Math 122 10 Nov 2003

§ Last Couple Lectures

Sylow Theorems

Let G be a gp of order $p^n m$ $(p \nmid m)$. Then:

(1) $n_p(G) = \#\{\text{Sylow } p\text{-subgroups}\} \equiv 1 \bmod p$, + $n_p(G) \mid m$

(2) all sylow p-subgroups are conjugate

Consequences

Let $H \subset G$ be a Sylow p-subgroup

(1) H is normal $\Leftrightarrow n_p(G) = 1$

(2) $[G : N_G(H)] = [G : \underbrace{\{\text{stab } H\}}_{\text{stab under the action of conjugation}}] = \#\{\text{orbit of } H \text{ under this action}\}$
$= n_p(G)$

(3) If $|G| = pm$, $p \nmid m$. Then $\#\{\text{elements of order } p \text{ in } G\}$
$= n_p(G)(p-1)$

§ Applications

(1) Groups of order pq ($p<q$ primes)

Must have a normal Sylow q-subgroup ($N_q(G) \mid p$, $N_q(G) \equiv 1 \bmod q$)

If $p \nmid q-1$ then $n_p(G) = 1$ ($\Rightarrow G \cong \mathbb{Z}/(pq)$)

If $p \mid q-1$, then $\exists!$ nonabelian gp. of order pq

(2) Groups of order 12

-Either G has a normal Sylow 3-subgp. of $G \simeq A_4$

(in which case G has normal 2-Sylow)

(3) Groups of order p^2q: $p>q \Rightarrow$ normal Sylow p-Sylow

$p<q$: either G has normal Sylow q-subgp. or $p^2q = 12$ + $G \simeq A_4$

§ Conjugacy in S_n (recall from 11/7)

§ Notes on A_5

Prop^n: A_5 is simple (optional HW - see website)

$\rightarrow$

Prop'n (converse) If G is a simple gp. of order 60, then $G \simeq A_5$

Pf: $n_2(G) \equiv 1 \mod 2$, $n_2(G) | 15$ $\therefore$ $n_2(G) = 3, 5$, or 15

Strategy (i) $n_2(G) \neq 3$ (ii) $\exists N \leq G$ of index 5
(iii) use action of G on G/N (coset space of order 5) w/ only even permutations

Claim 1: $n_2(G) \neq 3$. sufficient to show G has no proper subgps. of index < 5, since if P is a Sylow 2-subgp.,
$[G : N_G(P)] = n_2(G)$

Pf: Suppose $[G:H] = 2, 3$, or 4.
G acts on G/H by left mult. transitively
(every $g \in G$ induces perm. of G/H)

So we have a hom. $\varphi : G \longrightarrow Sym(G/H)$
$g \longmapsto$ perm induced by g

$\ker(\varphi) \subset H \subsetneq G$, $\ker(\varphi) \trianglelefteq G \Rightarrow \ker(\varphi) =$ trivial

$\therefore$ φ is injective. so $\underbrace{G}_{\text{order } 60} \subset Sym(G/H) \simeq \underbrace{S_{[G:H]}}_{\text{order } 2!, 3! \text{ or } 4!}$

Contradiction

Claim 2: $n_2(G) = 5 \Rightarrow G \simeq A_5$

Pf: P a Sylow 2-subgp. $N = N_G(P)$
$[G:N] = n_2(G) = 5$ G acts on G/N

Again have hom: $\varphi : G \rightarrow Sym(G/N) \simeq S_5$
know $N \subsetneq G$, so $\ker \varphi \neq G$ $\therefore$ as in claim 1, $\ker \varphi = \{e\}$
and so $G \subset S_5$.

Suppose $G \not\subset A_5$. Then $GA_5 = S_5$. Then
$[G : A_5 \cap G] = [S_5 : A_5] = 2 \Rightarrow A_5 \cap G \trianglelefteq G$ contradiction, since G is simple.

$\therefore$ $\underbrace{G}_{\text{order } 60} \subset \underbrace{A_5}_{\text{order } 60}$ $\therefore$ $G = A_5$ ∎

$\longrightarrow$

Claim 3: $n_2(G) \neq 15$

Pf: (by contradiction)

If we can show $\exists\, M < G$ w/ $[G:M] = 5$,

then as in claim 2 (replacing N w/ M), can show

$G \simeq A_5$, and $n_2(A_5) = 5 \Rightarrow\Leftarrow$

construction of M (details omitted)

show $\exists$ Sylow 2-subgroups $P + Q$ s.t. $|P \cap Q| = 2$ +

show that $|N_G(P \cap Q)| = 12$ $\quad M = N_G(P \cap Q)$

End of Proof that $G \simeq A_5$.

§ Notes on A_n $(n \geq 5)$

Thm: A_n is simple for $n \geq 5$

Pf: (by induction) A_5 ✓

$n \geq 6$: $G = A_n$. Suppose (for contr.)

that $\exists\, H \trianglelefteq G$, $H \neq 1, G$

$A_n = G$ acts on $\{1, \cdots, n\}$

$G_i :=$ Stab of $i \simeq A_{n-1}$

Claim 1: $\nexists\, \tau \in H_{\neq e}$ s.t. $\tau(i) = i$ $\quad$ ($\therefore$ $\tau_1(i) = \tau_2(i)$, $\tau_1, \tau_2 \in H$, $\Rightarrow \tau_1 = \tau_2$)

Pf: $G_j \subset H$ $\forall j$, $A_n = \langle G_1, \cdots, G_n \rangle \subset H$

Claim 2: Given $\tau \in H$, only 2-cycles can appear in its disj. cyc. decomp.

Claim 3: Given $\tau \in H$, τ doesn't just have 2-cycles