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EXTENSIONS OF LIPSCHITZ MAPPINGS INTO A HILBERT SPACE

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INTRODUCTION

In this note we consider the following extension problem for Lipschitz functions: Given a metric space X and $n = 2, 3, 4, \ldots$, estimate the smallest constant $L = L(X, n)$ so that every mapping f from every n-element subset of X into ℓ_2 extends to a mapping $\tilde{f}$ from X into ℓ_2 with

$$\|\tilde{f}\|_{\ell ip} \leq L\,\|f\|_{\ell ip} .$$

(Here $\|g\|_{\ell ip}$ is the Lipschitz constant of the function g.) A classical result of Kirszbraun's [14, p. 48] states that $L(\ell_2, n) = 1$ for all n, but it is easy to see that $L(X, n) \to \infty$ as $n \to \infty$ for many metric spaces X.

Marcus and Pisier [10] initiated the study of $L(X, n)$ for $X = L_p$. (For brevity, we will use hereafter the notation $L(p, n)$ for $L(L_p(0,1), n)$.) They prove that for each $1 < p < 2$ there is a constant $C(p)$ so that for $n = 2, 3, 4, \ldots$,

$$L(p, n) \leq C(p)\ (\mathrm{Log}\ n)^{1/p - 1/2} .$$

The main result of this note is a verification of their conjecture that for some constant C and all $n = 2, 3, 4, \ldots$,

$$L(X, n) \leq C(\mathrm{Log}\ n)^{1/2}$$

for all metric spaces X. While our proof is completely different from that of Marcus and Pisier, there is a common theme: Probabilistic techniques developed for linear theory are combined with Kirszbraun's theorem to yield extension theorems.

The main tool for proving Theorem 1 is a simply stated elementary geometric lemma, which we now describe: Given n points in Euclidean space, what

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is the smallest $k = k(n)$ so that these points can be moved into k-dimensional Euclidean space via a transformation which expands or contracts all pairwise distances by a factor of at most $1 + \varepsilon$? The answer, that $k \le C(\varepsilon)$ Log n, is a simple consequence of the isoperimetric inequality for the n-sphere in the form studied in [2].

It seems likely that the Marcus-Pisier result and Theorem 1 give the right order of growth for $L(p, n)$. While we cannot verify this, in Theorem 3 we get the estimate

$$L(p, n) \ge \delta \left(\frac{\operatorname{Log} n}{\operatorname{Log} \operatorname{Log} n}\right)^{1/p - 1/2} \qquad (1 \le p < 2)$$

for some absolute constant $\delta > 0$. (Throughout this paper we use the convention that Log x denotes the maximum of 1 and the natural logarithm of x.) This of course gives a lower estimate of

$$\delta \left(\frac{\operatorname{Log} n}{\operatorname{Log} \operatorname{Log} n}\right)^{1/2}$$

for $L(\infty, n)$. That our approach cannot give a lower bound of $\delta(\operatorname{Log} n)^{1/p - 1/2}$ for $L(p, n)$ is shown by Theorem 2, which is an extension theorem for mappings into ℓ_2 whose domains are ε-separated.

The minimal notation we use is introduced as needed. Here we note only that $B_Y(y, \varepsilon)$ (respectively, $b_Y(y, \varepsilon)$) is the closed (respectively, open) ball in Y about y of radius ε. If $y = 0$, we use $B_Y(\varepsilon)$ and $b_Y(\varepsilon)$, and we drop the subscript Y when there is no ambiguity. $S(Y)$ is the unit sphere of the normed space Y. For isomorphic normed spaces X and Y, we let

$$d(X,Y) = \inf \|T\| \, \|T^{-1}\|,$$

where the inf is over all invertible linear operators from X onto Y. Given a bounded Banach space valued function f on a set K, we set

$$\|f\|_\infty = \sup_{x \in K} \|f(x)\|.$$

1. THE EXTENSION THEOREMS

We begin with the geometrical lemma mentioned in the introduction.

LEMMA 1. For each $1 > \tau > 0$ there is a constant $K = K(\tau) > 0$ so that if $A \subset \ell_2^n$, $\overline{\overline{A}} = n$ for some $n = 2, 3, \ldots$, then there is a mapping f from A onto a subset of ℓ_2^k $(k = [K \log n])$ which satisfies

$$\|\tilde{f}\|_{\ell ip} \, \|\tilde{f}^{-1}\|_{\ell ip} \leq \frac{1 + \tau}{1 - \tau}.$$

PROOF. The proof will show that if one chooses at random a rank k orthogonal projection on ℓ_2^n, then, with positive probability (which can be made arbitrarily close to one by adjusting k), the projection restricted to A will satisfy the condition on $\tilde{f}$. To make this precise, we let Q be the projection onto the first k coordinates of ℓ_2^n and let σ be normalized Haar measure on $O(n)$, the orthogonal group on ℓ_2^n. Then the random variable

$$f : (O(n), \sigma) \to L(\ell_2^n)$$

defined by

$$f(u) = U^* QU$$

determines the notion of "random rank k projection." The applications of Levy's inequality in the first few self-contained pages of [2] make it easy to check that $f(u)$ has the desired property. For the convenience of the reader, we follow the notation of [2].

