Parts of the material in this article has appeared in converence articles. This material has been substantially extended and rewritten. In particular, the journal submission includes a more detailed theoretical analysis. Nevertheless, for complete information, we provide the conference articles in this supplement.

The articles are:

- S. Jegelka and J. Bilmes. Submodularity beyond Submodular Energies: Coupling Edges in Graph Cuts. *IEEE Conference of Computer Vision and Pattern Recognition (CVPR)*, 2011.
- S. Jegelka and J. Bilmes. Approximation Bounds for Inference using Cooperative Cut. *28th International Conference on Machine Learning (ICML)*, 2011.

Submodularity Beyond Submodular Energies: Coupling Edges in Graph Cuts

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Abstract

We propose a new family of non-submodular global energy functions that still use submodularity internally to couple edges in a graph cut. We show it is possible to develop an efficient approximation algorithm that, thanks to the internal submodularity, can use standard graph cuts as a subroutine. We demonstrate the advantages of edge coupling in a natural setting, namely image segmentation. In particular, for fine-structured objects and objects with shading variation, our structured edge coupling leads to significant improvements over standard approaches.

1. Introduction

For many years, Markov random fields (MRF) have been seen as a natural fit to solve various problems in computer vision [12]. In such models, finding a maximizing assignment of random variable values corresponds to minimizing a Gibbs energy. This minimization is in general not only NP-hard, but some models do not even admit any nontrivial approximation guarantees [5]. Consequently, for image processing, some early researchers considered MRFs destined for no more than a theoretical curiosity.

Recently, subclasses of MRFs were shown to be not only easy to exactly optimize (without a tree-width restriction) but also quite naturally applicable to many computer vision problems [4, 12, 23]. Specifically, finding the minimum energy configuration is very efficient for those Gibbs energy functions whose variable assignment costs correspond exactly to cut costs in an appropriate graph [8, 23]. Graph cuts are now successfully used in segmentation, stereo matching, and texture synthesis, among others.

Inspired by these results, a principal goal has become identifying the most general classes of energies that can be exactly optimized either directly or indirectly via graph cuts. For example, while some binary pairwise potential functions can be solved exactly using graph cuts, in many cases higher order (*e.g.*, k-ary) potential functions [8, 23, 33] and potentials functions over non-binary variables [4] can also be solved efficiently. In all cases, a critical property known as "regularity" [23] (more generally, submodularity [9]) is used.

Unfortunately, there are critical deficiencies when graph cuts are used in practice, partly stemming from their inability to represent more than only a limited class of energies [8, 10, 23, 33]. The core issue is that graph cuts model an energy that decomposes into *pairwise* terms with *nonnegative* weights. The direct use of such energies can cause insurmountable over-smoothing in image segmentation. While some higher order energies are graph-representable, this representation might regrettably require additional variables which also might not remain computationally feasible [33]. Recent research, therefore, has aimed to identify practically manageable higher order energies [15, 18, 19, 24], and to develop efficient optimization methods for non-submodular potentials [22].

In this work, we define a new powerful class of arbitrarily high order non-submodular energy functions that abandons neither the existence of an underlying graph, nor the use of submodularity, nor practical efficiency. This class is structurally and conceptually very different from the recently considered potentials in [15, 18, 24]. To wit, we make the following critical observation: graph cut based energy functions can be significantly enhanced if the cost of the edges that constitute a cut is measured **not** merely based on the sum of the edge weights. Rather, in our work, any or all the edges in a graph may interplay in complex ways. Formally, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph where each (s,t)-cut induces an assignment of pixel labels. We replace the usual cut cost (the sum of edge weights) by a submodular cut cost, and we therefore say that the edges themselves may *cooperate* [30]. Doing so introduces the following problem:

Definition 1 (Minimum Cooperative Cut)**.** *Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a nondecreasing submodular function $f : 2^{\mathcal{E}} \rightarrow \mathbb{R}_+$ defined on subsets of* **edges** *$\mathcal{E}$, find an (s,t)-cut $\Gamma \subseteq \mathcal{E}$ having minimum cost $f(\Gamma)$.*

As shown below, the equivalent energy functions are not in general submodular and cooperative cut is NP-hard, even though submodularity is, in a sense, "internal" to our problem as will be seen. The graph structure is key to obtain an efficient approximation algorithm.

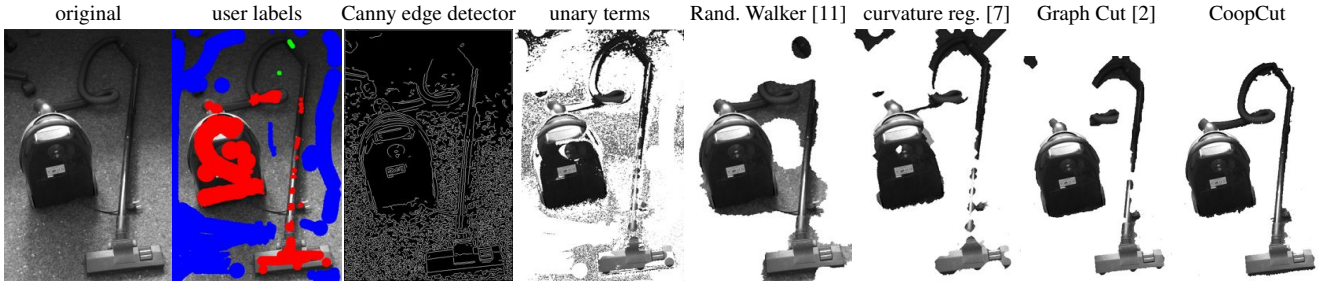


Figure 1. Segmentation results for an image with shading. The task is difficult despite many labels. All algorithms used the same unary terms, except for the Random Walker, which got enhanced seeds (green). Column 4 is the segmentation obtained from unary terms alone.

Edge cooperation naturally captures information that is missed by many existing approaches, such as global features of object boundaries in a segmentation. For example, consider the vacuum cleaner in Figure 1 (left): low contrast makes the tube difficult to identify, so that either background is included in the foreground, or parts of the tube are cut off. Such incorrect boundaries (chosen due to shortcutting) are qualitatively different from the correct boundary between the patterned carpet and the vacuum cleaner. We maintain that the boundary is "congruous": along it lies a repetitive pattern made up from the few adjoining texture types that, if properly represented, can help boundary identification. By globally coupling boundary sub-segments across lighted and shaded regions, such congruity can be exploited, and this is easily expressible by cooperative cuts as we show below.

More specifically, we show: 1) a new class of powerful energy functions having arbitrarily high order, where the potential functions and maximum order may automatically and efficiently adapt to each image; 2) an optimization method that is remarkably efficient and practical, and that uses standard graph cuts as a subroutine; 3) theoretical approximation guarantees for our optimization method; 4) a specific edge-cooperative potential for segmenting considerably difficult images that, compared to graph cuts, reduces the segmentation error by up to 70%. In particular, we show significant improvements on images having the potential for a severe shrinking bias problem, on images that possess light intensity gradients and shadows, and on images with both these difficulties simultaneously. Finally, we relate edge cooperation to other recent approaches in computer vision.

2. Background: Graph Cuts & Gibbs Energies

Before describing cooperative cuts, we recall the relationship between graph cuts and energies and in doing so define our notation. Labeling image pixels is often formulated as inference in an MRF. For each pixel i in an image I, a random variable takes values from a set $\mathcal{L}$ of labels. For simplicity, we consider only binary labels ($|\mathcal{L}| = 2$). A Gibbs energy $E(\mathbf{x};\bar{\mathbf{z}})$ over labelings $\mathbf{x} = \{x_1, \ldots, x_{|I|}\} \in \mathcal{L}^{|I|}$ defines the probability $p(\mathbf{x}|\bar{\mathbf{z}}) \propto \exp(-E(\mathbf{x};\bar{\mathbf{z}}))$ of a labeling given observed pixel values $\bar{\mathbf{z}}$. The energy decomposes into a sum of unary potential functions, making a connection to the image $\bar{\mathbf{z}}$, and a sum of clique potentials $\{\psi_C : C \in \mathcal{C}\}$, where $\mathcal{C}$ is the set of maximal cliques in the MRF. That is,

$$E(\mathbf{x};\bar{\mathbf{z}}) = \sum\nolimits_i \psi_i(x_i, \bar{\mathbf{z}}_i) + \sum\nolimits_{C\in\mathcal{C}} \psi_C(\mathbf{x}_C). \quad (1)$$

As $\bar{\mathbf{z}}$ is constant, using $E(\mathbf{x}) = E(\mathbf{x};\bar{\mathbf{z}})$ simplifies notation.

A pixel labeling is produced by finding a maximum a posteriori (MAP) (equivalently, energy minimizing) variable assignment, *i.e.*, $\mathbf{x}^* \in \operatorname{argmax}_{\mathbf{x}} p(\mathbf{x}|\bar{\mathbf{z}}) = \operatorname{argmin}_{\mathbf{x}} E(\mathbf{x})$. For a sub-family of energies, energy minimization is equivalent to a minimum (s,t)-cut in a corresponding graph [12]. A key graph-cut-enabling ingredient is "regularity," defined as follows in the pairwise case ($|C| = 2, \forall C$): for all $\{i,j\} = C \in \mathcal{C}$, we have

$$\psi_{i,j}(0,1) + \psi_{i,j}(1,0) \geq \psi_{i,j}(0,0) + \psi_{i,j}(1,1). \quad (2)$$

Graph cuts arise naturally via the relationship between energy functions and set functions on nodes of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Given a set $\mathcal{V} = \{v_1, v_2, \ldots, v_{|I|}\}$, one element per pixel, define the mapping $X(\mathbf{x}) = \{v_i \in \mathcal{V} : x_i = 1\}$ from labelings to sets. Then, the energy $E(\mathbf{x}) = \Psi(X(\mathbf{x}))$ has a corresponding set function Ψ, and regularity of E is equivalent to submodularity of $\Psi(X)$. A function $\Psi : 2^{\mathcal{V}} \rightarrow \mathbb{R}$ is *submodular* if for all $X, Y \subseteq \mathcal{V}$, we have $\Psi(X) + \Psi(Y) \geq \Psi(X \cap Y) + \Psi(X \cup Y)$ [9]. If this condition holds everywhere with equality, then Ψ is called *modular* (*i.e.*, $\Psi(X) = \sum_{x\in X} a_x$ for some $\mathbf{a} \in \mathbb{R}^{|I|}$).

