Laplacian eigenfunctions

October 2022

1 Basics

General solution of Helmholtz equation ($\nabla^2 f + k^2 f = 0$) in spherical polar coordinates

$$f(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[a_{lm} j_l(kr) + b_{lm} y_l(kr)\right] Y_{lm}(\hat{r}) \tag{1}$$

1.1 Inside a sphere $r \leq r_c$ with homogeneous Dirichlet BCs

$$f(\mathbf{r}) = \sum_{j=1}^{\infty} \sum_{m=-l}^{l} \sum_{n=0}^{\infty} a_{jlm} j_l(k_{jl} r) Y_{lm}(\hat{r}) + b_{jlm} y_k(k_{jl} r) Y_{lm}(\hat{r}) \tag{2}$$

where $k_{jl} = z_{jl}/r_c$ and z_{jl} is the jth zero of $j_l(z)$. When $f(\mathbf{r})$ is finite at the origin, the $\{b_{jlm}\}$ are zero.

1.2 Inside a spherical shell $r_1 \leq r \leq r_2$ with homogeneous Dirichlet BCs

The boundary conditions yield the system

$$j_l(kr_1) + c y_l(kr_1) = 0 j_l(kr_2) + c y_l(kr_2) = 0 \tag{3}$$

which has solutions for discrete values of k and c. After finding these values (numerically, I assume), one obtains new radial functions

$$R_{jl}(r) = j_l(k_{jl} r) + c_{jl} y_l(k_{jl} r) \tag{4}$$

such that the solution is

$$f(r) = \sum_{j=1}^{\infty} \sum_{m=-l}^{l} \sum_{n=0}^{\infty} d_{jlm} R_{jl}(r) Y_{lm}(\hat{r})$$