Let $|||\cdot|||$ denote the usual Euclidean norm on $\mathbb{R}^n$ and for $1 \leq k \leq n$ and $x \in \mathbb{R}^n$ set

$$r(x) = r_k(x) = \sqrt{n} \left(\sum_{i=1}^{k} x(i)^2 \right)^{1/2},$$

which is equal to

$$\sqrt{n} \; |||Qx|||$$

for our eventual choice of $k = [K \log n]$. Thus $r(\cdot)$ is a semi-norm on ℓ_2^n which satisfies

$$r(x) \leq \sqrt{n} \; |||x||| \qquad (x \in \ell_2^n).$$

(In [2], $r(\cdot)$ is assumed to be a norm, but inasmuch as the left estimate $a|||x||| \leq r(x)$ in formula (2.5) of [2] is not needed in the present situation, it is okay that $r(\cdot)$ is only a semi-norm.)

Setting

$$B = \left\{ \frac{x - y}{|||x-y|||} : x, y \in A; \; x \neq y \right\} \subset S^{n-1},$$

we want to select $U \in O(n)$ so that for some constant M,

$$M(1 - \tau) \le r(Ux) \le M(1 + \tau) \quad (x \in B) .$$

Let M_r be the median of $r(\cdot)$ on S^{n-1}, so that

$$\mu_{n-1}[x \in S^{n-1} : r(x) \ge M_r] \ge 1/2$$

and

$$\mu_{n-1}[x \in S^{n-1} : r(x) \le M_r] \le 1/2$$

where μ_{n-1} is normalized rotationally invariant measure on S^{n-1}.

We have from page 58 of [2] that for each $y \in S^{n-1}$ and $\varepsilon > 0$,

$$\sigma[U \in O(n) : M_r - \sqrt{n}\,\varepsilon \le r(Uy) \le M_r + \sqrt{n}\,\varepsilon] \ge 1 - 4 \exp\left(\frac{-n\varepsilon^2}{2}\right).$$

Hence

$$\sigma[U \in O(n) : M_r - \sqrt{n}\,\varepsilon \le r(Uy)) \le M_r + \sqrt{n}\,\varepsilon \text{ for all } y \in B] \ge \tag{1.1}$$

$$\ge 1 - 2n(n+1) \exp\left(\frac{-n\varepsilon^2}{2}\right).$$

By Lemma 1.7 of [2], there is a constant

$$C \le 4 \sum_{m=1}^{\infty} (m+1)\, e^{-m^2/2}$$

so that

$$\left|\int_{S_{n-1}} r(x)\, d\mu_{n-1}(x) - M_r\right| < C . \tag{1.2}$$

We now repeat a known argument for estimating $\int_{S_{n-1}} r(x)\, d\mu_{n-1}(x)$ which uses only Khintchine's inequality.

For $1 \le k \le n$ we have:

$$\operatorname*{Av}_{\pm} \int_{S^{n-1}} \left| \sum_{i=1}^{k} \pm x(i) \right| d\mu_{n-1}(x) =$$

$$= \operatorname*{Av}_{\pm} \int_{S^{n-1}} \left|\left< x, \sum_{i=1}^{k} \pm \delta_1 \right>\right| d\mu_{n-1}(x)$$

$$= \sqrt{k} \int_{S^{n-1}} |\langle x, \delta_1 \rangle|\, d\mu_{n-1}(x) \quad \left[\begin{array}{l}\text{by the rotational} \\ \text{invariance of } \mu_{n-1}\end{array}\right] .$$

Setting

$$\alpha_n = \int_{S^{n-1}} |\langle x, \delta_1 \rangle|\, d\mu_{n-1}(x) ,$$

we have from Khintchine's inequality that for each $1 \le k \le n$,

$$\sqrt{nk}\,\alpha_n \le \int_{S^{n-1}} r_k(x)\ d\mu_{n-1}(x) \le \sqrt{2nk}\,\alpha_n .$$

(We plugged in the exact constant of $\sqrt{2}$ in Khintchine's inequality calculated in [5] and [13], but of course any constant would serve as well.) Since obviously $r_n(x) = \sqrt{n}$, we conclude that for $1 \le k \le n$

$$\sqrt{k/\ } \le \int_{S^{n-1}} r_k(x)\ d\mu_{n-1}(x) \le \sqrt{k} . \tag{1.3}$$

Specializing now to the case $k = [K \log n]$, we have from (1.2) and (1.3) that

$$\sqrt{k/3} \le M_r$$

at least for $K \log n$ sufficiently large. Thus if we define

$$\varepsilon = \tau \sqrt{k/3n}$$

we get from (1.1) that

$$\sigma\,[U \in O(n) : (1-\tau)M_r \le r(Uy) \le (1+\tau)M_r \quad \text{for all} \quad y \in B]$$

$$\ge 1 - 2n(n+1)\exp\left(-\frac{\tau^2 k}{18}\right)$$

$$\ge 1 - 2n(n+1)\ \exp\left(-\frac{\tau^2 K \log n}{18}\right)$$

which is positive if, say,

$$K \ge (10/\tau)^2 . \qquad \square$$

It is easily seen that the estimate $K \log n$ in Lemma 1 cannot be improved. Indeed, in a ball of radius 2 in ℓ_2^k there are at most 4^k vectors $\{x_i\}$ so that $\|x_i - x_j\| \ge 1$ for every $i \ne j$ (see the proof of Lemma 3 below). Hence for τ sufficiently small there is no map F which maps an orthonormal set with more than 4^k vectors into a k-dimensional subspace of ℓ_2 with

$$\|F\|_{\ell ip}\ \|F^{-1}\|_{\ell ip} \le \frac{1+\tau}{1-\tau} .$$

We can now verify the conjecture of Marcus and Pisier [10].

