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Concerning the Applicability of Geometric Models to Similarity Data: The Interrelationship Between Similarity and Spatial Density

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In a recent article, Tversky questioned the application of geometric models to similarity data and proposed an alternative set-theoretic approach. He suggested that geometric models are inappropriate because the similarity data may violate the metric assumptions underlying such models. In addition, he demonstrated that the stimulus context and the nature of the experimental task can affect the similarity relations. The present article suggests that a geometric approach may be compatible with these effects if the traditional multidimensional scaling model is augmented by the assumption that spatial density in the configuration has an effect on the similarity measure. A distance–density model is outlined that assumes that similarity is a function of both interpoint distance and the spatial density of other stimulus points in the surrounding region of the metric space. The proposed relationship between similarity and spatial density is supported by empirical evidence. The distance–density model is shown to be able to account for violations of the metric axioms and certain context and task effects. A number of other issues are discussed with respect to geometric and set-theoretic models of similarity.

The concept of similarity has played a central role in a number of diverse areas of psychological investigation. One important approach to the problem of analyzing similarity has been multidimensional scaling (Shepard, 1962a, 1962b; Torgerson, 1958). This approach represents the similarity relations between objects in terms of a geometric model that consists of a set of points embedded in a dimensionally organized metric space, where the points correspond to the objects under consideration. The central assumption of this type of model is that the similarity data can be related by a linear or monotonic decreasing function to the interpoint distances in the metric space, that is, the larger the measure of similarity between two objects, the smaller the distance between the corresponding points in the metric space.

In a recent article, Tversky (1977) questioned both the metric and dimensional assumptions underlying geometric representations of similarity data and proposed an alternative set-theoretic approach, called the *feature matching model*. In this model, the similarity between two objects is expressed as a linear combination of the measures of the common and distinctive features of the two objects. When additional assumptions are made concerning the parameters of the model, the feature matching model is shown to be able to account for asymmetric similarity measures, certain effects of stimulus context on similarity, and discrepancies between similarity and difference judgments.

The present article suggests that some of the objections to geometric models raised by Tversky (1977) may be met if the traditional geometric model is augmented by the as-

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sumption that the similarity between objects is a function not only of interpoint distance in a metric space but also the spatial density of points in the surrounding configuration. A distance–density model, incorporating this assumption, is outlined and is shown to account for a number of effects that pose difficulties for the traditional multidimensional scaling model. The purpose of this article is not to argue that either the geometric or the set-theoretic approach is better able to account for similarity data in general. The choice between models would undoubtedly be influenced by the kind of objects under consideration, the investigator's reasons for applying a theoretical model of similarity, and possibly other factors. Rather, the discussion attempts to define more precisely some of the issues raised by Tversky and to clarify to what extent they pose real difficulties for geometric models.

Distance–Density Model

In this section, a general model will be proposed in which similarity is assumed to be a function of both interpoint distance and the density of stimulus points in a dimensionally organized metric space. The notion that spatial density may affect similarity measures is suggested by the work of Parducci and others (Birnbaum, 1974; Parducci, 1963, 1965, 1973; Parducci & Marshall, 1961; Parducci & Perrett, 1971) using categorical judgments on unidimensional stimuli. These investigators find that judges tend to employ the alternative categories with equal frequency. This finding implies that within dense subregions of the stimulus range, finer discriminations are made than within relatively less dense subregions. If the same principle applies to similarity data, two points in a relatively dense region of a stimulus space would have a smaller similarity measure than two points of equal interpoint distance but located in a less dense region of the space. This is the central assumption of the distance–density model. Empirical evidence in support of the proposed relationship between similarity and spatial density will be given in later sections of this article.

The traditional multidimensional scaling model (Shepard, 1962a, 1962b) assumes that the observed measure of similarity can be related by a monotonic decreasing function to interpoint distance in a metric space, that is, there exists a monotonic decreasing function, f, so that

$$s(x, y) = f[d(x, y)], \qquad (1)$$

where $s(x, y)$ denotes the observed similarity between x and y, and $d(x, y)$ denotes the distance between the corresponding points in the stimulus configuration. The basic relation between similarity and distance will be modified in the distance–density model by introducing a second distance function, $\bar{d}(x, y)$, which depends on both interpoint distance in the configuration and some measure of spatial density in the regions surrounding the points x and y. One possible form that this modified distance function, $\bar{d}(x, y)$, might take is

$$\bar{d}(x, y) = d(x, y) + \alpha\delta(x) + \beta\delta(y), \qquad (2)$$

where $d(x, y)$ is the interpoint distance, $\delta(x)$ and $\delta(y)$ are measures of spatial density in neighborhoods of x and y, and α and β are constants that reflect the relative weight given the densities $\delta(x)$ and $\delta(y)$. Finally, it is assumed, as in traditional geometric models, that the observed similarity measure $s(x, y)$ is related by a monotonic decreasing function, $\bar{f}$, to the modified distance function $\bar{d}(x, y)$, that is,

$$s(x, y) = \bar{f}[\bar{d}(x, y)]. \qquad (3)$$

The density measure $\delta(x)$ associates to each point in the spatial configuration a non-negative value that measures the density of points within the surrounding region. This density measure may be estimated in a variety of ways. Certain similarity measures, such as those derived from *same–different* discriminations, stimulus–response confusions, and under some conditions co-occurrence, yield a value of the similarity between an object and itself, $s(x, x)$. Evidence that the observed measure of self-similarity is related to spatial density in a region surrounding the corresponding point in the multidimensional configuration is given in the next section. This evidence suggests that it may be possible to estimate the spatial density function from the diagonal entries in the similarity matrix when these entries are available, that is,

$\delta(x)$ may be given by

$$\hat{\delta}(x) = g[s(x, x)], \quad (4)$$

where $s(x, x)$ is the observed diagonal entry, and g is some nonnegative monotonic decreasing function. If, in addition, it is assumed that the function that governs the relationship between the distance measure $\bar{d}$ and the observed measure of self-similarity $s(x, x)$ is the same function $\bar{f}$ that relates interobject distance to interobject similarity, then $\delta(x)$ can be written as

$$\hat{\delta}(x) = \frac{\bar{f}^{-1}[s(x, x)]}{(\alpha + \beta)}. \quad (5)$$

An alternative approach to determining the value of the density function $\delta(x)$ that does not depend on observed self-similarity is the computation of a measure of spatial density directly from the stimulus configuration. For example, $\delta(x)$ might be given by

$$\hat{\delta}(x) = \sum_{p \epsilon S} h[d(p, x)]$$

or $\quad (6)$

$$\int_S h[d(p, x)]dp,$$

where S is the stimulus domain under consideration, h is some monotonic decreasing function, and the two expressions correspond to discrete and continuous domains, respectively.[1] The stipulation that h be monotonic decreasing has the consequence that points near x add heavily to the measure of density relative to points far from x. Possible forms of the function h might be $h(d) = d^{-k}$ or $h(d) = \exp(-kd)$. Another possible expression for density in discrete stimulus domains, based on the number of points within a fixed radius, is

$$\hat{\delta}(x) = \sum_{p \epsilon S} i(p, x),$$

where

$$i(p, x) = \begin{cases} 1 \text{ if } d(p, x) \leq r \\ 0 \text{ if } d(p, x) > r \end{cases}. \quad (7)$$

Here, r is a fixed radius; for computational simplicity, it is this measure of density that will be used throughout this article.

