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MOST TENSOR PROBLEMS ARE NP HARD

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Abstract. The idea that one might extend numerical linear algebra, the collection of matrix computational methods that form the workhorse of scientific and engineering computing, to *numerical multilinear algebra*, an analogous collection of tools involving hypermatrices/tensors, appears very promising and has attracted a lot of attention recently. We examine here the computational tractability of some core problems in numerical multilinear algebra. We show that tensor analogues of several standard problems that are readily computable in the matrix (i.e. 2-tensor) case are NP hard. Our list here includes: determining the feasibility of a system of bilinear equations, determining an eigenvalue, a singular value, or the spectral norm of a 3-tensor, determining a best rank-1 approximation to a 3-tensor, determining the rank of a 3-tensor over $\mathbb{R}$ or $\mathbb{C}$. Hence making tensor computations feasible is likely to be a challenge.

1. Introduction

There has been a recent spade of work on 'tensor methods' in computer vision, data analysis, machine learning, scientific computing, and other areas (examples cited here include [7, 9, 10, 11, 12, 16, 17, 18, 19, 32, 34, 38, 39, 40, 41, 44, 45, 46]; see also the bibliography of the recent survey [31] and the 244 references therein). The idea of using tensors for numerical computing is appealing. Nearly all problems in computational science and engineering may eventually be reduced to one or more standard problems involving matrices: system of linear equations, least squares problems, eigenvalue problems, singular value problems, low-rank approximations, etc. If similar problems for tensors of higher order may be solved effectively, then one would have substantially enlarged the arsenal of fundamental tools in numerical computations.

We will show in this paper that tensor analogues of several problems that are readily computable in numerical linear algebra are NP hard[1]. More specifically consider the following problems in numerical linear algebra. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $A \in \mathbb{F}^{m\times n}$, $\mathbf{b} \in \mathbb{F}^m$, and $r \leq \min\{m, n\}$.

Rank and numerical rank: Determine $\operatorname{rank}(A)$.

Linear system of equations: Determine if $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{F}^n$.

Spectral norm: Determine the value of $\|A\|_{2,2} := \max_{\|\mathbf{x}\|_2=1}\|A\mathbf{x}\|_2 = \sigma_{\max}(A)$.

Eigenvalue problem: Determine if $\lambda \in \mathbb{F}$ has a non-zero $\mathbf{x} \in \mathbb{F}^n$ with $A\mathbf{x} = \lambda\mathbf{x}$ (assume $m = n$).

Singular value problem: Determine if $\sigma \in \mathbb{F}$ has non-zero $\mathbf{x} \in \mathbb{F}^n$, $\mathbf{y} \in \mathbb{F}^m$ with $A\mathbf{x} = \sigma\mathbf{y}$, $A^\top\mathbf{y} = \sigma\mathbf{x}$.

Low rank approximation: Determine $A_r \in \mathbb{F}^{m\times n}$ with $\operatorname{rank}(A_r) \leq r$ and $\|A - A_r\|_F = \min_{\operatorname{rank}(B)\leq r}\|A - B\|_F$.

We will examine generalizations of these problems to tensors of order 3 or higher and show that they all fall into one of the following categories: unsolvable (low rank approximation when rank $r > 1$), or NP hard (the rest).

Johan Håstad has shown that tensor rank is NP hard over $\mathbb{Q}$ and NP-complete over finite fields [24]. We will extend the NP hard list of tensor problems to include several other multilinear generalizations of standard problems in numerical linear algebra. Furthermore, as we will see in

[1]Here we limit our discussion of NP hardness to the traditional Cook-Karp-Levin sense [13, 29, 33]. Tractability of these tensor problems in the Blum-Shub-Smale sense [5] and the Valiant sense [43] is delayed to the full paper.

this paper, these multilinear problems are not just natural extensions of their linear cousins but have all surfaced in recent applications. Let $\mathcal{A} \in \mathbb{F}^{l\times m\times n}$ be a 3-tensor and $A_k, B_k, C_k \in \mathbb{F}^{n\times n}$, $k = 1,\dots,n$, be matrices. Our list includes the following:

Rank and numerical rank: Determine $\operatorname{rank}(\mathcal{A})$.

Bilinear system of equations: Determine if $\mathbf{x}^\top A_k\mathbf{y} = \alpha_k$, $\mathbf{y}^\top B_k\mathbf{z} = \beta_k$, $\mathbf{z}^\top C_k\mathbf{x} = \gamma_k$, $k = 1,\dots,n$, has a solution $(\mathbf{x},\mathbf{y},\mathbf{z}) \in \mathbb{F}^n \times \mathbb{F}^n \times \mathbb{F}^n$.

Spectral norm of a 3-tensor: Determine the value of $\sup_{\mathbf{x},\mathbf{y},\mathbf{z}\neq\mathbf{0}} |\mathcal{A}(\mathbf{x},\mathbf{y},\mathbf{z})|/\|\mathbf{x}\|_2\|\mathbf{y}\|_2\|\mathbf{z}\|_2$.

Eigenvalue a 3-tensor: Determine if $\lambda \in \mathbb{F}$ has a non-zero $\mathbf{x} \in \mathbb{F}^n$ with $A(\mathbf{x},\mathbf{x},I) = \lambda\mathbf{x}$ (assume $l = m = n$).

Singular value of a 3-tensor: Determine $\sigma \in \mathbb{F}$ has non-zero $\mathbf{x} \in \mathbb{F}^l$, $\mathbf{y} \in \mathbb{F}^m$, $\mathbf{z} \in \mathbb{F}^n$ with $\mathcal{A}(\mathbf{x},\mathbf{y},I) = \sigma\mathbf{z}$, $\mathcal{A}(\mathbf{x},I,\mathbf{z}) = \sigma\mathbf{y}$, $\mathcal{A}(I,\mathbf{y},\mathbf{z}) = \sigma\mathbf{x}$.

Best rank-1 approximation to a 3-tensor: Determine $(\mathbf{x},\mathbf{y},\mathbf{z}) \in \mathbb{F}^l \times \mathbb{F}^m \times \mathbb{F}^n$ that minimizes $\|\mathcal{A} - \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\|_F$.

We have adopted the shorthands $\mathcal{A}(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{i,j,k=1}^{l,m,n} a_{ijk}x_iy_jz_{jk}$ and $\mathcal{A}(\mathbf{x},\mathbf{y},I) = \sum_{i,j=1}^{l,m} a_{ijk}x_iy_j$ (see Section 2). As is the case for matrices, the definitions of eigenvalues and singular values for 3-tensors come from the stationary values of the cubic and multilinear Rayleigh quotient $\mathcal{A}(\mathbf{x},\mathbf{x},\mathbf{x})/\|\mathbf{x}\|_p^p$ and $\mathcal{A}(\mathbf{x},\mathbf{y},\mathbf{z})/\|\mathbf{x}\|_p\|\mathbf{y}\|_p\|\mathbf{z}\|_p$ where $p = 2$ or 3 (cf. Sections 5 and 6).

While Håstad's result applies to $\mathbb{Q}$ and $\mathbb{F}_q$, these choices of fields do not make sense for all but one of the above problems (the exception being the bilinear system of equations) as these are analytic problems only well-defined over a complete field of characteristic 0 with an absolute value. Among such fields, $\mathbb{R}$ and $\mathbb{C}$ are by far the most common in applications and so we shall restrict our discussions to these fields. As for the exceptional case, over $\mathbb{Q}$, the feasibility of a bilinear system of equations is undecidable, and over $\mathbb{F}_q$, they are NP hard in both the Blum-Shub-Smale and Cook-Karp-Levin sense.

In many cases, we will show that certain forms of *quadratic feasibility* stand in the way of solving many tensor problems. Such problems may be polynomially reducible to a specific type of quadratic feasibility problem over $\mathbb{R}$ or $\mathbb{C}$. To demonstrate NP hardness in the Cook-Karp-Levin sense, we show that these restricted forms of quadratic feasibility are at least as hard as 3-colorability or max-clique, two well-known NP hard problems [20].