THEOREM 1. $\sup_{n=2,3,\dots} (\log n)^{-1/2} L(\infty, n) < \infty$. In other words: there is a constant K so that for all metric spaces X and all finite subsets M of X (card $M = n$, say) every function f from M into ℓ_2 has a Lipschitz extension $\tilde{f} : X \to \ell_2$ which satisfies

$$\|\tilde{f}\|_{\ell ip} \le K \sqrt{\log n}\, \|f\|_{\ell ip} .$$

PROOF. Given X, $M \subset X$ with card $M = n$, and $f : M \to \ell_2$, set $A = f[M]$. We apply Lemma 1 with $\tau = 1/2$ to get a one-to-one function g^{-1} from A onto a subset $g^{-1}[A]$ of ℓ_2^k (where $k \le K \log n$) which satisfies

$$\|g^{-1}\|_{\ell ip} \le 1; \quad \|g\|_{\ell ip} \le 3 .$$

By Kirszbraun's theorem, we can extend g to a function $\tilde{g} : \ell_2^k \to \ell_2$ in such a way that

$$\|\tilde{g}\|_{\ell ip} \le 3 .$$

Let $I : \ell_2^k \to \ell_\infty^k$ denote the formal identity map, so that

$$\|I\| = 1, \quad \|I^{-1}\| = \sqrt{k} .$$

Then

$$h \equiv I g^{-1} f, \quad h : M \to \ell_\infty^k$$

has Lipschitz norm at most $\|f\|_{\ell ip}$, so by the non-linear Hahn-Banach theorem (see, e.g., p. 48 of [14]), h can be extended to a mapping

$$\tilde{h} : X \to \ell_\infty^k$$

which satisfies

$$\|\tilde{h}\|_{\ell ip} \le \|f\|_{\ell ip} .$$

Then

$$\tilde{f} \equiv \tilde{g}\, I^{-1}\, \tilde{h}; \quad \tilde{f} : X \to \ell_2$$

is an extension of f and satisfies

$$\|\tilde{f}\|_{\ell ip} \le 3\sqrt{k}\, \|f\|_{\ell ip} \le 3K \sqrt{\log n}\, \|f\|_{\ell ip} . \qquad \square$$

Next we outline our approach to the problem of obtaining a lower bound for $L(\infty,n)$. Take for f the inclusion mapping from an ε-net for S^{N-1} into ℓ_2^N, and consider ℓ_2^N isometrically embedded into L_∞. A Lipschitz extension of f to a mapping $\tilde{f} : L_\infty \to \ell_2$ should act like the identity ℓ_2^N, so the techniques of [8] should yield a linear projection from L_∞ onto ℓ_2^N whose norm is of order $\|\tilde{f}\|_{\ell ip}$. Since ℓ_2^N is complemented in L_∞ only of order $\sqrt{N}$ and there are ε-nets for S^{N-1} of cardinality $n \equiv [4/\varepsilon]^N$, we should get that

$$L(\infty,n) \geq \sqrt{N} \geq \delta \left(\frac{\operatorname{Log} n}{-\operatorname{Log} \varepsilon}\right)^{1/2} .$$

In Theorem 2 we make this approach work when ε is of order N^{-2}, so we get

$$L(\infty,n) \geq \delta' \left(\frac{\operatorname{Log} n}{\operatorname{Log}\operatorname{Log} n}\right)^{1/2} .$$

That the difficulties we incur with the outlined approach for larger values of ε are not purely technical is the gist of the following extension result.

(*)THEOREM 2. _Suppose that_ X _is a metric space,_ $A \subset X$, $f : A \to \ell_2$ _is Lipschitz and_ $d(x,y) \geq \varepsilon > 0$ _for all_ $x \neq y \in A$. _Then there is an extension_ $\tilde{f} : X \to \ell_2$ _of_ f _so that_

$$\|\tilde{f}\|_{\ell ip} \leq \frac{6D}{\varepsilon} \|f\|_{\ell ip} ,$$

where D _is the diameter of_ A.

PROOF. We can assume by translating f that there is a point $0 \in A$ so that $f(0) = 0$. Set $B = A \sim \{0\}$ and define

$$F : A \to \ell_1^B \text{ by}$$

$$F(b) = \begin{Bmatrix} \delta_b, & b \neq 0 \\ 0, & b = 0 \end{Bmatrix} .$$

Define

$$G : \ell_1^B \to \ell_2$$

by

$$G\Big(\sum_{b \in B} \alpha_b \delta_b\Big) = \sum_{b \in B} \alpha_b f(b) .$$

(*) See the appendix for a generalization of Theorem 2 proved by Yoav Benyamini.

