LECTURE 11

Oct. 8/2003

V, W finite dim v.s. over the field F

$T: V \to W$ linear operator

$$T(v+w) = T(v) + T(w)$$
$$T(cv) = c\,T(v)$$

<u>Ex 1</u>: $V = F[x]_{\deg \leq n}$ $\leftarrow$ dim $= n+1$

$W = F[x]_{\deg \leq n-1}$ $\leftarrow$ dim $= n$

$$T = \frac{d}{dx} : V \to W$$

$$(f+g)' = f' + g'$$
$$(cf)' = cf'$$

<u>Ex 2</u>: Total derivative of $F: \mathbb{R}^n \to \mathbb{R}^m$ at a pt $\underline{x}$ is a linear map

Ex 3 What is $\ker\left(\frac{d}{dx}\right)$ when $F = \mathbb{Z}/p\mathbb{Z}$?

$\ker T \supset F$

But if $n \geq p$, also have x^p in the kernel

$$\frac{d}{dx}(x^p) = px^{p-1} = 0.$$

Also x^{p-1} is not in the image

Recall: $\ker T = \{v \in V : Tv = 0\} \subset V$
$\operatorname{Im} T = \{Tv \in W : v \in V\} \subset W$

Dimension formula:

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$$

Pf: Let $\{v_1, \ldots, v_k\}$ be a basis of $\ker T$. Since linear indep., extend to a basis
$\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ of V

Then $\{w_i = T(v_{k+i}) \quad (i=1,\dots,n-k)\}$ is a basis for $\mathrm{im}(T)$.

They span since

$$w = T(v) = T\left(\sum_{i=1}^{k} a_i v_i + \sum_{k+1}^{n} b_i v_i\right)$$

$$= \sum_{i=k+1}^{n} b_i w_{i-k}.$$

They are linearly independent.

Assume they have a relation:

$$\sum b_i w_i = 0_W$$

Consider vector $v_0 = \sum_{1}^{n-k} b_i v_{i+k}$ in V

Claim: $T v_0 = 0_W$

So $v_0 \in \ker T$

$$\text{So } v_0 = \sum_{1}^{k} a_i v_i = \sum_{1}^{n-k} b_i v_{i+k}$$

Hence $\sum_{1}^{k} a_i v_i - \sum_{1}^{n-k} b_i v_{i+k} = 0$

This is a linear relation on our basis of V

So all $a_i = 0$ & all $b_i = 0$. $\boxtimes$

Corollary

If V is fin.-dim. and $W \subset V$, then $\dim W + \dim V/W = \dim V$.

Pf There is a homomorphism

$$T : V \longrightarrow V/W$$
$$v \longmapsto v + W$$

which is surjective w/ $\ker T = W$. $\boxtimes$

Notation: rank of $T := \dim(\operatorname{im} T)$
nullity of $T := \dim(\ker T)$.

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20 m. § Matrices

$V \quad \{v_1, \ldots, v_n\}$ — basis
$W \quad \{w_1, \ldots, w_m\}$

$V \longrightarrow F^n$ an isom. of vectorsps

$$v \longmapsto (a_1, \ldots, a_n)$$
$$\| $$
$$\sum a_i v_i$$

$W \longrightarrow F^m$ similar isom coming from $\{w_1, \dots, w_m\}$

What does $T: V \to W$ look like <u>once these isoms have been chosen?</u>

$$
\begin{tikzcd}
V \arrow[r, "\sim"] \arrow[d, "T"'] & F^n \arrow[d, dotted] \\
W \arrow[r, "\sim"] & F^m
\end{tikzcd}
$$

Each $T(v_j) = \sum_{i=1}^{m} \underbrace{a_{ij}}_{} w_i \qquad j = 1, \dots, n$

in F: determined by T & the choice of bases

Conversely, these scalars $\{a_{ij}\}$ determine T:

Write $v = \sum x_j v_j$

$$Tv = \sum_1^n x_j \left(\sum_1^m a_{ij} w_i \right)$$

$$= \sum y_j w_j$$

$A = m \times n$ matrix (a_{ij})

$$= \begin{array}{c} \qquad T v_j \qquad \\ \begin{pmatrix} \cdots & \Big| & \cdots \end{pmatrix} \end{array}$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = Y = AX = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

so:

$$\begin{tikzcd} V \arrow[r, "\sim"'] \arrow[d, "T"'] & F^n \arrow[d, "\text{left multiplication by the matrix } A."] \\ W \arrow[r, "\sim"'] & F^m \end{tikzcd}$$

<u>Ex:</u> $T : V \to W$

V : basis $\{v_1, v_2\}$
W : basis $\{w_1, w_2\}$

$$T v_1 = 2 w_1$$
$$T v_2 = 3 w_1 + 4 w_2.$$

$$A = \begin{array}{c} \; T v_1 \quad T v_2 \\ \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \end{array}$$

Suppose we want to calculate

$$v = 7v_1 + 8v_2$$

Well $\quad A\begin{pmatrix}7\\8\end{pmatrix} = \begin{pmatrix}38\\32\end{pmatrix}$

So $\quad Tv = 38w_1 + 32w_2.$

$V : \{v_1, \ldots, v_n\}$

$T : V \longrightarrow V$ endomorphism

A = matrix of T wrt given basis of V

$= n \times n$ matrix

$$V \xrightarrow[T]{} V \xrightarrow[S]{} V$$

matrix A $\qquad$ matrix B

What is the matrix of $S \circ T : V \to V$?

<u>Answer</u>: $B \cdot A$

matrix mult.

Prop The following are equivalent for $T: V \to V$:

1) T is an isom (automorphism) of v.s.
2) $\ker T = 0$
3) $\operatorname{Im} T = V$
4) If the matrix A of T wrt $\{v_1, \dots, v_n\}$ is A then A is invertible (as an $n \times n$ matrix)
5) $\det A \neq 0$ in F.

$$\left\{ \begin{array}{l} T: V \to V \\ \text{which are} \\ \text{isomorphisms} \end{array} \right\} =: GL(V)$$

$$\underset{\substack{\uparrow \\ \text{with a choice of basis}}}{\cong} GL_n(F)$$

Ex: $F = \mathbb{Z}/2\mathbb{Z}$

What is the group $GL_2(F)$?

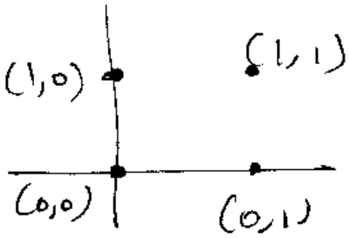
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftarrow$ 16 choices of 2×2 matrices

6 are invertible.

In fact $GL_2(F) \simeq S_3$.

How can we see this ?

Well, F^2 looks like



Now any linear operator will take $(0,0)$ to itself

The remaining three vectors $\{(1,0), (0,1), (1,1)\}$ can be permuted arbitrarily among themselves.

If A is the matrix of $T: V \to V$
wrt. $\{v_1, \ldots, v_n\}$.
What is matrix wrt a different basis $\{v_1', \ldots, v_n'\}$ A'

Answer: $A' = PAP^{-1}$
is the conjugate matrix
where P is the $n \times n$ matrix giving the change of basis.

The advantage of the (V, T) point of view over the (F^n, A) point of view:
is by <u>choosing a convenient basis</u> we can get a simpler form for the operator.

<u>Ex</u>: From our first proposition:
$T: V \to W$
$\exists$ basis ~~of~~ V $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$
basis of W $\{Tv_{k+1}, \ldots, Tv_n\}$
so that matrix of T is

$$[r = \operatorname{rank}(T)] \quad \left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

Suppose, though, we're constrained to choose a single basis for domain and target?

This brings us to theory of eigenvalues & eigenvectors.