PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 11, Pages 3469–3476
S 0002-9939(03)06917-X
Article electronically published on February 14, 2003

ON A SUBSPACE PERTURBATION PROBLEM

VADIM KOSTRYKIN, KONSTANTIN A. MAKAROV, AND ALEXANDER K. MOTOVILOV

(Communicated by Joseph A. Ball)

Abstract. We discuss the problem of perturbation of spectral subspaces for linear self-adjoint operators on a separable Hilbert space. Let A and V be bounded self-adjoint operators. Assume that the spectrum of A consists of two disjoint parts σ and Σ such that $d = \operatorname{dist}(\sigma, \Sigma) > 0$. We show that the norm of the difference of the spectral projections

$$\mathsf{E}_A(\sigma) \quad \text{and} \quad \mathsf{E}_{A+V}\big(\{\lambda \mid \operatorname{dist}(\lambda, \sigma) < d/2\}\big)$$

for A and $A+V$ is less than one whenever either (i) $\|V\| < \frac{2}{2+\pi} d$ or (ii) $\|V\| < \frac{1}{2} d$ and certain assumptions on the mutual disposition of the sets σ and Σ are satisfied.

1. Introduction

It is well known (see, e.g., [10]) that if A and V are bounded self-adjoint operators on a separable Hilbert space $\mathfrak{H}$, then (the perturbation) V does not close gaps of length greater than $2\|V\|$ in the spectrum of A. More precisely, if (a, b) is a finite interval and $(a, b) \subset \varrho(A)$, the resolvent set of A, then

$$(a + \|V\|, b - \|V\|) \subset \varrho(A + sV) \quad \text{for all } s \in [-1, 1]$$

whenever $2\|V\| < b - a$. Hence, under the assumption that A has an isolated part σ of the spectrum separated from its remainder by gaps of length greater than or equal to $d > 0$, the spectrum of the operators $A + sV$, $s \in [-1, 1]$, will also have separated components, provided that the condition

$$\|V\| < \frac{d}{2} \tag{1.1}$$

holds.

Our main concern is to study the variation of the corresponding spectral subspace associated with the isolated part σ of the spectrum of A under perturbations satisfying (1.1).

For notational setup we assume the following hypothesis.

Hypothesis 1. *Assume that A and V are bounded self-adjoint operators on a separable Hilbert space $\mathfrak{H}$. Suppose that the spectrum of A has a part σ separated from the remainder of the spectrum Σ in the sense that*

$$\operatorname{spec}(A) = \sigma \cup \Sigma \quad \textit{and} \quad \operatorname{dist}(\sigma, \Sigma) = d > 0.$$

Received by the editors March 29, 2002 and, in revised form, May 30, 2002.
2000 *Mathematics Subject Classification.* Primary 47A55, 47A15; Secondary 47B15.
Key words and phrases. Perturbation theory, spectral subspaces.

©2003 by the authors

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Introduce the orthogonal projections $P = \mathsf{E}_A(\sigma)$ *and* $Q = \mathsf{E}_{A+V}(U_{d/2}(\sigma))$*, where* $U_\varepsilon(\sigma)$*,* $\varepsilon > 0$*, is the open* ε*-neighborhood of the set* σ*. Here* $\mathsf{E}_A(\Delta)$ *and* $\mathsf{E}_{A+V}(\Delta)$ *denote the spectral projections for operators* A *and* $A+V$*, respectively, corresponding to a Borel set* $\Delta \subset \mathbb{R}$*.*

In this note we address the following question: *Assuming Hypothesis 1, does condition* (1.1) *imply*

$$\|P - Q\| < 1?$$

We give a partially affirmative answer to this question. The precise statement reads as follows.

Theorem 1. *Assume Hypothesis 1 and suppose that either*

(i) $\|V\| < \frac{2}{2+\pi}d$

or

(ii) $\|V\| < \frac{1}{2}d$

and

$$\operatorname{conv.hull}(\sigma) \cap \Sigma = \varnothing \quad \textit{or} \quad \operatorname{conv.hull}(\Sigma) \cap \sigma = \varnothing. \tag{1.2}$$

Then

$$\|P - Q\| < 1.$$

Our strategy of the proof of Theorem 1 does not allow us to relax the condition

$$\|V\| < \frac{2}{2+\pi}d \tag{1.3}$$

and just assume the natural condition (1.1) with no additional hypotheses. It is an *open problem* whether Hypothesis 1 alone and the bounds

$$\frac{2}{2+\pi} \le \frac{\|V\|}{d} < \frac{1}{2} \tag{1.4}$$

on the perturbation V imply $\|P - Q\| < 1$.

