LECTURE 14 Oct. 20/2003

Last time

F field

$O_n(F) \subset Gl_n(F) =$ Aut. group of vectorspace F^n (w/ v.s. struct)

set preserving the inner product $\langle v, w\rangle = \sum_1^n v_i w_i$

$\{A : A^t A = I\} = \{A : A^t = A^{-1}\}$

$\det(A)^2 = +1$

$X^2 - 1 = (X+1)(X-1) = 0$ has at most 2 roots
$\Rightarrow \det(A) = \pm 1$

If $-1 \neq 1$ in F, $SO_n(F) \underset{2}{\trianglelefteq} O_n(F)$

$\{A : A^t = A^{-1}$ & $\det A = +1\}$

For $F = \mathbb{R}$:

If A preserves inner product $\langle v, w\rangle$, it also preserves the Euclidean norm $\|v\| = \sqrt{\langle v, v\rangle} = \sqrt{\sum v_i^2} \geq 0$

& angle θ s.t. $\cos\theta = \frac{\langle v, w\rangle}{\|v\|\|w\|}$

General F:

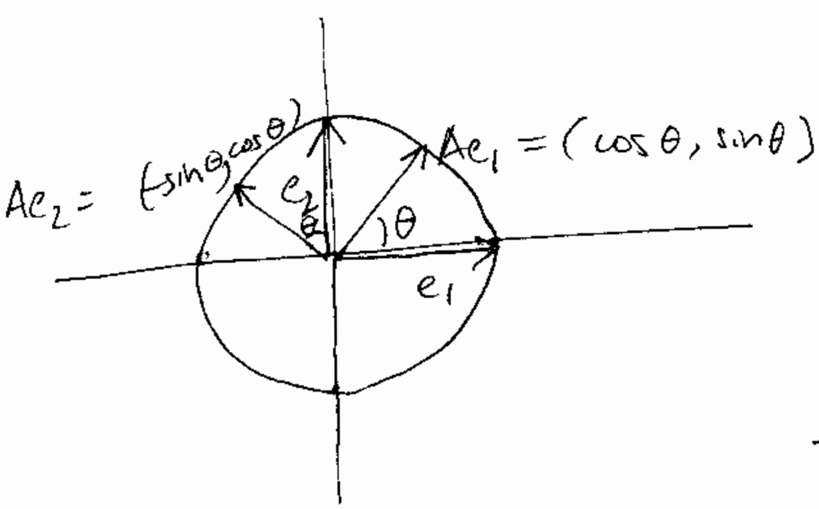
If v is an eigenvector for $A \in O_n(F)$ with eigenvalue λ, $Av = \lambda v$, & $\langle v, v \rangle \neq 0$, then $\lambda^2 = 1$, i.e. $\lambda = \pm 1$

$$\langle v, v \rangle = \langle Av, Av \rangle = \langle \lambda v, \lambda v \rangle = \lambda^2 \langle v, v \rangle$$

Now divide by $\langle v, v \rangle \neq 0$ to get $\lambda^2 = 1$

<u>Note</u>: $\langle v, v \rangle \neq 0$ automatically holds for eigenvectors $/ \mathbb{R}$, say.

What do transformations A in $SO(2)$ look like?



$\|e_1\| = \|e_2\| = 1$
$\langle e_1, e_2 \rangle = 0$

$\|Ae_1\| = 1$,
$\|Ae_2\| = 1$
$Ae_1 \perp Ae_2$
$\det A = +1$.

$$A = \begin{array}{cc} Ae_1 & Ae_2 \end{array}$$
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = rot(\theta) \quad (\text{rotation by } \theta.)$$

$$\mathrm{rot}(\theta) \circ \mathrm{rot}(\psi) = \mathrm{rot}(\theta + \psi)$$

Have isomorphism of (abelian groups)

$$SO(2) \xrightarrow[f]{} \{ z \in \mathbb{C}^\times : |z| = 1 \}$$

$$A = \mathrm{rot}(\theta) \longmapsto z = e^{i\theta} \text{ "} = f(e_1) \text{"}$$

WARNING: $O(2)$ is not abelian.

What are the transfo. $A \in O(2) - SO(2)$?

Each such A has two orthogonal eigenvectors v_1, v_2

$$Av_1 = v_1$$
$$Av_2 = -v_2$$

$$A = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \text{ wrt } (v_1, v_2)$$

Proof The char. poly. of A looks like

$$X^2 - \mathrm{Tr}(A)\, X \underbrace{- 1}_{\uparrow} = 0$$

determinant is necessarily -1 for $A \in O(2) - SO(2)$.

Claim: This has both roots $\in \mathbb{R}$

If not, the two roots are $\{z, \bar{z}\}$ conjugates in $\mathbb{C}$. (quadratic formula)

But $\det A = z\bar{z} \geq 0$

However $\det A = -1$

The contradiction shows that both roots are real: $\{\lambda_1, \lambda_2\}$

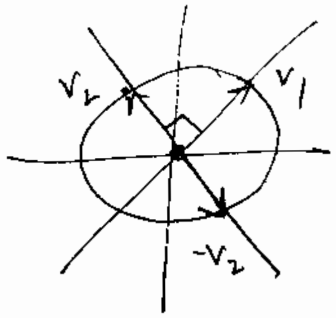
$$\lambda_1 \lambda_2 = -1$$

But these are eigenvalues for A & so $\lambda_1 = \pm 1$, $\lambda_2 = \pm 1$.