2. Tensors as hypermatrices

Let $V_1,\dots,V_k$ be vector spaces of dimensions $d_1,\dots,d_k$ respectively over a field $\mathbb{F}$. An element of the tensor product $V_1\otimes\cdots\otimes V_k$ is called an order-k *tensor* or k-*tensor* for short. For the purpose of this article and for notational simplicity, we will limit our discussion to 3-tensors

Up to a choice of bases on V_1, V_2, V_3, a 3-tensor in $V_1\otimes V_2\otimes V_3$ may be represented by a $l\times m\times n$ array of elements of $\mathbb{F}$,

$$\mathcal{A} = [\![a_{ijk}]\!]_{i,j,k=1}^{l,m,n} \in \mathbb{F}^{l\times m\times n}.$$

These are sometimes called *hypermatrices*[2] [21] and come equipped with some algebraic operations inherited from the algebraic structure of $V_1\otimes V_2\otimes V_3$:

- Outer Product Decomposition: every $\mathcal{A} = [\![a_{ijk}]\!] \in \mathbb{F}^{l\times m\times n}$ may be decomposed as

$$\mathcal{A} = \sum\nolimits_{\alpha=1}^{r} \lambda_\alpha\,\mathbf{x}_\alpha\otimes\mathbf{y}_\alpha\otimes\mathbf{z}_\alpha, \qquad a_{ijk} = \sum\nolimits_{\alpha=1}^{r} \lambda_\alpha x_{i\alpha}y_{j\alpha}z_{k\alpha}, \tag{2.1}$$

  with $\lambda_\alpha \in \mathbb{F}$, $\mathbf{x}_\alpha \in \mathbb{F}^l, \mathbf{y}_\alpha \in \mathbb{F}^l, \mathbf{z}_\alpha \in \mathbb{F}^n$. For $\mathbf{x} = [x_1,\dots,x_l]^\top$, $\mathbf{y} = [y_1,\dots,y_m]^\top$, $\mathbf{z} = [z_1,\dots,z_n]^\top$, the quantity $\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} = [\![x_iy_jz_k]\!]_{i,j,k=1}^{l,m,n} \in \mathbb{F}^{l\times m\times n}$.

[2]The subscripts and superscripts will be dropped when the range of i, j, k is obvious or unimportant.

- Multilinear Matrix Multiplication: every $\mathcal{A} = [\![a_{ijk}]\!] \in \mathbb{F}^{l \times m \times n}$ may be multiplied on its '3 sides' by matrices $X = [x_{i\alpha}] \in \mathbb{F}^{l \times p}$, $Y = [y_{j\beta}] \in \mathbb{F}^{m \times q}$, $Z = [z_{k\gamma}] \in \mathbb{F}^{m \times r}$,

$$\mathcal{A} \cdot (X, Y, Z) = [\![c_{\alpha\beta\gamma}]\!] \in \mathbb{F}^{p \times q \times r}, \qquad c_{\alpha\beta\gamma} = \sum\nolimits_{i,j,k=1}^{l,m,n} a_{ijk} x_{i\alpha} y_{j\beta} z_{k\gamma}. \tag{2.2}$$

A different choice of bases on $V_1, \dots, V_k$ would lead to a different hypermatrix representation of elements in $V_1 \otimes \cdots \otimes V_k$ — where the two hypermatrix representations would differ precisely by a multilinear matrix multiplication of the form (2.2) where X, Y, Z are the respective change of basis matrices. For the more pedantic readers, it is understood that what we call a tensor in this article really means a hypermatrix. In the context of matrices, $\mathbf{x} \otimes \mathbf{y} = \mathbf{x}\mathbf{y}^\top$ and $A \cdot (X, Y) = Y^\top A X$. When $p = q = r = 1$ in (2.2), i.e. the matrices X, Y, Z are vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$, we omit the $\cdot$ and write

$$\mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum\nolimits_{i,j,k=1}^{l,m,n} a_{ijk} x_i y_j z_{j_k}$$

for the associated trilinear functional. We also note the special case when one or more of the matrices X, Y, Z in (2.2) is the identity matrix. For example,

$$\mathcal{A}(\mathbf{x}, \mathbf{y}, I_n) = \sum\nolimits_{i,j=1}^{l,m} a_{ijk} x_i y_j \in \mathbb{F}^n \quad \text{and} \quad \mathcal{A}(\mathbf{x}, I_m, I_n) = \sum\nolimits_{i=1}^{l} a_{ijk} x_i \in \mathbb{F}^{m \times n}. \tag{2.3}$$

In particular, the (partial) gradient of the trilinear functional $\mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ may be expressed as

$$\nabla_{\mathbf{x}} \mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathcal{A}(I_l, \mathbf{y}, \mathbf{z}), \quad \nabla_{\mathbf{y}} \mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathcal{A}(\mathbf{x}, I_m, \mathbf{z}), \quad \nabla_{\mathbf{z}} \mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathcal{A}(\mathbf{x}, \mathbf{y}, I_n).$$

A 3-tensor $\mathcal{S} = [\![s_{ijk}]\!] \in \mathbb{F}^{n \times n \times n}$ is called *symmetric* if $s_{ijk} = s_{ikj} = s_{jik} = s_{jki} = s_{kij} = s_{kji}$.The homogeneous polynomial associated with $\mathcal{S}$ and its gradient can again be conveniently expressed via (2.2) as

$$\mathcal{S}(\mathbf{x}, \mathbf{x}, \mathbf{x}) = \sum\nolimits_{i,j,k=1}^{n} s_{ijk} x_i x_j x_k, \qquad \nabla \mathcal{S}(\mathbf{x}, \mathbf{x}, \mathbf{x}) = 3\mathcal{S}(\mathbf{x}, \mathbf{x}, I_n).$$

Observe that for a symmetric tensor, $\mathcal{S}(I_n, \mathbf{x}, \mathbf{x}) = \mathcal{S}(\mathbf{x}, I_n, \mathbf{x}) = \mathcal{S}(\mathbf{x}, \mathbf{x}, I_n)$. A special symmetric tensor is the *delta tensor* $\mathcal{I} = [\![\delta_{ijk}]\!]_{i,j,k=1}^n \in \mathbb{F}^{n \times n \times n}$ where $\delta_{ijk} = 1$ iff $i = j = k$ and 0 otherwise. The delta tensor generalizes the identity matrix in some restricted sense (unlike $\mathbb{F}^{n \times n}$, the set of 3-tensors $\mathbb{F}^{n \times n \times n}$ is not a ring and so there cannot be an 'identity tensor').

A tensor that can be expressed as an outer product of vectors is called a decomposable tensor and rank-1 if it is also nonzero. More generally, the *rank* of a tensor $\mathcal{A} = [\![a_{ijk}]\!]_{i,j,k=1}^{l,m,n} \in \mathbb{F}^{l \times m \times n}$, denoted $\operatorname{rank}(\mathcal{A})$, is defined as the minimum r for which $\mathcal{A}$ may be expressed as a sum of r rank-1 tensors [27, 28],

$$\operatorname{rank}(\mathcal{A}) := \min\Big\{ r \;\Big|\; \mathcal{A} = \sum\nolimits_{\alpha=1}^{r} \lambda_\alpha \, \mathbf{x}_\alpha \otimes \mathbf{y}_\alpha \otimes \mathbf{z}_\alpha \Big\}. \tag{2.4}$$

The definition of rank in (2.4) agrees with matrix rank when applied to an order-2 tensor.

