On Dynamic Range Reporting in One Dimension

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Abstract

We consider the problem of maintaining a dynamic set of integers and answering queries of the form: report a point (equivalently, all points) in a given interval. Range searching is a natural and fundamental variant of integer search, and can be solved using predecessor search. However, for a RAM with w-bit words, we show how to perform updates in $O(\lg w)$ time and answer queries in $O(\lg \lg w)$ time. The update time is identical to the van Emde Boas structure, but the query time is exponentially faster. Existing lower bounds show that achieving our query time for predecessor search requires doubly-exponentially slower updates. We present some arguments supporting the conjecture that our solution is optimal.

Our solution is based on a new and interesting recursion idea which is "more extreme" that the van Emde Boas recursion. Whereas van Emde Boas uses a simple recursion (repeated halving) on each path in a trie, we use a nontrivial, van Emde Boas-like recursion on every such path. Despite this, our algorithm is quite clean when seen from the right angle. To achieve linear space for our data structure, we solve a problem which is of independent interest. We develop the first scheme for dynamic perfect hashing requiring sublinear space. This gives a dynamic Bloomier filter (an approximate storage scheme for sparse vectors) which uses low space. We strengthen previous lower bounds to show that these results are optimal.

1 Introduction

Our problem is to maintain a set S under insertions and deletions of values, and a range reporting query. The query findany(a,b) should return an arbitrary value in $S \cap [a,b]$, or report that $S \cap [a,b] = \emptyset$. This is a form of existential range query. In fact, since we only consider update times above the predecessor bound, updates can maintain a linked list of the values in S in increasing order. Given a value $x \in S \cap [a,b]$, one can traverse this list in both directions starting from x and list all values in the interval [a,b] in constant time per value. Thus, the findany query is equivalent to one-dimensional range reporting.

The model in which we study this problem is the word RAM. We assume the elements of S are integers that fit in a word, and let w be the number of bits in a word (thus, the "universe size" is $u=2^w$). We let n=|S|. Our data structure will use Las Vegas randomization (through hashing), and the bounds stated will hold with high probability in n.

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Range reporting is a very natural problem, and its higher-dimensional versions have been studied for decades. In one dimension, the problem is easily solved using predecessor search. The predecessor problem has also been studied intensively, and the known bounds are now tight in almost all cases [2]. Another well-studied problem related to ours is the lookup problem (usually solved by hashing), which asks to find a key in a set of values. Our problem is more general than the lookup problem, and less general than the predecessor problem. While these two problems are often dubbed "the integer search problems", we feel range reporting is an equally natural and fundamental incarnation of this idea, and deserves similar attention.

The first to ask whether or not range reporting is as hard as finding predecessors were Miltersen et al in STOC'95 [10]. For the static case, they gave a data structure with space O(nw) and constant query time, which cannot be achieved for the predecessor problem with polynomial space. An even more surprising result from STOC'01 is due to Alstrup, Brodal and Rauhe [1], who gave an optimal solution for the static case, achieving linear space and constant query time. In the dynamic case, however, no solution better than the predecessor problem was known. For this problem, the fastest known solution in terms of w is the classic van Emde Boas structure [16], which achieves $O(\lg w)$ time per operation.

For the range reporting problem, we show how to perform updates in $O(\lg w)$ time, while supporting queries in $O(\lg \lg w)$ time. The space usage is optimal, i.e. O(n) words. The update time is identical to the one given by the van Emde Boas structure, but the query time is exponentially faster. In contrast, Beame and Fich [2, Theorem 3.7] show that achieving any query time that is $o(\lg w/\lg \lg w)$ for the predecessor problem requires update time $\Omega(2^{w^{1-\varepsilon}})$, which is doubly-exponentially slower than our update time. We also give an interesting tradeoff between update and query times; see theorem 4 below.

Our solution incorporates some basic ideas from the previous solutions to static range reporting in one dimension [10, 1]. However, it brings two important technical contributions. First, we develop a new and interesting recursion idea which is more advanced than van Emde Boas recursion (but, nonetheless, not technically involved). We describe this idea by first considering a simpler problem, the bit-probe complexity of the greater-than function. Then, the solution for dynamic range reporting is obtained by using the recursion for this simpler problem, on every path of a binary trie of depth w. This should be contrasted to the van Emde Boas structure, which uses a very simple recursion idea (repeated halving) on every root-to-leaf path of the trie. The van Emde Boas recursion is fundamental in the modern world of data structures, and has found many unrelated applications (e.g. exponential trees, integer sorting, cache-oblivious layouts, interpolation search trees). It will be interesting to see if our recursion scheme has a similar impact.

The second important contribution of this paper is needed to achieve linear space for our data structure. We develop a scheme for dynamic perfect hashing, which requires sublinear space. This can be used to store a sparse vector in small space, if we are only interested in obtaining correct results when querying non-null positions (the Bloomier filter problem). We also prove that our solution is optimal. To our knowledge, this solves the last important theoretical problem connected to Bloom filters. The stringent space requirements that our data structure can meet are important in data-stream algorithms and database systems. We mention one application below, but believe others exist as well.

1.1 Data-Stream Perfect Hashing and Bloomier Filters

The Bloom filter is a classic data structure for testing membership in a set. If a constant rate of false-positives is allowed, the space in bits can be made essentially linear in the size of the set. Optimal bounds for this problem are obtained in [11]. Bloomier filters, an extension of the classical Bloom filter with a catchy name, were defined and analyzed in the static case by Chazelle et al [4]. The problem is to represent a vector V[0..u-1] with elements from $\{0,\ldots,2^r-1\}$ which is nonzero in only n places (assume $n \ll u$, so the vector is sparse). Thus, we have a sparse set as before, but with values associated to the elements. The information theoretic lower bound for representing such a vector is $\Omega(n \cdot r + \lg \binom{u}{n}) \approx \Omega(n(r + \lg u))$ bits. However, if we only want correct answers when $V[x] \neq 0$, we can obtain a space usage of roughly O(nr) bits in the static case.

For the dynamic problem, where the values of V can change arbitrarily at any point, achieving such low space is impossible regardless of the query and update times. Chazelle et al. [4] proved that $\Omega(n(r + \min(\lg \lg \frac{u}{n^3}, \lg n)))$ bits are needed. No non-trivial upper bound was known. We give matching lower and upper bounds:

Theorem 1. The randomized space complexity of maintaining a dynamic Bloomier filter for $r \geq 2$ is $\Theta(n(r + \lg \lg \frac{u}{n}))$ bits in expectation. The upper bound is achieved by a RAM data structure that allows access to elements of the vector in worst-case constant time, and supports updates in amortized expected O(1) time.