To represent a pairwise submodular energy $E(\mathbf{x})$ as a graph cut, define a weighted directed graph $\mathcal{G} = (\mathcal{V} \cup \{s,t\}, \mathcal{E}, w)$ having a node $v_i \in \mathcal{V}$ for each image pixel, and two terminal nodes s, t. The edges $\mathcal{E}$ consist of inter-pixel edges $\mathcal{E}_n$ and terminal edges $\mathcal{E}_t$. Each potential $\psi_{ij}(x_i, x_j)$ corresponds to two edges $(v_i, v_j), (v_j, v_i) \in \mathcal{E}_n$, and each unary potential $\psi_i(x_i)$ corresponds to the edges $(s, v_i), (v_i, t) \in \mathcal{E}_t$ (although this is often done using undirected graphs, directed graphs better suit our needs). A minimal (s,t)-cut $\Gamma \subseteq \mathcal{E}$ defines a labeling by assigning 1 to x_i if v_i is uncut from s, and 0 otherwise. Equally, an assignment $\mathbf{x}$ defines an (s,t)-cut in $\mathcal{G}$. Let $X_1 = X(\mathbf{x}) \cup \{s\}$ and $X_0 = (\mathcal{V} \setminus X(\mathbf{x})) \cup \{t\}$, then $\Gamma(X(\mathbf{x})) = \mathcal{E} \cap (X_1 \times X_0)$

is a set of edges defining a cut. Given edge weights $w : 2^{\mathcal{E}} \to \mathbb{R}_+$, the cost of a cut Γ is usually the sum of weights $w(\Gamma) = \sum_{e\in\Gamma} w(e)$, which is a modular function of *edge* sets. This cut cost, if seen as a function of sets of *nodes*, is $\Psi_w(X) \triangleq w(\Gamma(X))$, a function well known to be submodular on $2^{\mathcal{V}}$. Moreover, for the pairwise regular energies $E(\mathbf{x})$ there exists a w such that [23]

$$E(\mathbf{x}) + \text{const} = w(\Gamma(X(\mathbf{x}))) = \sum_{e\in\Gamma(X(\mathbf{x}))} w(e). \quad (3)$$

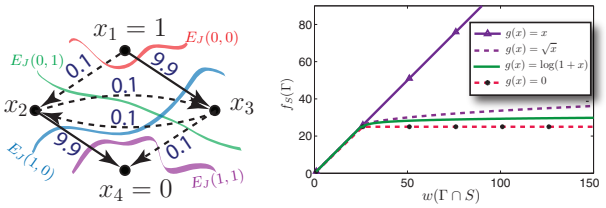
3. Cooperative Graph Cuts

The modularity of the edge weights w in the cut cost (3) is a critical structural limitation: cutting one edge has no effect on the cost of cutting a different edge. Modular edge weights allow efficient graph cut algorithms but can also have deleterious effects on computer vision results.

In our approach, by contrast, cutting an edge may influence the cost of cutting other edges. We express this influence by measuring the cost of a cut using a nondecreasing nonnegative submodular function $f : 2^{\mathcal{E}} \to \mathbb{R}_+$ defined on subsets of **edges** $\mathcal{E}$ (in stark contradistinction to Section 2, where submodular functions are defined on subsets of nodes $\mathcal{V}$). Because f is submodular and nonnegative, it is also *subadditive*: $f(A \cup B) \leq f(A) + f(B)$. If the inequality is strict, we will say that edges in A and B *cooperate* [30].

The weight of a *cooperative cut* between nodes X and $\mathcal{V} \setminus X$ can be expressed as the node function $\Psi_f(X) = f(\Gamma(X))$. Thus, cooperative cut leads to a family of energies of the form $E_f(\mathbf{x}) \triangleq f(\Gamma(X(\mathbf{x})))$. This has three consequences. First, MAP inference for E_f reduces to *Minimum Cooperative Cut*. Second, depending on f, there can be cooperation between arbitrarily large edge sets *anywhere* in the graph. Since this couples all nodes adjacent to the cooperating edges, E_f has arbitrarily high order. Third, E_f is not necessarily regular (equivalently, Ψ_f is not necessarily submodular). Figure 2(a) shows a cooperative energy E_f that violates regularity. A higher order energy E is *regular* [23] if all of its projections on any pair of variables i, j are regular. Let $J = I \setminus \{i, j\}$. A projection E_J of $E : \{0, 1\}^I \to \mathbb{R}_+$ on i, j is obtained by fixing the values of $\mathbf{x}_J$ to some $\bar{\mathbf{x}}_J$, and setting $E_J(x_i, x_j) = E(x_i, x_j, \bar{\mathbf{x}}_J)$.

Submodular functions can reward the *co-occurrence* of certain elements (here, edges of a graph cut). Useful submodular functions include (i) $f(A) = g(\sum_{e\in A} w(e))$ for nonnegative weights w and any concave, nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ [30]; (ii) cover-type functions $f(A) = |\bigcup_{e\in A} S_e|$, where each e has an associated set or area S_e; (iii) entropy; and (iv) neighborhood functions in bipartite graphs. Moreover, the sum of submodular functions is submodular. Additional flexibility is gained by the graph structure, as will be seen in Sections 5 and 6.



(a) non-regular E_f (b) examples of g

Figure 2. (a) Example of a non-regular energy E_f with $f(A) = \left(\sum_{e\in A} w(e)\right)^{1/2}$. Edge weights are as indicated. An edge is cut if its tail has label 1 and its head label 0. Consider the projection $E_J(x_2, x_3)$ for $J = (1, 4)$ and $\bar{\mathbf{x}}_J = (1, 0)$. Then $E_J(1,0) + E_J(0,1) = \sqrt{9.9 + 0.1 + 9.9} + \sqrt{0.1 + 0.1 + 0.1} < 5.01 < 6.32 < \sqrt{0.1 + 9.9} + \sqrt{0.1 + 9.9} = E_J(0,0) + E_J(1,1)$, violating regularity. (b) Effect of different gs in Eqn. (13).

4. Optimization

To minimize E_f, we must solve a Minimum Cooperative Cut. While coupling edges allows E_f to be non-regular, this also makes the problem NP-hard:

Theorem 1. *Minimum Cooperative Cut is NP-hard.*

The proof is a reduction from Graph Bisection [17]. On the other hand, the graph structure provides a definitive advantage over general higher order potentials — for some global energies, no algorithm can provide quality guarantees on the solutions it finds [5, 16]. In contrast, we now derive a practical and efficacious approximation algorithm for cooperative cuts that does have an approximation guarantee. It iteratively minimizes an upper bound on $E_f(\Gamma)$.

The simplest upper bound on a submodular function $f(A)$ is its modular counterpart $\hat{f}(A) = \sum_{e\in A} f(e)$, but this ignores all coupling inherent in f. We instead develop an *adjusting* bound that largely retains cooperation. Define, for $B \subseteq \mathcal{E}$ and an edge $e \in \mathcal{E}$, the *marginal cost* of e with respect to B as $\rho_e(B) = f(B \cup e) - f(B)$. Submodularity implies *diminishing marginal costs:* $\rho_e(B) \leq \rho_e(A)$ for all $A \subseteq B \subseteq \mathcal{E} \setminus \{e\}$.

Lemma 1. *For a submodular $f : 2^{\mathcal{E}} \to \mathbb{R}_+$, and an arbitrary $B \subseteq \mathcal{E}$, define $h_{B,f} : 2^{\mathcal{E}} \to \mathbb{R}_+$ as*

$$h_{B,f}(A) \triangleq f(B) + \sum_{e\in A\setminus B} \rho_e(B) - \sum_{e\in B\setminus A} \rho_e(\mathcal{E}\setminus\{e\}). \quad (4)$$

The function $h_{B,f}$ is a modular upper bound on f.

Proof. For any sets $A, B \subseteq \mathcal{E}$, it holds that [26]

$$f(A) \leq f(B) + \sum_{e\in A\setminus B} \rho_e(B) - \sum_{e\in B\setminus A} \rho_e((A \cup B) \setminus \{e\}). \quad (5)$$

Bound (4) follows by diminishing marginal costs: $\rho_e(\mathcal{E} \setminus \{e\}) \leq \rho_e((A \cup B) \setminus \{e\})$. Modularity is immediate. □

This bound adds an upper bound on the cost of $A \setminus B$ and subtracts a lower bound on the cost of $B \setminus A$. The bound, moreover, is tight at $A = B$, *i.e.*, $h_{B,f}(B) = f(B)$.

Importantly, the cut cost $h_{B,f}$ is efficient to minimize using standard minimum cut, thanks to its modularity. For $\mathcal{G} = (\mathcal{V}, \mathcal{E}, f)$, define $\mathcal{G}_B = (\mathcal{V}, \mathcal{E}, w_B)$ with edge weights

$$w_B(e) = \begin{cases} \rho_e(\mathcal{E} \setminus \{e\}) & \text{if } e \in B \\ \rho_e(B) & \text{otherwise.} \end{cases} \quad (6)$$

For a nondecreasing f, the weights w_B are nonnegative.

Lemma 2. *The minimum (s,t)-cut in $\mathcal{G}_B$ is a minimizing cut for the bound $h_{B,f}$.*

Proof. With weights w_B, the cost of a cut $\Gamma \subseteq \mathcal{E}$ is

$$\sum_{e \in \Gamma} w_B(e) = \sum_{e \in B \cap \Gamma} \rho_e(\mathcal{E} \setminus \{e\}) + \sum_{e \in \Gamma \setminus B} \rho_e(B) \quad (7)$$

$$= h_{B,f}(\Gamma) - f(B) + \sum_{e \in B} \rho_e(\mathcal{E} \setminus \{e\}). \quad (8)$$

Since $f(B)$ and the sum are constant for a fixed B, $w_B(\Gamma) = h_{B,f}(\Gamma) + const$ for any edge set $\Gamma \subseteq \mathcal{E}$. □

Using $h_{B,f}$, we derive an iterative minimization procedure (Algorithm 1). Given an initial reference set B, we find the minimum cut Γ with respect to $h_{B,f}$. Then we adjust the bound to be tight at Γ and repeat. Thus, $h_{B,f}$ is always tight at the currently best solution. The algorithm starts with an initial reference set $I_j \in \mathcal{I}$, the simplest case of which is $\mathcal{I} = \{\emptyset\}$. For further improvements, other options include setting $\mathcal{I}$ to the elements of a cut basis, *e.g.*, the cuts induced by cutting edges of a spanning tree. For our experiments in Section 8, however, $\mathcal{I} = \{\emptyset\}$ was sufficient, and the algorithm converged in less than 10 iterations.

Algorithm 1: Iterative bound minimization

Input: $G = (\mathcal{V}, \mathcal{E})$; submodular cost f: $2^{\mathcal{E}} \to \mathbb{R}_0^+$; reference initialization set $\mathcal{I} = \{I_1, \ldots, I_k\}$, $I_j \subseteq \mathcal{E}$; source /sink $s, t \in \mathcal{V}$
Output: cut $B \subseteq E$
for $j = 1$ **to** k **do**
  find (s,t)-mincut Γ for edge weights w_{I_j};
  repeat
    $B_j = \Gamma$;
    find (s,t)-mincut Γ for edge weights w_{B_j};
  until $f(\Gamma) \geq f(B_j)$;
return $B = \operatorname{argmin}_{B_1,\ldots,B_k} f(B_j)$;

As a result of Lemma 2, the algorithm alternates between adjusting weights and computing a minimum cut. Implementation efficiency can be improved by noting that the marginal cost of an edge e depends only on edges that cooperate with e. The weights w_B show how h_B captures the cost-reducing effect of f: $\rho_e(B) < f(e)$ if e cooperates with B. For a modular function $f = f_m$, $\rho_e(B) = f_m(e)$ and Algorithm 1 becomes the standard minimum cut.