We will now move on to define analytic notions. So in the following, $\mathbb{F}$ will denote either $\mathbb{R}$ or $\mathbb{C}$. The *Frobenius norm* or *F-norm* of a tensor $\mathcal{A} = [\![a_{ijk}]\!]_{i,j,k=1}^{l,m,n} \in \mathbb{F}^{l \times m \times n}$ is defined by

$$\|\mathcal{A}\|_F = \Big[\sum\nolimits_{i,j,k=1}^{l,m,n} |a_{ijk}|^2 \Big]^{\frac{1}{2}}. \tag{2.5}$$

Another common class of tensor norms generalizes operator norms of matrices:

$$\|\mathcal{A}\|_{p,q,r} := \sup_{\mathbf{x},\mathbf{y},\mathbf{z} \neq \mathbf{0}} \frac{|\mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z})|}{\|\mathbf{x}\|_p \|\mathbf{y}\|_q \|\mathbf{z}\|_r}$$

defines a norm for any $1 \le p, q, r \le \infty$. For $p = q = r = 2$, we call $\|\mathcal{A}\|_{2,2,2}$ the *spectral norm* of $\mathcal{A}$ in analogy with its matrix cousin.

The discussion in this section remains unchanged if $\mathbb{R}$ is replaced by $\mathbb{C}$ throughout. A minor caveat is that the tensor rank as defined in (2.4) depends on the choice of base fields (see [16] and Section 8 for a discussion).

3. Quadratic Feasibility

A basic problem in computational complexity is to decide whether a given set of quadratic equations has a solution over a field $\mathbb{F}$. It turns out that many questions about tensors can be reduced to this situation with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; thus, the following decision problem will play a large role in our complexity analysis of them.

Problem 3.1 (Real Quadratic Feasibility $\mathrm{QF}_{\mathbb{R}}$)**.** *Let $G_i(\mathbf{x}) = \mathbf{x}^\top A_i \mathbf{x}$ for $i = 1, \dots, n$ be n homogeneous, real quadratic forms; that is, $\mathbf{x} = (x_1, \dots, x_n)^\top$ is a column vector of indeterminates and each A_i is a real, symmetric $n \times n$ matrix. Determine if the system of equations $\{G_i(\mathbf{x}) = 0\}_{i=1}^n$ has a nontrivial real solution $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$.*

We may ask for the complexity of this problem in two ways. The *real complexity* of Problem 3.1 refers to number of field operations required to determine feasibility as a function of n. On the other hand, assuming that the matrices A_i have rational entries, the *bit complexity* of Problem 3.1 refers to the number of bit operations necessary as a function of the number of bits required to specify all of the A_i. For instance, the real complexity of multiplying two integers of size N (and thus of bit size $\log N$) is $O(1)$, whereas the bit complexity is $O(\log N \log \log N)$ (using the fast Fourier transform). Throughout this document, we shall refer to this second version of complexity, although our result will turn out to be independent of which one we use.

As the following theorem indicates, it is unlikely that there is a polynomial time algorithm to solve Problem 3.1. However, we remark that if we fix the number of equations to be m and allow the number of indeterminates n to vary, there is an algorithm which is polynomial in n due to Barvinok [2]. For an extension to sets of homogeneous bilinear equations, see Section 4.

Theorem 3.2. *Graph 3-coloring is polynomial reducible to Problem 3.1. Therefore, Problem 3.1 is NP hard.*

Remark 3.3. *It is well-known that inhomogeneous quadratic feasibility is NP hard over $\mathbb{R}$ and NP-complete over finite fields [4]. While we suspect that Theorem 3.2 is known in some form, we have been unable to locate an exact reference. For completeness (and to remain self-contained), we include a proof of Theorem 3.2 below. The reduction techniques employed here might also be of independent mathematical interest.*

We first describe some simple observations giving flexibility in specifying a quadratic feasibility problem. The proofs are immediate and therefore omitted.

Lemma 3.4. *Let $G_i(\mathbf{z})$ be as in Problem 3.1. Consider a new system $H_j(\mathbf{x}) = \mathbf{x}^\top B_j \mathbf{x}$ of $2n$ equations in $2n$ indeterminates given by:*

$$B_{2i} = \begin{bmatrix} A_i & 0 \\ 0 & -A_i \end{bmatrix}, \quad B_{2i+1} = \begin{bmatrix} 0 & A_i \\ A_i & 0 \end{bmatrix}, \quad i = 1, \dots, n.$$

Then the equations $\{H_j(\mathbf{x}) = 0\}_{j=1}^{2n}$ have a nontrivial real solution $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^{2n}$ if and only if the equations $\{G_i(\mathbf{z}) = 0\}_{i=1}^n$ have a nontrivial complex solution $\mathbf{0} \neq \mathbf{z} \in \mathbb{C}^n$.

Lemma 3.5. *Let $G_i(\mathbf{x}) = \mathbf{x}^\top A_i \mathbf{x}$ for $i = 1, \dots, m$ with each A_i being a real, symmetric $n \times n$ matrix. Consider a new system $H_j(\mathbf{x}) = \mathbf{x}^\top B_j \mathbf{x}$ of $r \geq m+1$ equations in $s \geq n$ indeterminates:*

$$B_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \ i = 1, \dots, m; \quad B_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ j = m+1, \dots, r-1; \quad B_r = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix};$$

in which I is the $(s-n) \times (s-n)$ identity matrix. Then $\{H_j(\mathbf{x}) = 0\}_{j=1}^r$ have a nontrivial real solution $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^s$ if and only if $\{G_i(\mathbf{x}) = 0\}_{i=1}^m$ have a nontrivial real solution $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$.

To prove that Problem 3.2 is NP hard, we shall reduce graph 3-colorability to quadratic feasibility over $\mathbb{C}$. The process of turning colorability problems into polynomial systems appears to originate with Bayer's thesis [3] although it has appeared in many other places, including the work of de Loera [14, 15] and Lovász [35]. For a recent discussion of these ideas, see the paper [26].

Let $G = (V, E)$ be a simple, undirected graph with vertices $V = \{1, \dots, n\}$ and m edges E. A (vertex) 3-*coloring* of G is a map $\nu : V \to \{1, \omega, \omega^2\}$, in which $\omega = e^{2\pi\sqrt{-1}/3} \in \mathbb{C}$ is a primitive cube root of unity. We say that a 3-coloring ν is *proper* if adjacent vertices receive different colors; otherwise ν is *improper*. The graph G is 3-*colorable* if there exists a proper 3-coloring of G.

Definition 3.6. *The color encoding of* G *is the set of* $3n+|E|$ *polynomials in* $2n+1$ *indeterminates:*

$$C_G := \begin{cases} x_i y_i - z^2, \quad y_i z - x_i^2, \quad x_i z - y_i^2, & i = 1, \dots, n, \\ x_i^2 + x_i x_j + x_i^2, & \{i, j\} \in E. \end{cases} \tag{3.1}$$

Lemma 3.7. C_G *has a common nonzero complex solution if and only if the graph* G *is* 3*-colorable.*

Proof. Suppose that G is 3-colorable and let $(x_1, \dots, x_n) \in \mathbb{C}^n$ be a proper 3-coloring of G. Set $z = 1$ and $y_i = 1/x_i$ for $i = 1, \dots, n$; we claim that these numbers are a common zero of C_G. It is clear that the first $3n$ polynomials in (3.1) evaluate to zero. Next consider any expression of the form $x_i^2 + x_i x_j + x_i^2$ in which $\{i, j\} \in E$. Since we have a 3-coloring, it follows that $x_i \neq x_j$; thus,

$$0 = x_i^3 - x_j^3 = (x_i^3 - x_j^3)/(x_i - x_j) = x_i^2 + x_i x_j + x_i^2.$$

Conversely, suppose that the set of polynomials C_G has a common nontrivial solution,

$$(x_1, \dots, x_n, y_1, \dots, y_n, z) \in \mathbb{C}^{2n+1} \setminus \{\mathbf{0}\}.$$

If $z = 0$, then all of the x_i and y_i must be zero as well. Thus, we have $z \neq 0$, and since the equations are homogenous, we may assume that our solution has $z = 1$. It follows that $x_i^3 = 1$ for all i so that $(x_1, \dots, x_n)$ is a 3-coloring of G. We are left with verifying that it is proper. If $\{i, j\} \in E$ and $x_i = x_j$, then $0 = x_i^2 + x_i x_j + x_i^2 = 3x_i^3$, so that x_i cannot be a root of unity. It follows that $x_i \neq x_j$ for all $\{i, j\} \in E$, and thus G is 3-colorable. $\square$

Remark 3.8. *The last* $|E|$ *polynomials in the definition* (3.1) *of* C_G *may be replaced by the following* $|V| = n$ *quadratics:* $p_i(\mathbf{x}) := \sum_{\{i,j\} \in E} (x_i^2 + x_i x_j + x_j^2)$, $i = 1, \dots, n$.

