Normalized Cuts and Image Segmentation
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Image segmentation: the problem

Image segmentation: partition digital image into multiple segments

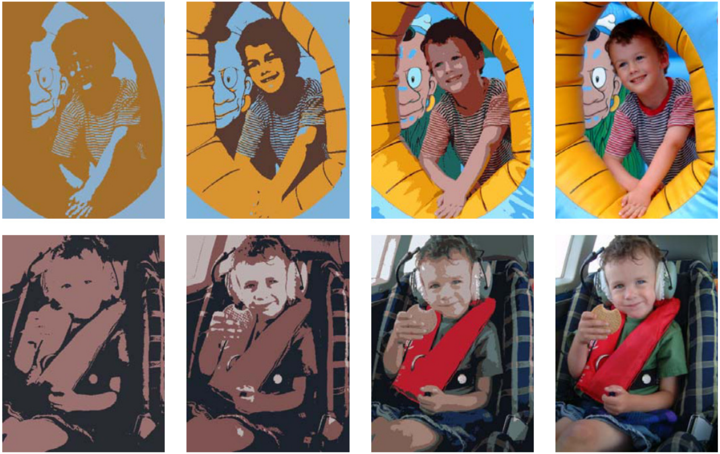

Image credit: Christopher Bishop

Image segmentation: applications

Image segmentation empowers

- Autonomous driving
- Object detection
- Video surveillance
- ...

Image credit: Tingwu (Wilson) Wang

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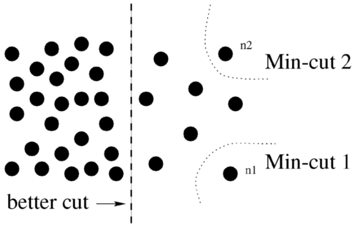
Graph partitioning

Graph partitioning: $G(V, E) \rightarrow A, B, A \cup B = V \ \& \ A \cap B = \phi$
Dissimilarity between A and B: *cut*

$$\text{cut}(A, B) = \sum_{u \in A, v \in B} w(u, v)$$

A first attempt: [Wu & Leahy, 1993] use min-cut to partition graph
Problem with the approach: small sets of isolated nodes



Min-cut can give bad solution; assuming weights invers prop. to distance

Graph partitioning: proposed approach

This work, [Shi & Malik] propose: *normalized cut*
Define a measure: association

$$\mathrm{assoc}(A, V) := \sum_{u \in A, t \in V} w(u, t)$$

Define a new criterion

$$\mathrm{Ncut}(A, B) := \frac{\mathrm{cut}(A, B)}{\mathrm{assoc}(A, V)} + \frac{\mathrm{cut}(A, B)}{\mathrm{assoc}(B, V)}$$

$\rightarrow$ minimizing $\mathrm{Ncut}(A, B)$ encourages A and B to be large

Min-Ncut as an eigenvalue problem

Let $\mathbf{x}$ be an indicator vector: $\mathbf{x}_i = 1$ if $i \in A$, $\mathbf{x}_i = 0$ if $i \in B$

$$\mathrm{cut}(A, B) = \sum_{\mathbf{x}_i = 1, \mathbf{x}_j = 0} w_{ij} = \mathbf{x}^T \mathbf{L} \mathbf{x}$$

because $\mathbf{x}^T \mathbf{L} \mathbf{x} = \sum_{ij} w_{ij} |\mathbf{x}_i - \mathbf{x}_j|$

$$\mathrm{assoc}(A, V) = \sum_{\mathbf{x}_i = 1} d_i = \mathbf{x}^T \mathbf{W} \mathbf{1} = \mathbf{x}^T \mathbf{D} \mathbf{1}$$

$$\mathrm{assoc}(B, V) = \sum_{\mathbf{x}_j = 0} d_j = (\mathbf{1} - \mathbf{x})^T \mathbf{W} \mathbf{1} = (\mathbf{1} - \mathbf{x})^T \mathbf{D} \mathbf{1}$$

