Engineering Mathematics I

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of
Advanced Engineering Mathematics

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Part C. Fourier Analysis

CHAPTER 11. Fourier Series, Integrals, and Transforms

- Periodic phenomena occur quite frequently --- think of motors, rotating machines, and the motion of planets, etc.
- We may represent the corresponding *periodic functions* in terms of a *series of cosine and sine functions*. These representations are called **Fourier series,** which is of most importance in the analysis of engineering systems.
- The corresponding ideas and techniques can also be extended to *non-periodic functions*, leading to **Fourier integrals** and **Fourier transforms**.

11.1 Fourier Series

- A function $f(t)$ is called *periodic* if there is some positive number T such that

$$f(t+T)=f(t) \text{ for all } t$$

==> T is called a period of $f(t)$.

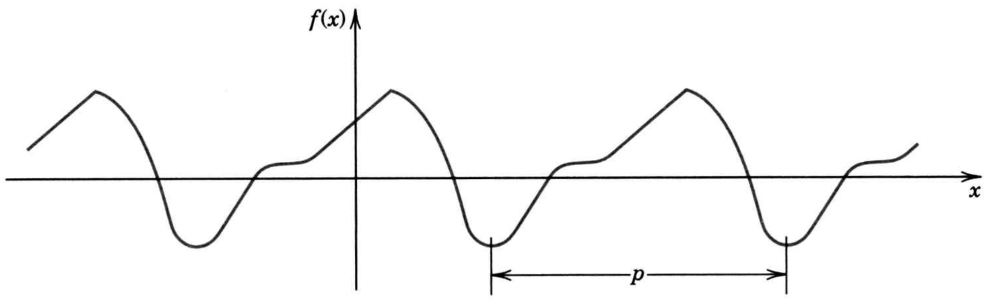


Fig. 258. Periodic function

□ Since $f(t+2T)=f((t+T)+T)=f(t+T)=f(t)$, for any integer n,

$$f(t+nT)=f(t)$$

==> $2T$, $3T$, $4T$,... are also periods of $f(t)$.
==> Smallest period T is called the *fundamental period*.

□ If $f(t)$ and $g(t)$ have a period T, then the function

$$h(t)=af(t)+bg(t)$$

also has the period T for any constants a, b.

Trigonometric Series, Fourier Series

□ We are concerned about the representation of various functions with period $T=2\pi$ in terms of a series of sinusoidal functions,

$$1,\ \cos t,\ \sin t,\ \cos 2t,\ \sin 2t,\ \cdots,\ \cos nt,\ \sin nt,\ \cdots$$

such that

$$f(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos nt+b_n\sin nt\right)$$

where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants.

⇨ Called the *trigonometric series when* $T=2\pi$.

⇨ For any T, called the *Fourier series*.

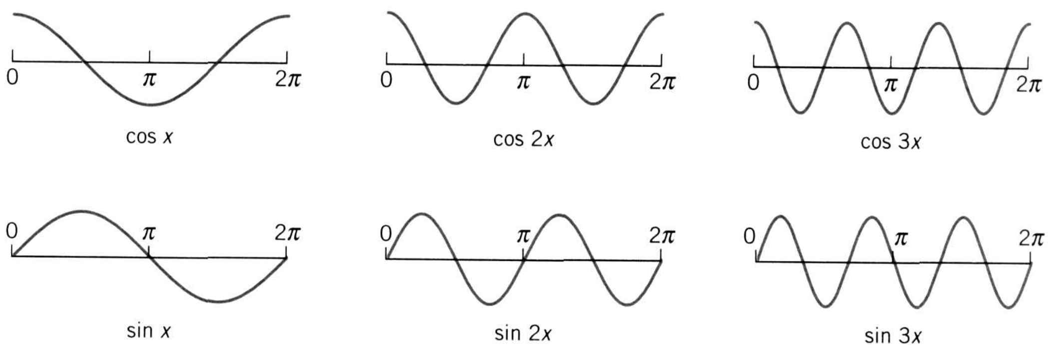


Fig. 259. Cosine and sine functions having the period 2π

□ If $f(t)$ is a function of period T, then it can be represented by

$$f(t)=a_0+\sum_{n=1}^{\infty}\left[a_n\cos\frac{2\pi nt}{T}+b_n\sin\frac{2\pi nt}{T}\right].$$

⇨ Fourier series representation of the function $f(t)$.

⇨ a_0, a_n, and b_n are called *Fourier coefficients*.

□ Given the function $f(t)$, we should determine the coefficients a_0, a_n, and b_n of the corresponding series.

Determination of the constant term a_0

Integrating on both sides from $-T/2$ to $T/2$, we have

$$\int_{-T/2}^{T/2}f(t)\,dt=\int_{-T/2}^{T/2}\left[a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi nt}{T}+b_n\sin\frac{2\pi nt}{T}\right)\right]dt$$

$$=\underbrace{\int_{-T/2}^{T/2}a_0\,dt}_{a_0T}+\sum_{n=1}^{\infty}\left(a_n\underbrace{\int_{-T/2}^{T/2}\cos\frac{2\pi nt}{T}dt}_{0}+b_n\underbrace{\int_{-T/2}^{T/2}\sin\frac{2\pi nt}{T}dt}_{0}\right).$$

Thus

$$a_0=\frac{1}{T}\int_{-T/2}^{T/2}f(t)\,dt.$$

Determination of the coefficients a_n

Multiplying both sides by $\cos\frac{2\pi mt}{T}$, where m is any fixed positive integer, and integrating from $-T/2$ to $T/2$, we have

$$\int_{-T/2}^{T/2}f(t)\cos\frac{2\pi mt}{T}dt=\int_{-T/2}^{T/2}a_0\cos\frac{2\pi mt}{T}dt+\sum_{n=1}^{\infty}a_n\int_{-T/2}^{T/2}\cos\frac{2\pi nt}{T}\cos\frac{2\pi mt}{T}dt$$

$$+\sum_{n=1}^{\infty}b_n\int_{-T/2}^{T/2}\sin\frac{2\pi nt}{T}\cos\frac{2\pi mt}{T}dt.$$

Since

$$\int_{-T/2}^{T/2} a_0 \cos\frac{2\pi mt}{T} dt = 0$$

$$\begin{aligned}\int_{-T/2}^{T/2} \cos\frac{2\pi nt}{T} \cos\frac{2\pi mt}{T} dt &= \frac{1}{2}\underbrace{\int_{-T/2}^{T/2} \cos\frac{2\pi(n+m)t}{T} dt}_{0} \\ &+ \frac{1}{2}\int_{-T/2}^{T/2} \cos\frac{2\pi(n-m)t}{T} dt \\ &= \begin{cases} \frac{T}{2}, & n = m \\ 0, & n \neq m \end{cases}\end{aligned}$$

$$\begin{aligned}\int_{-T/2}^{T/2} \sin\frac{2\pi nt}{T} \cos\frac{2\pi mt}{T} dt &= \frac{1}{2}\int_{-T/2}^{T/2} \sin\frac{2\pi(n+m)t}{T} dt \\ &+ \frac{1}{2}\int_{-T/2}^{T/2} \sin\frac{2\pi(n-m)t}{T} dt \\ &= 0.\end{aligned}$$

Thus

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos\frac{2\pi nt}{T} dt, \qquad n = 1,\ 2, \cdots$$

Determination of the coefficients b_n

Multiplying both sides by $\sin\frac{2\pi mt}{T}$, and integrating from $-T/2$ to $T/2$, we have

$$\begin{aligned}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi mt}{T} dt &= \int_{-T/2}^{T/2} a_0 \sin\frac{2\pi mt}{T} dt + \sum_{n=1}^{\infty} a_n \int_{-T/2}^{T/2} \cos\frac{2\pi nt}{T} \sin\frac{2\pi mt}{T} dt \\ &+ \sum_{n=1}^{\infty} b_n \int_{-T/2}^{T/2} \sin\frac{2\pi nt}{T} \sin\frac{2\pi mt}{T} dt.\end{aligned}$$

