A Practical Algorithm for Topic Modeling with Provable Guarantees

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August 17, 2018

Abstract

Topic models provide a useful method for dimensionality reduction and exploratory data analysis in large text corpora. Most approaches to topic model inference have been based on a maximum likelihood objective. Efficient algorithms exist that approximate this objective, but they have no provable guarantees. Recently, algorithms have been introduced that provide provable bounds, but these algorithms are not practical because they are inefficient and not robust to violations of model assumptions. In this paper we present an algorithm for topic model inference that is both provable and practical. The algorithm produces results comparable to the best MCMC implementations while running orders of magnitude faster.

1 Introduction

Topic modeling is a popular method that learns thematic structure from large document collections without human supervision. The model is simple: documents are mixtures of topics, which are modeled as distributions over a vocabulary [Blei, 2012]. Each word token is generated by selecting a topic from a document-specific distribution, and then selecting a specific word from that topic-specific distribution. Posterior inference over document-topic and topic-word distributions is intractable—in the worst case it is NP-hard even for just two topics [Arora et al., 2012b]. As a result, researchers have used approximate inference techniques such as singular value decomposition [Deerwester et al., 1990], variational inference [Blei et al., 2003], and MCMC [Griffiths & Steyvers, 2004].

Recent work in theoretical computer science focuses on designing *provably* efficient algorithms for topic modeling. These treat the topic modeling problem as one of *statistical recovery*: assuming the data was generated *perfectly* from the hypothesized model using an unknown set of parameter

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values, the goal is to recover the model parameters in polynomial time given a reasonable number of samples.

Arora et al. [2012b] present an algorithm that provably recovers the parameters of topic models provided that the topics meet a certain *separability* assumption [Donoho & Stodden, 2003]. Separability requires that every topic contains at least one *anchor word* that has non-zero probability only in that topic. If a document contains this anchor word, then it is guaranteed that the corresponding topic is among the set of topics used to generate the document. The algorithm proceeds in two steps: first it selects anchor words for each topic; and second, in the recovery step, it reconstructs topic distributions given those anchor words. The input for the algorithm is the second-order moment matrix of word-word co-occurrences.

Anandkumar et al. [2012] present a provable algorithm based on third-order moments that does not require separability, but, unlike the algorithm of Arora et al., assumes that topics are not correlated. Although standard topic models like LDA [Blei et al., 2003] assume that the choice of topics used to generate the document are uncorrelated, there is strong evidence that topics are dependent [Blei & Lafferty, 2007, Li & McCallum, 2007]: economics and politics are more likely to co-occur than economics and cooking.

Both algorithms run in polynomial time, but the bounds that have been proven on their sample complexity are weak and their empirical runtime performance is slow. The algorithm presented by Arora et al. [2012b] solves numerous linear programs to find anchor words. Bittorf et al. [2012] and Gillis [2012] reduce the number of linear programs needed. All of these algorithms infer topics given anchor words using matrix inversion, which is notoriously unstable and noisy: matrix inversion frequently generates negative values for topic-word probabilities.

In this paper we present three contributions. First, we replace linear programming with a combinatorial anchor selection algorithm. So long as the separability assumption holds, we prove that this algorithm is stable in the presence of noise and thus has polynomial sample complexity for learning topic models. Second, we present a simple probabilistic interpretation of topic recovery given anchor words that replaces matrix inversion with a new gradient-based inference method. Third, we present an empirical comparison between recovery-based algorithms and existing likelihood-based topic inference. We study both the empirical sample complexity of the algorithms on synthetic distributions and the performance of the algorithms on real-world document corpora. We find that our algorithm performs as well as collapsed Gibbs sampling on a variety of metrics, and runs at least an order of magnitude faster.

Our algorithm both inherits the provable guarantees from Arora et al. [2012a,b] and also results in simple, practical implementations. We view our work as a step toward bridging the gap between statistical recovery approaches to machine learning and maximum likelihood estimation, allowing us to circumvent the computational intractability of maximum likelihood estimation yet still be robust to model error.

2 Background

We consider the learning problem for a class of admixture distributions that are frequently used for probabilistic topic models. Examples of such distributions include latent Dirichlet allocation [Blei et al., 2003], correlated topic models [Blei & Lafferty, 2007], and Pachinko allocation [Li & McCallum, 2007]. We denote the number of words in the vocabulary by V and the number of topics by K. Associated with each topic k is a multinomial distribution over the words in the vocabulary, which we will denote as the column vector A_k of length V. Each of these topic models postulates a particular prior distribution τ over the topic distribution of a document. For example,

in latent Dirichlet allocation (LDA) τ is a Dirichlet distribution, and for the correlated topic model τ is a logistic Normal distribution. The generative process for a document d begins by drawing the document's topic distribution $W_d \sim \tau$. Then, for each position i we sample a topic assignment $z_i \sim W_d$, and finally a word $w_i \sim A_{z_i}$.

We can combine the column vectors A_k for each of the K topics to obtain the word-topic matrix A of dimension $V \times K$. We can similarly combine the column vectors W_d for M documents to obtain the topic-document matrix W of dimension $K \times M$. We emphasize that W is unknown and stochastically generated: we can never expect to be able to recover it. The learning task that we consider is to find the word-topic matrix A. For the case when τ is Dirichlet (LDA), we also show how to learn hyperparameters of τ .

Maximum likelihood estimation of the word-topic distributions is NP-hard even for two topics [Arora et al., 2012b], and as a result researchers typically use approximate inference. The most popular approaches are variational inference [Blei et al., 2003], which optimizes an approximate objective, and Markov chain Monte Carlo [McCallum, 2002], which asymptotically samples from the posterior distribution but has no guarantees of convergence.

Arora et al. [2012b] present an algorithm that provably learns the parameters of a topic model given samples from the model, provided that the word-topic distributions satisfy a condition called *separability*:

Definition 2.1. The word-topic matrix A is p-separable for p > 0 if for each topic k, there is some word i such that $A_{i,k} \geq p$ and $A_{i,k'} = 0$ for $k' \neq k$.

Such a word is called an anchor word because when it occurs in a document, it is a perfect indicator that the document is at least partially about the corresponding topic, since there is no other topic that could have generated the word. Suppose that each document is of length $D \geq 2$, and let $R = \mathbb{E}_{\tau}[WW^T]$ be the $K \times K$ topic-topic covariance matrix. Let α_k be the expected proportion of topic k in a document generated according to τ . The main result of Arora et al. [2012b] is:

Theorem 2.2. There is a polynomial time algorithm that learns the parameters of a topic model if the number of documents is at least

$$M = \max \left\{ O\left(\frac{\log V \cdot a^4 K^6}{\epsilon^2 p^6 \gamma^2 D}\right), O\left(\frac{\log K \cdot a^2 K^4}{\gamma^2}\right) \right\},\,$$

where p is defined above, γ is the condition number of R, and $a = \max_{k,k'} \alpha_k/\alpha_{k'}$. The algorithm learns the word-topic matrix A and the topic-topic covariance matrix R up to additive error ϵ .

Unfortunately, this algorithm is not practical. Its running time is prohibitively large because it solves V linear programs, and its use of matrix inversion makes it unstable and sensitive to noise. In this paper, we will give various reformulations and modifications of this algorithm that alleviate these problems altogether.

3 A Probabilistic Approach to Exploiting Separability

The Arora et al. [2012b] algorithm has two steps: anchor selection, which identifies anchor words, and recovery, which recovers the parameters of A and of τ . Both anchor selection and recovery take as input the matrix Q (of size $V \times V$) of word-word co-occurrence counts, whose construction is described in the supplementary material. Q is normalized so that the sum of all entries is 1. The high-level flow of our complete learning algorithm is described in Algorithm 1, and follows the same two steps. In this section we will introduce a new recovery method based on a probabilistic

Algorithm 1. High Level Algorithm

Input: Textual corpus \mathcal{D} , Number of anchors K, Tolerance parameters $\epsilon_a, \epsilon_b > 0$.

Output: Word-topic matrix A, topic-topic matrix R

 $Q \leftarrow \text{Word Co-occurences}(\mathfrak{D})$

Form $\{\bar{Q}_1, \bar{Q}_2, ... \bar{Q}_V\}$, the normalized rows of Q.

 $\mathbf{S} \leftarrow \text{FastAnchorWords}(\{\bar{Q}_1, \bar{Q}_2, ... \bar{Q}_V\}, K, \epsilon_a) \text{ (Algorithm 4)}$

 $A, R \leftarrow \text{RecoverKL}(Q, \mathbf{S}, \epsilon_b) \text{ (Algorithm 3)}$

return A, R

Algorithm 2. Original Recover [Arora et al., 2012b]

Input: Matrix Q, Set of anchor words **S**

Output: Matrices A,R

Permute rows and columns of Q

Compute $\vec{p}_{\mathbf{S}} = Q_{\mathbf{S}} \vec{1}$

Solve for \vec{z} : $Q_{S,S}\vec{z} = \vec{p}_S$

Solve for $A^T = (Q_{\mathbf{S},\mathbf{S}} \operatorname{Diag}(\vec{z}))^{-1} Q_{\mathbf{S}}^T$

Solve for $R = \text{Diag}(\vec{z})Q_{S,S}\text{Diag}(\vec{z})$

return A, R

framework. We defer the discussion of anchor selection to the next section, where we provide a purely combinatorial algorithm for finding the anchor words.

