LECTURE 12

Oct. 10/2003

$T: V \longrightarrow V$

Can we find a good basis so that the matrix A of T is in simple form?

Invariant subspace $W \subset V$,
i.e. subspace such that $T(W) \subset W$
then can write A in the form

$$\left(\begin{array}{c|c} A_1 & A_2 \\ \hline & A_3 \end{array}\right) \longleftarrow A_1 = \text{matrix of } T \circlearrowleft W$$

If have an invariant complement $W' \subset V$, i.e.
$V = W \oplus W'$ (so $v = w + w'$ uniquely)
& $T(W) \subset W$ & $T(W') \subset W'$
then can write A in the form

$$\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) \longleftarrow \begin{array}{l} A_1 = \text{matrix of } T \circlearrowleft W \\ A_2 = \text{matrix of } T \circlearrowleft W' \end{array}$$

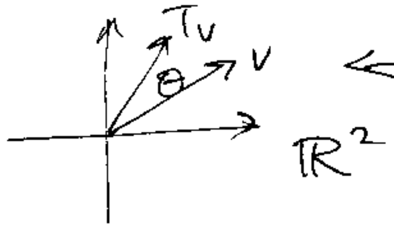
If W is 1-dimensional, i.e.
$W = Fw = \{cw : c \in F\}$ is invariant;
or equivalently: $Tw = cw$ for some $c \in F$;
then say w is an eigenvector
c is an eigenvalue

If there is a basis $\{v_1, \ldots, v_n\}$ of V consisting of eigenvectors, with $Tv_i = c_i v_i$

$$A = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{pmatrix}$$

Cannot always write A in diagonal form, i.e. cannot always find basis of eigenvectors:

<u>Ex 1</u>



$T =$ Rotation by angle θ

$(\theta \neq 0, \pi)$

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Has <u>no</u> eigenvectors $v \neq 0$ in $\mathbb{R}^2$

<u>Ex2</u> Another problem

$T : F^2 \to F^2$

(has one eigenvalue) $\to$ $Te_1 = e_1$ $\qquad c = 1$

$Te_2 = e_1 + e_2$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

<u>Claim</u> There is no basis of eigenvectors
(i.e. there is no complement W' for $W = F \cdot e_1$ which is T-invariant)

Suppose otherwise:

$$v = ae_1 + be_2$$
$$Tv = ae_1 + b(e_1 + e_2)$$
$$\stackrel{?}{=} cv = cae_1 + cbe_2.$$

$$\Rightarrow \quad ca = a + b$$
$$cb = b$$

$$b \neq 0 \Rightarrow c = 1 \quad \text{so } a = a + b$$
$$\text{so } b = 0 \quad \rightarrow\leftarrow$$

$$\Rightarrow b = 0, \; c = 1$$

(If $b = 0$ $\quad v = ae_1 \in W$.) $\boxtimes$

Given T, what are the possible eigenvalues c?

If $Tw = cw$, i.e. $(T - cI)w = 0$

i.e. $w \in \ker(T - cI)$

then $T - cI$ is not invertible.

Conversely, if $T - cI$ has a kernel (or just is not invertible) then c is an eigenvalue for T.

Conclusion: $\{\text{Set of eigenvalues}\}$

$$\begin{aligned} &\{\text{Set of eigenvalues}\} \\ &\qquad \| \\ &\{c \in F : T - cI \text{ is not invertible}\}. \\ &\qquad \| \\ &\{c \in F : \det(A - cI) = 0\} \end{aligned}$$

(where A is the matrix of T w.r.t. some basis)

Consider the determinant

$\det(t \cdot I - A)$

$$= \det \begin{pmatrix} t-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & t-a_{22} & \cdots & \\ \vdots & & & \\ -a_{n1} & & & t-a_{nn} \end{pmatrix}$$

$$= t^n - (a_{11} + a_{22} + \text{---} + a_{nn}) t^{n-1} + \text{———} + (-1)^n \det A \quad =: f(t)$$