Since

$$\int_{-T/2}^{T/2} a_0 \sin\frac{2\pi mt}{T} dt = 0$$

$$\int_{-T/2}^{T/2} \cos\frac{2\pi nt}{T} \sin\frac{2\pi mt}{T} dt = 0$$

$$\int_{-T/2}^{T/2} \sin\frac{2\pi nt}{T}\sin\frac{2\pi mt}{T}dt = \frac{1}{2}\int_{-T/2}^{T/2}\cos\frac{2\pi(n-m)t}{T}dt - \underbrace{\frac{1}{2}\int_{-T/2}^{T/2}\cos\frac{2\pi(n+m)t}{T}dt}_{0}$$

$$= \begin{cases} \frac{T}{2}, & n = m \\ 0, & n \neq m. \end{cases}$$

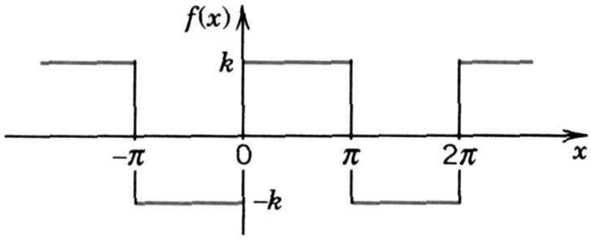
Thus

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi nt}{T}dt, \qquad n = 1,\ 2,\cdots$$

Example 1. Periodic Rectangular Wave

Find the Fourier coefficients of the periodic function $f(t)$, which is given by

$$f(t) = \begin{cases} -k & \text{if } -\pi < t < 0 \\ k & \text{if } 0 < t < \pi \end{cases} \text{ and } f(t+2\pi) = f(t) \implies T = 2\pi.$$



Solution.

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = 0,$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi nt}{T}dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos ntdt + \int_{0}^{\pi} k\cos ntdt\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nt}{n}\Big|_{-\pi}^{0} + k\frac{\sin nt}{n}\Big|_{0}^{\pi}\right] = 0,$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi nt}{T}dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin ntdt + \int_0^\pi k\sin ntdt\right]$$

$$= \frac{1}{\pi}\left[k\frac{\cos nt}{n}\Big|_{-\pi}^{0} - k\frac{\cos nt}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2k}{n\pi}(1-\cos n\pi).$$

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n \\ \ \ 1 & \text{for even } n \end{cases} \quad ==> \ 1-\cos n\pi = \begin{cases} 2 & \text{for odd } n \\ 0 & \text{for even } n. \end{cases}$$

Hence the Fourier coefficients b_n are

$$b_1 = \frac{4k}{\pi}, \ \ b_2 = 0, \ \ b_3 = \frac{4k}{3\pi}, \ \ b_4 = 0, \ \ b_5 = \frac{4k}{5\pi}, \cdots$$

Finally we have

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt = \frac{4k}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right).$$

- Consider the partial sums of the series given by

$$S_1 = \frac{4k}{\pi}\sin t, \quad S_2 = \frac{4k}{\pi}\left(\sin t + \frac{1}{3}\sin 3t\right), \ \text{etc.}$$

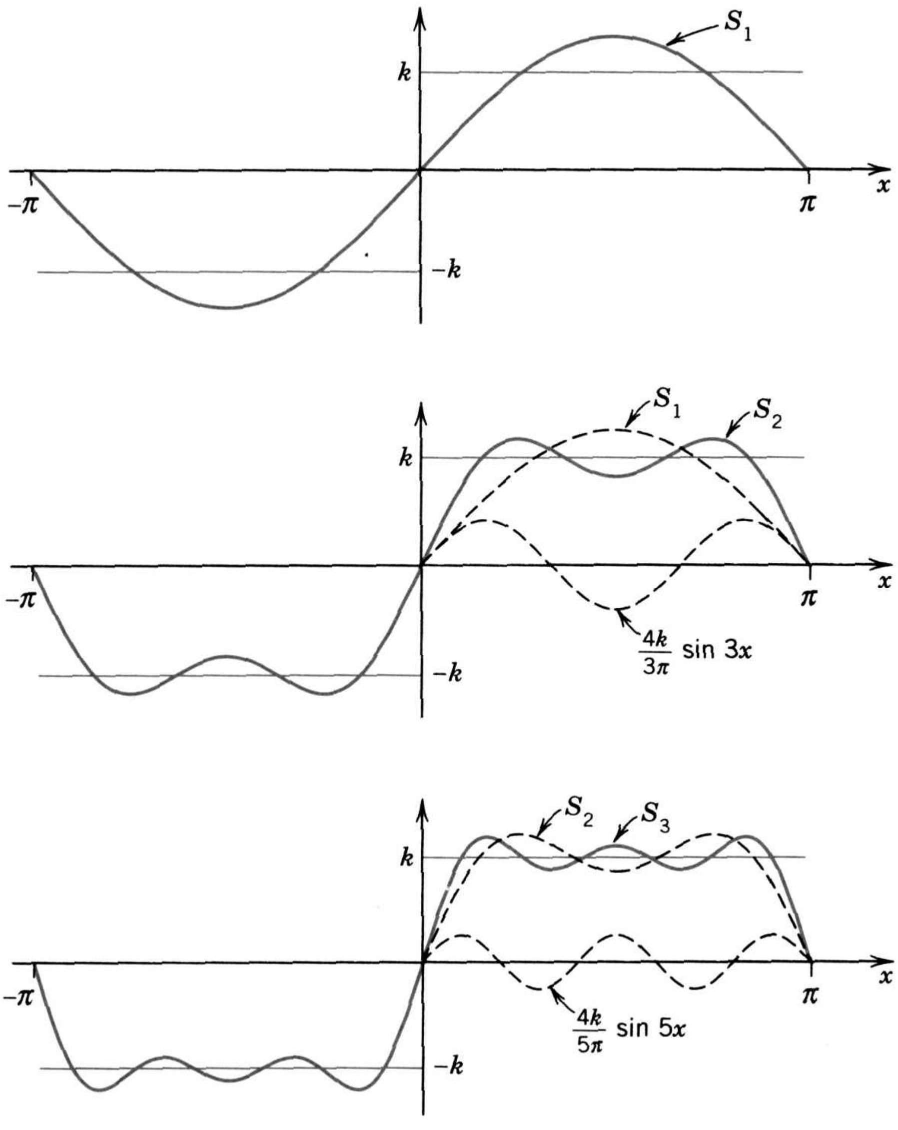
- Fig. 261 indicates that the series is *convergent* and has the sum $f(t)$. Note that at $t=0$ and $t=\pi$, the points of discontinuity of $f(t)$, all partial sums have the value zero.
- Furthermore, setting $t = \pi/2$,

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots\right)$$

  thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

□ This is a famous result by Leibniz, illustrating that the values of various series can be obtained by evaluating Fourier series at specific points.



(b) The first three partial sums of the corresponding Fourier series

Fig. 261. Example 1

Orthogonality of the Trigonometric System

□ By definition, two functions are called *orthogonal* on the interval $0 \le t \le T$ if

$$\int_0^T f(t)g(t)\,dt = 0.$$

Theorem 1. The following functions are orthogonal to each other on the interval $-\pi \le t \le \pi$

$$1,\ \cos t,\ \sin t,\ \cos 2t,\ \sin 2t,\ \cdots,\ \cos nt,\ \sin nt,\ \cdots$$

Proof.

$$\int_{-\pi}^{\pi} \cos mt \cdot \cos nt\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)t\,dt + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)t\,dt$$
$$= 0 \ (m \ne n)$$

$$\int_{-\pi}^{\pi} \sin mt \cdot \sin nt\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)t\,dt - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)t\,dt$$
$$= 0 \ (m \ne n)$$

$$\int_{-\pi}^{\pi} \cos mt \cdot \sin nt\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)t\,dt + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)t\,dt$$
$$= 0, \quad \text{for all } m,\ n$$

Convergence and Sum of a Fourier Series

Theorem 2. Representation by a Fourier Series

If a function $f(t)$ with period 2π is *piecewise continuous* in the interval $-\pi \le t \le \pi$ and has a *left-hand derivative* and *right-hand derivative* at each point of that interval, then the Fourier series of $f(t)$ is *convergent*.