Proof of Theorem 3.2. Given a graph G, construct the color encoding C_G. Lemma 3.7 says that the set of homogeneous quadratic polynomials C_G has a nonzero complex solution if and only if G is 3-colorable. Adding dummy indeterminates to make the system square by Lemma 3.5, it has a nonzero complex solution iff the system given by Lemma 3.4 has a nonzero real solution. $\square$

4. Systems of Bilinear Equations

Multilinear systems of equations have been studied as early as in the early 19th century. The following result was known to Cayley [8].

Example 4.1 ($2 \times 2 \times 2$ hyperdeterminant)**.** *For* $\mathcal{A} = [\![a_{ijk}]\!] \in \mathbb{C}^{2\times2\times2}$, *we define*

$$\operatorname{Det}_{2,2,2}(\mathcal{A}) = \frac{1}{4}\left[\det\left(\begin{bmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{bmatrix} + \begin{bmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{bmatrix}\right) - \det\left(\begin{bmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{bmatrix} - \begin{bmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{bmatrix}\right)\right]^2$$
$$- 4\det\begin{bmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{bmatrix} \det\begin{bmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{bmatrix}.$$

A result that parallels the matrix case is the following: the system of bilinear equations

$$\begin{aligned} a_{000}x_0y_0 + a_{010}x_0y_1 + a_{100}x_1y_0 + a_{110}x_1y_1 &= 0, & a_{001}x_0y_0 + a_{011}x_0y_1 + a_{101}x_1y_0 + a_{111}x_1y_1 &= 0, \\ a_{000}x_0z_0 + a_{001}x_0z_1 + a_{100}x_1z_0 + a_{101}x_1z_1 &= 0, & a_{010}x_0z_0 + a_{011}x_0z_1 + a_{110}x_1z_0 + a_{111}x_1z_1 &= 0, \\ a_{000}y_0z_0 + a_{001}y_0z_1 + a_{010}y_1z_0 + a_{011}y_1z_1 &= 0, & a_{100}y_0z_0 + a_{101}y_0z_1 + a_{110}y_1z_0 + a_{111}y_1z_1 &= 0, \end{aligned}$$

has a non-trivial solution iff $\operatorname{Det}_{2,2,2}(\mathcal{A}) = 0$.

More generally, given $3n$ coefficient matrices $A_k, B_k, C_k \in \mathbb{F}^{n\times n}$, $k = 1, \dots, n$, we would like to solve the system of bilinear equations

$$\mathbf{x}^\top A_k \mathbf{y} = \alpha_k, \quad \mathbf{y}^\top B_k \mathbf{z} = \beta_k, \quad \mathbf{z}^\top C_k \mathbf{x} = \gamma_k, \qquad k = 1, \dots, n. \tag{4.1}$$

We will restrict our attention to the homogenous case where $\alpha_k = \beta_k = \gamma_k = 0$ for $k = 1, \dots, n$. We will show that the associated feasibility problem is NP hard.

Our definition of a bilinear system in (4.1) is not arbitrary. If the coefficient matrices are obtained from a tensor $\mathcal{A} = [\![a_{ijk}]\!]_{i,j,k=1}^n \in \mathbb{F}^{n\times n\times n}$ as $A_k = [a_{ijk}]_{i,j=1}^n$, $B_j = [a_{ijk}]_{i,k=1}^n$, $C_i = [a_{ijk}]_{j,k=1}^n$, $i, j, k = 1, \dots, n$, then there exists a non-zero solution in the homogeneous case iff the $n \times n \times n$ hyperdeterminant of $\mathcal{A}$ is zero [21, 22] — exactly as in the matrix case.

In this section, we consider some natural bilinear extensions to the quadratic feasibility problems encountered earlier. The main result of this section is Theorem 4.6, which shows that the following bilinear feasibility problem (*tensor quadratic feasibility over* $\mathbb{R}$) is NP hard. In Section 6, we use this result to show that certain singular value problems for tensors are NP hard.

Problem 4.2 ($\mathrm{TQF}_{\mathbb{R}}$)**.** *Let $\mathcal{A} = [\![a_{ijk}]\!] \in \mathbb{R}^{l\times m\times n}$ be a real tensor, and let $A_i(j,k) = a_{ijk}$, $B_j(i,k) = a_{ijk}$, and $C_k(i,j) = a_{ijk}$ be all the slices of $\mathcal{A}$. Determine if the following set of equations*

$$\mathbf{v}^\top A_i \mathbf{w} = 0, \quad i = 1, \dots, l; \quad \mathbf{u}^\top B_j \mathbf{w} = 0, \quad j = 1, \dots, m; \quad \mathbf{u}^\top C_k \mathbf{v} = 0, \quad k = 1, \dots, n; \tag{4.2}$$

has a solution $\mathbf{u} \in \mathbb{R}^l$, $\mathbf{v} \in \mathbb{R}^m$, $\mathbf{w} \in \mathbb{R}^n$ with $\mathbf{u} \neq 0$, $\mathbf{v} \neq 0$ and $\mathbf{w} \neq 0$.

When $\mathbb{R}$ is replaced by $\mathbb{C}$, we call the corresponding problem *tensor quadratic feasibility over* $\mathbb{C}$. Before approaching Problem 4.2, we consider the closely related *triple quadratic feasibility.*

Problem 4.3 ($3\mathrm{QF}_{\mathbb{R}}$)**.** *Let A_i, B_i, C_i be real $n \times n$ matrices for each $i = 1, \dots, n$. Determine if the following set of equations*

$$\mathbf{v}^\top A_i \mathbf{w} = 0, \quad i = 1, \dots, n; \quad \mathbf{u}^\top B_j \mathbf{w} = 0, \quad j = 1, \dots, n; \quad \mathbf{u}^\top C_k \mathbf{v} = 0, \quad k = 1, \dots, n; \tag{4.3}$$

has a solution $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ with $u \neq 0$, $v \neq 0$ and $w \neq 0$.

Problem 4.3 is NP hard even on the subclass of problems where $B_i = C_i$ for all i. As we explain later, if one can show that triple quadratic feasibility is NP hard for A_i, B_i, and C_i coming from a tensor $\mathcal{A}$ with $l = m = n$, then checking if 0 is the hyperdeterminant of a tensor is also NP hard. This is still an open problem. The proof of the following theorem is in the appendix.

Theorem 4.4. *Problem 4.3 is NP hard.*

We close this section with a proof that tensor quadratic feasibility is a difficult problem. However, first we verify that the complexity of the problem does not change when passing to the $\mathbb{C}$.