The choice of the form of the function $\bar{d}(x, y)$ as a linear combination of the interpoint distance $d(x, y)$ and the spatial densities $\delta(x)$ and $\delta(y)$ was based on the description of the range–frequency theory given by Birnbaum (1974). In terms of the range–frequency theory, the judgment function is assumed to be a linear combination of the cumulative density function on the stimulus dimension and the psychophysical function. It follows that the difference between responses to two different objects, which can be taken as a measure of interobject distance, takes the form of a linear function of the distance between the objects on the psychological continuum and the density of stimuli lying between the two objects on the continuum. The distance–density model proposed here, then, is closely related to the range–frequency theory for unidimensional stimuli.

The modified distance function $\bar{d}(x, y)$ need not satisfy the metric axioms. The distance between a point and itself, $\bar{d}(x, x) = (\alpha + \beta)\delta(x)$, will in general be greater than zero and will depend on the spatial density of points surrounding the point x, so the minimality axiom will not hold in general. In addition, the symmetry axiom need not hold. Explicitly, $\bar{d}(x, y)$ and $\bar{d}(y, x)$ will be equal if and only if $\alpha = \beta$ or $\delta(x) = \delta(y)$. In a directional similarity task, greater emphasis may be placed on one member of the object pair than on the other (Tversky, 1977). In terms of the distance–density model, this emphasis may be reflected in the weights α and β, with the spatial density surrounding one point affecting the distance measure $\bar{d}$ more than the density surrounding the second point. If $\alpha > \beta$, then $d(x, y) > \bar{d}(y, x)$ if and only if $\delta(x) > \delta(y)$; some empirical results supporting this relation will be given later. However, the triangle inequality axiom must hold for $\bar{d}$. Since the modified distance function

[1] For discrete unidimensional stimulus domains, the measure of density given in Equation 6 is in some sense inversely related to Murdock's (1960) measure of distinctiveness, which expresses distinctiveness as the sum of the distances between the stimulus and the other stimuli along the psychological scale. Murdock's measure, however, differs from the density measure proposed in Equation 6 in one important respect. While Murdock's measure implies that the stimuli at the extremes of the range are necessarily more distinctive than any other stimuli in the set, it need not be the case that the densities of extreme points are less than the densities of interior points; although, in general, this will tend to be true for homogeneously distributed stimulus domains.

$\bar{d}$ is assumed to be a linear combination of interpoint distance and spatial density, $\bar{d}(x, y) + \bar{d}(y, z) \geq \bar{d}(x, z)$.

Metric Assumptions: Minimality

Prototypicality and Distinctiveness

In this section, the possibility that certain objects within a stimulus set may be more prototypical or central or more distinctive or extreme than others will be discussed, together with possible implications for similarity models. A number of investigators have noted the special status of certain elements or objects, and the work in this area will only be briefly reviewed here. The idea that particular perceptual stimuli serve a special function as "ideal types" was suggested by Wertheimer (1938). Goldmeier (1936/1972) noted the special role of certain properties (in particular, symmetry, perpendicularity, and parallelism) and suggested that objects possessing these properties have special status within sets of otherwise similar stimuli. Garner (1962, 1966) investigated the notion that patterns vary in terms of "goodness." Primarily using geometric stimuli, he found that rated pattern goodness correlated negatively with the number of different patterns that could be generated by rotating and reflecting the stimulus. Rosch (1975a) showed that in such stimulus domains as colors, lines differing in orientation, and numbers, certain elements of these sets, namely, focal colors, horizontal and vertical lines, and multiples of 10, play the role of "reference points." These reference points are those objects to which other objects in the domain are seen "in relation to." Typicality of category members has been investigated in numerous studies (Rips, Shoben, & Smith, 1973; Rosch, 1974, 1975b, 1975c; Smith, Shoben, & Rips, 1974) and has variously been related to similarity to a prototype (Garner, 1962; Posner, Goldsmith, & Welton, 1967; Posner & Keele, 1968; Rosch, Simpson, & Miller, 1976), closeness to the average values of category attributes (Reed, 1972; Rips et al., 1973; Rosch et al., 1976), and degree of family resemblance (overlap of features) among category members (Rosch et al., 1976; Rosch & Mervis, 1975). These structural descriptions of typicality suggest that special status is associated with objects that are in some sense central to the stimulus domain or have the greatest number of features in common with the other objects in the domain. On the other hand, it may be that certain objects in the domain have special status in that they are particularly salient or distinctive. Objects that are unusual or extreme in terms of their features or values along underlying dimensions may be distinctive in this way. For example, Durlach and Braida (1969) and Weber, Green, and Luce (1977) have found higher accuracy for absolute identification of tone intensity for tones near the extremes of the intensity range. The distinction suggested here between prototypicality or centrality on the one hand and distinctiveness or extremality on the other hand may be reflected in similarity judgments about these objects, as will be discussed later.

Given the wide variety of evidence that within categories certain elements have special status, a question of interest is whether such structure affects measures of similarity. Similarity data have been derived using a number of different tasks, some of which yield a measure of how similar an object is to itself. If this measure of self-similarity is found to vary from object to object within a stimulus domain, the variation may be related to the prototypicality or the distinctiveness of the objects. Although this hypothesis has not been systematically investigated, there is some supporting evidence from earlier studies. Rothkopf (1957) used a *same–different* task to investigate similarities between Morse code signals. In this study, more correct *same* judgments were found for particularly simple signals, such as those consisting of a single dash or dot or two dashes or two dots, than for more complex stimuli. Attneave (1950) found in a learning paradigm that the correct response was given more frequently to the objects at the extremes of the ranges of stimulus parameters employed (rectangle size and reflectance and triangle size and angle) than to objects with intermediate values along these dimensions. A similar effect was also found in a study of stimulus–response errors by Shepard (1957) and in studies of absolute identification of tones varying in intensity (Durlach & Braida, 1969; Weber et al., 1977). Recently, Balzano (1977, Experiments 1, 2, and 3) found shorter correct *same* latencies for musical intervals that play a central or important role in musical composition, such as perfect fourths, fifths, octaves, and major thirds, than for other intervals.