Its sum is $f(t)$ except at a *discontinuity point* t_0 and the sum of the series at t_0 is the average of the *left-* and *right-hand limits* of $f(t)$.

Definition: A function $f(t)$ is called **piecewise continuous** on a finite interval $a \le t \le b$ if it is defined on that interval and the interval can be subdivided into finitely many intervals in each of which $f(t)$ is *continuous* and has *finite limits* as t approaches either endpoint of the interval of subdivision.

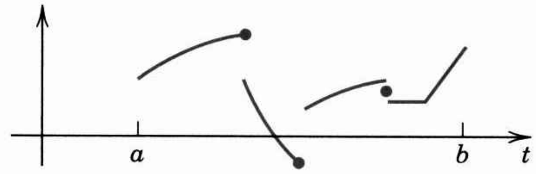


Fig. 107. Example of a piecewise continuous function $f(t)$ (The dots mark the function values at the jumps.)

Proof. We prove the convergence for a continuous function $f(t)$ having *continuous first and second derivatives*. Using the integration by parts,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(t)}_{u}\underbrace{\cos nt}_{v'}\,dt = \underbrace{\left.\frac{f(t)\sin nt}{n\pi}\right|_{-\pi}^{\pi}}_{0} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(t)\sin nt dt$$

Another integration by parts gives

$$a_n = \underbrace{\left.\frac{f'(t)\cos nt}{n^2\pi}\right|_{-\pi}^{\pi}}_{0} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(t)\cos nt dt.$$

Since $f''(t)$ is continuous in the interval of integration, we have

$$|f''(t)| < M$$

where M is a constant. Also since $|\cos nt| \le 1$, it follows that

$$|a_n| = \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(t)\cos nt dt\right| < \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M dt = \frac{2M}{n^2}.$$

Similarly, $|b_n| < 2M/n^2$ for all n.

Hence the sum of absolute values of each term of the Fourier series is at most equal to

$$\left|a_0\right| + 2M\left(1+1+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\cdots\right)$$ ==> Convergent

□ The left- and right-hand derivatives of $f(t)$ at t_0 are defined as

$$\lim_{h\to+0}\frac{f(t_0-h)-f(t_0-0)}{-h} \quad \text{and} \quad \lim_{h\to+0}\frac{f(t_0+h)-f(t_0+0)}{h}.$$

Example 2. Convergence at a jump

The square wave in Example 1 has a jump at $t=0$. Its left-hand limit is $-k$ and right-hand limit is k. Note that the Fourier series of this square wave converges to *zero*, average of two limits, at $t=0$.

11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions T

Example 1. Periodic Rectangular Wave

Find the Fourier series of the function

$$f(t)=\begin{cases}0 & \text{if } -2<t<-1\\ k & \text{if } -1<t<1\\ 0 & \text{if } \ 1<t<2\end{cases} \quad \text{==>} \quad T=4.$$

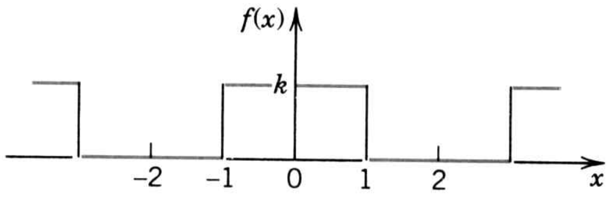


Fig. 263. Example 1

Solution.

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{4}\int_{-2}^{2} f(t)\,dt$$
$$= \frac{1}{4}\int_{-1}^{1} k\,dt = \frac{k}{2}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2n\pi t}{T}\,dt = \frac{1}{2}\int_{-2}^{2} f(t)\cos\frac{n\pi t}{2}\,dt$$
$$= \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi t}{2}\,dt = \frac{k}{n\pi}\sin\frac{n\pi t}{2}\bigg|_{-1}^{1} = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

==> $a_n = 0$ if n is even

$a_n = 2k/n\pi$ if n = 1, 5, 9, …

$a_n = -2k/n\pi$ if n = 3, 7, 11, …

Also $b_n = 0$. ($\because f(t)$ is even function)

Hence the result is

$$f(t) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}t - \frac{1}{3}\cos\frac{3\pi}{2}t + \frac{1}{5}\cos\frac{5\pi}{2}t - + \cdots\right).$$

Example 2. Periodic Rectangular Wave

Find the Fourier series of the function

$$f(t) = \begin{cases} -k & \text{if } -2 < t < 0 \\ k & \text{if } \ \ 0 < t < 2 \end{cases} \quad ==> \ T = 4$$

Solution.

$$a_0 = 0,$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{0} (-k)\cos\frac{n\pi t}{2}\,dt + \int_{0}^{2} k\cos\frac{n\pi t}{2}\,dt\right]$$
$$= \frac{1}{2}\left[-\frac{2k}{n\pi}\sin\frac{n\pi t}{2}\bigg|_{-2}^{0} + \frac{2k}{n\pi}\sin\frac{n\pi t}{2}\bigg|_{0}^{2}\right] = 0,$$

$$b_n = \frac{1}{2}\left[\int_{-2}^{0}(-k)\sin\frac{n\pi t}{2}dt + \int_{0}^{2}k\sin\frac{n\pi t}{2}dt\right]$$

$$= \frac{1}{2}\left[\frac{2k}{n\pi}\cos\frac{n\pi t}{2}\Big|_{-2}^{0} - \frac{2k}{n\pi}\cos\frac{n\pi t}{2}\Big|_{0}^{2}\right]$$

$$= \frac{k}{n\pi}\left(1 - \cos n\pi - \cos n\pi + 1\right) = \begin{cases} 4k/n\pi & \text{if } n = 1,\ 3,\ldots \\ 0 & \text{if } n = 2,\ 4,\ldots \end{cases}$$

Hence the Fourier series of $f(t)$ is

$$f(t) = \frac{4k}{\pi}\left(\sin\frac{\pi t}{2} + \frac{1}{3}\sin\frac{3\pi t}{2} + \frac{1}{5}\sin\frac{5\pi t}{2} + \cdots\right).$$

Example 3. Half-Wave Rectifier

A sinusoidal voltage $E\sin\omega t$ is passed through a *half-wave rectifier*. Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0 & \text{if } -\pi/\omega < t < 0 \\ E\sin\omega t & \text{if } 0 < t < \pi/\omega \end{cases} \quad \text{==>} \quad T = \frac{2\pi}{\omega}.$$

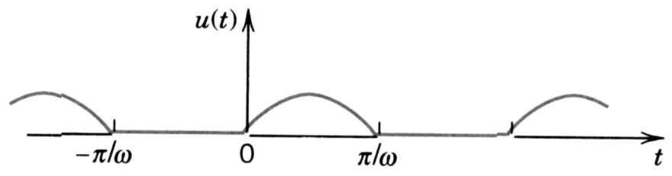


Fig. 265. Half-wave rectifier

Solution. We have

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{\omega}{2\pi}\int_{0}^{\pi/\omega} E\sin\omega t dt$$

$$= -\frac{\omega E}{2\pi}\frac{1}{\omega}\cos\omega t\Big|_{0}^{\pi/\omega} = \frac{E}{\pi},$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi nt}{T}dt = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\sin\omega t\cdot\cos n\omega t dt$$
$$= \frac{\omega E}{2\pi}\int_0^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt.$$

If $n=1$, $a_n = 0$.

If $n = 2, 3, 4, \cdots$, we obtain

$$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$
$$= \frac{E}{2\pi}\left(\frac{-\cos(1+n)\pi+1}{1+n} + \frac{-\cos(1-n)\pi+1}{1-n}\right).$$

If n is odd, this is equal to zero, and for even n we have

$$a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{2E}{(n-1)(n+1)\pi}, \qquad n = 2, 4, \cdots$$

In a similar fashion, we have $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, 4, \cdots$.

