rho_xtregar: a new command to improve the estimation of rho in AR(1) panels.

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23 septembre 2019

Version 1.0.6 - comments are welcome !

Résumé

This paper proposes new estimators of the autocorrelation parameter in fixed- and random-effects models with serial correlation of order one in the idiosyncratic perturbations. In balanced panels, it indeed shows that current estimators are biased and quantifies their bias. Then it proposes an estimator that is consistent and asymptotically unbiased in both balanced and unbalanced panels. It also proposes additional estimators which are asymptotically equivalent in long panels and very simple to compute. Monte-Carlo simulations eventually confirm that these new estimators are much more reliable than all estimates currently provided by **xtregar**. A Stata command, which computes these new estimators, is provided. The use of the command is illustrated by revisiting the example of Baltagi et Wu (1999).

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1 Introduction

This articles improves the estimation of the autocorrelation coefficient in linear unobserved effects panel data models [1] with AR(1) disturbances, that refer to processes of the form :

$$y_{it} = x'_{it}\beta + \nu_i + \varepsilon_{it} \tag{1}$$

$$\varepsilon_{it} = \rho\varepsilon_{it-1} + \eta_{it} \tag{2}$$

where $|\rho| < 1$ and the η_{it}'s are i.i.d. disturbances with mean 0 and variance σ^2_η.

Such estimations are common in the wage equation literature (with even higher order of correlation), and are not infrequent in the general economics literature. They are performed in Stata with the command *xtregar*, which has been used in influential and recent economics articles such as Dafny (2010) or Hau *et al.* (2013).

The estimation of the autocorrelation parameter ρ is key in such models : First, these models are often specifically chosen for their explicit modeling of serial correlations in error terms, something which is accounted for by ρ. Second, the other parameters of the model are usually estimated based on a Cochrane–Orcutt transformation using the given estimate of ρ.

In xtregar [2], ρ is estimated by default with the Durbin-Watson statistic d :

$$\hat{\rho} = \rho_d = 1 - \frac{d}{2} \tag{3}$$

Other estimates are proposed but it is claimed that *"dw is the default because it performs well in Monte Carlo simulations"*. This article challenges this claim in Section 2. Notably, in balanced panels, this standard estimator is shown to be biased towards zero in $O(\frac{1}{T})$, where T is the number of observations per individual.
In section 3, an alternative estimate of ρ, that we call ρ_{BFN} since it was suggested with much intuition by Bhargava *et al.* (1982), is shown to be consistent and asymptotically unbiased as N becomes large. We also generalize it to the unbalanced setting, where it keeps these desirable properties under reasonable hypotheses. This

1. That is in both fixed and random effects models.
2. In the whole article, xtregar refers to its version 1.6.6 of March 2018, as consulted on the 31^{st} of August 2019.

new estimator eventually performs much better than the existing estimates in Monte Carlo simulations both in balanced and unbalanced panels.
We further define two additional estimators that can yield advantages over this estimator in long panels. In balanced panels, just dividing ρ_d estimator by $1 - \frac{2}{T}$, with T the number of periods, is enough to get an estimator whose performances are almost indiscernible to the estimator initially suggested by Bhargava *et al.* (1982) (the bias is in $\frac{1}{T^2}$). In unbalanced panels, a similar approximation yields a less precise estimator whose bias tends however to zero when the minimal number of period per individual tends to infinity, something which does not seem guaranteed by current estimators.
Section 4 describes the *rho_xtregar* command, which implements these estimators. A brief example on how to apply it is exposed in Section 5. Section 6 concludes.

2 Overlooked consequences of Bhargava *et al.* (1982).

2.1 The balanced case

2.1.1 Definitions

In a perfectly balanced panel (Bhargava *et al.*, 1982), the Durbin-Watson statistics (used by the Stata command *xtregar* to estimate ρ) writes :

$$d_p = \frac{\sum_{i=1}^{N}\sum_{t=2}^{T}(\tilde{u}_{it} - \tilde{u}_{it-1})^2}{\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{u}_{it}^2} \tag{4}$$

where $\tilde{u}_{it}$ are the residuals of the within-estimation of model (1) - the residuals of the OLS regression of $y_{it} - \overline{y}_i$ on $x_{it} - \overline{x}_i$ with $\overline{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{it}$ and $\overline{x}_i = \frac{1}{T}\sum_{t=1}^{T} x_{it}$ -, N is the number of individuals and T the number of (equally spaced) periods.

Bhargava *et al.* (1982) have generalized the Durbin-Watson statistics to get a Uniformly Most Powerful test that ρ is significantly different from zero. However, they first note that one can use equation (3) with this formula to generate a consistent estimate of ρ as $T \to \infty$. This is what *xtregar* currently performs.

2.1.2 An estimate of the bias

In their MonteCarlo simulations (with finite $T = 10$), Bhargava *et al.* (1982) however point out that the corresponding estimates ρ_d can be significantly different from the true value of ρ (see their Table IV, page 541 ; or our own Table 1). This bias can be high in magnitude : in their simulations, it can amount to -0.26 for a value of $\rho = 0.9$.

Although they do not highlight it, the existence of such a bias theoretically derives from the below relation (see Bhargava *et al.* (1982)) [3] :

$$\mathbb{E}(\rho_d) = 1 - \frac{(1-\rho)(T-1)}{T - \frac{1}{T}\sum_{i,j=1}^{T} \rho^{|i-j|}} \tag{5}$$

The bias does not depend on the number of individuals N, as equation (5) does not depend on N [4]. It does however depend on the number of period T. We therefore provide an approximation of the bias when $T \to \infty$. One first notices (as demonstrated by Appendix B of Bhargava *et al.* (1982)) :

$$\sum_{j,k=1}^{T} \rho^{|j-k|} = \frac{1+\rho}{1-\rho}T - \frac{2\rho}{1-\rho}\frac{1-\rho^T}{1-\rho} \tag{6}$$

which leads to :

$$\mathbb{E}(\rho_d) = 1 - \frac{(1-\rho)\frac{(T-1)}{T}}{1 - \frac{1+\rho}{1-\rho}\frac{1}{T} + \frac{2\rho(1-\rho^T)}{(1-\rho)^2}\frac{1}{T^2}} \tag{7}$$

3. Indeed, if ρ_d were unbiased, then $\mathbb{E}(\rho_d) = \rho$ for any ρ. From equation (5), this would imply, for any ρ :

$$\rho = 1 - \frac{(1-\rho)(T-1)}{T - \frac{1}{T}\sum_{i,j=1}^{T} \rho^{|i-j|}}$$

which contradicts the fact this equation has a finite number of roots.

4. Note that relation (5) is however valid for N large enough.

We develop this equation at the second order in $\frac{1}{T}$ [5] :

$$\mathbb{E}(\rho_d) = \rho - \frac{2\rho}{T} + O(\frac{1}{T^2}) \tag{9}$$

Hence, in the balanced case, the bias is of the order of $\frac{1}{T}$ [6]. For instance, for $T = 10$, it is of the order of 0.1. Which is high in magnitude if, for instance, ρ equals 0.5. This bias tends to 0 as the time dimension gets larger. For T = 50, it is of the order of 0.02, which might be perceived (or not) as negligible.

2.2 A generalization to the unbalanced case

In the unbalanced case, an estimator of $d_p = 2(1 - \rho)$ is given by :

$$d_p = \frac{\sum_{i=1}^{N} \frac{1}{K_i+1} \sum_{j=2}^{n_i} (\tilde{u}_{it_{ij}} - \tilde{u}_{it_{ij-1}})^2 \mathbb{1}_{t_{ij}-t_{ij-1}=1}}{\sum_{i=1}^{N} \frac{1}{n_i} \sum_{j=1}^{n_i} \tilde{u}^2_{it_{ij}}} \tag{10}$$

where $t_{i1} < ... < t_{in_i}$ denote the dates at which individual i is observed and $K_i = \sum_{j=2}^{n_i} \mathbb{1}_{t_{ij}-t_{ij-1}=1}$ the number of observations separated by one period for the individual i. It is easy to see that, in the balanced case, one gets equation (4) back [7].

We explain in Annex A why d_p can be perceived as a natural estimator of ρ also in the unbalanced case. However, we show in Annex C that $\rho_d = 1 - \frac{d_p}{2}$ is asymptotically close to its expectancy as $N \to \infty$ [8], and in Annex B that the generalization of relation (5) writes :

5. More precisely :

$$\mathbb{E}(\rho_d) = \rho - \frac{2\rho}{T} + \frac{1}{T^2} \frac{\frac{2\rho^2}{(1-\rho)}(2\frac{1-\rho^T}{T(1-\rho)} - 1 - \rho^{T-1})}{1 - \frac{1+\rho}{1-\rho}\frac{1}{T} + \frac{2\rho(1-\rho^T)}{(1-\rho)^2}\frac{1}{T^2}} \tag{8}$$

6. And, as the coefficient in front of $\frac{1}{T}$ is negative, it is probably negative (if ρ positive) even for moderate values of T, as could be already observed in the Monte Carlo simulations of Baltagi et Wu (1999).

7. Note that such generalization (in the unbalanced case) of the Durbin-Watson statistics of Bhargava *et al.* (1982) differs from the d1-statistics of Baltagi et Wu (1999) (which is another generalization to the unbalanced case of the the Durbin-Watson statistics of Bhargava *et al.* (1982)). In this latter, the dummy variable in the numerator is straightly in the parentheses along with $\tilde{u}_{it_{ij-1}}$, and there is no mention of K_i.

