7 November 2003

Math 122

HW for Mon., 11 Nov: §6.6, problems 5, 6, 10, + 15

The Symmetric Group

S_n symmetric group on n elements: permutations of $\{1, \dots, n\}$

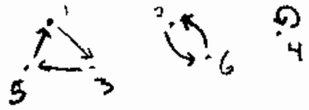
- as in Artin, permutations act on right

$(i)p.$ $(ip)q = (i)(pq)$ $\quad p, q \in S_n,\ i \in \{1, \dots, n\}$

Notation: (1) $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 6 & 5 & 4 & 1 & 2 \end{pmatrix}$

$q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 & 3 \end{pmatrix}$

$pq = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 5 & 2 & 1 \end{pmatrix}$

(2) Cycle notation $\quad p:$ 

$p = (1\ 3\ 5)(2\ 6)(4)$ or $(5\ 1\ 3)(2\ 6)(4)$
or $(5\ 1\ 3)(2\ 6)$ or $(2\ 6)(5\ 1\ 3)$

- Any permutation can be written uniquely as a product of disjoint cycles, up to reordering of cycles / dropping 1-cycles / cyclicly permuting the elements in a cycle.

q (from above): $q = (12)(3456)$

$pq = (1\ 3\ 5)(2\ 6)(1\ 2)(3\ 4\ 5\ 6)$ $\qquad 1 \xmapsto{p} 3,\ 3 \xmapsto{q} 4:\ 1 \xmapsto{pq} 4$ etc.

$= (1\ 4\ 5\ 2\ 3\ 6)$ ~ same as $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 5 & 2 & 1 \end{pmatrix}$ as above

$p^6 = e$ $\quad$ - disjoint cycles commute.

so $p^6 = (135)^6 (26)^6 (4)^6 = e \cdot e \cdot e$ $\quad$ since $(1\ 3\ 5)^3 = (2\ 6)^2 = e$

§ Conjugation

$q^{-1} p q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 & 3 \end{pmatrix}^{-1} (135)(26)(4) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 & 3 \end{pmatrix}$

$= (2\ 4\ 6)(1\ 3)(5)$

$\longrightarrow$

- Cycle shape is the same $(x\ x\ x)(x\ x)(x)$

$q^{-1}pq = (2\ 4\ 6)(1\ 3)(5)$

$= ((1)q\ (3)q\ (5)q)((2)q\ (6)q)((4)q)$. More generally,

Suppose $ip = j$ (i.e. p sends $i \mapsto j$)

then $(iq)(q^{-1}pq) = (ip)q$

so if $p = (i_1\ i_2 \cdots i_r)(i_1' \cdots i_s') \cdots\cdots$

then $q^{-1}pq = ((i_1q)(i_2q)\cdots(i_rq))((i_1'q)(i_2'q)\cdots(i_s'q))\cdots\cdots$

<u>Prop'n</u>. Two permutations which are conjugate in S_n have the same cycle shape (i.e. when expressed as a product of disjoint cycles). Conversely, if two permutations have the same cycle shape they are conjugate.

<u>Ex</u>: does $q^{-1}(135)(26)(4)q = (654)(32)(1)$

take $q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 1 & 4 & 2 \end{pmatrix}$ - can always find such a q

§ S_5 Classes

S_5 - order 120

Conj. classes -
- 5-cycles $= 4! = 24$
- 4-cycles $= 30$
- 3-cycles $= 20$
- 2-cycles $= 10$
- 1-cycle (e) $= 1$
- 2-cycle $\times$ disjoint 3-cycle $= 20$
- 2-cycle $\times$ disjoint 2-cycle $= 15$

- Read about S_p for p prime