LECTURE 9 Oct. 3/2003

Last time: F field (e.g. $\mathbb{R}, \mathbb{Z}/p\mathbb{Z}, \ldots$)
V vectorspace $/F$
$T : V \longrightarrow W$ homomorphism of F-vectorspaces

§ Spans, Linear independence, bases

$(v_1, \ldots, v_n)$ ordered finite set of vectors in V

$S = \{v_1, \ldots, v_n\}$ set of vectors

linear combination:

$$w = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \qquad a_i \in F$$

all such $w =:$ span of $S =: W$

Fact: this is a subspace of V

(since $w + w' = (a_1 + b_1) v_1 + \cdots + (a_n + b_n) v_n$

$ew = (ea_1) v_1 + \cdots + (ea_n) v_n$.

Convention

$$S = \emptyset$$
$$\text{Span of } S = \{0\} \subset V$$

Def'n V is finite dim'l if there is a finite set S of vectors in V with span of $S = V$

Ex $V = F^n$ is finite dim'l

$$v_1 = (1, 0, \ldots, 0)$$
$$v_2 = (0, 1, \ldots, 0)$$
$$\vdots$$
$$v_n = (0, 0, \ldots, 0, 1)$$
$$(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i v_i$$

Non-example

$V = F[x]$ is not finite dim'l
(consider maximum degree of polynomials occurring in a putative spanning set)

Linear independence

$\{v_1, \dots, v_n\}$ is *linearly independent* if the relation

$$a_1 v_1 + \cdots + a_n v_n = 0_V$$

only holds when

$$a_1 = a_2 = \cdots = a_n = 0.$$

Example:

$V = \mathbb{R}^3$:

$v_1 = (1,0,0)$

$v_2 = (1,1,0)$

$v_3 = (1,2,3)$

$$\operatorname{Span}\{v_1, v_2\} = \{\underset{\substack{\shortparallel \\ bv_2 + (a-b)v_1}}{(a,b,0)} : a, b \in \mathbb{R}\}$$

(v_1, v_2, v_3) are linearly independent since if

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$$

then $3a_3 = 0 \Rightarrow a_3 = 0$

we're working in $\mathbb{R}$.

likewise $a_2 = 0$

$a_1 = 0.$

Def'n

We say an ordered set $(v_1, \ldots, v_n)$ is a basis of V if it spans V and is linearly independent.

What this means:

Every vector $w \in V$ is uniquely expressed as a linear combination

$$w = a_1 v_1 + \cdots + a_n v_n$$

(Because suppose also

$$w = b_1 v_1 + \cdots + b_n v_n$$

Then $0 = (a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n$

so $a_1 - b_1 = \cdots = a_n - b_n = 0$)

A basis gives rise to an isomorphism of vectorspaces

$$V \xrightarrow[f]{} F^n$$

$$f(w) = (a_1, \ldots, a_n)$$

where $w = a_1 v_1 + \cdots + a_n v_n$ is unique expression in given basis.

Easy to verify it is a homomorphism of vectorspaces

Onto $\Leftrightarrow$ spans & 1-to-1 $\Leftrightarrow$ lin. indep.

Theorem

If $S = \{v_1, \dots, v_n\}$ is a finite set which spans V then a subset of S gives a basis for V.

Proof If the elements of S are linearly independent, then we're done.

If not, we have a relation

$$a_1 v_1 + \cdots + a_n v_n = 0$$

with some $a_i \neq 0$

Can reorder so that $a_n \neq 0$ & thus a_n^{-1} exists.

Then $a_n v_n = -(a_1 v_1 + \cdots + a_{n-1} v_{n-1})$ in V;

multiply by a_n^{-1}:

$$v_n = \left(-\frac{a_1}{a_n}\right) v_1 + \cdots + \left(-\frac{a_{n-1}}{a_n}\right) v_{n-1}$$

Hence

$$V = \text{Span } S = \text{Span}\{v_1, \dots, v_{n-1}\}$$

If this new set is linearly independent, we're done.

If not, we repeat until we're done (we must finish in a finite # of steps because S was finite to start with). $\boxtimes$

Theorem

If L is a linearly independant set of vectors, it can be extended to form a basis of V

Proof:

If L spans V, then done.

If not, let S be a finite set spanning V.

There must be some $v \in S$ s.t. $v \notin \operatorname{Span} L$
(since otherwise $\operatorname{Span} L = \operatorname{Span} S = V$)

Then we claim $L' = L \cup \{v\}$ is <u>linearly independent</u>

Why? Suppose $L = \{w_1, \ldots, w_n\}$
& $\sum a_i w_i + bv = 0$
Then either $b = 0$ or
$v = -\frac{1}{b} \sum a_i w_i) \in \operatorname{Span}(L)$
(contradiction)
Hence $\sum a_i w_i = 0 \Rightarrow a_i = 0$ since
L was lin indep.

If L spans, we're done.
Otherwise we keep adjoining vectors from the finite set S until done. $\boxtimes$

Main Theorem

If $S = \{v_1, \ldots, v_n\}$ spans V
$L = \{w_1, \ldots, w_m\}$ is linearly indep. in V
then $n \geq m$.

Proof

Since S spans, we may write each element of L:

$$w_j = \sum_{i=1}^{n} a_{ij} v_i$$

Try to make a non-trivial linear relation on w_j:

$$0_V = \sum_{j=1}^{m} c_j w_j$$

$$= \sum_{j=1}^{m} c_j \left(\sum_{i=1}^{n} a_{ij} v_i \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} c_j \right) v_i$$

If we can arrange that

$$\sum a_{ij} c_j = 0 \text{ for all } i$$

with some $c_j \neq 0$, then the w_j would not be lin. indep.

We are dealing with a system of n linear equations and m unknowns

If $m > n$ (more unknowns j than the equations indexed by i) we can find a nontrivial solution.

This is reviewed in Chapter 1 of Artin. $\boxtimes$

Corollary

1) All bases of V have the same number of elements $=: \dim(V)$

2) All spanning sets S have $\#S \geq \dim V$

3) All linearly independent sets L have $\#L \leq \dim V$

$\dim\{0\} = 0$ & $\dim(F^n) = n$.

Proof of cor :

1) Two bases B & B'
Since B spans & B' is lin. ind.,
$$\# B \geq \# B'$$
Since B' spans & B is lin ind.,
$$\# B' \geq \# B$$
so $\# B = \# B'$.

2) & 3) are immediate. $\boxtimes$

Proposition
Suppose $W \subset V$ finite dim'l and $\{w_1, \ldots, w_m\}$ is a basis for W
Then we may extend this to a basis for V

Pf The basis is lin. indep in V so can be extended to a basis. $\boxtimes$

$W \subset V$ gives $V \longrightarrow V/W$
quotient vectorspace.

Fact $(f(v_1), -, f(v_n))$ gives a basis for V/W.

Hence $\dim V = \dim W + \dim(V/W)$

$W' =$ Span of $\{v_{n+1}, \ldots, v_n\}$ is a subspace of V mapping isomorphically to V/W

<u>Warning</u> This is not true for general groups.

$$\mathbb{Z}/2\mathbb{Z} \simeq \underset{\overset{\parallel}{H}}{2\mathbb{Z}/4\mathbb{Z}} \subset \underset{\overset{\parallel}{G}}{\mathbb{Z}/4\mathbb{Z}}$$

$\nexists\, H' \subset G$ mapping isomorphically to G/H