8. Note that this extends Bhargava *et al.* (1982) work in the balanced case as they only noted that ρ_d was consistent when $T \to \infty$. True, when only $N \to \infty$, ρ_d is consistent towards something else as ρ, which may be why Bhargava *et al.* (1982) have not noticed it.

$$\mathbb{E}(\rho_d) = 1 - \frac{(1-\rho)\sum_{i=1}^{N}\frac{K_i}{1+K_i}}{N - \sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}} \tag{11}$$

Unlike what was observed in the balanced case, such bias does not necessarily tend towards zero if the number of observations per individual tends to infinity. This depends on the structure of the missing observations.

Indeed, since $|t_{ij} - t_{ik}| \geq |j-k|$ and $|\rho| < 1$:

$$|\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}| \leq \sum_{j,k=1}^{n_i}|\rho|^{|j-k|} = \frac{1+|\rho|}{1-|\rho|}n_i - \frac{2|\rho|}{1-|\rho|}\frac{1-|\rho|^{n_i}}{1-|\rho|}$$

and using a development of order 1 in $\frac{1}{m}$ with $m = \min(n_i)$, we notice :

$$\begin{aligned}
\mathbb{E}(\rho_d) &= 1 - \frac{\frac{1-\rho}{N}\sum_{i=1}^{N}\frac{K_i}{1+K_i}}{1+O(\frac{1}{m})} \\
&= 1 - \frac{1-\rho}{N}\sum_{i=1}^{N}\frac{K_i}{1+K_i}(1+O(\frac{1}{m})) \\
&= (1 - \frac{1}{N}\sum_{i=1}^{N}\frac{K_i}{1+K_i}) + \frac{\rho}{N}\sum_{i=1}^{N}\frac{K_i}{1+K_i} + O(\frac{1}{m})
\end{aligned} \tag{12}$$

For instance, consider a dataset where we observe the two first consecutive observations, and only every other observation thereafter. Hence $K_i = 1$ and $\mathbb{E}(\rho_d) \xrightarrow[m\to\infty]{} \frac{1+\rho}{2} \neq \rho$.

In the unbalanced case, however, *xtregar* does not use this generalization from the ρ_d estimate of Bhargava *et al.* (1982). Yet we already showed that it provides a biased estimate of ρ in the balanced case - a specific and simpler instance of the general case. So it seems to us unlikely that its theoretical properties would improve in the unbalanced setting. Moreover, in Section 3.2, we perform MonteCarlo simulations in which all the estimates of ρ provided by *xtregar* prove highly biased in short panels, both in the balanced and the unbalanced case ; and in which the default estimate of *xtregar* proves highly biased even in very long panels.

3 The ρ_{BFN} estimate of ρ and its approximations

Bhargava *et al.* (1982) also suggest to estimate ρ by solving for equation (7) after substituting $\mathbb{E}(\rho_d)$ with ρ_d. Using rather equation (11), ρ_{BFN} is thus implicitly defined in the general case by :

$$\rho_d = g_N(\rho_{BFN}) \tag{13}$$

with :

$$g_N : r \mapsto 1 - \frac{(1-r)\sum_{i=1}^{N} \frac{K_i}{1+K_i}}{N - \sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|}} \tag{14}$$

Note that in the balanced case, g_N does not depend on N, in which case we write it f with :

$$f : r \mapsto 1 - \frac{(1-r)(T-1)}{T - \frac{1}{T}\sum_{i,j=1}^{T} r^{|i-j|}}$$

In the balanced case, Bhargava *et al.* (1982) perform Monte Carlo simulations to provide an assessment of this method : it seems to deliver unbiased estimates of ρ. We build on this remark to improve the estimation of ρ.

3.1 Formal definition and theoretical properties

As noted by Bhargava *et al.* (1982), this correction procedure for estimating ρ is "*somewhat unconventional*". Even though their Monte Carlo study is somewhat conclusive, they do not establish the theoretical properties of their estimator, such as whether it is unbiased or consistent.

We first check that the definition of the ρ_{BFN} is unambiguous, in that there is at most one solution to the defining equation (13). We establish it formally in the case $0 < \rho < 1$. In both the balanced and unbalanced case, this new estimator is consistent under reasonable hypotheses.

We further show that ρ_{BFN} is not biased for large N in both balanced an unba-

lanced panels - unlike the bias in ρ_d which is, for instance, of order $\frac{1}{T}$ whatever the value of N in balanced panel. This confirms that this estimator of ρ should be considered instead of ρ_d.
As their is some ambiguity regarding the definition of ρ_{BFN} when $\rho < 0$, and as the computation of ρ_{BFN} may become numerically intractable as the time dimension of the panel increases, we also provide two approximations of ρ_{BFN} : one in the balanced case, and one in the unbalanced case, which are well-defined and have good theoretical properties in long panels.

3.1.1 Unambiguity of the definition of ρ_{BFN}

The following lemma shows there exists at most one estimator taking values between 0 and 1 that solves equation (13) (see Annex D for a demonstration).

Lemma 1 : $g_N : r \to 1 - \frac{(1-r)\sum_{i=1}^{N} \frac{K_i}{1+K_i}}{N - \sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|}}$ establishes a bijection from $[0,1]$ to $[1 - \frac{\sum_i^N \frac{K_i}{1+K_i}}{N - \sum_i^N \frac{1}{n_i}}, g_N(1)]$.

Section (3.1.4) deals with the case $\rho < 0$.

3.1.2 Consistency of ρ_{BFN}

We want to prove :

$$\rho_{BFN} \xrightarrow{\mathbb{P}} \rho \tag{15}$$

Under mild hypotheses, we have established in Annex C the following convergence as N grows to $+\infty$:

$$\rho_d - g_N(\rho) \xrightarrow{\mathbb{P}} 0$$

which implies, from the defining equation (13) :

$$g_N(\rho_{BFN}) - g_N(\rho) \xrightarrow{\mathbb{P}} 0$$

Balanced case :

In the balanced case, the above convergence writes :

$$f(\rho_{BFN}) \xrightarrow{\mathbb{P}} f(\rho)$$

In Annex B, we show that f is continuous and bijective over $[0,1]$. Hence, there exists a continuous inverse function f^{-1}. From the continuous mapping theorem :

$$\rho_{BFN} = f^{-1}(f(\rho_{BFN})) \xrightarrow{\mathbb{P}} f^{-1}(f(\rho)) = \rho$$

Unbalanced case :

We shall prove the convergence (15) under the following assumption :

Assumption 3 : For all $r \in [0,1]$, both sequences $\frac{1}{N}\sum_{i=1}^{N} \frac{K_i}{1+K_i}$ and $\frac{1}{N}\sum_{i=1}^{N} \frac{1}{n_i^2}\sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|}$ converge as $N \to \infty$.

One notices that, in the general case, this assumption holds for $\frac{1}{N}\sum_{i=1}^{N} \frac{K_i}{1+K_i}$, provided the K_i's are realizations of any distribution. Furthermore, this assumption holds in the balanced case for both sequences.

Such an assumption ensures the pointwise convergence of g_N to some function g_∞ on the segment $[0,1]$. Note that, in Annex B, we show that g_N is bijective over $[0,1]$, and its inverse g_N^{-1} is C^1 over its image [9]. Hence, g_N admits a C^1 inverse function g_N^{-1}. g_∞ is assumed to verify the following assumption :

Assumption 4 : g_∞ is bijective from $[0,1]$ to its image and its inverse g_∞^{-1} is C^1 over its image.

Lemma 1 above shows that g_N is increasing on $[0,1]$, hence according to the second Dini theorem, g_N converges uniformly to g_∞. Moreover, g_∞ is continuous as it is the case for g_∞^{-1}. The rest of the proof is now straightforward.

First, we observe that, for all $0 < \rho < 1$, $g_N(\rho_{BFN})$ converges to $g_\infty(\rho)$ in probability.

9. As soon there is at least one individual that is observed at least three times.

Second, the following inequality shows $g_\infty(\rho_{BFN}) \xrightarrow{\mathbb{P}} g_\infty(\rho)$:

$$\begin{aligned}|g_\infty(\rho_{BFN}) - g_\infty(\rho)| &\leq |g_\infty(\rho_{BFN}) - g_N(\rho_{BFN})| + |g_N(\rho_{BFN}) - g_\infty(\rho)| \\ &\leq ||g_\infty - g_N||_\infty + |g_N(\rho_{BFN}) - g_\infty(\rho)|\end{aligned}$$

Besides, applying the mean value theorem to $h = g_\infty^{-1}$, we may write

$$h(g_\infty(\rho_{BFN})) = h(g_\infty(\rho)) + (g_\infty(\rho_{BFN}) - g_\infty(\rho))h'(c_N)$$

for some scalar c_N lying between $g_\infty(\rho)$ and $g_\infty(\rho_{BFN})$. This relation writes down to

$$\rho_{BFN} = \rho + (g_\infty(\rho_{BFN}) - g_\infty(\rho))h'(c_N)$$

Since $g_\infty(\rho_{BFN}) \xrightarrow{\mathbb{P}} g_\infty(\rho)$ and h' is bounded, we have $\rho_{BFN} \xrightarrow{\mathbb{P}} \rho$.

3.1.3 The asymptotic bias of ρ_{BFN}

Here, we establish the following convergence :

$$\mathbb{E}(\rho_{BFN}) \xrightarrow[N\to\infty]{} \rho \tag{16}$$

That is, ρ_{BFN} is asymptotically unbiased as the number of individuals is becoming large.