(equals $DR\vec{1}$)

(Diag(\vec{z}) equals D^{-1})

The original recovery procedure (which we call "Recover") from Arora et al. [2012b] is as follows. First, it permutes the Q matrix so that the first K rows and columns correspond to the anchor words. We will use the notation $Q_{\mathbf{S}}$ to refer to the first K rows, and $Q_{\mathbf{S},\mathbf{S}}$ for the first K rows and just the first K columns. If constructed from infinitely many documents, Q would be the second-order moment matrix $Q = \mathbb{E}[AWW^TA^T] = A\mathbb{E}[WW^T]A^T = ARA^T$, with the following block structure:

$$Q = ARA^T = \begin{pmatrix} D \\ U \end{pmatrix} R \begin{pmatrix} D & U^T \end{pmatrix} = \begin{pmatrix} DRD & DRU^T \\ URD & URU^T \end{pmatrix}$$

where D is a diagonal matrix of size $K \times K$. Next, it solves for A and R using the algebraic manipulations outlined in Algorithm 2.

The use of matrix inversion in Algorithm 2 results in substantial imprecision in the estimates when we have small sample sizes. The returned A and R matrices can even contain small negative values, requiring a subsequent projection onto the simplex. As we will show in Section 5, the original recovery method performs poorly relative to a likelihood-based algorithm. Part of the problem is that the original recover algorithm uses only K rows of the matrix Q (the rows for the anchor words), whereas Q is of dimension $V \times V$. Besides ignoring most of the data, this has the additional complication that it relies completely on co-occurrences between a word and the anchors, and this estimate may be inaccurate if both words occur infrequently.

Here we adopt a new probabilistic approach, which we describe below after introducing some notation. Consider any two words in a document and call them w_1 and w_2 , and let z_1 and z_2 refer to their topic assignments. We will use $A_{i,k}$ to index the matrix of word-topic distributions, i.e. $A_{i,k} = p(w_1 = i|z_1 = k) = p(w_2 = i|z_2 = k)$. Given infinite data, the elements of the Q matrix can be interpreted as $Q_{i,j} = p(w_1 = i, w_2 = j)$. The row-normalized Q matrix, denoted \bar{Q} , which plays a role in both finding the anchor words and the recovery step, can be interpreted as a conditional probability $Q_{i,j} = p(w_2 = j|w_1 = i)$.

Denoting the indices of the anchor words as $S = \{s_1, s_2, ..., s_K\}$, the rows indexed by elements

Algorithm 3. RecoverKL

```
Input: Matrix Q, Set of anchor words \mathbf{S}, tolerance parameter \epsilon. Output: Matrices A,R

Normalize the rows of Q to form \bar{Q}.

Store the normalization constants \vec{p}_w = Q\vec{1}.

\bar{Q}_{s_k} is the row of \bar{Q} for the k^{th} anchor word.

for i=1,...,V do

Solve C_i. = \operatorname{argmin}_{\bar{C}_i} D_{KL}(\bar{Q}_i||\sum_{k\in\mathbf{S}} C_{i,k}\bar{Q}_{s_k})

Subject to: \sum_k C_{i,k} = 1 and C_{i,k} \geq 0

With tolerance: \epsilon

end for

A' = \operatorname{diag}(\bar{p}_w)C

Normalize the columns of A' to form A.

R = A^{\dagger}QA^{\dagger T}

return A, R
```

of **S** are special in that every other row of \bar{Q} lies in the convex hull of the rows indexed by the anchor words. To see this, first note that for an anchor word s_k ,

$$\bar{Q}_{s_k,j} = \sum_{k'} p(z_1 = k'|w_1 = s_k) p(w_2 = j|z_1 = k')$$
(1)

$$=p(w_2=j|z_1=k), (2)$$

where (1) uses the fact that in an admixture model $w_2 \perp w_1 \mid z_1$, and (2) is because $p(z_1 = k | w_1 = s_k) = 1$. For any other word i, we have

$$\bar{Q}_{i,j} = \sum_{k} p(z_1 = k | w_1 = i) p(w_2 = j | z_1 = k).$$

Denoting the probability $p(z_1 = k|w_1 = i)$ as $C_{i,k}$, we have $\bar{Q}_{i,j} = \sum_k C_{i,k} \bar{Q}_{s_k,j}$. Since C is non-negative and $\sum_k C_{i,k} = 1$, we have that any row of \bar{Q} lies in the convex hull of the rows corresponding to the anchor words. The mixing weights give us $p(z_1|w_1 = i)!$ Using this together with $p(w_1 = i)$, we can recover the A matrix simply by using Bayes' rule:

$$p(w_1 = i|z_1 = k) = \frac{p(z_1 = k|w_1 = i)p(w_1 = i)}{\sum_{i'} p(z_1 = k|w_1 = i')p(w_1 = i')}.$$

Finally, we observe that $p(w_1 = i)$ is easy to solve for since $\sum_j Q_{i,j} = \sum_j p(w_1 = i, w_2 = j) = p(w_1 = i)$.

Our new algorithm finds, for each row of the empirical row normalized co-occurrence matrix, \hat{Q}_i , the coefficients $p(z_1|w_1=i)$ that best reconstruct it as a convex combination of the rows that correspond to anchor words. This step can be solved quickly and in parallel (independently) for each word using the exponentiated gradient algorithm. Once we have $p(z_1|w_1)$, we recover the A matrix using Bayes' rule. The full algorithm using KL divergence as an objective is found in Algorithm 3. Further details of the exponentiated gradient algorithm are given in the supplementary material.

One reason to use KL divergence as the measure of reconstruction error is that the recovery procedure can then be understood as maximum likelihood estimation. In particular, we seek the parameters $p(w_1)$, $p(z_1|w_1)$, $p(w_2|z_1)$ that maximize the likelihood of observing the word co-occurrence

counts, \hat{Q} . However, the optimization problem does not explicitly constrain the parameters to correspond an admixture model.

We can also define a similar algorithm using quadratic loss, which we call RecoverL2. This formulation has the extremely useful property that both the objective and gradient can be kernelized so that the optimization problem is independent of the vocabulary size. To see this, notice that the objective can be re-written as

$$||\overline{Q}_i - C_i^T \overline{Q}_{\mathbf{S}}||^2 = ||\overline{Q}_i||^2 - 2C_i(\overline{Q}_{\mathbf{S}} \overline{Q}_i^T) + C_i^T(\overline{Q}_{\mathbf{S}} \overline{Q}_{\mathbf{S}}^T)C_i,$$

where $\overline{Q}_{\mathbf{S}}\overline{Q}_{\mathbf{S}}^T$ is $K \times K$ and can be computed once and used for all words, and $\overline{Q}_{\mathbf{S}}\overline{Q}_i^T$ is $K \times 1$ and can be computed once prior to running the exponentiated gradient algorithm for word i.

To recover the R matrix for an admixture model, recall that $Q = ARA^T$. This may be an overconstrained system of equations with no solution for R, but we can find a least-squares approximation to R by pre- and post-multiplying Q by the pseudo-inverse A^{\dagger} . For the special case of LDA we can learn the Dirichlet hyperparameters. Recall that in applying Bayes' rule we calculated $p(z_1) = \sum_{i'} p(z_1|w_1=i')p(w_1=i')$. These values for p(z) specify the Dirichlet hyperparameters up to a constant scaling. This constant could be recovered from the R matrix [Arora et al., 2012b], but in practice we find it is better to choose it using a grid search to maximize the likelihood of the training data.

We will see in Section 5 that our nonnegative recovery algorithm performs much better on a wide range of performance metrics than the recovery algorithm in Arora et al. [2012b]. In the supplementary material we show that it also inherits the theoretical guarantees of Arora et al. [2012b]: given polynomially many documents, our algorithm returns an estimate \hat{A} at most ϵ from the true word-topic matrix A.

4 A Combinatorial Algorithm for Finding Anchor Words

Here we consider the anchor selection step of the algorithm where our goal is to find the anchor words. In the infinite data case where we have infinitely many documents, the convex hull of the rows in \overline{Q} will be a simplex where the vertices of this simplex correspond to the anchor words. Since we only have a finite number of documents, the rows of \overline{Q} are only an approximation to their expectation. We are therefore given a set of V points $d_1, d_2, ...d_V$ that are each a perturbation of $a_1, a_2, ...a_V$ whose convex hull P defines a simplex. We would like to find an approximation to the vertices of P. See Arora et al. [2012a] and Arora et al. [2012b] for more details about this problem.

Arora et al. [2012a] give a polynomial time algorithm that finds the anchor words. However, their algorithm is based on solving V linear programs, one for each word, to test whether or not a point is a vertex of the convex hull. In this section we describe a purely combinatorial algorithm for this task that avoids linear programming altogether. The new "FastAnchorWords" algorithm is given in Algorithm 4. To find all of the anchor words, our algorithm iteratively finds the furthest point from the subspace spanned by the anchor words found so far.

Since the points we are given are perturbations of the true points, we cannot hope to find the anchor words exactly. Nevertheless, the intuition is that even if one has only found r points S that are close to r (distinct) anchor words, the point that is furthest from $\mathrm{span}(S)$ will itself be close to a (new) anchor word. The additional advantage of this procedure is that when faced with many choices for a next anchor word to find, our algorithm tends to find the one that is most different than the ones we have found so far.

