# LECTURE 4

Sept. 22/2003

## § Review

- $G \xrightarrow[f]{} G'$ homomorphism

  $\Longrightarrow f(g \cdot h) = f(g) \cdot f(h)$

  $\qquad\qquad\quad\uparrow \qquad\qquad \uparrow$

  $\qquad\quad$ in $G$ $\qquad\quad$ in $G'$

- $f(e) = e'$
- $f(g^{-1}) = f(g)^{-1}$
- composition

  $$G \xrightarrow[f]{} G' \xrightarrow[h]{} G''$$

  $h \circ f$ is homomorphism if $h$ and $f$ are

- Image $= \{ g' = f(g) \text{ in } G' \} \subset G'$
- Kernel $= \{ g : f(g) = e' \} \subset G$
- If Image $= G'$ & Kernel $= \{e\}$
  we say $f$ is an isomorphism
- If $G = G'$ and $f$ is an
  isomorphism, we say $f$ is <u>automorphism</u>

- Kernel & Image are subgroups but Kernel is also a special kind of subgroup:

  it's a <u>normal</u> subgroup

  (<u>Notation</u>: $H \lhd G$)

  this means that

  $\forall g \in G \quad gHg^{-1} = H.$

  Another way of saying it:
  H is "closed under conjugation"

- Verification that Kernels are normal:

  $h \in$ Kernel, $g \in G$.

  $$f(ghg^{-1}) = f(g)f(h)f(g)^{-1}$$
  $$= f(g) \cdot e \cdot f(g)^{-1} = e$$

  $\therefore \; ghg^{-1} \in$ Kernel.

- Not all subgroups are normal!

<u>Ex</u>: $G = S_3$
$H = \{e, \tau\}$ $\quad \tau: \begin{array}{c} 1 \to 2 \\ 2 \to 1 \\ 3 \to 3 \end{array}$

This is not <u>normal</u>:
conjugate by

$\tau': \begin{array}{c} 1 \to 1 \\ 2 \to 3 \\ 3 \to 2 \end{array}$

since

$\tau' \tau (\tau')^{-1} = \tau'': \begin{array}{c} 1 \to 3 \\ 2 \to 2 \\ 3 \to 1 \end{array}$.

So, for example, $H$ is not the kernel of a homomorphism

- Eventually we will see (big thm):

if $H \lhd G$ normal subgroup
then there is homomorphism
$f: G \longrightarrow Q$ such that
$\mathrm{Kernel}(f) = H$.

## § Examples

(1) $G = GL_n(\mathbb{R}) \xrightarrow[f]{} G' = \mathbb{R}^\times = GL_1(\mathbb{R})$

$f(A) = \det(A)$.

$\det(A \cdot B) = \det(A) \cdot \det(B)$

$\text{Image}(f) = \mathbb{R}^\times \left( \det\begin{pmatrix} \lambda & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} = \lambda \right)$

$\text{Kernel}(f) = \{A : \det A = 1\}$

$=: SL_n(\mathbb{R}) \triangleleft GL_n(\mathbb{R})$

Note: Set of matrices with any fixed determinant value is closed under conjugation.

② $f: S_n \to GL_n(\mathbb{R})$

$f(\sigma) = A_\sigma =$ "permutation matrix assoc'd to $\sigma$

$$= \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & 1 & \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix}$$

in the $\sigma(j)^{th}$ place.

$j^{th}$ column

Ex: $G = S_3$

$\sigma: \quad 1 \mapsto 2, \quad 2 \mapsto 3, \quad 3 \mapsto 1$

$$\Rightarrow A_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Book verifies: $f(\sigma\tau) = A_\sigma A_\tau$

Image $(f)$ = "set of permutation matrices"

$\mathrm{Ker}(f) = \{e\}$.

Fact
$\det(f(\sigma)) = \pm 1$
$\forall \sigma \in S_n$.

and both values occur

③ Composition of ① & ②

$$\underbrace{S_n \xrightarrow[f]{} GL_n(\mathbb{R}) \xrightarrow[\det]{} \mathbb{R}^\times}$$

Image $= \{\pm 1\} \subset \mathbb{R}^\times$

Kernel $= \{\sigma : \det(f(\sigma)) = +1\}$
$\lhd S_n$

called "Alternating group"

Will show $|A_n| = |S_n|/2$
$= n!/2$

This is "sign map"

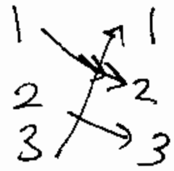
(Note: to avoid circularity, define det independently of formula involving signs of permutations)

The elements of $A_n$ are called "even permutations"

<u>Note</u>: There is also an alternative definition of this "sign map" (write permutation as product of transpositions; if even # of them $\mapsto +1$
if odd ———— $\mapsto -1$

<u>Ex</u>: In $S_3$ order 6

$e \qquad \sigma \qquad \sigma' = \sigma^2$



$\left.\begin{matrix} e & \sigma & \sigma' = \sigma^2 \end{matrix}\right\}$ even perms., so $A_3$ has order 3

$\tau \qquad \tau' \qquad \tau''$ transpositions = odd permutations

## § Centers and Inner Automorphisms

$G$

$Z(G) =$ "center of $G$"

$= \{z \in G : zg = gz \ \forall g \in G\}$

Commute with everything in $G$

This is a normal subgroup; it is abelian.

Ex:
- $G = Z(G) \iff G$ abelian
- $G = S_n \Rightarrow Z(G) = \{e\}$
- $G = GL_n(\mathbb{R}) \Rightarrow$

$$Z(G) = \{\lambda I : \lambda \in \mathbb{R}^\times\}$$

Another homomorphism:

$$G \xrightarrow[f]{} \mathrm{Aut}(G)$$

$= \{$all isomorphisms $h : G \to G\}$

Defined by $f(g)(h) = ghg^{-1}$

$=$ conjugation by $g$

Can verify $f(g) \in \mathrm{Aut}(G)$ easily:

- $f(g)(hh') = ghg^{-1} \cdot gh'g^{-1}$
  $= f(g)(h) \cdot f(g)(h')$

- $f(g)$ is bijective because can exhibit inverse:
  $f(g) \cdot f(g^{-1}) = id$
  & $f(g^{-1}) \circ f(g) = id.$

Also $f$ is a homomorphism:

$$f(gg') = f(g) f(g')$$

$$\mathrm{Kernel}(f) = \{ g \in G : ghg^{-1} = f(g)(h) = h \ \forall h \in G \}$$

$$= Z(G)$$

Is $f$ surjective?

<u>Ex</u>:
$G =$ Klein 4-group $= \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$.

Abelian $\therefore$ $Z(G) = G$

so Image$(f)$ in Aut$(G)$ is $\{e\}$

However Aut$(G)$ is nontrivial!

In fact Aut$(G) \cong S_3$.

If $g : G \to G$ is an automorphism then it permutes

$$\left\{ \tau_1 = \begin{pmatrix} -1 & \\ & +1 \end{pmatrix},\ \tau_2 = \begin{pmatrix} +1 & \\ & -1 \end{pmatrix},\ \tau_3 = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \right\}.$$

This gives a homomorphism
$$\text{Aut } G \longrightarrow S_3$$
with trivial <u>kernel</u>.

This map $\mathrm{Aut}\, G \longrightarrow S_3$
is also surjective
(full Image)

---

In general:

$$\mathrm{Image}\left( \begin{array}{rl} f: & G \longrightarrow \mathrm{Aut}\, G \\ & g \longmapsto [h \mapsto ghg^{-1}] \end{array} \right)$$

is called $\mathrm{Inn}(G)$ = "group of inner automorphisms"