As $\rho_d = g_N(\rho_{BFN})$ and $\mathbb{E}(\rho_d) = g_N(\rho)$, we have :

$$\mathbb{E}(g_N(\rho_{BFN})) = g_N(\rho) \tag{17}$$

Denoting $A_N = \frac{1}{N}\sum_{i=1}^N \frac{K_i}{1+K_i}$ which satisfies $\frac{1}{2} \leq |A_N| \leq 1$, we notice :

$$g_N(r) = 1 - A_N + rA_N - A_N p_N(r) \tag{18}$$

where :

$$p_N(r) = \frac{1-r}{\frac{N}{\sum_{i=1}^N \frac{1}{n_i^2}\sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|}} - 1}$$

We treat the balanced and the unbalanced case separately.

The balanced case :

In the balanced case, relation (17) translates into :

$$\mathbb{E}(f(\rho_{BFN})) = f(\rho) \tag{19}$$

Relation (18) becomes :

$$f(r) = 1 - \frac{T-1}{T} + r\frac{T-1}{T} - \frac{T-1}{T}q(r) \tag{20}$$

where

$$q(r) = \frac{1-r}{\frac{1}{\frac{1}{T^2}\sum_{j,k=1}^{T} r^{|j-k|}} - 1}$$

Combining relations (19) and (20), we get :

$$\mathbb{E}(\rho_{BFN} - \rho) = \mathbb{E}(q(\rho_{BFN}) - q(\rho))$$

It is easy to show that the right hand side tends to zero as N grows to infinity, which establishes (16). Indeed, as shown in Annex E, q may be continously prolounged over the segment $[0, 1]$, hence q is bounded by some constant $B > 0$ and, according to the continuous mapping theorem, $q(\rho_{BFN}) \xrightarrow{\mathbb{P}} q(\rho)$. The rest of the proof is straightforward. Let $\varepsilon > 0$. There exists $N \geq 0$ such that for all $n \geq N$, $\mathbb{P}(|q(\rho_{BFN}) - q(\rho)| > \varepsilon) \leq \varepsilon$. Then,

$$\begin{aligned}
|\mathbb{E}(q(\rho_{BFN}) - q(\rho))| &\leq \mathbb{E}(|q(\rho_{BFN}) - q(\rho)|) \\
&= \mathbb{E}(|q(\rho_{BFN}) - q(\rho)|\mathbb{1}_{|q(\rho_{BFN})-q(\rho)|>\varepsilon)} + \\
&\mathbb{E}(|q(\rho_{BFN}) - q(\rho)|\mathbb{1}_{|q(\rho_{BFN})-q(\rho)|\leq\varepsilon)} \\
&\leq 2B\varepsilon + \varepsilon
\end{aligned}$$

The convergence (16) is then established in the balanced case.

The unbalanced case

By applying equations (17) and (18), and considering that A_N is deterministic [10], it comes :

$$\mathbb{E}(\rho_{BFN} - \rho) = \mathbb{E}(p_N(\rho_{BFN}) - p_N(\rho))$$

10. That is : the pattern of missing data is supposed to be given here, just like N, m or ρ.

Contrary to the balanced case, the continuous mapping theorem does not apply here, as p_N depends on N. However, the above demonstration in the balanced case would apply if we were able to show that : $p_N(\rho_{BFN}) - p_N(\rho) \xrightarrow{\mathbb{P}} 0$.

We notice :

$$g_N(\rho) - g_N(\rho_{BFN}) = A_N(\rho - \rho_{BFN}) - A_N(p_N(\rho) - p_N(\rho_{BFN}))$$

which may be rewritten

$$p_N(\rho) - p_N(\rho_{BFN}) = \rho - \rho_{BFN} - \frac{1}{A_N}(g_N(\rho) - g_N(\rho_{BFN}))$$

As we have :

$$\rho_{BFN} - \rho \xrightarrow{\mathbb{P}} 0$$

$$g_N(\rho_{BFN}) - g_N(\rho) \xrightarrow{\mathbb{P}} 0$$

$$\frac{1}{2} \leq A_N$$

We indeed get :

$$p_N(\rho_{BFN}) - p_N(\rho) \xrightarrow{\mathbb{P}} 0$$

Besides, Annex (E) shows that p_N is bounded uniformly in N :

$$0 \leq p_N(r) \leq \frac{1}{1 - \frac{1}{m}}$$

Hence, we get the announced result (16).

This result is of first-order importance. Section 2.2 showed that the bias of ρ_d could heavily depend on the patterns of the missing values and does not necessarily converge to zero, even when the minimal number of periods m is becoming large. All the opposite, ρ_{BFN} is asymptotically unbiased as $N \to \infty$ even for small values of m.

3.1.4 Two approximations of ρ_{BFN} : ρ_{BFN2B} and ρ_{BFN2U}

What is the rationale behind additional estimates of ρ ?
When $-1 \leq \rho \leq 0$, we do not provide a demonstration that ρ_{BFN} is well defined. In

the remaining cases, one might have a use of an estimator of ρ_{BFN} to determine which root is the most appropriate estimator of ρ. Such estimate ρ_{BFN2U} can be obtained thanks to Formula (12). If its last term can be neglected, Formula (12) [11] becomes indeed linear in ρ and provides a unique root when one replaces $E(\rho_d)$ with ρ_d (see formulas (21) and (22) below, respectively for the balanced and unbalanced case).

One may even consider ρ_{BFN2B} and ρ_{BFN2U} rather than ρ_{BFN} as estimators of ρ. Note these approximations could also be interesting when $\rho > 0$ as their computation is numerically almost as simple as the estimation of ρ_d.

Definition :

$$\rho_{BFN2B} = \frac{\rho_d}{(1 - \frac{2}{T})} \tag{21}$$

$$\rho_{BFN2U} = \frac{\frac{1}{N}\sum_i^N \frac{K_i}{1+K_i} - 1 + \rho_d}{\frac{1}{N}\sum_i^N \frac{K_i}{1+K_i}} \tag{22}$$

Asymptotic behavior :

In the balanced case, from the asymptotic behavior of ρ_d as $N \to \infty$, it comes :

$$\rho_{BFN2B} \xrightarrow{\mathbb{P}} \frac{f(\rho)}{1 - \frac{2}{T}}$$

where :

$$\frac{f(\rho)}{1 - \frac{2}{T}} = \rho + O(\frac{1}{T^2})$$

In the unbalanced case, from the asymptotic behavior of ρ_d as $N \to \infty$, it comes :

$$\rho_{BFN2U} - \frac{\frac{1}{N}\sum_i^N \frac{K_i}{1+K_i} - 1 + g_N(\rho)}{\frac{1}{N}\sum_i^N \frac{K_i}{1+K_i}} \xrightarrow{\mathbb{P}} 0$$

where :

$$\frac{\frac{1}{N}\sum_i^N \frac{K_i}{1+K_i} - 1 + g_N(\rho)}{\frac{1}{N}\sum_i^N \frac{K_i}{1+K_i}} = \rho + O(\frac{1}{m})$$

11. Or Formula (9) in the balanced case. In which case we call the new estimator ρ_{BFN2B}.

Expectations :
By dividing equation (9) by $1 - \frac{2}{T}$, one gets in the balanced case :

$$E(\rho_{BFN2B}) = \rho + O(\frac{1}{T^2})$$

Similarly, it comes in the unbalanced case :

$$E(\rho_{BFN2U}) = \rho + O(\frac{1}{m})$$

The above properties show that ρ_{BFN2B} and ρ_{BFN2U} have good properties in long panels. Both their limit and their asymptotic expectancy (for large N) are a term whose difference to ρ tends to 0 when the time dimension of the panel (T in the balanced case, or for instance m in the unbalanced case) tends to $+\infty$.

Yet, when the panel is short (in the balanced case) or only moderately long (in the unblanced case), ρ_{BFN} should be strictly preferred when it is possible. For instance, MonteCarlo simulations show that ρ_{BFN} is a much better estimator than ρ_{BFN2U}, even for relatively long panels (e.g. $T = 50$). Hence, the former should be preferred to the later.

3.2 Monte Carlo simulations

3.2.1 In the balanced case

Monte Carlo simulations confirm that, in a given balanced setting ($N = 500$; $T = 10$; $\rho = 0.6$; $\sigma_\eta = 0.3$; $\sigma_\nu = 0.35$), ρ_{BFN} and ρ_{BFN2U} are unbiased [12], unlike all other estimators currently provided by the *xtregar* command (see Table 1). Estimates of σ_η derived from these two estimators seem also unbiased. Yet, they do not seem to improve the estimation of σ_ν.

Comparing the two alternative estimators ρ_{BFN2B} and ρ_{BFN2U} that we suggested in Section 3.1.4, simulation results follow the theoretical properties : the alternative estimator adapted to the balanced case ρ_{BFN2B} performs as well as ρ ; the alternative estimator built for the unbalanced case has a bad performance, due to the small time

12. For ρ_{BFN}, this was already observed by Bhargava *et al.* (1982).

dimension of the panel (T=10).

3.2.2 In the unbalanced case

Similarly to the balanced case, ρ_{BFN} provides a much better estimate of ρ than all other alternatives currently provided by *xtregar*. However, it does not improve the estimates of σ_η and of σ_ν that are highly biased in all cases.

As in the balanced case, the approximate estimator ρ_{BFN2U} has a very poor performance due to the low time dimension of the panel (T=10). We study in section 3.2.3 what happens when the time dimension of the panel increases.