Although the evidence for the hypothesis that the observed measure of similarity between an object and itself is related to the status of the object within the domain is sketchy, at least there is some support from a variety of different tasks and stimulus domains. How, then, might models of similarity account for variation in self-similarity? This issue will be discussed in the next two subsections with respect to Tversky's feature matching model and geometric models of similarity.

Feature Matching Model

The feature matching model proposed by Tversky (1977) assumes that the similarity between two objects is a linear combination of the measures of the features shared by the two objects and the features associated with one of the objects but not the other. Shared features are assumed to add to, and distinctive features are assumed to subtract from, the overall similarity between two objects. When applied to the special case of a single object, this model reduces to the statement that the similarity of an object to itself is related to the measure of the features possessed by the object. The model predicts that the similarity between an object and itself should be an increasing function of the number of known features or of the salience of the features. It seems possible, for example, that certain countries, such as the United States, would have a greater number of features or more salient features than other countries, such as Belgium. Similarly, some block letters, for example, E, have more features than other letters, for example, C. For geometric forms, the measure of the features of "good" figures may be relatively large since these figures have the additional salient feature of symmetry, parallelism, or perpendicularity. Although variation in measured self-similarity is consistent with the feature matching model, Tversky does not investigate the relationship between self-similarity and the measure of the stimulus features.

Geometric Models of Similarity

Traditional geometric similarity models assume that the similarity between objects can be represented by interpoint distance in some dimensionally organized metric space. By definition, a metric space must satisfy the minimality axiom, namely, that the distance between a point and itself must be zero and less than the distance between any two distinct points. In terms of similarities, this means that the similarity between an object and itself must be larger than the similarity between any two different objects. However, this constraint does not always hold in similarity data, that is, it may happen that some off-diagonal entries in the matrix exceed some diagonal entries. Although cases like this do violate the minimality axiom and may raise questions about the applicability of a simple geometric model to the similarity data, the minimality axiom does not necessarily imply that the measure of self-similarity needs to be the same for all objects. For nonmetric multidimensional scaling models, the relation between similarity and distance is assumed to be a monotonic decreasing function. Since the function does not need to be strictly monotonic decreasing, the model is compatible with similarity data in which the similarity between an object and itself varies from object to object as long as the off-diagonal entries do not exceed the diagonal entries.

Even though it might be argued that variations in diagonal entries in a similarity matrix merely represent noise in the data and that all objects are really equally similar to themselves, it is interesting to consider the possibility that self-similarity is related to spatial density in the configuration. For example, large self-similarity values may be associated with particularly distinctive objects, that is, those objects that lie at the extremes of the stimulus range or are otherwise unusual in the object domain. This suggests the hypothesis that those objects that are most similar to themselves lie at the boundary of the stimulus configuration or occupy otherwise less dense regions of the space.

The proposed relationship between the similarity of an object to itself and the density of points in a neighborhood of the corresponding point in the geometric representation can be tested when the measure of similarity is derived from stimulus–response errors, or *same–different* latencies or error rates. By necessity, when the similarity measure is based on stimulus–response errors, a stimulus that is frequently associated with the correct response will infrequently be associated with incorrect responses. Stated in terms of similarity, an object that is very similar to itself cannot also be very similar to other objects (Shepard, 1957). However, as a consequence of other constraints in the similarity data, it need not be the case that a multidimensional solution would be able to locate high self-similarity stimuli in less dense regions of the object space. In addition, similarity data based on *same–different* responses would not be subject to the trade-off between self-similarity and similarity to other objects.

The correspondence suggested here between the observed measure of self-similarity and the spatial density in the region surrounding the stimulus point is reminiscent of the correspondence between how well an object can be identified and how discriminable it is from other objects in the domain. A number of studies reviewed by Smith (1968) have shown that discriminative reaction times increase as the similarity between objects in the domain increases (Bindra, Donderi, & Nishisato, 1968; Crossman, 1955; Chase & Posner, Note 1). It might be argued that the same relationship would hold for *same–different* error data, since an object with few similar alternatives may be correctly identified even when information about its features or properties is incomplete or somewhat inaccurate. The result would be few errors on *same* trials for these stimuli. What is being suggested here is that this factor of discriminability may operate locally within a stimulus domain, that is, if an object has few close neighbors, then it will have a relatively large measure of self-similarity.

Those available studies having both on-diagonal similarity measures and multidimensional scaling solutions were considered in order to test the proposed relationship. Unfortunately, only a few studies have met these requirements, but the results seem promising. The *same–different* error data for Morse code signals collected by Rothkopf (1957) were analyzed by Shepard (1963) using the nonmetric multidimensional scaling technique. For these data, a significant negative correlation was found between the percentage of correct responses on *same* trials and the number of objects within a certain fixed radius of the corresponding point in the scaling solution. Two different radii were somewhat arbitrarily chosen, roughly corresponding to the distance between the letters R and U and the distance between R and G in the published solution; the correlations were significant in both cases ($r = -.54$, $r = -.66$; $p < .01$ for both). The stimuli with few close neighbors tended to be signals consisting of a few dots and dashes or longer signals that consisted of a homogeneous string of either dots or dashes. In addition, Rothkopf (1958) collected learning data for a subset of 12 Morse code signals, which were also scaled by Shepard (1963). For these data, there was a negative relationship found between the number of correct responses and the number of points within a fixed radius in the stimulus configuration ($r = -.95$, $p < .01$). Here, the radius corresponded approximately to the distance between the letters S and H in the solution.

As mentioned earlier, in the learning data collected by Attneave (1950), the stimuli with the fewest errors were those stimuli that lay at the extremes of the ranges of the dimensions used to generate the stimulus set. Although multidimensional scaling was not applied to these data, the stimuli that were extreme in terms of the physical parameters would probably also fall at the extremes of the multidimensional solution were such an analysis performed. Since extreme points would in general have fewer close neighbors than points falling at interior positions, these data are consistent with the proposed relationship. The same argument would apply to the data from other studies showing an advantage for extreme stimuli (Durlach & Braida, 1969; Shepard, 1957; Weber et al., 1977).

Finally, Balzano (1977) collected *same–different* reaction times for musical intervals. The latency data from Experiment 1 and

Experiments 2 and 3 combined were correlated with the number of objects within a radius corresponding to the distance between the tritone and the major sixth in the two-dimensional solutions. In both cases, there was a significant positive correlation between latency and the number of other points within the fixed radius (rs = .68 and .64; $p < .05$ for both).

The data from these studies, then, support the hypothesis that objects with large measures of self-similarity are located in relatively less dense regions of the multidimensional scaling solutions. In this way, the distance–density model may be able to account for variation in the diagonal entries in a similarity matrix. In addition, these findings also support the idea that spatial density, like dimensional organization and clustering patterns, is an important feature of geometric representations.