Polynomial in t of degree n with coeffs in the field F

This is called the "characteristic polynomial of T"

<u>Remark</u>: The char poly $f(t)$ depends only on T, not on the basis of V used to obtain the matrix A

If we use a different basis, have $A^* = PAP^{-1}$

$$f^*(t) = \det(tI - A^*) = \det(tI - PAP^{-1})$$
$$= \det(P(tI-A)P^{-1}) = (\det P)\det(tI-A)(\det P)^{-1}$$
$$= f(t).$$

The roots of the characteristic polynomial are exactly the eigenvalues of T.

<u>Lemma</u> If polynomial $f(t)$ has degree n over <u>field</u> F, then it has at most n distinct roots c in F.

(root $\Leftrightarrow f(c)=0$)

<u>Pf</u>: Use division algorithm:

$$f(t) = (t-c)g(t) + d$$

remainder is a constant $\in F$.

$\deg(g(t)) = n-1$ out of necessity

If $f(c)=0$ then $d=0$ so

$$f(t) = (t-c)g(t)$$

If c' is another root, $g(c')=0$.

Proceed by induction. $\boxtimes$

<u>Note</u>: We need to assume F is a field: e.g. $\mathbb{Z}/8\mathbb{Z}$ $f(t)=t^2-1$ has <u>4</u> distinct roots $(1,3,5,7)$

Now returning to

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \neq 0, \pi$$

$$f(t) = \det(tI - A) = t^2 - 2\cos\theta \cdot t + 1$$

Since $\theta \neq 0, \pi$: $|2\cos\theta| < 2$

$\Rightarrow b^2 - 4ac < 0$

Thus $f(t)$ has no real roots.

General 2×2 characteristic polynomial:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$f(t) = t^2 - (a+d)t + (ad - bc)$$

<u>Notation</u> Call $a_{11} + a_{22} + \cdots + a_{nn}$ the "Trace of A"

So e.g. $f(t) = t^2 - (\mathrm{Tr}\,A)t + (\det A)$

<u>Ex</u>. $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ $\quad f(t) = \cancel{t^2 - 7t} + 10$

$= (t-5)(t-2)$

$c = 5, 2$ eigenvalues.

So there are v_1, v_2 s.t. $Tv_1 = 5v_1$

$Tv_2 = 2v_2$.

If $2 \neq 5$ in F, we get a basis of eigenvectors, $\{v_1, v_2\}$

However in $F = \mathbb{Z}/3\mathbb{Z}$ (for example) this argument fails.

Char. poly. Roots are eigenvalues, so at most n distinct eigenvalues.

But if $f(t) = (t - c_1)(t - c_2) \cdots (t - c_n)$ with $c_1, \ldots, c_n$ distinct roots, then have a basis of eigenvectors.

Thm: $T: V \to V$ gives a vector in $\mathrm{Hom}(V, V)$

F-vectorspace of dim n^2

$$\{I, T, T^2, T^3, \ldots, T^{n^2}\}$$

n^2+1 vectors in that vectorspace

So there must be a linear relation

$$a_0 I + a_1 T + a_2 T^2 + \cdots + a_{n^2} T^{n^2} = 0$$

so: T satisfies a poly. of deg. $\leq n^2$ with coeffs in F.

<u>Cayley - Hamilton Theorem:</u>

T always satisfies its own characteristic polynomial (of degree n).

Can easily verify this, e.g. for 2×2 matrices.

<u>Pf in case where $f(t) = (t-c_1) \cdots (t-c_n)$</u> ($c_i$ distinct):

In that case $A = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$

w.r.t. basis of eigenvectors

$$f(A) = (A - c_1 I)(A - c_2 I) \cdots (A - c_n I)$$

$A - c_i I$ is diagonal with a 0 in the ith place so

$f(A) = 0$. $\boxtimes$