For compact perturbations V satisfying inequality (1.1) we can however state that the pair (P, Q) of the orthogonal projections is a Fredholm pair with zero index. Recall that the pair (P, Q) of orthogonal projections is called Fredholm if the operator QP viewed as a map from $\operatorname{Ran} P$ to $\operatorname{Ran} Q$ is a Fredholm operator [3]. The index of this operator is called the index of the pair (P, Q).

Theorem 2. *Assume Hypothesis 1 and suppose that* V *is a compact operator satisfying* (1.1)*. Then the pair* (P, Q) *is Fredholm with zero index. In particular, the subspaces* $\operatorname{Ker}(PQ^\perp - I)$ *and* $\operatorname{Ker}(P^\perp Q - I)$ *are finite-dimensional and*

$$\dim \operatorname{Ker}(PQ^\perp - I) = \dim \operatorname{Ker}(P^\perp Q - I).$$

In the "overcritical" case $\|V\| > d/2$, the perturbed operator $A + V$ may not have separated parts of the spectrum at all. In this case we give an example where the spectral measure of the perturbed operator $A+V$ is "concentrated" on the unit sphere in the space of bounded operators $\mathcal{B}(\mathfrak{H})$ centered at the point $P = \mathsf{E}_A(\sigma)$, with the norm of the perturbation being arbitrarily close to $d/2$. That is, given $d > 0$, for any $\varepsilon > 0$ one can find a self-adjoint operator A satisfying Hypothesis 1 and a self-adjoint perturbation V with $\|V\| = d/2 + \varepsilon$ such that

$$\|\mathsf{E}_A(\sigma) - \mathsf{E}_{A+V}(\Delta)\| = 1$$

for any Borel set $\Delta \subset \mathbb{R}$.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

2. Proof of Theorem 1

Our proof of Theorem 1 is based on the following sharp result (see [9] and references cited therein) taken from geometric perturbation theory initiated by C. Davis [6] and developed further in [4], [5], [7], [8], [10].

Proposition 2.1. *Let A and B be bounded self-adjoint operators and δ and Δ two Borel sets on the real axis $\mathbb{R}$. Then*

$$\operatorname{dist}(\delta,\Delta)\|\mathsf{E}_A(\delta)\mathsf{E}_B(\Delta)\| \le \frac{\pi}{2}\|A-B\|.$$

If, in addition, the convex hull of the set δ does not intersect the set Δ, or the convex hull of the set Δ does not intersect the set δ, then one has the stronger result

$$\operatorname{dist}(\delta,\Delta)\|\mathsf{E}_A(\delta)\mathsf{E}_B(\Delta)\| \le \|A-B\|.$$

We split the proof of Theorem 1 into the following two lemmas.

Lemma 2.2. *Assume Hypothesis 1. Assume, in addition, that* (1.3) *holds. Then*

$$\|P-Q\|<1.$$

Proof. Clearly $\operatorname{spec}(A+V) \subset \overline{U_{\|V\|}(\sigma\cup\Sigma)}$, where the bar denotes the (usual) closure in $\mathbb{R}$, and then

$$Q^\perp = \mathsf{E}_{A+V}\big(\overline{U_{\|V\|}(\Sigma)}\big).$$

By the first claim of Proposition 2.1,

$$\|PQ^\perp\| \le \frac{\pi}{2}\frac{\|V\|}{\operatorname{dist}(\sigma, U_{\|V\|}(\Sigma))}. \tag{2.1}$$

The distance between the set σ and the $\|V\|$-neighborhood of the set Σ can be estimated from below as follows:

$$\operatorname{dist}(\sigma, U_{\|V\|}(\Sigma)) \ge d-\|V\|>0.$$

Then (2.1) implies the inequality

$$\|PQ^\perp\| \le \frac{\pi}{2}\frac{\|V\|}{d-\|V\|}.$$

Hence, from inequality (1.3) it follows that

$$\|PQ^\perp\| \le \frac{\pi}{2}\frac{\|V\|}{d-\|V\|}<1. \tag{2.2}$$

Interchanging the roles of σ and Σ one obtains the analogous inequality

$$\|P^\perp Q\|<1. \tag{2.3}$$

Since

$$\|P-Q\| = \max\{\|PQ^\perp\|, \|P^\perp Q\|\} \tag{2.4}$$

(see, e.g., [2, Ch. III, Section 39]), inequalities (2.2) and (2.3) prove the assertion. □

Under additional assumptions on mutual disposition of the parts σ and Σ of the spectrum of A one can relax the condition (1.3) on the norm of perturbation and replace it by the natural condition (1.1).

