Homework (*required*):

Prepare for the midterm:

Reread the assigned sections of Artin.

Review assigned homework problems and solutions.

Try the practice midterm included in this document (and check your answers against the solutions also included in this document).

Try other problems from the relevant sections of Artin.

Do exercises 4.3.5, 4.3.10, 4.4.2, 4.4.7.

Read §4.5 for next lecture.

Practice Midterm 1

Math 122/E222

(1) Define a *normal subgroup*. Show that any normal subgroup is the kernel of a homomorphism.

(2) Enumerate the 6 subgroups of S_3 and identify which are normal.

(3) Let Z be the center of a group G. Show that Z is a normal subgroup. Suppose G/Z is cyclic. Show that G is abelian.

(4) Let G be the group $(\mathbb{Z}/p\mathbb{Z})^2$ under addition. Calculate the automorphism group of G.

(5) Let F be a field and let $E = \{e_1, e_2\}$ be the standard basis of F^2. When is $S = \{e_1 + e_2, e_1 - e_2\}$ a basis?
Let T be the linear operator $F^2 \to F^2$ whose matrix with respect to E is
$$\begin{pmatrix} 5 & 2 \\ 1 & 8 \end{pmatrix}.$$
If S is a basis, write the matrix of T with respect to it.

(6) Let V be the vectorspace of polynomials of degree ≤ 2 over $\mathbb{Z}/3\mathbb{Z}$ with indeterminate X. Show that $B = \{1 + X, X + X^2, 1 + X^2\}$ is a basis for V. Let $T : V \to V$ be the linear map $T(f(X)) = \frac{d}{dx} f(X)$. Write T with respect to the basis B and calculate the image of T.

Solutions

(1) A normal subgroup H of a group G is a subgroup such that for every $g \in G$ and $h \in H$, $ghg^{-1} \in H$. Given a normal subgroup $H \trianglelefteq G$, we know that the set of left cosets G/H forms a group under $(xH)(yH) = (xy)H$. The map $f : G \to G/H$ taking $x \mapsto xH$ is a well-defined homomorphism with kernel H.

(2) Let e denote the identity in S_3, let τ_{ij} be the transposition exchanging i and j, and let σ_{ijk} be the permutation taking $i \mapsto j$, $j \mapsto k$, $k \mapsto i$. Then

$$S_3 = \{e, \tau_{12}, \tau_{23}, \tau_{13}, \sigma_{123}, \sigma_{321}\}$$

and the six subgroups are:

$$\{e\}\,,\ \{e, \tau_{12}\}\,,\ \{e, \tau_{23}\}\,,\ \{e, \tau_{13}\}\,,\ \{e, \sigma_{123},\ \sigma_{321}\}\,,\ S_3.$$

Only three of them are normal:

$$\{e\}\,,\ \{e, \sigma_{123},\ \sigma_{321}\}\,,\ S_3.$$

since the transpositions get taken to different transpositions by conjugation, and the cycles σ get permuted by conjugation.

(3) By definition, every element of Z commutes with every element of G so $gzg^{-1} = z \in Z$ for every $g \in G$, $z \in Z$. Assume now G/Z is cyclic. Then there is $x \in G$ such that every element of G/Z is of the form $x^n Z$ ($n \in \mathbb{Z}$), and hence every element of G can be written in the form $x^n z$ with $n \in \mathbb{Z}$ and $z \in Z$. Since $(x^n z)(x^m z') = x^{n+m} z z' = (x^m z')(x^n z)$ for all $n, m \in \mathbb{Z}$ and $z, z' \in Z$, we conclude that G is abelian.

(4) Suppose $f : (\mathbb{Z}/p\mathbb{Z})^2 \to (\mathbb{Z}/p\mathbb{Z})^2$ is a group isomorphism. Then for every $\overline{c} \in \mathbb{Z}/p\mathbb{Z}$,

$$f(\overline{c}(\overline{x}, \overline{y})) = f((\overline{x}, \overline{y}) + \cdots + (\overline{x}, \overline{y}))$$
$$= f((\overline{x}, \overline{y})) + \cdots + f((\overline{x}, \overline{y})) = \overline{c} f((\overline{x}, \overline{y}))$$

(where each sum is taken to be c-fold, $c \in \mathbb{Z}$). Thus f is in fact a linear map, and so $f \in GL((\mathbb{Z}/p\mathbb{Z})^2)$. We conclude that

$$\mathrm{Aut}(G) = GL_2(\mathbb{Z}/p\mathbb{Z})$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \, : \, a, b, c, d \in \mathbb{Z}/p\mathbb{Z},\ ad - bc \neq 0 \right\}.$$

(5) Let $v_1 = e_1 + e_2$ and $v_2 = e_1 - e_2$. Note that if $2 = 0$ in F then $v_1 + v_2 = 0$ so S is not linearly independent. However, if $2 \neq 0$ in F, then 2 is invertible so $e_1 = 2^{-1}(v_1 + v_2)$ and $e_2 = 2^{-1}(v_1 - v_2)$. Hence S spans, and since V is 2-dimensional, the dimension formula implies that S must in fact be a basis.

By the theory of change of basis, the matrix of T with respect to S is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 2 \\ 1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ -1 & 5 \end{pmatrix}.$$

(6) Write $v_1 = 1 + X$, $v_2 = X + X^2$, $v_3 = 1 + X^2$. Since

$$1 = -v_1 + v_2 - v_3 \quad \text{(note that } -2 = 1\text{)},$$
$$X = v_1 - 1, \quad \text{and} \quad X^2 = v_3 - 1,$$

we see that $B = \{v_1, v_2, v_3\}$ spans V. Since V is 3-dimensional, the dimension formula implies that B is a basis. Noting carefully the arithmetic of $\mathbb{Z}/3\mathbb{Z}$, we calculate

$$T(v_1) = 1 = -v_1 + v_2 - v_3$$
$$T(v_2) = 1 + 2X = 1 - (v_1 - 1) = -v_1 - (-v_1 + v_2 - v_3) = -v_2 + v_3$$
$$T(v_3) = 2X = -(v_1 - 1) = -v_1 + (-v_1 + v_2 - v_3) = v_1 + v_2 - v_3.$$

Thus the matrix of T with respect to B is

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}.$$

Since $\frac{d}{dX}(a + bX + cX^2) = b + 2cX$, it follows that the image of V consists of all polynomials of degree ≤ 1 (and hence in terms of B, is spanned by $\{-v_1 + v_2 - v_3, -v_1 - v_2 + v_3)\}$).