

# Introduction to Graph Theory

Fourth edition

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*Go forth, my little book! pursue thy way!*  
*Go forth, and please the gentle and the good.*  
 William Wordsworth

# Index of definitions

*Only a little more*

*I have to write,*

*Then I'll give o'er*

*And bid the world Goodnight.*

Robert Herrick

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*Here is my journey's end.*

William Shakespeare (Macbeth)