Metric Axioms: Symmetry

Empirical Evidence for Asymmetry

Asymmetries in similarity data may arise when the similarity task is in some way directional, that is, places the two objects to be compared in different roles. This is the case when subjects judge how similar one object, a, is to another object, b. Object a takes the role of subject of the sentence, while b takes the role of referent. Or, in a *same–different* task, one stimulus may precede the other stimulus temporally. Asymmetries might also be found in confusion data. For example, b might be given as the response to a more frequently than a is given as the response to b.

Tversky (1977) systematically investigated asymmetries in similarity judgments using two different directional similarity tasks. In one task, subjects judged for each object pair (a, b) which of the following two similarity statements was preferred: "a is similar to b" or "b is similar to a." In a second task, subjects gave numerical judgments of how similar one object, a, was to a second object, b. Using countries as stimuli, Tversky found that the sentence in which the more prominent stimulus took the role of the referent was preferred to the sentence in which the more prominent stimulus took the role of the subject. In terms of similarity judgments, the less prominent country was seen as more similar to the more prominent country than the more prominent country was seen to the less prominent country. Similar results were found for geometric forms, where good forms were seen as more prominent. In another experiment, Tversky used letters in a *same–different* task in which one of the letters, called the *standard*, was known to the subject in advance of the trial and always appeared in a given spatial position. He found that the proportion of incorrect *same* responses was larger when the features of the unknown letter were a subset of the features of the standard letter than when the opposite relation held. Tversky also analyzed Rothkopf's (1957) *same–different* error data on Morse code signals and found that on *different* trials, more incorrect *same* responses occurred when the first signal had fewer components than the second signal, particularly when the first signal was a proper subset of the second signal.

Rosch (1975a) found that subjects tend to place reference point stimuli (focal colors, horizontal and vertical line segments, and multiples of 10) in the referent position of such sentence frames as "— is essentially —" and "— is virtually —" more often than stimuli that were not reference point stimuli (nonfocal colors, oblique line segments, and numbers not even multiples of 10). Also, Rosch had subjects place a stimulus in physical space to represent its psychological distance from a stimulus in a fixed position. The measured distances were smaller when the fixed position contained a reference point stimulus than when it contained a nonreference stimulus. Garner (1966; Handel & Garner, 1965) found that good patterns were selected as associates of less good patterns more frequently than less good patterns were given as associates of good patterns. Finally, Rips (1975) conducted a study in which each subject was told that a hypothetical island was populated by eight different species of animals and that all of the animals in one of the species had a contagious disease. The subject then estimated the percentage of animals in each of the remaining species that had also contracted the disease. If the species with the disease was more typical, subjects produced a higher estimate for an atypical species than if the atypical species had the disease and they were estimating the percentage for the typical species.

Before considering how feature-based and geometric models of similarity might account for such asymmetries, the general notion that asymmetries may be related to one of the two processes, hypothesis confirmation and normalizing operations, will briefly be discussed. Sjöberg (1972) suggested that similarity judgments involve an active search for ways in which the two objects are similar, that is, that subjects look for features or properties shared by the objects. It may be that when the similarity task is directional, as when the subject is asked to judge how similar one object, a, is to a second object, b, the features of the first object are seen as given or fixed, and the subject then actively searches for features of the second object to confirm the hypothesis that a is like b. Inasmuch as the features of the first object are also associated with the second object, the hypothesis is confirmed. Owing to the bias for hypothesis confirmation, those features of the second object that are not possessed by the first object are not weighted heavily and do not detract much from the similarity measure. Such a process could account for the finding that when the features of the first object are a subset of the features of the second object, the perceived similarity is large relative to the case when the first object has many features not shared by the second object. This is closely related to the way in which Tversky's feature matching model accounts for asymmetries, as will be discussed in the next subsection.

Concerning normalizing operations, asymmetric similarity measures may result if similarity judgments are based on the ease with which one object can be transformed to come into correspondence with the other. If rounding off, normalizing, or regularizing operations are easier cognitive transformations than their inverses, then this may account for asymmetries found for numbers, lines varying in orientation, and geometric figures (Rosch, 1975a). A number of authors (Cooper & Shepard, in press; Foster, 1972a, 1972b; Hoffman, 1966; Julesz, 1971; Metzler & Shepard, 1974) have stressed the transformational nature of perception. It seems plausible that such transformations may also play a role in determining similarity measures. This notion is supported in a study by Imai (1977). He found that the judged similarity between two objects was directly related to the transformations that bring one object into correspondence with the other. Similarity judgments were largest for configurations that could be made identical by two or more different basic transformations, followed by configurations that could be made identical by only one basic transformation. The configurations that could be made identical only by successive application of more than one transformation were even less similar, and the configurations that could not be transformed into one another were the least similar.

Feature Matching Model

The feature matching model (Tversky, 1977) assumes that the similarity between objects a and b, $s(a, b)$, is monotonically related to the expression $S(a, b) = \theta f(A \cap B) - \alpha f(A - B) - \beta f(B - A)$, where A and B are the features of a and b, f is the measure or weight assigned to the features, and θ, α, and β are nonnegative constants. In terms of this formulation, differences between $s(a, b)$ and $s(b, a)$ can be accounted for if it is assumed that in a directional similarity task, the coefficients α and β differ, reflecting a greater focus on one of the objects than on the other. If the focusing hypothesis holds (i.e., $\alpha > \beta$),[2] then those features of a not shared by b detract more from the similarity of a to b, $s(a, b)$, than those features of b not shared by a. If $\alpha > \beta$, it follows that $s(a, b)$ will be greater than $s(b, a)$ if and only if $f(B)$ is larger than $f(A)$. That is, if the features of the two objects are given unequal weight, then asymmetries are expected when one object has more features or more salient features than the other. In this way, the feature matching model is able to account for asymmetric similarity data.

[2] It should be noted that the focusing hypothesis can be tested directly from a full similarity matrix. If in such a matrix the rows correspond to the first stimulus and the columns to the second stimulus, then more variance should be found in the row sums than in the column sums if the focusing hypothesis is true.

The way in which the feature matching model accounts for asymmetries is closely related to the idea suggested earlier that subjects may consider the features of the first object a as given or fixed and tend to ignore those features of the second object b that do not confirm the hypothesis that a is like b. A bias for hypothesis confirmation would mean that the effective size of the set $B - A$ is smaller when B is the second or referent stimulus than when it is the first stimulus. It may be difficult to empirically distinguish between these closely related accounts of asymmetry.

Geometric Models of Similarity

In simple geometric models, the similarity between two objects is assumed to be related to the distances between the corresponding points in a metric space. By definition, distances in a metric space must satisfy the symmetry axiom, namely, that the distance from point a to point b in the space is the same as the distance from b to a for all pairs of points. Thus, this kind of simple geometric model has difficulty accounting for asymmetric similarity values.