To detect whether V[x] = 0 with probability of correctness at least $1 - \varepsilon$, one can use a Bloom filter on top. This requires space $\Theta(n \lg(1/\varepsilon))$, and also works in the dynamic case [11]. Note that even for $\varepsilon = 1$, randomization is essential, since any deterministic solution must use $\Omega(n \lg(u/n))$ bits of space, i.e. it must essentially store the set of nonzero entries in the vector.

With marginally more space, $O(n(r + \lg \lg u))$, we can make the space and update bounds hold with high probability. To do that, we analyze a harder problem, namely maintaining a perfect hash function dynamically using low space. The problem is to maintain a set S of keys from $\{0, \ldots, u-1\}$ under insertions and deletions, and be able to evaluate a perfect hash function (i.e. a one-to-one function) from S to a small range. An element needs to maintain the same hash value while it is in S. However, if an element is deleted and subsequently reinserted, its hash value may change.

Theorem 2. We can maintain a perfect hash function from a set $S \subset \{0, \ldots, u-1\}$ with $|S| \le n$ to a range of size n + o(n), under $n^{O(1)}$ insertions and deletions, using $O(n \lg \lg u)$ bits of space w.h.p., plus a constant number of machine words. The function can be evaluated in worst-case constant time, and updates take constant time w.h.p.

This is the first dynamic perfect hash function that uses less space than needed to store S ($\lg \binom{u}{n}$) bits). Our space usage is close to optimal, since the problem is harder than dynamic Bloomier filtering. These operating conditions are typical of data-stream computation, where one needs to support a stream of updates and queries, but does not have space to hold the entire state of the data structure. Quite remarkably, our solution can achieve this goal without introducing errors (we use only Las Vegas randomization).

We mention an independent application of Theorem 2. In a database we can maintain an index of a relation under insertions of tuples, using internal memory per tuple which is logarithmic in the length of the key for the tuple. If tuples have fixed length, they can be placed directly in the hash table, and need only be moved if the capacity of the hash table is exceeded.

1.2 Tradeoffs and the scheme of things

We begin with a discussion of the greater-than problem. Consider an infinite memory of bits, initialized to zero. Our problem has two stages. In the update stage, the algorithm is given a number $a \in [0..n-1]$. After seeing a, the algorithm is allowed to flip $O(T_u)$ bits in the memory. In the query stage, the algorithm is given a number $b \in [0..n-1]$. Now the algorithm may inspect $O(T_q)$ bits, and must decide whether or not b > a. The problem was previously studied by Fredman [7], who showed that $\max(T_u, T_q) = \Omega(\lg n / \lg \lg n)$. It is quite tempting to believe that one cannot improve past the trivial upper bound $T_u = T_q = O(\lg n)$, since, in some sense, this is the complexity of "writing down" a. However, as we show in this paper, Fredman's bound is optimal, in the sense that it is a point on our tradeoff curve. We give upper and lower bounds that completely characterize the possible asymptotic tradeoffs:

Theorem 3. The bit-probe complexity of the greater-than function satisfies the tight tradeoffs:

$$T_q \ge \lg \lg n, \ T_u \le \lg n$$
 : $T_u = \Theta(\lg_{T_q} n)$
 $T_q \le \lg \lg n, \ T_u \ge \lg n$: $2^{T_q} = \Theta(\lg_{T_u} n)$

As mentioned already, we use the same recursion idea as in the previous algorithm for dynamic range reporting, except that we apply this recursion to every root-to-leaf path of a binary trie of depth w. Quite remarkably, these structures can be made to overlap in-as-much as the paths overlap, so only one update suffices for all paths going through a node. Due to this close relation, we view the lower bounds for the greater-than function as giving an indication that our range reporting data structure is likewise optimal. In any case, the lower bounds show that markedly different ideas would be necessary to improve our solution for range reporting.

Let T_{pred} be the time needed by one update and one query in the dynamic predecessor problem. The following theorem summarizes our results for dynamic range reporting:

Theorem 4. There is a data structure for the dynamic range reporting problem, which uses O(n) space and supports updates in time $O(T_u)$, and queries in time $O(T_q)$, $(\forall)T_u, T_q$ satisfying:

$$T_q \ge \lg \lg w, \quad \frac{\lg w}{\lg \lg w} \le T_u \le \lg w \quad : \quad T_u = O(\lg_{T_q} w) + T_{pred}$$

$$T_q \le \lg \lg w, \qquad T_u \ge \lg w \quad : \quad 2^{T_q} = O(\lg_{T_u} w)$$

Notice that the most appealing point of the tradeoff is the cross-over of the two curves: $T_u = O(\lg w)$ and $T_q = O(\lg \lg w)$ (and indeed, this has been the focus of our discussion). Another interesting point is at constant query time. In this case, our data structure needs $O(w^{\varepsilon})$ update time. Thus, our data structure can be used as an optimal static data structure, which is constructed in time $O(nw^{\varepsilon})$, improving on the construction time of $O(n\sqrt{w})$ given by Alstrup et al [1].

The first branch of our tradeoff is not interesting with $T_{pred} = \Theta(\lg w)$. However, it is generally believed that one can achieve $T_{pred} = \Theta(\lg w / \lg \lg w)$, matching the optimal bound for the static case. If this is true, the T_{pred} term can be ignored. In this case, we can remark a very interesting relation between our problem and the predecessor problem. When $T_u = T_q$, the bounds we achieve are identical to the ones for the predecessor problem, i.e. $T_u = T_q = O(\lg w / \lg \lg w)$. However, if we are interested in the possible tradeoffs, the gap between range reporting and the predecessor problem quickly becomes huge. The same situation appears to be true for deterministic dictionaries

with linear space, though the known tradeoffs are not as general as ours. We set forth the bold conjecture (the proof of which requires many missing pieces) that all three search problems are united by an optimal time of $\Theta(\lg w/\lg \lg w)$ in this point of their tradeoff curves.

We can achieve bounds in terms of n, rather than w, by the classic trick of using our structure for small w and a fusion tree structure [8] for large w. In particular, we can achieve $T_q = O(\lg \lg n)$ and $T_u = O\left(\frac{\lg n}{\lg \lg n}\right)$. Compared with the optimal bound for the predecessor problem of $\Theta\left(\sqrt{\frac{\lg n}{\lg \lg n}}\right)$, our data structure improves the query time exponentially by sacrificing the update time quadratically.