Table 1. Monte Carlo simulations on a balanced panel

	ρ	σ_η	σ_ν
true values	0.6	0.3	0. 35
dw (default)	.464***	.293**	.415***
	(.012)	(3.0e-03)	(.012)
regress	.389***	.292**	.415***
	(.015)	(3.0e-03)	(.012)
freg	.389***	.292**	.415***
	(.014)	(3.0e-03)	(.012)
tscorr	.341***	.293**	.415***
	(.013)	(3.0e-03)	(.012)
theil	.379***	.292**	.415***
	(.014)	(3.0e-03)	(.012)
nagar	.464***	.293**	.415***
	(.012)	(3.0e-03)	(.012)
onestep	.379***	.292**	.415***
	(.014)	(3.0e-03)	(.012)
ρ_{BFN}	**.598**	**.299**	.415***
as in (Bhargava *et al.*, 1982))	**(.017)**	**(3.0e-03)**	(.012)
ρ_{BFN2B}	**.597**	**.299**	.415***
	(.016)	**(3.0e-03)**	(.012)
ρ_{BFN2U}	.405***	.292**	.415***
	(.014)	(3.0e-03)	(.012)

Legend : The average estimators should not be significantly different from the true values. It is the case only for those in bold. Significance levels for the differences with the true values are otherwise pinpointed by stars : * ($p < 0.10$), ** ($p < 0.05$), *** ($p < 0.01$)

Note : This Monte Carlo simulation was performed on a panel of 500 individuals observed each over 10 periods, with 50 replications.

The estimates of σ_η and of σ_ν in the last lines are obtained by estimating first ρ_{BFN}, and then by imposing it as the estimate of ρ in *xtregar*.

Table 2. Monte Carlo simulations on an unbalanced panel

	ρ	σ_η	σ_ν
true valuess	0.6	0.3	0. 35
dw (default)	.691***	.555***	.442***
	(.013)	(.014)	(.013)
regress	.271***	.561***	.442***
	(.027)	(.025)	(.013)
freg	.272***	.562***	.442***
	(.029)	(.025)	(.013)
tscorr	.115***	.396***	.442***
	(.013)	(.017)	(.013)
theil	.248***	.541***	.442***
	(.028)	(.027)	(.013)
nagar	.691***	.555***	.442***
	(.013)	(.014)	(.013)
onestep	.248***	.541***	.442***
	(.028)	(.027)	(.013)
ρ_{BFN}	**.601**	.616***	.442***
(our generalization)	**(.035)**	(.021)	(.013)
ρ_{BFN2U}	.326***	.603***	.442***
	(.032)	(.022)	(.013)

Legend : The average estimators should not be significantly different from the true values. It is the case only for those in bold. Significance levels for the differences with the true values are otherwise pinpointed by stars : * ($p < 0.10$), ** ($p < 0.05$), *** ($p < 0.01$)

Note : Approximately half of a panel of 500 individuals observed each over 10 periods has been randomly deleted, before the Monte Carlo process has been implemented with 50 replications.

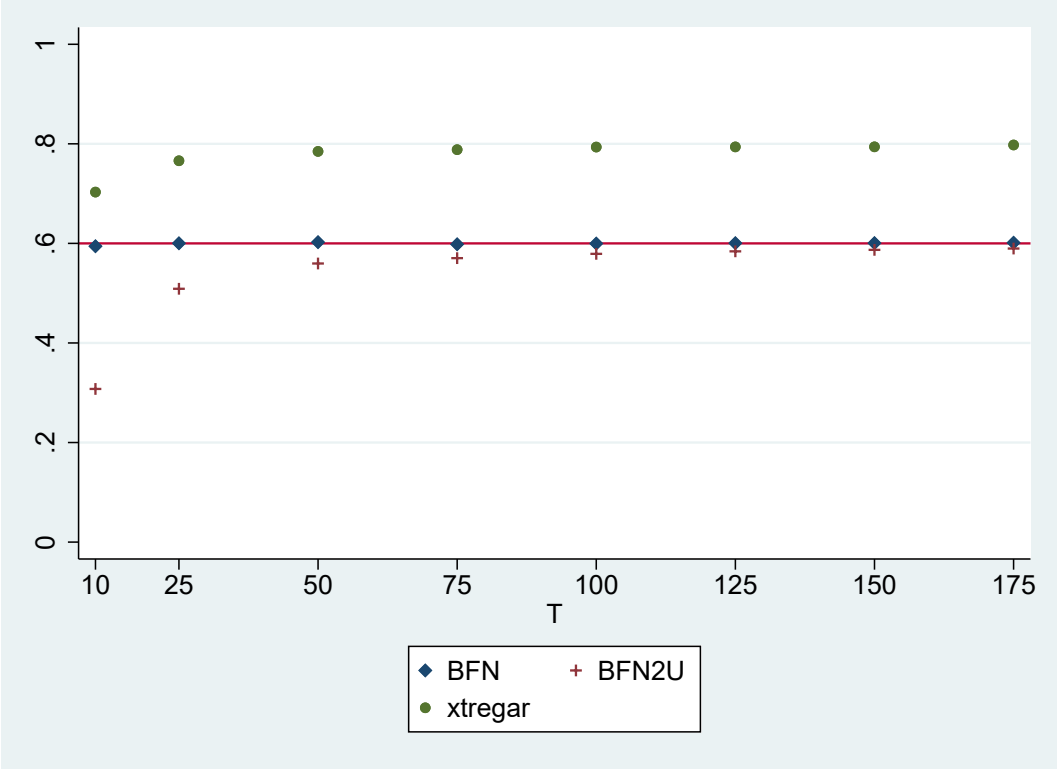
The estimates of σ_η and of σ_ν in the two last lines are obtained by estimating first ρ_{BFN} (or ρ_{BFN2U}), and then by imposing it as the estimate of ρ in *xtregar*.

3.2.3 Monte Carlo simulations when $T \to \infty$

In our previous Monte Carlo simulations with $T = 10$, we observed that ρ_{BFN} (and ρ_{BFN2B} in the balanced case) provided much better estimates than all other existing alternatives.

However, the performances of ρ_{BFN2U} was poor, even relative to currently existing alternatives. As we establish that : $\mathbb{E}(\rho_{BFN2U}) = \rho + O(\frac{1}{m})$, we empirically study how this relative performance evolves when $T \to \infty$, where T is the time span of the panel before we drop observations [13].

Figure 1.
Estimations of ρ with Monte Carlo simulations on an unbalanced panel with $T \to \infty$



Note : This Monte Carlo simulation was performed on a panel of 500 individuals observed each over T periods, with 5 replications. Around half of all observations are each time randomly dropped to get an unbalanced panel. True value of the parameter ρ is 0.6.

13. Another way could have been to both increase T and decrease the probability that an observation goes missing (currently 0.5).

Unsurprisingly, simulations show that ρ_{BFN2U} converges as T tends to infinity. For T=50, it starts delivering a prediction relatively close to the true value of ρ. More surprisingly, the current estimation of $\hat{\rho}$ does not seem to converge to the true value of ρ. If anything, it would rather converge towards 0.8. Hence, even for $\rho > 0$, if the computations for estimating ρ_{BFN} were computationally intractable due to a too large value of T, ρ_{BFN2U} should be preferred to the current implemented estimator of xtregar.

4 The rho_xtregar command

The *rho_xtregar* command is compatible with Stata 15.1 and later versions. It uses the *moremata* Stata command (Jann, 2005). If this command is not already installed, one must type *ssc install moremata* in Stata's command line.

4.1 Syntax

The syntax of *rho_xtregar* is as follows :

rho_xtregar depvar [indepvars] [if] [using *filename*] [, approx approx_balanced approx_unbalanced option_nodisplay]

rho_xtregar requires that the data are *xtset* before estimation. Moreover, in very long panels (T>70), computation of rho_BFN may require an increase in *maxvar* (set it higher than T^2 where T is the maximum number of observations for an individual, over all individuals).

4.2 Options

approx computes the appropriate approximation of ρ_{BFN}, taking account whether the panel is balanced (it computes ρ_{BFN2B}) or unbalanced (it computes ρ_{BFN2U}).
approx_balanced (seldom used) Force the command to compute ρ_{BFN2B}, even in the unbalanced case.
approx_unbalanced (seldom used) Force the command to compute ρ_{BFN2U}, even in the balanced case.
Only one of these three options should be specified.
option_nodisplay Do not display the message regarding the number of units with

at least two successive observations.

4.3 Description

rho_xtregar estimates the autoregressive parameter for cross-sectional time-series regression models when the disturbance term is first-order autoregressive.
It implements the method initially exposed in Bhargava *et al.* (1982) and generalizes it to the unbalanced case.

depvar is the dependent variable. **indepvars** are the explanatory variables.

4.4 Stored results

The **rho_xtregar** command saves the following in **r()** :

r(rho_BFN) a scalar containing ρ_{BFN} (or ρ_{BFN2B} or ρ_{BFN2U} depending on the specified option).

5 An application

To illustrate the use of rho_xtregar, we use the same dataset as Baltagi et Wu (1999) to study the dynamics of firms' investments. It is the Grunfeld data on investment, consisting in a balanced sample of 10 large US manufacturing firms observed from 1935 to 1954 : T=19 ; N = 10. We load the dataset and indicate the time and firm indicators.