$\mathbf{d}_i = \sum_j w_{ij}, \quad \mathbf{D} = \mathrm{diag}(\mathbf{d})$

$$\min\ \mathrm{Ncut}(A, B) \Leftrightarrow \min_{\mathbf{x}_i \in \{0,1\}} \frac{\mathbf{x}^T \mathbf{L} \mathbf{x}}{\mathbf{x}^T \mathbf{D} \mathbf{1}} + \frac{\mathbf{x}^T \mathbf{L} \mathbf{x}}{(1 - \mathbf{x})^T \mathbf{D} \mathbf{1}}$$

Min-Ncut as an eigenvalue problem

Define $b = \frac{\text{assoc}(A,V)}{\text{assoc}(B,V)} = \frac{\mathbf{x}^\top \mathbf{D1}}{(1-\mathbf{x})^\top \mathbf{D1}}$, $\mathbf{y} = \mathbf{x} - b(1-\mathbf{x})$

$$\min_{\mathbf{y}_i \in \{0,-b\}} \frac{\mathbf{y}^\top \mathbf{L}\mathbf{y}}{\mathbf{y}^\top \mathbf{D}\mathbf{y}}$$
$$\text{s.t. } \mathbf{y}^\top \mathbf{D1} = 0$$

Change variable: $\mathbf{z} = \mathbf{D}^{1/2}\mathbf{y}$, relax

$$\min_{\mathbf{z}} \frac{\mathbf{z}^\top \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}\mathbf{z}}{\mathbf{z}^\top \mathbf{z}}$$
$$\text{s.t. } \mathbf{z}^\top \mathbf{D}^{1/2}\mathbf{1} = 0$$

- $\mathbf{z}_1 = \mathbf{D}^{1/2}\mathbf{1} \rightarrow$ smallest eigvec of $\mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$
- Constraint $\mathbf{z}^\top \mathbf{D}^{1/2}\mathbf{1} = 0 \rightarrow$ solution is $\mathbf{z}_2$

Binarize the result $\mathbf{y}$ by grid-searching a threshold

Extension to multi-segments

Recursive two-way Ncut

- Repeat bi-partitioning on the resulted segments

Simultaneous K-way partition

- Use several smallest eigenvectors
- Run K-means clustering algorithm to get partitioning

Transform images to graphs

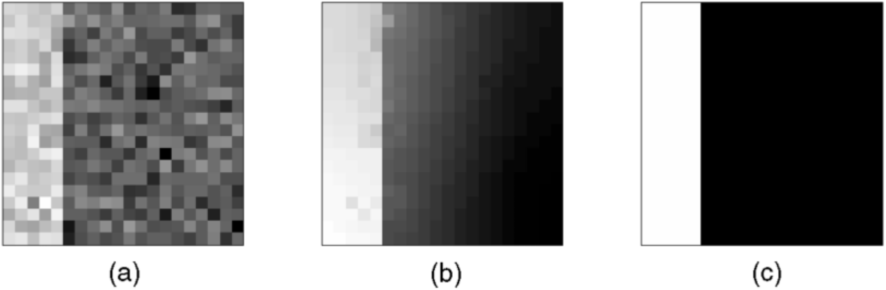
Pixels $\rightarrow$ graph nodes
Define edge weights

$$w_{ij} = e^{\frac{-\|F_i - F_j\|_2^2}{\sigma_I^2}} \times \begin{cases} e^{\frac{-\|X_i - X_j\|_2^2}{\sigma_X^2}} & \text{if } \|X_i - X_j\|_2 < r \\ 0 & \text{otherwise} \end{cases}$$

F_i: feature value (e.g. brightness); X_i: spatial position

- Number of nodes can be large, e.g. 1M for 1000×1000
- $\boldsymbol{W}$ is highly sparse

