LECTURE 15

Oct. 23/03

Last time:

Group G of motions $m: \mathbb{R}^n \to \mathbb{R}^n$, which preserve $d(v,w) = \|v-w\|$

i.e. $d(v,w) = d(m(v), m(w))$ $\forall v, w$.

Have subgroup $\simeq (\mathbb{R}^n, +)$ of translations:

$$m = t_b(v) = v + b$$

$$G = \mathbb{R}^n \cdot G_0$$

motions which preserve 0

(Pf $m(0) = b$

$(t_{-b} \cdot m)(0) = 0$.)

$m \in G_0 \simeq O(n)$ is the group of orthog. trans.

To prove that this really is an isomorphism:

Preserves inner product:
$$\begin{cases} \langle m(v), m(w) \rangle = \langle v, w \rangle \\ d(v,w)^2 = d(v,0)^2 + d(w,0)^2 - 2\langle v,w \rangle \end{cases}$$

Linearity:

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$

$\langle e_i, e_i \rangle = 1$

$\langle e_i, e_j \rangle = 0$ $i \neq j$.

Then $m(e_1), \ldots, m(e_n)$ is another orthonormal basis

Let A be the element in $O(n)$ with column vectors

Claim $m = A$ as a transf. of $\mathbb{R}^n$

$$A = \begin{pmatrix} m(e_1) & \cdots & m(e_n) \\ | & \cdots & | \end{pmatrix}$$

$$A^t A = I$$

Pf: Consider the motion

$$m \circ A^{-1} = m' \in G_0$$

$$m'(e_i) = e_i \quad \forall i$$

Claim $m'(v) = v \quad \forall v \in \mathbb{R}^n$

The i^{th} coord of $m'(v)$ is

$$\langle m'(v), e_i \rangle = \langle m'(v), m'(e_i) \rangle$$

$$= \langle v, e_i \rangle = v_i = i^{th} \text{ coord.}$$

Thus $m' = id \Rightarrow m = A$ $\boxtimes$

<u>Note</u>: In decomposition $G = \mathbb{R}^n \cdot O(n)$:

$$
\begin{aligned}
&(b, A) \cdot (b', A')(v) \\
&= (b, A)(A'v + b) \\
&= A(A'v + b') + b \\
&= AA'(v) + (A(b') + b) \\
&= (b + A(b'), AA')(v)
\end{aligned}
$$

So G is not product of $\mathbb{R}^n \cdot O(n)$, but a sort of twisted product.

$$
\left.\begin{aligned} f : G &\to O(n) \\ (b, A) &\mapsto A \end{aligned}\right\} \text{ is a surjective hom with kernel}
$$

$\mathbb{R}^n \simeq \{(b, I)\}$

$(b, I)(v) = v + b$
$= t_b(v)$

normal subgroup.

We get a very rich theory already in the $n=2$ case:

$$G = \mathbb{R}^2 \cdot O(2)$$

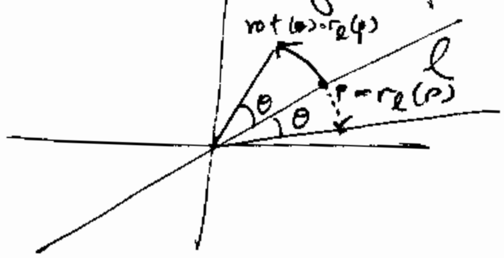
$$G_0 = O(2) = SO(2) \cup \underbrace{SO(2) r_\ell}_{\text{order 2}}$$

reflections across a line through the origin

$$r_\ell \circ rot(\theta) \circ r_\ell^{-1} = r_\ell \circ rot(\theta) \circ r_\ell$$

$= rot(\theta')$ for some θ' since it is orientation-preserving and it fixes a point.

Thus to determine it, it suffices to figure out what happens to a non-origin point on ℓ:



So $\theta' = -\theta$:

$$r_\ell \circ rot(\theta) \circ r_\ell^{-1} = rot(-\theta) = rot(\theta)^{-1}$$

So we have a nice description of $G_0 = O(2)$ in terms of $SO(2)$ & any fixed reflection.

Have map $G = \mathbb{R}^2 \cdot O(2) \to O(2) \xrightarrow[\det]{} \{\pm 1\}$ determining orientation.