Lemma 3 gives an approximation bound for the initial solution $\Gamma_\emptyset$ for $h_{\emptyset,f}$, which improves in subsequent iterations.

Lemma 3. *Let $\Gamma_\emptyset \in \operatorname{argmin}\{h_{\emptyset,f}(\Gamma) \mid \Gamma \subseteq \mathcal{E}$ an (s,t)-cut$\}$ be a minimum cut for $h_{\emptyset,f}$, and $\Gamma^* \in \operatorname{argmin}\{f(\Gamma) \mid \Gamma \subseteq \mathcal{E}$ an (s,t)-cut$\}$ an optimal solution. Let $\nu(\Gamma^*) = \min_{e \in \Gamma^*} \rho_e(\Gamma^* \setminus \{e\}) / \max_{e \in \Gamma^*} f(e)$. Then*

$$f(\Gamma_\emptyset) \leq \frac{|\Gamma^*|}{1 + (|\Gamma^*| - 1)\nu(\Gamma^*)} f(\Gamma^*) \quad \leq \quad |\Gamma^*| f(\Gamma^*).$$

The proof is deferred to [16]. For the functions we use in Section 5, the term $\nu(\Gamma^*)$ is always nonzero and the second inequality is strict. Lemma 3 is a worst case bound and holds for *any* nondecreasing submodular f. In practice, the algorithm usually performs much better [17].

5. Structured cooperation for segmentation

We now apply edge cooperation to interactive figure-ground segmentation, where, given initial user input, the remaining pixels are to be labeled as object or background. In particular, we address the problems shown in Figure 1: while graph cuts have been used successfully for this task, they are known to shortcut elongated boundaries, especially in low contrast, shaded regions (see also Figs. 3, 4, 5). These failures are caused by the commonly used pairwise energy inherent to standard grid-structured MRFs:

$$E(\mathbf{x}) = \sum_{i \in I} \psi_i(x_i) \quad + \quad \lambda \sum_{(i,j) \in \mathcal{E}_n} \psi_{ij}(x_i, x_j) \quad (9)$$

$$= w(\Gamma(X(\mathbf{x})) \cap \mathcal{E}_t) + \lambda w(\Gamma(X(\mathbf{x})) \cap \mathcal{E}_n) + \text{const.}$$

The associated graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has terminal edges $\mathcal{E}_t$, and a grid of inter-pixel edges $\mathcal{E}_n$ expressing the pairwise potentials. While the former integrate user interaction, the latter enforce smoothness and coherency. The edge weights on $\mathcal{E}_n$ are a function of the intensity gradient; their sum may be seen as the weighted length of the boundary. This penalty favors short boundaries, and thus results in the aforementioned shortcutting. Lowering the coefficient λ is not a solution since boundaries become noisy and true background is included into the hypothesized foreground (Fig. 5).

Instead, we utilize edge cooperation to selectively reward global features of true boundaries. Specifically, we retain $\mathcal{G}$ and replace only the over-smoothing inter-pixel cut by a cooperative cut:

$$E_f(\mathbf{x}) = w(\Gamma(X(\mathbf{x})) \cap \mathcal{E}_t) + \lambda f(\Gamma(X(\mathbf{x})) \cap \mathcal{E}_n) \quad (10)$$

$$= \sum\nolimits_{i \in I} \psi_i(\mathbf{x}_i) \quad + \lambda f(\Gamma(X(\mathbf{x})) \cap \mathcal{E}_n). \quad (11)$$

Since an object boundary consists of cut edges in $\mathcal{G}$, we desire a submodular edge cost f that captures desirable boundary features. We observe that, along true object boundaries, many images possess a certain *congruence*, and this may be true globally throughout the image. Boundary congruity materializes in a number of contexts. For example, of the many inter-pixel color gradients in Figures 3, 5, only few occur along the true boundary in difficult regions; and shortcutting introduces new, incongruous, boundary types. Moreover, the repetitiveness of patterned backgrounds retains congruity to a large extent. Similarly, there is congruity between lighted and shaded regions in Figure 1, if shade is neutralized. In this latter case, we thus need a *shade-invariant* congruity criterion.

Consequently, f should (i) decrease the penalty for globally congruous boundaries, (ii) retain the common smoothing effect of pairwise potentials for incongruous boundaries, and (iii) allow automatic and efficient adaptation of the congruence criterion to each image.

We define congruity in terms of classes of similar edges, $\mathcal{S}(\bar{\mathbf{z}}) = \{S_1, S_2, \ldots, S_\ell\}$, $S_i \subseteq \mathcal{E}_n$ and $\mathcal{E}_n = \bigcup_i S_i$. Congruous boundaries use few classes. Submodularity may selectively reward congruity since it possesses diminishing marginal costs. We make f submodular, however, only *within* classes, and modular across classes:

$$f(\Gamma) = \sum\nolimits_{S \in \mathcal{S}(\bar{\mathbf{z}})} f_S(\Gamma \cap S). \tag{12}$$

As a result, (i) the marginal cost of an edge decreases only when enough edges from the same class are cut. The discount increases with the number of edges included from that class. On the other hand, (ii) there is no discount for cuts that use edges from many classes, *i.e.*, incongruous cuts. The class costs f_S are thresholded discount functions,

$$f_S(\Gamma) = \begin{cases} w(\Gamma \cap S) & \text{if } w(\Gamma \cap S) \leq \theta_S \\ \theta_S + g(w(\Gamma \cap S) - \theta_S) & \text{if } w(\Gamma \cap S) > \theta_S \end{cases}, \tag{13}$$

for any nondecreasing, nonnegative concave function $g : \mathbb{R} \to \mathbb{R}$. For our experiments, we chose $g(x) = \sqrt{x}$. Alternatives include $g(x) = \log(1+x)$ or roots $g(x) = x^{1/p}$ (Figure 2(b)). The modular case (9) corresponds to $g(x) = x$.

To adapt f, we infer the classes by clustering edges $\mathcal{E}_n$ for each image. Furthermore, the discount only sets in after a threshold θ_S is reached, and we adapt θ_S to the total weight of the class, *i.e.*, $\theta_S = \vartheta w(S)$ for $\vartheta \in [0, 1]$, which improves scale-invariance. For large objects or images, more edges are in a class, requiring more cutting to observe a discount. The factor ϑ trades off between completely modular cuts ($\vartheta = 1$) and completely cooperative cuts ($\vartheta = 0$).

The quantitative gauge of "congruence" depends on the distance measure used to cluster the edges. For an edge $e = (v_i, v_j)$ with observed pixel values z_i, z_j, we define two possible feature vectors $\phi(e)$: (i) for uniformly lit images, we use linear color gradients, $\phi_l(e) = z_j - z_i$, and squared Euclidean distance for clustering; and (ii) for shading, we use log intensity ratios $\phi_r(e) = \log(z_j/z_i)$ (channel-wise for color images) which are approximately invariant to shading, and ℓ_1 distance for clustering. In each case, we use the features ($\phi_l(e)$, or $\phi_r(e)$) for clustering edges, and use the standard weights $w(e)$ inside of f.

potential	cooperating edges	$f(\Gamma)$
Graph Cut	$\mathcal{E}$	$w(\Gamma)$
congruence (Sec. 5)	groups of $\mathcal{E}_n$	$\sum_S g_\theta(w(\Gamma \cap S))$
(binary) P^n [19]	$\mathcal{E}$ in $\mathcal{G}'$	$g(\lvert\Gamma\rvert)$
rand. walker [11]	$\mathcal{E}$	$\sqrt{w^2(\Gamma)}$
ℓ_∞ [31]	$\mathcal{E}_n$	$\max_{e \in \Gamma \cap \mathcal{E}_n} w(e)$
class labels [25]	$\mathcal{E}_t$	$f_L(\bigcup_{e \in \Gamma} l(e))$

Table 1. Examples of cooperative cuts; $l(e)$ is the label of edge e [16], and $w(\Gamma)$ ($w^2(\Gamma)$) the sum of (squared) weights.

6. The expressive power of cooperative cuts

Cooperative cuts cover (and strictly generalize) a number of recent approaches in computer vision (summarized in Table 1). Note, however, that cooperative cut is not a special case of any these methods (*e.g.*, some are not NP-hard).

Kohli *et al.* [19] consider $\mathcal{P}^{|C|}$ potentials of the form $\psi_C(\mathbf{x}_C) = g(\sum_{i,j \in C} \tilde{\psi}_{C,i,j}(x_i, x_j))$ for a concave, nondecreasing function g and clique C. Because of the structure of their ψ_C, in the binary case, the sum of pairwise potentials $\tilde{\psi}_{C,i,j}$ is representable as cuts in a graph. These potentials are special cases of cooperative cut potentials that remain regular [19], unlike the example in Figure 2. Similarly, the $\alpha\beta$ swap for a multi-label ψ_C is a cooperative cut, as is the α expansion if $\tilde{\psi}$ is a metric [16]. The $\mathcal{P}^n$ *Potts model* [19] and robust P^n potentials [20] are regular special cases of cooperative cut as well [16].

In class-based image segmentation, each pixel must be labeled by an object class. Ladický *et al.* [25] suggest a global potential $f_L(L(\mathbf{x}))$ on the set of class labels $L(\mathbf{x})$ used in $\mathbf{x}$. If $f_L : 2^{\mathcal{L}} \to \mathbb{R}_+$ is nondecreasing and submodular, then the α expansion can be formulated as a cooperative cut on $\mathcal{E}_t$ [16]. The co-occurrence function $f_L(L)$ in [25] is not submodular with respect to class labels. An alternative, submodular f_L could count the number of training images whose labels do not contain the entire set $L(\mathbf{x})$. The label-cost function in [6] is submodular and thus a cooperative cut on $\mathcal{E}_t$, as it corresponds to a neighborhood function in a bipartite graph [16].

Lastly, Sinop and Grady [31] express an objective for variants of the Random Walker algorithm [11] as $E(\mathbf{x}) = (\sum_{(i,j) \in \mathcal{E}} w_{ij}^q |x_i - x_j|^q)^{1/q}$. In a discrete version, $|x_i - x_j| = 1$ if and only if the edge (v_i, v_j) is cut. Since the qth root is concave for $q \geq 1$, $f(\Gamma) = (w^q(\Gamma))^{1/q}$ is submodular. The same holds for the $q \to \infty$ version $E(\mathbf{x}) = \max_{(i,j) \in \mathcal{E}} w_{ij} |x_i - x_j|$. Therefore, the discrete

case of [31] is a cooperative cut as well.

7. Other related work

Starting with [12], graph cuts have become standard in computer vision, with many applications [2, 3, 4, 14, 13]. In the standard case, the cut represents a pairwise, regular energy, but graph cuts can also be used for non-submodular pairwise potentials [22] and ratio problems [21].