Hence $\lambda_1 = \pm 1$ & $\lambda_2 = -\lambda_1$.

Remains to show v_1 & v_2 are orthogonal $\leftarrow$ EXERCISE. for the reader.

So A looks like this:



reflection around line $\mathbb{R} \cdot v_1$

or, alternatively,

Proof that $v_1 \perp v_2$:

$$\langle v_1, v_2 \rangle = \langle A v_1, A v_2 \rangle$$
$$= \langle v_1, -v_2 \rangle = -\langle v_1, v_2 \rangle$$
$$\therefore \langle v_1, v_2 \rangle = 0$$

<u>Note</u>: $\mathrm{Refl}(v_1) \circ \mathrm{Refl}(v_2) = \mathrm{Rot}(\theta)$.

Euler's Theorem

Any A in $SO(3)$ has an eigenvalue of $+1$ so there is a $v \in \mathbb{R}^3$: $Av = v$.

(Or in Euler's terminology:
Any motion preserving $S^2 \subset \mathbb{R}^3$
has an axis of rotation)
if it preserves orientation)

Pf $f(x)$ has degree 3, so 3 roots in $\mathbb{C}$. Possibilities:

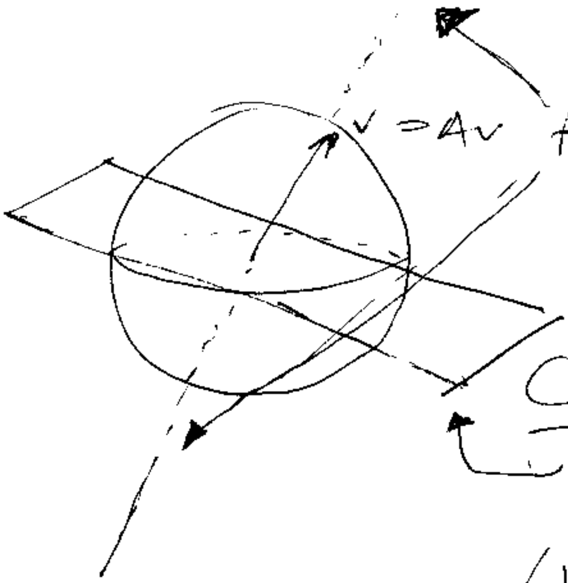
$\{\lambda_1, \lambda_2, \lambda_3\}$ all real

$\{\lambda, z, \bar{z}\}$ $z, \bar{z}$ complex conj.

(these are the only possibilities because Intermediate value theorem $\Rightarrow$ every poly of odd degree $/\mathbb{R}$ has a real root)

Know $\prod$ roots $= +1$ (we're in $SO(3)$)

It follows that one of the roots must be $+1$.



<u>Claim</u> A preserves the plane $\perp$ to v

(If $\langle v, w\rangle = 0$ then $\langle Av, Aw\rangle = \langle v, Aw\rangle = 0$)

e_1, e_2 orthogonal basis of plane $\perp$ to v

$$A = \begin{array}{c} \begin{array}{ccc} v & e_1 & e_2 \end{array} \\ \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{\qquad\qquad}} \\ 0 & & \end{array}\right) \end{array}$$

something preserving inner product on the span of $\{e_1, e_2\}$; so is element of $SO(2)$

But we know the elements of $SO(2)$: they are all of the form $\mathrm{rot}(\theta)$

So A just rotates around $\mathbb{R}v$ by some angle θ.

Rigid motions:

All motions $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ which preserve the distance between any two points:
$$d(v, w) = \|v - w\|$$
are called rigid motions.
(here motion just means function)

Claim If m is a rigid motion and $m(\mathbf{0}) = 0$, then m is a linear transformation in $O(n)$.

Pf — that $\langle m(v), m(w) \rangle = \langle v, w \rangle$:
$$\|v - w\|^2 = \langle v-w, v-w \rangle$$
$$= \underbrace{\|v\| + \|w\|^2 - 2\langle v, w\rangle}$$
all of these are preserved by m by definition.

However to prove the result, we also need that m is linear.

At the very least, the above shows $O(n) \subset G$ = group of rigid motions.

Another subgroup of G:
Translations by a fixed vector b

$$t_b(v) = v + b \quad ; \quad t_b \circ t_{b'} = t_{b+b'}$$

This subgroup is isomorphic to $\mathbb{R}^n$

Can show:

G = The group of rigid motions
$= \underbrace{\mathbb{R}^n \cdot O(n)}$

in the sense that any rigid motion can be written uniquely as the composite of ~~an~~ a translation with an element of $O(n)$.

<u>Note</u>: this copy of $\mathbb{R}^n$ is normal in G.