However, the distance–density model, which assumes that similarity is a function of interpoint distance and the spatial densities in the two regions surrounding the points, may be able to account for asymmetric similarity data in the following way. Suppose, as suggested by Tversky (1977), that in a directional similarity task, the subject focuses more on one of the objects than on the other. This differential focusing may have the effect that the spatial density surrounding one point affects the similarity measure more than the spatial density surrounding the other point. In terms of Equation 2, this would mean that the parameters α and β are unequal. If $\alpha > \beta$, then $s(x, y) > s(y, x)$ if and only if $\delta(x) < \delta(y)$, that is, in directional similarity tasks, asymmetries would be expected to be associated with differences in the densities in the regions surrounding the two points in the geometric configuration.

In order to test the proposed relationship between asymmetries and differences in spatial densities, the data from Rothkopf (1957) were again considered in conjunction with the multidimensional scaling solution published by Shepard (1963). Those pairs of Morse code signals were selected for which large asymmetries were found (larger than a somewhat arbitrary criterion of 20% difference in the confusion probabilities). Altogether, there were 39 such pairs. For each pair, the distance in the solution between the two points was defined to be the critical distance. For each point a, let $n(a)$ denote the number of points falling within a circle with radius equal to the critical distance centered at a. For 26 of the asymmetric pairs, it was found that $s(a, b) < s(b, a)$, when $n(a) > n(b)$, where $s(a, b)$ denotes the percentage of incorrect *same* responses when the first stimulus was signal a and the second stimulus was signal b. These cases are consistent with the hypothesized relation. There were five cases for which the opposite relation held, and eight cases for which $n(a)$ and $n(b)$ were equal. Considering only those cases for which $n(a)$ and $n(b)$ were unequal, the hypothesis was confirmed in 26 out of the 31 cases ($p < .01$).

The relationship between densities and asymmetries may also be supported by the finding that less prototypical objects are seen as more similar to prototypical objects than the reverse order (e.g., Rosch, 1975a). When multidimensional scaling was applied to a number of object domains, Rips et al. (1973) found that more prototypical items were generally scaled near the center of the resulting spatial configuration. This finding is consistent with the idea (Rosch & Mervis, 1975) that prototypical objects have the greatest overlap of features with the other objects in the domain. Thus, prototypical objects would be similar to a relatively large number of objects and would tend to be located at interior positions in a spatial configuration. Since, in general, central items occupy more dense regions of the space than peripheral items, the hypothesized relation between differential spatial densities and asymmetry would predict that less prototypical objects would be more similar to prototypical items than the prototypical items would be to the less prototypical items. Rosch's (1975a) results are consistent with this prediction.

The results of Tversky's (1977) experiments using countries and letters may also be consistent with the proposed relationship. He found that less prominent objects are more similar to more prominent objects than the reverse order. If prominent countries and letters are those stimuli having relatively many features, then these objects have features in common with a larger number of different objects than objects with fewer features. Objects that share features with relatively many objects would tend to occupy relatively dense regions of the stimulus configuration, and thus, the data are consistent with the hypothesis that asymmetric similarities are related to differences in spatial densities.

In the study of musical intervals by Balzano (1977), subjects were first presented with a name of an interval followed by a sounded interval, and *same–different* reaction times were measured. As mentioned earlier, those intervals that play an important or distinctive role in musical contexts were associated with faster *same* reaction times and also occupied less dense regions of the stimulus configuration. The proposed relationship between asymmetry and spatial density would predict that the time required to decide that a less distinctive interval is not a more distinctive interval would be shorter than the time required to decide that a more distinctive interval is not a less distinctive interval. Some of the largest asymmetries in discriminative reaction times found in his study could be accounted for in this way.

The proposed relationship, however, may not be supported by similarity data based on the number of stimulus–response errors. In this case, those items that are frequently correctly identified would, according to the distance–density model, tend to fall in less spatially dense regions of the configuration than objects with many stimulus–response errors. Due to constraints in the data, those objects that are frequently identified correctly are infrequently confused with other objects, so it is unlikely that the predicted pattern of asymmetries would be found in such data. Clearly, the relationship between asymmetries and differences in spatial densities needs to be tested further but should be tested using tasks that are clearly directional. While this requirement was met in the studies of Rosch (1975a) and Tversky (1977), the roles of the two stimuli in the studies of Rothkopf (1957) and Balzano (1977) were not clearly differentiated.

Metric Axioms: Triangle Inequality

Features, Dimensions, and Context

When an object is similar to another object, it must be similar with respect to certain features or properties. However, which features or properties are to be considered relevant to a similarity task are rarely explictly specified and may depend on the stimulus context and possibly even the particular pair of objects under consideration. Although the concept of similarity is "clear enough when closely confined by context and circumstance in ordinary discourse, it is hopelessly ambiguous when torn loose" (Goodman, 1972, p. 444).

Torgerson (1965) made a distinction between similarity as a basic, possibly perceptual, relation between instances of a multidimensional attribute and similarity as a derivative, cognitive relation between stimuli varying on several dimensions. In the latter case, similarity judgments may be based on complex cognitive processes and may be subject to changes in strategy depending on the context. He suggests that as the contribution of cognition goes up, the appropriateness of the multidimensional representation goes down. Using stimuli that varied in terms of two physical parameters only, Shepard (1964) showed that subjects do shift emphasis from one dimension to the other. Variability in similarity criteria may occur because, when subjects are faced with a large or heterogeneous universe of objects, they try to simplify the task by focusing on a subset of the relevant object characteristics at any one time. Gregson (1975) has also considered the variable nature of similarity, suggesting that subjects may change from one level of analysis to another at different points in time or may exploit characteristics of the stimulus set, such as the extent to which physical stimulus dimensions are intercorrelated over the set of stimuli.

Tversky (1977) suggested that features may vary in terms of salience and that the salience of a particular feature is determined by two types of factors: intensive and diagnostic.

The former refers to attributes that may be inherently obvious or striking, such as tone loudness, color saturation, and figure size. The weight given such properties would be, he argues, independent of context. However, Gravetter and Lockhead (1973), Parducci (1965), and others have demonstrated context effects in such stimulus domains as tones varying in loudness and squares varying in size, suggesting that the salience of these intensive properties may in fact vary as a function of stimulus context. The second factor, diagnosticity, refers to the classificatory significance of the features and is assumed by Tversky to depend on stimulus context. Those features that can be used as a convenient basis for classifying objects in the domain would be relatively heavily weighted in the similarity judgment. Tversky suggests that the diagnosticity principle reflects the tendency of subjects to sort the collection of objects into subgroups in order to reduce the information load and facilitate further processing. Sjöberg (1972) has made a similar proposal.