Beyond pairwise energies, efficiently optimizable higher order potentials have been the subject of many recent studies ([15, 18, 24] and references therein), but the structure of those potentials is very different from edge cooperation. Examples of higher order constraints include single global constraints such as connectivity [27], statistical constraints [25], or clique potentials enforcing homogeneity for groups of nodes [19]. While user-interactive connectivity has been used to tackle shrinking bias [32], it may become tedious for trees (Fig. 5), and does not address holes (Fig. 4).

8. Experiments

For the task of interactive figure-ground segmentation, our experiments address three main questions: (i) What is the effect of coupling edges, and does this strengthen correct boundaries? We compare E_f (CoopCut) from Section 5 to the standard graph cut (GC) [2] for pairwise potentials. (ii) What is the effect of the structure of coupling, *i.e.*, the classes S_i? (iii) Does edge cooperation harm the segmentation of objects requiring standard smoothing?

We use color and grayscale images of complicated objects, with and without shading. Since, to our knowledge, no public database exists for such images, we created our own hand-labeled collection. Images and code are available at `ssli.ee.washington.edu/~jegelka/cc`. For (iii), we use the Grabcut data [1, 28].

Both methods use the same 8-neighbor graph structure, the same unary potentials and inter-pixel edge weights $w(e) = 2.5 + 47.5\exp(-0.5\|z_i - z_j\|^2/\sigma)$ for edges $e = (v_i, v_j)$ and variance σ of color gradients (parameters as in [32]). The unary potentials are either based on color histograms [2] or on Gaussian mixture models (GMMs) with 5 components [28, 32]. Edge classes are inferred by k-means clustering, and edges between identically colored pixels form an extra class S' with no discount ($\vartheta_{S'} = 1$). Errors are the percentage of wrongly assigned unlabeled pixels. To quantify the recovery of fine object parts that only make up a small fraction of the pixels, we compute the "twig error" on these delicate parts only. We chose good parameters for each method. In Tables 2, 3, parameters are the same on all images. All algorithms were implemented in C++, using the graph cut code [3], OpenCV, and some Matlab pre-processing. For details and more results, see [16].

The results show that (i) cooperation helps to track boundaries into shaded regions, and preserves fine segments; (ii) what matters is the *structure of cooperation*; and (iii) the improvements on complicated boundaries do not harm the results for "standard" boundaries.

Experiment 1: Shading gradient. Table 2 shows segmentation errors on shaded objects in (a) 8 grayscale and (b) 7 color images. On such images, the unary terms are very noisy (Figs. 1,3). Coupling edges using ϕ_r reduces the error, compared to GC, by up to two thirds. Figures 3 and 4 show that CoopCut recovers the object shape much better. To ensure that not the mere ratio information but *cooperation* makes the difference, we ran GC with ("log") edge weights derived from ϕ_r: the errors improve only slightly. To probe the effect of the classes S_i, we compare against a cooperative cut with only one class (plus the class S'). Such uniform, unstructured coupling is much less effective, *i.e.*, the structure implied by the S_i is crucial.

CoopCut does not model shading explicitly, but cancels shading effects via ϕ_r. Thus, it also improves results if the shade varies locally with higher frequency. We artificially shaded images from Expt. 2, by multiplying the pixel at location (x, y) by $0.4(1 + \sin(2\pi y/\gamma))$ ($\gamma \in [10, 120]$). Unary terms were computed from the modified image. Figure 4 shows an example, and Table 2 lists average errors over 18 such images. Indeed, CoopCut halves the total error of GC, and preserves delicate structures much better than GC.

Experiment 2: Thin, elongated parts and holes. To examine the effect of coupling in uniformly lit images, we compute the total and twig error for 17 images with delicate objects. Table 2(d) and Figure 5 show results for two parameter settings: (1) low overall error, and (2) low twig error. Graph cut roughly recovers fine structures if the smoothness term is reduced, but at the price of a high overall segmentation error. CoopCut preserves fine parts without including pieces of background. Total and twig error are minimized *simultaneously*. In comparison, curvature regularization (as in [7], with our unary terms) is more sensitive to noise in the unary terms (which are less noisy in [7, 29]).

Experiment 3: Grabcut data. As a "sanity check", we address the effect of cooperation with objects that are rounder and need regularization. Table 3 displays the errors for GC and CoopCut on the 50 images of the Grabcut data set [1, 28] with the "Lasso" labeling. Even here, CoopCut slightly improves the results on both color models. Figure 3 shows segmentations for two objects where GC faces the shrinking bias and CoopCut recovers the shape.

The optimal parameter choice varies slightly with the setting, like with standard graph cuts, but the errors show that one choice is reasonable for a wide range of images.

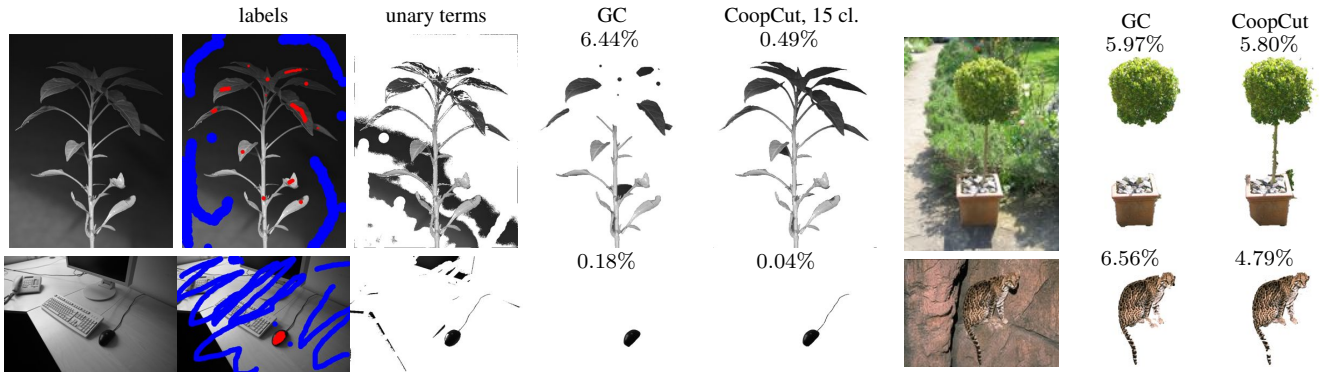


Figure 3. Example results and errors for Expt. 1 (left) and the Grabcut data (right). GC has minimum-error parameters $\lambda = 1.2, 1.0$; CoopCut $(\lambda, 10^4\vartheta) = (8, 5)$ on both images. Grabcut data: GC $\lambda = 1.3, 0.05$, CoopCut (15 & 10 classes): $(\lambda, 10^4\vartheta) = (12, 3), (0.4, 7)$.

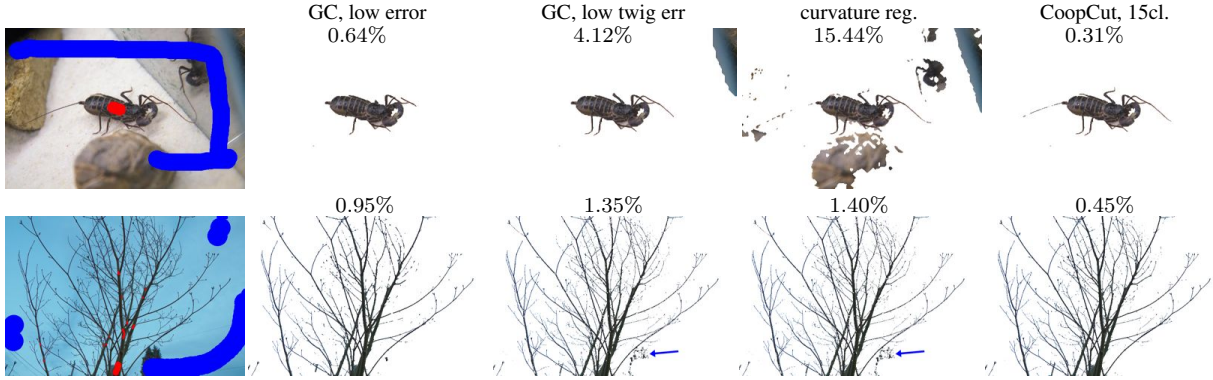


Figure 5. Example results for Experiment 2. Cooperation preserves legs and fine twigs without including pieces of background (arrows). Parameters: GC low err $\lambda = 1.5, 0.05$, GC low err_{twig} $\lambda = 1.0, 0.001$; curv. $\lambda = 0.03, 0.002$, CoopCut: $(\lambda, 10^4\vartheta) = (1.5, 9), (1.8, 10)$.

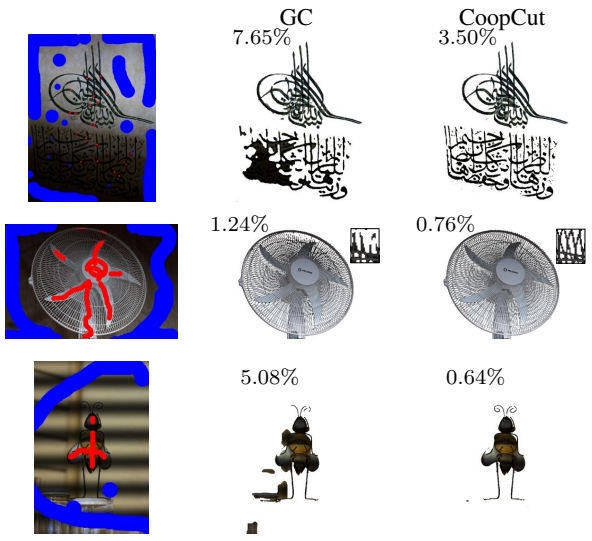


Figure 4. Results on shaded color images for 15, 20 and 25 classes (top to bottom). Parameters chosen for low total and twig error; GC: $\lambda = 0.1, 0.05, 0.1$; CoopCut: $(\lambda, 10^4\vartheta) = (4.5, 6), (7.0, 3), (1.5, 50)$. The zoom-in shows a part of the grid.

	GMM	hist.
GC	5.33 ± 3.7	6.88 ± 5.0
CoopCut ϕ_l, 20 cl.	4.95 ± 3.2	6.25 ± 4.3
CoopCut ϕ_r, 20 cl.	4.79 ± 3.1	6.12 ± 4.0

Table 3. Errors on the Grabcut data with both feature types.

9. Discussion

We introduced a general model, cooperative cuts, that can express a family of global potentials and reward co-occurrences, while still being approximable efficiently. We demonstrated its effectiveness for image segmentation, where we reward the co-occurrence of local boundary features. Key to this is a new class-structured cooperation that drives to globally cut *similar* edges, instead of merely *few* edges. Our approach can thus be viewed as a discrete structured sparsity. Furthermore, it can be extended to multiple labels. Swap or expansion moves then become cooperative cuts. Finally, the relations to other recent models imply that segmentation is only one possible application of the rich modeling capabilities of cooperative cuts.