Finally

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right).$$

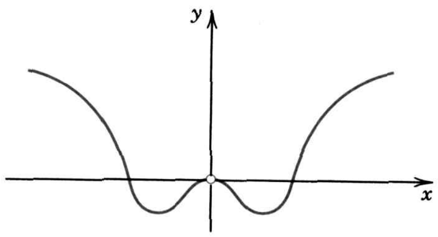
Simplifications: Even and Odd Functions.

- □ Unnecessary work in determining Fourier coefficients can be avoided if a function is *even* or *odd*.
- □ Remember that a function $y = g(t)$ is even if

$$g(-t) = g(t) \text{ for all } t$$

and a function $h(t)$ is odd if

$$h(-t) = -h(t) \text{ for all } t.$$



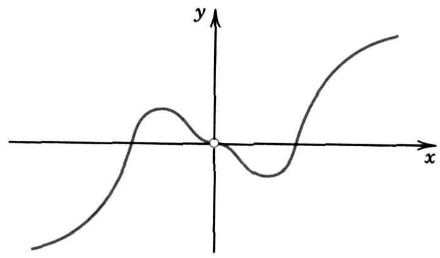


Fig. 266. Even function   Fig. 267. Odd function

Three Key Facts for the Present Discussion

1. If $g(t)$ is an *even* function, then

$$\int_{-L}^{L} g(t)dt = 2\int_{0}^{L} g(t)dt.$$

2. If $h(t)$ is an *odd* function, then

$$\int_{-L}^{L} h(t)dt = 0.$$

3. The **product** of an **even** and **odd** function is **odd**. Let $q(t) = g(t)h(t)$. Then

$$q(-t) = g(-t)h(-t) = g(t)\big(-h(t)\big) = -g(t)h(t) = -q(t)$$

==> $q(t)$ is an odd function.

□ *The Fourier series of an* ***even*** *function* of period T is a **"Fourier cosine series"**

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T}$$

with coefficients

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)dt, \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos\frac{2\pi nt}{T}dt, \quad n = 1,\ 2,\cdots$$

□ *The Fourier series of an* ***odd*** *function* of period T is a **"Fourier sine series"**

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}$$

with coefficients

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\frac{2\pi nt}{T}\,dt.$$

The case of period 2π. If $f(t)$ is even, we have

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(t)dt, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos ntdt, \quad n = 1,\ 2,\cdots$$

If $f(t)$ is odd, we have

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin ntdt.$$

Theorem 1 Sum and Scaler Multiple

The Fourier coefficients of a sum $f_1(t) + f_2(t)$ are the sums of the corresponding Fourier coefficients of $f_1(t)$ and $f_2(t)$. The Fourier coefficients of $cf(t)$ are c times the corresponding Fourier coefficients of $f(t)$.

Example 4 Rectangular Pulse

The function $f^*(t)$ is the sum of the function $f(t)$ in Example 1 of Sec. 11.1 and the constant k. Hence, we have

$$f^*(t) = k + \frac{4k}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right).$$

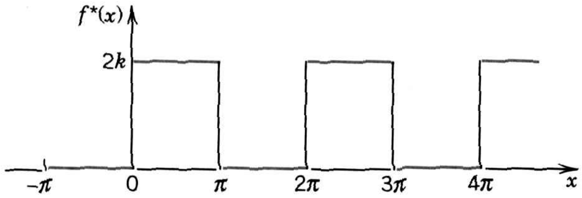


Fig. 244. Example 1

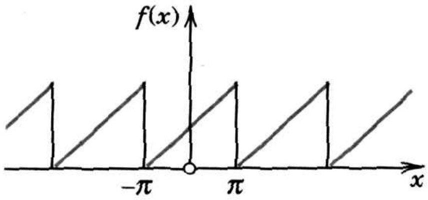
Example 2 Half-Wave Rectifier

The function $u(t)$ in Example 3 of Sec. 11.2 has a Fourier cosine series plus a single term $v(t) = (E/2)\sin\omega t$. Therefore, $u(t) - v(t)$ is an even function.

Example 5 Sawtooth Wave

Find the Fourier series of the function

$$f(t) = t + \pi \ \text{ if } -\pi < t < \pi \ \text{ and } f(t + 2\pi) = f(t).$$



(a) The function f(x)

Solution. We may write

$$f = f_1 + f_2, \text{ where } f_1 = t \text{ and } f_2 = \pi.$$

Since f_1 is odd, $a_n = 0$ for all n, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(t)\sin nt dt = \frac{2}{\pi}\int_0^{\pi} \underbrace{t}_{u} \underbrace{\sin nt}_{v'} dt$$

$$= \frac{2}{\pi}\left[\frac{-t\cos nt}{n}\bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nt dt \right] = -\frac{2}{n}\cos n\pi$$

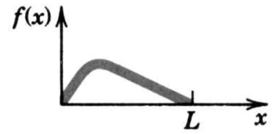
$$\Longrightarrow b_1 = 2,\ b_2 = -2/2,\ b_3 = 2/3,\ b_4 = -2/4, \cdots$$

Finally the Fourier series of $f(t)$ is

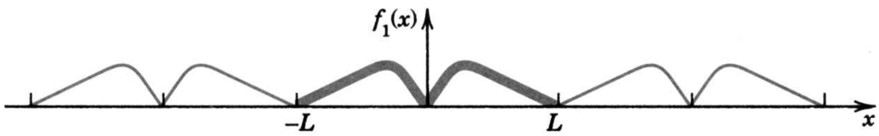
$$f(t) = \pi + 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - + \cdots\right).$$

Half-Range Expansions

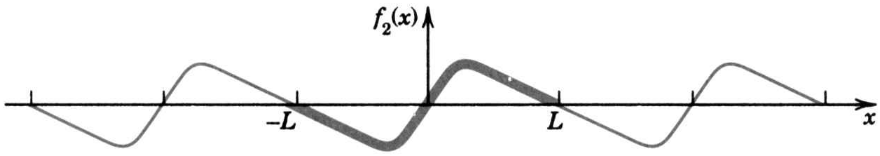
- In some applications it is often necessary to employ a Fourier series for a function that is defined only on some interval $0 \le t \le L$.
- We could extend the function as a function of period L and develop the extended function into a Fourier series.
- However, we can obtain simpler form by taking either the **even periodic extension** or **odd periodic extension** according to analytical convenience.



(a) The given function $f(x)$



(b) $f(x)$ extended as an even periodic function of period $2L$



(c) $f(x)$ extended as an odd periodic function of period $2L$

Example 6 "Triangle" and its Half-Range Expansions

Find the two half-range expansions of the function

$$f(t) = \begin{cases} \dfrac{2k}{L}t & \text{if } 0 < t < \dfrac{L}{2} \\ \dfrac{2k}{L}(L-t) & \text{if } \dfrac{L}{2} < t < L \end{cases}$$

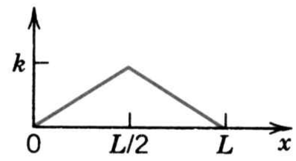


Fig. 271.

Solution. (a) Even periodic extension. Since $T = 2L$, we have

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} t\,dt + \frac{2k}{L}\int_{L/2}^{L}(L-t)\,dt\right] = \frac{k}{2}$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} t\cos\frac{n\pi}{L}t\,dt + \frac{2k}{L}\int_{L/2}^{L}(L-t)\cos\frac{n\pi}{L}t\,dt\right].$$

Using integration by parts

$$\int_0^{L/2} \underbrace{\underset{u}{t}\cos\frac{n\pi}{L}t}_{v'}\,dt = \frac{Lt}{n\pi}\sin\frac{n\pi}{L}t\Big|_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\frac{n\pi}{L}t\,dt$$

$$= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right).$$

Similarly, for the second integration we obtain

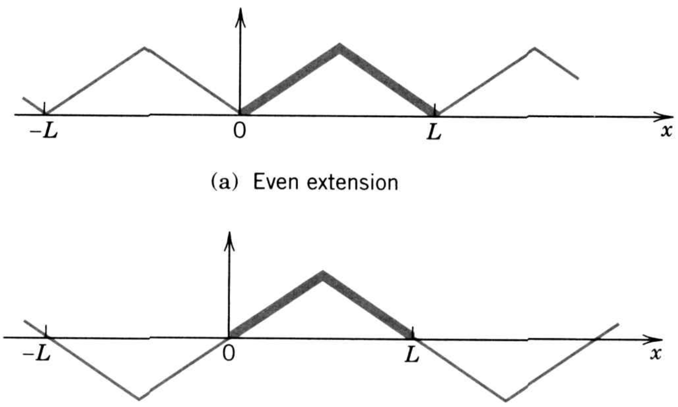
$$\int_{L/2}^{L}(L-t)\cos\frac{n\pi}{L}t\,dt = \frac{Lt}{n\pi}(L-t)\sin\frac{n\pi}{L}t\Big|_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\frac{n\pi}{L}t\,dt$$

$$= 0 - \frac{L}{n\pi}\left(L - \frac{L}{2}\right)\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right).$$

Thus

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

==> $a_2 = -16k/(2\pi)^2,\ a_6 = -16k/(6\pi)^2,\ a_{10} = -16k/(10\pi)^2, \cdots$



(a) Even extension

(b) Odd extension

Fig. 272.