Then

$$G\,F = f, \quad G \text{ is linear with}$$

$$\|G\| \le D\,\|f\|_{\ell ip}, \quad \text{and} \quad \|F\|_{\ell ip} \le 2/\varepsilon .$$

A weakened form of Grothendieck's inequality (see section 2.6 in [9]) yields that G (as any bounded linear operator from an L_1 space into a Hilbert space) factors through an $\ell_\infty(\aleph)$ space:

$$G = H\,J, \ \|J\| = 1, \ \|H\| \le 3\,\|G\|,$$

$$J : \ell_1^B \to \ell_\infty(\aleph), \quad H : \ell_\infty(\aleph) \to \ell_2 .$$

By the non-linear Hahn-Banach Theorem the mapping $J\,F$ has an extension

$$E : X \to \ell_\infty(\aleph) \quad \text{which satisfies}$$

$$\|E\|_{\ell ip} \le \|J\,F\|_{\ell ip} \le 2/\varepsilon .$$

Then $\tilde{f} \equiv H\,E$ extends f and $\|\tilde{f}\| \le \frac{6D}{\varepsilon}\,\|f\|_{\ell ip}$, as desired. □

For the proof of Theorem 3, we need three well known facts which we state as lemmas.

LEMMA 2. *Suppose that* Y, X *are normed spaces and* $f : S(Y) \to X$ *is Lipschitz with* $f(0) = 0$. *Then the positively homogeneous extension of* f, *defined for* $y \in Y$ *by*

$$\tilde{f}(y) = \|y\|\, f\left(\frac{y}{\|y\|}\right), \ (y \ne 0); \quad \tilde{f}(0) = 0$$

is Lipschitz and

$$\|\tilde{f}\|_{\ell ip} \le 2\,\|f\|_{\ell ip} + \|f\|_\infty .$$

PROOF. Given $y_1, y_2 \in Y$ with $0 < \|y_1\| \le \|y_2\|$,

$$\begin{aligned}\|\tilde{f}(y_1) - \tilde{f}(y_2)\| &\le \left\| \|y_1\|\, f\left(\frac{y_1}{\|y_1\|}\right) - \|y_2\|\, f\left(\frac{y_1}{\|y_1\|}\right) \right\| + \|y_2\| \left\| f\left(\frac{y_1}{\|y_1\|}\right) - f\left(\frac{y_2}{\|y_2\|}\right) \right\| \\ &\le \Big(\|y_2\| - \|y_1\|\Big) \left\| f\left(\frac{y_1}{\|y_1\|}\right) \right\| + \|y_2\|\,\|f\|_{\ell ip} \left\| \frac{y_1}{\|y_1\|} - \frac{y_2}{\|y_2\|} \right\| \\ &\le \|y_1 - y_2\|\,\|f\|_\infty + \|f\|_{\ell ip} \left\| \frac{\|y_2\|}{\|y_1\|}\, y_1 - y_2 \right\|\end{aligned}$$

$$\le \|f\|_\infty \|y_1 - y_2\| + \|f\|_{\ell ip} \left[\left(\frac{\|y_2\|}{\|y_1\|} - 1\right) \|y_1\| + \|y_1 - y_2\|\right]$$

$$\le \left(\|f\|_\infty + 2\|f\|_{\ell ip}\right) \|y_1 - y_2\|. \qquad \square$$

LEMMA 3. <u>If</u> Y <u>is an n-dimensional Banach space and</u> $0 < \varepsilon$, <u>then</u> $S(Y)$ <u>admits an ε-net of cardinality at most</u> $(1 + 4/\varepsilon)^n$.

PROOF. Let M be a subset of $S(Y)$ maximal with respect to "$\|x-y\| \ge \varepsilon$ for all $x \ne y \in M$".

Then the sets

$$b(y, \varepsilon/2) \cap S(Y), \quad (y \in M)$$

are pairwise disjoint hence so are the sets

$$b(y, \varepsilon/4), \quad (y \in M).$$

Since these last sets are all contained in $b(1 + \varepsilon/4)$, we have that

$$\text{card } M \cdot \text{vol } b(\varepsilon/4) \le \text{vol } b(1 + \varepsilon/4)$$

so that

$$\text{card } M \le \left[\frac{4}{\varepsilon}(1 + \varepsilon/4)\right]^n. \qquad \square$$

LEMMA 4. <u>There is a constant</u> $\delta > 0$ <u>so that for each</u> $1 \le p < 2$ <u>and each</u> $N = 1, 2, \ldots$, L_p <u>contains a subspace</u> E <u>such that</u>

$$d(E, \ell_2^N) \le 2$$

<u>and every projection from</u> L_p <u>onto</u> E <u>has norm at least</u>

$$\delta N^{1/p - 1/2}.$$

PROOF. Given a finite dimensional Banach space X and $1 \le p < \infty$, let

$$\gamma_p(x) = \inf\{\|T\|\,\|S\| : T : X \to L_p,\ S : L_p \to X,\ S\,T = I_X\}.$$

So $\gamma_\infty(X)$ is the projection constant of X, hence by [4], [12]

$$\gamma_1(\ell_2^N) = \gamma_\infty(\ell_2^N) = \sqrt{2n/\pi}.$$

This gives the $p = 1$ case.