Thus any $g \in G$ is one of 4 types, geometrically:

- $\det g = +1$ $\qquad t_b \circ \mathrm{rot}(\theta)$

  orientation-preserving

  $\Longrightarrow$ (i) translations t_b

  fix no points

  (or all pts if $b=0$)

  (ii) fix a single pt. p

  & consist of rotations

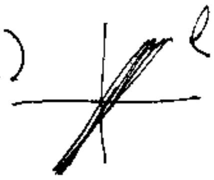
  around p

- $\det g = -1$ $\qquad t_b \circ \mathrm{rot}(\theta) \circ \mathrm{ref}(l)$

  orientation-reversing

  $\Rightarrow$ (iii) reflection in a line

  $\mathrm{ref}(l)$



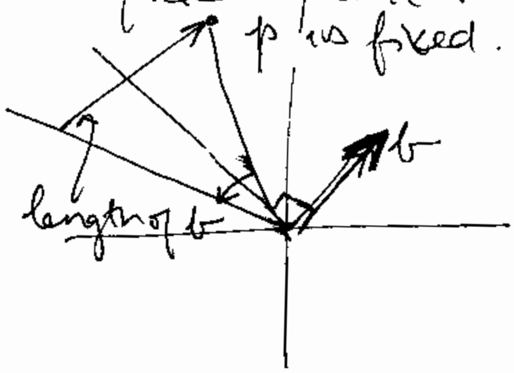
  (iv) glide reflection

  $\mathrm{ref}(l)(v)+b$;

  fixes a line

Breakdown of the cases:
(see textbook for details)

(i) is the case where $\theta = 0$
(ii) is the case where $\theta \neq 0$
If $b = 0$ then $p = 0$ and in $G_0 = O(2)$ & since $\det = +1$, is in $SO(2)$.
Assume $b \neq 0$. How do we find the fixed point?



Can also see geometrically that p is the only fixed pt.

(iii), (iv) similar geometric arguments

We now make a study of finite subgroups of $G = \mathbb{R}^2 \cdot O(2)$:

Let Γ be a fin. subgp.
- It contains no translations

<u>Theorem</u> Γ fixes a point $p \in \mathbb{R}^2$.
(i.e. there is $p \in \mathbb{R}^2$ s.t.
$\gamma(p) = p$ for every $\gamma \in \Gamma$).

Abstract recipe for fixed pt. p:
Let $s \in \mathbb{R}^2$ be any vector
Consider the set of vectors
$\{\gamma(s) : \gamma \in \Gamma\}$. ← finite set $\subset \mathbb{R}^2$.
Let $n = \#\Gamma$

and set $p = \frac{1}{n} \sum_{\gamma \in \Gamma} \gamma(s)$.

<u>Claim</u> p is a fixed point.

If $g \in G$:
$$g(p) = \frac{1}{n} \sum_{\gamma} g\gamma(s)$$

Then check separately for
g a translation
& $g \in O(2)$ ~~to~~
verify $\forall g \in G = \mathbb{R}^2 \cdot O(2)$
$g(p) = p$

Note $\Gamma \subset G_p \subset G$

$$G_p = \{g : g(p) = p\} = t_p G_0 t_p^{-1}$$ a conjugate of G_0.

So: $t_p^{-1} \Gamma t_p \subset G_0 = O(2)$

$t_p^{-1} \Gamma t_p = \Gamma^*$ finite subgroup iso to Γ.

Classify $\Gamma \underset{\text{finite}}{\subset} O(2)$

① $\Gamma \subset SO(2)$ $\quad (\det = +1)$

② $\Gamma \cap SO(2) = \Gamma_+$ has index 2 in Γ (normal subgp)

$(\det = \pm 1)$

In first case:

every $\gamma = \mathrm{rot}(\theta)$ $\quad 0 \leq \theta \leq 2\pi$

Let θ be the smallest angle of rotation for $\gamma \in \Gamma$ $\quad \theta > 0$.

Then Γ is a <u>cyclic group</u>, generated by this element

If $|\Gamma| = n$, $\theta = 2\pi/n$.

So for ①: every Γ is cyclic & all cyclic groups occur.

Have in second case,

no reflection in $\Gamma - \Gamma_+$.

$\Gamma_+ =$ case ① example, so is cyclic

$\therefore$

$\Gamma = \langle$ rotation by $\frac{2\pi}{n}$ & reflection $\rangle$

$\leftarrow$ dihedral group ...