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Lemma 2.3. *Assume Hypothesis 1 and suppose that condition* (1.1) *holds.*

(i) *If either* $\sigma \cap \mathrm{conv.hull}(\Sigma) = \varnothing$ *or* $\mathrm{conv.hull}(\sigma) \cap \Sigma = \varnothing$, *then*

$$\|P - Q\| < 1. \tag{2.5}$$

(ii) *If in addition the sets* σ *and* Σ *are subordinated, that is,*

$$\mathrm{conv.hull}(\sigma) \cap \mathrm{conv.hull}(\Sigma) = \varnothing,$$

then the following sharp estimate holds:

$$\|P - Q\| < \frac{\sqrt{2}}{2}. \tag{2.6}$$

Proof. (i) The proof follows that of Lemma 2.2. Applying the second assertion of Proposition 2.1 instead of inequality (2.1), one derives the estimates

$$\|PQ^{\perp}\| \le \frac{\|V\|}{\mathrm{dist}(\sigma, U_{\|V\|}(\Sigma))} \le \frac{\|V\|}{d - \|V\|} < 1, \tag{2.7}$$

under hypothesis (1.4), and then the inequality $\|P^{\perp}Q\| < 1$, proving assertion (2.5) using (2.4).

(ii) First assume that V is off-diagonal, that is,

$$\mathsf{E}_A(\sigma) V \mathsf{E}_A(\sigma) = \mathsf{E}_A(\sigma)^{\perp} V \mathsf{E}_A(\sigma)^{\perp} = 0.$$

Then the inequality $\|P - Q\| < \frac{\sqrt{2}}{2}$ follows from the $\tan 2\Theta$-Theorem proven first by C. Davis (see, e.g., [8])

$$\|P - Q\| \le \sin\left(\frac{1}{2}\arctan\frac{2\|V\|}{d}\right) < \frac{\sqrt{2}}{2}.$$

A related result can be found in [1].

The general case can be reduced to the off-diagonal one by the following trick. Assume that V is not necessarily off-diagonal. Decomposing the perturbation V into the diagonal V_{diag} and off-diagonal V_{off} parts with respect to the orthogonal decomposition $\mathfrak{H} = \mathrm{Ran}\,\mathsf{E}_A(\sigma) \oplus \mathrm{Ran}\,\mathsf{E}_{\mathbf{A}}(\sigma)^{\perp}$ associated with the range of the projection $\mathsf{E}_A(\sigma)$

$$V = V_{\mathrm{diag}} + V_{\mathrm{off}},$$

one concludes that

$$\mathsf{E}_{A+V_{\mathrm{diag}}}(U_{d/2}(\sigma)) = \mathsf{E}_A(\sigma).$$

Moreover, the distance between the spectrum of the part of $A+V_{\mathrm{diag}}$ associated with the invariant subspace $\mathrm{Ran}\,\mathsf{E}_{A+V_{\mathrm{diag}}}(U_{d/2}(\sigma))$ and the remainder of the spectrum of $A + V_{\mathrm{diag}}$ does not exceed $d - 2\|V_{\mathrm{diag}}\| > 0$. Using the $\tan 2\Theta$-Theorem then yields

$$\begin{aligned}\|P - Q\| &\le \sin\left(\frac{1}{2}\arctan\frac{2\|V_{\mathrm{off}}\|}{d - 2\|V_{\mathrm{diag}}\|}\right)\\ &\le \sin\left(\frac{1}{2}\arctan\frac{2\|V\|}{d - 2\|V\|}\right) < \frac{\sqrt{2}}{2},\end{aligned}$$

completing the proof. □

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

The sharpness of estimate (2.6) is shown by the following example.