We want to regress with the AR(1) model real gross investment of firm i in year t **invest[i,t]** on the real value of the firm **mvalue[i,t]** and the real value of the capital stock **kstock[i,t]**. As in Baltagi et Wu (1999), this example is merely provided for illustrative purposes. We first execute the rho_xtregar command, and we apply the estimated ρ_{BFN} to the *xtregar* command :

```
. rho_xtregar invest mvalue kstock, option_nodisplay
. local rhoBFN = r(rho_BFN)
```

```
. xtregar invest mvalue kstock, fe rhof('rhoBFN')

FE (within) regression with AR(1) disturbances  Number of obs     =       190
Group variable: company                         Number of groups  =        10

R-sq:                                           Obs per group:
     within  = 0.5489                                         min =        19
     between = 0.7981                                         avg =      19.0
     overall = 0.7897                                         max =        19

                                                F(2,178)          =    108.30
corr(u_i, Xb)  = -0.0292                        Prob > F          =    0.0000

------------------------------------------------------------------------------
      invest |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      mvalue |   .0938027   .0089244    10.51   0.000     .0761915    .1114139
      kstock |   .3490061   .0334632    10.43   0.000     .2829706    .4150417
       _cons |  -64.42704   5.244971   -12.28   0.000    -74.77737   -54.07672
-------------+----------------------------------------------------------------
      rho_ar |     .74097
     sigma_u |  91.619229
     sigma_e |  41.074805
     rho_fov |  .83264534   (fraction of variance because of u_i)
------------------------------------------------------------------------------
F test that all u_i=0: F(9,178) = 7.83                     Prob > F = 0.0000
```

We compare it to a direct application of the *xtregar* command :

```
. xtregar invest mvalue kstock, fe
```

```
FE (within) regression with AR(1) disturbances  Number of obs     =       190
Group variable: company                         Number of groups  =        10

R-sq:                                           Obs per group:
     within  = 0.5927                                         min =        19
     between = 0.7989                                         avg =      19.0
     overall = 0.7904                                         max =        19

                                                F(2,178)          =    129.49
corr(u_i, Xb)  = -0.0454                        Prob > F          =    0.0000
```

invest	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
mvalue	.0949999	.0091377	10.40	0.000	.0769677	.113032
kstock	.350161	.0293747	11.92	0.000	.2921935	.4081286
_cons	-63.22022	5.648271	-11.19	0.000	-74.36641	-52.07402
rho_ar	.67210608					
sigma_u	91.507609					
sigma_e	40.992469					
rho_fov	.8328647	(fraction of variance because of u_i)				

```
F test that all u_i=0: F(9,178) = 11.53                     Prob > F = 0.0000
```

The difference is substantial : $\rho_{BFN} = 0.74$ to be compared with 0.67 when d_p is straightly used by *xtregar*. It is a 10% error.

Some caution is necessary here, as the properties of ρ_{BFN} are established for N large enough, and N is only 10 here. We therefore perform a short Monte-Carlo study to see how **rho_xtregar** perform in this unfavorable setting.

```
quietly {
scalar the_rho = e(rho_ar)
scalar the_sigma_eta = e(sigma_e)
scalar the_sigma_c_i = e(sigma_u)
scalar the_sigma_epsilon = the_sigma_eta / sqrt(1-the_rho*the_rho)
matrix the_sd = (the_sigma_eta,the_sigma_epsilon,the_sigma_c_i)
        gen rho_emp = 0
        gen rho_emp2 = 0
set seed 89
        forvalues i = 1/10 {
        drawnorm eta epsilon0 c_i, means(0,0,0) sds(the_sd)
```

```
        bysort company: gen epsilon= epsilon0 if _n==1
        bysort company: replace epsilon=eta + the_rho * epsilon[_n-1] if _
        bysort company: replace c_i = c_i[1]
        gen y = c_i + epsilon
        xtregar y, fe
                replace rho_emp = e(rho_ar) if _n==`i'
        rho_xtregar y, option_nodisplay
                replace rho_emp2 = r(rho_BFN) if _n==`i'
        drop epsilon epsilon epsilon0 y eta c_i
    }
        keep if _n<=10
        collapse (mean)rho_emp rho_emp2
        }
display "true value: " the_rho "; xtregar gives: " rho_emp[1] "; rho_xtr
```

One gets :

```
true value: .67210608; xtregar gives: .58660442 ; rho_xtregar gives: .6585713
```

Hence, with **rho_xtregar**, the error is only of -0.014 (that is a 2% error) while it is of -0.086 (that is a 13% error) with **xtregar**. Even in this limit case with a very low N, **rho_xtregar** is to be preferred to get an estimate of ρ.

6 Conclusion

We built upon the work and an intuition of Bhargava *et al.* (1982) to provide a new estimator of the autocorrelation parameter in fixed- or random-effects models with AR(1) disturbances. We show that the suggested estimation method defines at most one estimator, denoted ρ_{BFN}, which is consistent and less biased than current estimates of ρ. For $0 < \rho < 1$, it is asymptotically unbiased as $N \to \infty$, with N the number of individuals.

To take into account specific situations (e.g. when the computation of ρ_{BFN} is numerically too demanding), we defined two additional estimators of ρ that approximate ρ_{BFN} in long panels : ρ_{BFN2B} in balanced panels, and ρ_{BFN2U} in unbalanced panels. Their bias tends however to zero when the time dimension of the panel (i.e. the minimal number of observations per individual) tends to infinity. They are easier to compute than ρ_{BFN}, and perform as well as ρ_{BFN} in our Monte Carlo simulations in long and very long panels. In case computations of ρ_{BFN} are numerically demanding, ρ_{BFN2B} and ρ_{BFN2U} would be our preferred choice in long balanced and very long unbalanced panels respectively.

Monte-Carlo simulations highlight these estimators are usually much better than the current methods provided by the dedicated command xtregar of the Stata software. A new Stata command **rho_xtregar** is then implemented to enable other researchers to benefit from these improvements in the estimation of ρ.

As a first caveat, in unbalanced panels, this approach heavily depends on consecutive observations. In sparse datasets with no consecutive observations, it cannot be implemented. In such cases, other methods like those of Magnac *et al.* (2018) are to be preferred.

As a second caveat, our MonteCarlo simulations show there is room for improvement in the estimations of other possible parameters of interest such as the variance of the perturbations in unbalanced panels. This is let for further research.

As a final thought, our work has a different focus than the canonical works by Bhargava *et al.* (1982) and Baltagi et Wu (1999). While they were mostly interested in testing the nullity (or equality to 1) of the autocorrelation coefficient, this article

focuses on its estimation performance. These topics are related but distinct as clearly shown here.

Références

Baltagi, B. H. et Wu, P. X. (1999). Unequally spaced panel data regressions with AR (1) disturbances. *Econometric Theory*, 15(6):814–823.

Bhargava, A., Franzini, L. et Narendranathan, W. (1982). Serial correlation and the fixed effects model. *The Review of Economic Studies*, 49(4):533–549.

Cazenave-Lacroutz, A., Godet, F. et Lin, V. (2019). Modélisation des trajectoires de revenus d'activité pour le modèle destinie 2.

Dafny, L. S. (2010). Are health insurance markets competitive? *American Economic Review*, 100(4):1399–1431.

Hau, H., Langfield, S. et Marques-Ibanez, D. (2013). Bank ratings: what determines their quality? *Economic Policy*, 28(74):289–333.

Jann, B. (2005). moremata: Stata module (Mata) to provide various functions.

Magnac, T., Pistolesi, N. et Roux, S. (2018). Post-Schooling Human Capital Investments and the Life Cycle of Earnings. *Journal of Political Economy*, 126(3): 1219–1249.

Nagar, A. L. (1959). The bias and moment matrix of the general k-class estimators of the parameters in simultaneous equations. *Econometrica*, pages 575–595.

Annexes

A Intuitions behind the generalisation of d_p to the unbalanced case

We give here the intuition lying behind formula (10) that defines d_p.

Firstly :

$$\mathbb{E}(\sum_{j=2}^{n_i}(u_{it_{ij}}-u_{it_{ij-1}})^2\mathbb{1}_{t_{ij}-t_{ij-1}=1}) = \sum_{j=2}^{n_i}\mathbb{E}((u_{it_{ij}}-u_{it_{ij-1}})^2)\mathbb{1}_{t_{ij}-t_{ij-1}=1} = K_i((1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2) \tag{23}$$

hence $\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i+1}\sum_{j=2}^{n_i}(\tilde{u}_{it_{ij}}-\tilde{u}_{it_{ij-1}})^2\mathbb{1}_{t_{ij}-t_{ij-1}=1}$ is a natural estimate of $(1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2$. Let us note A this estimate.

Secondly :

$$\mathbb{E}(\frac{1}{n_i}\sum_{j=1}^{n_i}(u_{it_{ij}}-\overline{u}_i)^2) = \mathbb{E}(\frac{1}{n_i}\sum_{j=1}^{n_i}u_{it_{ij}}^2-(\overline{u}_i)^2) = \sigma_u^2-\mathbb{E}(\overline{u}_i^2) \tag{24}$$

One develops $\mathbb{E}(\overline{u}_i^2)$:

$$\mathbb{E}(\overline{u}_i^2) = \frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\mathbb{E}(u_{it_{ij}}u_{it_{ik}}) = \frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\frac{\rho^{|t_{ij}-t_{ik}|}}{1-\rho^2}\sigma_\varepsilon^2 = \frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}\sigma_u^2 \tag{25}$$

hence :

$$\mathbb{E}(\frac{1}{n_i}\sum_{j=1}^{n_i}(u_{it_{ij}}-\overline{u}_i)^2) = \sigma_u^2(1-\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}) \tag{26}$$

By using relation (6), we now show that, as $n_i\to\infty$, $\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}\to 0$.
We indeed notice that : $|t_{ij}-t_{ik}|\geq|j-k|$; hence $|\rho|^{|t_{ij}-t_{ik}|}\leq|\rho|^{|j-k|}$ as $|\rho|\leq 1$.
Therefore :

$$|\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}| \leq \sum_{j,k=1}^{n_i}|\rho|^{|t_{ij}-t_{ik}|} \leq \sum_{j,k=1}^{n_i}|\rho|^{|j-k|} = \mathcal{O}(n_i) \tag{27}$$

hence $\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|} = \mathcal{O}(\frac{1}{n_i})$. The later terms tends to zero when $n_i\to\infty$.
Therefore $\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i}\sum_{j=1}^{n_i}\tilde{u}_{it_{ij}}^2$ is a natural estimate of σ_u^2. We note B this estimateur.

