A Couple of Proofs of De Moivre's Theorem

& My Favourite Piece of Maths

De Moivre's Theorem says:

$$(\cos(\theta) + i\sin(\theta))^n = \cos n\theta + i\sin n\theta$$

where, we remember, $i^2 = -1$. I'll prove this in two ways. Firstly by induction (which is the way the syllabus says to prove it), and secondly using the exponential form of a complex number (which I'll also discuss). At the end I will show what I think is one of the most beautiful results in mathematics.

Proof of De Moivre's Theorem by Induction

We just show De Moivre for positive integer values of n. Remember we need our starting step for induction proofs. For $n = 1$ we have:

$$(\cos(\theta) + i\sin(\theta))^1 = \cos 1\theta + i\sin 1\theta.$$

So De Moivre holds for $n = 1$. Assume now it's true for $n = k$. I.e.:

$$(\cos(\theta) + i\sin(\theta))^k = \cos k\theta + i\sin k\theta. \tag{1}$$

We want to show it is also true for $n = k + 1$, and then we'll be done. Consider then:

$$(\cos(\theta) + i\sin(\theta))^{k+1} = (\cos(\theta) + i\sin(\theta))(\cos(\theta) + i\sin(\theta))^k.$$

We now use equation (1), and sub in for $(\cos(\theta) + i\sin(\theta))^k$ on the right hand side. Then we get

$$(\cos(\theta) + i\sin(\theta))^{k+1} = (\cos(\theta) + i\sin(\theta))(\cos(k\theta) + i\sin(k\theta)).$$

$$= \cos(\theta)\cos(k\theta) + i^2\sin(\theta)\sin(k\theta) + i\sin(\theta)\cos(k\theta) + i\sin(k\theta)\cos(\theta)$$

$$= \cos(\theta)\cos(k\theta) - \sin(\theta)\sin(k\theta) + i(\sin(\theta)\cos(k\theta) + \sin(k\theta)\cos(\theta))$$

after expanding the brackets. Now, as you all know (without having to look it up in your formula booklet, of course), $\sin(a + b) = \sin a\cos b + \sin b\cos a$ and $\cos(a + b) = \cos a\cos b - \sin a\sin b$. So we have

$$(\cos(\theta) + i\sin(\theta))^{k+1} = \cos(k + 1)\theta + i\sin(k + 1)\theta.$$

But this is just what we need, right? So by induction we've shown De Moivre to be true for all positive integers n.

Exponential Form of a Complex Number

So we know that we can write any complex number z as $z = a + ib$, where a and b are real numbers. From drawing a triangle in the argand diagram, we know it can also be written as $z = r(\cos\theta + i\sin\theta)$, where $r \geq 0$ and $0 \leq \theta \leq 2\pi$. In fact, r is the length of the line joining z with the origin (r is called the *modulus*), so by Pythagoras' Theorem, $r = \sqrt{a^2 + b^2}$. Also, θ is the angle this line makes with the x-axis (going anticlockwise from the x-axis, by convention)(θ is called the *argument*). So, by trigonometry, $\theta = \tan^{-1}(b/a)$. In fact, with the same r and θ, we can write $z = re^{i\theta}$, (and I'll leave you to try to make any sense of what it means to have a power that is a complex number). So let's try to show we can write z in this way.

Firstly, recall the Taylor (or Maclaurin) series for $\sin x$, $\cos x$ and e^x:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + ...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + ...$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + ...$$

These are also in the formula booklet. Let's plug in θ for x in the expressions for $\sin x$ and $\cos x$. And lets multiply the expression for $\sin x$ by i, just for fun. We get:

$$i\sin\theta = i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - i\frac{\theta^7}{7!} + i\frac{\theta^9}{9!} + ...$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + ...$$

Remember that $i^2 = -1$. So, we see $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$, etc. Notice the pattern? So we can write the above expressions as:

$$i\sin\theta = i\theta + i^3\frac{\theta^3}{3!} + i^5\frac{\theta^5}{5!} + i^7\frac{\theta^7}{7!} + i^9\frac{\theta^9}{9!} + ...$$

$$\cos\theta = 1 + i^2\frac{\theta^2}{2!} + i^4\frac{\theta^4}{4!} + i^6\frac{\theta^6}{6!} + i^8\frac{\theta^8}{8!} + ...$$

Keeping up? Make sure you're happy with what I've just done before moving on. We can write these as:

$$i\sin\theta = (i\theta) + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + \frac{(i\theta)^9}{9!} + ...$$

$$\cos\theta = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^8}{8!} + ...$$

Adding together the two expressions above (we'll assume it's OK to add together infinite sums of numbers), we get:

$$\cos\theta + i\sin\theta = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + ...$$

But if you look back at the Taylor series for e^x, we see that this is just $e^{i\theta}$, i.e.

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + ...$$

So we've deduced that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which is called *Euler's formula*, (unsurprisingly, this is also in the formula booklet) and if we want, we can times through by r:

$$re^{i\theta} = r(\cos\theta + i\sin\theta).$$

So we can indeed write complex numbers as $re^{i\theta}$.

Another proof of De Moivre's Theorem

From Euler's formula, which we described above, we know:

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

Taking both sides to the power n gives us:

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n,$$

and remembering that taking a power of a power, we multiply the powers (i.e. $(a^b)^c = a^{bc}$), we get:

$$(\cos\theta + i\sin\theta)^n = e^{i(n\theta)}.$$

We use Euler's formula again:

$$e^{i(n\theta)} = \cos n\theta + i\sin n\theta,$$

and therefore we have De Moivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

In fact, we've shown here that De Moivre is true for any number n, not just positive integers, which is cool.

The bit you've been waiting for... Euler's identity

Let's use Euler's formula again:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Put $\theta = \pi$, and remember that we have to use RADIANS here. Then $\sin\pi = 0$ and $\cos\pi = -1$, so we conclude that

$$e^{i\pi} = -1,$$

so taking the -1 over to the left hand side...

$$e^{i\pi} + 1 = 0,$$

which is called Euler's identity. Pretty cool, right? Who would've thought that the numbers 0, 1, π, e and i would be so intimately related? This one equation uses the five most important numbers in maths (well, I think they're the five most important numbers), and the three most important operations (add, times, powers), together with an equals sign. Amazing.