LECTURE 13 Oct. 17/2003.

Some subgroups of the general linear group we will introduce:

$$SO_n(F) \underset{②}{\triangleleft} O_n(F) \subset GL_n(F)$$

② ← assuming $1 \neq -1$ in F

F any field.

When $F = \mathbb{R}$, this is related to Euclidean geometry.

To define these subgroups:
Put additional structure on $V = F^n$:

Inner product $\langle v, w \rangle \in F$
if $v = (a_1, \ldots, a_n)$
$w = (b_1, \ldots, b_n)$
$\langle v, w \rangle = a_1 b_1 + \cdots + a_n b_n$.
"bilinear form"

Let $O_n(F) = \{ A \in GL_n(F) : \langle Av, Aw \rangle = \langle v, w \rangle \ \forall v, w \in V \}$

Can check $O_n(F)$ is actually a subgroup of $GL_n(F)$.

What does a matrix $A \in O_n(F)$ look like?

$$A = \begin{array}{cc} Ae_1 \ \cdots & Ae_n \\ \left(\begin{array}{c} | \\ | \end{array} \right. & \left. \begin{array}{c} | \\ | \end{array} \right) \end{array}$$

$\left(e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \text{ std basis} \right)$

$\langle e_1, e_1 \rangle = 1, \dots, \langle e_n, e_n \rangle = 1$

& $\langle e_i, e_j \rangle = 0$ if $i \neq j$.

$\Rightarrow \langle Ae_i, Ae_j \rangle =$ inner product of i^{th} & j^{th} columns of A

$$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This is equivalent to:

$$A^t \cdot A = I$$

$$(\Leftrightarrow A^t = A^{-1})$$

<u>So</u>: $O_n(F) = \{ A \in GL_n(F) : A^t = A^{-1} \}$

<u>Note</u>: $\langle v, w \rangle = v^t \cdot w$

write as column vectors

So $\langle Av, Aw \rangle = v^t (A^t A) w$

<u>Note</u>: $1 = \det I = \det A^t A = \det A^t \cdot \det A$
$= (\det A)^2$

So $\det A = \pm 1$.

(This is <u>not</u> an equivalent cond'n)

$SO_n(F) :=$
$\{A \in O_n(F) : \det A = +1\}$
$=$ kernel of $O_n(F) \xrightarrow[\det]{} \{\pm 1\}$

<u>Claim</u> The permutation matrices are in $O_n(F)$; this gives injective hom. $f : S_n \hookrightarrow O_n(F)$

$$\begin{array}{ccc} S_n & \hookrightarrow & O_n(F) \\ \triangledown & & \triangledown \\ A_n & \hookrightarrow & SO_n(F) \end{array}$$

(if $1 \neq -1$ in F)

Now take $F = \mathbb{R}$

$$\langle v, v \rangle = \sum_{i=1}^{n} a_i^2 \geq 0$$

this makes sense specifically because we're working in <u>$\mathbb{R}$</u>.

and $\langle v, v \rangle = 0 \iff v = 0$.

$\|v\| = \sqrt{\langle v, v \rangle}$ positive square root.

"norm" or "length of v"

<u>Cauchy-Schwartz Inequality</u>

$(v, w \neq 0)$: $$-1 \leq \left[\frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}\right] \leq +1$$

so $\exists \theta \in [0, \pi]$ s.t. $\cos\theta =$
$\theta =$ "angle between v & w".

$O_n(\mathbb{R})$ acts linearly on $\mathbb{R}^n$
& preserves lengths & angles
(i.e. preserves Euclidean geometry
of $\mathbb{R}^n$