ is symmetric. The proof that $A$ is symmetric positive definite is complete.

### 1.5.2 Part B

We must find the $LDU$ factorization of $A$. To this end we will first find the $LU$ factorization and then extract $D$.

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix} \tag{12}$$

We must find the $L$ matrices that transform $A$ to $U$.

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1.5 & 0 & 1 \end{bmatrix} \tag{13}$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix} \tag{14}$$

$$A_1 = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix} \tag{15}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{16}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix} \tag{17}$$

$$A_2 = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1.5 \end{bmatrix} \tag{18}$$

Now we gather the terms from $L_1$ and $L_2$ (flipping signs to obtain the inverse) for our matrix $L$. The matrix $U$ is simply $A_2$. In total,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix} U = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1.5 \end{bmatrix} \tag{19}$$

Now, we factor $D$ from $U$. We simply pull out the diagonal and scale each row of $U$ by the diagonal element of $D$. This changes $U$ so we are left with,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix} D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} U = \begin{bmatrix} 1 & 1 & 1.5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

### 1.5.3 Part C

The matrix $A$ is positive definite. This is because all the diagonal entries in $D$ are positive which suffices to prove a matrix is positive definite.

### 1.5.4 Part D

To find a Cholesky factorization of $A$ we will use the $LDU$ factorization. We will split $D$ into two equal matrices by taking the square root's of the diagonal entries. Therefore,

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{1.5} \end{bmatrix}}_{\text{C'}} \underbrace{\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{1.5} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1.5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{C}} \tag{21}$$

$$A = \underbrace{\begin{bmatrix} 1.4142 & 0 & 0 \\ 1.4142 & 1.4142 & 0 \\ 2.1213 & 1.4142 & 1.2247 \end{bmatrix}}_{\text{C'}} \underbrace{\begin{bmatrix} 1.4142 & 1.4142 & 2.1213 \\ 0 & 1.4142 & 1.4142 \\ 0 & 0 & 1.2247 \end{bmatrix}}_{\text{C}} \tag{22}$$

Listing 11: Matlab Commands

```
A=[2,2,3;2,4,5;3,5,8];
C=chol(A)
C =

    1.4142    1.4142    2.1213
         0    1.4142    1.4142
         0         0    1.2247

C'
ans =

    1.4142         0         0
    1.4142    1.4142         0
    2.1213    1.4142    1.2247
```

## 1.6 Question 6

The matrix $A$ is singular. Note, this means this does not contradict the theorem that proves a solution is unique because we will solve an underdetermined system which gives us a free variable. We can find solutions to $Ax = b$ through the normal equations solution. Therefore, we will solve $A'Ax = A'b$.

$$
\begin{aligned}
A'Ax &= A'b && (23)\\
LUx &= A'b && (24)\\
Ly &= A'b && (25)\\
y &= L\backslash(A'b) && (26)\\
x &= U\backslash x && (27)
\end{aligned}
$$

Listing 12: Matlab Commands

```
A=[1,2,3;4,5,6;7,8,9];
b=[1,0,1]';
% LU factor A'A
[L,U] = lu(A'*A);
% Solve y = L\(A'b)
y=L\(A'*b)

y =

   12.0000
   -0.8000
    0.0000

diary off
```

Now we will solve,

$$
\begin{aligned}
Ux &= y && (28)\\
\begin{bmatrix} 90 & 108 & 126 \\ 0 & -1.2 & -2.4 \\ 0 & 0 & 0 \end{bmatrix} x &= \begin{bmatrix} 12 \\ -0.8 \\ 0 \end{bmatrix} && (29)
\end{aligned}
$$

This is an underdetermined system so we set $x_3 = c$ and solve. This yields $x_2 = -2c + 0.6667$ and $x_1 = c - 0.6667$. Now we simply choose three values for $c$.

$$
c = 1 \Rightarrow \begin{bmatrix} 0.3333 \\ -1.3333 \\ 1 \end{bmatrix} \; c = 2 \Rightarrow \begin{bmatrix} 1.3333 \\ -3.3333 \\ 2 \end{bmatrix} \; c = 3 \Rightarrow \begin{bmatrix} 2.3333 \\ -5.3333 \\ 3 \end{bmatrix} \quad (30)
$$