Acknowledgments. We thank Sebastian Nowozin, Peter Gehler and Christoph Lampert for comments, and Richard

	(a) shading graysc.		(b) shading color		(c) high-frequ. shading (GMM)				(d) Expt. 2 (GMM)			
	GMM	hist	GMM	hist	(1) tot	twig	(2) tot	twig	(1) tot	twig	(2) tot	twig
unary terms	15.66	17.42	4.42	8.18	5.50	14.55	5.50	14.55	5.73	15.47	5.73	15.47
GC	14.03	14.71	3.41	6.49	2.56	20.96	3.43	13.54	2.10	34.40	3.78	18.08
GC,log weights	13.67	14.13	3.63	6.54	2.58	23.21	4.11	13.52	n/a	n/a	n/a	n/a
CoopCut,1	11.58	10.61	2.95	5.31	1.49	33.03	3.10	12.53	1.25	34.35	4.73	15.60
CoopCut,10	4.39	5.02	1.67	3.05	**1.26**	**14.79**	1.65	12.47	1.01	18.27	1.17	16.43
CoopCut,15	**3.63**	**4.27**	1.69	**2.94**	1.27	14.69	1.73	12.39	1.01	26.32	1.02	16.36
CoopCut,20	4.33	4.48	**1.62**	3.00	1.29	18.10	**1.62**	**12.01**	**0.98**	**17.78**	**1.16**	**15.91**
curvature reg.	17.40	19.48	3.93	7.37	3.38	34.50	4.70	14.08	3.82	56.09	5.73	16.00

Table 2. Average error (percent mispredicted pixels) for Expt. 1 and 2. GC: Graph Cut, Coopcut: Cooperative Cut with 1-20 classes. (a), (b) total error across 8 and 7 images; (c), (d) total and twig error across 18 and 17 images, respectively; (c), (d) results for parameters with (1) minimum total error, and (2) minimum joint error ($2\text{err}_{\text{total}} + \text{err}_{\text{twig}}$). CoopCut achieves low total *and* twig error, whereas GC can only minimize one of those. Twig error is overall higher since it counts fewer pixels. Results with histogram unary terms are similar [16].

Karp for the name "cooperative cut."

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Approximation Bounds for Inference using Cooperative Cuts

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Abstract

We analyze a family of probability distributions that are characterized by an embedded combinatorial structure. This family includes models having arbitrary treewidth and arbitrary sized factors. Unlike general models with such freedom, where the "most probable explanation" (MPE) problem is inapproximable, the combinatorial structure within our model, in particular the indirect use of submodularity, leads to several MPE algorithms that all have approximation guarantees.

1. Introduction

Our interest is in the "most probable explanation" (MPE) problem: given a probability distribution $p(x) = \frac{1}{Z}\exp(-E(x))$ where $x = (x_1, x_2, \ldots, x_n) \in \mathcal{D}^n$ for some discrete domain $\mathcal{D}$, find

$$x^* \in \operatorname*{argmax}_x p(x), \text{ or equivalently, } x^* \in \operatorname*{argmin}_x E(x),$$

where $E(x)$ is an "energy" function. In this work, we assume all variables are binary, i.e., $\mathcal{D} = \{0, 1\}$.

Without any restrictions placed on E, it is easy to see that there is not much hope for efficient inference, even if we consider bounded approximations. For example, assume that E is given by an oracle, and let $y \in \{0,1\}^n$ be an unknown vector. Consider the energy $E(x) = 1$ if $x = y$, and $E(x) = \gamma(n)$ otherwise, where $\gamma(n) > 1$ could be any (polynomial-time) computable function of n. With only polynomially many queries to E, it is exponentially unlikely to identify y, and since $\gamma(n)$ is almost arbitrary, no approximation guarantee of any form is possible in polynomial time. The exponential difficulty of approximate inference in such unrestricted models, therefore, is worse than that implied by the well known fact that MPE is NP-hard and not constant-factor approximable (Abdelbar & Hedetniemi, 1998).

Thus, model restrictions are often applied to allow for exact or good approximate inference in polynomial time. These are either structural, such as treewidth or factor size, or functional, such as submodularity. There are often problems with such restrictions, however, such as the well known drawbacks of local pairwise random fields in computer vision. Our work herein is motivated by finding new combinatorial structures that go beyond the previous restrictions but still, as opposed to the introductory example, enable inference with a *bounded approximation factor*. Thus, we devote a major part of this paper to algorithms and approximation bounds. The model we address indeed includes non-local and rich energy functions, and consequently improves, e.g., image segmentation results substantially (Jegelka & Bilmes, 2011).

The common structural restrictions for tractability correspond to factorizations of p. Let p factor with respect to a graphical model $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ comprising $n = |\mathsf{V}|$ nodes and edge set E. The decisive parameter indicating the complexity of MPE in G is the *treewidth* (Chandrasekaran et al., 2008). The treewidth is one less than the size of the maximum clique in a minimum triangulation. Generally, finding the MPE takes time exponential in the treewidth when it is known.

In general, we write $E(x) = \sum_{\phi\in\Phi} E_\phi(x_\phi)$ where Φ corresponds to the set of factors comprising the distribution. Viewed as a bipartite (factor) graph, each $\phi \in \Phi$ is the subset of nodes $\phi \subseteq \mathsf{V}$ involved in a factor. Many approximate inference algorithms rely on $\max_{\phi\in\Phi} |\phi|$ being small. For example, the cost of sending messages even in loopy belief propagation is exponential in $|\phi|$. Therefore, $\max_{\phi\in\Phi} |\phi|$ (which we call the *factorwidth*) may also be seen as a complexity parameter for certain *approximate* inference algorithms.

Appearing in *Proceedings of the 28th International Conference on Machine Learning*, Bellevue, WA, USA, 2011.

Nevertheless, treewidth and factorwidth are not the only characterizations of tractability. In fact, exact polynomial-time MPE is possible even with maximum treewidth and factorwidth if E is restricted in other ways. A recent class of energy functions having received attention in the vision community is that of submodular functions. A function $g : 2^{\mathsf{V}} \to \mathbb{R}$ is submodular if for all $A, B \subseteq \mathsf{V}$, $g(A) + g(B) \geq g(A \cup B) + g(A \cap B)$. The following notation should cause no confusion: let $X(x) \subseteq \mathsf{V}$ be the set of nodes $v_i \in \mathsf{V}$ whose corresponding variable x_i is one, i.e., $X(x) = \{v_i \in \mathsf{V} : x_i = 1\}$. We can then define an energy function via g as $E(x) = g(X(x))$, and finding an assignment that minimizes the energy is equivalent to finding the subset $X \subseteq \mathsf{V}$ that minimizes g. When g is submodular, this can be done in polynomial time (Fujishige, 2005). As an example of a submodular g that places restrictions neither on treewidth nor factorwidth, consider the submodular function $g(S) = -\sum_i \prod_{v \in S} w_{i,v}$, where $0 \leq w_{i,v} \leq 1$ is a set of coefficients $\forall i$ and $v \in \mathsf{V}$.

Submodular function minimization is not currently a low-order polynomial time algorithm. In some cases, however, much faster inference is possible. For example, if g may be written as $g(S) = \sum_{(v,v') \in \mathcal{N}} g_{v,v'}(S \cap \{v, v'\})$, where each $g_{v,v'}(\cdot)$ is a submodular function over a size-2 ground set $\{v, v'\}$, and $\mathcal{N}$ is a set of node-pairs, then MPE reduces to a minimum (s,t)-cut (Boykov & Jolly, 2001; Kolmogorov & Zabih, 2004) on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We call $\mathcal{G}$ the *structure graph* to clearly distinguish it from the graphical model G. In particular, $\mathcal{G}$ has terminal nodes s, t, and a node v_i for each variable x_i. For a set of nodes $X \subset \mathcal{V}$, we define its cut as $\delta(X) = \{(u, v) \in \mathcal{E} | u \in X \cup \{s\}, v \in \mathcal{V} \setminus X\}$. A labeling x induces a partition of V and thus an (s,t)-cut $\delta(X(x))$. The graph $\mathcal{G}$ has weights $w : \mathcal{E} \to \mathbb{R}_+$ and is designed such that its cut equals the energy:

$$E(x) = \sum_{e \in \delta(X(x))} w(e) \triangleq w(\delta(X(x))). \qquad (1)$$

Let $C^* \subseteq \mathcal{E}$ be the optimal cut, and X^* the nodes reachable from s after removal of C^*; then $C^* = \delta(X^*)$ and $x_i^* = 1$ if and only if $v_i \in X^*$. To achieve such efficiency, the construction must be limited to pairwise energies (a factorwidth of 2). Higher order models may be obtained by adding variables, but at additional cost.

Even though submodular energies are widely applicable, there are still cases where submodularity can be limiting. For example, applications that traditionally have been well suited to submodular functions (such as information cascades) sometimes have exceptions to their submodularity (Sheldon, 2010).

In this paper, we define a class of energies that are neither submodular nor restricted in treewidth/factorwidth, but still have limited generality in order to retain approximate optimizability. While an application of these energies is addressed in (Jegelka & Bilmes, 2011), we here focus on the theoretical aspects of the problem and propose a set of approximation algorithms for finding MPE, by finding a minimum of the energy. The key feature for deriving approximation bounds is a new structural characterization that relies on a generalization of graph cuts. Although the energies are not submodular, our construction exploits submodularity indirectly. Theorem 7 backs up our approximation factors by giving a lower bound.

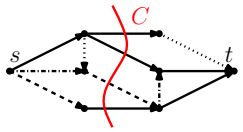

Figure 1. Illustration of a label cost function. Labels are indicated by line style. Cut C cuts four edges with two different labels, hence the cost is $f(C) = 2$.

1.1. The generalized cut model

Similar to the graph cut analogy (1), we define a set $\mathcal{F}_{coop}$ of energies that are representable by generalized cuts, i.e., *cooperative cuts* in a structure graph $\mathcal{G}$.

Definition 1 (Cooperative Cut)**.** *Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with nodes $s, t \in \mathcal{V}$, the cost of an (s,t)-cut $C \subseteq \mathcal{E}$ is measured by a nondecreasing submodular function $f : 2^{\mathcal{E}} \to \mathbb{R}_+$ defined on sets of edges.*

Note that f is defined on **edges**, not nodes. Here, a cut is a set of edges whose removal disconnects s and t. A *submodular* function satisfies diminishing marginal costs: for all $A \subseteq B \subseteq E \setminus \{e\}$, it holds that $f(A \cup \{e\}) - f(A) \geq f(B \cup \{e\}) - f(B)$. In common minimum cut, f would be the sum of edge weights. This is a *modular* function, that is, it satisfies diminishing costs with equality. A cooperative cut replaces the usual sum of weights by a more general submodular cost function f that *couples* the edge weights.

A simple illustrative example of a submodular function is the following: each edge has a label, and the cost of a set of edges is the number of distinct edge labels in the set. Figure 1 illustrates this cost. Other submodular functions include entropy, matroid rank functions and concave functions of sums.