Lemma 4.5. *Let $\mathcal{A} \in \mathbb{R}^{l\times m\times n}$ be a real tensor. There is a tensor $\mathcal{B} \in \mathbb{R}^{2l\times 2m\times 2n}$ such that tensor quadratic feasibility over $\mathbb{R}$ for $\mathcal{B}$ is the same as tensor quadratic feasibility over $\mathbb{C}$ for $\mathcal{A}$*

Proof. Consider the tensor $\mathcal{B} = [\![b_{ijk}]\!] \in \mathbb{R}^{2l\times 2m\times 2n}$ given by setting its slices $B_i(j,k) = b_{ijk}$ as follows:

$$B_i = \begin{bmatrix} A_i & 0 \\ 0 & -A_i \end{bmatrix}, \; B_{l+i} = \begin{bmatrix} 0 & A_i \\ A_i & 0 \end{bmatrix}, \quad i = 1, \dots, l.$$

It is straightforward to check that real solutions to (4.2) for the tensor $\mathcal{B}$ correspond in a one-to-one manner with complex solutions to (4.2) for the tensor $\mathcal{A}$. $\square$

Theorem 4.6. *Graph 3-colorability is polynomial reducible to Problem 4.2. Thus, Problem 4.2 is NP Hard.*

Proof. Given a graph $G = (V, E)$ with $|V| = k$, we shall form a tensor $\mathcal{A} = \mathcal{A}(G) \in \mathbb{Z}^{l\times m\times n}$ with $l = k(2k+5)$ and $m = n = (2k+1)$ having the property that system (4.2) has a nonzero complex solution if and only if G has a proper 3-coloring. An appeal to Lemma 4.5 completes the proof.

Consider vectors $\mathbf{v} = (x_1, \dots, x_k, y_1, \dots, y_k, t)^\top$ and $\mathbf{w} = (\hat{x}_1, \dots, \hat{x}_k, \hat{y}_1, \dots, \hat{y}_k, \hat{t})^\top$ of indeterminates. The 2×2 minors of the matrix formed by placing $\mathbf{v}$ and $\mathbf{w}$ side-by-side are $k(2k+1)$ quadratics. Thus, the set of complex numbers determined by the vanishing of all these polynomials can be described by equations $\mathbf{v}^\top A_i \mathbf{w} = 0$, $i = 1, \dots, k(2k+1)$, for matrices $A_i \in \mathbb{Z}^{(2k+1)\times(2k+1)}$ with entries in $\{-1, 0, 1\}$. By construction, if these equations are satisfied for some $\mathbf{v}, \mathbf{w} \neq 0$, then there is a $c \in \mathbb{C} \setminus \{0\}$ such that $\mathbf{v} = c\mathbf{w}$.

Next, we write down the $3k$ equations $\mathbf{v}^\top A_i \mathbf{w} = 0$ for $i = k(2k+1)+1, \dots, k(2k+1)+3k$ whose vanishing (along with the equations above) implies that the x_i are cubic roots of unity; see (3.1). We also encode k equations $\mathbf{v}^\top A_i \mathbf{w}$ for $i = k(2k+4)+1, \dots, k(2k+4)+k$ whose vanishing says that x_i and x_j are different if $\{i, j\} \in E$ (c.f. Remark 3.8). Finally, consider the tensor $\mathcal{A}(G) = [\![a_{ijk}]\!] \in \mathbb{Z}^{l\times m\times n}$ given by $a_{ijk} = A_i(j, k)$.

We verify that $\mathcal{A}$ has the claimed property. Suppose that there are three nonzero complex vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ which satisfy tensor quadratic feasibility (over $\mathbb{C}$). Then, from construction, $\mathbf{v} = c\mathbf{w}$ and also $\mathbf{v}$ encodes a proper 3-coloring of the graph G. On the other hand, suppose that G is 3-colorable with a coloring represented using roots of unity $(x_1, \dots, x_k) \in \mathbb{C}^k$ (as discussed in Section 3). Then, the vectors $\mathbf{v} = \mathbf{w} = (x_1, \dots, x_k, x_1^{-1}, \dots, x_k^{-1}, 1)$ satisfy the first set of equations in (4.2). The other sets of equations define a linear system for the vector $\mathbf{u}$ consisting of $4k+2$ equations in $l = k(2k+5) > 4k+2$ unknowns. In particular, there is always a $\mathbf{u}$ solving them, proving that complex tensor quadratic feasibility is true for $\mathcal{A}$. □

5. Tensor eigenvalue is NP hard

The eigenvalues and eigenvectors of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ are the stationary values and stationary points of its Rayleigh quotient $q(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}/\mathbf{x}^\top\mathbf{x}$. Equivalently, one may look at the quadratic form $\mathbf{x}^\top A\mathbf{x} = \sum_{i,j=1}^n a_{ij}x_ix_j$ constrained to the unit l^2-sphere,

$$\|\mathbf{x}\|_2^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = 1. \tag{5.1}$$

The stationary conditions of the Lagrangian $L(\mathbf{x}, \lambda) = \mathbf{x}^\top A\mathbf{x} - \lambda(\|\mathbf{x}\|_2^2 - 1)$ at a stationary point $(\lambda, \mathbf{x})$ yields the familiar eigenvalue equation $A\mathbf{x} = \lambda\mathbf{x}$ which is then used to define eigenvalue/eigenvector pairs for any square (not necessarily symmetric) matrices.

The above discussion may be extended to yield a notion of eigenvalues and eigenvectors for a 3-tensor $\mathcal{A} \in \mathbb{R}^{n\times n\times n}$. We start from suitably constrained stationary values and stationary points of the cubic form $\mathcal{A}(\mathbf{x}, \mathbf{x}, \mathbf{x}) = \sum_{i,j,k=1}^n a_{ijk}x_ix_jx_k$ associated with a 3-tensor $\mathcal{A} \in \mathbb{R}^{n\times n\times n}$. What is not so certain is the appropriate generalization of the (5.1). One may retain the constraint (5.1). Alternatively, one may view (5.1) as defining either the l^2-sphere or a unit sum-of-squares with a corresponding generalization to the l^3-sphere,

$$\|\mathbf{x}\|_3^3 = |x_1|^3 + |x_2|^3 + \cdots + |x_n|^3 = 1, \tag{5.2}$$

or a unit sum-of-cubes,

$$x_1^3 + x_2^3 + \cdots + x_n^3 = 1. \tag{5.3}$$

Each of these choices has its advantage: condition (5.2) defines a compact set while condition (5.3) defines an algebraic set — both result in eigenvectors that are scale-invariant (i.e. if $\mathbf{x}$ is an eigenvector, then so is $c\mathbf{x}$ for any $c \neq 0$); condition (5.1) defines a set that is both compact and algebraic but results in eigenvectors that are not scale-invariant. These were proposed independently and studied in [34, 38].

If we examine the stationary condition of the Lagrangian $L(\mathbf{x}, \lambda) = \mathcal{A}(\mathbf{x}, \mathbf{x}, \mathbf{x}) - \lambda c(\mathbf{x})$ where $c(\mathbf{x})$ is one of the conditions in (5.1), (5.1), or (5.3), we obtain the following definition.

Definition 5.1. *Let $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$. λ is said to be an l^2-**eigenvalue** and $\mathbf{v}$ an l^2-**eigenvector** of $\mathcal{A} \in \mathbb{R}^{n\times n\times n}$ if*

$$\sum\nolimits_{i,j=1}^{n} a_{ijk}x_ix_j = \lambda x_k, \quad k = 1,\dots,n, \tag{5.4}$$

*λ is said to be an l^3-**eigenvalue** and $\mathbf{v}$ an l^3-**eigenvector** of $\mathcal{A}$ if*

$$\sum\nolimits_{i,j=1}^{n} a_{ijk}x_ix_j = \lambda x_k^2, \quad k = 1,\dots,n. \tag{5.5}$$

By (2.3), shorthands for (5.4) and (5.5) are $\mathcal{A}(\mathbf{x},\mathbf{x},I_n) = \lambda\mathbf{x}$ and $\mathcal{A}(\mathbf{x},\mathbf{x},I_n) = \lambda\mathcal{I}(\mathbf{x},\mathbf{x},I_n)$.

Problem 5.2 (Tensor eigenvalue)**.** *Given a 3-tensor $\mathcal{A} \in \mathbb{R}^{n\times n\times n}$. Determine $\lambda \in \mathbb{R}$ such that (5.4) or (5.5) has a solution for some $\mathbf{x} \in \mathbb{R}^n\backslash\{\mathbf{0}\}$.*

By quadratic feasibility (Theorem 3.1), we obtain the following result.