2 Data-Stream Perfect Hashing

We denote by S be the set of values that we need to hash at present time. Our data structure has the following parts:

- A hash function $\rho: \{0, \dots, u-1\} \to \{0, 1\}^v$, where $v = O(\lg n)$, from a family of universal hash functions with small representations (for example, the one from [5]).
- A hash function $\phi: \{0,1\}^v \to \{1,\ldots,r\}$, where $r = \lceil n/\lg^2 n \rceil$, taken from Siegel's class of highly independent hash functions [15].
- An array of hash functions $h_1, \ldots, h_r : \{0,1\}^v \to \{0,1\}^s$, where $s = \lceil (6+2c) \lg \lg u \rceil$, chosen independently from a family of universal hash functions; c is a constant specified below.
- A high performance dictionary [6] for a subset S' of the keys in S. The dictionary should have a capacity of $O(\lceil n/\lg u \rceil)$ keys (but might expand further). Along with the dictionary we store a linked list of length $O(\lceil n/\lg u \rceil)$, specifying certain vacant positions in the hash table.
- An array of dictionaries D_1, \ldots, D_r , where D_i is a dictionary that holds $h_i(\rho(k))$ for each key $k \in S \setminus S'$ with $\phi(\rho(k)) = i$. A unique value in $\{0, \ldots, j-1\}$, where $j = (1+o(1)) \lg^2 n$, is associated with each key in D_i . A bit vector of j bits and an additional string of $\lg n$ bits is used to keep track of which associated values are in use. We will return to the exact choice of j and the implementation of the dictionaries.

The main idea is that all dictionaries in the construction assign to each of their keys a unique value within a subinterval of [1..m]. Each of the dictionaries D_1, \ldots, D_r is responsible for an interval of size j, and the high performance dictionary is responsible for an interval of size $O(n/\lg u) = o(n)$.

The hash function ρ is used to reduce the key length to v. The constant in $v = O(\lg n)$ can be chosen such that with high probability, over a polynomially bounded sequence of updates, ρ will never map two elements of S to the same value (the conflicts, if they occur, end up in S' and are handled by the high performance dictionary).

When inserting a new value k, the new key is included in S' if either:

- There are j keys in D_i , where $i = \phi(\rho(k))$, or
- There exists a key $k' \in S$ where $\phi(\rho(k)) = \phi(\rho(k')) = i$ and $h_i(\rho(k)) = h_i(\rho(k'))$.

Otherwise k is associated with the key $h_i(\rho(k))$ in D_i . Deletion of a key k is done in S' if $k \in S'$, and otherwise the associated key in the appropriate D_i is deleted.

To evaluate the perfect hash function on a key k we first see whether k is in the high performance dictionary. If so, we return the value associated with k. Otherwise we compute $i = \phi(\rho(k))$ and look up the value Δ associated with the key $h_i(\rho(k))$ in D_i . Then we return $(i-1)j + \Delta$, i.e., position Δ within the i-th interval.

Since D_1, \ldots, D_r store keys and associated values of $O(\lg \lg u)$ bits, they can be efficiently implemented as constant depth search trees of degree $w^{\Omega(1)}$, where each internal node resides in a single machine word. This yields constant time for dictionary insertions and lookups, with an optimal space usage of $O(\lg^2 n \lg \lg u)$ bits for each dictionary. We do not go into details of the implementation as they are standard; refer to [9] for explanation of the required word-level parallelism techniques.

What remains to describe is how the dictionaries keep track of vacant positions in the hash table in constant time per insertion and deletion. The high performance dictionary simply keeps a linked list of all vacant positions in its interval. Each of D_1, \ldots, D_r maintain a bit vector indicating vacant positions, and additional $O(\lg n)$ summary bits, each taking the or of an interval of size $O(\lg n)$. This can be maintained in constant time per operation, employing standard techniques.

Only o(n) preprocessing is necessary for the data structure (essentially to build tables needed for the word-level parallelism). The major part of the data structure is initialized lazily.

2.1 Analysis

Since evaluation of all involved hash functions and lookup in the dictionaries takes constant time, evaluation of the perfect hash function is done in constant time. As we will see below, the high performance dictionary is empty with high probability unless $n/\lg u > \sqrt{n}$. This means that it always uses constant time per update with high probability in n. All other operations done for update are easily seen to require constant time w.h.p.

We now consider the space usage of our scheme. The function ρ can be represented in O(w) bits. Siegel's highly independent hash function uses o(n) bits of space. The hash functions h_1, \ldots, h_r use $O(\lg n + \lg \lg u)$ bits each, and $o(n \lg \lg u)$ bits in total. The main space bottleneck is the space for D_1, \ldots, D_r , which sums to $O(n \lg \lg u)$.

Finally, we show that the space used by the high performance dictionary is O(n) bits w.h.p. This is done by showing that each of the following hold with high probability throughout a polynomial sequence of operations:

- 1. The function ρ is one-to-one on S.
- 2. There is no i such that $S_i = \{k \in S \mid \phi(\rho(k)) = i\}$ has more than j elements.
- 3. The set S' has $O(\lceil n/\lg u \rceil)$ elements.

That 1. holds with high probability is well known. To show 2. we use the fact that, with high probability, Siegel's hash function is independent on every set of $n^{\Omega(1)}$ keys. We may thus employ Chernoff bounds for random variables with limited independence to bound the probability that any i has $|S_i| > j$, conditioned on the fact that 1. holds. Specifically, we can use [14, Theorem 5.I.b] to argue that for any l, the probability that $|S_i| > j$ for $j = \lceil \lg^2 n + \lg^{5/3} n \rceil$ is $n^{-\omega(1)}$, which is negligible. On the assumption that 1. and 2. hold, we finally consider 3. We note that every key $k' \in S'$ is involved in an h_i -collision in S_i for $i = \phi(\rho(k'))$, i.e. there exists $k'' \in S_i \setminus \{k'\}$ where $h_i(k') = h_i(k'')$. By universality, for any i the expected number of h_i -collisions in

 S_i is $O(\lg^4 n/(\lg u)^{6+2c}) = O((\lg u)^{-(2+2c)})$. Thus the probability of one or more collisions is $O((\lg u)^{-(2+2c)})$. For $\lg u \geq \sqrt{n}$ this means that there are no keys in S' with high probability. Specifically, c may be chosen as the sum of the constants in the exponents of the length of the operation sequence and the desired high probability bound. For the case $\lg u < \sqrt{n}$ we note that the expected number of elements in S' is certainly $O(n/\lg u)$. To see than this also holds with high probability, note that the event that one or more keys from S_i end up in S' is independent among the i's. Thus we can use Chernoff bounds to get that the deviation from the expectation is small with high probability.

3 Lower Bound for Bloomier Filters

For the purpose of the lower bound, we consider the following two-set distinction problem, following [4]. The problem has the following stages:

- 0. a random string R is drawn, which will be available to the data structure throughout its operation. This is equivalent to drawing a deterministic algorithm from a given distribution, and is more general than assuming each stage has its own random coins (we are giving the data structure free storage for its random bits).
- 1. the data structure is given $A \subset [u], |A| \leq n$. It must produce a representation $f_R(A)$, which for any A has size at most S bits, in expectation over all choices of R. Here S is a function of n and u, which is the target of our lower bound.
- 2. the data structure is given $B \subset [u]$, such that $|B| \leq n, A \cap B = \emptyset$. Based on the old state $f_R(A)$, it must produce $g_R(B, f_R(A))$ with expected size at most S bits.
- 3. the data structure is given $x \in [u]$ and its previously generated state, i.e. $f_R(A)$ and $g_R(B, f_R(A))$. Now it must answer as follows with no error allowed: if $x \in A$, it must answer "A"; if $x \in B$, it must answer "B"; if $x \notin A \cup B$, it can answer either "A" or "B". Let $h_R(x, f, g)$ be the answer computed by the data structure, when the previous state is f and g.