The main contribution of this section is a proof that the FastAnchorWords algorithm succeeds in finding K points that are close to anchor words. To precisely state the guarantees, we recall the following definition from [Arora et al., 2012a]:

Algorithm 4. FastAnchorWords

Input: V points $\{d_1, d_2, ... d_V\}$ in V dimensions, almost in a simplex with K vertices and $\epsilon > 0$ **Output:** K points that are close to the vertices of the simplex.

```
Project the points d_i to a randomly chosen 4 \log V/\epsilon^2 dimensional subspace S \leftarrow \{d_i\} s.t. d_i is the farthest point from the origin. for i=1 TO K-1 do

Let d_j be the point in \{d_1,\ldots,d_V\} that has the largest distance to \mathrm{span}(S). S \leftarrow S \cup \{d_j\}. end for S = \{v_1',v_2',\ldots v_K'\}. for i=1 TO K do

Let d_j be the point that has the largest distance to \mathrm{span}(\{v_1',v_2',\ldots,v_K'\}\setminus\{v_i'\}) Update v_i' to d_j end for Return \{v_1',v_2',\ldots,v_K'\}.
```

Notation: $\operatorname{span}(S)$ denotes the subspace spanned by the points in the set S. We compute the distance from a point x to the subspace $\operatorname{span}(S)$ by computing the norm of the projection of x onto the orthogonal complement of $\operatorname{span}(S)$.

Definition 4.1. A simplex P is γ -robust if for every vertex v of P, the ℓ_2 distance between v and the convex hull of the rest of the vertices is at least γ .

In most reasonable settings the parameters of the topic model define lower bounds on the robustness of the polytope P. For example, in LDA, this lower bound is based on the largest ratio of any pair of hyper-parameters in the model [Arora et al., 2012b]. Our goal is to find a set of points that are close to the vertices of the simplex, and to make this precise we introduce the following definition:

Definition 4.2. Let $a_1, a_2, ...a_V$ be a set of points whose convex hull P is a simplex with vertices $v_1, v_2, ...v_K$. Then we say $a_i \epsilon$ -covers v_j if when a_j is written as a convex combination of the vertices as $a_i = \sum_j c_j v_j$, then $c_j \geq 1 - \epsilon$. Furthermore we will say that a set of K points ϵ -covers the vertices if each vertex is ϵ covered by some point in the set.

We will prove the following theorem: suppose there is a set of points $\mathcal{A} = a_1, a_2, ... a_V$ whose convex hull P is γ -robust and has vertices $v_1, v_2, ... v_K$ (which appear in \mathcal{A}) and that we are given a perturbation $d_1, d_2, ... d_V$ of the points so that for each i, $||a_i - d_i|| \le \epsilon$, then:

Theorem 4.3. There is a combinatorial algorithm that runs in time $\tilde{O}(V^2 + VK/\epsilon^2)$ and outputs a subset of $\{d_1, \ldots, d_V\}$ of size K that $O(\epsilon/\gamma)$ -covers the vertices provided that $20K\epsilon/\gamma^2 < \gamma$.

This new algorithm not only helps us avoid linear programming altogether in inferring the parameters of a topic model, but also can be used to solve the nonnegative matrix factorization problem under the separability assumption, again without resorting to linear programming. Our analysis rests on the following lemmas, whose proof we defer to the supplementary material. Suppose the algorithm has found a set S of k points that are each δ -close to distinct vertices in $\{v_1, v_2, ..., v_K\}$ and that $\delta < \gamma/20K$.

Lemma 4.4. There is a vertex v_i whose distance from span(S) is at least $\gamma/2$.

¹In practice we find setting dimension to 1000 works well. The running time is then $O(V^2 + 1000VK)$.

The proof of this lemma is based on a volume argument, and the connection between the volume of a simplex and the determinant of the matrix of distances between its vertices.

Lemma 4.5. The point d_i found by the algorithm must be $\delta = O(\epsilon/\gamma^2)$ close to some vertex v_i .

This lemma is used to show that the error does not accumulate too badly in our algorithm, since δ only depends on ϵ , γ (not on the δ used in the previous step of the algorithm). This prevents the error from accumulating exponentially in the dimension of the problem, which would be catastrophic for our proof.

After running the first phase of our algorithm, we run a cleanup phase (the second loop in Alg. 4) that can reduce the error in our algorithm. When we have K-1 points close to K-1 vertices, only one of the vertices can be far from their span. The farthest point must be close to this missing vertex. The following lemma shows that this cleanup phase can improve the guarantees of Lemma A.2:

Lemma 4.6. Suppose |S| = K-1 and each point in S is $\delta = O(\epsilon/\gamma^2) < \gamma/20K$ close to some vertex v_i , then the farthest point v_i' found by the algorithm is $1 - O(\epsilon/\gamma)$ close to the remaining vertex.

This algorithm is a greedy approach to maximizing the volume of the simplex. The larger the volume is, the more words per document the resulting model can explain. Better anchor word selection is an open question for future work. We have experimented with a variety of other heuristics for maximizing simplex volume, with varying degrees of success.

Related work. The separability assumption has also been studied under the name "pure pixel assumption" in the context of hyperspectral unmixing. A number of algorithms have been proposed that overlap with ours – such as the VCA [Nascimento & Dias, 2004] algorithm (which differs in that there is no clean-up phase) and the N-FINDR [Gomez et al., 2007] algorithm which attempts to greedily maximize the volume of a simplex whose vertices are data points. However these algorithms have only been proven to work in the infinite data case, and for our algorithm we are able to give provable guarantees even when the data points are perturbed (e.g., as the result of sampling noise). Recent work of Thurau et al. [2010] and Kumar et al. [2012] follow the same pattern as our paper, but use non-negative matrix factorization under the separability assumption. While both give applications to topic modeling, in realistic applications the term-by-document matrix is too sparse to be considered a good approximation to its expectation (because documents are short). In contrast, our algorithm works with the Gram matrix Q so that we can give provable guarantees even when each document is short.

5 Experimental Results

We compare three parameter recovery methods, Recover [Arora et al., 2012b], RecoverKL and RecoverL2 to a fast implementation of Gibbs sampling [McCallum, 2002]. Linear programming-based anchor word finding is too slow to be comparable, so we use FastAnchorWords for all three recovery algorithms. Using Gibbs sampling we obtain the word-topic distributions by averaging over 10 saved states, each separated by 100 iterations, after 1000 burn-in iterations.

5.1 Methodology

We train models on two synthetic data sets to evaluate performance when model assumptions are correct, and real documents to evaluate real-world performance. To ensure that synthetic

²We were not able to obtain Anandkumar et al. [2012]'s implementation of their algorithm, and our own implementation is too slow to be practical.

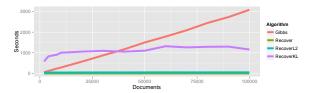


Figure 1: Training time on synthetic NIPS documents.

documents resemble the dimensionality and sparsity characteristics of real data, we generate *semi-synthetic* corpora. For each real corpus, we train a model using MCMC and then generate new documents using the parameters of that model (these parameters are *not* guaranteed to be separable).

We use two real-world data sets, a large corpus of **New York Times** articles (295k documents, vocabulary size 15k, mean document length 298) and a small corpus of **NIPS** abstracts (1100 documents, vocabulary size 2500, mean length 68). Vocabularies were pruned with document frequency cutoffs. We generate semi-synthetic corpora of various sizes from models trained with K = 100 from NY Times and NIPS, with document lengths set to 300 and 70, respectively, and with document-topic distributions drawn from a Dirichlet with symmetric hyperparameters 0.03.

We use a variety of metrics to evaluate models: For the semi-synthetic corpora, we can compute **reconstruction error** between the true word-topic matrix A and learned topic distributions. Given a learned matrix \hat{A} and the true matrix A, we use an LP to find the best matching between topics. Once topics are aligned, we evaluate ℓ_1 distance between each pair of topics. When true parameters are not available, a standard evaluation for topic models is to compute **held-out probability**, the probability of previously unseen documents under the learned model. This computation is intractable but there are reliable approximation methods [Buntine, 2009, Wallach et al., 2009]. Topic models are useful because they provide interpretable latent dimensions. We can evaluate the **semantic quality** of individual topics using a metric called *Coherence*. Coherence is based on two functions, D(w) and $D(w_1, w_2)$, which are number of documents with at least one instance of w, and of w_1 and w_2 , respectively [Mimno et al., 2011]. Given a set of words \mathcal{W} , coherence is

$$Coherence(W) = \sum_{w_1, w_2 \in W} \log \frac{D(w_1, w_2) + \epsilon}{D(w_2)}.$$
 (3)

The parameter $\epsilon=0.01$ is used to avoid taking the log of zero for words that never co-occur [Stevens et al., 2012]. This metric has been shown to correlate well with human judgments of topic quality. If we perfectly reconstruct topics, all the high-probability words in a topic should co-occur frequently, otherwise, the model may be mixing unrelated concepts. Coherence measures the quality of individual topics, but does not measure redundancy, so we measure **inter-topic similarity**. For each topic, we gather the set of the N most probable words. We then count how many of those words do not appear in any other topic's set of N most probable words. Some overlap is expected due to semantic ambiguity, but lower numbers of unique words indicate less useful models.

5.2 Efficiency

The Recover algorithms, in Python, are faster than a heavily optimized Java Gibbs sampling implementation [Yao et al., 2009]. Fig. 1 shows the time to train models on synthetic corpora on a single machine. Gibbs sampling is linear in the corpus size. RecoverL2 is also linear ($\rho = 0.79$), but

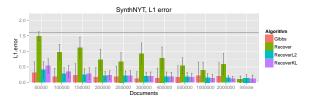


Figure 2: ℓ_1 error for a semi-synthetic model generated from a model trained on NY Times articles with K = 100. The horizontal line indicates the ℓ_1 error of K uniform distributions.