Theorem 5.3. *The tensor l^2- and l^3-eigenvalue problems are NP hard.*

Specifically, our methods show that determining if $\lambda = 0$ is an eigenvector of a tensor is an NP hard problem. We remark that the situation is the same for determining whether any fixed rational number λ is an l^3-eigenvalue, which is easily seen from the form of equations (5.5).

6. Tensor singular value is NP hard

The argument for deriving the singular value equations for a 3-tensor is similar to that for the eigenvalue equation in the previous section. It is easy to verify that the singular values and singular vectors of a matrix $A \in \mathbb{R}^{m\times n}$ are the stationary values and stationary points of the quotient $\mathbf{x}^\top A\mathbf{y}/\|\mathbf{x}\|_2\|\mathbf{y}\|_2$. Indeed, at a stationary point $(\mathbf{x},\mathbf{y},\sigma)$, the first order condition of the associated Lagrangian $L(\mathbf{x},\mathbf{y},\sigma) = \mathbf{x}^\top A\mathbf{y} - \sigma(\|\mathbf{x}\|_2\|\mathbf{y}\|_2 - 1)$ yields the familiar singular value equations $A\mathbf{v} = \sigma\mathbf{u}$, $A^\top\mathbf{u} = \sigma\mathbf{v}$ where $\mathbf{u} = \mathbf{x}/\|\mathbf{x}\|_2$ and $\mathbf{v} = \mathbf{y}/\|\mathbf{y}\|_2$.

This derivation has been extended to higher-order tensors to define a notion of singular values and singular vectors for tensors [34]. For $\mathcal{A} \in \mathbb{R}^{l\times m\times n}$, the Lagrangian $L(\mathbf{x},\mathbf{y},\mathbf{z},\sigma) = \mathcal{A}(\mathbf{x},\mathbf{y},\mathbf{z}) - \sigma(\|\mathbf{x}\|_p\|\mathbf{y}\|_p\|\mathbf{z}\|_p - 1)$ is straightforward to define. The only ambiguity is in the choice of p.

Definition 6.1. *Let $\sigma \in \mathbb{R}$, $\mathbf{u} \in \mathbb{R}^l\backslash\{\mathbf{0}\}$, $\mathbf{v} \in \mathbb{R}^m\backslash\{\mathbf{0}\}$, $\mathbf{w} \in \mathbb{R}^n\backslash\{\mathbf{0}\}$. σ is said to be an l^2-**singular value** and $\mathbf{u},\mathbf{v},\mathbf{w}$ l^2-**singular vectors** of $\mathcal{A}$ if*

$$\sum\nolimits_{i,j=1}^{l,m} a_{ijk}u_iv_j = \sigma w_k, \quad k = 1,\dots,n, \tag{6.1}$$

*λ is said to be an l^3-**singular value** and $\mathbf{x}$ an l^3-**singular vector** of $\mathcal{A}$ if*

$$\sum\nolimits_{i,j=1}^{l,m} a_{ijk}u_iv_j = \sigma w_k^2, \quad k = 1,\dots,n. \tag{6.2}$$

Note that the spectral norm $\|\mathcal{A}\|_{2,2,2}$ is either the maximum or minimum value of $\mathcal{A}(\mathbf{x},\mathbf{y},\mathbf{z})$ constrained to the set $\{(\mathbf{x},\mathbf{y},\mathbf{z}) \mid \|\mathbf{x}\|_2\|\mathbf{y}\|_2\|\mathbf{z}\|_2 = 1\}$ and thus an l^2-singular value of $\mathcal{A}$.

We shall prove the following theorems in this section.

Theorem 6.2. *Determining whether $\sigma = 1$ is an l^2 singular value of a tensor $\mathcal{A}$ is NP hard.*

Theorem 6.3. *Determining whether $\sigma = 0$ is an l^2 or l^3 singular value of a tensor $\mathcal{A}$ is NP hard.*

By homogeneity and the observation that $\sigma_{\max}(A) = \|\mathcal{A}\|_{2,2,2}$, the following is immediate.

Corollary 6.4. *Fix $\sigma \in \mathbb{Q}$. Determining if σ is an l^2 singular value of a tensor $\mathcal{A}$ is NP hard.*

Corollary 6.5 (He-Li-Zhang [25])**.** *Tensor spectral norm is NP hard.*

We employ two very different methods to prove Theorems 6.2 and 6.3. For the first theorem, we shall reduce the decision problem to that of computing the max-clique number for a graph, extending some ideas from [37] and [25]. The proof of the second theorem, on the other hand, follows directly from Theorem 4.6 which was proved by a reduction to 3-colorability.

We begin by describing the setup required to prove Theorem 6.2. Let $G = (V, E)$ be a simple graph on vertices $V = \{1, \dots, n\}$ with m edges E, and let $\omega = \omega(G)$ be the clique number of G (that is, the number of vertices in a largest clique of G). An important result linking an optimization problem to the number ω is the following classical theorem of Motzkin and Straus [36, 1].

Theorem 6.6 (Motzkin-Straus). *Let $\Delta_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i = 1\}$ and let $G = (V, E)$ be a graph on n vertices with clique number $\omega(G)$. Then,*

$$1 - \frac{1}{\omega(G)} = 2 \cdot \max_{\mathbf{x} \in \Delta_n} \sum\nolimits_{\{i,j\} \in E} x_i x_j.$$

Let A_G be the adjacency matrix of the graph G. For each positive integer ℓ, define $Q_\ell := A_G + \frac{1}{\ell} J$, in which J is the all ones matrix. Also, let

$$m_\ell := \max_{\mathbf{x} \in \Delta_n} \mathbf{x}^\top Q_\ell \mathbf{x} = 1 + \frac{\omega - \ell}{\ell \omega}.$$

By the Motzkin-Straus theorem, we have $m_\omega = 1$ and also that

$$m_\ell < 1 \ \text{ if } \ \ell > \omega; \quad m_\ell > 1 \ \text{ if } \ \ell < \omega. \tag{6.3}$$

For $k = 1, \dots, m$, let $E_k = \frac{1}{2} E_{i_k j_k} + \frac{1}{2} E_{j_k i_k}$ in which $\{i_k, j_k\}$ is the kth edge of G. Here, the $n \times n$ matrix E_{ij} has a 1 in the (i, j)th spot and zeroes elsewhere. For each positive integer ℓ, consider the following optimization problem (having rational input):

$$M_\ell := \max_{\|\mathbf{u}\|_2 = 1} \left\{ \sum\nolimits_{i=1}^{\ell} (\mathbf{u}^\top \ell^{-1} I \mathbf{u})^2 + \sum\nolimits_{k=1}^{m} (\mathbf{u}^\top E_k \mathbf{u})^2 + \sum\nolimits_{k=1}^{m} (\mathbf{u}^\top E_k \mathbf{u})^2 \right\}.$$

Lemma 6.7. *For any graph G, we have $M_\ell = m_\ell$.*

Proof. By construction, $M_\ell = \frac{1}{\ell} + 2 \cdot \max_{\|\mathbf{u}\|_2 = 1} \sum_{\{i,j\} \in E} u_i^2 u_j^2$, which is easily seen to equal m_ℓ. $\square$

Using an observation of [25], we further note the following.

