LECTURE 16 Oct. 24/2003

So far:

$G = \mathbb{R}^n \cdot O(n)$ group of motions of $\mathbb{R}^n$ preserving $d(v,w)$

If $\Gamma \subset G$ is a finite subgroup, then Γ fixes a point $p \in \mathbb{R}^n$, so is conjugate to a finite subgroup.

Cases:

- $\Gamma \subset O(n) = G_0$ ← fixing 0
- finite $\Gamma \subset O(2)$

$$\Gamma_+ = \{\gamma \in \Gamma : \det \gamma = 1\}$$

1) $\Gamma = \Gamma_+$ cyclic of order $n \geq 1$ generated by $\mathrm{rot}(\theta)$, θ as small as possible.

2) $\Gamma \underset{2}{\trianglerighteq} \Gamma_+$ dihedral of order $2n$

Γ_+: cyclic of order n

$r\,\mathrm{rot}(\theta) r^{-1} = \mathrm{rot}(\theta)^{-1}$

$r \in \Gamma \setminus \Gamma_+$

$r^2 = 1$

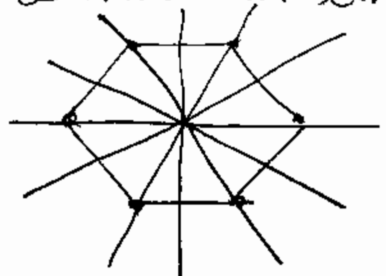
(r = reflection)

For example:

Actions on regular polygons by subgroups of G.

For the regular hexagon, $n = 6$, have action of dihedral group D_{12} consisting of

- 6 rotations
  $$T_+ = \langle \mathrm{rot}\left(\tfrac{2\pi}{6}\right) \rangle$$
- 6 reflections in lines:



- whether a reflection is through a vertex or a side impacts conjugacy — we'll see this in greater detail later.

WARNING: In our notation, D_n is the dihedral group of order n (so n is necessarily even!)

In Artin: D_n is the dihedral group of order $\boxed{2n}$!

Discrete subgroups of G

(this is a generalization of ~~our~~ discussion of finite subgroups of G)

In $G = \mathbb{R}^2 \cdot O(2)$, discrete means that Γ does not contain arbitrarily small rotations or translations, i.e. $\exists\, \varepsilon > 0$ s.t.

if $t_b \in \Gamma$, $|b| \geq \varepsilon$

if $rot(\theta) \in \Gamma$, $|\theta| \geq \varepsilon$.

Thus finite subgroups are necessarily discrete (just choose smallest b, θ)
But the converse is false:

Ex An infinite discrete $\Gamma \subset G = \mathbb{R}^n \cdot O(n)$:
Let $b \neq 0$ in $\mathbb{R}^n$
& set $\Gamma :=$ cyclic group generated by t_b.

We will now proceed to classify the discrete subgroups of $G = \mathbb{R}^2 \cdot O(2)$

Let Γ be a discrete subgroup of G

Consider

① $L = \Gamma \cap \mathbb{R}^2 = \{t_b \in \Gamma\}$

② Image $\bar{\Gamma}$ of G in $O(2)$

$\bar{\Gamma} = \Gamma/L$, $O(2) = G/\mathbb{R}^2$

(recall $O(2) \cong G/\mathbb{R}$ by 1st iso theorem & have natural map $\Gamma/L \longrightarrow G/\mathbb{R}^2$)

First step What are the possibilities for L?

① $L = \{0\}$

This is the case when Γ is finite! $(\bar{\Gamma} = \Gamma)$

② $L = \mathbb{Z}b$ $b \neq 0$ in $\mathbb{R}^2$

③ $L = \mathbb{Z}a + \mathbb{Z}b$ where $\{a, b\}$ is a basis of $\mathbb{R}^2$

here we call L a "lattice".

Observation:

If $b, b' \in L$, then $b - b' \in L$ so $|b - b'| \geq \varepsilon$ for some fixed ε

i.e. vectors can't get too close together

Q: Why are the above possibilities ①, ②, ③ the only ones?

A: If $L = 0$, ① is clear, so assume $L \neq 0$.

~~Suppose~~ all vectors in L lie on a fixed line $\ell \subset \mathbb{R}^2$

Let b be in L & closest to 0

If $b' \in L$,

$$b' = nb + r_0 b$$

$$n \in \mathbb{Z} \ \& \ 0 \leq r_0 < 1$$

If $r_0 \neq 0$, then $r_0 b \in L$ & $|r_0 b| < |b|$, a contradiction

So $b' \in \mathbb{Z} b$

& $L = \mathbb{Z} b$

$\Rightarrow$ Case ②.