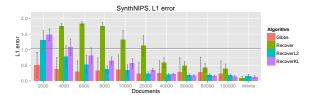


Figure 3: ℓ_1 error for a semi-synthetic model generated from a model trained on NIPS papers with K = 100. Recover fails for D = 2000.

only varies from 33 to 50 seconds. Estimating Q is linear, but takes only 7 seconds for the largest corpus. FastAnchorWords takes less than 6 seconds for all corpora.

5.3 Semi-synthetic documents

The new algorithms have good ℓ_1 reconstruction error on semi-synthetic documents, especially for larger corpora. Results for semi-synthetic corpora drawn from topics trained on NY Times articles are shown in Fig. 2 for corpus sizes ranging from 50k to 2M synthetic documents. In addition, we report results for the three Recover algorithms on "infinite data," that is, the true Q matrix from the model used to generate the documents. Error bars show variation between topics. Recover performs poorly in all but the noiseless, infinite data setting. Gibbs sampling has lower ℓ_1 with smaller corpora, while the new algorithms get better recovery and lower variance with more data (although more sampling might reduce MCMC error further).

Results for semi-synthetic corpora drawn from NIPS topics are shown in Fig. 3. Recover does poorly for the smallest corpora (topic matching fails for D = 2000, so ℓ_1 is not meaningful), but achieves moderate error for D comparable to the NY Times corpus. RecoverKL and RecoverL2 also do poorly for the smallest corpora, but are comparable to or better than Gibbs sampling, with much lower variance, after 40,000 documents.

5.4 Effect of separability

The non-negative algorithms are more robust to violations of the separability assumption than the original Recover algorithm. In Fig. 3, Recover does not achieve zero ℓ_1 error even with noiseless "infinite" data. Here we show that this is due to lack of separability. In our semi-synthetic corpora, documents are generated from the LDA model, but the topic-word distributions are learned from data and may not satisfy the anchor words assumption. We test the sensitivity of algorithms to violations of the separability condition by adding a synthetic anchor word to each topic that is by construction unique to the topic. We assign the synthetic anchor word a probability equal to the

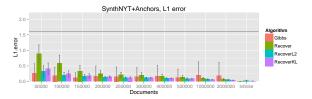


Figure 4: When we add artificial anchor words before generating synthetic documents, ℓ_1 error goes to zero for Recover and close to zero for RecoverL2.

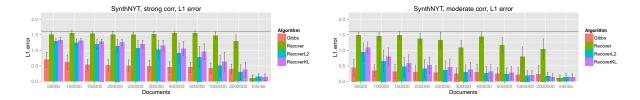


Figure 5: ℓ_1 error increases as we increase topic correlation. We use the same K = 100 topic model from NY Times articles, but add correlation: TOP $\rho = 0.05$, BOTTOM $\rho = 0.1$.

most probable word in the original topic. This causes the distribution to sum to greater than 1.0, so we renormalize. Results are shown in Fig. 4. The ℓ_1 error goes to zero for Recover, and close to zero for RecoverKL and RecoverL2. The reason RecoverKL and RecoverL2 do not reach exactly zero is because we do not solve the optimization problems to perfect optimality.

5.5 Effect of correlation

The theoretical guarantees of the new algorithms apply even if topics are correlated. To test how algorithms respond to correlation, we generated new synthetic corpora from the same K=100 model trained on NY Times articles. Instead of a symmetric Dirichlet distribution, we use a logistic normal distribution with a block-structured covariance matrix. We partition topics into 10 groups. For each pair of topics in a group, we add a non-zero off-diagonal element to the covariance matrix. This block structure is not necessarily realistic, but shows the effect of correlation. Results for two levels of covariance ($\rho=0.05, \rho=0.1$) are shown in Fig. 5. Results for Recover are much worse in both cases than the Dirichlet-generated corpora in Fig. 2. The other three algorithms, especially Gibbs sampling, are more robust to correlation, but performance consistently degrades as correlation increases, and improves with larger corpora. With infinite data ℓ_1 error is equal to ℓ_1 error in the uncorrelated synthetic corpus (non-zero because of violations of the separability assumption).

5.6 Real documents

The new algorithms produce comparable quantitative and qualitative results on real data. Fig. 6 shows three metrics for both corpora. Error bars show the distribution of log probabilities across held-out documents (top panel) and coherence and unique words across topics (center and bottom panels). Held-out sets are 230 documents for NIPS and 59k for NY Times. For the small NIPS corpus we average over 5 non-overlapping train/test splits. The matrix-inversion in Recover failed for the smaller corpus (NIPS). In the larger corpus (NY Times), Recover produces noticeably worse held-out log probability per token than the other algorithms. Gibbs sampling produces the best

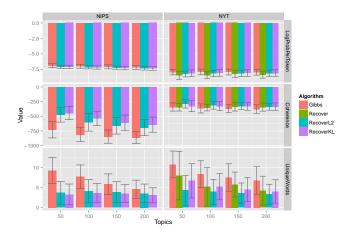


Figure 6: Held-out probability (per token) is similar for RecoverKL, RecoverL2, and Gibbs sampling. RecoverKL and RecoverL2 have better coherence, but fewer unique words than Gibbs. (Up is better for all three metrics.)

Table 1: Example topic pairs from NY Times (closest ℓ_1), anchor words in bold. All 100 topics are in suppl. material.

RecoverL2	run inning game hit season zzz_anaheim_angel
Gibbs	run inning hit game ball pitch
RecoverL2	father family zzz_elian boy court zzz_miami
Gibbs	zzz_cuba zzz_miami cuban zzz_elian boy protest
RecoverL2	file sport read internet email zzz_los_angeles
Gibbs	web site com www mail zzz_internet

average held-out probability (p < 0.0001 under a paired t-test), but the difference is within the range of variability between documents. We tried several methods for estimating hyperparameters, but the observed differences did not change the relative performance of algorithms. Gibbs sampling has worse coherence than the Recover algorithms, but produces more unique words per topic. These patterns are consistent with semi-synthetic results for similarly sized corpora (details are in supplementary material).

For each NY Times topic learned by RecoverL2 we find the closest Gibbs topic by ℓ_1 distance. The closest, median, and farthest topic pairs are shown in Table 1.³ We observe that when there is a difference, recover-based topics tend to have more specific words (*Anaheim Angels* vs. *pitch*).

6 Conclusions

We present new algorithms for topic modeling, inspired by Arora et al. [2012b], which are efficient and simple to implement yet maintain provable guarantees. The running time of these algorithms is effectively independent of the size of the corpus. Empirical results suggest that the sample complexity of these algorithms is somewhat greater than MCMC, but, particularly for the ℓ_2 variant, they provide comparable results in a fraction of the time. We have tried to use the output of our algorithms as initialization for further optimization (e.g. using MCMC) but have not yet found a hybrid that out-performs either method by itself. Finally, although we defer parallel implementations to future work, these algorithms are parallelizable, potentially supporting web-scale topic inference.

³The UCI NY Times corpus includes named-entity annotations, indicated by the zzz prefix.

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A Proof for Anchor-Words Finding Algorithm

Recall that the correctness of the algorithm depends on the following Lemmas:

Lemma A.1. There is a vertex v_i whose distance from span(S) is at least $\gamma/2$.

Lemma A.2. The point Δ_j found by the algorithm must be $\delta = O(\epsilon/\gamma^2)$ close to some vertex v_i .

In order to prove Lemma A.1, we use a volume argument. First we show that the volume of a robust simplex cannot change by too much when the vertices are perturbed.

Lemma A.3. Suppose $\{v_1, v_2, ..., v_K\}$ are the vertices of a γ -robust simplex S. Let S' be a simplex with vertices $\{v'_1, v'_2, ..., v'_K\}$, each of the vertices v'_i is a perturbation of v_i and $\|v'_i - v_i\|_2 \leq \delta$. When $10\sqrt{K}\delta < \gamma$ the volume of the two simplices satisfy

$$vol(S)(1 - 2\delta/\gamma)^{K-1} \le vol(S') \le vol(S)(1 + 4\delta/\gamma)^{K-1}.$$

Proof: As the volume of a simplex is proportional to the determinant of a matrix whose columns are the edges of the simplex, we first show the following perturbation bound for determinant.

Claim A.4. Let A, E be $K \times K$ matrices, the smallest eigenvalue of A is at least γ , the Frobenius norm $||E||_F \leq \sqrt{K}\delta$, when $\gamma > 5\sqrt{K}\delta$ we have

$$\det(A+E)/\det(A) \ge (1-\delta/\gamma)^K.$$

Proof: Since $\det(AB) = \det(A)\det(B)$, we can multiply both A and A + E by A^{-1} . Hence $\det(A + E)/\det(A) = \det(I + A^{-1}E)$.