Hence the even periodic extension of $f(t)$ is

$$f(t)=\frac{k}{2}-\frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}t+\frac{1}{6^2}\cos\frac{6\pi}{L}t+\cdots\right).$$

(b) Odd periodic extension. Similarly, we obtain

$$b_n=\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Thus the odd periodic extension of $f(t)$ is

$$f(t)=\frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}t-\frac{1}{3^2}\sin\frac{3\pi}{L}t+\frac{1}{5^2}\sin\frac{5\pi}{L}t-+\cdots\right).$$

11.7 Fourier Integral

- Recall that Fourier series are powerful tools in treating various problems involving *periodic functions.*
- The Fourier series concept can be extended to *Fourier integrals* to handle *non-periodic functions.*

Example 1 Rectangular Wave

Consider the periodic rectangular wave $f_L(t)$ of period $2L > 2$ given by

$$f_L(t) = \begin{cases} 0 & \text{if} \quad -L < t < -1 \\ 1 & \text{if} \quad -1 < t < 1 \\ 0 & \text{if} \quad 1 < t < L. \end{cases}$$

If we let $L \to \infty$, we obtain a non-periodic function $f(t)$ so that

$$f(t) = \lim_{L\to\infty} f_L(t) = \begin{cases} 1 & \text{if} \quad -1 < t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $f_L(t)$ is even, $b_n = 0$ for all n and

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dt = \frac{1}{L}$$

$$a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi t}{L} dt = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi t}{L} dt = \frac{2}{L}\frac{\sin(n\pi/L)}{n\pi/L}$$

$$= \frac{2}{L}\operatorname{sinc}(n/L)$$

- The sequence of Fourier coefficients are called the **amplitude spectrum** of $f_L(t)$ because $|a_n|$ is the amplitude of the wave $a_n \cos(n\pi t/L)$.

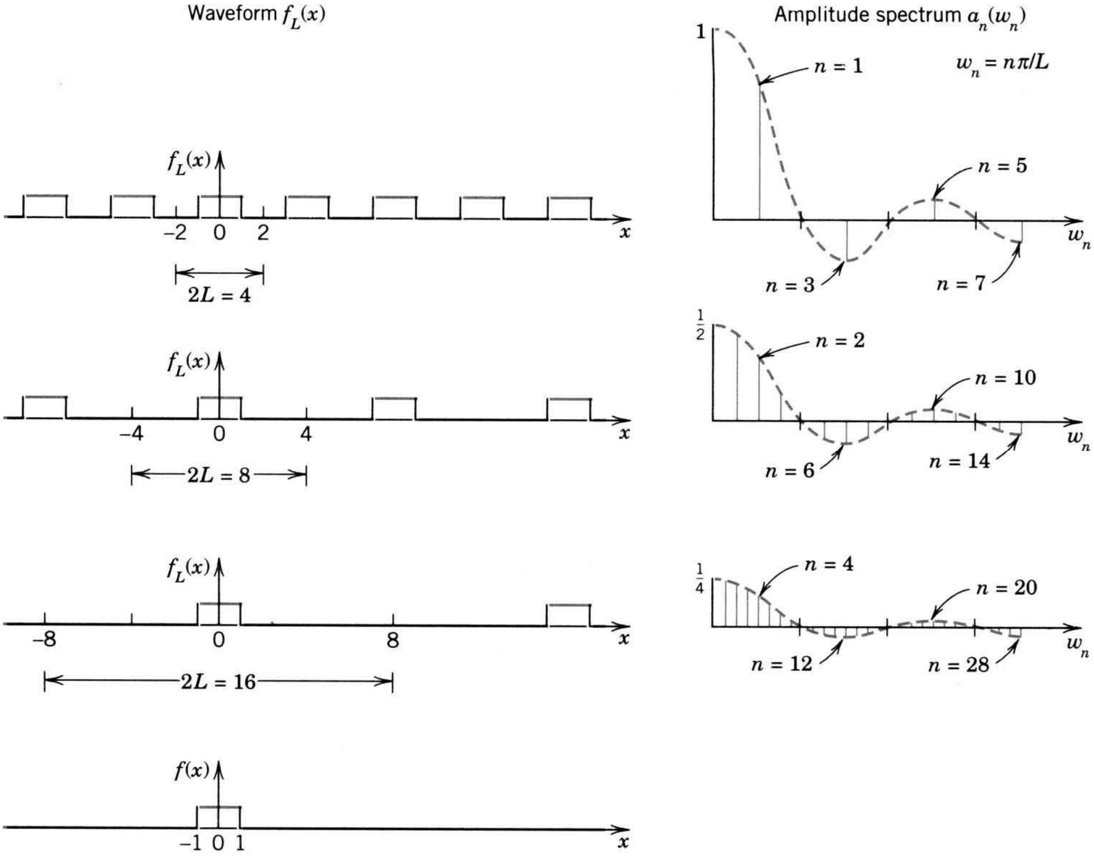


Fig. 254. Waveforms and amplitude spectra in Example 1

==> As L increases, the amplitude spectrum becomes more and more dense.

From Fourier Series to Fourier Integral

Any periodic function of period T=2L can be represented by a Fourier series

$$f_L(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\omega_n t + b_n \sin\omega_n t\right), \quad \omega_n = \frac{2\pi n}{T} = 2\pi f_n = \frac{\pi n}{L}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_L(t)dt$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f_L(t)\cos\omega_n t dt$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f_L(t)\sin\omega_n t dt.$$

Thus

$$f_L(t) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}\Big[\cos\omega_n t\cdot\int_{-L}^{L} f_L(v)\cos\omega_n v dv$$
$$+\sin\omega_n t\cdot\int_{-L}^{L} f_L(v)\sin\omega_n v dv\Big].$$

Let $\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$, then $1/L = \Delta\omega/\pi$.

Therefore,

$$f_L(t) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[\left(\cos\omega_n t\right)\int_{-L}^{L} f_L(v)\cos\omega_n v dv$$
$$+\left(\sin\omega_n t\right)\int_{-L}^{L} f_L(v)\sin\omega_n v dv\Big]\Delta\omega.$$

We now let $L\to\infty$, then

$$f(t) = \lim_{L\to\infty} f_L(t).$$

We assume that $f(t)$ is **absolutely integrable** so that

$$\int_{-\infty}^{\infty}\left|f(t)\right|dt < \infty.$$

As $L\to\infty$, $1/L\to 0$ and $\Delta\omega = \pi/L\to 0$. Thus

$$f(t) = \frac{1}{\pi}\int_0^{\infty}\Big[\cos\omega t\cdot\int_{-\infty}^{\infty} f(v)\cos\omega v dv + \sin\omega t\cdot\int_{-\infty}^{\infty} f(v)\sin\omega v dv\Big]d\omega.$$

If we introduce the notations

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v dv,\quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v dv,$$

we have

$$f(t) = \int_0^{\infty}\left[A(\omega)\cos\omega t + B(\omega)\sin\omega t\right]d\omega$$

==> Called a *Fourier integral representation* of $f(t)$.