By making the ratio $\frac{A}{B}$, one gets formula (10). Therefore, it provides a natural estimate of $\frac{(1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2}{\sigma_u^2} = 2 - 2\rho$.

B Derivation of the expectancy of ρ_d in the unbalanced case

We follow the demonstration of Bhargava *et al.* (1982). For N large enough :

$$\mathbb{E}(d_p) \simeq \mathbb{E}\left(\frac{\sum_{i=1}^{N}\frac{1}{K_i+1}\sum_{j=2}^{n_i}(u_{it_{ij}}-u_{it_{ij-1}})^2\mathbb{1}_{t_{ij}-t_{ij-1}=1}}{\sum_{i=1}^{N}\frac{1}{n_i}\sum_{j=1}^{n_i}(u_{it_{ij}}-\overline{u}_i)^2}\right) \tag{28}$$

Adapting to our data pattern the Nagar (1959) approximation used by Bhargava *et al.* (1982), one gets :

$$\mathbb{E}(d_p) \simeq \frac{\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i+1}\sum_{j=2}^{n_i}(u_{it_{ij}}-u_{it_{ij-1}})^2\mathbb{1}_{t_{ij}-t_{ij-1}=1}\right)}{\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i}\sum_{j=1}^{n_i}(u_{it_{ij}}-\overline{u}_i)^2\right)} \tag{29}$$

On the one hand, the numerator is :

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i+1}\sum_{j=2}^{n_i}(u_{it_{ij}}-u_{it_{ij-1}})^2\mathbb{1}_{t_{ij}-t_{ij-1}=1}\right) = \\
&\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i+1}\sum_{j=2}^{n_i}\mathbb{E}((u_{it_{ij}}-u_{it_{ij-1}})^2)\mathbb{1}_{t_{ij}-t_{ij-1}=1} = \\
&\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i+1}\sum_{j=2}^{n_i}((1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2)\mathbb{1}_{t_{ij}-t_{ij-1}=1} = \\
&\frac{(1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2}{N}\sum_{i=1}^{N}\frac{K_i}{1+K_i}
\end{aligned} \tag{30}$$

On the other hand, the denominator is :

$$
\begin{aligned}
\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i}\sum_{j=1}^{n_i}(u_{it_{ij}}-\overline{u}_i)^2\right) &= \\
\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\frac{1}{n_i}\sum_{j=1}^{n_i}(u_{it_{ij}}-\overline{u}_i)^2\right) &= \\
\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\frac{1}{n_i}\sum_{j=1}^{n_i}u_{it_{ij}}^2-(\overline{u}_i)^2\right) &= \\
\frac{1}{N}\sum_{i=1}^{N}(\sigma_u^2-\mathbb{E}(\overline{u}_i^2)) &= \\
\frac{1}{N}\sum_{i=1}^{N}\left(\sigma_u^2-\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}\sigma_u^2\right) &= \\
\sigma_u^2\left(1-\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}\right) &
\end{aligned}
\tag{31}
$$

Hence :

$$
\begin{aligned}
\mathbb{E}(d_p) &\simeq \frac{\frac{(1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2}{N}\sum_{i=1}^{N}\frac{K_i}{1+K_i}}{\sigma_u^2\left(1-\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}\right)} \\
&= \frac{2(1-\rho)}{N}\frac{\sum_{i=1}^{N}\frac{K_i}{1+K_i}}{1-\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}}
\end{aligned}
\tag{32}
$$

Therefore

$$
\begin{aligned}
\mathbb{E}(\rho_d) &= 1-\frac{1}{2}\mathbb{E}(d_p) \\
&\simeq 1-\frac{(1-\rho)\sum_{i=1}^{N}\frac{K_i}{1+K_i}}{N-\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}}
\end{aligned}
\tag{33}
$$

Note this relation is valid provided N is large enough.

C Asymptotic behavior of ρ_d

We study the asymptotic behavior of the estimator ρ_d defined via formula (10). More precisely, we show that ρ_d is close to the right member of formula (11) as $N \to \infty$. In what follows, we denote $\ddot{y}_{it} = y_{it} - \overline{y}_i$, $\ddot{x}_{it} = x_{it} - \overline{x}_i$ and $\ddot{u}_{it} = u_{it} - \overline{u}_i$. We denote by $\tilde{u}_{it}$ the OLS residuals from the regression of $\ddot{y}_{it}$ on $\ddot{x}_{it}$.

C.1 The case of the numerator of d_p

Writing

$$\begin{aligned}(\tilde{u}_{it_{ij}} - \tilde{u}_{it_{ij-1}})^2 &= (u_{it_{ij}} - u_{it_{ij-1}} + (\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})'(\beta - \hat{\beta}))^2 \\ &= (u_{it_{ij}} - u_{it_{ij-1}})^2 \\ &+ (\beta - \hat{\beta})'(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})'(\beta - \hat{\beta}) \\ &+ 2(u_{it_{ij}} - u_{it_{ij-1}})(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})'(\beta - \hat{\beta})\end{aligned} \tag{34}$$

one has

$$\begin{aligned}&\frac{1}{K_i+1}\sum_{j=2}^{n_i}(\tilde{u}_{it_{ij}} - \tilde{u}_{it_{ij-1}})^2 \mathbb{1}_{t_{ij}-t_{ij-1}=1} \\ &= \frac{1}{K_i+1}\sum_{j=2}^{n_i}(u_{it_{ij}} - u_{it_{ij-1}})^2 \mathbb{1}_{t_{ij}-t_{ij-1}=1} \\ &+ \frac{1}{K_i+1}\sum_{j=2}^{n_i}(\beta - \hat{\beta})'(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})'(\beta - \hat{\beta})\mathbb{1}_{t_{ij}-t_{ij-1}=1} \\ &+ \frac{2}{K_i+1}\sum_{j=2}^{n_i}(u_{it_{ij}} - u_{it_{ij-1}})(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})'(\beta - \hat{\beta})\mathbb{1}_{t_{ij}-t_{ij-1}=1}\end{aligned} \tag{35}$$

Denoting by A_i, B_i, C_i the first, the second and the third term of the right hand side of this equality respectively, we shall prove the following probability convergences, as $N \to \infty$:

$$\frac{1}{N}\sum_{i=1}^{N} A_i - \frac{(1-\rho)^2\sigma_u^2 + \sigma_\varepsilon^2}{N}\sum_{i=1}^{N}\frac{K_i}{K_i+1} \to 0 \tag{36}$$

$$\frac{1}{N}\sum_{i=1}^{N} B_i \to 0 \tag{37}$$

$$\frac{1}{N}\sum_{i=1}^{N} C_i \to 0 \tag{38}$$

which will prove

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i+1}\sum_{j=2}^{n_i}(\tilde{u}_{it_{ij}}-\tilde{u}_{it_{ij-1}})^2\mathbb{1}_{t_{ij}-t_{ij-1}=1}-\frac{(1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2}{N}\sum_{i=1}^{N}\frac{K_i}{K_i+1}\to 0 \quad (39)$$

We make the following assumption

Assumption 1 : There exists $\delta>0$ such that for all i and t, $\mathbb{E}(|u_{it}|^{2+\delta})<\infty$.

C.1.1 Proof of Assertion (36)

Since $\mathbb{E}((u_{it_{ij}}-u_{it_{ij-1}})^2)\mathbb{1}_{t_{ij}-t_{ij-1}=1}=((1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2)\mathbb{1}_{t_{ij}-t_{ij-1}=1}$, one has

$$\mathbb{E}\left(\frac{1}{K_i+1}\sum_{j=2}^{n_i}(u_{it_{ij}}-u_{it_{ij-1}})^2\mathbb{1}_{t_{ij}-t_{ij-1}=1}\right)=\frac{K_i}{K_i+1}((1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2)$$

.

Using the independence of the A_i's and assumption 1, we may apply some version of the law of large numbers (see below) to obtain assertion (36).

Law of large number for independent non-identically distributed random variables : Let (X_i) be a sequence of independent random variables such that $\mathbb{E}[|X_i|^{1+\delta}]<\infty$ for some $\delta>0$ and all i. Then, almost surely,

$$\frac{1}{N}\sum_{i=1}^{N}X_i-\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(X_i)\xrightarrow[N\to\infty]{}0$$

C.1.2 Proof of Assertion (37)

We make the following assumption

Assumption 2 : Let K denote the number of regressors. Then the $K\times K$ semi-definite positive matrix

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i+1}\sum_{j=2}^{n_i}(\ddot{x}_{it_{ij}}-\ddot{x}_{it_{ij-1}})(\ddot{x}_{it_{ij}}-\ddot{x}_{it_{ij-1}})'\mathbb{1}_{t_{ij}-t_{ij-1}=1}$$

is uniformly bounded in N. That is, if M_N denotes this matrix, then there exists

$C > 0$ such that for all $N \geq 0$,

$$||M_N|| \leq C,$$

where $||.||$ denotes some norm over the $K \times K$ matrix space.

Under this assumption and recalling that $\hat{\beta}$ is consistent, we get assertion (37).

