LECTURE 18

Oct. 29/2003.

G acts on a set S

$$G \times S \longrightarrow S$$
$$(g.s) \longmapsto g \cdot s$$

$s \in S$; orbit $O_s \subset S$

stabilizer $G_s \subset G$

- $G/G_s \xrightarrow{\sim} O_s$ as a G-set
  $gG_s \longmapsto g \cdot s$

- $s' = g \cdot s$
  $G_{s'} = g G_s g^{-1}$ conjugate subgroup

Suppose G acts on S, with orbits $O_{s_1}, O_{s_2}, \ldots, O_{s_n}$, $|G|$ & $|S|$ both finite.

Then
$$|S| = |O_{s_1}| + |O_{s_2}| + \cdots + |O_{s_n}|$$
$$= |G/G_{s_1}| + |G/G_{s_2}| + \cdots + |G/G_{s_n}|$$
$$= |G| \cdot \sum_{i=1}^{n} \frac{1}{|G_{s_i}|}$$

<u>Ex</u>:

$$\begin{array}{ccccc} & & H & & \\ & \overset{\text{finite}}{\supset} & & \supset & \\ G & & & & H\cap K \\ & \supset & & \supset & \\ & & K & & \end{array}$$

<u>Prop</u> $[G:K] \geq [H:H\cap K]$

<u>Pf</u>) Let $S = G/K$ $\qquad |S| = [G:K]$

G acts transitively on S

Can restrict the action of G on S to the subgroup H

This action is a union of H-orbits.

Consider the orbit of the coset $s = eK$ under the action of H

$$\underset{\text{under } H}{O_s} \simeq H/H_s \qquad H_s = \{h \in H : hK = K\} = H \cap K$$

$\boxtimes$

<u>Ex</u> G acts transitively on $S = G$ by left mult.

$g \cdot s = gs$

$O_e = G$

<u>Ex</u> But, there is a more interesting action of G on $S = G$ — by conjugation:

$g \cdot s = gsg^{-1}$

easy to verify $(gh)\cdot s = g\cdot(h \cdot s)$

$O_e = \{e\}$ & $G_e = G$ so unless G is trivial, this action is <u>not transitive</u>.

O_s = Orbit of s $\underset{\text{defn}}{=}:$ "Conjugacy class of s"

$$|G| = \sum_{\text{conj classes}} |O_s| = \sum_{\substack{\text{conj} \\ \text{Classes} \\ \text{of } G}} \frac{|G|}{|Z_s|}$$

"class equation"

where Z_s = "centralizer of s"
$= \{g \in G : gs = sg\}$.
$= G_s$

When G is abelian, each
$|O_s| = 1$ & $G_s = G$
so this formula is not very interesting

<u>Ex</u> $G = S_3$

The conjugacy classes:

- $\{e\}$
- $\{3$ elements of order $2\}$
- $\{2$ elements of order $3\}$

class equation $6 = 1 + 3 + 2$

$Z_e = G$	$Z_\tau = \{e, \tau\}$	$Z_\sigma = \{1, \sigma, \sigma^2\}$
	τ order 2	σ order 3
$O_e = \{e\}$	$O_\tau = \{3$ elts of order $2\}$	$O_\sigma = \{2$ elements of order $3\}$

Monster group

$|G| \sim 10^{47}$

< 200 conjugacy classes
(highly nonabelian)

Q For conjugation action, when is $|O_s| = 1$?
i.e. when is $G_s = G$?

$$G_s = \{g : gs = sg\}$$

A: Exactly when $s \in$ center of G.
$= Z(G)$

Thm If $|G| = p^n$, with p prime
then $Z \neq \{e\}$

Pf) $|G| / |Z_s| = p^k$ with $0 \leq k \leq n$
So all divisible by p except when
$k = 0$ & $|O_s| = 1$
$\Rightarrow \# Z$ is divisible by p & $\neq 1$. $\boxtimes$

$\longrightarrow$ If G is of prime power order
$Z =$ center $\neq \{e\}$
$G_1 = G/Z$ has prime power order $< |G|$
$Z_1 =$ its center $\neq \{e\}$
$G_2 = G_1 / Z_1$ $\quad (< |G_1|)$
$Z_2 =$ its center

$\vdots$

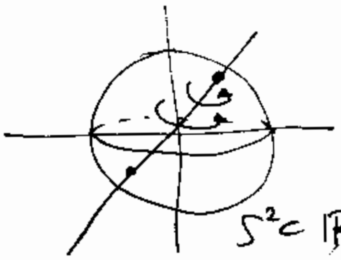
inductively break down to trivial group.

Consequence of earlier result:

G is not simple unless $|G| = p$.

Finite subgroups $G \subset SO(3)$ preserving the regular solids in $\mathbb{R}^3$.

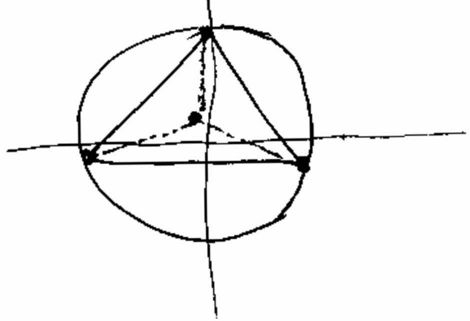
Every $g \in G$ is rotation around an axis



$S^2 \subset \mathbb{R}^3$

$$A = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathrm{rot}(\theta)$}} \\ 0 & & \end{array}\right)$$

<u>Ex</u> Tetrahedron



4 vertices

6 edges

4 faces, each one equilateral triangle

G preserving tetrahedron is a subgroup of S_4 = permutations of the vertices.

G acts transitively on the set S of vertices $\#S = 4 = |G| / |G_{vertex}|$

$= |G|/3$ (G_{vertex} = {rotations preserving fixed vertex}.)

So $|G| = 12$

$\Rightarrow$ $G \simeq A_4$ since A_4 is the only subgroup of S_4 of order 12 (we'll see this later...)

Recall: $A_4 = \{ g \in S_4 : \text{sign of } g = 1 \}$

Ex Can repeat this argument on the octahedron $\longrightarrow$ $|G| = 24$; $G \simeq S_4$. (same as for cube)

Ex G acts on $G/H = S$
Suppose $H \triangleleft G$ is normal
Then $G_s = H$ for every s!
(So the stabilizers do not pick out the element!)

<u>Ex</u> Will study icosahedron later
(object with 12 vertices,
20 faces, 30 edges)

<u>A physical conjecture</u>

Universe = Poincaré 3-sphere
$= SO_3/A_5$

← icosahedral group.

SO_3 acts on S^2, with stabilizer
of a pt $\simeq SO_2$.