According to Fillenbaum and Rapoport (1974), the similarity criterion employed may even depend on the particular object pair under consideration. Two words may be judged similar if they are synonyms, but two other words may be similar if they are antonyms. This is closely related to the idea discussed earlier that given a pair of objects, subjects actively search for features or properties to justify high similarity ratings (Sjöberg, 1972), and thus, different features will be considered important depending on the particular stimulus pair under consideration. This kind of criterion shift may lead to examples like the one given by James (1890) in which the moon is similar to a gas jet (with respect to luminosity) and also similar to a football (with respect to roundness), but a gas jet and a football are not at all similar. The implication of such criterion shifts for feature-based and geometric models of similarity will be discussed in the next two subsections.

Feature Matching Model

The feature matching model (Tversky, 1977) can account for context effects on similarity judgments by assuming that the measure or weight given the various features is different in different contexts. That is, in one stimulus context, a given feature may be weighted heavily; in another context, the same feature may be weighted less heavily or possibly given no weight at all. The extreme flexibility of the feature matching model in this way would make it well suited for situations in which criterion shifts are expected to occur. The model itself does not specify what factors influence how the weights are assigned. Additional assumptions need to be made about how context affects the assigned weights. One such assumption, the diagnosticity principle, was suggested by Tversky (1977) and will be discussed again in the next section on context effects.

It should be noted, however, that this kind of set-theoretic model can account for examples such as the one given above without assuming criterion shifts. In the example, one object was seen as similar to each of two others, but the two objects were not at all similar to each other. In terms of feature sets, even if the first object has features in common with both the second and the third objects, it need not be the case that the second and the third objects share any features.

Geometric Models of Similarity

Geometric models assume that similarities are monotonically related to distances in a metric space. Distances in a metric space must satisfy the triangle inequality axiom, which says that the distance between any two points must be smaller than the sum of the distances between each of the two points and any third point. In terms of similarities, this means that if an object is similar to each of two other objects, the two objects must be at least fairly similar to each other. The type of example posed by James (1890) above would seem to violate this constraint and would thus present problems for geometric models.

It is possible, however, that similarity judgments may reflect an emphasis on features or dimensions in terms of which the objects are similar. In geometric terms, the similarity between two objects would be large if there is some lower dimensional projection of the higher dimensional psychological space in which the corresponding points are close. This would happen if the lower dimensional projection were just that space defined by the dimensions in terms of which the objects were similar. Projections onto certain subspaces may make more conceptual sense than projections onto other subspaces. Subspaces defined by obvious stimulus dimensions would seem to be likelier projections than subspaces not corresponding to such dimensions. This kind of approach may be able to account for violations of the triangle inequality axiom. One object might be quite close to a second object in one projection and to a third object in some other projection, yet this need not imply that there exists a projection that is conceptually reasonable in which the second and third objects are close. Indeed, the example given by James (1890) suggests this kind of explanation.

It might be interesting to consider the possibility of constructing higher dimensional representations from the configurations found in lower dimensional spaces, subject to the constraint that the lower dimensional configurations correspond to projections from the higher dimensional space. The different lower dimensional representations would presumably be generated in different stimulus contexts or when subjects are explicitly instructed as to which dimensions should be considered relevant for the similarity judgment. Fillenbaum and Rapoport (1974) suggested that if similarity structures found using a variety of distinct criteria were compounded into a single overall arrangement, the resulting structure might correspond to the structure yielded by the use of an unspecified or global similarity criterion. In an unpublished study, Shoben (cited in Smith, Rips, & Shoben, 1974) successfully applied this approach to at least one stimulus domain. He had subjects rate the similarity of pairs of objects with respect to single dimensions. The resulting similarity matrices were then scaled. Two of these solutions were found to correlate well with the two-dimensional solution arrived at when subjects were uninstructed as to the relevant dimensions. Compounding lower dimensional solutions into higher dimensional configurations may result in configurations of rather high dimensionality; thus, one of the advantages of low dimensionality, namely, visualizability, is lost. However, such higher dimensional representations would be able to account for a wider range of similarity data, particularly if stimulus context is found to have a large and lawful effect on the dimensions considered relevant to the similarity task.

The idea that similarity judgments are based on lower dimensional projections of a higher dimensional configuration is a special case of the model underlying the individual differences scaling technique, INDSCAL, developed by Carroll and Chang (1970). The technique uses multiple similarity matrices generated by different subjects or in different experimental conditions. The technique accounts for differences between the matrices by assuming that the stimulus dimensions are differentially weighted by the individual subjects or in the different conditions. That is, differences between the similarity matrices are assumed to reflect shifts in the emphasis given the various dimensions. The method has been shown to yield interpretable shifts in the weighting of the dimensions when applied to a variety of stimulus domains (Wish & Carroll, 1974). Owing to the additional information contained in the multiple matrices, the INDSCAL method is typically able to support higher dimensional solutions than standard multidimensional scaling methods.

Other Context Effects

Diagnosticity and Density

In the previous section, the idea was discussed that a feature or dimension may be given different weights in different stimulus contexts. Tversky (1977) suggested that these weights are determined in part by how diagnostic the feature is for the particular set of objects under consideration, that is, how significant the feature is for classifying the objects into subclasses. In order to test the diagnosticity principle, Tversky used pairs of four object sets of the form $\{a, b, c, p\}$ and $\{a, b, c, q\}$. For each set of objects, subjects were to decide which of the last three members was most like the first. The sets were constructed so that in the first set, the object p was similar to one of the common alternatives, say, the object b. In the second set, the object q was similar to the other common alternative, c. In the first set, subjects tended to choose object c over object b as being most similar to object a, but in the second set, they tended to choose b over c. It is interesting to note that the effect found by Tversky in the similarity task is somewhat analogous to an effect found in choice behavior by him also (Tversky, 1972). In terms of the probability that an object is chosen from a set, the effect of adding an alternative to an offered set "hurts" alternatives that are similar to the added alternative more than those that are dissimilar to it.

The feature matching model (Tversky, 1977) is able to account for this result in the following way. In the first context, it is assumed that some feature becomes diagnostic that classifies objects p and b together, and in the second context, some other feature becomes diagnostic that classifies objects q and c together. Since diagnostic features are assumed to be given relatively heavy weight, the model can account for shifts in the rank order of similarities.

This kind of result poses problems for traditional geometric models. Such models would be unable to account for shifts in the rank order of similarity judgments as a function of the rest of the stimulus context in which the objects appeared. However, the hypothesized relationship between similarity and the density of the object configuration may be useful in explaining Tversky's results. According to the hypothesis, as the density of the configuration increases, interobject similarity decreases. In the first context, the item p is added to be similar to the object b; according to the hypothesis, this would tend to decrease the similarity between b and a. In the second context, the item q is added to be similar to c, and this would tend to decrease the similarity between c and a. In this way, the context effects demonstrated by Tversky (1977) can be accounted for by the distance–density model.