Results

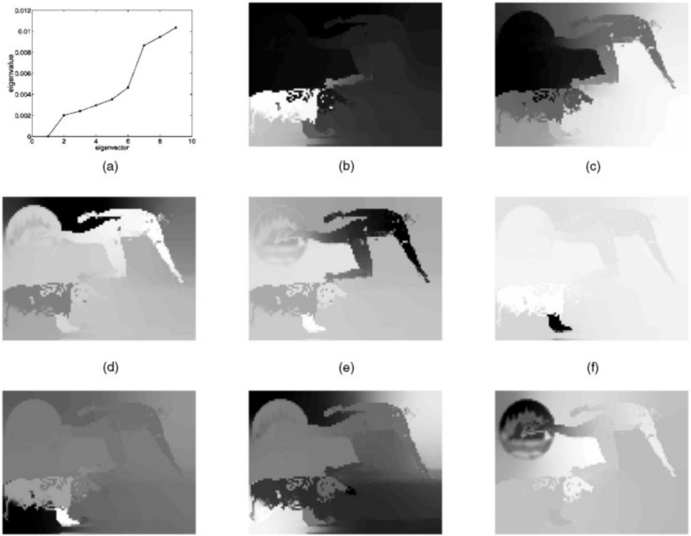


(a) Image, (b) Second smallest eigenvector, (c) partition

Results (cont.)

A image to segment

Results (cont.)



The smallest eigenvectors

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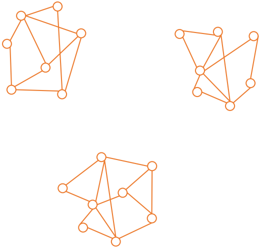
Spectral Clustering: Overview

The recipe

1. A collection of data samples $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$
2. *Construct an undirected graph, calculate similarity* $\mathbf{W}$

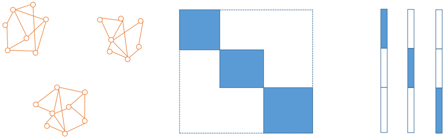


3. Construct a graph Laplacian, e.g. $\mathbf{L} = \mathbf{D} - \mathbf{W}$
4. Perform K-means clustering on smallest eigenvectors of $\mathbf{L}$

Spectral Clustering: Details

Facts of graph Laplacian

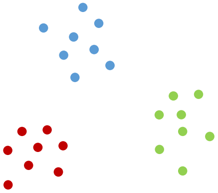
- $\mathbf{L}$ is PSD, $\lambda_n \geqslant \lambda_{n-1} \geqslant \cdots \lambda_1 \geqslant 0$
- The smallest eigenvalue of $\mathbf{L}$ is 0, corresponding eigenvector is $\mathbf{1}$, since $\mathbf{L1} = \mathbf{D1} - \mathbf{W1} = 0 \times \mathbf{1}$



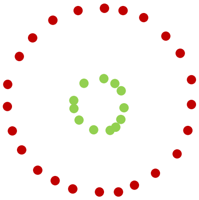
- Algebraic multi. of eigenvalue $0 = \#$ connected components + 1
- Easy to perform clustering on the rows of eigenvectors corresponding to the small eigenvalues

Spectral Clustering: Summary

Compare to K-means



K-means ✓
Spectral ✓

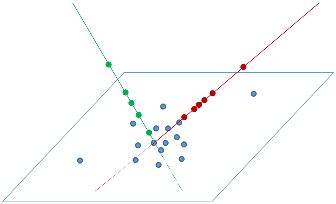


K-means ×
Spectral (kNN)✓

Spectral clustering is general

- Can construct many clustering algorithms, if one can define a "good" similarity graph
- Can work with any data object, e.g. images, text, etc.

The subspace clustering problem



- Separate data according to their *subspace membership*
- Question: how to construct the graph?

References

- Shi J., Malik J. . "Normalized cuts and image segmentation." IEEE Transactions on Pattern Analysis and Machine Intelligence, 2000.
- Von Luxburg, U. "A tutorial on spectral clustering." Statistics and Computing, 2007
- Elhamifar E., Vidal R. Sparse subspace clustering: Algorithm, theory, and applications. IEEE Transactions on Pattern Analysis and Machine Intelligence, 2013.