It is easy to see that a solution for dynamic Bloomier filters supporting ternary associated data, using expected space $o(n \lg \lg \frac{u}{n})$, yields a solution to the two-set distinction problem with $S = o(n \lg \lg \frac{u}{n})$. We will prove such a solution does not exist.

Since a solution to the distinction problem is not allowed to make an error we can assume w.l.o.g. that step 3 is implemented as follows. If there exist appropriate $A, B \subset [u]$, with $x \in A$ such that $f_R(A) = f_0$ and $g_R(B, f_0) = g_0$, then $h_R(x, f_0, g_0)$ must be "A". Similarly, if there exists a plausible scenario with $x \in B$, the answer must be "B". Otherwise, the answer can be arbitrary.

Assume that the inputs $A \times B$ are drawn from a given distribution. We argue that if the expected sizes of f and g are allowed to be at most 2S, the data structure need not be randomized. This uses a bicriteria minimax principle. We have $E_{R,A,B}\left[\frac{|f|}{S} + \frac{|g|}{S}\right] \leq 2$, where |f|, |g| denote the length of the representations. Then, there exists a random string R_0 such that $E_{A,B}\left[\frac{|f|}{S} + \frac{|g|}{S}\right] \leq 2$. Since $|f|, |g| \geq 0$, this implies $E_{A,B}[|f|] \leq 2S$, $E_{A,B}[|g|] \leq S$. The data structure can simply use the deterministic sequence R_0 as its random bits; we drop the subscript from f_R, g_R when talking about this deterministic data structure.

3.1 Lower Bound for Two-Set Distinction

Assume $u = \omega(n)$, since a lower bound of $\Omega(n)$ is trivial for universe $u \geq 2n$. Break the universe into n equal parts U_1, \ldots, U_n ; w.l.o.g. assume n divides u, so $|U_i| = \frac{n}{u}$. The hard input distribution chooses A uniformly at random from $U_1 \times \cdots \times U_n$. We write $A = \{a_1, \ldots, a_n\}$, where a_i is a random variable drawn from U_i . Then, B' is chosen uniformly at random from the same product space; again $B' = \{b_1, \ldots, b_n\}, b_i \leftarrow U_i$. We let $B = B' \setminus A$. Note that $E[|B|] = n \cdot \Pr[A_1 \neq B_1] = (1 - \frac{n}{u}) \cdot n = (1 - o(1)) \cdot n$.

Let A_i^p be the plausible values of A_i after we see f(A); that is, A_i^p contains all $a \in U_i$ for which there exists a valid A' with $a \in A'$ and f(A') = f(A). Intuitively speaking, if f(A) has expected size $o(n \lg \lg \frac{u}{n})$, it contains on average $o(\lg \lg \frac{u}{n})$ bits of information about each a_i . This is much smaller than the range of a_i , which is $\frac{u}{n}$, so we would expect that the average $|A_i^p|$ is quite large, around $\frac{u}{n}/(\lg \frac{u}{n})^{o(1)}$. This intuition is formalized in the following lemma:

Lemma 5. With probability at least a half over a uniform choice of A and i, we have $|A_i^p| \geq \frac{u/n}{2O(S/n)}$.

Proof. The Kolmogorov complexity of A is $n \lg \frac{u}{n} - O(1)$; no encoding for A can have an expected size less than this quantity. We propose an encoding for A consisting of two parts: first, we include f(A); second, for each i we include the index of a_i in $|A_i^p|$, using $\lceil \lg |A_i^p| \rceil$ bits. This is easily decodable. We first generate all possible A' with f(A') = f(A), and thus obtain the sets A_i^p . Then, we extract from each plausible set the element with the given index. The expected size of the encoding is $2S + \sum_i E_A[\lg |A_i^p|] + O(n)$, which must be $\geq n \lg \frac{u}{n} - O(1)$. This implies $\lg \frac{u}{n} - E_{i,A}[\lg |A_i^p|] \leq \frac{2S}{n} + O(1)$. By Markov's inequality, with probability at least a half over i and A, $\lg \frac{u}{n} - \lg |A_i^p| \leq \frac{4S}{n} + O(1)$, so $\lg |A_i^p| \geq \lg \frac{u}{n} - O(\frac{S}{n})$.

We now make a crucial observation which justifies our interest in A_i^p . Assume that $b_i \in A_i^p$. In this case, the data structure must be able to determine b_i from f(A) and g(B, f(A)). Indeed, suppose we compute h(x, f, g) for all $x \in |A_i^p|$. If that data structure does not answer "B" when $x = b_i$, it is obviously incorrent. On the other hand, if it answers "B" for both $x = b_i$ and some other $x' \in A_i^p$, it also makes an error. Since x' is plausible, there exist A' with $x' \in A'$ such that f(A') = f(A). Then, we can run the data structure with A' as the first set and B as the second set. Since f(A') = f(A), the data structure will behave exactly the same, and will incorrectly answer "B" for x'.

To draw our conclusion, we consider another encoding argument, this time in connection to the set B'. The Kolmogorov complexity of B' is $n \lg \frac{u}{n} - O(1)$. Consider a randomized encoding, depending on a set A drawn at random. First, we encode an n-bit vector specifying which indices i have $a_i = b_i$. It remains to encode $B' \setminus A = B$. We encode another n-bit vector, specifying for which positions i we have $b_i \in A_i^p$. For each $b_i \notin A_i^p$, we simply encode B_i using $\lceil \lg \frac{u}{n} \rceil$ bits. Finally, we include in the encoding g(B, f(A)). As explained already, this is enough to recover all b_i which are in A_i^p . Note that we do not need to encode f(A), since this depends only on our random coins, and the decoding algorithm can reconstruct it.