For $1 < p < 2$ we reduce to the case $p = 1$ by using Example 3.1 of [2], which asserts that there is a constant $C < \infty$ so that for $1 \le p < 2$ ℓ_p^{CN} contains a subspace E with $d(E, \ell_2^N) \le 2$. Since, obviously,

$$d(\ell_p^{CN}, \ell_1^{CN}) \le (CN)^{1 - 1/p}$$

we get that if E is K-complemented in ℓ_p^{CN}, then

$$\pi^{-1/2} (2n)^{1/2} = \gamma_1(\ell_2^N) \le d(E, \ell_2^N)\, d(\ell_p^{CN}, \ell_1^{CN})\, K$$

$$\le 2\, (CN)^{1 - 1/p}\, K. \qquad \square$$

The next piece of background information we need for Theorem 3 is a linearization result which is an easy consequence of the results in [8].

PROPOSITION 1. <u>Suppose</u> $X \subset Y$ <u>and</u> Z <u>are Banach spaces,</u> $f : Y \to Z$ <u>is Lipschitz, and</u> $U : X \to Z$ <u>is bounded, linear. Then there is a linear operator</u> $G : Z^* \to Y^*$ <u>so that</u> $\|G\| \le \|f\|_{\ell ip}$ <u>and</u>

$$\|R_2 G - U^*\| \le \|f_{|X} - U\|_{\ell ip},$$

<u>where</u> R_2 <u>is the natural restriction map from</u> Y^* <u>onto</u> X^*.

REMARK. Note that if Z is reflexive, the mapping $F \equiv G^*_{|Y} : Y \to Z$ satisfies $\|F\| \le \|f\|_{\ell ip}$ and $\|F_{|X} - U\| \le \|f_{|X} - U\|_{\ell ip}$.

PROOF. We first recall some notation from [8]. If Y is a Banach space, $Y^{\#}$ denotes the Banach space of all scalar valued Lipschitz functions $y^{\#}$ from Y for which $y^{\#}(0) = 0$, with the norm $\|y^{\#}\|_{\ell ip}$. There is an obvious isometric inclusion from Y^* into $Y^{\#}$. For a Lipschitz mapping $f : Y \to Z$, Z a normed space, we can define a <u>linear</u> mapping

$$f^{\#} : Z^* \to Y^{\#} \text{ by}$$

$$f^{\#} z^* = z^* f.$$

Given Banach spaces $X \subset Y$, Theorem 2 of [8] asserts that there are norm one linear projections

$$P_Y : Y^{\#} \to Y^*, \quad P_X : X^{\#} \to X^*$$

so that

$$P_X R_1 = R_2 P_Y,$$

where R_1 is the restriction mapping from $Y^{\#}$ onto $X^{\#}$. Thus if $X \subset Y$, f, U, Z are as in the hypothesis of Proposition 1, the linear mapping $P_Y f^{\#}$ satisfies

$$\|P_Y f^{\#}\| \le \|f\|_{\ell ip}, \quad R_2 P_Y f^{\#} = P_X R_1 f^{\#}.$$

Since $U: X \to Z$ is linear,

$$U^* = P_X U^{\#}$$

so

$$\|R_2 P_Y f^{\#} - U^*\| = \|P_X(R_1 f^{\#} - U^{\#})\|$$

$$\le \|R_1 f^{\#} - U^{\#}\| = \sup_{z^* \in S(Z^*)} \|R_1 f^{\#} z^* - U^{\#} z^*\|$$

$$= \sup_{z^* \in S(Z^*)} \|(z^* f)_{|X} - z^* U\| \le \|f_{|X} - U\|_{\ell ip}. \qquad \square$$

The final lemma we use in the proof of Theorem 3 is a smoothing result for homogeneous Lipschitz functions.

LEMMA 5. <u>Suppose</u> $X \subset Y$ <u>and</u> Z <u>are Banach spaces with</u> $\dim X = k < \infty$, $F: Y \to Z$ <u>is Lipschitz with</u> F <u>positively homogeneous</u> (i.e. $F(\lambda y) = \lambda F(y)$ <u>for</u> $\lambda \ge 0$, $y \in Y$) <u>and</u> $U : X \to Z$ <u>is linear. Then there is a positively homogeneous Lipschitz mapping</u>

$$\tilde{F} : Y \to Z \quad \underline{\text{which satisfies}}$$

(1) $\|\tilde{F}_{|X} - U\|_{\ell ip} \le (8k + 2) \quad \|F_{|S(X)} - U_{|S(X)}\|_\infty$

(2) $\|\tilde{F}\|_{\ell ip} \le 4 \|F\|_{\ell ip}$.