The Frobenius norm of $A^{-1}E$ is bounded by

$$\left\|A^{-1}E\right\|_F \leq \left\|A^{-1}\right\|_2 \left\|E\right\|_F \leq \sqrt{K}\delta/\gamma.$$

Let the eigenvalues of $A^{-1}E$ be $\lambda_1, \lambda_2, ..., \lambda_K$, then by definition of Frobenius Norm $\sum_{i=1}^K \lambda_i^2 \le \|A^{-1}E\|_F^2 \le K\delta^2/\gamma^2$. The eigenvalues of $I + A^{-1}E$ are just $1 + \lambda_1, 1 + \lambda_2, ..., 1 + \lambda_K$, and the determinant $\det(I + A^{-1}E) = \prod_{i=1}^K (1 + \lambda_i)$. Hence it suffices to show

$$\min \prod_{i=1}^K (1+\lambda_i) \ge (1-\delta/\gamma)^K \text{ when } \sum_{i=1}^K \lambda_i^2 \le K\delta^2/\gamma^2.$$

To do this we apply Lagrangian method and show the minimum is only obtained when all λ_i 's are equal. The optimal value must be obtained at a local optimum of

$$\prod_{i=1}^{K} (1 + \lambda_i) + C \sum_{i=1}^{K} \lambda_i^2.$$

Taking partial derivatives with respect to λ_i 's, we get the equations $-\lambda_i(1+\lambda_i) = -\prod_{i=1}^K (1+\lambda_i)/2C$ (here using $\sqrt{K}\delta/\gamma$ is small so $1+\lambda_i > 1/2 > 0$). The right hand side is a constant, so each λ_i must be one of the two solutions of this equation. However, only one of the solution is larger than 1/2, therefore all the λ_i 's are equal.

For the lower bound, we can project the perturbed subspace to the K-1 dimensional space. Such a projection cannot increase the volume and the perturbation distances only get smaller.

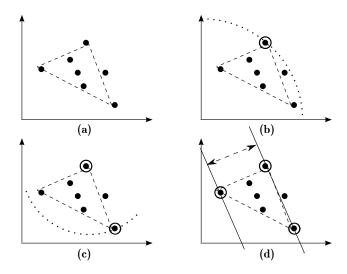


Figure 7: Illustration of the Algorithm

Therefore we can apply the claim directly, the columns of A are just $v_{i+1} - v_1$ for i = 1, 2, ..., K-1; columns of E are just $v'_{i+1} - v_{i+1} - (v'_1 - v_1)$. The smallest eigenvalue of A is at least γ because the polytope is γ robust, which is equivalent to saying after orthogonalization each column still has length at least γ . The Frobenius norm of E is at most $2\sqrt{K-1}\delta$. We get the lower bound directly by applying the claim.

For the upper bound, swap the two sets S and S' and use the argument for the lower bound. The only thing we need to show is that the smallest eigenvalue of the matrix generated by points in S' is still at least $\gamma/2$. This follows from Wedin's Theorem [Wedin, 1972] and the fact that $||E|| \leq ||E||_F \leq \sqrt{K}\delta \leq \gamma/2$.

Now we are ready to prove Lemma A.1.

Proof: The first case is for the first step of the algorithm, when we try to find the farthest point to the origin. Here essentially $S = \{\vec{0}\}$. For any two vertices v_1, v_2 , since the simplex is γ robust, the distance between v_1 and v_2 is at least γ . Which means $\operatorname{dis}(\vec{0}, v_1) + \operatorname{dis}(\vec{0}, v_2) \geq \gamma$, one of them must be at least $\gamma/2$.

For the later steps, recall that S contains vertices of a perturbed simplex. Let S' be the set of original vertices corresponding to the perturbed vertices in S. Let v be any vertex in $\{v_1, v_2, ..., v_K\}$ which is not in S. Now we know the distance between v and S is equal to $\operatorname{vol}(S \cup \{v\})/(|S|-1)\operatorname{vol}(S)$. On the other hand, we know $\operatorname{vol}(S' \cup \{v\})/(|S'|-1)\operatorname{vol}(S') \ge \gamma$. Using Lemma A.3 to bound the ratio between the two pairs $\operatorname{vol}(S)/\operatorname{vol}(S')$ and $\operatorname{vol}(S \cup \{v\})/\operatorname{vol}(S' \cup \{v\})$, we get:

$$\operatorname{dis}(v, S) \ge (1 - 4\epsilon'/\gamma)^{2|S| - 2}\gamma > \gamma/2$$

when $\gamma > 20K\epsilon'$.

Lemma A.2 is based on the following observation: in a simplex the point with largest ℓ_2 is always a vertex. Even if two vertices have the same norm if they are not close to each other the vertices on the edge connecting them will have significantly lower norm.

Proof: (Lemma A.2)

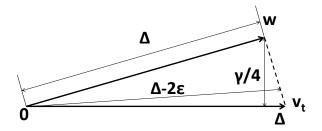


Figure 8: Proof of Lemma A.2, after projecting to the orthogonal subspace of span(S).

Since d_j is the point found by the algorithm, let us consider the point a_j before perturbation. The point a_j is inside the simplex, therefore we can write a_j as a convex combination of the vertices:

$$a_j = \sum_{t=1}^K c_t v_t$$

Let v_t be the vertex with largest coefficient c_t . Let Δ be the largest distance from some vertex to the space spanned by points in S ($\Delta = \max_l \operatorname{dis}(v_l, \operatorname{span}(S))$). By Lemma A.1 we know $\Delta > \gamma/2$. Also notice that we are not assuming $\operatorname{dis}(v_t, \operatorname{span}(S)) = \Delta$.

Now we rewrite a_j as $c_t v_t + (1 - c_t)w$, where w is a vector in the convex hull of vertices other than v_t . Observe that a_j must be far from span(S), because d_j is the farthest point found by the algorithm. Indeed:

$$\operatorname{dis}(a_j,\operatorname{span}(S)) \ge \operatorname{dis}(d_j,\operatorname{span}(S)) - \epsilon \ge \operatorname{dis}(v_l,\operatorname{span}(S)) - 2\epsilon \ge \Delta - 2\epsilon$$

The second inequality is because there must be some point d_l that correspond to the farthest vertex v_l and have $\operatorname{dis}(d_l, \operatorname{span}(S)) \geq \Delta - \epsilon$. Thus as d_j is the farthest point $\operatorname{dis}(d_j, \operatorname{span}(S)) \geq \operatorname{dis}(d_l, \operatorname{span}(S)) \geq \Delta - \epsilon$.

The point a_j is on the segment connecting v_t and w, the distance between a_j and span(S) is not much smaller than that of v_t and w. Following the intuition in ℓ_2 norm when v_t and w are far we would expect a_j to be very close to either v_t or w. Since $c_t \geq 1/K$ it cannot be really close to w, so it must be really close to v_t . We formalize this intuition by the following calculation (see Figure 8):

Project everything to the orthogonal subspace of span(S) (points in span(S) are now at the origin). After projection distance to span(S) is just the ℓ_2 norm of a vector. Without loss of generality we assume $||v_t||_2 = ||w||_2 = \Delta$ because these two have length at most Δ , and extending these two vectors to have length Δ can only increase the length of d_j .

The point v_t must be far from w by applying Lemma A.1: consider the set of vertices $V' = \{v_i : v_i \text{ does not correspond to any point in } S \text{ and } i \neq t\}$. The set $V' \cup S$ satisfy the assumptions in Lemma A.1 so there must be one vertex that is far from $\text{span}(V' \cup S)$, and it can only be v_t . Therefore even after projecting to orthogonal subspace of span(S), v_t is still far from any convex combination of V'. The vertices that are not in V' all have very small norm after projecting to orthogonal subspace (at most δ_0) so we know the distance of v_t and w is at least $\gamma/2 - \delta_0 > \gamma/4$.

Now the problem becomes a two dimensional calculation. When c_t is fixed the length of a_j is strictly increasing when the distance of v_t and w decrease, so we assume the distance is $\gamma/4$. Simple calculation (using essentially just pythagorean theorem) shows

$$c_t(1-c_t) \le \frac{\epsilon}{\Delta - \sqrt{\Delta^2 - \gamma^2/16}}.$$

The right hand side is largest when $\Delta = 2$ (since the vectors are in unit ball) and the maximum value is $O(\epsilon/\gamma^2)$. When this value is smaller than 1/K, we must have $1 - c_t \leq O(\epsilon/\gamma^2)$. Thus $c_t \geq 1 - O(\epsilon/\gamma^2)$ and $\delta \leq (1 - c_t) + \epsilon \leq O(\epsilon/\gamma^2)$.

The cleanup phase tries to find the farthest point to a subset of K-1 vertices, and use that point as the K-th vertex. This will improve the result because when we have K-1 points close to K-1 vertices, only one of the vertices can be far from their span. Therefore the farthest point must be close to the only remaining vertex. Another way of viewing this is that the algorithm is trying to greedily maximize the volume of the simplex, which makes sense because the larger the volume is, the more words/documents the final LDA model can explain.

The following lemma makes the intuitions rigorous and shows how cleanup improves the guarantee of Lemma A.2.

Lemma A.5. Suppose |S| = K - 1 and each point in S is $\delta = O(\epsilon/\gamma^2) < \gamma/20K$ close to some vertex v_i , then the farthest point v_i' found by the algorithm is $1 - O(\epsilon/\gamma)$ close to the remaining vertex.

Proof: We still look at the original point a_j and express it as $\sum_{t=1}^K c_t v_t$. Without loss of generality let v_1 be the vertex that does not correspond to anything in S. By Lemma A.1 v_1 is $\gamma/2$ far from span(S). On the other hand all other vertices are at least $\gamma/20r$ close to span(S). We know the distance $\operatorname{dis}(a_j,\operatorname{span}(S)) \geq \operatorname{dis}(v_1,\operatorname{span}(S)) - 2\epsilon$, this cannot be true unless $c_t \geq 1 - O(\epsilon/\gamma)$.