Example 2.4. Let $\mathfrak{H} = \mathbb{C}^2$. For an arbitrary $\varepsilon \in (0, 3/4)$ consider the 2×2 matrices

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1/2-\varepsilon & \sqrt{\varepsilon}/2 \\ \sqrt{\varepsilon}/2 & -1/2+\varepsilon \end{pmatrix}.$$

Let $\sigma = \{0\}$ and $\Sigma = \{1\}$. Obviously, $\operatorname{dist}(\sigma, \Sigma) = 1$. Since

$$\|V\| = \frac{1}{2}\sqrt{1-3\varepsilon+4\varepsilon^2} < \frac{1}{2},$$

the perturbation V satisfies the hypotheses of Lemma 2.3. Simple calculations yield

$$\begin{aligned} Q &= \mathsf{E}_{A+V}\big(U_{1/2}(\sigma)\big) = \mathsf{E}_{A+V}\big((-1/2, 1/2)\big) \\ &= \frac{1}{1+(2\sqrt{\varepsilon}+\sqrt{1+4\varepsilon})^2}\begin{pmatrix} (2\sqrt{\varepsilon}+\sqrt{1+4\varepsilon})^2 & -2\sqrt{\varepsilon}-\sqrt{1+4\varepsilon} \\ -2\sqrt{\varepsilon}-\sqrt{1+4\varepsilon} & 1 \end{pmatrix}, \end{aligned}$$

and hence,

$$\|P-Q\| = \big[1+(2\sqrt{\varepsilon}+\sqrt{1+4\varepsilon})^2\big]^{-1/2} < \frac{\sqrt{2}}{2}.$$

Taking ε sufficiently small, the norm $\|P-Q\|$ can be made arbitrarily close to $\sqrt{2}/2$.

3. Proof of Theorem 2

Lemma 3.1. *Assume Hypothesis 1 and suppose, in addition, that V is a compact operator satisfying condition* (1.1). *Then there is a unitary W such that $Q = WPW^*$ and $W - I$ is compact.*

Proof. Fix $\varepsilon > 0$ such that $(1+\varepsilon)\|V\| < d/2$ and introduce the family of spectral projections

$$\mathcal{P}(s) = \mathsf{E}_{A+sV}(U_{d/2}(\sigma)), \quad s \in (-\varepsilon, 1+\varepsilon).$$

Clearly, $\mathcal{P}(0) = P$ and $\mathcal{P}(1) = Q$. From the analytical perturbation theory (see [10]) one concludes that the operator-valued function $\mathcal{P}(s)$ is real-analytic on $(-\varepsilon, 1+\varepsilon)$. Moreover (see [10, Section II.4.2]),

$$\mathcal{P}(s) = X(s)\mathcal{P}(0)X(s)^*, \quad s \in [0,1],$$

where $X(s)$ is the unique unitary solution to the initial value problem

$$\begin{aligned} X'(s) &= H(s)X(s), \quad s \in [0,1], \\ X(0) &= I, \end{aligned}$$

with $H(s) = \mathcal{P}'(s)\mathcal{P}(s) - \mathcal{P}(s)\mathcal{P}'(s)$.

Let Γ be a Jordan counterclockwise oriented contour encircling $U_{d/2}(\sigma)$ in a way such that no point of $U_{d/2}(\Sigma)$ lies within Γ. Then

$$\mathcal{P}(s) = -\frac{1}{2\pi i}\int_\Gamma (A+sV-z)^{-1}dz, \quad s \in [0,1],$$

and hence,

$$\mathcal{P}'(s) = \frac{1}{2\pi i}\int_\Gamma (A+sV-z)^{-1}V(A+sV-z)^{-1}dz, \quad s \in [0,1].$$

By the hypothesis V is compact, and hence, $\mathcal{P}'(s)$, $s \in [0,1]$, is also compact, which implies that $H(s)$ is a compact operator for $s \in [0,1]$.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Applying the successive approximation method

$$X_n(s) = I + \int_0^s H(t)X_{n-1}(t)dt, \quad X_0(s) = I,$$

yields that $X_n(s)$ converges to $X(s)$, $s \in [0,1]$, in the norm topology and $X_n(s) - I$ is compact for all $n \in \mathbb{N}$. Thus, $X(s) - I$ is a compact operator for all $s \in [0,1]$. Taking $W = X(1)$ yields $Q = WPW^*$, completing the proof. □

Lemma 3.1 implies that the operator PWP viewed as a map from $\operatorname{Ran} P$ to $\operatorname{Ran} P$ is Fredholm with zero index. By Theorem 5.2 of [3] it follows that the pair (P,Q) is Fredholm and $\operatorname{index}(P,Q) = \operatorname{index}(PW|_{\operatorname{Ran} P}) = 0$, proving Theorem 2.