C.1.3 Proof of Assertion (38)

Using strict exogeneity, we have

$$\begin{aligned}
&\mathbb{E}\left((u_{it_{ij}} - u_{it_{ij-1}})(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})|x_{it_{i1}}, ..., x_{it_{in_i}}\right) \\
&= \mathbb{E}\left((u_{it_{ij}} - u_{it_{ij-1}})|x_{it_{i1}}, ..., x_{it_{in_i}}\right)(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}}) \\
&= 0.
\end{aligned}$$

hence $\mathbb{E}\left((u_{it_{ij}} - u_{it_{ij-1}})(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})\right) = 0$.
Then, using the law of large numbers as previously,

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i + 1}\sum_{j=2}^{n_i}(u_{it_{ij}} - u_{it_{ij-1}})(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})'\mathbb{1}_{t_{ij}-t_{ij-1}=1} \to 0.$$

We conclude by recalling that $\hat{\beta}$ is consistent.

C.2 The case of the denominator of d_p

Writing

$$\begin{aligned}
\tilde{u}^2_{it_{ij}} &= ((\ddot{x}_{it_{ij}})'(\beta - \hat{\beta}) + \ddot{u}_{it_{ij}})^2 \\
&= \ddot{u}^2_{it_{ij}} \\
&+ (\beta - \hat{\beta})'(\ddot{x}_{it_{ij}})(\ddot{x}_{it_{ij}})'(\beta - \hat{\beta}) \\
&+ 2\ddot{u}_{it_{ij}}(\ddot{x}_{it_{ij}})'(\beta - \hat{\beta})
\end{aligned}$$

one has

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i}\sum_{j=1}^{n_i}\tilde{u}_{it_{ij}}^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i}\sum_{j=1}^{n_i}\ddot{u}_{it_{ij}}^2 \\
&+ \frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i}\sum_{j=1}^{n_i}(\beta-\hat{\beta})'(\ddot{x}_{it_{ij}})(\ddot{x}_{it_{ij}})'(\beta-\hat{\beta}) \\
&+ \frac{1}{N}\sum_{i=1}^{N}\frac{2}{n_i}\sum_{j=1}^{n_i}\ddot{u}_{it_{ij}}(\ddot{x}_{it_{ij}})'(\beta-\hat{\beta})
\end{aligned}$$

Following the same arguments as for the proofs of (37) and (38), we show that the second and the third term of the right hand side of the equality converge to 0 as $N \to \infty$. Using the same argument as in the proof of (36) and the equality (31), we get, as $N \to \infty$,

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i}\sum_{j=1}^{n_i}\tilde{u}_{it_{ij}}^2 - \sigma_u^2\left(1-\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}\right) \to 0, \tag{40}$$

the convergence holding in probability.

C.3 Reconciliation

Let us now write the result of section C.1 with obvious notations :

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i+1}\sum_{j=2}^{n_i}(\tilde{u}_{it_{ij}}-\tilde{u}_{it_{ij-1}})^2\mathbb{1}_{t_{ij}-t_{ij-1}=1}-\frac{(1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2}{N}\sum_{i=1}^{N}\frac{K_i}{K_i+1} = u_N - v_N \to 0$$

And the result of section C.2 :

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i}\sum_{j=1}^{n_i}\tilde{u}_{it_{ij}}^2 - \sigma_u^2\left(1-\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\rho^{|t_{ij}-t_{ik}|}\right) = w_N - x_N \xrightarrow{\mathbb{P}} 0$$

Writing between braces differences that converge towards zero, we notice :

$$\frac{u_N}{w_N} = \frac{v_N}{x_N}\left(1+\frac{\{u_N-v_N\}}{v_N}-\frac{\{w_N-x_N\}}{x_N+\{w_N-x_N\}}-\frac{\{u_N-v_N\}\{w_N-x_N\}}{v_N(x_N+\{w_N-x_N\})}\right)$$

We use the following lemma :

Lemma 2 : *There exists three numbers strictly positive K_1, K_2, and N_0 such as for all $N \geq N_0$:*

$$\textit{such as : } K_1 \leq x_N$$

$$\textit{and such as : } K_1 \leq v_N \leq K_2$$

From this lemma, we get immediately :

$$\frac{u_N}{w_N} - \frac{v_N}{x_N} \xrightarrow{\mathbb{P}} 0$$

That is :

$$\rho_d - \left(1 - \frac{(1-\rho)\sum_{i=1}^{N} \frac{K_i}{1+K_i}}{N - \sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} \rho^{|t_{ij}-t_{ik}|}}\right) = \rho_d - g_N(\rho) \xrightarrow{\mathbb{P}} 0 \tag{41}$$

Note also that from Annex (B), one has :

$$g_N(\rho) - \mathbb{E}(\rho_d) \xrightarrow{\mathbb{P}} 0$$

Hence, we can even conclude :

$$\rho_d - \mathbb{E}(\rho_d) \xrightarrow{\mathbb{P}} 0 \tag{42}$$

C.4 Justification of Assumption 2 and demonstration of Lemma 2

We recall Assumption 2 and Lemma 2.

Assumption 2 : Let K denote the number of regressors. Then the $K \times K$ semi-definite positive matrix

$$\frac{1}{N}\sum_{i=1}^{N} \frac{1}{K_i+1} \sum_{j=2}^{n_i} (\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})(\ddot{x}_{it_{ij}} - \ddot{x}_{it_{ij-1}})' \mathbb{1}_{t_{ij}-t_{ij-1}=1}$$

is uniformly bounded in N. That is, if M_N denotes this matrix, then there exists $C > 0$ such that for all $N \geq 0$,

$$||M_N|| \leq C,$$

where $||.||$ denotes some norm over the $K \times K$ matrix space.

This assumption is reasonnable because it is fair to assume that the vector of

covariables x_{it} is bounded. For instance, in a wage equation model, x_{it} is typically made of variables such as experience, age, level of education, spell of unemployment, which obviously take a finite number of values.

Lemma 2 : *There exists three numbers strictly positive K_1, K_2 and N_0 such as for all $N \geq N_0$:*

$$\textit{such as : } K_1 \leq x_N = \sigma_u^2 \left(1 - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} \rho^{|t_{ij} - t_{ik}|} \right)$$

$$\textit{and such as : } K_1 \leq v_N = \frac{(1-\rho)^2 \sigma_u^2 + \sigma_\varepsilon^2}{N} \sum_{i=1}^{N} \frac{K_i}{K_i + 1} \leq K_2$$

The existence of K_1 and K_2 in the second equation is obvious. It is sufficient to choose $K_1 = \frac{(1-\rho)^2 \sigma_u^2 + \sigma_\varepsilon^2}{2}$ as we restricted ourselves to individuals with $K_i \geq 1$, and $K_2 = (1-\rho)^2 \sigma_u^2 + \sigma_\varepsilon^2$.
For the first equation, we notice

$$\frac{1}{N} \sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} \rho^{|t_{ij} - t_{ik}|} \leq \frac{1}{N} \sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} |\rho|^{|t_{ij} - t_{ik}|} \leq \frac{1}{N} \sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} |\rho|^{|j-k|}$$

Hence :

$$x_N = \sigma_u^2 \left(1 - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} \rho^{|t_{ij} - t_{ik}|} \right) \geq \sigma_u^2 \frac{1}{N} \sum_{i=1}^{N} h(|\rho|, n_i)$$

with :

$$h(r, n_i) = 1 - \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} r^{|j-k|}$$

and with :

$$\frac{\partial h(r, n_i)}{\partial r} = -\frac{1}{n_i^2} \sum_{|j-k| \geq 1}^{n_i} |j-k| r^{|j-k|-1} \leq 0$$

and $h(1, n_i) = 0$. Hence, for all n_i, for every $-1 < \rho < 1 : h(|\rho|, n_i) > h(1, n_i) = 0$.

The integer n_i takes only a finite number of values : those between 2 and T (defined here as the maximal length of the time dimension). For a given $|\rho| < 1$, let us write $M(\rho) = min_i(h(|\rho|, n_i)) = h(|\rho|, n_I) > h(1, n_I) = 0$.
Then, for every $-1 < \rho < 1 : x_N \geq \sigma_u^2 M(\rho) > 0$.

We may choose $K_1 = min(\frac{(1-\rho)^2\sigma_u^2+\sigma_\varepsilon^2}{2}, \sigma_u^2 M(\rho)) > 0$.

D Lemma 1 : unicity of the ρ_{BFN} estimator

We want to prove :

Lemma 1 : $g_N : r \to 1 - \frac{(1-r)\sum_{i=1}^{N} \frac{K_i}{1+K_i}}{N - \sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|}}$ establishes a bijection from $[0,1]$ to $[1 - \frac{\sum_i^N \frac{K_i}{1+K_i}}{N - \sum_i^N \frac{1}{n_i}}, g(1)]$.

It is clear that the denominator of the above function g_N is not nul over [0-1[; hence g_N is derivable. One derives it. The derivative has the same sign as :

$$h(r) = N - \sum_i^N \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} r^{|t_{i,j}-t_{i,k}|} - (1-r) \sum_i^N \frac{1}{n_i^2} \sum_{|t_{i,j}-t_{i,k}|\geq 1}^{n_i} |t_{i,j} - t_{i,k}| r^{|t_{i,j}-t_{i,k}|-1}$$

with :

$$h(1) = 0$$

Its derivative is negative over $[0,1]$, and even strictly negative over $[0,1[$ (as soon as there is at least one individual with at least three observations, which we assume from now on) :

$$h'(r) = -(1-r) \sum_i^N \frac{1}{n_i^2} \sum_{|t_{i,j}-t_{i,k}|\geq 2}^{n_i} |t_{i,j} - t_{i,k}|(|t_{i,j} - t_{i,k}| - 1) r^{|t_{i,j}-t_{i,k}|-2} < 0$$

Hence, h is strictly decreasing over $[0;1]$; h is always positive ; g' is always positive ; g is strictly increasing and goes from g(0) to g(1).