Theorem 1 (Fourier Integral)

If $f(t)$ is *piecewise continuous* in every finite interval and has a *right-hand derivative* and *left-hand derivative* at every point and *absolutely integrable*, then $f(t)$ can be represented by a Fourier integral. At a point where $f(t)$ is discontinuous, the value of the Fourier integral equals the average of the left- and right-hand limits of $f(t)$ at that point.

Applications of the Fourier Integral

- □ The Fourier integral is mainly used in solving *differential equations* (Ch.12).
- □ It also can be used in solving some *integration* problems.

Example 2 Single Pulse, Sine Integral

Find the Fourier integral representation of the function

$$f(t) = \begin{cases} 1 & \text{if} \quad |t| < 1 \\ 0 & \text{if} \quad |t| > 1. \end{cases}$$

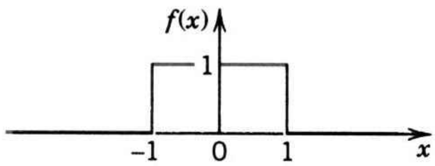


Fig. 281. Example 2

Solution. We have

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v dv = \frac{1}{\pi}\int_{-1}^{1}\cos\omega v dv = \left.\frac{\sin\omega v}{\pi\omega}\right|_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}$$

$$B(\omega) = \frac{1}{\pi}\int_{-1}^{1}\sin\omega v dv = 0.$$

Thus

$$f(t)=\int_0^{\infty}\left[A(\omega)\cos\omega t+B(\omega)\sin\omega t\right]d\omega=\frac{2}{\pi}\int_0^{\infty}\frac{\cos\omega t\sin\omega}{\omega}d\omega.$$

The average of the left- and right-hand limits of $f(t)$ at $t=\pm 1$ is equal to 1/2. From Theorem 1 we obtain

$$\int_0^{\infty}\frac{\cos\omega t\sin\omega}{\omega}d\omega=\begin{cases}\pi/2 & \text{if } -1<t<1\\ \pi/4 & \text{if } t=\pm 1\\ 0 & \text{if } t>1\end{cases}$$

Let us consider the case $t=0$.

$$\int_0^{\infty}\frac{\sin\omega}{\omega}d\omega=\frac{\pi}{2}$$

==> limit of the sine integral

$$Si(u)=\int_0^{u}\frac{\sin\omega}{\omega}d\omega.$$

The graph of $Si(u)$ is shown in Fig. 282.

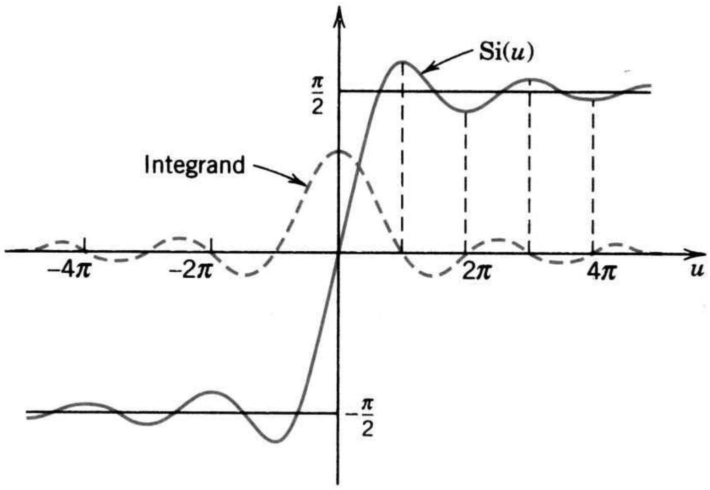


Fig. 282. Sine integral Si(u) and integrand

As a increases, the following integral

$$\frac{2}{\pi}\int_0^{a}\frac{\cos\omega t\sin\omega}{\omega}d\omega$$

approximates $f(t)$.

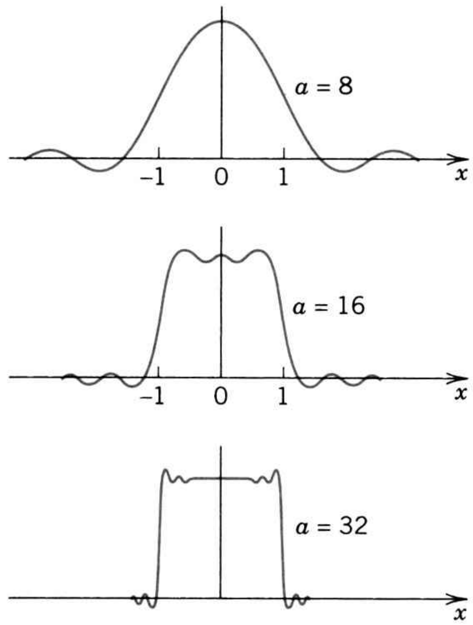


Fig. 283.

Fourier Cosine and Sine Integrals

- For an *even* or *odd* function, the Fourier integral becomes simpler. That is,

  If $f(t)$ is an *even* function, then $B(\omega)=0$ and

$$A(\omega)=\frac{2}{\pi}\int_0^\infty f(v)\cos\omega v dv.$$

  In this case we have

$$f(t)=\int_0^\infty A(\omega)\cos\omega t d\omega$$

==> Fourier cosine integral

  Similarly, if $f(t)$ is an *odd* function, then $A(\omega)=0$ and

$$B(\omega)=\frac{2}{\pi}\int_0^\infty f(v)\sin\omega v dv.$$

  Thus

$$f(t)=\int_0^\infty B(\omega)\sin\omega t d\omega$$

==> Fourier sine integral

Evaluation of Integrals

The main application of the Fourier integral is in differential equations but it also helps in evaluating certain integrals.

Example 3 Laplace Integrals

Find the Fourier cosine and sine integrals of

$$f(t) = e^{-kt} \qquad (t > 0,\ k > 0).$$

Solution. Taking an *even extension* for $f(t)$, we have

$$A(\omega) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos\omega v\,dv.$$

Now, using integration by parts twice,

$$\int_0^\infty \underbrace{e^{-kv}}_{u}\underbrace{\cos\omega v}_{v'}\,dv = \frac{k}{k^2+\omega^2}$$

$$==> A(\omega) = \frac{2k/\pi}{k^2+\omega^2}.$$

Thus we obtain the **Fourier cosine integral representation**

$$f(t) = e^{-kt} = \frac{2k}{\pi}\int_0^\infty \frac{\cos\omega t}{k^2+\omega^2}\,d\omega \qquad (t > 0,\ k > 0).$$

From this representation we see that

$$\int_0^\infty \frac{\cos\omega t}{k^2+\omega^2}\,d\omega = \frac{\pi}{2k}e^{-kt}, \qquad (t > 0,\ k > 0)$$

==> *Laplace integral*

(b) Similarly, by taking an *odd extension* for $f(t)$, we have

$$B(\omega) = \frac{2}{\pi}\int_0^\infty e^{-kv}\sin\omega v\,dv.$$

Using integration by parts twice,

$$\int_0^\infty \underbrace{e^{-kv}}_{u} \underbrace{\sin \omega v}_{v'} dv = \frac{\omega}{k^2 + \omega^2}$$

$$\text{==>} \quad B(\omega) = \frac{2\omega / \pi}{k^2 + \omega^2}.$$

We thus obtain the Fourier sine integral representation

$$f(t) = e^{-kt} = \frac{2}{\pi} \int_0^\infty \frac{\omega \sin \omega t}{k^2 + \omega^2} d\omega, \quad (t > 0,\ k > 0).$$

From this we see that

$$\int_0^\infty \frac{\omega \sin \omega t}{k^2 + \omega^2} d\omega = \frac{\pi}{2} e^{-kt}, \qquad (t > 0,\ k > 0)$$

==> *Laplace integral*

11.9 Fourier Transform

- □ The Fourier transform can be extended from the complex Fourier series representation or Fourier integral.
- □ Recall that the Fourier integral representation of $f(t)$ is given by

$$f(t) = \int_0^\infty \left[A(\omega) \cos \omega t + B(\omega) \sin \omega t \right] d\omega$$

where

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos \omega v dv, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin \omega v dv.$$

Substituting $A(\omega)$ and $B(\omega)$, we have

$$f(t) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(v) \left[\cos \omega v \cos \omega t + \sin \omega v \sin \omega t \right] dv d\omega$$

$$= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(v) \cos(\omega t - \omega v) dv \right] d\omega.$$

Since the integral in brackets is an even function of ω, we can rewrite $f(t)$ as

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(\omega t-\omega v)\,dv\right]d\omega.$$

It is also obvious that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\sin(\omega t-\omega v)\,dv\right]d\omega=0.$$

Therefore, we have

$$\begin{aligned} f(t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\left\{\cos(\omega t-\omega v)+j\sin(\omega t-\omega v)\right\}dv\right]d\omega \\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)e^{-j\omega v}dv\right]e^{j\omega t}d\omega. \end{aligned}$$

If we let

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt,$$

then we have

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega.$$

==> $F(\omega)$ is called the *Fourier transform* of $f(t)$ and $f(t)$ is called the *Fourier inverse transform* of $F(\omega)$.

