Math 122 10 October 2003

Recall: for a fin.-dim'l v.s. V and $S = \{v_1, \cdots, v_n\}$ L.I.,

S can be extended to a basis of V, $\{v_1, \cdots, v_n, v_{n+1}, \cdots, v_m\}$

Consequences of this result:

1) Let $W = \mathrm{span}\{v_1, \cdots, v_n\}$ be a subspace of V and $W' = \mathrm{span}\{v_{n+1}, \cdots, v_m\}$. Then $W \cap W' = \{0\}$ + $\exists$ a linear isomorphism

$$W \times W' = \{(w, w') \; w \in W, w' \in W'\}$$

$$\downarrow \wr \varphi \qquad \qquad (w,w') \mapsto w + w'$$

$$V$$

b/c if $w = \sum_1^n a_i v_i$, $w' = \sum_{n+1}^m a_i v_i$ + $w = w'$,

$0 = w - w' = \sum_1^n a_i v_i + \sum_{n+1}^m (-a_i) v_i \Rightarrow a_i = 0$ by lin. independence

$\Rightarrow w = w' = 0$

φ is a linear transform - clear from defs

- surjective b/c v_i span
- injective: $w_1 + w_1' = w_2 + w_2' \Rightarrow \underset{\in W}{(w_1 - w_2)} = \underset{\in W'}{(w_2' - w_1')}$

$W \cap W' = 0 \Rightarrow w_1 = w_2 \quad w_1' = w_2'$.

2) If $W \subset V$ subspace then $\exists$ another $W' \subset V$ s.t. $W' \hookrightarrow V \underset{\text{canonical quotient}}{\twoheadrightarrow} V/W$ the composite map is an isomorphism

B/c take $\{v_1, \cdots, v_n\}$ basis for W + extend to a basis $\{v_1, \cdots, v_m\}$ for V, + let $W' = \mathrm{span}\{v_{n+1}, \cdots, v_m\}$ (straightforward)

3) for $W \subset V$ subspace, $V \xrightarrow{\sim} W \times V/W$

(combine 1 + 2) $\Rightarrow \dim(V) = \dim(W) + \dim(V/W)$

4) If $f: V \to U$ is a lin. transf., then

$V \xrightarrow{\sim} \ker(f) \times \mathrm{Im}(f)$ and

$\dim(V) = \dim(\ker(f)) + \dim(\mathrm{Im}(f))$ $\longrightarrow$

why? the Isomorphism thm. for groups implies: given $f: V \to U$ linear transf.
the map $\bar{f}: V/_{ker(f)} \longrightarrow Im(f)$
$$v + ker(f) \longmapsto f(v)$$
is a well-defined linear isomorphism

§ General Problem for Groups

Given $f: G \to G'$, find $ker(f)$ & $Im(f)$ (generally difficult)

- but we can solve for vector spaces.

Tools: Bases, Matrices, Matrix Algebras

Some correspondences

I. We saw last lecture V (n-dim'l v.s./F)

→ there is a correspondence

$$\left\{\begin{matrix}\text{ordered bases}\\ \text{of } V\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{lin. isomorphisms}\\ F^n \to V\end{matrix}\right\}$$

$$B = (v_1, \dots, v_n) \longmapsto \left[\begin{matrix}\rho_B: F^n \to V \\ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mapsto a_1 v_1 + \cdots + a_n v_n\end{matrix}\right]$$

$$\left(\rho\begin{pmatrix}1\\ \vdots \\ 0\end{pmatrix}, \rho\begin{pmatrix}0\\1\\ \vdots\\ 0\end{pmatrix}, \dots, \rho\begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix}\right) \longleftarrow\!\!\shortmid \rho$$

II. Basic Linear Algebra

$$\left\{\begin{matrix}\text{Lin. Transf.}\\ F^n \to F^m\end{matrix}\right\} \longleftrightarrow M_{m\times n}(F)$$

$$f \longmapsto [f] = \left(f\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} \cdots\cdots f\begin{pmatrix}0\\ \vdots\\0\\1\end{pmatrix}\right)$$

$$f: F^n \to F^m \longleftarrow\!\!\shortmid A$$
$$v \longmapsto Av$$

III. Correspondences I + II together:

Given $f: V \to V'$ lin tranf.

dim: n m

bases: B B'

→

- have matrix of f w.r.t. B, B'

$$[f]_B^{B'} = [\rho_{B'}^{-1} f \rho_B]$$

Explicitly if $B = (v_1, \ldots, v_n)$ $B' = (w_1, \ldots, w_m)$

$$f(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad [f]_B^{B'} = (a_{ij})$$

$$\mathrm{Hom}(V, V') = \{\text{lin transf. } V \to V'\}$$
$$\simeq M_{m\times n}(F)$$

$$[c_1 f_1 + c_2 f_2]_B^{B'} = c_1 [f_1]_B^{B'} + c_2 [f_2]_B^{B'}$$

$f: V \to V'$, $g: V' \longrightarrow V''$

bases B B' B''

$$[g \circ f]_B^{B''} = [g]_{B'}^{B''} [f]_B^{B'}$$

<u>Change of Basis</u>

Suppose V — bases B_1, B_2

V' — bases B_1', B_2'

$$[f]_{B_2}^{B_2'} = [\rho_{B_2'}^{-1} f \rho_{B_2}]$$
$$= [\rho_{B_2'}^{-1} \rho_{B_1'} \rho_{B_1'}^{-1} f \rho_{B_1} \rho_{B_1}^{-1} \rho_{B_2}]$$

$$= [\rho_{B_2'}^{-1} \rho_{B_1'}][\rho_{B_1'}^{-1} f \rho_{B_1}][\rho_{B_1}^{-1} \rho_{B_2}]$$
$$= \underbrace{[\rho_{B_2'}^{-1} \rho_{B_1'}]}_{\text{change of basis matrix, e.g.}} [f]_{B_1}^{B_1'} [\rho_{B_1}^{-1} \rho_{B_2}]$$

If $V = V'$

$$[f]_{B_1}^{B_2} = [\rho_{B_1}^{-1} \rho_{B_2}]^{-1} [f]_{B_1}^{B_1} [\rho_{B_1}^{-1} \rho_{B_2}] = P^{-1} [f]_{B_1}^{B_1} P$$

$B_1 = (v_1, \ldots, v_n)$ $B_2 = (w_1, \ldots, w_n)$

$[\rho_{B_1}^{-1} \rho_{B_2}] = (c_{ij})$ where $w_j = \sum_{i=1}^{n} c_{ij} v_i$

<u>GL(V)</u>

$$\mathrm{Aut}(V, +) \supset GL(V) := \{\text{lin. iso. } V \to V\}$$
$$= \{\text{invertible lin transfs. } V \to V\}$$
$$= \mathrm{Hom}(V, V)^\times$$

Given a basis B of V, $\xrightarrow{\sim}$ $\{$invertible matrices in $M_{n\times n}(F)\}$

$GL_n(F) \cong$