Convex Functions and Optimization Methods on Riemannian Manifolds

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Convex Functions and Optimization Methods on Riemannian Manifolds

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To Aneta, Daniel and Sorin

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PREFACE

The object of this book is to present the basic facts of convex functions, standard dynamical systems, descent numerical algorithms and some computer programs on Riemannian manifolds in a form suitable for applied mathematicians, scientists and engineers. It contains mathematical information on these subjects and applications distributed in seven chapters whose topics are close to my own areas of research: Metric properties of Riemannian manifolds, First and second variations of the p-energy of a curve; Convex functions on Riemannian manifolds; Geometric examples of convex functions; Flows, convexity and energies; Semidefinite Hessians and applications; Minimization of functions on Riemannian manifolds. All the numerical algorithms, computer programs and the appendices (Riemannian convexity of functions $f:\mathbb{R} \longrightarrow \mathbb{R}$, Descent methods on the Poincaré plane, Descent methods on the sphere, Completeness and convexity on Finsler manifolds) constitute an attempt to make accesible to all users of this book some basic computational techniques and implementation of geometric structures. To further aid the readers,this book also contains a part of the folklore about Riemannian geometry, convex functions and dynamical systems because it is unfortunately "nowhere" to be found in the same context; existing textbooks on convex functions on Euclidean spaces or on dynamical systems do not mention what happens in Riemannian geometry, while the papers dealing with Riemannian manifolds usually avoid discussing elementary facts.

Usually a convex function on a Riemannian manifold is a real-valued function whose restriction to every geodesic arc is convex. When we refer to a subset A of a Riemannian manifold, this definition of the convexity of $f:A \longrightarrow \mathbb{R}$ requires a definition of the convexity of the subset A. Only for a C^2 function it is possible to give a generalized definition which does not require the presence of geodesics: a C^2 function is convex if Hess f is positive semidefinite. This coordinate-free description of convexity can be easily connected with symbolic computation.

Convex functions occur abundantly, have many structural implications on manifolds and form an important link in the modern analysis and geometry. These implications do not occur in the extensive theory of convex functions on Euclidean spaces because of the particularity of these spaces.

From 1964 new insights have been gained in old problems combined with new ones, and great coherence has been achieved in understanding the role of the Riemannian convexity in science.For example, Gordon has found two applications of convexity to mechanics surprisingly opposite in consequence: a trajectory cannot stay in a compact domain supporting a function whose Hessian with respect to the Jacobi metric is positive definite; and (much deeper) if a potential function has positive semidefinite Hessian which is positive definite on a geodesic through a minimum point, then every neighborhood of that point has nontrivial closed trajectories. Also some recent papers suggest that a reasonable general mathematical approach to thermodynamics, which has not yet been given, will involve ideas from differential-geometric facts concerning extrema, convexity and dynamical systems. Therefore our goal to answer questions about specific dynamical systems on Riemannian manifolds, and interactions between numerical computation, vizualization and mathematical theory of the orbits, geodesics and optimization is justified .

An optimization (minimization or maximization) problem is specified by a set C, called the feasible set, and a real-valued function f on C, called the objective function. Often the feasible set C is a Riemannian manifold described by equality constraints and inequality constraints on $\mathbb{R}^n$,and the optimization problem resides in the optimal value of f (a number), or the optimal solutions (distinguished elements of the feasible set), or both, depending on circumstances.

The concept of convexity plays a very important part in the theory of optimization, firstly because many objective functions are convex in a sufficiently small neighborhood of a local minimum point, and secondly because the convergence of numerical methods for estimating local minimum points can be established for convex objective functions. The numerical methods and the computer programs presented in this book for the finding of a critical point of a function are intimately related to the Riemannian structure of the manifold and are independent of the choice of coordinate systems.

The topics of this book have been selected in order to give the readers a feeling for the way in which the theory of convex functions on Riemannian manifolds has developed and is developing, and to make available a battery of procedures which can be used to solve real-life problems in a resonable amount of time. Consequently we hope that our book will give the reader an initial perspective on this subject and make it easier for him to approach the specialized literature.

From the point of view of the mathematical expression, one has considered that accessible reassessments are more useful than maintaining of a tight language, accepting that mathematics is not an isolated intellectual structure but a part of the general process of scientific modelling. In this sense one has prefered the topics and notations that do not embed the mathematical drafts into a lot of unessential data, and the technique of begining each chapter by a short introduction which suggests the contents and the importance of the respective chapter.

Many mathematicians have asked me why do we study the convexity on Riemanian manifolds. Is it not enough to study the convexity on Euclidean spaces ? Has this theory any applications which justify it ? We consider that the present work contains the necessary reasons and the clear answer that in fact the theory of convex functions reveals all its power and consequences only when it is conceived on a Riemannian structure.

The book will be of interest especially to applied mathematicians (working in convexity, differential geometry, dynamical systems, optimization and numerical methods), to scientists and engineers as well. By virtue of the elementary nature of the analytical tools, it can also be used as a text for undergraduate and graduate students with a good background in Riemannian Geometry and Analysis.

Bucharest, Romania, 1994 Constantin Udriste

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In time, first there are the geometers who have accepted discussions on the subject at certain Conferences and Seminaries on Differential Geometry, where I was a participant or organiser. Their suggestions have been invaluable. Second I have benefitted from the criticisms of a reviewer of USA who helped me to overcome the difficult task of writing and re-writing. Third, there are Dr. Vladimir Balan and Dr. Mihai Postolache in our Department of Mathematics who have designed the computer programs in this book and Mrs.Monica Ionescu in Department of Foreign Languages, who has overseen the entire manuscript from the linguistic point of view. Fourth, a very careful reading of the text by Dr. Vladimir Balan led to the elimination of misprints and other errors. Fifth, I thank to Mrs. Camelia Rosca who has typed the manuscript.

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