These lemmas directly lead to the following theorem:

Theorem A.6. FastAnchorWords algorithm runs in time $\tilde{O}(V^2 + VK/\epsilon^2)$ and outputs a subset of $\{d_1, ..., d_V\}$ of size K that $O(\epsilon/\gamma)$ -covers the vertices provided that $20K\epsilon/\gamma^2 < \gamma$.

Proof: In the first phase of the algorithm, do induction using Lemma A.2. When $20K\epsilon/\gamma^2 < \gamma$ Lemma A.2 shows that we find a set of points that $O(\epsilon/\gamma^2)$ -covers the vertices. Now Lemma A.5 shows after cleanup phase the points are refined to $O(\epsilon/\gamma)$ -cover the vertices.

B Proof for Nonnegative Recover Procedure

In order to show RecoverL2 learns the parameters even when the rows of \bar{Q} are perturbed, we need the following lemma that shows when columns of \bar{Q} are close to the expectation, the posteriors c computed by the algorithm is also close to the true value.

Lemma B.1. For a γ robust simplex S with vertices $\{v_1, v_2, ..., v_K\}$, let v be a point in the simplex that can be represented as a convex combination $v = \sum_{i=1}^K c_i v_i$. If the vertices of S are perturbed to $S' = \{..., v'_i, ...\}$ where $||v'_i - v_i|| \leq \delta_1$ and v is perturbed to v' where $||v - v'|| \leq \delta_2$. Let v^* be the point in S' that is closest to v', and $v^* = \sum_{i=1}^K c'_i v_i$, when $10\sqrt{K}\delta_1 \leq \gamma$ for all $i \in [K]$ $|c_i - c'_i| \leq 4(\delta_1 + \delta_2)/\gamma$.

Proof: Consider the point $u = \sum_{i=1}^K c_i v_i'$, by triangle inequality: $||u - v|| \le \sum_{i=1}^K c_i ||v_i - v_i'|| \le \delta_1$. Hence $||u - v'|| \le ||u - v|| + ||v - v'|| \le \delta_1 + \delta_2$, and u is in S'. The point v^* is the point in S' that is closest to v', so $||v^* - v'|| \le \delta_1 + \delta_2$ and $||v^* - u|| \le 2(\delta_1 + \delta_2)$.

Then we need to show when a point (u) moves a small distance, its representation also changes by a small amount. Intuitively this is true because S is γ robust. By Lemma A.1 when $10\sqrt{K}\delta_1 < \gamma$,

the simplex S' is also $\gamma/2$ robust. For any i, let $Proj_i(v^*)$ and $Proj_i(u)$ be the projections of v^* and u in the orthogonal subspace of span $(S' \setminus v'_i)$, then

$$|c_i - c_i'| = \|Proj_i(v^*) - Proj_i(u)\| / \operatorname{dis}(v_i, \operatorname{span}(S' \setminus v_i')) \le 4(\delta_1 + \delta_2) / \gamma$$

and this completes the proof.

With this lemma it is not hard to show that RecoverL2 has polynomial sample complexity.

Theorem B.2. When the number of documents M is at least

$$\max\{O(aK^3\log V/D(\gamma p)^6\epsilon), O((aK)^3\log V/D\epsilon^3(\gamma p)^4)\}$$

our algorithm using the conjunction of FastAnchorWords and RecoverL2 learns the A matrix with entry-wise error at most ϵ .

Proof: (sketch) We can assume without loss of generality that each word occurs with probability at least $\epsilon/4aK$ and furthermore that if M is at least $50 \log V/D\epsilon_Q^2$ then the empirical matrix \tilde{Q} is entry-wise within an additive ϵ_Q to the true $Q = \frac{1}{M} \sum_{d=1}^{M} AW_dW_d^TA^T$ see [Arora et al., 2012b] for the details. Also, the K anchor rows of \bar{Q} form a simplex that is γp robust.

The error in each column of \bar{Q} can be at most $\delta_2 = \epsilon_Q \sqrt{4aK/\epsilon}$. By Theorem A.6 when $20K\delta_2/(\gamma p)^2 < \gamma p$ (which is satisfied when $M = O(aK^3 \log V/D(\gamma p)^6\epsilon)$), the anchor words found are $\delta_1 = O(\delta_2/(\gamma p))$ close to the true anchor words. Hence by Lemma B.1 every entry of C has error at most $O(\delta_2/(\gamma p)^2)$.

With such number of documents, all the word probabilities p(w=i) are estimated more accurately than the entries of $C_{i,j}$, so we omit their perturbations here for simplicity. When we apply the Bayes rule, we know $A_{i,k} = C_{i,k}p(w=i)/p(z=k)$, where p(z=k) is α_k which is lower bounded by 1/aK. The numerator and denominator are all related to entries of C with positive coefficients sum up to at most 1. Therefore the errors δ_{num} and δ_{denom} are at most the error of a single entry of C, which is bounded by $O(\delta_2/(\gamma p)^2)$. Applying Taylor's Expansion to $(p(z=k,w=i)+\delta_{num})/(\alpha_k+\delta_{denom})$, the error on entries of A is at most $O(aK\delta_2/(\gamma p)^2)$. When $\epsilon_Q \leq O((\gamma p)^2 \epsilon^{1.5}/(aK)^{1.5})$, we have $O(aK\delta_2/(\gamma p)^2) \leq \epsilon$, and get the desired accuracy of A. The number of document required is $M = O((aK)^3 \log V/D\epsilon^3(\gamma p)^4)$.

The sample complexity for R can then be bounded using matrix perturbation theory.

C Empirical Results

This section contains plots for ℓ_1 , held-out probability, coherence, and uniqueness for all semi-synthetic data sets. Up is better for all metrics except ℓ_1 error.

C.1 Sample Topics

Tables 2, 3, and 4 show 100 topics trained on real NY Times articles using the RecoverL2 algorithm. Each topic is followed by the most similar topic (measured by ℓ_1 distance) from a model trained on the same documents with Gibbs sampling. When the anchor word is among the top six words by probability it is highlighted in bold. Note that the anchor word is frequently not the most prominent word.

Table 2: Example topic pairs from NY Times sorted by ℓ_1 distance, anchor words in bold.

RecoverL2	run inning game hit season zzz_anaheim_angel
Gibbs	run inning hit game ball pitch
RecoverL2 Gibbs	king goal game team games season point game team play season games
RecoverL2	yard game play season team touchdown
Gibbs RecoverL2	yard game season team play quarterback point game team season games play
Gibbs	point game team season games play point game team play season games
RecoverL2	zzz_laker point zzz_kobe_bryant zzz_o_neal game
Gibbs	team point game team play season games
RecoverL2	point game team play season games point game team season player zzz_clipper
Gibbs	point game team season play zzz_usc
RecoverL2 Gibbs	ballot election court votes vote zzz_al_gore election ballot zzz_florida zzz_al_gore votes vote
RecoverL2	game zzz_usc team play point season
Gibbs	point game team season play zzz_usc
RecoverL2 Gibbs	company billion companies percent million stock company million percent billion analyst deal
RecoverL2	car race team season driver point
Gibbs	race car driver racing zzz_nascar team
RecoverL2 Gibbs	zzz_dodger season run inning right game season team baseball game player yankees
RecoverL2	palestinian zzz_israeli zzz_israel official attack
	zzz_palestinian
Gibbs	palestinian zzz_israeli zzz_israel attack zzz_palestinian zzz_yasser_arafat
RecoverL2	zzz_tiger_wood shot round player par play
Gibbs	zzz_tiger_wood shot golf tour round player
RecoverL2 Gibbs	percent stock market companies fund quarter percent economy market stock economic growth
RecoverL2	zzz_al_gore zzz_bill_bradley campaign president
an i	zzz_george_bush vice
Gibbs	zzz_al_gore zzz_george_bush campaign presidential republican zzz_john_mccain
RecoverL2	zzz_george_bush zzz_john_mccain campaign repub-
G:1.1	lican zzz_republican voter
Gibbs	zzz_al_gore zzz_george_bush campaign presidential republican zzz_john_mccain
RecoverL2	net team season point player zzz_jason_kidd
Gibbs RecoverL2	point game team play season games yankees run team season inning hit
Gibbs	season team baseball game player yankees
RecoverL2	zzz_al_gore zzz_george_bush percent president cam-
Gibbs	paign zzz_bush zzz_al_gore zzz_george_bush campaign presidential
	republican zzz_john_mccain
RecoverL2	zzz_enron company firm zzz_arthur_andersen
Gibbs	companies lawyer zzz_enron company firm accounting
	zzz_arthur_andersen financial
RecoverL2 Gibbs	team play game yard season player yard game season team play quarterback
RecoverL2	film movie show director play character
Gibbs	film movie character play minutes hour
RecoverL2	zzz_taliban zzz_afghanistan official zzz_u_s govern- ment military
Gibbs	zzz_taliban zzz_afghanistan zzz_pakistan afghan
- D	zzz_india government
RecoverL2	palestinian zzz_israel israeli peace zzz_yasser_arafat leader
Gibbs	palestinian zzz_israel peace israeli zzz_yasser_arafat
RecoverL2	leader point team game shot play zzz_celtic
Gibbs	point team game snot play zzz_centic point game team play season games
RecoverL2	zzz_bush zzz_mccain campaign republican tax
Gibbs	zzz_republican zzz_al_gore zzz_george_bush campaign presidential
	republican zzz_john_mccain
RecoverL2 Gibbs	zzz_met run team game hit season
RecoverL2	season team baseball game player yankees team game season play games win
Gibbs	team coach game player season football
RecoverL2	government war zzz_slobodan_milosevic official court president
Gibbs	government war country rebel leader military
RecoverL2	game set player zzz_pete_sampras play won
Gibbs RecoverL2	player game match team soccer play zzz_al_gore campaign zzz_bradley president demo-
	cratic zzz_clinton
Gibbs	zzz_al_gore zzz_george_bush campaign presidential
RecoverL2	republican zzz_john_mccain team zzz_knick player season point play
Gibbs	point game team play season games
RecoverL2 Gibbs	com web www information sport question palm beach com statesman daily american
Gibbs	pain beach com statesman dany american

Table 3: Example topic pairs from NY Times sorted by ℓ_1 distance, anchor words in bold.