PROOF. For $y \in S(Y)$ define

$$\hat{F}y = \int_{B_X(1)} F(y+x)\, d\mu(x)$$

where $\mu(\cdot)$ is Haar measure on $X\ (= \mathbb{R}^k)$ normalized so that

$$\mu(B_X(1)) = 1.$$

For $y_1, y_2 \in S(Y)$ we have

$$\|\hat{F}y_1 - \hat{F}y_2\| \le \int_{B_X(1)} \|F(y_1 + x) - F(y_2 + x)\| \, d\mu(x)$$

$$\le \|F\|_{\ell ip} \, \|y_1 - y_2\|$$

so

$$\|\hat{F}\|_{\ell ip} \le \|F\|_{\ell ip}.$$

For $x_1, x_2 \in S(X)$ with $\|x_1 - x_2\| = \delta > 0$ we have, since U is linear, that

$$\|(\hat{F} - U)x_1 - (\hat{F} - U)x_2\| =$$

$$\|\int_{B_X(1)} F(x_1 + x)\,d\mu(x) - \int_{B_X(1)} U(x_1 + x)\,d\mu(x) - \int_{B_X(1)} F(x_2 + x)\,d\mu(x) +$$

$$\int_{B_X(1)} U(x_2 + x)\,d\mu(x)\| \le$$

$$\le \int_{B_X(x_1;\, 1) \,\Delta\, B_X(x_2;\, 1)} \|Fx - Ux\| \, d\mu(x) \le$$

$$\le \sup_{x \in B_X(2)} \|Fx - Ux\| \; \mu\,[B_X(x_1;\, 1) \,\Delta\, B_X(x_2;\, 1)]$$

$$= 2 \sup_{x \in B_X(1)} \|Fx - Ux\| \; \mu\,[B_X(x_1;\, 1) \,\Delta\, B_X(x_2;\, 1)] \qquad \left[\text{since } F \text{ is positively homogeneous}\right].$$

Since

$$B_X(x_1;\, 1) \,\Delta\, B_X(x_2;\, 1) \subset [B_X(x_1;\, 1) \sim B_X(x_1;\, 1-\delta)] \cup [B_X(x_2;\, 1) \sim B_X(x_2;\, 1-\delta)]$$

we have if $\delta \le 1$ that

$$\mu[B_X(x_2;\, 1) \,\Delta\, B_X(x_2;\, 1)] \le 2[1 - (1-\delta)^k]$$

$$\le 2\,k\,\delta$$

and hence for all $x_1, x_2 \in S(X)$ that

$$\|(\hat{F} - U)\,x_1 - (\hat{F} - U)\,x_2\| \le 4k\,\|F_{|S(X)} - U_{|S(X)}\|\,\|x_1 - x_2\|$$

whence

$$\|\hat{F}_{|S(X)} - U_{|S(X)}\|_{\ell ip} \le 4k \, \|F_{|S(X)} - U_{|S(X)}\|_{\infty}.$$

Finally, note that the positive homogeniety of F implies that

$$\|\hat{F}\|_{\infty} \le 2 \, \|F\|_{\ell ip} \quad \text{and} \quad \|\hat{F}_{|S(X)} - U_{|S(X)}\|_{\infty} \le 2 \, \|F_{|S(X)} - U_{|S(X)}\|_{\infty}.$$

It now follows from Lemma 2 that the positively homogeneous extension $\tilde{F}$ of $\hat{F}$ satisfies the conclusions of Lemma 5. □

THEOREM 3. There is a constant $\tau > 0$ so that for all $n = 2, 3, 4, \ldots$ and all $1 \le p < 2$,

$$L(p,n) \ge \tau \left(\frac{\text{Log } n}{\text{Log Log } n}\right)^{1/p - 1/2}.$$

REMARK. Since $L(\infty,n) \ge L(1,n)$, we get the lower estimate for $L(\infty,n)$ mentioned in the introduction.

PROOF. Given p and n, for a certain value of $N = N(n)$ to be specified later choose a subspace E of L_p with $d(E, \ell_2^N) \le 2$ and E only $\delta N^{1/p - 1/2}$-complemented in L_p (Lemma 4). For a value $\varepsilon = \varepsilon(n) > 0$ to be specified later, let A be a minimal ε-net of $S(E)$, so, by Lemma 3,

$$\text{card } A \le (1 + 4/\varepsilon)^N.$$

One relation among n, N, ε we need is

$$(1.4) \qquad (1 + 4/\varepsilon)^N + 1 \le n.$$

Let $f : A \cup \{0\} \to E$ be the identify map. Since $d(E, \ell_2^N) \le 2$, we can by Lemma 2 get a positively homogeneous extension $\tilde{f} : L_p \to E$ of f so that

$$\|\tilde{f}\|_{\ell ip} \le 6 \, L(p,n).$$

Since $\tilde{f}(a) = f(a) = a$ for $a \in A$ and A is an ε-net for $S(E)$, we get that for $x \in S(E)$,

$$\|\tilde{f}(x) - x\| \le (6 \, L(p,n) + 1) \, \varepsilon.$$

Therefore, from Lemma 5 we get a Lipschitz mapping $\hat{f} : L_p \to E$ which satisfies

$$\|\hat{f}\|_{\ell ip} \le 24 \, L(p,n)$$

$$(1.5) \qquad \|\hat{f}_{|E} - I_E\| \le (8N + 2)(6 \, L(p,n) + 1)\varepsilon.$$

