LECTURE 17 Oct. 27/2003

Discrete subgroups $\Gamma \subset G = \mathbb{R}^2 \cdot O(2)$

normal in Γ → $L := \Gamma \cap \mathbb{R}^2 \subset \mathbb{R}^2$

<u>possibilities:</u>

- $L = \{0\}$
- $L = \mathbb{Z}a$
- $L = \mathbb{Z}a + \mathbb{Z}b$

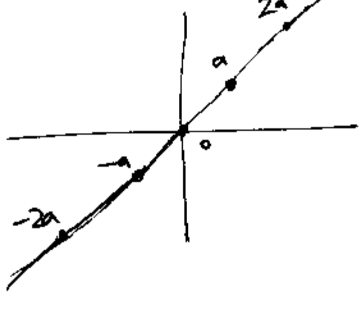
  $\{a,b\}$ basis of $\mathbb{R}^2$

$\overline{\Gamma}$ = image of Γ in $G/\mathbb{R}^2 \cong O(2)$
= image of Γ in Γ/L

$\overline{\Gamma}$ is finite & preserves L in the action of $O(2) \circlearrowright \mathbb{R}^2$

Saw:
- If $L = \{0\}$ then $\Gamma = \overline{\Gamma} = C_n$ or D_{2n}, $n \geq 1$

  (& all these occur!)
- If $L = \mathbb{Z}a$ then $\overline{\Gamma} = C_1, C_2, D_2,$ D_4 = Klein 4-group



$\gamma \in \overline{\Gamma}$
$\gamma(a) = \pm a$ as in $\mathbb{Z}a$ & same length

If $\gamma \in SO(2)$, then $\gamma = I$ or $-I$.

• If $L = \mathbb{Z}a + \mathbb{Z}b$ then $\overline{\Gamma} = C_n, D_{2n}$
$n = 1, 2, 3, 4$, or 6

Pf) Suff. as before prove that the only possible rotations in $\overline{\Gamma}$ have order 1, 2, 3, 4 or 6.
If $\gamma \in \overline{\Gamma}$ is a rotation of angle θ,
$f(x) =$ char. poly. of γ
$= x^2 - 2\cos(\theta)\, x + 1$
so $|2\cos(\theta)| \leq 1$
& since matrix must be of integral trace,
$2\cos\theta$ is an integer.
This only occurs for $\theta = \frac{2\pi}{n}$
$n \in \{1, 2, 3, 4, 6\}$. $\boxtimes$

What is the classification of lattices $L = \mathbb{Z}a + \mathbb{Z}b$ in $\mathbb{R}^2$?

Changing L to γL with $\gamma \in O(2)$ just conjugates $\overline{\Gamma} = \mathrm{Aut}(L)$ by γ. Changing L to $c \cdot L$ doesn't change $\overline{\Gamma}$ $(c \in \mathbb{R}^\times)$

L up to action of $O(2)$ and $\mathbb{R}^\times$ on $\mathbb{R}^2$:
Can assume shortest a has length 1 $\longleftarrow \mathbb{R}^\times_+$ -action
Can rotate a so that $a = (1, 0)$ $\longleftarrow SO(2)$-action

~~Where~~ What is second basis vector b?

Know $|b| \geq 1$ (a was shortest, $|a|=1$)

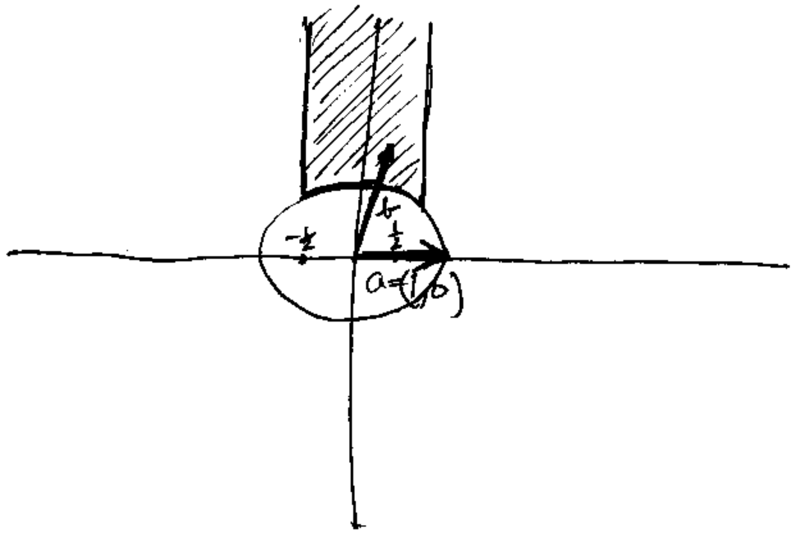
y-coord of b is nonzero ($\{a,b\}$ basis)

Replace b by $-b$ if nec. to make y-coord of b positive.

