Assignment 1 Solutions

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1 Solutions

1.1 Question 1

1.1.1 Question 1.1

1.1.2 Part A

1. double column 1 (postmultiply):

$$X\{1\} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

2. halve row 3 (premultiply):

$$X\{2\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

3. add row 3 to row 1 (premultiply):

$$X\{3\} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

4. interchange columns 1 and 4 (postmultiply):

$$X\{4\} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

5. subtract row 2 from each of the other rows (premultiply):

$$X\{5\} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{5}$$

6. replace column 4 by column 3 (postmultiply):

$$X\{6\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

7. delete column 1 (postmultiply):

$$X\{7\} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

The result as a product of eight matrices is as follows:

$$\boxed{M = X\{5\} * X\{3\} * X\{2\} * B * X\{1\} * X\{4\} * X\{6\} * X\{7\}} \tag{8}$$

1.1.3 Part B

(8) can be rewritten as a product of three matrices (same B) as follows:

$$A = X\{5\} * X\{3\} * X\{2\} \tag{9}$$

$$A = \begin{pmatrix} 1 & -1 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0.5 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{10}$$

$$C = X\{1\} * X\{4\} * X\{6\} * X\{7\} \tag{11}$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{12}$$

This results in:

$$\boxed{M = A * B * C} \tag{13}$$

1.1.4 Matlab Code for Parts A & B

Listing 1: Matlab Commands

```
% Part A
X={};
X{1}=eye(4);
X{1}(1,1)=2;
X{2}=eye(4);
X{2}(3,3)=.5;
X{3}=eye(4);
X{3}(1,3)=1;
X{4}=eye(4);
X{4}(1,1)=0;
X{4}(4,1)=1;
X{4}(4,4)=0;
X{4}(1,4)=1;
X{5}=eye(4);
X{5}(1,2)=-1;
X{5}(3:4,2)=-1;
X{6}=eye(4);
X{6}(4,4)=0;
X{6}(3,4)=1;
X{7}=eye(4);
X{7}=X{7}(:,2:end);

B=rand(4);
ANS1=X{5}*X{3}*X{2}*B*X{1}*X{4}*X{6}*X{7};

% Part B
A=X{5}*X{3}*X{2};
C=X{1}*X{4}*X{6}*X{7};
ANS2=A*B*C;

% Check equivalence
isequal(ANS1,ANS2)
```

1.1.5 Question 2.3

Let $A \in \mathbb{C}^{m\times m}$ be hermitian. By definition $A = A^*$.

1.1.6 Part A

Prove that all eigenvalues of A are real. Assume λ is an eigenvalue of A and v is eigenvector associated with λ. The proof is as follows:

$$\lambda \langle v,v\rangle = \langle \lambda v,v\rangle = \langle Av,v\rangle = \langle v,A^*v\rangle = \langle v,Av\rangle = \langle v,\lambda v\rangle = \bar{\lambda}\langle v,v\rangle \tag{14}$$

$$\lambda \langle v,v\rangle = \bar{\lambda}\langle v,v\rangle \tag{15}$$

Positive-definiteness says,

$$\langle x,x\rangle \geq 0 \tag{16}$$

$$\langle x,x\rangle = 0 \Rightarrow x = 0 \tag{17}$$

By definition eigenvectors cannot be the zero vector. Therefore,

$$\langle v,v\rangle \neq 0 \tag{18}$$

This means we can divide both sides of (15) by $\langle v,v\rangle$. This leaves,

$$\lambda = \bar{\lambda} \tag{19}$$

By definition, for a real scalar z, $\bar{z} = z$. Therefore, λ is real valued and since we choose λ to be an arbitrary eigenvalue of A, all eigenvalues of A are real.

1.1.7 Part B

Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal. Note, a pair of vectors w_1 and w_2 are orthogonal if $\langle w_1, w_2 \rangle = 0$. We will therefore prove orthogonality of x and y by proving the equivalent statement: $\langle x, y \rangle = 0$. The proof is as follows:

Assume λ is the eigenvalue associated with x, μ is the eigenvalue associated with y, and $\lambda \neq \mu$. Then,

$$\lambda \langle x, y \rangle = \langle \lambda x, y \rangle = \langle Ax, y \rangle = \langle x, A^* y \rangle = \langle x, Ay \rangle = \langle x, \mu y \rangle = \bar{\mu} \langle x, y \rangle \tag{20}$$