4. Overcritical perturbations

If the perturbation V closes a gap between the separated parts σ and Σ of the spectrum of the unperturbed operator A, then, necessarily, we are dealing with the case $\|V\| \geq d/2$. In this case one encounters a new phenomenon: It may happen that any invariant subspace of the operator $A+V$ contains a nontrivial element orthogonal to $\operatorname{Ran} P = \operatorname{Ran} \mathsf{E}_A(\sigma)$.

To illustrate this phenomenon we need the following abstract result.

Lemma 4.1. *Let A and V be bounded self-adjoint operators and $\sigma \neq \varnothing$ be a finite set consisting of isolated eigenvalues of A of finite multiplicity. Assume that the spectrum of the operator $A+V$ has no pure point component. Then for the orthogonal projection Q onto an arbitrary invariant subspace of the operator $A+V$, the subspace $\operatorname{Ker}(P^\perp Q - I)$, where $P = \mathsf{E}_A(\sigma)$, is infinite-dimensional. In particular,*

$$\|P - Q\| = 1. \tag{4.1}$$

Proof. Since $A+V$ has no eigenvalues, $\operatorname{Ran} Q$ is an infinite-dimensional subspace. By hypothesis, $\operatorname{Ran} P$ is a finite-dimensional subspace. Thus, there exists an orthonormal system $\{f_n\}_{n\in\mathbb{N}}$ in $\operatorname{Ran} Q$ such that f_n is orthogonal to $\operatorname{Ran} P$ for any $n \in \mathbb{N}$ and hence $P^\perp Q f_n = f_n$, $n \in \mathbb{N}$, proving $\dim\big(\operatorname{Ker}(P^\perp Q - I)\big) = \infty$. Now equality (4.1) follows from representation (2.4). □

The next lemma shows that an isolated *eigenvalue* of the unperturbed operator A separated from the remainder of the spectrum of A by a gap of length 1 may "dissolve" in the essential spectrum of the perturbed operator $A+V$ turning into a "*resonance*", with the norm of the perturbation being larger but arbitrarily close to 1/2.

Lemma 4.2. *Let $\varepsilon > 0$. Let A and V be 2×2 operator matrices in $\mathfrak{H} = L^2(0,1)\oplus\mathbb{C}$,*

$$A = \begin{pmatrix} M & 0 \\ 0 & -I_{\mathbb{C}} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -\left(\frac12 + \varepsilon\right) I_{L^2(0,1)} & \sqrt{\varepsilon} v \\ \sqrt{\varepsilon} v^* & (\frac12 + \varepsilon) I_{\mathbb{C}} \end{pmatrix}$$

with respect to the decomposition $\mathfrak{H} = L^2(0,1)\oplus\mathbb{C}$. Here M denotes the multiplication operator in $L^2(0,1)$,

$$(Mf)(\mu) = \mu f(\mu), \quad 0 < \mu < 1, \quad f \in L^2(0,1),$$

and $v \in \mathcal{B}\big(\mathbb{C}, L^2(0,1)\big)$

$$(vg)(\mu) = w(\mu) g, \quad \mu \in (0,1), \quad g \in \mathbb{C}, \qquad w(\mu) = \sqrt{\mu(1-\mu)}.$$

If $\varepsilon < 2/5$, then the operator $A+V$ has no eigenvalues.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

Proof. Assume to the contrary that $\lambda \in \mathbb{R}$ is an eigenvalue of the perturbed operator $A+V$, that is,

$$(\mu - 1/2 - \varepsilon)f(\mu) + \sqrt{\varepsilon}w(\mu)g = \lambda f(\mu) \quad \text{a.e. } \mu \in (0,1)$$

and

$$\sqrt{\varepsilon}\int_0^1 d\mu f(\mu)w(\mu) + (-1/2+\varepsilon)g = \lambda g$$

for some $f \in L^2(0,1)$ and $g \in \mathbb{C}$. In particular,

$$f(\mu) = \sqrt{\varepsilon}\frac{w(\mu)}{\lambda - (\mu - \frac{1}{2} - \varepsilon)}g,$$

and hence $f \notin L^2(0,1)$ whenever $\lambda \in [-1/2-\varepsilon, 1/2-\varepsilon]$ (unless $f=0$ and $g=0$). Thus, the interval $[-1/2-\varepsilon, 1/2-\varepsilon]$ does not intersect the point spectrum of $A+V$. Moreover, $\lambda \in (-\infty, -1/2-\varepsilon) \cup (1/2-\varepsilon, \infty)$ is an eigenvalue of $A+V$ if and only if

$$\lambda + \frac{1}{2} - \varepsilon + \varepsilon\int_0^1 d\mu \frac{\mu(1-\mu)}{\mu - \frac{1}{2} - \varepsilon - \lambda} = 0. \tag{4.2}$$