Lemma 6.8.

$$M_\ell = \max_{\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1} \left\{ \sum\nolimits_{i=1}^{\ell} (\mathbf{u}^\top \ell^{-1} I \mathbf{v})^2 + \sum\nolimits_{k=1}^{m} (\mathbf{u}^\top E_k \mathbf{v})^2 + \sum\nolimits_{k=1}^{m} (\mathbf{u}^\top E_k \mathbf{v})^2 \right\}.$$

Proof. The proof can be found in [25]. The argument uses the KKT conditions to find a maximizer with $\mathbf{v} = \pm \mathbf{u}$ for any optimization problem involving sums of squares of bilinear forms in $\mathbf{u}, \mathbf{v}$. $\square$

Next, let

$$T_\ell := \max_{\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = \|\mathbf{w}\|_2 = 1} \left\{ \sum\nolimits_{i=1}^{\ell} (\mathbf{u}^\top \ell^{-1} I \mathbf{v}) w_i + \sum\nolimits_{k=1}^{m} (\mathbf{u}^\top E_k \mathbf{v}) w_{\ell+k} + \sum\nolimits_{k=1}^{m} (\mathbf{u}^\top E_k \mathbf{v}) w_{m+\ell+k} \right\}. \tag{6.4}$$

Lemma 6.9. *The optimization problem T_ℓ has*

$$T_\ell = 1 \ \textit{iff} \ \ell = \omega; \quad T_\ell > 1 \ \textit{iff} \ \ell < \omega; \quad T_\ell < 1 \ \textit{iff} \ \ell > \omega.$$

Proof. We shall prove that $T_\ell = M_\ell^{1/2}$, from which the lemma follows (using (6.3)). If we fix real numbers $\mathbf{a} = (a_1, \dots, a_s)^\top$, then by the Cauchy-Schwarz inequality, a sum of the form $\sum_{i=1}^s a_i w_i$ with $\|\mathbf{w}\|_2 = 1$ achieves a maximum value of $\|\mathbf{a}\|_2$ (with $\mathbf{w}_i = \mathbf{a}_i / \|\mathbf{a}\|_2$ if $\|\mathbf{a}\|_2 \neq 0$). A straightforward computation using this observation now shows that $T_\ell = M_\ell^{1/2}$. $\square$

Proof of Theorem 6.2. We cast (6.4) in the form of a tensor optimization problem. Set $\mathcal{A}_\ell$ to be the three dimensional tensor with a_{ijk} equal to the coefficient of the term $u_i v_j w_k$ in the multilinear form (6.4). Then, T_ℓ is just the maximum l^2-singular value of $\mathcal{A}_\ell$. We show that if we could decide whether $\sigma = 1$ is an l^2 singular value of $\mathcal{A}_\ell$, then we could solve the max-clique problem (a well-known NP-Complete problem). Given a graph G, construct the tensor $\mathcal{A}_\ell$ for each integer $\ell \in \{1, \dots, n\}$. The largest value of ℓ for which 1 is a singular value of $\mathcal{A}_\ell$ is $\omega(G)$. To see this, notice that if ℓ is larger than $\omega(G)$, the maximum singular value of $\mathcal{A}_\ell$ is smaller than 1 by Lemma 6.9. Therefore, $\sigma = 1$ cannot be a singular value of $\mathcal{A}_\ell$ in these cases. However, $\sigma = 1$ is a singular value of the tensor $\mathcal{A}_\omega$. □

7. Rank-1 tensor approximation is NP hard

For $r > 1$, the best rank-r approximation problem for tensors does not have a solution in general

$$\min_{\mathbf{x}_i,\mathbf{y}_i,\mathbf{z}_i} \|\mathcal{A} - \lambda_1 \mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{z}_1 - \cdots - \lambda_r \mathbf{x}_r \otimes \mathbf{y}_r \otimes \mathbf{z}_r\|_F$$

because the set $\{\mathcal{A} \in \mathbb{F}^{l \times m \times n} \mid \operatorname{rank}(\mathcal{A}) \le r\}$ is in general not a closed set when $r > 1$. The following is a simple example based on an exercise in [30].

Examples 7.1. *Let $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{F}^m$, $i = 1, 2, 3$. Let $A := \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{y}_3 + \mathbf{x}_1 \otimes \mathbf{y}_2 \otimes \mathbf{x}_3 + \mathbf{y}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3$ and for $n \in \mathbb{N}$, let*

$$A_n := \mathbf{x}_1 \otimes \mathbf{x}_2 \otimes (\mathbf{y}_3 - n\mathbf{x}_3) + \left(\mathbf{x}_1 + \frac{1}{n}\mathbf{y}_1\right) \otimes \left(\mathbf{x}_2 + \frac{1}{n}\mathbf{y}_2\right) \otimes n\mathbf{x}_3.$$

One may show that $\operatorname{rank}(A) = 3$ *iff $\mathbf{x}_i, \mathbf{y}_i$ are linearly independent, $i = 1, 2, 3$. Since it is clear that* $\operatorname{rank}(A_n) \le 2$ *and* $\lim_{n\to\infty} A_n = A$*, the rank-3 tensor A has no best rank-2 approximation.*

This phenomenon where a tensor fails to have a best rank-r approximation is much more widespread than one might imagine, occurring over a wide range of dimensions, orders, and ranks; happens regardless of the choice of norm (or even Brègman divergence) used. These counterexamples occur with positive probability and in some cases with certainty (in $\mathbb{F}^{2\times 2\times 2}$, no tensor of rank-3 has a best rank-2 approximation). We refer the reader to [16] for further details.

The set of rank-1 tensors (together with zero), $\{\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} \mid \mathbf{x} \in \mathbb{F}^l, \mathbf{y} \in \mathbb{F}^m, \mathbf{z} \in \mathbb{F}^n\}$, is on the other hand closed. In fact the set is precisely the Segre variety in classical algebraic geometry. We would like to find the best rank-1 approximation

$$\min_{\mathbf{x},\mathbf{y},\mathbf{z}} \|\mathcal{A} - \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}\|_F.$$

By introducing an additional parameter $\sigma \ge 0$, we may write the rank-1 term in a form where $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = \|\mathbf{z}\|_2 = 1$. Then

$$\|\mathcal{A} - \sigma \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}\|_F^2 = \|\mathcal{A}\|_F^2 - 2\sigma\langle \mathcal{A}, \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}\rangle + \sigma^2.$$

This is minimized when $\sigma = \max_{\|\mathbf{x}\|_2=\|\mathbf{y}\|_2=\|\mathbf{z}\|_2=1}\langle \mathcal{A}, \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}\rangle$. But $\langle \mathcal{A}, \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}\rangle = \mathcal{A}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and so $\sigma = \|\mathcal{A}\|_{2,2,2}$, which we already know is NP hard to compute.

8. Tensor rank is NP hard over $\mathbb{R}$ and $\mathbb{C}$

Johan Håstad [24] has shown that any 3SAT boolean formula in n variables and m clauses has an associated $(n + 2m + 2) \times 3n \times (3n + m)$ tensor and that the boolean formula is satisfiable iff the associated tensor has rank not more than $4n + 2m$. A consequence is that tensor rank is NP hard over $\mathbb{Q}$ and NP-complete over finite fields.

Since the majority of recent applications of tensor methods are over $\mathbb{R}$ and $\mathbb{C}$, a natural question is whether tensor rank is also NP hard over these fields for, say, tensors with rational entries. In other words, given a tensor $\mathcal{A}$ with rational entries and $r \in \mathbb{N}$, check if $\operatorname{rank}_{\mathbb{R}}(\mathcal{A}) \le r$ or if

$\operatorname{rank}_{\mathbb{C}}(\mathcal{A}) \leq r$ (Håstad's result implies that checking $\operatorname{rank}_{\mathbb{Q}}(\mathcal{A}) \leq r$ is NP hard). We assume, as in previous sections, access to an oracle capable of performing real and complex arithmetic internally.

Note that tensor rank depends on the base field; an example of a tensor with real entries whose rank over $\mathbb{C}$ is strictly less than its rank over $\mathbb{R}$ may be found in [16]. The same can happen for tensors with rational entries; the argument can be found in the appendix.

Proposition 8.1. *The rank over $\mathbb{R}$ of a tensor with rational entries can be strictly less than its rank over $\mathbb{Q}$.*

We would like to provide here an addendum to Håstad's result by pointing out that his proof contains the following stronger result. In particular, tensor rank is also NP hard over $\mathbb{R}$ and $\mathbb{C}$.