==> More frequently used expressions in engineering applications are

$$F\{f(t)\}=\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt,$$

$$f(t)=\int_{-\infty}^{\infty} F\{f(t)\}e^{j2\pi ft}df$$

with $\omega=2\pi f$.

Easy to memorize it!!!

Theorem 1 Existence of the Fourier Transform

If $f(t)$ is *absolutely integrable* and *piecewise continuous* on every finite interval, then its Fourier transform exists.

Example 1 Fourier Transform

Find the Fourier transform of $f(t)=1$ if $|t|<1$ and $f(t)=0$ otherwise.

Solution.

$$\begin{aligned} F\{f(t)\} &= \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt = \int_{-1}^{1} e^{-j2\pi ft}dt \\ &= \frac{1}{-j2\pi f}e^{-j2\pi ft}\Big|_{-1}^{1} = \frac{1}{-j2\pi f}\left(e^{-j2\pi f} - e^{j2\pi f}\right) \\ &= \frac{\sin(2\pi f)}{\pi f}. \end{aligned}$$

Example 2 Fourier Transform

Find the Fourier transform of $f(t)=e^{-at},\ t>0\quad (a>0)$.

Solution.

$$\begin{aligned} F\left\{e^{-at}\right\} &= \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt \\ &= \int_{0}^{\infty} \exp\left[-at - j2\pi ft\right]dt \\ &= \frac{-1}{a+j2\pi f}\exp\left[-at - j2\pi ft\right]\Big|_{0}^{\infty} \\ &= \frac{1}{a+j2\pi f}. \end{aligned}$$

Physical Interpretation: Spectrum

Consider the Fourier transform representation

$$f(t) = \int_{-\infty}^{\infty} F\{f(t)\}e^{j2\pi ft}df.$$

Then, $f(t)$ can be interpreted as a *superposition of sinusoidal waveforms*. We call

$$\int_{-\infty}^{\infty} \left| F\{f(t)\} \right|^2 df = \int_{-\infty}^{\infty} f^2(t) dt$$

total energy of the signal $f(t)$ and therefore call

$$\left| F\{f(t)\} \right|^2$$

energy spectral density or energy spectrum.

Theorem 2 (Linearity of the Fourier Transform)

The Fourier transform is a *linear operation*; that is, for any functions $f(t)$ and $g(t)$ whose Fourier transforms exist and any constants a and b,

$$F\{af(t) + bg(t)\} = aF\{f(t)\} + bF\{g(t)\}.$$

Proof.

$$\begin{aligned} F\{af(t) + bg(t)\} &= \int_{-\infty}^{\infty} \left[af(t) + bg(t) \right] e^{-j2\pi ft} dt \\ &= a\int_{-\infty}^{\infty} f(t) e^{-j2\pi ft} dt + b\int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \\ &= aF\{f(t)\} + bF\{g(t)\}. \end{aligned}$$

Theorem 3 (Fourier Transform of the Derivative of $f(t)$)

Let $f(t)$ be continuous and $f(t) \to 0$ as $|t| \to \infty$. Furthermore, let $f'(t)$ is absolutely integrable. Then

$$F\{f'(t)\} = j2\pi f \cdot F\{f(t)\}.$$

Proof. By definition

$$F\{f'(t)\} = \int_{-\infty}^{\infty} f'(t) e^{-j2\pi ft} dt.$$

Using integration by parts, we have

$$F\{f'(t)\} = \left[f(t)e^{-j2\pi ft}\Big|_{-\infty}^{\infty} - (-j2\pi f)\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt \right]$$
$$= j2\pi f \cdot F\{f(t)\}.$$

Similarly,

$$F\{f''(t)\} = j2\pi f \cdot F\{f'(t)\} = (j2\pi f)^2 F\{f(t)\}$$
$$= -(2\pi f)^2 F\{f(t)\}.$$

Example 3 An application of the operational formula

Find the Fourier transform of te^{-t^2}.

Solution.

$$F\left\{te^{-t^2}\right\} = F\left\{-\frac{1}{2}\left(e^{-t^2}\right)'\right\} = -\frac{1}{2}F\left\{\left(e^{-t^2}\right)'\right\}$$
$$= -\frac{1}{2}j2\pi f \cdot F\left\{e^{-t^2}\right\}.$$

$$F\left\{e^{-t^2}\right\} = \sqrt{\pi}e^{-(\pi f)^2}.$$ (See Table III)

Thus

$$F\left\{te^{-t^2}\right\} = -\frac{1}{2}j2\pi f \cdot \sqrt{\pi}e^{-(\pi f)^2}$$
$$= -j(\pi)^{3/2} fe^{-(\pi f)^2}.$$

Convolution

The convolution of the functions $f(t)$ and $g(t)$ is defined by

$$h(t) = f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$
$$= \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau.$$

Theorem 3 (Convolution Theorem)

Suppose that $f(t)$ and $g(t)$ are Fourier transformable.

Then

$$F\{f(t)\otimes g(t)\}=F\{f(t)\}F\{g(t)\}.$$

Proof. By definition,

$$\begin{aligned}
F\{f(t)\otimes g(t)\} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right]e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} g(t-\tau)e^{-j2\pi ft}dt\right]d\tau \quad (\text{let } t-\tau=\tilde{t}) \\
&= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} g(\tilde{t})e^{-j2\pi f(\tilde{t}+\tau)}d\tilde{t}\right]d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)e^{-j2\pi f\tau}d\tau\cdot\int_{-\infty}^{\infty} g(\tilde{t})e^{-j2\pi f\tilde{t}}d\tilde{t} \\
&= F\{f(t)\}F\{g(t)\}.
\end{aligned}$$

Thus

$$f(t)\otimes g(t)=\int_{-\infty}^{\infty} F\{f(t)\}F\{g(t)\}e^{j2\pi ft}df.$$

Theorem 4 (Time Scaling)

If $F\{g(t)\}=G(f)$, then $F\{g(at)\}=\dfrac{1}{|a|}G\left(\dfrac{f}{a}\right)$.

Proof. By definition,

$$F\{g(at)\}=\int_{-\infty}^{\infty} g(at)e^{-j2\pi ft}dt \quad (\text{let } \tilde{t}=at)$$

If $a>0$, then

$$\begin{aligned}
F\{g(at)\} &= \frac{1}{a}\int_{-\infty}^{\infty} g(\tilde{t})e^{-j2\pi f\tilde{t}/a}d\tilde{t}=\frac{1}{a}\int_{-\infty}^{\infty} g(\tilde{t})e^{-j2\pi(f/a)\tilde{t}}d\tilde{t} \\
&= \frac{1}{a}G\left(\frac{f}{a}\right).
\end{aligned}$$

If $a<0$, then

$$\begin{aligned}
F\{g(at)\} &= \frac{1}{a}\int_{\infty}^{-\infty} g(\tilde{t})e^{-j2\pi f\tilde{t}/a}d\tilde{t}=-\frac{1}{a}\int_{-\infty}^{\infty} g(\tilde{t})e^{-j2\pi(f/a)\tilde{t}}d\tilde{t} \\
&= -\frac{1}{a}G\left(\frac{f}{a}\right).
\end{aligned}$$

Thus

$$F\{g(at)\} = \frac{1}{|a|} G\left(\frac{f}{a}\right).$$

Theorem 5 Duality

If $F\{g(t)\} = G(f)$, then $F\{G(t)\} = g(-f)$.

