LECTURE 7

Sept. 29/2003

§ Recall

$(\mathbb{Z}, +) = G$

All the subgroups have the form

$n\mathbb{Z} = H \qquad n \geq 0$

Associated to each $H \subset G$, have a new group $\mathbb{Z}/n\mathbb{Z}$

$\bar{a} \in \mathbb{Z}/n\mathbb{Z}$ depends on the remainder of the integer a after division by n

$a, b \in \mathbb{Z}$

$$\bar{a} = \bar{b} \text{ in } \mathbb{Z}/n\mathbb{Z} \iff a \equiv b \ (\mathrm{mod}\ n)$$
$$\Updownarrow$$
$$n \mid (a-b)$$

$\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$

cyclic group of order n gen'd by $\bar{1}$

$$\bar{a} + \bar{b} = \overline{a+b}$$

$$f : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$$
$$a \longmapsto \bar{a}$$

is a group hom., which is <u>surjective</u> with kernel $n\mathbb{Z} = H$.

Also introduced multiplication

$\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are "rings": "group w/ multiplication"

§ Quotient groups

When can we put a group structure on the set of cosets $\{aH\}$ for a subgroup $H \subset G$?

① Suppose $H = \ker(f)$ $\quad f : G \to G'$
Then the set of cosets $\{aH\}$

$\updownarrow$

fibres of the map f

$\updownarrow$

pts $\bar{a}$ in the Image $\subset G'$

But the image of f is
a <u>subgroup of G'</u>

By transport of structure, we get a group structure on the set G/H of cosets, with

$$aH \cdot bH = (ab)H$$

$$\text{(i.e.: } \bar{a} \cdot \bar{b} = f(a) \cdot f(b) = f(ab)$$
$$= \overline{ab} \text{ !)}$$

This makes the map

$$F : G \to G/H$$
$$a \mapsto aH$$

a surjective group homomorphism

On G/H:

Identity: $\bar{e} = eH = H$

Inverses: $(aH)^{-1} = a^{-1}H$.

(2) More general, we might try:
Let $H \subset G$ be <u>any</u> subgroup.

G/H = set of all cosets aH

$\boxed{\text{Try}}$ to define a group structure by setting
$aH \cdot bH = abH$.

<u>This will fail</u>

Is this well-defined?

i.e.: If $aH = a'H$
$bH = b'H$

is $abH = a'b'H$?

Suppose $aHa^{-1} \neq H$, i.e. $aH \neq Ha$ for some $a \in G$

$(aH)(a^{-1}H) = eH = H$ under this defn.

By hyp. $\exists h \in H$ s.t. $aha^{-1} \notin H$.
so $(ah)(a^{-1}e) \notin H$

Things worked in case ①
because

$$aHa^{-1} = H \quad \text{for all } a \in G$$

since there H was a normal subgroup.

③ Assume $H \lhd G$ (normal subgroup)

So $aHa^{-1} = H$ for all $a \in G$

(i.e. for every $h \in H$, $a \in G$ there is $h' \in H$ s.t. $ah = h'a$)

In this case, the naive multiplication law on cosets actually is well-defined and defines a group structure on G/H

Check this by calculating the set of all products

$$(aH)\cdot(bH) = \{ahbh' \in G : h, h' \in H\}$$

$$\overset{\shortparallel}{(Ha)}(bH) = H(ab)H \quad \text{(associativity)}$$

$$= (ab)H \cdot H = (ab)H$$

(naturally inherited from G)

So: we can put a <u>group structure</u> on the set of all cosets $\{aH\}$ for a subgroup $H \subset G$ if and only if H is <u>normal</u>.

We also get a surjective group homomorphism

$$f : G \longrightarrow G/H$$
$$a \longmapsto aH$$

with kernel $f^{-1}(eH) = H$

<u>Corollary</u> Every normal subgroup $H \triangleleft G$ is the kernel of a group homomorphism.

<u>Isomorphism theorem</u>

If $f : G \to G'$ is a surjective group homomorphism with kernel H, then f induces an <u>isomorphism</u>

$$\overline{f} : G/H \xrightarrow{\sim} G'$$

$$\bar{f}(aH) = f(a)$$

this is well-defined since $f(a) = f(a')$ if $a' \in aH$.

Certainly this is surjective; also $\ker(\bar{f}) = \{\bar{e} = eH\}$

Put another way the Isomorphism theorem says that any homomorphism factors through the quotient by its kernel:

$$\begin{tikzcd}
G \arrow[rr, "f"] \arrow[dr, twoheadrightarrow, "F"'] & & G' \\
& G/\text{kernel} \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

Short exact sequences of groups:

$$1 \longrightarrow H \xrightarrow{g} G \xrightarrow{f} G' \longrightarrow 1$$

f and g are group homs.

w/
- g injective
- f surjective
- Image (g) = Kernel (f).

Isomorphism theorem says then

$$G' \simeq G/H.$$

WARNING:

H and G' do not determine the group G

Example:

$S_3 \not\simeq \mathbb{Z}/6\mathbb{Z}$.

$$\left\{\begin{array}{l} 1 \longrightarrow A_3 \xrightarrow{\text{inclusion}} S_3 \xrightarrow{\text{sign}} \{\pm 1\} \to 1 \\ \qquad\quad\ \ \wr\!\wr \qquad\qquad\qquad\qquad\qquad \wr\!\wr \\ 1 \to \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2 \to 1 \\ \qquad\ \ \ 2\mathbb{Z}/6\mathbb{Z} \qquad \begin{pmatrix}\bar{a} \\ \text{mod } 6\end{pmatrix} \longmapsto \begin{pmatrix}\bar{a} \\ \text{mod } 2\end{pmatrix} \end{array}\right.$$

with $\mathbb{Z}/3\mathbb{Z} = 2\mathbb{Z}/6\mathbb{Z}$.