The expected size of this encoding will be $O(n+S)+n\cdot \Pr_{A,B',i}[b_i\notin A_i^p]\cdot\lg\frac{u}{n}$. We know that with probability a half over A and i, we have $|A_i^p|\geq \frac{u/n}{2^{O(S/n)}}$. Thus, $\Pr_{A,B',i}[b_i\in A_i^p]\geq \frac{1}{2}\cdot 2^{-O(S/n)}$. Thus, the expected size of the encoding is at most $O(n+S)+(1-2^{-O(S/n)})\cdot n\lg\frac{u}{n}$. Note that by the minimax principle, randomness in the encoding is unessential and we can always fix A guaranteeing the same encoding size, in expectation over B. We now get the bound:

$$O(n+S) + (1 - 2^{-O(S/n)}) \cdot n \lg \frac{u}{n} \ge n \lg \frac{u}{n} - O(1)$$

$$\Rightarrow O\left(\frac{S}{n}\right) \ge 2^{-O(S/n)} \lg \frac{u}{n} - O(1) \Rightarrow 2^{O(S/n)} O(S/n) \ge \lg \frac{u}{n} \Rightarrow \frac{S}{n} = \Omega\left(\lg \lg \frac{u}{n}\right)$$

4 A Space-Optimal Bloomier Filter

It was shown in [3] that the approximate membership problem (i.e., the problem solved by Bloom filters) can be solved optimally using a reduction to the exact membership problem. The reduction uses universal hashing. In this section we extend this idea to achieve optimal dynamic Bloomier filters.

Recall that Bloomier filters encode sparse vectors with entries from $\{0, \ldots, 2^r - 1\}$. Let $S \subseteq [u]$ be the set of at most n indexes of nonzero entries in the vector V. The data structure must encode a vector V' that agrees with V on indexes in S, and such that for any $x \notin S$, $\Pr[V'[x] \neq 0] \leq \varepsilon$, where $\varepsilon > 0$ is the error probability of the Bloomier filter. Updates to V are done using the following operations:

- INSERT(x, a). Set V[x] := a, where $a \neq 0$.
- Delete(x). Set V[x] := 0.

The data structure assumes that only valid updates are performed, i.e. that inserts are done only in situations where V[x] = 0 and deletions are done only when $V[x] \neq 0$.

Theorem 6. Let positive integers n and r, and $\varepsilon > 0$ be given. On a RAM with word length w we can maintain a Bloomier filter V' for a vector V of length $u = 2^{O(w)}$ with at most n nonzero entries from $\{0, \ldots, 2^r - 1\}$, such that:

- Insert and Delete can be done in amortized expected constant time. The data structure assumes all updates are valid.
- Computing V'[x] on input x takes worst case constant time. If $V[x] \neq 0$ the answer is always V'[x]. If V[x] = 0 the answer is '0' with probability at least 1ε .
- The expected space usage is $O(n(\lg \lg(u/n) + \lg(1/\varepsilon) + r))$ bits.

4.1 The Data Structure

Assume without loss of generality that $u \ge 2n$ and that $\varepsilon \ge u/n$. Let $v = \max(n \log(u/n), n/\varepsilon)$, and choose $h : \{0, \dots, u-1\} \to \{0, \dots, v-1\}$ as a random function from a universal class of hash functions. The data structure maintains information about a minimal set S' such that h is 1-1 on $S \setminus S'$. Specifically, it consists of two parts:

- 1. A dictionary for the set S', with corresponding values of V as associated information.
- 2. A dictionary for the set $h(S \setminus S')$, where the element h(x), $x \in S \setminus S'$, has V[x] as associated information.

Both dictionaries should succinct, i.e., use space close to the information theoretic lower bound. Raman and Rao [13] have described such a dictionary using space that is 1+o(1) times the minimum possible, while supporting lookups in O(1) time and updates in expected amortized O(1) time.

To compute V'[x] we first check whether $x \in S'$, in which case V'[x] is stored in the first dictionary. If this is not the case, we check whether $h(x) \in h(S \setminus S')$. If this is the case we return the information associated with h(x) in the second dictionary. Otherwise, we return '0'.

INSERT(x, a). First determine whether $h(x) \in h(S \setminus S')$, in which case we add x to the set S', inserting x in the first dictionary. Otherwise we add h(x) to the second dictionary. In both cases, we associate a with the inserted element.

DELETE(x) proceeds by deleting x from the first dictionary if $x \in S'$, and otherwise deleting h(x) from the second dictionary.

4.2 Analysis

It is easy to see that the data structure always return correct function values on elements in S, given that all updates are valid. When computing V'[x] for $x \notin S$ we get a nonzero result if and only if there exists $x' \in S$ such that h(x) = h(x'). Since h was chosen from a universal family, this happens with probability at most $n/v \le \varepsilon$.

It remains to analyze the space usage. Using once again that h was chosen from a universal family, the expected size of S' is $O(n/\log(u/n))$. This implies that the expected number of bits necessary to store the set S' is $\log \binom{u}{O(n/\log(u/n))} = O(n)$, using convexity of the function $x \mapsto \binom{u}{x}$ in the interval $0 \dots u/2$. In particular, the first dictionary achieves an expected space usage of O(n) bits. The information theoretical minimum space for the set $h(S \setminus S')$ is bounded by $\log \binom{r}{n} = O(n\log(r/n)) = O(n\log\log(u/n) + n\log(1/\varepsilon))$ bits, matching the lower bound. We disregarded is the space for the universal hash function, which is $O(\log u)$ bits. However, this can be reduced to $O(\log n + \log \log u)$ bits, which is vanishing, by using slightly weaker universal functions and doubling the size r of the range. Specifically, 2-universal functions suffice; see [12] for a construction. Using such a family requires preprocessing time $(\log u)^{O(1)}$, expected.

5 Upper Bounds for the Greater-Than Problem

We start with a simple upper bound of $T_u = O(\lg n)$, $T_q = O(\lg \lg n)$. Our upper bound uses a trie structure. We consider a balanced tree with branching factor 2, and with n leaves. Every possible value of the update parameter a is represented by a root-to-leaf path. In the update stage, we mark this root-to-leaf path, taking time $O(\lg n)$. In the query stage, we want to find the point where b's path in the trie would diverge from a's path. This uses binary search on the $\lg n$ levels, as follows. To test if the paths diverge on a level, we examine the node on that level on b's path. If the node is marked, the paths diverge below; otherwise they diverge above. Once we have found the divergence point, we know that the larger of a and b is the one following the right child of the lowest common ancestor.

For the full tradeoff, we consider a balanced tree with branching factor B. In the update stage, we need to mark a root-to-leaf path, taking time $\lg_B n$. In the query stage, we first find the point where b's path in the trie would diverge from a's path. This uses binary search on the $\lg_B n$ levels, so it takes time $O(\lg \lg_B n)$. Now we know the level where the paths of a and b diverge. The nodes on that level from the paths of a and b must be siblings in the tree. To test whether b > a, we must

find the relative order of the two sibling nodes. There are two strategies for this, giving the two branches of the tradeoff curve. To achieve small update time, we can do all work at query time. We simply test all siblings to the left of b's path on the level of divergence. If we find a marked one, then a's path goes to the left of b's path, so a < b; otherwise a > b. This stragegy gives $T_u = O(\lg_B n)$ and $T_q = O(\lg(\lg_B n) + B)$, for any $B \ge 2$. For $T_q > \Omega(\lg\lg n)$, we have $T_q = \Theta(B)$, so we have achieved the tradeoff $T_u = O(\lg_{T_q} n)$.