So now suppose not all $b \in L$ lie on a line

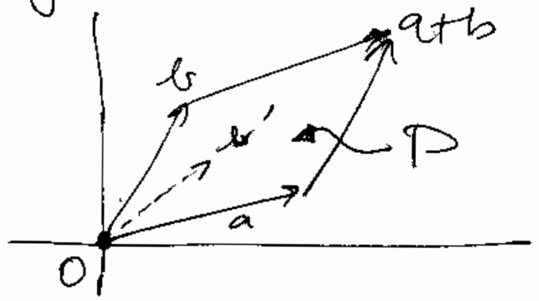
then L contains a basis of $\mathbb{R}^2$, say $\{a, b\}$.

Without loss of generality, we may assume that

a is a shortest vector in $\mathbb{R}a \cap L$
& b ——————————— $\mathbb{R}b \cap L$

<u>Lemma</u>
If S is a bounded subset of $\mathbb{R}^2$, then $S \cap L$ is finite.
<u>Pf</u>: If infinite, then choose an infinite subsequence which is convergent. But this is impossible since no sequence of element in L is Cauchy (i.e. get arb. close together). $\boxtimes$

Apply lemma to this picture:



There are only finitely many points inside $P \cap L$ (by lemma). Suppose there is one, & b' is the one closest to a.

Replace b with b'.
So now b is the point closes to a

in the new version of the parallelogram P, which must thus be empty interiorly.

<u>Claim</u> $L = \mathbb{Z}a + \mathbb{Z}b$.

<u>Pf:</u> Any $v \in \mathbb{R}^2$ can be written as $v = ra + sb$ $(r, s \in \mathbb{R})$

$$= (na + r_0 a) + (mb + s_0 b)$$

$$(n, m \in \mathbb{Z}, \quad 0 \leq r_0, s_0 < 1)$$

Now $na + mb$ if $v \in L$, so then also

$$r_0 a + s_0 b \in L$$

But then also

$r_0 a + s_0 b$ is in the parallelogram P (strictly inside ~~or~~ 0)

so $r_0 a + s_0 b = 0$,

& $v \in \mathbb{Z}a + \mathbb{Z}b$ $\boxtimes$

Now consider image $\overline{\Gamma}$ of Γ in $O(2)$

<u>Lemma</u> $\overline{\Gamma}$ <u>preserves</u> the subgroup $L < O(2)$.

<u>Pf</u> If $b \in L$, i.e. $t_b \in \Gamma$, and take say $\bar{\gamma} \in \overline{\Gamma}$ lifting to $\gamma \in O(2) \subset G$, in fact choose with $\gamma \in \Gamma$.

Consider $\gamma t_b \gamma^{-1}$ in Γ:

$\gamma t_b \gamma^{-1}$ = translation by $\bar{\gamma}(b)$

$= t_{\bar{\gamma}(b)} \Rightarrow \bar{\gamma}(b) \in L$

$\boxtimes$

Hi $L = \{0\}$, then $\bar{\Gamma} = C_n$ or D_{2n}

for some $n \geq 1$, and any such is possible

But suppose we are in case ③.

Then $L = \mathbb{Z}a + \mathbb{Z}b$ (a, b lin. indep.)

Then $\bar{\Gamma} = C_n$ or D_n with

$n = 1, 2, 3, 4$ or 6

$(\# \bar{\Gamma} \leq 12)$.

<u>Proof of limitation in case ③:</u>

$A \in \bar{\Gamma}$ rotation

Want: order of A is $1, 2, 3, 4$ or 6

Consider the charpoly of A

$$X^2 - tX + 1$$

$$t^2 - 4 \leq 0$$

$$t = \mathrm{Tr}\,A.$$

<u>Claim</u>: $Tr(A) \in \mathbb{Z}$

This is true because the matrix of A wrt to the basis $\{a, b\}$ must have <u>integer entries</u>.

Hence $t = \pm 2, \pm 1$ or 0.

$$\underbrace{\underset{\substack{\text{orders} \\ 3 \text{ or } 6}}{\pm 2} \qquad \underset{\substack{\text{orders} \\ 1 \text{ or } 2}}{\pm 1} \qquad \underset{\substack{\text{order} \\ 4.}}{0}}_{\text{order of } A.}$$