So far have place several conds on $L = \mathbb{Z}a + \mathbb{Z}b$ $\longleftarrow$ Now: Replace b by $b + na$ so that $-\frac{1}{2} \leq x\text{-coord} \leq \frac{1}{2}$

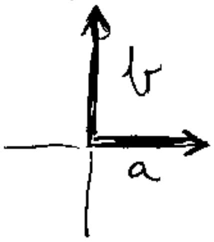
Conclusion

$L \simeq L_b := \mathbb{Z} + \mathbb{Z}b$ for b in the following region:



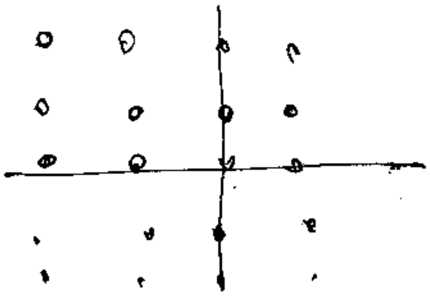
If b is <u>strictly inside</u> of region then only possibilities for $\bar{\Gamma}$ are 1 or $C_2 = \langle -I \rangle$.

If $b \perp a$
(but $|b| \neq 1$)



then $\bar{\Gamma}$ is one of $1, C_2, D_2$ or D_4

If square lattice



$a = (1,0)$
$b = (0,1)$

then $\bar{\Gamma}$ is one of $1, C_2, C_4, D_4, D_8$

In case of "hexagonal lattice" ($a = (1,0)$, b = one of the "corner pts" )
then $\bar{\Gamma}$ is one of
$1, C_2, C_3, C_6, \quad D_6, D_{12}.$

We've been discussing group
$G = \mathbb{R}^2 \cdot O(2)$ of motions on $\mathbb{R}^2 = S$
$(b, A)(s) \longmapsto As + b$

Can abstract this notion to general group G and set S:

"Group action"

$G \times S \longrightarrow S$
$(g, s) \longmapsto g \cdot s$

Demand: 1) $e \cdot s = s \quad \forall s \in S$

2) $(gh) \cdot s = g \cdot (h \cdot s)$
$\forall g, h \in G,\ s \in S$

Ex: Action of G on the set of lines $l \subset \mathbb{R}^2$, $(G = \mathbb{R}^2 \cdot O(2))$

Ex: Action of G on the set of all triangles in $\mathbb{R}^2$ $(G = \mathbb{R}^2 \cdot O(2))$

Two concepts:

Given $s \in S$:

- Orbit: $O_s := \{g \cdot s : g \in G\} \subset S$
- Stabilizer: $G_s := \{g \in G : g \cdot s = s\} \subset S$

In examples:

- If ℓ is any line $O_\ell = S$
- If Δ is a triangle $O_\Delta \neq$ all triangles.

If $O_s = S$ for some (or equivalently all) $s \in S$, then we say G acts transitively on S.

Model of a transitive G-action

G any group
H any subgroup of G
$S = G/H = \{aH \subset G\}$

G acts on S by
$$g \cdot (aH) \stackrel{\text{def}}{=} gaH$$

This is transitive since $a \cdot H = aH$
$$\Rightarrow O_H = S.$$

Q: What is G_H?
A: $G_H = \{g : gH = H\} = H$.

Q: What is G_{aH}?
A:
$$\begin{aligned} G_{aH} &= \{g : gaH = aH\} \\ &= \{g : a^{-1}gaH = H\} \\ &= a\{g' : g'H = H\}a^{-1} = aHa^{-1}! \end{aligned}$$

More "generally", if G acts transitively on S: for any fixed $s \in S$, and any $s' \in S$, $\exists g \in G$ s.t. $g \cdot s = s'$ &

$$G_{s'} = g G_s g^{-1} \subset G$$

so: all stabilizers are conjugate

In fact, the general transitive action is no more general than the coset action:

<u>Prop</u> If G acts transitively on S and $s \in S$, then there is a bijection

$$\varphi : G/G_s \xrightarrow{\sim} S$$

(G_s = stab. of s)

$$g \longmapsto g \cdot s$$

(G/G_s = coset space viewed as G-<u>set</u>)

s.t. $\varphi(g \cdot x) = g \cdot \varphi(x)$

"map of G-sets"
= "map of sets w/ G-action"

Thus: Transitive actions of G

$$\updownarrow$$

Conjugacy classes of subgroups of G

In cases where G & S are finite, ~~get~~ many interesting combinatorial formulas...

<u>Ex</u>: G acts on $S = G$ by conjugation

$$g \cdot s := gsg^{-1}$$

Orbits = conjugacy classes.

G_s = centralizer of s

$= \{g \in G : gs = sg\}$.