Elementary analysis of the graph of the function on the left-hand side of (4.2) then yields that under the condition $0 < \varepsilon < 2/5$ there is no solution of equation (4.2) in $(-\infty, -1/2-\varepsilon) \cup (1/2-\varepsilon, \infty)$. Thus, the point spectrum of $A+V$ is empty. □

Remark 4.3. We note that $\operatorname{spec}(A) = \{-1\} \cup [0,1]$ and hence $\operatorname{spec}(A)$ has two components separated by a gap of length one, and the norm of the perturbation V may be arbitrarily close to $1/2$ (from above):

$$\|V\| = \sqrt{\left(\frac{1}{2}+\varepsilon\right)^2 + \frac{1}{6}\varepsilon} = \frac{1}{2} + \frac{7}{6}\varepsilon + \mathcal{O}(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0. \tag{4.3}$$

Using scaling arguments, Remark 4.3 combined with the result of Lemma 4.1 shows that given $d > 0$, for any $\varepsilon > 0$ one can find a self-adjoint operator A satisfying Hypothesis 1 and a self-adjoint perturbation V with $\|V\| = d/2 + \varepsilon$ such that

$$\|\mathsf{E}_A(\sigma) - Q\| = 1$$

for the orthogonal projection Q onto an arbitrary invariant subspace of the operator $A+V$.

Acknowledgments

V. Kostrykin is grateful to V. Enss for useful discussions. K. A. Makarov is grateful to F. Gesztesy for critical remarks. A. K. Motovilov acknowledges the great hospitality and financial support by the Department of Mathematics, University of Missouri–Columbia, Missouri. He was also supported in part by the Russian Foundation for Basic Research within the RFBR Project 01-01-00958.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

References

[1] V. Adamyan and H. Langer, *Spectral properties of a class of rational operator valued functions*, J. Operator Theory **33** (1995), 259 – 277. MR **96i:**47023

[2] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Dover Publications, New York, 1993. MR **94i:**47001

[3] J. Avron, R. Seiler, and B. Simon, *The index of a pair of projections*, J. Funct. Anal. **120** (1994), 220 – 237. MR **95b:**47012

[4] R. Bhatia, C. Davis, and A. McIntosh, *Perturbation of spectral subspaces and solution of linear operator equations*, Linear Algebra Appl. **52/53** (1983), 45 – 67. MR **85a:**47020

[5] R. Bhatia, C. Davis, and P. Koosis, *An extremal problem in Fourier analysis with applications to operator theory*, J. Funct. Anal. **82** (1989), 138 – 150. MR **91a:**42006

[6] C. Davis, *Separation of two linear subspaces*, Acta Scient. Math. (Szeged) **19** (1958), 172 – 187. MR **20:**5425

[7] C. Davis, *The rotation of eigenvectors by a perturbation. I and II*, J. Math. Anal. Appl. **6** (1963), 159 – 173; **11** (1965), 20 – 27. MR **26:**6799; MR **31:**5082

[8] C. Davis and W. M. Kahan, *The rotation of eigenvectors by a perturbation. III*, SIAM J. Numer. Anal. **7** (1970), 1 – 46. MR **41:**9044

[9] R. McEachin, *Closing the gap in a subspace perturbation bound*, Linear Algebra Appl. **180** (1993), 7 – 15. MR **94c:**47017

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer–Verlag, Berlin, 1966. MR **34:**3324

Fraunhofer-Institut für Lasertechnik, Steinbachstrasse 15, D-52074, Aachen, Germany

E-mail address: `kostrykin@ilt.fhg.de`
E-mail address: `kostrykin@t-online.de`

Department of Mathematics, University of Missouri, Columbia, Missouri 65211
E-mail address: `makarov@math.missouri.edu`

Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia
E-mail address: `motovilv@thsun1.jinr.ru`
Current address: Department of Mathematics, University of Missouri, Columbia, Missouri 65211

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use