RecoverL2	season team game coach play school
Gibbs RecoverL2	team coach game player season football air shower rain wind storm front
Gibbs	water fish weather storm wind air
RecoverL2	book film beginitalic enditalic look movie
Gibbs RecoverL2	film movie character play minutes hour zzz_al_gore campaign election zzz_george_bush
	zzz_florida president
Gibbs	zzz_al_gore zzz_george_bush campaign presidential
RecoverL2	republican zzz_john_mccain race won horse zzz_kentucky_derby win winner
Gibbs	horse race horses winner won zzz_kentucky_derby
RecoverL2	company companies zzz_at percent business stock
Gibbs RecoverL2	company companies business industry firm market company million companies percent business cus-
	tomer
Gibbs RecoverL2	company companies business industry firm market
Gibbs	team coach season player jet job team player million season contract agent
RecoverL2	season team game play player zzz_cowboy
Gibbs	yard game season team play quarterback
RecoverL2	zzz_pakistan zzz_india official group attack zzz_united_states
Gibbs	zzz_taliban zzz_afghanistan zzz_pakistan afghan
Dans IC	zzz_india government
RecoverL2 Gibbs	show network night television zzz_nbc program film movie character play minutes hour
RecoverL2	com information question zzz_eastern commentary
C;LL.	daily
Gibbs	com question information zzz_eastern daily commen- tary
RecoverL2	power plant company percent million energy
Gibbs	oil power energy gas prices plant
RecoverL2 Gibbs	cell stem research zzz_bush human patient cell research human scientist stem genes
RecoverL2	zzz_governor_bush zzz_al_gore campaign tax presi-
Gibbs	dent plan
GIDDS	zzz_al_gore zzz_george_bush campaign presidential republican zzz_john_mccain
RecoverL2	cup minutes add tablespoon water oil
Gibbs RecoverL2	cup minutes add tablespoon teaspoon oil family home book right com children
Gibbs	film movie character play minutes hour
RecoverL2	zzz_china chinese zzz_united_states zzz_taiwan
Gibbs	official government
Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official
RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas
RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor
RecoverL2 Gibbs RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas
RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox
RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour
RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour
RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack
RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour
RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies companiy companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton
RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_blush zzz_george_bush president administration
RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies companiy companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton
RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bish zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest
RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_elian boy court zzz_miami zzz_cush zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states
RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bish zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest
RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cus zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software
RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bush zzz_george_bush president administration zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at
RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cus zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_internet
RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official tetter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_internet government election zzz_mexico political zzz_vicente_fox president
RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cus zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_internet
RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official tetter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_internet government election zzz_mexico political zzz_vicente_fox president election political campaign zzz_party democratic voter fight zzz_mike_tyson round right million champion
RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_internet government election zzz_mexico political zzz_vicente_fox president election political campaign zzz_party democratic voter fight zzz_mike_tyson round right million champion fight zzz_mike_tyson ring fighter champion round
RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official tetter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_internet government election zzz_mexico political zzz_vicente_fox president election political campaign zzz_party democratic voter fight zzz_mike_tyson round right million champion
RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_internet government election zzz_mexico political zzz_vicente_fox president election political campaign zzz_party democratic voter fight zzz_mike_tyson round right million champion fight zzz_mike_tyson ring fighter champion round right law president zzz_george_bush zzz_senate zzz_john_ashcroft election political campaign zzz_party democratic
RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company computer system window software zzz_niternet government election zzz_mexico political zzz_vicente_fox president election political campaign zzz_party democratic voter fight zzz_mike_tyson round right million champion fight zzz_mike_tyson ring fighter champion round right law president zzz_george_bush zzz_senate zzz_john_ashcroft election political campaign zzz_party democratic voter
RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_internet government election zzz_mexico political zzz_vicente_fox president election political campaign zzz_party democratic voter fight zzz_mike_tyson round right million champion fight zzz_mike_tyson ring fighter champion round right law president zzz_george_bush zzz_senate zzz_john_ashcroft election political campaign zzz_party democratic
RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_nicente_fox president election political campaign zzz_party democratic voter fight zzz_mike_tyson round right million champion fight zzz_mike_tyson ring fighter champion round right law president zzz_george_bush zzz_senate zzz_john_ashcroft election political campaign zzz_party democratic voter com home look found show www film movie character play minutes hour car driver race zzz_dale_earnhardt racing
RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cus zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_internet government election zzz_mexico political zzz_vicente_fox president election political campaign zzz_party democratic voter fight zzz_mike_tyson round right million champion fight zzz_mike_tyson ring fighter champion round right law president zzz_george_bush zzz_senate zzz_john_ashcroft election political campaign zzz_party democratic voter com home look found show www film movie character play minutes hour car driver race zzz_dale_earnhardt racing
RecoverL2 Gibbs RecoverL2	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official letter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_bush zzz_george_bush president administration zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_nicente_fox president election political campaign zzz_party democratic voter fight zzz_mike_tyson round right million champion fight zzz_mike_tyson ring fighter champion round right law president zzz_george_bush zzz_senate zzz_john_ashcroft election political campaign zzz_party democratic voter com home look found show www film movie character play minutes hour car driver race zzz_dale_earnhardt racing zzz_nascar night hour room hand told morning book women family called author woman
RecoverL2 Gibbs	official government zzz_china chinese zzz_beijing zzz_taiwan government official death court law case lawyer zzz_texas trial death prison case lawyer prosecutor company percent million sales business companies company companies business industry firm market dog jump show quick brown fox film movie character play minutes hour shark play team attack water game film movie character play minutes hour anthrax official mail letter worker attack anthrax official tetter mail nuclear chemical president zzz_clinton zzz_white_house zzz_bush official zzz_bill_clinton zzz_white_house zzz_dick_cheney father family zzz_elian boy court zzz_miami zzz_cuba zzz_miami cuban zzz_elian boy protest oil prices percent million market zzz_united_states oil power energy gas prices plant zzz_microsoft company computer system window software zzz_microsoft company companies cable zzz_at zzz_internet government election zzz_mexico political zzz_vicente_fox president election political campaign zzz_party democratic voter fight zzz_mike_tyson round right million champion fight zzz_mike_tyson ring fighter champion round right law president zzz_george_bush zzz_senate zzz_john_ashcroft election political campaign zzz_party democratic voter com home look found show www film movie character play minutes hour car driver race zzz_dale_earnhardt racing zzz_nascar night hour room hand told morning

Table 4: Example topic pairs from NY Times sorted by ℓ_1 distance, anchor words in bold.

RecoverL2	tax bill zzz_senate billion plan zzz_bush
Gibbs	bill zzz_senate zzz_congress zzz_house legislation
	zzz_white_house
RecoverL2	company francisco san com food home
Gibbs RecoverL2	palm beach com statesman daily american
Gibbs	team player season game zzz_john_rocker right season team baseball game player yankees
RecoverL2	zzz_bush official zzz_united_states zzz_u_s president
	zzz_north_korea
Gibbs	zzz_united_states weapon zzz_iraq nuclear zzz_russia
RecoverL2	zzz_bush zzz_russian zzz_russia official military war attack
Gibbs	government war country rebel leader military
RecoverL2	wine wines percent zzz_new_york com show
Gibbs	film movie character play minutes hour
RecoverL2 Gibbs	police zzz_ray_lewis player team case told
RecoverL2	police officer gun crime shooting shot government group political tax leader money
Gibbs	government war country rebel leader military
RecoverL2	percent company million airline flight deal
Gibbs	flight airport passenger airline security airlines
RecoverL2 Gibbs	book ages children school boy web
RecoverL2	book author writer word writing read corp group president energy company member
Gibbs	palm beach com statesman daily american
RecoverL2	team tour zzz_lance_armstrong won race win
Gibbs	zzz_olympic games medal gold team sport
RecoverL2 Gibbs	priest church official abuse bishop sexual church religious priest zzz_god religion bishop
RecoverL2	human drug company companies million scientist
Gibbs	scientist light science planet called space
RecoverL2	music zzz_napster company song com web
Gibbs	palm beach com statesman daily american
RecoverL2	death government case federal official zzz_timothy_mcveigh
Gibbs	trial death prison case lawyer prosecutor
RecoverL2	million shares offering public company initial
Gibbs	company million percent billion analyst deal
RecoverL2	buy panelist thought flavor product ounces food restaurant chef dinner eat meal
Gibbs RecoverL2	school student program teacher public children
Gibbs	school student teacher children test education
RecoverL2	security official government airport federal bill
Gibbs	flight airport passenger airline security airlines
RecoverL2 Gibbs	company member credit card money mean zzz_enron company firm accounting
GIDDS	zzz_arthur_andersen financial
RecoverL2	million percent bond tax debt bill
Gibbs	million program billion money government federal
RecoverL2	million company zzz_new_york business art percent
Gibbs RecoverL2	art artist painting museum show collection percent million number official group black
Gibbs	palm beach com statesman daily american
RecoverL2	company tires million car zzz_ford percent
Gibbs	company companies business industry firm market
RecoverL2 Gibbs	article zzz_new_york misstated company percent com palm beach com statesman daily american
RecoverL2	company million percent companies government
	official
Gibbs	company companies business industry firm market
RecoverL2	official million train car system plan
Gibbs RecoverL2	million program billion money government federal test student school look percent system
Gibbs	patient doctor cancer medical hospital surgery
RecoverL2	con una mas dice las anos
Gibbs	fax syndicate article com information con
RecoverL2	por con una mas millones como
Gibbs RecoverL2	fax syndicate article com information con las como zzz_latin_trade articulo telefono fax
Gibbs	fax syndicate article com information con
RecoverL2	los con articulos telefono representantes
G".	zzz_america_latina
Gibbs RecoverL2	fax syndicate article com information con file sport read internet email zzz_los_angeles
Gibbs	web site com www mail zzz_internet