The second strategy is to do all work at update time. For every node on a's path, we mark all left siblings of the node as such. Then to determine if b's path is to the left or to the right of a's path, we can simply query the node on b's path just below the divergence point, and see if it is marked as a left sibling. This strategy gives $T_u = O(B \lg_B n)$ and $T_q = O(\lg(\lg_B n))$. For small enough B (say $B = O(\lg n)$), this strategy gives $T_q = O(\lg\lg n)$ regardless of B and T_u . For $B = \Omega(\lg n)$, we have $\lg B = \Theta(\lg T_u)$. Therefore, we can express our tradeoff as: $2^{T_q} = O(\lg_{T_u} n)$.

6 Dynamic Range Reporting

We begin with the case $T_u = O(\lg w)$, $T_q = O(\lg \lg w)$. Let S be the current set of values stored by the data structure. Without loss of generality, assume w is a power of two. For an arbitrary $t \in [0, \lg w]$, we define the trie of order t, denoted T_t , to be the trie of depth $w/2^t$ and alphabet of 2^t bits, which represents all numbers in S. We call T_0 the primary trie (this is the classic binary trie with elements from S). Observe that we can assign distinct names of O(w) bits to all nodes in all tries. We call active paths the paths in the tries which correspond to elements of S. A node vfrom T_t corresponds to a subtree of depth v0 in the primary trie; we denote the root of this subtree by v0 and v0. A node from v0 corresponds to a 2-level subtree in v0, we call such a subtree a natural subtree. Alternatively, a 2-level subtree of any trie is natural iff its root is at an even depth.

A root-to-leaf path in the primary trie is seen as the leaves of the tree used for the greater-than problem. The paths from the primary trie are broken into chunks of length 2^t in the trie of order t. So T_t is similar to the t-th level (counted bottom-up) of the greater-than tree. Indeed, every node on the t-th level of that tree held information about a subtree with 2^t leaves; here one edge in T_t summarizes a segment of length 2^t bits. Also, a natural subtree corresponds to two siblings in the greater-than structure. On the next level, the two siblings are contracted into a node; in the trie of higher order, a natural subtree is also contracted into a node. It will be very useful for the reader to hold these parallels in mind, and realize that the data structure from this section is implementing the old recursion idea on every path.

The root-to-leaf paths corresponding to the values in S determine at most n-1 branching nodes in any trie. By convention, we always consider roots to be branching nodes. For every branching node from T_0 , we consider the extreme points of the interval spanned by the node's subtree. By doubling the universe size, we can assume these are never elements of S (alternatively, such extreme points are formal rationals like $x+\frac{1}{2}$). We define \overline{S} to be the union of S and the two special values for each branching node in the primary trie; observe that $|\overline{S}| = O(n)$. We are interested in holding \overline{S} for navigation purposes: it gives a way to find in constant time the maximum and minimum element from S that fits under a branching node (because these two values should be the elements from S closest to the special values for the branching node).

Our data structure has the following components:

1. a linked list with all elements of S in increasing order, and a predecessor structure for S.

- 2. a linked list with all elements of \overline{S} in increasing order, accompanied by a navigation structure which enables us to find in constant time the largest value from S smaller than a given value from $\overline{S} \setminus S$. We also hold a predecessor structure for \overline{S} .
- 3. every branching node from the primary trie holds pointers to its lowest branching ancestor, and the two branching descendants (the highest branching nodes from the left and right subtrees; we consider leaves associated with elements from S as branching descendants). We also hold pointers to the two extreme values associated with the node in the list in item 2. Finally, we hold a hash table with these branching nodes.
- 4. for each t, and every node v in T_t , which is either a branching node or a child of a branching node on an active path, we hold the depth of the lowest branching ancestor of $r_0(v)$, using a Bloomier filter.

We begin by showing that this data structure takes linear space. Items 1-3 handle O(n) elements, and have constant overhead per element. We show below that the navigation structure from 2. can be implemented in linear space. The predecessor structure should also use linear space; for van Emde Boas, this can be achieved through hashing [17].

In item 4., there are O(n) branching nodes per trie. In addition, there are O(n) children of branching nodes which are on active paths. Thus, we consider $O(n \lg w)$ nodes in total, and hold $O(\lg w)$ bits of information for each (a depth). Using our solution for the Bloomier filter, this takes $O(n(\lg w)^2 + w)$ bits, which is o(n) words. Note that storing the depth of the branching ancestor is just a trick to reduce space. Once we have a node in T_0 and we know the depth of its branching ancestor, we can calculate the ancestor in O(1) time (just ignore the bits below the depth of the ancestor). So in essence these are "compressed pointers" to the ancestors.

We now sketch the navigation structure from item 2. Observe that the longest run in the list of elements from $\overline{S} \setminus S$ can have length at most 2w. Indeed, the leftmost and rightmost extreme values for the branching nodes form a parenthesis structure; the maximum depth is w, corresponding to the maximum depth in the trie. Between an open and a closed parenthesis, there must be at least one element from S, so the longest uninterrupted sequence of parenthesis can be w closed parenthesis and w open parenthesis.

The implementation of the navigation structure uses classic ideas. We bucket $\Theta(\sqrt{w})$ consecutive elements from the list, and then we bucket $\Theta(\sqrt{w})$ buckets. Each bucket holds a summary word, with a bit for each element indicating whether it is in S or not; second-order buckets hold bits saying whether first order buckets contain at least one element from S or not. There is also an array with pointers to the elements or first order buckets. By shifting, we can always insert another summary bit in constant time when something is added. However, we cannot insert something in the array in constant time; to fix that, we insert elements in the array on the next available position, and hold the correct permutation packed in a word (using $O(\sqrt{w} \lg w)$ bits). To find an element from S, we need to walk O(1) buckets. The time is O(1) per traversed bucket, since we can use the classic constant-time subroutine for finding the most significant bit [8].

We also describe a useful subroutine, test-branching(v), which tests whether a node v from some T_t is a branching node. To do that, we query the structure in item 4. to find the lowest branching ancestor of $r_0(v)$. This value is defined if v is a branching node, but the Bloomier filter may return an arbitrary result otherwise. We look up the purported ancestor in the structure of item 3. If the node is not a branching node, the value in the Bloom filter for v was bogus, so v is

not a branching node. Otherwise, we inspect the two branching descendants of this node. If v is a branching node, one of these two descendants must be mapped to v in the trie of order t, which can be tested easily.

6.1 Implementation of Updates

We only discuss insertions; deletions follow parallel steps uneventfully. We first insert the new element in S and \overline{S} using the predecessor structures. Inserting a new element creates exactly one branching node v in the primary trie. This node can be determined by examining the predecessor and successor in S. Indeed, the lowest common ancestor in the primary trie can be determined by taking an xor of the two values, finding the most significant bit, and them masking everything below that bit from the original values [1].

