LECTURE 19 Oct. 31/2003

Finite groups associated to regular solids S in $\mathbb{R}^3$

$\{\text{rotations preserving } S\} = \Gamma \subset SO_3$

Ex: • S = tetrahedron
$\Rightarrow \Gamma$ order 12, $\simeq A_4$
permuting vertices

• S = octahedron or cube
$\Rightarrow \Gamma$ order 24, $\simeq S_4$
permuting diagonals of cube

• S = icosahedron or dodecahedron
$\Rightarrow \Gamma$ order 60, $\simeq A_5$

A_5: non-abelian simple group — no nontrivial normal subgroups

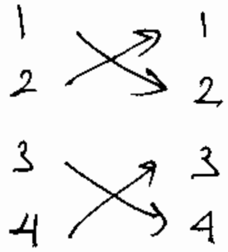
(Recall: the only abelian simple groups are $\mathbb{Z}/p\mathbb{Z}$ cyclic of order p prime.)

<u>Ex</u>: The elements of Γ for $S =$ tetrahedron:

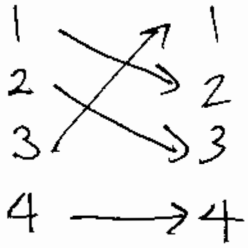
- 1 identity element e
- 8 rotations of order 3, fixing a vertex
- 3 rotations of order 2, switching pairs of vertices.

Notation of permutations:

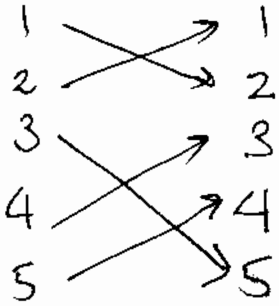
"Follow where the elements go":



$$\longleftrightarrow \quad (12)(34)$$



$$\longleftrightarrow \quad (123)(4)$$



$$\longleftrightarrow \quad (12)(354)$$

So now in this notation:
the elements of Γ for
$S=$ tetrahedron are exactly: —

- the identity is $(1)(2)(3)(4)$
- the 8 rotations fixing a vertex:

$$(123)(4) \quad (132)(4)$$
$$(1)(234) \quad (1)(243)$$
$$(2)(134) \quad (2)(143)$$
$$(3)(124) \quad (3)(142)$$

- the 3 rotations of order 2:

$$(12)(34)$$
$$(13)(24)$$
$$(14)(23).$$

<u>Ex</u>: $S=$ cube $\left\{ \begin{array}{ll} 8 & \text{vertices} \\ 6 & \text{faces} \\ 12 & \text{edges} \end{array} \right.$

Let S_4 act on "diagonals" of the cube (faithful action!)
Can permute the diagonals any way you want
$\therefore \Gamma = S_4$.

Ex: Dodecahedron: $\begin{cases} 12 \text{ faces (pentagons)} \\ 20 \text{ vertices} \\ 30 \text{ edges} \end{cases}$

See Artin for explanation of why $\Gamma = A_5$ ("inscribed cubes")

Conjugacy classes in Γ ($\cong A_5$)

Γ = symmetries of dodecahedron in $SO(3)$

Note:
- 12 faces give 6 different subgroups $\cong \mathbb{Z}/5\mathbb{Z}$
  $\Rightarrow$ 24 elements of order 5
- 30 edges give 15 different subgroups $\cong \mathbb{Z}/2\mathbb{Z}$
  $\Rightarrow$ 15 elements of order 2
- 20 vertices give 10 different subgroups $\cong \mathbb{Z}/3\mathbb{Z}$
  $\Rightarrow$ 20 elements of order 3

Argue each of these by non-intersection of subgroups of prime order!

Thus these plus identity are complete list of elements of Γ:

$$60 = \underset{\text{ident}}{1} + \underset{\text{order } 2}{15} + \underset{\text{order } 3}{20} + \underset{\text{order } 5}{24}$$

<u>Rmk</u>: Conjugate elements have the same order; in fact,

$$(hgh^{-1})^m = hg^m h^{-1}.$$

<u>Q</u> Is it possible that
$$60 = 1 + 15 + 20 + 24$$
is the <u>class equation</u> for I?

<u>A</u> No!
$$\#\text{ conj class} = \frac{\#G}{\#Z_g}$$
so $\#\text{ conj class} \mid \#G$.
But $24 \nmid 60$.

It turns out the elements of order 5 split into 2 conjugacy classes; and class equation is in fact
$$60 = 1 + 15 + 20 + 12 + 12$$

Let's show elements of order 2 are conjugate:-

Say g of order 2 is element fixing pair of edges (12)
and g' of order 2 is element fixing pair of edges (34)

Now let $h \in G$ take edge 1 to edge 3 & edge 2 to edge 4

Then $g' = h g h^{-1}$
(verify by looking at behavior on edges!)

For showing conjugacy of elements of order 3 & two sets of conjugacy of elements of order 5:
use geometric argument (see Artin)

So indeed there are 5 conjugacy classes: $60 = 1 + 15 + 20 + 12 + 12$.

<u>Proposition</u> T' is a simple group.

<u>Pf)</u> Let $H \triangleleft T'$ with $H \neq \{e\}$
Then since $gHg^{-1} = H$ $\forall g$,
H is a union of conjugacy classes in G
so $\# H = 1 +$ (some of the terms in the class equation)

But $\#H \mid \#G$ and no combination of $15, 20, 12, 12$ gives divisor except

$$1 + 15 + 20 + 12 + 12 = 60.$$

So $\#H = 60$ & $H = G$. $\boxtimes$.

In our notation for permutations

$g = (12345)$

$g' = g^2 = (13524)$

are conjugate in S_5 but not in A_5

Aside on finite simple groups

Refer to Atlas of Finite Groups

Recall If $\#G = p^n$ then $Z_G \neq \{e\}$

Pf) Consider the class eq.

$$\# G = p^n = \underset{\substack{\uparrow \\ \text{conj class of } e}}{1} + \underset{\substack{g \neq e \\ \text{conj classes.}}}{\sum} \frac{\#G}{\#Z_g}$$

Then $p^n = 1 + $ (things divisible by p)

if $Z_g \neq G \ \forall g$

$\therefore Z_g = G$ ~~for~~ some $g \neq e$

$\therefore \exists g$ nonidentity in the center $\boxtimes$

Corollary • If $\# G = p$ (prime), then G is cyclic & generated by any $g \neq e$

• If $\# G = p^2$ (p prime), then G is abelian.

Pf) • First part obvious

• Second part:

$Z_G \neq \{e\}$ so has order p or p^2.

If Z_G has order p^2, we are done.

If Z_G has order p, then

take $g \in G - Z_G$

& consider $\underset{\substack{\uparrow \\ \text{order } p^2}}{Z_g} \supset \langle Z, g \rangle \supsetneq Z$

$\Rightarrow g \in Z_G$

contradiction.

$\therefore$ Z_G must have order p^2 $\boxtimes$.

A non-abelian group of order p^3

$$G \subset GL_3(\mathbb{Z}/p\mathbb{Z})$$

$$\| $$

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\} \leftarrow Z_G \text{ has order } p$$

$$\left(= \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \right)$$