Eventually :

$$g_N(0) = 1 - \frac{\sum_i^N \frac{K_i}{1+K_i}}{N - \sum_i^N \frac{1}{n_i}}$$

Note also that **in the balanced case** :

$$g_N(0) = f(0) = 0$$

$$g_N(1) = f(1) = 1 - \frac{1}{1 + \frac{T-2}{3}}$$

Moreover, Annex E shows that p_N (see Formula (45)) is bounded on $[0,1]$. It follows, from Formula (18), that g_N is also capped. As g_N is monotonous and continuous on $[0,1[$, it may thus be continuously extended on $[0,1]$.

Besides, note eventually that g_N is C^1 over $[0,1[$ and that $\lim_{r\to 1} g_N^{'}(r)$ exists (see Annex F). Hence, g_N is C^1 over $[0,1]$.

Hence, there exists a continuous $(g_N)^{-1}$ over the image of g_N. It also quickly follows that (g_N^{-1}) is C^1 over $[(g_N)(0),(g_N(1)]$.

Indeed, with the notations introduced above :

$$g'_N(r) = (\sum_i^N \frac{K_i}{1+K_i}) \frac{(h(r)}{(N - \sum_{i=1}^N \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|})^2}$$

As both the numerator and the denominator are strictly positive on $[0,1[$ (see above ; a sufficient and necessary condition for h to be strictly positive is that there is at least one individual with at least three observations, which we assumed to be the case) : for all $0 \leq r < 1 : g'_N(r) > 0$.

For all $0 \leq r < 1$, one gets :

$$(g_N^{-1})'(g_N(r)) = \frac{1}{g'_N(r)}$$

Hence : g_N^{-1} is C^1 over the interior of the image of g_N.

Note that the above demonstration also shows that : $(g_N^{-1})'(g_N(0)) > 0$.

Note also that Annex F shows that :

$$g'_N(1) = A_N \frac{\frac{1}{N} \sum_{i=1}^N \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} \frac{|t_{ij}-t_{ik}|(|t_{ij}-t_{ik}|-1)}{2}}{(\frac{1}{N} \sum_{i=1}^N \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} |t_{ij}-t_{ik}|)^2}$$

This latter term is strictly positive as soon as there is at least one observation with at least three observations. In those cases, we have also that $(g_N^{-1})'(g_N(1)) > 0$.

E Lemma 3 : Capping $p_N(r)$

Lemma 3 : for $m \geq 3$, for all $0 < r < 1$: $|p(r)| \leq \frac{1}{1-\frac{1}{m}}$ where :

$$p_N(r) = \frac{1-r}{\frac{N}{\sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k}^{n_i} r^{|t_{i,j}-t_{i,k}|}} - 1}$$

For $0 < r < 1$:

$$\frac{1}{n_i^2} \sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|} \leq \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} r^{|j-k|} = \frac{1+r}{(1-r)n_i} - \frac{2r}{(1-r)n_i} \frac{1-r^{n_i}}{(1-r)n_i} = l(r, n_i)$$

Hence :

$$0 \leq p_N(r) \leq \frac{1-r}{\frac{1}{\frac{1}{N}\sum_{i=1}^{N} l(r,n_i)} - 1} \tag{43}$$

$$= \frac{(1-r)\frac{1}{N}\sum_{i=1}^{N} l(r, n_i)}{\frac{1}{N}\sum_{i=1}^{N}(1 - l(r, n_i))} \tag{44}$$

We consider a given $M = n_i$. We apply in Annex G the Taylor's theorem, about the mean-value form of the remainder, at order 2, to $x \mapsto (1-x)^M$, where $1 - r = x$. One gets :

$$1 - l(r, M) \geq (1 - \frac{1}{M})(1 - r)$$

We notice :

$$l(r, M) \leq 1$$

$$1 - l(r, M) \geq (1 - \frac{1}{M})(1 - r)$$

Hence :

$$\frac{1}{N} \sum_{i=1}^{N} l(r, n_i) \leq 1$$

$$\frac{1}{N} \sum_{i=1}^{N} (1 - l(r, n_i)) \geq \frac{1}{N} \sum_{i=1}^{N} (1 - \frac{1}{n_i})(1 - r)$$

$$\frac{(1-r)\frac{1}{N}\sum_{i=1}^{N} l(r, n_i)}{\frac{1}{N}\sum_{i=1}^{N} 1 - l(r, n_i)} \leq \frac{1-r}{\frac{1}{N}\sum_{i=1}^{N}(1 - \frac{1}{n_i})(1-r)}$$

We can conclude that over [0,1] :

$$0 \leq p_N(r) = \frac{1-r}{\frac{N}{\sum_{i=1}^{N} \frac{1}{n_i^2} \sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|}} - 1} \leq \frac{1}{\frac{1}{N} \sum_{i=1}^{N} (1 - \frac{1}{n_i})} \leq \frac{1}{1 - \frac{1}{m}} \tag{45}$$

F Existence of a limit for g'_N in 1

We recall that :

$$g_N(r) = 1 - \frac{(1-r)A_N}{1 - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|}}$$

where : $\frac{1}{2} \leq A_N = \frac{1}{N}\sum_{i=1}^{N}\frac{K_i}{1+K_i} \leq 1$

$$g'_N(r) = A_N \frac{1 - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|} - (1-r)\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{|t_{ij}-t_{ik}|\geq 1}^{n_i} |t_{ij}-t_{ik}| r^{|t_{ij}-t_{ik}|-1}}{(1 - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i} r^{|t_{ij}-t_{ik}|})^2}$$

$$g'_N(r) = A_N \frac{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}(1 - r^{|t_{ij}-t_{ik}|} - (1-r)|t_{ij}-t_{ik}| r^{|t_{ij}-t_{ik}|-1})}{(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}(1 - r^{|t_{ij}-t_{ik}|}))^2}$$

$$g'_N(r) = A_N \frac{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\lambda(|t_{ij}-t_{ik}|, r)}{(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}(1 - r^{|t_{ij}-t_{ik}|}))^2} \tag{46}$$

where we note $\lambda(m,r) = 1 - r^m - (1-r)mr^{m-1}$, which satisfies $\lambda(0,r) = \lambda(1,r) = 0$. Let $m = |t_{ij} - t_{ik}|$. By using Taylor developments around 1, we get

$$\lambda(m,r) = \frac{m(m-1)}{2}(1-r)^2 + \ o((1-r)^2)$$

and

$$1 - r^m = m(1-r) + \ o(1-r)$$

which yields :

$$g'_N(r) = A_N \frac{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\frac{|t_{ij}-t_{ik}|(|t_{ij}-t_{ik}|-1)}{2} + \ o(1)}{(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}|t_{ij}-t_{ik}| + \ o(1))^2}$$

$$g'_N(r) \xrightarrow[r\to 1]{} A_N \frac{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}\frac{|t_{ij}-t_{ik}|(|t_{ij}-t_{ik}|-1)}{2}}{(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{n_i^2}\sum_{j,k=1}^{n_i}|t_{ij}-t_{ik}|)^2} \tag{47}$$

G A Taylor development

We consider : $l(r,n) = \frac{1}{n^2}\sum_{j,k=1}^{n} r^{|j-k|} = \frac{1+r}{(1-r)n} - \frac{2r}{(1-r)n}\frac{1-r^n}{(1-r)n}$

We consider a given $M = n_i$. We apply the Taylor's theorem, about the mean-value form of the remainder to $x->(1-x)^M$, where $1-r = x; r = 1-x$. It shows : There exists $0 < \zeta < 1$ such that :

$$r^M = (1-x)^M = 1 - Mx + \frac{M(M-1)}{2}x^2 - \frac{M(M-1)(M-2)}{6}(1-\zeta)^{M-3}x^3$$

$$1 - r^M = Mx - \frac{M(M-1)}{2}x^2 + \frac{M(M-1)(M-2)}{6}(1-\zeta)^{M-3}x^3$$

$$\frac{1-r^M}{(1-r)M} = \frac{1-(1-x)^M}{Mx} = 1 - \frac{(M-1)}{2}x + \frac{(M-1)(M-2)}{6}(1-\zeta)^{M-3}x^2 \quad (48)$$

$$\begin{aligned}
&1 + r - 2r\frac{1-r^M}{(1-r)M} \\
&= 1 + r - 2r + 2r\frac{(M-1)}{2}x - 2r\frac{(M-1)(M-2)}{6}(1-\zeta)^{M-3}x^2 \\
&= x + (1-x)(M-1)x - (1-x)\frac{(M-1)(M-2)}{3}(1-\zeta)^{M-3}x^2 \\
&= Mx - (M-1)x^2 - (1-x)\frac{(M-1)(M-2)}{3}(1-\zeta)^{M-3}x^2
\end{aligned}$$

Hence :

$$l(r,M) = \frac{1}{(1-r)M}(1 + r - 2r\frac{1-r^M}{(1-r)M}) = 1 - (M-1)x - (1-x)\frac{(M-1)(M-2)}{3M}(1-\zeta)^{M-3}x$$

It comes :

$$1 - l(r,M) \geq (1 - \frac{1}{M})(1-r) \quad (49)$$