We calculate the extreme values for the new branching node v, and insert them in \overline{S} using the predecessor structure. Finding the branching ancestor of v is equivalent to finding the enclosing parentheses for the pair of parentheses which was just inserted. But \overline{S} has a special structure: a pair of parentheses always encloses two subexpressions, which are either values from S, or a parenthesized expression (i.e., the branching nodes from T_0 form a binary tree structure). So one of the enclosing parentheses is either immediately to the left, or immediately to the right of the new pair. We can traverse a link from there to find the branching ancestor. Once we have this ancestor, it is easy to update the local structure of the branching nodes from item 3. Until now, the time is dominated by the predecessor structure.

It remains to update the structure in item 4. For each t > 0, we can either create a new branching node in T_t , or the branching node existed already (this is possible for t > 0 because nodes have many children). We first test whether the branching node existed or not (using the test-branching subroutine). If we need to introduce a branching node, we simply add a new new entry in the Bloomier filter with the depth of the branching ancestor of v. It remains to consider active children of branching nodes, for which we must store the depth of v. If we have just introduced a branching node, it has exactly two active children (if there exist more than two children on active paths, the node was a branching node before). These children are determined by looking at the branching descendants of v; these give the two active paths going into v. Both descendants are mapped to active children of the new branching node from T_t . If the branching node already existed, we must add one active child, which is simply the child that the path to the newly inserted value follows. Thus, to update item 4., we spend constant time per T_t . In total, the running time of an update is $T_{pred} + O(\lg w) = O(\lg w)$.

6.2 Implementation of Queries

Remember that a query receives an interval [a, b] and must return a value in $S \cap [a, b]$, if one exists. We begin by finding the node v which is the lowest common ancestor of a and b in the primary trie; this takes constant time [1]. Note that v spans an interval which includes [a, b]. The easiest case is when v is a branching node; this can be recognized by a lookup in the hash table from item 3. If so, we find the two branching descendants of v; call the left one v_L and the right one v_R . Then, if $S \cap [a, b] \neq \emptyset$, either the rightmost value from S that fits under v_L or the leftmost value from S that in fits under v_R must be in the interval [a, b]. This is so because [a, b] straddles the middle point of the interval spanned by v. The two values mentioned above are the two values from S closest (on both sides) to this middle point, so if [a, b] is non-empty, it must contain one of these

two. To find these two values, we follow a pointer from v_L to its left extreme point in \overline{S} . Then, we use the navigation structure from item 2., and find the predecessor from S of this value in constant time. The rightmost value under v_R is the next element from S. Altogether, the case when v is a branching node takes constant time.

Now we must handle the case when v is not a branching node. If $S \cap [a, b] \neq \emptyset$, it must be the case that v is on an active path. Below we describe how to find the lowest branching ancestor of v, assuming that v is on an active path. If this assumption is violated, the value returned can be arbitrary. Once we have the branching ancestor of v, we find the branching descendant w which is in v's subtree. Now it is easy to see, by the same reasoning as above, that if $[a, b] \cap S \neq \emptyset$ either the leftmost or the rightmost value from S which is under w must be in [a, b]. These two values are found in constant time using the navigation structure from item 2., as described above. So if $[a, b] \cap S \neq \emptyset$, we can find an element inside [a, b]. If none of these two elements were in [a, b] it must be the case that [a, b] was empty, because the algorithm works correctly when $[a, b] \cap S \neq \emptyset$.

It remains to show how to find v's branching ancestor, assuming v is on an active path, but is not a branching node. If for some t > 0, v is mapped to a branching node in T_t , it will also be mapped to a branching node in tries of higher order. We are interested in the smallest t for which this happens. We find this t by binary search, taking time $O(\lg \lg w)$. For some proposed t, we check whether the node to which v is mapped in T_t is a branching node (using the test-branching subroutine). If it is, we continue searching below; otherwise, we continue above.

Suppose we found the smallest t for which v is mapped to a branching node. In T_{t-1} , v is mapped to some z which is not a branching node. Finding the lowest branching ancestor of v is identical to finding the lowest branching ancestor of $r_0(z)$ in the primary trie (since z is a not a branching node, there is no branching node in the primary trie in the subtree corresponding to z). Since in T_t z gets mapped to a branching node, its natural subtree in T_{t-1} must contain at least one branching node. We have two cases: either z is the root or a leaf of the natural subtree (remember that a natural subtree has two levels). These can be distinguished based on the parity of z's depth. If z is a leaf, the root must be a branching node (because there is at least another active leaf). But then z is an active child of a branching node, so item 4. tells us the branching ancestor of $r_0(z)$. Now consider the case when z is the root of the natural subtree. Then z is above any branching node in its natural subtree, so to find the branching ancestor of $r_0(z)$ we can find the branching ancestor of the node from T_t to which the natural subtree is mapped. But this is a branching node, so the structure in item 4. gives the desired branching ancestor. To summarize, the only super-constant cost is the binary search for t, which takes $O(\lg \lg w)$ time.

7 Tradeoffs from Dynamic Range Reporting

Fix a value $B \in [2, \sqrt{w}]$; varying B will give our tradeoff curve. For an arbitrary $t \in [0, \lg_B w]$, we define the trie of order t to be the trie of depth w/B^t and alphabet of B^t bits, which represents all numbers in S. We call the trie for t = 0 the primary trie. A node v in a trie of order t is represented by a subtree of depth B^t in the primary trie; we say that the root of this subtree "corresponds to" the node v. A node from a trie of order t is represented by a subtree of depth B in the trie of order t - 1; we call such a subtree a "natural depth-B subtree". Alternatively, a depth-B subtree is natural if it starts at a depth divisible by B.

The root-to-leaf paths from the primary trie are boken into chunks of length B^t in the trie of order t. A trie of order t is similar to the t-th level (counted bottom-up) of the tree used for the

greater-than problem, since a path in the primary trie is seen as the leaves of that tree. Indeed, every node on the t-th level of that tree held information about a subtree with B^t leaves; here one edge in a trie of order t summarizes a segment of length B^t bits. Also, a natural depth-B subtree corresponds to B siblings in the old structure. On the next level, the B siblings are contracted into a node; in the trie of higher order, a natural depth-B subtree is also contracted into a node.

Our data structure has the following new components:

- 5A. choose this for the first branch of the tradeoff (faster updates, slower queries): hold the same information as in 4. for each t, and every node v in the trie of order t which is not a branching node, is on an active path, and is the child of a branching node in the trie of order t.
- 5B. choose this for the second branch of the tradeoff: hold the same information as above for each t, and every node v which is not a branching node, is on an active path, and has a branching ancestor in the same natural depth-B subtree.