Note that if

$$(8N + 2)(6\,L(p,n) + 1)\varepsilon \le 1/2, \tag{1.6}$$

(1.5) implies that there is a linear projection from L_p onto E with norm at most $48\,L(p,n)$, so we can conclude that

$$L(p,n) > \delta/48\; N^{1/p - 1/2}.$$

Finally, we just need to observe that (1.4) and (1.6) are satisfied (at least for sufficiently large n) if we set

$$\varepsilon = \operatorname{Log}^{-2} n, \quad N = \frac{\operatorname{Log} n}{2 \operatorname{Log} \operatorname{Log} n}. \qquad \square$$

2. OPEN PROBLEMS.

Besides the obvious question left open by the preceding discussion (i.e. whether the estimate for $L(\infty,n)$ given in Theorem 1 is indeed the best possible), there are several other problems which arise naturally in the present context. We mention here only some of them.

PROBLEM 1. *Is it true that for* $1 < p < 2$, *every subset* X *of* $L_p(0,1)$, *and every Lipschitz map* f *from* X *into* ℓ_2^k *there is an extension* $\tilde{f}$ *of* f *from* $L_p(0,1)$ *into* ℓ_2^k *with*

$$\|\tilde{f}\|_{\ell ip} \le C(p)\, \|f\|_{\ell ip}\, k^{1/p - 1/2} \tag{2.1}$$

where $C(p)$ *depends only on* p?

A positive answer to problem 1 combined with Lemma 1 above will of course provide an alternative proof to the result of Marcus and Pisier [10] mentioned in the introduction. The linear version of problem 1 (where X is a subspace and f a linear operator) is known to be true (cf. [7] and [3]).

PROBLEM 2. *What happens in the Marcus-Pisier theorem if* $2 < p < \infty$? *Is the Lipschitz analogue of Maurey's extension theorem* [11] (*cf. also* [3]) *true? In other words, is it true that for* $2 < p < \infty$ *there is a* $c(p)$ *such that for every Lipschitz map* f *from a subset* X *of* $L_p(0,1)$ *into* ℓ_2 *there is a Lipschitz extension* $\tilde{f}$ *from* $L_p(0,1)$ *into* ℓ_2 *with*

$$\|\tilde{f}\|_{\ell ip} \le c(p)\|f\|_{\ell ip}?$$

PROBLEM 3. *What are the analogues of Lemma 1 in the setting of Banach spaces different from Hilbert spaces?* The most interesting special case seems to be concerning the spaces ℓ_∞^n. It is well known that every finite metric space $X = \{x_i\}_{i=1}^n$ embeds isometrically into ℓ_∞^n (the point x_i is mapped to the n-tuple $\{d(x_1, x_i), d(x_2, x_i), \ldots, d(x_n, x_i)\}$ in ℓ_∞^n). Hence in view of Lemma 1 it is quite natural to ask the following. *Does there exist for all* $\varepsilon > 0$ *(or alternatively for some* $\varepsilon > 0$) *a constant* $K(\varepsilon)$ *so that for every metric space* X *with cardinality* n *there is a Banach space* Y *with* $\dim Y \le K(\varepsilon)\log n$ *and a map* f *from* X *into* Y *so that* $\|f\|_{\ell ip}\,\|f^{-1}\|_{\ell ip} \le 1 + \varepsilon$?

A weaker version of Problem 3 is

PROBLEM 4. *It is true that for every metric space* X *with cardinality* n *there is a subset* $\tilde{X}$ *in* ℓ_2 *and a Lipschitz map* F *from* X *onto* $\tilde{X}$ *so that*

$$\|F\|_{\ell ip}\,\|F^{-1}\|_{\ell ip} \le K\sqrt{\log n} \tag{2.2}$$

for some absolute constant K?

Since for every Banach space Y with $\dim Y = k$ we have $d(Y, \ell_2^k) \le \sqrt{k}$ (cf. [6]) it is clear that a positive answer to problem 3 implies a positive answer to problem 4. V. Milman pointed out to us that it follows easily from an inequality of Enflo (cf. [1]) that (2.2), if true, gives the best possible estimate. (In the notation of [1], observe that the "m-cube"

$$x_\theta = (\theta_1, \theta_2, \ldots, \theta_m) \quad (\theta \in \{-1, 1\}^m)$$

in ℓ_1^m has all "diagonals" of length $2m$ and all "edges" of length 2, so that if F is any Lipschitz mapping from these 2^m points in ℓ_1^m into a Hilbert space, the corollary in [1] implies that

$$\|F\|_{\ell ip}\,\|F^{-1}\|_{\ell ip} \ge m^{1/2}.)$$

3. APPENDIX.

After this note was written, Yoav Benyamini discovered that Theorem 2 remains valid if ℓ_2 is replaced with any Banach space. He kindly allowed us to reproduce here his proof. The main lemma Benyamini uses is:

LEMMA 6. *Let* Γ *be an indexing set and let* $\{e_\gamma\}_{\gamma\in\Gamma}$ *be the unit vector basis for* $c_o(\Gamma)$. *Set*

$$A = \{\alpha e_\gamma : 0 \le \alpha \le 1;\ \gamma \in \Gamma\}$$

$$B = \overline{\text{conv}}\ A\ (= \text{positive part of } B_{\ell_1}(\Gamma)).$$

Then

(i) there is a retraction G from $\ell_\infty(\Gamma)$ onto B which satisfies $\|G\|_{\ell ip} \le 2$

(ii) there is a mapping H from $\ell_\infty(\Gamma)$ into A which satisfies $\|H\|_{\ell ip} \le 4$ and $He_\gamma = e_\gamma$ for all $\gamma \in \Gamma$.

PROOF. Since the mapping $x \to x^+$ is a contractive retraction from $\ell_\infty(\Gamma)$ onto its positive cone, $\ell_\infty(\Gamma)^+$; to prove (i) it is enough to define G only on $\ell_\infty(\Gamma)^+$.

For $y \in \ell_\infty(\Gamma)^+$, let

$$g(y) = \inf\{t : \|(y - te)^+\|_1 \le 1\}$$

where $e \in \ell_\infty(\Gamma)$ is the function identically equal to one and $\|\cdot\|_1$ is the usual norm in $\ell_1(\Gamma)$. Clearly the inf is actually a minimum and $0 \le g(y) \le \|y\|_\infty$. Note that

$$|g(y) - g(z)| \le \|y-z\|_\infty.$$

Indeed, assume that $g(y) \ge g(z)$. Then

$$y - [g(z) + \|y-z\|_\infty e] \le y - g(z)e + z - y \le z - g(z)e$$

and hence

$$\|(y-[g(z) + \|y-z\|_\infty]e)^+\|_1 \le 1;$$

that is

$$g(y) \le g(z) + \|y-z\|_\infty.$$

Now set for $y \in \ell_\infty(\Gamma)^+$

$$G(y) = (y - g(y)e)^+.$$

To prove (ii), it is enough, in view of (i), to define H on B with $\|H_{|B}\|_{\ell ip} \le 2$. For $y \in B$, $y = \{y(\gamma)\}_{\gamma \in \Gamma}$, defined Hy by

$$Hy(\gamma) = (2y(\gamma) - 1)^+.$$

For $y \in B$, there is at most one $\gamma \in \Gamma$ for which $y(\gamma) > \frac{1}{2}$, hence $HB \subset A$. Evidently $He_\gamma = e_\gamma$ for $\gamma \in \Gamma$ and $\|H_{|B}\|_{\ell ip} \le 2$.

THEOREM 2 (Y. Benyamini). Suppose that X is a metric space, Y is a subset of X with $d(x,y) \ge \varepsilon > 0$ for all $x \ne y \in Y$, Z is a Banach space, and $f : Y \to Z$ is Lipschitz. Then there is an extension $\tilde{f} : X \to Z$ of f so that

$$\|\tilde{f}\|_{\ell ip} \le (4D/\varepsilon)\|f\|_{\ell ip}$$

where D is the diameter of Y.

PROOF. Represent

$$Y = \{0\} \cup \{y_\gamma : \gamma \in \Gamma\}$$

and assume, by translating f, that $f(0) = 0$. We can factor f through the subset $C = \{0\} \cup \{e_\gamma : \gamma \in \Gamma\}$ of $\ell_\infty(\Gamma)$ by defining $g : Y \to C$, $h : C \to Z$ by

$$g(y_\gamma) = e_\gamma,\ g(0) = 0$$

$$h(e_\gamma) = f(y_\gamma),\ h(0) = 0.$$

Evidently,

$$\|g\|_{\ell ip} \le 1/\varepsilon,\ \|h\|_{\ell ip} \le D\|f\|_{\ell ip}.$$

By the non-linear Hahn-Banach theorem, g has an extension to a function $\tilde{g} : X \to \ell_\infty(\Gamma)$ with $\|\tilde{g}\|_{\ell ip} = \|g\|_{\ell ip}$, so to complete the proof, it suffices to extend h to a function $\tilde{h} : B \to Z$ with $\|\tilde{h}\|_{\ell ip} = \|h\|_{\ell ip}$ and apply Lemma 6(ii).

Define for $0 \le t \le 1$ and $\gamma \in \Gamma$

$$\tilde{h}(te_\gamma) = th(e_\gamma).$$

If $1 \ge t \ge s \ge 0$ and $\gamma \ne \Delta \in \Gamma$ then

$$\|\tilde{h}(te_\gamma) - \tilde{h}(se_\Delta)\| \le (t-s)\|h(e_\gamma)\| + s\,\|h(e_\Delta) - h(e_\gamma)\|$$

$$\le (t-s)\|h\|_{\ell ip} + s\|h\|_{\ell ip} = \|h\|_{\ell ip}\|te_\gamma - se_\Delta\|_\infty,$$

so $\|\tilde{h}\|_{\ell ip} = \|h\|_{\ell ip}$. □

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