Range and Frequency

In order to account for effects on categorical judgments of the distribution of the stimulus set along a single dimension, Parducci (1965) proposed the range–frequency theory. The theory is based on two principles. The first principle is that the judge tends to divide his psychological range into a fixed number of subranges of equal size. This implies that if the context is enlarged by extending the range over which the stimuli vary, the judge will adjust his category responses to accommodate the enlarged range of stimuli within the fixed range of responses. Support for this prediction is found in the work of Gravetter and Lockhead (1973). The second principle is that the judge employs the alternative categories with equal frequency. That is, if within a subrange of the stimulus range there are relatively many stimuli, then finer discriminations are made within that subregion than in less dense subregions. Parducci and others (Birnbaum, 1974; Parducci, 1965; Parducci & Perrett, 1971) have shown that this theory is able to account for categorical judgments using a variety of unidimensional stimuli.

Although this theory is stated in terms of categorical judgments, analogous principles can be stated for similarity judgments. In terms of similarity judgments, the first principle would be that if the range of stimuli is increased by adding more extreme stimuli, then the similarity judgments for stimuli that are common to the original and extended stimulus sets should increase. The second principle would be that the similarity between two objects in a relatively dense subregion of the stimulus domain should be judged to be smaller than the similarity between two objects that differ an equivalent amount but occupy a less dense subregion. This second principle is identical to the density principle proposed in this article, and the evidence for it has already been discussed.

In addition, there is some evidence from similarity tasks for the first principle. Tversky (1977) and Sjöberg (1972) demonstrated using countries, musical instruments, and animals that broadening or extending the stimulus domain resulted in greater average similarity judgments among the objects from the original set. Thus, subjects do tend to standardize or normalize the response scale to fit the stimulus range. Torgerson (1965) showed using multidimensional objects that this range effect can operate with respect to one of the dimensions independently of other dimensions. He used kite-shaped objects varying in size and bottom heaviness. In one set, the range of variation of physical size to bottom heaviness was twice that of the other set. The multidimensional scaling solutions of the two sets of stimuli were, however, almost identical. Thus, the range and density effects found in similarity tasks have close correspondence to those found in categorical judgments.

Spatial Inhomogeneity and Density

In this last subsection on context effects, an effect predicted by the density hypothesis will be discussed. Essentially, the density hypothesis states that if two objects occupy a relatively less spatially dense region of the stimulus domain, they will be seen as more similar than two other objects that differ an equivalent amount but lie in a more spatially dense region of the domain. Since points at the edge of a configuration generally have fewer neighbors than points interior to the configuration, this hypothesis predicts that interobject similarity for points at the edge of the configuration should be larger than points with equal interpoint distances but lying at interior positions.

In order to test this hypothesis, the similarity data on the Morse code signals (Rothkopf, 1957; Shepard, 1963) were again analyzed. The hypothesis was tested by first finding points of approximately equal distance in the scaling solution. Two different distances were chosen. The first distance corresponded approximately to the distance between the letters A and M in the published solution, the second to the distance between D and R. For each of these fixed distances, all pairs of points that were separated by the critical distance were considered. There were 23 such pairs for the first distance and 22 pairs for the second distance. In addition, each pair of points was classified according to whether both points fell at the boundary of the solution (these pairs will be called *exterior pairs*) or whether at least one of the points fell at an interior position in the configuration (these pairs will be called *interior pairs*). For each pair, the similarity score used was the average number of incorrect *same* responses for the two orders of presentation. As predicted, the average similarity measure for exterior pairs was larger than the average similarity measure for interior pairs [the t values were $t(21) = 3.687$ and $t(20) = 3.327$, $p < .001$, for the two distances, respectively].

This result suggests that context, that is, the entire stimulus set under consideration, may introduce inhomogeneities into the psychological space so that a given change in one part of the space carries a different meaning in terms of similarity than an equivalent change in another part of the space. Further investigation is needed to determine whether systematic effects are found in other stimulus domains. The work of Rumelhart and Abrahamson (1973), however, suggests that the effects may be small, since their subjects were able to solve analogy problems with some accuracy where the solution was determined by transposition of a vector distance from one part of the multidimensional space to another. In addition, Shepard (1963) compared the spatial solution for similarity data on a subset of the Morse code signals with the solution for the full set of signals and concluded that the structure was not seriously distorted by the additional signals.

Similarity Versus Difference

Tversky (1977) demonstrated that similarity judgments and difference judgments may not be perfectly negatively correlated. That is, for some objects, a, b, c, and e, the judged similarity between a and b is larger than the similarity between c and e, and the judged difference between a and b is also larger than the difference between c and e. In order to account for this pattern, he suggested that common features are given heavier weight relative to distinctive features in similarity judgments than in difference judgments. The feature matching model predicts that the observed pattern would be expected to occur when a and b have both more common and distinctive features than c and e. As predicted, using countries as stimuli, Tversky found that when a and b were prominent countries with many features, they tended to be judged both more similar and more different than the pairs of less prominent countries, c and e.

This kind of result is also consistent with the distance–density model if it is assumed that the weights given the distance and density components of Equation 2 vary as a function of task. For example, more emphasis may be given to distance in a similarity task and to density in a difference task. This means that in a difference task, subjects attend to other similar objects more than in a similarity task. In terms of the distance–density model, similarity in a nondirectional task can be written as a monotonic decreasing function of $\theta d(x, y) + \delta(x) + \delta(y)$, and difference judgments can be written as a monotonic increasing function of $\lambda d(x, y) + \delta(x) + \delta(y)$. If $\theta > \lambda$, then a and b will be both more similar and more different than c and e if and only if $\lambda[d(c, e) - d(a, b)] < [\delta(a) + \delta(b)] - [\delta(c) + \delta(e)] < \theta[d(c, e) - d(a, b)]$. That is, this pattern may occur when the interpoint distance between a and b is less than the distance between c and e, but the density in the regions around a and b is greater than the density near c and e. In this way, the distance–density model is consistent with similarity and difference data that are not perfectly negatively correlated.

Dimensions Versus Features: Dimensional Assumptions

In geometric models of similarity, points corresponding to the objects in the stimulus domain are embedded in a dimensionally organized metric space.[3] One interpretation of such a representation is that the stimuli can be described as varying along a number of underlying dimensions. Indeed, one of the applications of multidimensional scaling techniques has been to discover the dimensions in terms of which the objects were seen as varying. The technique has been applied to a wide variety of stimulus domains with considerable success, that is, the resulting geometric configurations were organized within a framework of interpretable dimensions.