From the previous proof we know that $\bar{\mu} = \mu$. Therefore we are left with the following equation,

$$\lambda \langle x, y \rangle = \mu \langle x, y \rangle \tag{21}$$

We can then reorder terms,

$$\lambda \langle x, y \rangle - \mu \langle x, y \rangle = 0 \tag{22}$$

This reduces to,

$$(\lambda - \mu) \langle x, y \rangle = 0 \tag{23}$$

Allowing $(\lambda - \mu) = 0$ would imply $\lambda = \mu$. However, this contradicts our assumption that λ and μ are distinct; $\lambda \neq \mu$. Therefore, $\langle x, y \rangle = 0$. This proves the equivalent statement that eigenvectors x and y are orthogonal.

1.2 Question 2

Let Q by an $m \times m$ real matrix that satisfies, for every vector $x \in \mathbb{R}^m$,

$$||Qx|| = ||x|| \tag{24}$$

1.2.1 Part A

Show that 1 is the only eigenvalue of $Q'Q$, i.e., show that $\sigma(Q'Q) = \{1\}$.

To prove this we will show that $Q'Q = I$, i.e., $Q'Q$ is the identity matrix. This is equivalent because 1 is the only eigenvalue of any indentity matrix. To find the eigenvalues of I we must satisfy,

$$\det(I - \lambda I) = 0 \tag{25}$$

Since I is a diagonal matrix the matrix $I - \lambda I$ will also be diagonal. The diagonal entries will be of the form $1 - \lambda$. The determinant of $I - \lambda I$ is the product of its diagonal entries. Which means the characteristic polynomial is of the form,

$$(1-\lambda)_1(1-\lambda)_2 \ldots (1-\lambda)_m = 0 \tag{26}$$

Therefore, the λ's all equal 1 and these are the eigenvalues of I. Concisely,

$$\boxed{p(I) = \{1\}} \tag{27}$$

Now we will prove $Q'Q = I$. We know that $||x|| = \sqrt{\langle x,x \rangle}$. We will assume $x \neq 0$, as this is a degenerate case. The proof is as follows,

$$\begin{aligned} ||Qx|| &= ||x|| && (28) \\ \sqrt{\langle Qx, Qx \rangle} &= \sqrt{\langle x,x \rangle} && (29) \end{aligned}$$

We can drop the square roots on both sides of the equation after the arrow and continue,

$$\begin{aligned} \langle Qx, Qx \rangle &= \langle x,x \rangle && (30) \\ x'Q'Qx &= x'x && (31) \end{aligned}$$

In order to satisfy (31), $Q'Q$ must equal the identity matrix, i.e., $Q'Q = I$. We have already shown the only eigenvalue of the identity matrix is 1. Therefore, (27) implies,

$$\boxed{p(Q'Q) = \{1\}} \tag{32}$$

This concludes the proof.

1.2.2 Part B

We must first prove that $Q'Q$ is symmetric for any matrix Q. A matrix is symmetric if $A = A'$. To do this we will define $B = Q'Q$. Then,

$$
\begin{aligned}
B' &= (Q'Q)' && (33) \\
&= Q'Q'' && (34) \\
&= Q'Q && (35) \\
&= B && (36)
\end{aligned}
$$

We have shown $B = B'$ which is the definition of symmetric. Therefore, $Q'Q$ is symmetric for any Q.

Using the previous proof, we must now show that Q from Section 1.2.1 is orthogonal. A matrix A is orthogonal if $A' = A^{-1}$.

In Section 1.2.1 we showed that $Q'Q = I$. The proof is as follows,

$$
\begin{aligned}
Q'Q &= I = Q^{-1}Q && (37) \\
Q'Q &= Q^{-1}Q && (38)
\end{aligned}
$$

Therefore, $Q' = Q^{-1}$. Since the transpose of Q is equal to the inverse of Q, by definition Q is orthogonal.

1.3 Question 3

The *sloppy_qr.m* algorithm executes a loop n times. Within the loop we have operations that have complexity n, $2n$, and $2n^3$. The complexity of the algorithm is,

$$n(2n^3 + 3n) \Rightarrow \boxed{2n^4 + 3n^2} \tag{39}$$

Therefore, the big-O complexity is $2n^4$ or O(n^4).

The code for the experiments:

Listing 2: Matlab Commands

```
M = zeros(1,100);
for i=1:100,
    A=rand(i);
    sloppy_qr;
    M(1,i) = ops(1);
end
polyfit(1:99,M(1:end-1),4)
```