

Convex Functions and Optimization Methods on Riemannian Manifolds

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Convex Functions and Optimization Methods on Riemannian Manifolds

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To Aneta, Daniel and Sorin

CONTENTS

Preface	xiii
Chapter 1. Metric properties of Riemannian manifolds	1
§1. Riemannian metric	1
Examples of Riemannian metrics	2
§2. Riemannian connection	4
§3. Differential operators	8
§4. Definite symmetric tensor fields of order two	13
§5. Geodesics and exponential map	15
§6. Metric structure of a Riemannian manifold	21
§7. Completeness of Riemannian manifolds	23
Gordon completeness criterion	24
Nomizu-Ozeki theorem	26
Applications to Hamiltonian systems	26
§8. Minimum points of a real function	28
Chapter 2. First and second variations of the p-energy of a curve	34
§1. Preliminaries	35
§2. The p-energy and the first variation formula	36
§3. Second variation of the p-energy	38
§4. Null space of the Hessian of the p-energy	41
§5. Index theorem	46
§6. Distance from a point to a closed set	51
§7. Distance between two closed sets	54
Chapter 3. Convex functions on Riemannian manifolds	56
§1. Convex sets in Riemannian manifolds	57
§2. Convex functions on Riemannian manifolds	60

§3. Basic properties of convex functions	66
§4. Directional derivatives and subgradients	71
§5. Convexity of functions of class C^1	77
§6. Convexity of functions of class C^2	81
Convexity of Rosenbrock banana function	83
Examples on the sphere S^2	84
Examples on Poincaré plane	86
Linear affine functions	88
§7. Convex programs on Riemannian manifolds	90
§8. Duality in convex programming	93
§9. Kuhn-Tucker theorem on Riemannian manifolds	95
§10. Quasiconvex functions on Riemannian manifolds	97
Nontrivial examples of quasiconvex functions which are not convex	100
§11. Distance from a point to a closed totally convex set	101
§12. Distance between two closed totally convex sets	105
Chapter 4. Geometric examples of convex functions	108
§1. Example of Greene and Shiohama	109
§2. Example of Wu	110
§3. Examples of Bishop and O'Neill	113
§4. Convexity of Busemann functions	116
§5. Construction of Cheeger and Gromoll	121
§6. Preserving the completeness and the convexity	122
Chapter 5. Flows, convexity and energies	128
§1. Flows and energies on Riemannian manifolds	129
§2. General properties of the gradient flow	135
§3. Gradient flow of a convex function	137

§4. Diffeomorphisms imposed by a convex function	141
§5. Energy and flow of an irrotational vector field	146
§6. Energy and flow of a Killing vector field	152
§7. Energy and flow of a conformal vector field	157
Examples of vector fields with dense orbits	164
§8. Energy and flow of an affine vector field	165
§9. Energy and flow of a projective vector field	168
§10. Energy and flow of a torse forming vector field	170
§11. Runge-Kutta approximation of the orbits	173
TPascal program for Runge-Kutta approximation of the orbits	176
Chapter 6. Semidefinite Hessians and applications	186
§1. Strongly convex functions on Riemannian manifolds	187
§2. Convex hypersurfaces in Riemannian manifolds	192
§3. Convex functions on Riemannian submanifolds	199
Gradient and Hessian on submanifolds	199
Case of tangent bundle	204
Obata theorem	206
Special hypersurfaces of constant level	207
§4. Convex functions and harmonic maps	208
Examples and applications	211
§5. G-connected domains	214
Preliminaries	214
Connected domains	215
Examples	219
§6. Linear complementarity problems	220
§7. Conservative dynamical systems with convex potential	223

Chapter 7. Minimization of functions on Riemannian manifolds	226
§1. Special properties of the minus gradient flow	227
Minus gradient flow	227
Runge-Kutta approximation of a minus gradient line	232
TC program for gradient lines in 3-dimensional space	234
TC program for gradient lines in Poincaré plane	237
§2. Numerical approximation of geodesics	238
Approximate solution of Cauchy problem	239
Case of Poincaré plane	240
TC program for Poincaré geodesics	241
Case of hypersurfaces described by Cartesian implicit equations	242
TC program for spherical geodesics	244
Approximate solution of boundary value problem	246
TC program for geodesics by boundary conditions	249
§3. General descent algorithm on Riemannian manifolds	252
Descent directions and criteria of stopping	252
Convergence of $\{\text{grad } f(x_1)\}$ to zero	256
Convergence of $\{x_1\}$ to a critical point	260
§4. Gradient methods on Riemannian manifolds	262
Method of steepest descent	262
Convergence of $\{\text{grad } f(x_1)\}$ to zero	264
Convergence of $\{x_1\}$ to a critical point	265
Variants of the gradient method	269
Examples	272
Other gradient methods	276
§5. Generalized Newton method on Riemannian manifolds	279
Radial approximations	279
First construction of the method	279
Second construction of the method	281