In item 5A., notice that for every t there are at most 2n-2 children of branching nodes which are on active paths. We store $O(\lg w)$ bits for each, and there are $O(\lg_B w)$ values of t, so we can store this in a Bloomier filter with o(n) words of space. In item 5B., the number of interesting nodes blows up by at most B compared to 5A., and since $B \leq \sqrt{w}$, we are still using o(n) words of space.

Updates. For each t > 0, we can either create a new branching node in the trie of order t, or the branching node existed already. We first test whether the branching node existed or not. If we just introduced a branching node, it has at most two children which are not branching nodes and are on active paths (if more than two such children exist, the node was a branching node before). If the branching node was old, we might have added one such child. These children are determined by looking at the branching descendents of v (these give the two active paths going into v, one or both of which are new active paths going into the node in the subtrie of order t). For such children, we add the depth of v in the structure from item 5A. If we are in case 5B, we follow both paths either until we find a branching node, or the border of the natural depth-B subtree. For of these O(B) positions, we add the depth of v in item b. To summarize, the running time is b. To summarize, the running time is b.

Queries. We need to show how to find v's branching ancestor, assuming v is on an active path, but is not a branching node. For some t > 0, and all t's above that value, v will be mapped in the trie of order t to some branching node. That is the smallest t such that the depth- B^t natural subtree containing v contains some branching node. We find this t by binary search, taking time $O(\lg(\lg_B w))$. For some proposed t, we check if the node to which v is mapped is a branching node in the trie of order t (using the subroutine described above). If it is, we continue searching below; otherwise, we continue above.

Say we found the smallest t for which v is mapped to a branching node. In the trie of order t-1, v is mapped to some w which is not a branching node. Finding the lowest branching ancestor of v is identical to finding the lowest branching ancestor of the node corresponding to v in the primary trie (since v is a not a branching node, there is no branching node in the primary trie in the subtree represented by v). In the trie of order v, v gets mapped to a branching node, so the natural depth-v subtree of v contains at least one branching node. The either: (1) there is some

branching node above w in its natural depth-B subtree, or (2) w is on the active path going to the root of this natural subtree (it is above any branching node).

We first deal with case (2). If w is above any branching node in its natural subtree, to find w's branching ancestor we can find the branching ancestor of the node from the trie of order t, to which this subtree is mapped. But this is a branching node, so the structure in item 4. gives the branching ancestor z. We can test that we are indeed in case (2), and not case (1), by looking at the two branching descendents of z, and checking that one of them is strictly under v.

Now we deal with case (1). If we have the structure 5B., this is trivial. Because w is on an active path and has a branching ancestor in its natural depth-B subtree, it records the depth of the branching ancestor of the node corresponding to w in the primary trie. So in this case, the only super-constant cost is the binary search for t, which is $O(\lg(\lg_B w))$. If we only have the structure 5A., we need to walk up the trie of order t-1 starting from w. When we reach the child of the branching node above w, the branching node from the primary trie is recorded in item 5A. Since the branching node is in the same natural depth-B subtree as w, we reach this point after O(B) steps. One last detail is that we do not actually know when we have reached the child of a branching node (because the Bloomier filter from item 5A. can return arbitrary results for nodes not satisfying this property). To cope with this, at each level we hope that we have reached the destination, we query the structure in item 5A., we find the purported branching ancestor, and check that it really is the lowest branching acestor of v. This takes constant time; if the result is wrong, we continue walking up the trie. Overall, with the structure of 5A. we need query time $O(\lg(\lg_B w) + B)$.

We have shown how to achieve the same running times (as functions of B) as in the case of the greater-than function. The same calculation establishes our tradeoff curve.

8 Lower Bounds for the Greater-Than Problem

A lower bound for the first branch of the tradeoff can be obtained based on Fredman's proof idea [7]. We ommit the details for now. To get a lower bound for the second case $(T_q < O(\lg \lg n))$, we use the sunflower lemma of Erdős and Rado. A sunflower is collection of sets (called petals) such that the intersection of any two of the sets is equal to the intersection of all the sets.

Lemma 7 (Sunflower Lemma). Consider a collection of n sets, of cardinalities at most s. If $n > (p-1)^{s+1}s!$, the collection contains as a subcollection a sunflower with p petals.

For every query parameter in [0, n-1], the algorithm performs at most T_q probes to the memory. Thus, there are 2^{T_q} possible execution paths, and at most $2^{T_q} - 1$ bit cells are probed on at least some execution path. This gives n sets of cells of sizes at most $s = O(2^{T_q})$; we call these sets query schemes. By the sunflower lemma, we can find a sunflower with p petals, if p satisfies: $n > (p-1)^{s+1}s! \Rightarrow \lg n > \Theta(s(\lg p + \lg s))$. If $T_q < (1-\varepsilon) \lg \lg n$, we have $s \lg s = o(\lg n)$, so our condition becomes $\lg n > \Theta(s \lg p)$. So we can find a sunflower with p petals such that $\lg p = \Omega((\lg n)/s)$. Let P be the set of query parameters whose query schemes are these p petals.

The center of the sunflower (the intersection of all sets) obviously has size at most s. Now consider the update schemes for the numbers in P. We can always find $T \subset P$ such that $|T| \ge |P|/2^s$ and the update schemes for all numbers in T look identical if we only inspect the center of the sunflower. Thus $\lg |T| = \lg |P| - s = \Omega(\frac{\lg n}{s} - s)$. If $T_u < (\frac{1}{2} - \varepsilon) \lg \lg n$, we have $s = o(\frac{\lg n}{s})$, so we obtain $\lg |T| = \Omega(\frac{\lg n}{s})$.

Now we restrict our attention to numbers in T for both the update and query value. The cells in the center of the sunflower are thus fixed. Define the natural result of a certain query to be the result (greater than vs. not greater than) of the query if all bit cells read by the query outside the center of the sunflower are zero. Now pick a random $x \in T$. For some y in the middle third of T (when considering the elements of T in increasing order), we have $\Pr[y \leq x] \geq \frac{1}{3}, \Pr[y > x] \geq \frac{1}{3}$, so no matter what the natural result of querying y is, it is wrong with probability at least $\frac{1}{3}$. So for a random x, at least a fraction of $\frac{1}{9}$ of the natural results are wrong. Consider an explicit x with this property. The update scheme for x must set sufficiently many cells to change these natural results. But these cells can only be in the petals of the queries whose natural results are wrong, and the petals are disjoint except for the center, which is fixed. So the update scheme must set at least one cell for every natural result which is wrong. Hence $T_u \geq |T|/9 \Rightarrow \lg T_u = \Omega(\lg |T|) = \Omega(\frac{\lg n}{s}) = \Omega(\frac{\lg n}{2^Tq}) \Rightarrow 2^{T_q} = \Omega(\lg |T|)$.

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