The family $\mathcal{F}_{coop}$ contains all energy functions that can be represented as a cooperative cut in an appropriate structure graph $\mathcal{G}$ with a submodular f such that

$$E_f(x) = f(\delta(X(x))). \qquad (2)$$

(a) structure graph (b) factor graph

Figure 2. Effect of coupling edges. (a) structure graph $\mathcal{G}$, where coupled edges have the same color; (b) factor graph corresponding to $\mathcal{G}$. Coupled edges couple their incident nodes, and lead to large factors.

Eq. 1 is a special case of Eq. 2, because sums of weights are also submodular. In general, the submodular cost f *couples* sets of edges, such that $f(A)+f(B) > f(A\cup B)$ (in Fig. 1, this happens if A and B contain edges with the same label). As a result, E_f usually has neither of the common simplifying properties mentioned in the introduction, and thus imposes neither of those restrictions on models: (1) all nodes incident to coupled edges are coupled, too, and hence also their corresponding random variables (Fig. 2). Since f can couple up to all edges, the corresponding graphical model can have *arbitrarily large treewidth and factorwidth.* (2) The energy E_f in Eq. 2 is in general *not submodular.* It is subadditive, but subadditivity alone is not enough for tractability: our introductory intractable example is in fact subadditive. In applications, coupling occurs if variables belong to the same greater structure, e.g., in images, to the same object boundary.

In view of (1) and (2), it becomes decisive that $\mathcal{G}$ endows the potentials in $\mathcal{F}_{coop}$ with structure. Before we explain how to exploit the structure, we remark that MPE inference for distributions in $\mathcal{F}_{coop}$ is equivalent to cooperative cut in $\mathcal{G}$, thanks to Eq. 2. Thus, we strive to solve the following problem:

$$\min f(C) \quad \text{s.t. } C \subseteq \mathcal{E} \text{ is an } (s,t)\text{-cut in } \mathcal{G}. \qquad (3)$$

For ease of notation, we proceed using the cut formulation (3). Since the cut cost $f(C)$ is equivalent to the potential, all guarantees transfer to the potential.

1.2. Preliminaries and notation

We are given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $m = |\mathcal{E}|$ edges. We assume the submodular cost function $f : 2^{\mathcal{E}} \to \mathbb{R}_+$ to be normalized, $f(\emptyset) = 0$, and nondecreasing: if $A \subseteq B$, then $f(A) \leq f(B)$. We note that a nonnegative submodular set function is also *subadditive*: $f(A) + f(B) \geq f(A \cup B)$. A *matroid rank function* is an integral submodular function with $f(e) \in \{0, 1\}$ for all $e \in \mathcal{E}$. The convolution of two submodular functions f, g is defined as $(f * g)(B) = \min_{A\subseteq B} f(A) + g(B \setminus A)$. More details about submodular functions can be found in (Fujishige, 2005). We denote the feasible set of all cuts by $\mathcal{C} \subset 2^{\mathcal{E}}$. Sometimes, we consider a set $A \subseteq \mathcal{E}$ by its indicator $\chi_A \in \{0, 1\}^{\mathcal{E}}$, where $\chi_A(e) = 1$ if and only if $e \in A$. In the sequel, $C^* = \operatorname{argmin}_{C\in\mathcal{C}} f(C)$ is the optimal solution of Problem (3). An algorithm with approximation factor $\alpha \geq 1$ finds a solution $\widehat{C}$ that satisfies $f(\widehat{C}) \leq \alpha f(C^*)$. For simplicity, we state the results mostly in terms of directed graphs – they do extend to undirected graphs as well, however.

In the next several sections, we give a variety of algorithms that approximately solve Problem (3) and also provide approximation bounds. In Section 3, which gives a lower bound for approximability, we see that there can be no constant factor approximation for this problem. On the other hand, the class of problems is still within the realm of approximability, unlike the more general case mentioned in Section 1.

2. Algorithms

We aim for approximation algorithms for Problem (3). First, we build on a generalized maxflow-mincut duality, and later show two alternative techniques. The first algorithm differs from any of the algorithms for related submodular-cost problems. The main idea is to replace f by a tractable approximation $\hat{f}$ whose deviation from f is limited. We will use the following lemma.

Lemma 1. *Let* $\widehat{C} \in \operatorname{argmin}_{C\in\mathcal{C}} \hat{f}(C)$ *for an approximation* $\hat{f}$ *of* f*, with* $f(A) \leq \hat{f}(A)$ *for all* $A \subseteq \mathcal{E}$*, and* $\hat{f}(C^*) \leq \alpha f(C^*)$ *for* $C^* \in \operatorname{argmin}_{C\in\mathcal{C}} f(C)$*. Then* $f(\widehat{C}) \leq \alpha f(C^*)$.

Proof. Since $\hat{f}(\widehat{C}) \leq \hat{f}(C^*)$, we have that $f(\widehat{C}) \leq \hat{f}(\widehat{C}) \leq \hat{f}(C^*) \leq \alpha f(C^*)$. □

Lemma 1 immediately gives a bound for the simple linearization $\hat{f}_{\text{add}}(A) = \sum_{e\in A} f(e)$. Thanks to the subadditivity of f, $f(A) \leq \hat{f}_{\text{add}}(A)$. To derive α, consider the extreme case of a label cost where all edges have the same label. Then $f(A) = 1$ for all $A \subseteq \mathcal{E}$, but $\hat{f}_{\text{add}}(A) = |A|$. Thus, α can be as large as $|C^*| = O(m)$.

2.1. Approximation with polymatroidal network flows

We now find a tractable approximation $\hat{f}$ of f that is better than $\hat{f}_{\text{add}}$. Note that Problem (3) is hard because f is globally non-separable: the cost of remote edges e_1, e_2 can interact, so that $f(\{e_1, e_2\}) \ll f(e_1) + f(e_2)$. In contrast, the standard minimum cut with a *separable* sum of edge weights is solvable efficiently.

Therefore, we design $\hat{f}$ to be globally separable, but a locally tight approximation. To measure the cost

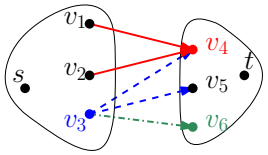

Figure 3. Approximation of a cut cost. Red edges are in $C^{\Pi}_{v_4}$ (head), blue dashed edges in $C^{\Pi}_{v_3}$ (tail), and the green dash-dotted edge in $C^{\Pi}_{v_6}$ (head).

of an edge set $C \subseteq \mathcal{E}$, we partition C into groups $\Pi(C) = \{C^{\Pi}_v\}_{v \in V}$, where the edges in C^{Π}_v must be incident to v. That is, we assign each edge either to its head or to its tail node (Fig. 3). Let $\mathcal{P}_C$ be the family of all such partitions (which vary in the head or tail assignment of each edge). We define an approximation

$$\hat{f}(C) = \min_{\Pi(C) \in \mathcal{P}_C} \sum\nolimits_{v \in V} f(C^{\Pi}_v) \tag{4}$$

that decomposes across node neighborhoods, but is accurate within a group C^{Π}_v. Thanks to the subadditivity of f, $\hat{f}$ is an upper bound on f, and we use the tightest possible such approximation. Instead of Problem (3), we now solve a different optimization problem:

$$\min \hat{f}(C) \;\; \text{s.t. } C \subseteq \mathcal{E} \;\; \text{is an } (s,t)\text{-cut.} \tag{5}$$

To solve Problem (5) exactly, we use its analogy to a generalized maxflow problem. This analogy only holds for cuts, but that suffices here. We first introduce the flow problem.

2.1.1. Polymatroidal network flows

Polymatroidal network flows (Lawler & Martel, 1982) generalize the capacity of traditional flow problems. A function $\varphi : E \to \mathbb{R}_+$ is a flow if the inflow at each node $v \in \mathcal{V} \setminus \{s,t\}$ equals the outflow, and if the flow on an edge does not exceed its capacity: $\varphi(e) \leq \text{cap}(e)$ for all $e \in \mathcal{E}$, given a capacity function $\text{cap} : \mathcal{E} \to \mathbb{R}_+$. Polymatroidal flows replace the usual additive capacities by submodular ones at each node v: cap^{in}_v for incoming edges, and $\text{cap}^{\text{out}}_v$ for outgoing edges. Let $\delta^- v$ be the incoming edges of v, and $\delta^+ v$ its outgoing edges. Then the capacity constraints are, at each $v \in \mathcal{V}$:

$$\varphi(A) \leq \text{cap}^{\text{in}}_v(A) \quad \text{for all } A \subseteq \delta^- v,$$

and equivalently for $\text{cap}^{\text{out}}_v$ on $\delta^+ v$. The maximum flow with such constraints is solved exactly in polynomial time by a layered augmenting paths algorithm (Tardos et al., 1986). The algorithm involves submodular function minimization (SFM) only on the sets $\delta^+ v$, $\delta^- v$ that are much smaller than $\mathcal{E}$. It takes time $O(m^4 T)$, where T is the time for SFM on any $\delta^+ v$, $\delta^- v$.

2.1.2. Analogy

The next lemma relates Problem (5) to polymatroidal flows. For ease of notation, we explicitly write restrictions here, but drop them later.

Lemma 2. *Minimum (s,t)-cut with cost function $\hat{f}$ is dual to a polymatroidal network flow with capacities $\text{cap}^{in}_v = f|_{\delta^- v}$ and $\text{cap}^{out}_v = f|_{\delta^+ v}$ at each node $v \in \mathcal{V}$.*

Proof. First, we restate the dual of a polymatroidal flow. Let $\text{cap}^{\text{in}} : 2^{\mathcal{E}} \to \mathbb{R}_+$ be the joint incoming capacity, $\text{cap}^{\text{in}}(C) = \sum_{v \in V} \text{cap}^{\text{in}}_v(C \cap \delta^- v)$, and equivalently cap^{out} the joint outgoing capacity. The dual of the polymatroidal maxflow is a mincut problem whose cost is a convolution of edge capacities: $\text{cap}(C) = (\text{cap}^{\text{in}} * \text{cap}^{\text{out}})(C) \triangleq \min_{A \subseteq C} \text{cap}^{\text{in}}(A) + \text{cap}^{\text{out}}(C \setminus A)$ (Lovász, 1983).