Properties of the method	282
§6. General descent algorithm for a constrained minimum	284
Appendices	287
1. Riemannian convexity of functions $f: \mathbb{R} \longrightarrow \mathbb{R}$	287
§0. Introduction	287
§1. Geodesics of (\mathbb{R}, g)	287
§2. Geodesics of $(\mathbb{R} \times \mathbb{R}, g_{11} + 1)$	289
§3. Convex functions on (\mathbb{R}, g)	291
2. Descent methods on the Poincaré plane	297
§0. Introduction	297
§1. Poincaré plane	297
§2. Linear affine functions on the Poincaré plane	298
§3. Quadratic affine functions on the Poincaré plane	299
§4. Convex functions on the Poincaré plane	300
Examples of hyperbolic convex functions	301
§5. Descent algorithm on the Poincaré plane	301
TC program for descent algorithm on Poincaré plane (I)	303
TC program for descent algorithm on Poincaré plane (II)	305
3. Descent methods on the sphere	311
§1. Gradient and Hessian on the sphere	311
§2. Descent algorithm on the sphere	312
Critical values of the normal stress	313
Critical values of the shear stress	314
TC program for descent method on the unit sphere	316
4. Completeness and convexity on Finsler manifolds	318
§1. Complete Finsler manifolds	318
§2. Analytical criterion for completeness	323
§3. Warped products of complete Finsler manifolds	326
§4. Convex functions on Finsler manifolds	326
References	329
Bibliography	331
Index	341

P R E F A C E

The object of this book is to present the basic facts of convex functions, standard dynamical systems, descent numerical algorithms and some computer programs on Riemannian manifolds in a form suitable for applied mathematicians, scientists and engineers. It contains mathematical information on these subjects and applications distributed in seven chapters whose topics are close to my own areas of research: Metric properties of Riemannian manifolds, First and second variations of the p-energy of a curve; Convex functions on Riemannian manifolds; Geometric examples of convex functions; Flows, convexity and energies; Semidefinite Hessians and applications; Minimization of functions on Riemannian manifolds. All the numerical algorithms, computer programs and the appendices (Riemannian convexity of functions $f:\mathbb{R} \rightarrow \mathbb{R}$, Descent methods on the Poincaré plane, Descent methods on the sphere, Completeness and convexity on Finsler manifolds) constitute an attempt to make accesible to all users of this book some basic computational techniques and implementation of geometric structures. To further aid the readers, this book also contains a part of the folklore about Riemannian geometry, convex functions and dynamical systems because it is unfortunately "nowhere" to be found in the same context; existing textbooks on convex functions on Euclidean spaces or on dynamical systems do not mention what happens in Riemannian geometry, while the papers dealing with Riemannian manifolds usually avoid discussing elementary facts.

Usually a convex function on a Riemannian manifold is a real-valued function whose restriction to every geodesic arc is convex. When we refer to a subset A of a Riemannian manifold, this definition of the convexity of $f:A \rightarrow \mathbb{R}$ requires a definition of the convexity of the subset A . Only for a C^2 function it is possible to give a generalized definition which does not require the presence of geodesics: a C^2 function is convex if $\text{Hess } f$ is positive semidefinite. This coordinate-free description of convexity can be easily connected with symbolic computation.

Convex functions occur abundantly, have many structural

implications on manifolds and form an important link in the modern analysis and geometry. These implications do not occur in the extensive theory of convex functions on Euclidean spaces because of the particularity of these spaces.

From 1964 new insights have been gained in old problems combined with new ones, and great coherence has been achieved in understanding the role of the Riemannian convexity in science. For example, Gordon has found two applications of convexity to mechanics surprisingly opposite in consequence: a trajectory cannot stay in a compact domain supporting a function whose Hessian with respect to the Jacobi metric is positive definite; and (much deeper) if a potential function has positive semidefinite Hessian which is positive definite on a geodesic through a minimum point, then every neighborhood of that point has nontrivial closed trajectories. Also some recent papers suggest that a reasonable general mathematical approach to thermodynamics, which has not yet been given, will involve ideas from differential-geometric facts concerning extrema, convexity and dynamical systems. Therefore our goal to answer questions about specific dynamical systems on Riemannian manifolds, and interactions between numerical computation, visualization and mathematical theory of the orbits, geodesics and optimization is justified.

An optimization (minimization or maximization) problem is specified by a set C , called the feasible set, and a real-valued function f on C , called the objective function. Often the feasible set C is a Riemannian manifold described by equality constraints and inequality constraints on \mathbb{R}^n , and the optimization problem resides in the optimal value of f (a number), or the optimal solutions (distinguished elements of the feasible set), or both, depending on circumstances.

The concept of convexity plays a very important part in the theory of optimization, firstly because many objective functions are convex in a sufficiently small neighborhood of a local minimum point, and secondly because the convergence of numerical methods for estimating local minimum points can be established for convex objective functions. The numerical methods and the computer programs presented in this book for the finding of a critical point of a function are intimately

related to the Riemannian structure of the manifold and are independent of the choice of coordinate systems.

The topics of this book have been selected in order to give the readers a feeling for the way in which the theory of convex functions on Riemannian manifolds has developed and is developing, and to make available a battery of procedures which can be used to solve real-life problems in a reasonable amount of time. Consequently we hope that our book will give the reader an initial perspective on this subject and make it easier for him to approach the specialized literature.

From the point of view of the mathematical expression, one has considered that accessible reassessments are more useful than maintaining of a tight language, accepting that mathematics is not an isolated intellectual structure but a part of the general process of scientific modelling. In this sense one has preferred the topics and notations that do not embed the mathematical drafts into a lot of unessential data, and the technique of beginning each chapter by a short introduction which suggests the contents and the importance of the respective chapter.

Many mathematicians have asked me why do we study the convexity on Riemannian manifolds. Is it not enough to study the convexity on Euclidean spaces ? Has this theory any applications which justify it ? We consider that the present work contains the necessary reasons and the clear answer that in fact the theory of convex functions reveals all its power and consequences only when it is conceived on a Riemannian structure.

The book will be of interest especially to applied mathematicians (working in convexity, differential geometry, dynamical systems, optimization and numerical methods), to scientists and engineers as well. By virtue of the elementary nature of the analytical tools, it can also be used as a text for undergraduate and graduate students with a good background in Riemannian Geometry and Analysis.

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