Tversky (1977) argued, however, that while the dimensional assumption may be appropriate for certain kinds of perceptual stimuli, that is, those that vary continuously along one or more quantifiable dimensions, it may be inappropriate for more semantic stimuli, which vary in terms of discrete qualitative features. For such stimuli, the similarity structure might better be described by set-theoretic relations, where the similarity between objects is accounted for in terms of the categories to which the objects belong or the properties or features associated with the objects. A number of authors have proposed semantic theories based on set-theoretic relations. Basic to the work of Meyer (1970) is the idea that semantic stimuli can be described in terms of category membership, and the interrelationship between category membership and similarity has been suggested by Wallach (1958) and Handel and Garner (1965). Feature-based descriptions of semantic objects have been proposed by Katz (1972), and the similarity models of Sjöberg (1972) and Tversky (1977) are based on the featural properties of the objects.

However, theories of semantics based on category membership may be unable to account for a number of effects. Boundaries between categories may not be clearly defined and category membership may be a matter of degree. Evidence for the "fuzzy" character of category boundaries (Zadeh, 1965) is found in the work of Lakoff (1972, 1973), Labov (1973), and Lehrer (1970). Smith, Shoben, and Rips (1974) and Rosch et al. (1976) have found large variation across individual items in the time required to determine whether the item is a member of a specified category. Rosch and Mervis (1975) suggested that category membership is best described in terms of family resemblance. According to this approach, category membership arises through a network of overlapping attributes, and it need not be the case that any feature or set of features can adequately distinguish between category members and nonmembers. From this network of attributes, a measure of family resemblance can be derived; in terms of this measure, certain objects are "better" category members than others. This measure of family resemblance is able to predict performance in categorization tasks. Thus, models

[3] One exception is the method of maximum variance nondimensional scaling developed by Cunningham and Shepard (1974), which derives interpoint distances from similarity data without specifying a particular underlying dimensionally organized metric space.

based solely on category membership are unable to account for a variety of effects, suggesting that similarity models based on category membership will also be incomplete.

A number of difficulties are also associated with feature-based models of similarity. In such approaches, two objects are seen as similar inasmuch as they have features in common. It might be argued, however, that the extent to which an object possesses or is associated with a feature may be a matter of degree. For example, Halff, Ortony, and Anderson (1976) have illustrated the case in which some red objects are "redder" than other red objects. Even if some of these differences can be accounted for in terms of the physical shade of red typically associated with the object, it may be true that objects of identical physical color differ in terms of how salient or important the color feature is for the description of the object. Thus, a given property may be more central or important to the meaning or appearance of one object than another. Another problem with feature-based descriptions, suggested by the work of Smith, Shoben, and Rips (1974), is that it may be necessary to distinguish between defining and characteristic features and to determine to what extent they are involved in similarity judgments. It may happen that a defining feature of an object, while necessary, may be less salient than a characteristic feature of the object. For example, that penguins have feathers may be less salient than the fact that they typically are found in antarctic localities. Therefore, the extent to which a given feature or property is associated with an object may be a matter of degree, and feature-based models of similarity may have to take into account variations in the associative strength between objects and their features. A similar proposal was made by Lehrer (1970).

Although geometric models of similarity may not seem well suited for stimuli that vary in terms of discrete properties, multidimensional scaling methods have been applied to such stimuli with considerable success (Shepard, 1963, 1972, 1974; Torgerson, 1965). In these solutions, objects were grouped together in the spatial representation according to the discrete features that they share. It appears, then, that the discrete nature of stimulus features is not a major difficulty for geometric models. In this regard, it might be useful to distinguish between the continuous nature of the space underlying the geometric representation and the continuous nature of the stimulus domain itself. Although the underlying metric space is assumed to be continuous, it need not be the case that for every point in the underlying space there corresponds a possible stimulus. It may be, for example, that the full set of stimuli corresponds only to disconnected subregions or a finite set of points in the underlying space (Goldmeier, 1936/1972; Torgerson, 1965). The geometric model does not strictly imply that the stimulus domain needs to be continuous, although Rumelhart and Abrahamson (1973) have shown that in one semantic domain, it may be possible to create a new object to correspond to an arbitrary point in the metric space in which the object configuration is embedded.

A similar distinction may also be useful concerning the dimensional nature of the underlying space. While the space in which the configuration is embedded is assumed to be organized in terms of coordinate axes, the coordinate axes need not have particular meaning with respect to the objects themselves. Multidimensional scaling techniques have yielded solutions in which the axes of the underlying space bear no particular relationship to the object configuration. Rather, the structure underlying the similarity relations could better be interpreted in terms of clusters or placement around a circle or some other subspace (Levelt, Van de Geer, & Plomp, 1966; Shepard, 1962b, 1974). While such applications do not aid the discovery of underlying stimulus dimensions, it might be argued that the discovery of clusters or other nonlinear patterns is of comparable interest. Finally, Goldmeier (1936/1972) has even suggested that the dimensionality of a stimulus domain may vary from one part of the stimulus space to another. In terms of the geometric representation, this may be true even though the stimulus configuration is embedded in some metric space of constant dimensionality.

Two final points concerning set-theoretic and geometric similarity models will be mentioned briefly. First, one advantage of set-theoretic models is that hierarchical relations among the objects can conveniently be represented as subset relations within such a framework. Although geometric models incorporate no mechanism for representing such structure, Shepard (1974) has suggested that a hierarchical clustering solution (Johnson, 1967) may be superimposed on the multidimensional solution for this purpose. A second advantage for set-theoretic models is that such models appear well suited for describing objects that vary in terms of a large number of features or properties. While in theory, geometric models impose no restriction on the number of dimensions in the underlying space, in practice, multidimensional scaling techniques do tend to yield the most satisfactory and interpretable solutions in spaces of relatively low dimensionality. Thus, the number of relevant features or properties may be a factor to consider in choosing between the two general types of models when applying a similarity model to a particular stimulus domain.

Conclusions

The article by Tversky (1977) raised a number of extremely interesting issues concerning the nature of similarity, some of which directly call into question the applicability of geometric models to similarity data. In the present article, these issues have been considered in some depth within the context of geometric models. The main proposal was that the similarity between objects may be a function not only of interpoint distance in a metric space but also the spatial density of points in the configuration. A distance-density model was proposed that modifies the traditional multidimensional scaling model to take into account the effect on similarity of spatial density in the stimulus configuration. Such a model may be able to account for variations in how similar an object is to itself (violation of the minimality axiom), asymmetric similarity measures in directional similarity tasks (violations of the symmetry axiom), certain effects found when the explicit stimulus context is manipulated, and the effect of task (similarity vs. difference). In addition, the idea was discussed that subjects may weight dimensions differently depending on stimulus context and possibly even the specific object pair under consideration. In this connection, it was suggested that similarity judgments may involve an active search for dimensions or features in terms of which the objects are similar and that the judgments may be made with respect to these dimensions. In this way, geometric models may be consistent with violations of the triangle inequality axiom. Thus, geometric representations of similarity relations may be able to account for a wide range of effects if a number of assumptions are made about the kind of geometrically represented information that is relevant to a particular experimental task and the way in which this information interacts with the stimulus context.

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