Corollary 8.2. *Tensor rank is NP hard over any field as long as one is able to compute field arithmetic in that field.*

Proof. Observe that the matrices V_i, S_i, M_i, C_i in [24] that comprise the 'slices' of Håstad's tensor are defined with only the additive identity 0, the multiplicative identity 1, and the additive inverse of the multiplicative identity -1. Håstad's argument does not depend on any special properties of $\mathbb{Q}$ or finite fields but only on the field axioms and that $0 \neq 1$ (his argument also works in characteristic 2 when $-1 = 1$). ☐

Appendix

In this section, we include proofs of theorems that were omitted in the main body of text because of space considerations.

Proof of Proposition 4.4. By Theorem 3.2, it is enough to prove that a given quadratic feasibility problem can be polynomially reduced to this one. Therefore, suppose that A_i are given $n \times n$ real matrices for which we would like to determine if $\mathbf{x}^\top A_i \mathbf{x} = 0$ $(i = 1, \dots, n)$ has a solution $0 \neq \mathbf{x} \in \mathbb{R}^n$. Let E_{ij} denote the matrix with a 1 in the (i,j) entry and 0's elsewhere. Consider a system S as in (4.3) in which we also define

$$B_1 = E_{11} \text{ and } B_i = E_{1i} - E_{i1} \text{ for } i = 2, \dots, n;$$

$$C_1 = E_{11} \text{ and } C_i = E_{1i} - E_{i1} \text{ for } i = 2, \dots, n.$$

We shall construct a decision tree based on the answer to feasibility questions involving systems having the form of S. This will give us an algorithm to determine whether the original quadratic problem is feasible.

Consider changing system S by replacing B_1 and C_1 with matrices consisting of all zeroes, and call this system S'. We first make two claims about solutions to S and S'.

Claim 1: If S has a solution, then $(u_1, v_1) = (0, 0)$. Moreover, in this case, $w_1 = 0$.

To show this, suppose first that S has a solution with $u_1 = 0$ and $v_1 \neq 0$. Then the form of the matrices C_i forces $u_2 = \cdots = u_n = 0$ as well. But then $\mathbf{u} = 0$, which contradicts S having a solution. A similar examination with $u_1 \neq 0$ and $v_1 = 0$ proves the claim. It is also easy to see that $w_1 = 0$ in this case.

Claim 2: Suppose that S has no solution. If S' has a solution, then $\mathbf{v} = c\mathbf{u}$ and $\mathbf{w} = d\mathbf{u}$ for some $0 \neq c, d \in \mathbb{R}$. Moreover, if S' has no solution, then the original quadratic problem has no solution.

To verify Claim 2, suppose first that S' has a solution $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ but S does not. In that case we must have $u_1 \neq 0$. Also, $v_1 \neq 0$ since otherwise the third set of equations $\{u_1 v_i - u_i v_1 = 0\}_{i=2}^n$ would force $\mathbf{v} = 0$. But then $\mathbf{v} = c\mathbf{u}$ for $c = \frac{v_1}{u_1}$ and $\mathbf{w} = d\mathbf{u}$ for $d = \frac{w_1}{u_1}$ as desired. On the other hand, suppose that both S and S' have no solution. We claim that $\mathbf{x}^\top A_i \mathbf{x} = 0$ $(i = 1, \dots, n)$ has no solution $\mathbf{x} \neq 0$ either. Indeed, if it did, then setting $\mathbf{u} = \mathbf{v} = \mathbf{w} = \mathbf{x}$, we would get a solution to S' (as one can easily check), a contradiction.

We are now prepared to give our method for solving quadratic feasibility using at most $n+2$ queries to the restricted version ($B_i = C_i$ for all i) of Problem 4.3.

First check if S has a solution. If it does not, then ask if S' has a solution. If it does not, then output "INFEASIBLE". This answer is correct by Claim 2. If S has no solution but S' does, then there is a solution with $\mathbf{v} = c\mathbf{u}$ and $\mathbf{w} = d\mathbf{u}$, both c and d nonzero. But then $\mathbf{x}^\top A_i \mathbf{x} = 0$ for $\mathbf{x} = \mathbf{u}$ and each i. Thus, we output "FEASIBLE".

If instead, S has a solution, then the solution necessarily has $(u_1, v_1, w_1) = (0,0,0)$. Consider now the $n-1$-dimensional system T in which A_i becomes the lower-right $(n-1) \times (n-1)$ block of A_i, and C_i and D_i are again of the same form as the previous ones. This is a smaller system with 1 less indeterminate. We now repeat the above examination inductively with starting system T replacing S.

If we make it to the final stage of this process without outputting an answer, then the original system S has a solution with

$$u_1 = \cdots = u_{n-1} = v_1 = \cdots = v_{n-1} = w_1 = \cdots = w_{n-1} = 0 \text{ and } u_n, v_n, w_n \text{ are all nonzero.}$$

It follows that the (n,n) entry of each A_i $(i = 1\ldots, n)$ is zero. Thus, it is clear that there is a solution $\mathbf{x}$ to the quadratic feasibility problem, and so we output "FEASIBLE".

We have therefore verified the algorithm terminates with the correct answer and it does so in polynomial time with an oracle that can solve Problem 4.3. □

Proof of Proposition 8.1. We will construct such a tensor explicitly. Let $\mathbf{x}$ and $\mathbf{y}$ be linearly independent over $\mathbb{Q}$. First observe that

$$\overline{\mathbf{z}}_1 \otimes \mathbf{z}_2 \otimes \overline{\mathbf{z}}_3 + \mathbf{z}_1 \otimes \overline{\mathbf{z}}_2 \otimes \mathbf{z}_3 = 2\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \mathbf{x}_3 - 4\mathbf{y}_1 \otimes \mathbf{y}_2 \otimes \mathbf{x}_3 + 4\mathbf{y}_1 \otimes \mathbf{x}_2 \otimes \mathbf{y}_3 - 4\mathbf{x}_1 \otimes \mathbf{y}_2 \otimes \mathbf{y}_3$$

where $\overline{\mathbf{z}} = \mathbf{x} - \sqrt{2}\mathbf{y}$ for $\mathbf{z} = \mathbf{x} + \sqrt{2}\mathbf{y}$. For simplicity, we choose $\mathbf{x}_i = [1,0]^\top, \mathbf{y}_i = [0,1]^\top \in \mathbb{Q}^2$ for $i = 1,2,3$. Let A be the tensor above. It is clear that $\operatorname{rank}_{\mathbb{R}}(A) \leq 2$. We claim that $\operatorname{rank}_{\mathbb{Q}}(A) > 2$. Suppose not and that there exist $\mathbf{u}_i = [a_i, b_i]^\top$, $\mathbf{v}_i = [c_i, d_i]^\top \in \mathbb{Q}^2$, $i = 1,2,3$, with

$$A = \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \mathbf{u}_3 + \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3.$$

This yields the following polynomial equations for unknown real quantities a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, and d_1, d_2, d_3:

$$\begin{aligned}
&a_1a_2a_3 + c_1c_2c_3 = 2,\ a_1b_2a_3 + c_1d_2c_3 = 0, b_1a_2a_3 + d_1c_2c_3 = 0,\\
&b_1b_2a_3 + d_1d_2c_3 = 4, a_1a_2b_3 + c_1c_2d_3 = 0,\ a_1b_2b_3 + c_1d_2d_3 = 4,\\
&b_1a_2b_3 + d_1c_2d_3 = 4,\ b_1b_2b_3 + d_1d_2d_3 = 0.
\end{aligned} \tag{8.1}$$

It is easily checked using Gröbner basis methods that the ideal (over $\mathbb{Q}$) generated by the polynomials defining these equations contains: $2c_1^2 - d_1^2$, $c_1d_2d_3 - 2$. It follows that in order for there to be rational solutions, we must have $c_1 = d_1 = 0$, a clear contradiction to the second equation. Thus, there cannot be any rational solutions to (8.1). □

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