We relate this dual to our approximation $\hat{f}$. Given a minimal[1] (s,t)-cut C, let $\Pi(C)$ be a partition of C, and $C^{\text{in}}_v = C^{\Pi}_v \cap \delta^-_v$, $C^{\text{out}}_v = C^{\Pi}_v \cap \delta^+_v$. Since C is a minimal directed cut, it contains only edges from the s side to the t side of the graph. In consequence, $C^{\text{in}}_v = \emptyset$ if v is on the s side, and $C^{\text{out}}_v = \emptyset$ otherwise. Hence, $f(C^{\text{in}}_v \cup C^{\text{out}}_v) = f(C^{\text{in}}_v) + f(C^{\text{out}}_v)$. Then

$$\begin{aligned}
\hat{f}(C) &= \min_{\Pi(C) \in \mathcal{P}_C} \sum\nolimits_{v \in \mathcal{V}} f(C^{\Pi}_v) && (6)\\
&= \min_{\Pi(C) \in \mathcal{P}_C} \sum\nolimits_{v \in \mathcal{V}} f(C^{\text{in}}_v \cup C^{\text{out}}_v) && (7)\\
&= \min_{\{(C^{\text{in}}_v, C^{\text{out}}_v)\}_v} \sum\nolimits_{v \in \mathcal{V}} (f(C^{\text{in}}_v) + f(C^{\text{out}}_v)) && (8)\\
&= \min_{\{(C^{\text{in}}_v, C^{\text{out}}_v)\}_v} \sum_{v \in \mathcal{V}} (\text{cap}^{\text{in}}_v(C^{\text{in}}_v) + \text{cap}^{\text{out}}_v(C^{\text{out}}_v))\\
&= \min_{C^{\text{in}}, C^{\text{out}}} (\text{cap}^{\text{in}}(C^{\text{in}}) + \text{cap}^{\text{out}}(C^{\text{out}})) && (9)\\
&= \min_{C^{\text{in}} \subseteq C} (\text{cap}^{\text{in}}(C^{\text{in}}) + \text{cap}^{\text{out}}(C \setminus C^{\text{in}})) && (10)\\
&= (\text{cap}^{\text{in}} * \text{cap}^{\text{out}})(C). && (11)
\end{aligned}$$

Eq. 6 is the definition of $\hat{f}$. The minimum in Eq. 8 is taken over all feasible partitions $\Pi(C)$ and their intersections with the $\delta^+ v, \delta^- v$. Then we use the notation $C^{\text{in}} = \bigcup_{v \in \mathcal{V}} C^{\text{in}}_v$ for all edges assigned to their head nodes, and $C^{\text{out}} = \bigcup_{v \in \mathcal{V}} C^{\text{out}}_v$. The minima in Eqs. 9 and 10 are again over all partitions in $\mathcal{P}_C$. The final equality follows from the above definition of a convolution of submodular functions. □

2.1.3. Approximation factor

Section 2.1.2 shows that Problem (5) can be solved exactly. With Lemma 1, we bound the approximation factor by a quantity that depends on the graph

[1] If a cut C is minimal, then no subset $A \subset C$ is a cut.

structure. Let C^* be the optimal cut for cost f. We define Δ_s to be the tail nodes of the edges in C^*: $\Delta_s = \{v \in \mathcal{V} \mid \exists (v,u) \in C^*\}$. These are still reachable from s. Similarly, Δ_t contains all nodes on the t side that are the head of an edge in C^*.

Theorem 3. *Let $\widehat{C}$ be the minimum cut for cost $\hat{f}$, and C^* the optimal cut for cost f. Then $f(\widehat{C}) \leq \min\{|\Delta_s|, |\Delta_t|\} f(C^*) \leq |\mathcal{V}| f(C^*)/2$.*

Proof. To use Lemma 1, we need to show that $f(C) \leq \hat{f}(C)$ for all $C \subseteq \mathcal{E}$, and find an α such that $\hat{f}(C^*) \leq \alpha f(C^*)$. We already argued for the first condition using subadditivity. It remains to bound α. We do so by referring to the flow analogy with capacities set to f:

$$\hat{f}(C^*) = (\mathrm{cap}^{\mathrm{in}} * \mathrm{cap}^{\mathrm{out}})(C^*) \tag{12}$$

$$\leq \min\{\mathrm{cap}^{\mathrm{in}}(C^*),\ \mathrm{cap}^{\mathrm{out}}(C^*)\} \tag{13}$$

$$\leq \min\left\{ \sum\nolimits_{v\in\Delta_s} f(C^* \cap \delta^+ v),\ \sum\nolimits_{v\in\Delta_t} f(C^* \cap \delta^- v) \right\}$$

$$\leq \min\left\{ |\Delta_s| \max_{v\in\Delta_s} f(C^* \cap \delta^+ v),\ |\Delta_t| \max_{v\in\Delta_t} f(C^* \cap \delta^- v) \right\}$$

$$\leq \min\left\{ |\Delta_s|,\ |\Delta_t| \right\} f(C^*). \tag{14}$$

Thus, Lemma 1 implies an approximation bound $\alpha \leq \min\left\{ |\Delta_s|,\ |\Delta_t| \right\} \leq |\mathcal{V}|/2$. □

2.2. Alternative approximations

Minimizing $\hat{f}$ instead of f yields in general a good solution — on dense graphs, $n/2 = O(\sqrt{m})$. However, the approximation bound depends on the graph structure. Thus, we add three complementary algorithms.

2.2.1. Global approximation of f

Instead of $\hat{f}$ from Section 2.1, any other approximation of f can be used in Lemma 1, as long as it makes the minimum cut problem tractable. Goemans et al. (2009) approximate a submodular function by a square root $\hat{f}_{\mathrm{ell}}(C) = \sqrt{\sum_{e\in C} w_f(e)}$. This function stems from the submodular polyhedron. The *submodular polyhedron* is a subset of $\mathbb{R}^{\mathcal{E}}$ and defined as $P_f = \{x \in \mathbb{R}^{\mathcal{E}} \mid \sum_{e\in A} x(e) \leq f(A)\ \forall\, A \subseteq \mathcal{E}\}$. For the function f, it holds that

$$f(A) = \max\nolimits_{y\in P_f} y \cdot \chi_A. \tag{15}$$

Replacing P_f in (15) by a certain ellipsoid yields $\hat{f}_{\mathrm{ell}}$. Computing the ellipsoid, i.e., the weights w_f, is the bottleneck of this approximation, and takes $O(m^4 \log^2 m)$ time. For matroid rank functions, $\hat{f}_{\mathrm{ell}}$ guarantees an approximation factor $\alpha = \sqrt{m+1}$, and otherwise $\alpha = O(\sqrt{m} \log m)$. This leads to the following bound:

Lemma 4. *Let $\widehat{C} = \mathrm{argmin}_{C\in\mathcal{C}} \hat{f}_{ell}(C)$ be the minimum cut for cost $\hat{f}_{ell}$, and $C^* = \mathrm{argmin}_{C\in\mathcal{C}} f(C)$. Then $f(\widehat{C}) = O(\sqrt{m} \log m) f(C^*)$.*

Algorithm 1 Greedy randomized path cover

Input: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $s, t \in \mathcal{V}$, f
$C = \emptyset$, $x = 0$
while $\sum_{e\in P_{\min}} x(e) < 1$ for shortest path $P_{\min}$ **do**
  let $\beta \in (0, \min_{e\in P_{\min}} \rho_e(C)]$
  for e in $P_{\min}$ **do**
    with probability $\beta/\rho_e(C)$, set $C = C \cup \{e\}$, $x(e) = 1$.
  end for
end while
prune C to C' and return C'

In comparison to $\hat{f}$, the approximation $\hat{f}_{\mathrm{ell}}$ is harder to compute, but the optimization is easier: minimizing $\hat{f}^2_{\mathrm{ell}}$ is equivalent to minimizing $\hat{f}_{\mathrm{ell}}$, and corresponds to a sum-of-weights minimum cut.

2.2.2. Cuts via greedy covers

Our final strategy relates cuts to covers. An (s,t)-cut is also a hitting set: a cut "hits" (intersects) or "covers" each (s,t)-path. Therefore, we write Problem (3) as

$$\begin{aligned} \min\ & f(x) \\ \text{s.t.}\ & \sum\nolimits_{e\in P} x(e) \geq 1 \quad \forall (s,t)\text{-paths } P \subseteq \mathcal{E} \\ & x \in \{0,1\}^E. \end{aligned} \tag{16}$$

Here, with a little abuse of notation, we write f as a function on binary indicator vectors, $f(\chi_A) = f(A)$. The constraints imply that Problem (16) is a minimum-cost cover problem. There can be exponentially many constraints, one for each path. Luckily, it is not hard to find a violated constraint. We merely compute the shortest path $P_{\min}$, using x as the edge lengths. If $P_{\min}$ is longer than one, then x is feasible, otherwise not.

Owing to the form of the constraints, we can adapt a randomized greedy cover-algorithm (Koufogiannakis & Young, 2009) to Problem (16) and obtain Algorithm 1. In each step, we compute the shortest path with weights x to find a possibly uncovered path. Ties are resolved arbitrarily. To cover the path, we randomly pick edges from $P_{\min}$, with probabilities inversely proportional to the marginal cost $\rho_e(C) \triangleq f(C \cup \{e\}) - f(C)$. We must also specify an appropriate β. With the maximum possible β we select the cheapest edge deterministically, and others randomly. To pick exactly one edge in each iteration, we set $\beta = (\sum_{e\in P_{\min}} \rho_e(C)^{-1})^{-1}$, and then sample one edge from $P_{\min}$, with probabilities $p(e) = \beta/\rho_e(C)$. Since C grows by at least one edge in each iteration, the algorithm terminates after at most m iterations.

Finally, the algorithm may return a set C that is

feasible but not a minimal cut. Then we prune C to a minimal cut $C' \subseteq C$. Since f is nondecreasing, $f(C') \leq f(C)$. We assign infinite weight to all edges in $\mathcal{E} \setminus C$, and weight $f(e)$ to each edge $e \in C$ (or contract nodes accordingly). The standard minimum (s,t)-cut in the resulting graph is the desired C'.

The last important question is the approximation bound. Lemma 5 implies that Algorithm 1 returns at least an $O(n)$-approximation, because the longest path spans at most $|\mathcal{V}| - 1$ edges.

Lemma 5. $\mathbb{E}[f(\widehat{C'})] \leq |P_{\max}| f(C^*)$, *where* $P_{\max}$ *is the longest simple path in* $\mathcal{G}$.

Proof. We already argued that the pruned C' can only be better than C. By Theorem 7 in (Koufogiannakis & Young, 2009), a greedy randomized procedure like Algorithm 1 gives a Δ-approximation for a cover, where Δ is the maximum number of variables in any constraint. In (16), Δ is the maximum number of edges in any simple path, i.e., the length of the longest path. This implies that $f(C') \leq f(C) \leq |P_{\max}| f(C^*)$. □

2.2.3. Relaxation

An alternative to the greedy randomized algorithm is to solve a relaxation of Problem (16). For the relaxation, we need to extend f from a set function to a function on a continuous domain. We view f as a function on binary indicator vectors, $\{0,1\}^{\mathcal{E}}$, and extend it to $[0,1]^{\mathcal{E}}$ via its *Lovász extension* $\tilde{f} : [0,1]^{\mathcal{E}} \to \mathbb{R}_+$,

$$\tilde{f}(x) = \max_{y \in P_f} y \cdot x.$$

The maximization over the submodular polyhedron P_f takes $O(m \log m)$ time (Edmonds, 1970). Furthermore, a submodular function satisfies $f(\chi_A) = \max_{y \in P_f} y \cdot \chi_A = \tilde{f}(\chi_A)$. The Lovász extension is convex and piecewise linear. We substitute $\tilde{f}$ for f in Program (16), and allow $x \in [0,1]^{\mathcal{E}}$. The result is a non-smooth convex program with exponentially many constraints. The constraints can be summarized by the $m+1$ constraints of a standard linear program for minimum cut (Papadimitriou & Steiglitz, 1998):

$$\begin{aligned} \min \quad & \tilde{f}(x) \\ \text{s.t.} \quad & x(e) \geq \pi(v) - \pi(u) \quad \forall (u,v) \in \mathcal{E} \\ & \pi(t) - \pi(s) \geq 1 \\ & \pi \in [0,1]^{\mathcal{V}},\ x \in [0,1]^{\mathcal{E}} \end{aligned} \tag{17}$$

The node variables π essentially indicate membership of a node in the s side ($\pi(v) = 0$) or t side ($\pi(v) = 1$) of the cut. The constraints demand that an edge e from a label-zero node to a label-one node should be selected, that is, $x(e) = 1$. These edges will eventually make up the cut. At closer inspection, the label $\pi(v)$ indicates the length of the shortest path from s to v, measured by additive distances x. Program (17) can be solved using any solver for non-smooth convex problems, or by adapting the approach in (Chudak & Nagano, 2007).

