LECTURE 29

Nov. 26/2003

Domains & fields of fractions

A commutative ring is an <u>integral domain</u> (also called just a <u>domain</u>) if it has the property that

if $a \cdot b = 0$ then $a = 0$ or $b = 0$ in R.

<u>Ex</u>: • $R = F$ a field

is a domain

($a \cdot b = 0$ & $a \neq 0$, then multiply by a^{-1} to get $b = 0$)

• $R = \mathbb{Z}$ is a domain.

• $R = \mathbb{Z}[i]$ is a domain

<u>Non-ex</u>: • $R = \mathbb{Z}/4\mathbb{Z}$ is <u>not</u> a domain

$2 \not\equiv 0 \pmod 4$

$2 \cdot 2 \equiv 0 \pmod 4$.

<u>Rmk</u> If R is a domain, then $R[X]$, $R[X_1, \ldots, X_n]$ are domains.

(in part. if F is a field $F[X]$ is a domain)

<u>Prop</u> If $a \cdot b = a \cdot c$ in R and $a \neq 0$ then $b = c$.

<u>Pf</u>) $0 = ab - ac = a(b-c) \Rightarrow b - c = 0$ $\boxtimes$.

Rmk If $R \xhookrightarrow{\text{embeds}} F$ field (i.e. R is a subring of a field F) then R is a domain.

Big Theorem
If R is a domain, then there is a field F we can construct from R (field of fractions) and an inclusion of rings $R \hookrightarrow F$, and we can choose F minimally.

Ex: • $R = \mathbb{Z} \Rightarrow F = \mathbb{Q}$
• $R = \mathbb{Z}[i] \Rightarrow F = \mathbb{Q}(i) = \{\alpha + \beta i : \alpha, \beta \in \mathbb{Q}\}$
• $R = k[X]$, k a field
$\Rightarrow F = k(X) = \{$"rational fns" $\frac{f(X)}{g(X)} : g(X) \neq 0\}$.

Idea behind "Big Theorem":
Formally invert elements; i.e. add $1/a$ for all $a \neq 0$ in R
($a \cdot 1/a = 1$ in F).
but do it so that $b/ab = 1/a$

Pf) of "Big Theorem":
Construction of F from R.
Start w/ set of symbols
$S = \{a/b : a \in R,\ b \neq 0 \text{ in } R\}$
& define equiv. relation
$a/b \sim a'/b' \iff ab' = ba'$ in R.

Not hard to verify that $\sim$ is indeed an equivalence relation (need to use the fact that R is a domain to do this!)

Define: $F = S/\sim$.

Then define $\left[\frac{a}{b}\right] + \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right]$

$$\left[\frac{a}{b}\right] \times \left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right]$$

$$\left[\frac{a}{b}\right]^{-1} = \left[\frac{b}{a}\right]$$

(when $a \neq 0$)

($\left[\frac{a}{b}\right]$ denotes class of $\frac{a}{b}$ modulo $\sim$)

and verify that this is well-defined (i.e. these definitions are compatibly given wrt. $\sim$):

e.g. if $\frac{a}{b} \sim \frac{a'}{b'}$ & $\frac{c}{d} \sim \frac{c'}{d'}$

then $\frac{ad+bc}{bd} \sim \frac{a'd'+b'c'}{b'd'}$

(this requires just an easy verification)

Also $R \hookrightarrow F$

$$a \longmapsto \left[\frac{a}{1}\right]$$

Moreover F is the smallest field containing R — this will follow from below. $\boxtimes$

Universal Property of field of fractions

If
$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr] & k \\ & F \arrow[u, "\text{(hom. of fields)}"'] \end{tikzcd}$$
is any ring inclusion into a field, then factors through.

field of fractions

<u>Note</u>: Any homomorphism $F \to k$ of fields is injective since its kernel must be (0) or (1), but 1 can't be in the kernel since $1 \mapsto 1$; hence the kernel is (0) & the map is injective.

<u>Pf)</u> of universal property.

Given $h: R \hookrightarrow k$,
define $h^*: F \to k$
by $h^*([a/b]) = h(a)\, h(b)^{-1}$

$b \neq 0 \quad h(b) \neq 0$ so $h(b)^{-1}$ exists in k

$\boxtimes$