Proof. By definition,

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df.$$

Exchanging the variables t and f, we have

$$g(f) = \int_{-\infty}^{\infty} G(t) e^{j2\pi ft} dt$$

$$==> \quad g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi ft} dt.$$

Thus

$$F\{G(t)\} = g(-f).$$

Example 4

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = e^{-j2\pi f0} \int_{-\infty}^{\infty} \delta(t) dt$$
$$= 1.$$

Thus

$$F\{1\} = \delta(-f) = \delta(f).$$

Theorem 6 Time Shifting

If $F\{g(t)\} = G(f)$, then $F\{g(t-t_0)\} = G(f)\exp(-j2\pi ft_0)$.

Proof. By definition,

$$
\begin{aligned}
F\{g(t-t_0)\} &= \int_{-\infty}^{\infty} g(t-t_0)e^{-j2\pi ft}dt \quad \left(\text{let } \tilde{t} = t - t_0\right) \\
&= \int_{-\infty}^{\infty} g(\tilde{t})e^{-j2\pi f(\tilde{t}+t_0)}d\tilde{t} \\
&= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} g(\tilde{t})e^{-j2\pi f\tilde{t}}d\tilde{t} \\
&= G(f)\exp(-j2\pi ft_0).
\end{aligned}
$$

Theorem 7 Frequency Shifting

If $F\{g(t)\} = G(f)$, then $F\{\exp(j2\pi f_c t)g(t)\} = G(f - f_c)$.

Proof. By definition,

$$
\begin{aligned}
F\{\exp(j2\pi f_c t)g(t)\} &= \int_{-\infty}^{\infty} e^{j2\pi f_c t}g(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} g(t)e^{-j2\pi(f-f_c)t}dt \\
&= G(f-f_c).
\end{aligned}
$$

Example 5 Find the Fourier transform

$$
g(t) = A\mathrm{rect}\left(\frac{t}{T}\right)\cos(2\pi f_c t).
$$

Solution.

$$
\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| < T/2 \\ 0 & \text{otherwise.} \end{cases}
$$

Thus

$$
\begin{aligned}
F\left\{A\mathrm{rect}\left(\frac{t}{T}\right)\right\} &= \int_{-T/2}^{T/2} Ae^{-j2\pi ft}dt = 2A\int_0^{T/2}\cos(2\pi ft)dt \\
&= \frac{2A}{2\pi f}\sin(2\pi ft)\Big|_0^{T/2} = \frac{2A}{2\pi f}\sin(\pi fT) \\
&= AT\frac{\sin(\pi fT)}{\pi fT} = AT\,\mathrm{sinc}(fT).
\end{aligned}
$$

Also since,

$$
\cos(2\pi f_c t) = \frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2},
$$

we have

$$g(t) = A\text{rect}\left(\frac{t}{T}\right)\cos(2\pi f_c t) = A\text{rect}\left(\frac{t}{T}\right)\left(\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}\right).$$

Finally, by frequency shifting theorem, we have

$$F\{g(t)\} = \frac{AT}{2}\left[\operatorname{sinc}\{(f - f_c)T\} + \operatorname{sinc}\{(f + f_c)T\}\right].$$

Theorem 8 Integration in the time domain

If $F\{g(t)\} = G(f)$ and $G(0) = 0$, then $F\left\{\int_{-\infty}^{t} g(\tau)d\tau\right\} = \frac{1}{j2\pi f}G(f)$.

Proof. Since

$$g(t) = \frac{d}{dt}\int_{-\infty}^{t} g(\tau)d\tau,$$

we have

$$G(f) = j2\pi f \cdot F\left\{\int_{-\infty}^{t} g(\tau)d\tau\right\}.$$

Thus,

$$F\left\{\int_{-\infty}^{t} g(\tau)d\tau\right\} = \frac{1}{j2\pi f}G(f).$$

Theorem 9 Multiplication in the time domain

If $F\{g_1(t)\} = G_1(f)$ and $F\{g_2(t)\} = G_2(f)$, then

$$F\{g_1(t)g_2(t)\} = G_1(f) \otimes G_2(f).$$

Proof. By definition,

$$
\begin{aligned}
F\left\{g_1(t)g_2(t)\right\} &= \int_{-\infty}^{\infty} g_1(t)g_2(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} g_1(t)\left[\int_{-\infty}^{\infty} G_2(\tilde{f})e^{j2\pi\tilde{f}t}d\tilde{f}\right]e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} g_1(t)\left[\int_{-\infty}^{\infty} G_2(\tilde{f})e^{-j2\pi(f-\tilde{f})t}d\tilde{f}\right]dt, \quad (\lambda = f-\tilde{f}) \\
&= \int_{-\infty}^{\infty} g_1(t)\left[\int_{-\infty}^{\infty} G_2(f-\lambda)e^{-j2\pi\lambda t}d\lambda\right]dt \\
&= \int_{-\infty}^{\infty} G_2(f-\lambda)\left[\int_{-\infty}^{\infty} g_1(t)e^{-j2\pi\lambda t}dt\right]d\lambda \\
&= \int_{-\infty}^{\infty} G_2(f-\lambda)G_1(\lambda)d\lambda \\
&= G_1(f)\otimes G_2(f).
\end{aligned}
$$

11.10 Tables of Transforms

Table A6.2 Fourier-Transform Pairs.

Time Function	Fourier Transform
$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}(fT)$
$\mathrm{sinc}(2Wt)$	$\frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), \quad a>0$	$\frac{1}{a+j2\pi f}$
$\exp(-a\lvert t\rvert), \quad a>0$	$\frac{2a}{a^2+(2\pi f)^2}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$\begin{cases} 1-\frac{\lvert t\rvert}{T}, & \lvert t\rvert<T \\ 0, & \lvert t\rvert\geq T \end{cases}$	$T\,\mathrm{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f-f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)]$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\,\mathrm{sgn}(f)$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\sum_{i=-\infty}^{\infty}\delta(t-iT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)$

Summary of Chapter 11

- □ Every periodic function $f(t)$ of period T can be represented by **a Fourier series**, which is of the form

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}\right]$$

with

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi nt}{T}dt, \qquad n = 1,\ 2,\cdots$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi nt}{T}dt, \qquad n = 1,\ 2,\cdots$$

- □ If $f(t)$ is even, the series reduces to Fourier cosine series and if $f(t)$ is odd, the series reduces to Fourier sine series.

- □ If $f(t)$ is given for $0 \le t \le L$ only, it has two **half-range expansions** of period $2L$, namely, a cosine and a sine series.

- □ The **orthogonality** of the trigonometric system:

$$\int_0^T f(t)g(t)\,dt = 0 \Rightarrow \text{ orthogonal}$$

$$\int_{-\pi}^{\pi} \cos mt \cdot \cos nt dt = 0,\ \text{for } m \ne n$$

$$\int_{-\pi}^{\pi} \sin mt \cdot \sin nt dt = 0,\ \text{for } m \ne n$$

$$\int_{-\pi}^{\pi} \cos mt \cdot \sin nt dt = 0,\ \text{for all } m,\ n$$

- □ Ideas and techniques of Fourier series extend to a non-periodic function $f(t)$, which lead to the **Fourier integral**

$$f(t)=\int_0^{\infty}\left[A(\omega)\cos\omega t+B(\omega)\sin\omega t\right]d\omega$$

with

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\cos\omega v dv,\ \ B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\sin\omega v dv.$$

- □ The Fourier integral representation can be used to obtain the Fourier transform pair of a non-periodic function:

$$F\{f(t)\}=\int_{-\infty}^{\infty}f(t)e^{-j2\pi ft}dt,$$

$$f(t)=\int_{-\infty}^{\infty}F\{f(t)\}e^{j2\pi ft}df$$

- □ Finally, various theorems on Fourier transform have been studied.