Homework (*required*):

Read §§ 10.1, 10.2 for next lecture.

Homework (*optional, not for extra credit*):

The purpose of this assignment is to prove that A_5 is simple using Sylow theory. Notice that $\langle(12345)\rangle$ and $\langle(13245)\rangle$ are distinct Sylow 5-subgroups of A_5. Thus it suffices to prove:

If $|G| = 60$ and G has more than one Sylow 5-subgroup, then G is simple.

We proceed by contradiction. Let G be a group of order 60 and suppose

$$n_5(G) := \#\{\text{Sylow 5-subgroups}\} > 1,$$

and $H \lhd G$, $H \neq 1, G$.

(1) Show that $n_5(G) = 6$.

Let P be a Sylow 5-subgroup. Since $[G : N_G(P)] = n_5(G) = 6$, we have $|N_G(P)| = 10$.

Claim 1. $5 \nmid |H|$

We prove the claim by contradiction. Suppose $5|\#H$.

(2) Show that this implies H contains every Sylow 5-subgroup of G.
(3) Show that H must contain at least 24 elements of order 5.
(4) Show that $\#H = 30$.

The Sylow theorem shows that $n_3(H)$ must be 1 or 10.

(5) Show that $n_3(H)$ cannot equal 10 (count elements).

Thus $n_3(H)$ is 1 and we may let $Q \lhd H$ be the Sylow 3-subgroup of H. Recall that also $P \subset H$.

(6) Show that PQ is a subgroup of H of order 15.

Since $[H : PQ] = 2$, $PQ \lhd H$. The Sylow theorem implies that $n_5(PQ) = 1$, and since $P \subset PQ$ is a Sylow 5-subgroup, we must have $P \lhd PQ$.

(7) Show that P must therefore actually be normal in H.

This is a contradiction since $n_5(H) = 6$, and the claim is proved.

So now we know $5 \nmid \#H$. The claim shows that there is no nontrivial proper normal subgroup of G with order divisible by 5. To get a contradiction, we will construct from H another normal subgroup $H_2 \lhd G$, $H_2 \neq 1, G$ such that $5 | \#H_2$.

Claim 2. G contains a normal subgroup of order 2, 3 or 4.

Since $5 \nmid \#H$, the only possible orders for H are $2, 3, 4, 6, 12$.

(8) Show that if H has order 6 then G contains a normal subgroup of order 3.

(9) Show that if H has order 12 then G contains a normal subgroup of order 3 or 4.

Thus indeed there is $H_1 \lhd G$ of order 2, 3 or 4, proving the claim.

Let $\overline{G} = G/H_1$, which must have order 30, 20 or 15.

(10) Show that in any of these cases $\overline{G}$ contains a normal 5-Sylow subgroup $\overline{P}$.

Let $H_2 = \left\{ g \in G \, : \, g \bmod H \in \overline{P} \subset \overline{G} \right\}$.

(11) Show that H_2 is a normal subgroup of G which is neither 1 nor G, and that $5 | \#H_2$.

This contradicts *Claim 1.* Hence A_5 is simple.