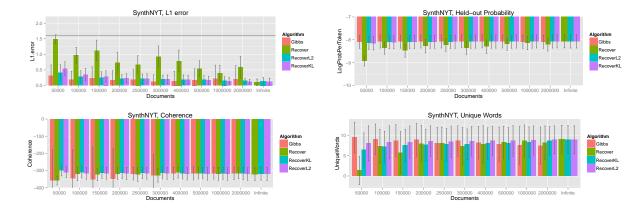


Figure 9: Results for a semi-synthetic model generated from a model trained on NY Times articles with K = 100.

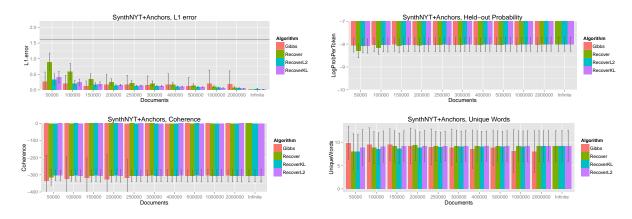


Figure 10: Results for a semi-synthetic model generated from a model trained on NY Times articles with K = 100, with a synthetic anchor word added to each topic.

D Algorithmic Details

D.1 Generating Q matrix

For each document, let H_d be the vector in \mathbb{R}^V such that the *i*-th entry is the number of times word *i* appears in document *d*, n_d be the length of the document and W_d be the topic vector chosen according to Dirichlet distribution when the documents are generated. Conditioned on W_d 's, our algorithms require the expectation of Q to be $\frac{1}{M} \sum_{d=1}^{M} AW_dW_d^T A^T$.

In order to achieve this, similar to [Anandkumar et al., 2012], let the normalized vector $\tilde{H}_d = \frac{H_d}{\sqrt{n_d(n_d-1)}}$ and diagonal matrix $\hat{H}_d = \frac{\mathrm{Diag}(H_d)}{n_d(n_d-1)}$. Compute the matrix

$$\tilde{H}_d \tilde{H}_d^T - \hat{H}_d = \frac{1}{n_d(n_d - 1)} \sum_{i \neq j, i, j \in [n_d]} e_{z_{d,i}} e_{z_{d,j}}^T.$$

Here $z_{d,i}$ is the *i*-th word of document d, and $e_i \in \mathbb{R}^V$ is the basis vector. From the generative model, the expectation of all terms $e_{z_{d,i}}e_{z_{d,j}}^T$ are equal to $AW_dW_d^TA^T$, hence by linearity of expectation we know $\mathbf{E}[\tilde{H}_d\tilde{H}_d^T - \hat{H}_d] = AW_dW_d^TA^T$.

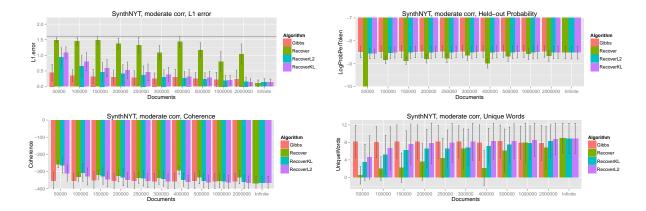


Figure 11: Results for a semi-synthetic model generated from a model trained on NY Times articles with K = 100, with moderate correlation between topics.

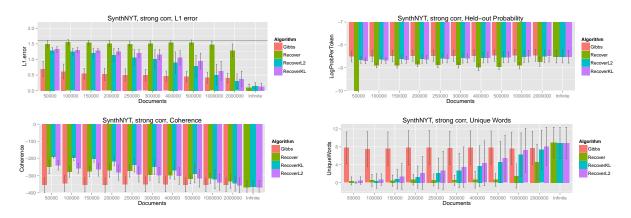


Figure 12: Results for a semi-synthetic model generated from a model trained on NY Times articles with K = 100, with stronger correlation between topics.

If we collect all the column vectors \tilde{H}_d to form a large sparse matrix \tilde{H} , and compute the sum of all \hat{H}_d to get the diagonal matrix \hat{H} , we know $Q = \tilde{H}\tilde{H}^T - \hat{H}$ has the desired expectation. The running time of this step is $O(MD^2)$ where D^2 is the expectation of the length of the document squared.

D.2 Exponentiated gradient algorithm

The optimization problem that arises in RecoverKL and RecoverL2 has the following form,

$$\label{eq:minimize} \begin{aligned} & \text{minimize } d(b, Tx) \\ & \text{subject to: } x \geq 0 \text{ and } x^T \mathbf{1} = 1 \end{aligned}$$

where $d(\cdot, \cdot)$ is a Bregman divergence, x is a vector of length K, and T is a matrix of size $V \times K$. We solve this optimization problem using the Exponentiated Gradient algorithm [Kivinen & Warmuth, 1995], described in Algorithm 5. In our experiments we show results using both squared Euclidean distance and KL divergence for the divergence measure. Stepsizes are chosen with a line search to find an η that satisfies the Wolfe and Armijo conditions (For details, see Nocedal & Wright

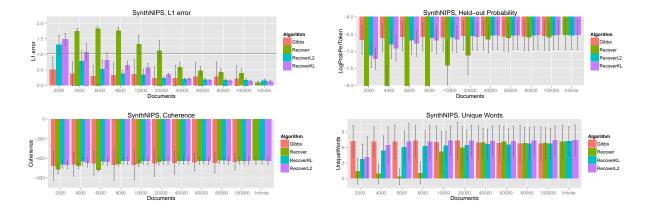


Figure 13: Results for a semi-synthetic model generated from a model trained on NIPS papers with K = 100. For $D \in \{2000, 6000, 8000\}$, Recover produces log probabilities of $-\infty$ for some held-out documents.

[2006]). We test for convergence using the KKT conditions. Writing the KKT conditions for our constrained minimization problem:

1. Stationarity: $\nabla_x d(b, Tx^*) - \vec{\lambda} + \mu \mathbf{1} = 0$

2. Primal Feasibility: $x^* \ge 0$, $|x^*|_1 = 1$

3. Dual Feasibility: $\lambda \geq 0$

4. Complementary Slackness: $\lambda_i x_i^* = 0$

For every iterate of x generated by Exponentiated Gradient, we set λ, μ to satisfy conditions 1-3. This gives the following equations:

$$\lambda = \nabla_x d(b, Tx^*) + \mu \mathbf{1}$$
$$\mu = -(\nabla_x d(b, Tx^*))_{\min}$$

By construction conditions 1-3 are satisfied (note that the multiplicative update and the projection step ensure that x is always primal feasible). Convergence is tested by checking whether the final KKT condition holds within some tolerance. Since λ and x are nonnegative, we check complimentary slackness by testing whether $\lambda^T x < \epsilon$. This convergence test can also be thought of as testing the value of the primal-dual gap, since the Lagrangian function has the form: $L(x, \lambda, \mu) = d(b, Tx) - \lambda^T x + \mu(x^T \mathbf{1} - 1)$, and $(x^T \mathbf{1} - 1)$ is zero at every iteration.

The running time of RecoverL2 is the time of solving V small $(K \times K)$ quadratic programs. Especially when using Exponentiated Gradient to solve the quadratic program, each word requires O(KV) time for preprocessing and $O(K^2)$ per iteration. The total running time is $O(KV^2 + K^2VT)$ where T is the average number of iterations. The value of T is about 100 - 1000 depending on data sets.

Algorithm 5. Exponentiated Gradient

```
Input: Matrix T, vector b, divergence measure d(\cdot,\cdot), tolerance parameter \epsilon

Output: non-negative normalized vector x close to x^*, the minimizer of d(b,Tx)

Initialize x \leftarrow \frac{1}{K}\mathbf{1}

Initialize Converged \leftarrow False

while not Converged do

p = \nabla d(b,Tx)

Choose a step size \eta_t

x \leftarrow xe^{-\eta_t p} (Gradient step)

x \leftarrow \frac{x}{|x|_1} (Projection onto the simplex)

\mu \leftarrow \nabla d(b,Tx)_{\min}

\lambda \leftarrow \nabla d(b,Tx) - \mu

Converged \leftarrow \lambda^T x < \epsilon

end while
```