The nonlinear Program (17) usually does not have an integral solution, and thus we must round appropriately. The rounding procedure, shown in Algorithm 2, will determine the approximation guarantee. Let x^* be the optimal solution of Program (17). We test the values of $x^*(e)$ as thresholds θ_i in decreasing order (or by binary search). If the set C_i of edges e with $x^*(e) \geq \theta_i$ contains a cut, we stop and prune C_i to a minimal cut.

Algorithm 2 Rounding procedure given x^*

```
order E such that x*(e_1) ≥ x*(e_2) ≥ ... ≥ x*(e_m)
for i = 1, ..., m do
  let C_i = {e_j | x*(e_j) ≥ x*(e_i)}
  if C_i is a cut then
    prune C_i to Ĉ and return Ĉ
  end if
end for
```

A faster, cruder rounding uses a threshold that is at most as large as the inverse of the length of the longest path in the graph (threshold $(n-1)^{-1}$ always works). The reason for this quantity becomes clear in the proof of the following lemma, the approximation bound.

Lemma 6. *Let* $\widehat{C}$ *be the rounded solution returned by Algorithm 2, and* C^* *the optimal cut. Then* $f(\widehat{C}) \leq |P_{\max}| f(C) \leq (n-1) f(C)$, *where* $P_{\max}$ *is the longest simple path in the graph.*

Proof. Program (16) is a submodular covering program. Thus, thresholded rounding is possible, similar to the case of cover problems (Iwata & Nagano, 2009). Let θ be the rounding threshold that implied the final C_i. In the worst case, x^* is uniformly distributed along the longest path, and then θ must be $|P_{\max}|^{-1}$ to include at least one of the edges in $P_{\max}$. Since $\tilde{f}$ is nondecreasing like f and also positively homogeneous, it holds that

$$\begin{aligned} f(\widehat{C}) &\leq f(C_i) = \tilde{f}(\chi_{C_i}) \\ &\leq \tilde{f}(\theta^{-1} x^*) \leq \theta^{-1} \tilde{f}(x^*) \leq \theta^{-1} \tilde{f}(\chi_{C^*}) = \theta^{-1} f(C^*). \end{aligned}$$

The first inequality follows from monotonicity of f and the fact that $\widehat{C} \subseteq C_i$. Similarly, the relation between $\tilde{f}(\chi_{C_i})$ and $\tilde{f}(\theta^{-1} x^*)$ holds because $\tilde{f}$ is nondecreasing: by construction, $x^*(e) \geq \theta \chi_{C_i}(e)$ for all $e \in E$, and hence $\chi_{C_i}(e) \leq \theta^{-1} x^*(e)$. Finally, we use the optimality of x^* to relate the cost to $f(C^*)$ (χ_{C^*} is also feasible, but x^* optimal). The lemma follows since $\theta^{-1} \leq |P_{\max}|$. □

2.3. Discussion

We presented four methods to solve Problem (3). Beyond inference, a special case of Problem (3) arises is the analysis of attack graphs in computer security. Zhang et al. (2011) propose an algorithm for this special case, but their method does not apply to general nondecreasing submodular functions. Other applications are based on mean-risk minimization in discrete stochastic optimization (Atamtürk & Narayanan, 2008).

Which of our algorithms performs best depends on the problem at hand. At first sight, all guarantees might appear as $O(\sqrt{m})$ or $O(n)$ for $n = |\mathcal{V}|$, and almost equivalent. Still, the exact structural terms can make a difference. For sparse graphs with $m = O(n)$, the approximation $\hat{f}_{ell}$ is theoretically the best. On dense graphs, the flow-based approximation dominates theoretically. As an illustrative example, consider a chain of $\sqrt{n}$ cliques between s and t, each clique consisting of roughly $\sqrt{n}$ nodes. Two adjacent cliques intersect at one node. Then the longest path has length $n-1$, whereas $|\Delta_s| \leq \sqrt{n}$ and $\sqrt{m} \approx n^{3/4}$. In any case, it is important to note that the theoretical factors are *worst-case* approximation bounds – on many examples, the algorithms perform much better, as we demonstrate in the next section.

From an implementation viewpoint, the greedy cover is the simplest, and often fast. Since it is randomized, its solution quality in single runs can vary. Often, a heuristic to include the edge with the lowest marginal cost works well, too.

2.4. Experiments

As a proof of concept, Figure 4(a) shows an example graph with coupled edges. It is a complete graph, and the minimum cut contains the maximum possible number of edges. We compare the described algorithms to a minimum cut with $\hat{f}_{\text{add}}$. The cost function is

$$f(A) = \mathbf{1}[|A \cap E_k| \geq 1] + \sum_{i=1}^{n/2-1} \frac{n}{2} \cdot \mathbf{1}[|A \cap E_i| \geq 1],$$

where E_k is the set of black edges, and the E_i are the other sets of edges with identical color.

The proposed algorithms all find the optimal solution. A standard minimum cut with $\hat{f}_{\text{add}}$ yields a solution with an approximation factor of $\Omega(n^2/4)$ – its worst case. The cost of its solution is larger than permissible with the approximation factors of the other algorithms. Thus, the example illustrates that approximation bounds do indeed matter.

Most inputs, however, are more benign. Therefore, we

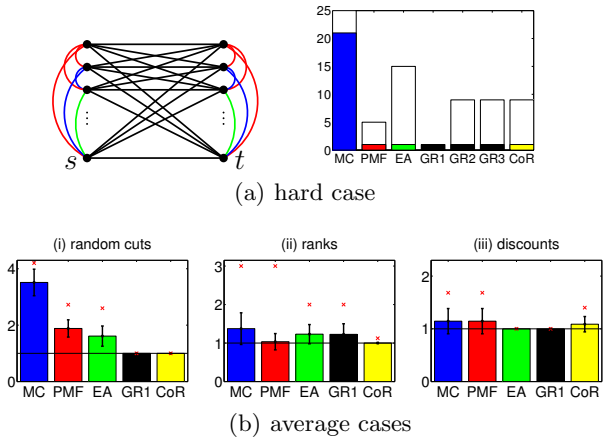

Figure 4. Empirical approximation factors for (a) the shown structure graph ($n = 10$); (b) more common cases. MC: mincut with $\hat{f}_{\text{add}}$, PMF: $\hat{f}$ via polymatroidal flows, EA: approximation with $\hat{f}_{\text{ell}}$, GR: greedy cover (1) picking an edge with minimum marginal cost, (2) sampling one edge, (3) maximum β, CoR: convex relaxation. White bars in (a) indicate theoretical bounds where applicable, red crosses in (b) worst empirical results. (Figure best viewed in color.)

show empirical approximation bounds on three other classes of cost functions on clustered graphs ($n = 30$, $m = 90$): (i) functions similar to the worst case, where the optimal cut was picked randomly and the function designed to make it optimal; (ii) matroid rank functions and sums thereof; (iii) concave functions (log and square root) of a sum of weights. Figure 4(b) shows averages over 45, 100 and 50 instances for computing the minimum cut by a sequence of (s,t)-cuts. The approximation factors are in general between 1 and 2, and much better than the theoretical bounds. For more detailed experiments, see (Jegelka & Bilmes, 2010).

3. Hardness

The approximation factors that we derived in the previous section are put into context by the following lower bound. It assumes oracle access to f.

Theorem 7. *No polynomial-time algorithm can solve Problem (3) to an approximation factor of* $o(\sqrt{m/\log m})$.

Theorem 7 implies that the best possible approximation factor in the general case is on the order of $\sqrt{|\mathcal{E}|}$. The proof is information-theoretic.

Proof. The key idea is to construct two submodular cost functions f, h with different minima that are almost indistinguishable. With high probability they cannot be discriminated within a polynomial number of function queries. If the optima of h and f differ by

a factor larger than α, then any solution for f within a factor α of the optimum would be enough evidence to discriminate f and h. Hence, a polynomial-time algorithm with an approximation factor α would lead to a contradiction. The proof technique is similar to (Goemans et al., 2009; Svitkina & Fleischer, 2008).

The function f depends on a hidden random set $R \subseteq \mathcal{E}$ that will be its optimal cut. Construct a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with ℓ parallel disjoint paths from s to t; each path has k edges. Let the random set $R \subset \mathcal{E}$ be a cut consisting of $|R| = \ell$ edges. The cut contains one edge from each path uniformly at random. We define $\beta = 8\ell/k < \ell$ (for $k > 8$), and, for any $C \subseteq \mathcal{E}$,

$$h(C) = \min\{|C|, \ell\} \tag{18}$$

$$f(C) = \min\{|C \setminus R| + \min\{|C \cap R|,\ \beta\},\ \ell\}. \tag{19}$$

The functions differ only for the relatively few sets C with $|C \cap R| > \beta$ and $|C \setminus R| < \ell - \beta$. Define ε such that $\varepsilon^2 = \omega(\log m)$, and set $k = 8\sqrt{m}/\varepsilon$ and $\ell = \varepsilon\sqrt{m}$. By a Chernoff bound, one can show that the probability (over all choices of R) that f and h differ for a given C is very small:

$$\begin{aligned} P\big(\, f(C) \neq h(C)\big) \;&\leq\; P\big(\, |C \cap R| \geq 8\ell/k \,\big) \\ &\leq\; 2^{-8\ell/k} \;=\; 2^{-\varepsilon^2} = 2^{-\omega(\log m)} = m^{-\omega(1)}. \end{aligned}$$

By a union bound, the probability of distinguishing f and h with a polynomial number of queries C still vanishes as m grows.

As argued above, the bound will be the ratio of optima of h and f. The minimum cooperative-cost cut for f is R with $f(R) = \beta$, and h has uniform cost $h(C) = \ell$ for all minimal cuts C. Hence, the ratio is $h(R)/f(R) = \ell/\beta = \sqrt{m}/\varepsilon = o(\sqrt{m/\log m})$. □

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