Mathematical Logic (VII)

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1 Completeness

1.1 Henkin's Theorem

Recall that we fix a set Φ of S-formulas.

Definition 1.1. Let $t_1, t_2 \in T^S$. Then $t_1 \sim t_2$ if $\Phi \vdash t_1 \equiv t_2$. $\dashv$

Lemma 1.2. *(i)* $\sim$ *is an* ***equivalence*** *relation.*

(ii) $\sim$ *is a* ***congruence*** *relation. That is:*

- *For every* n*-ary function symbol* $f \in S$ *and* $2 \cdot n$ *S-terms* $t_1 \sim t_1', \ldots, t_n \sim t_n'$*, we have*

$$ft_1 \cdots t_n \sim ft_1' \cdots t_n'.$$

- *For every* n*-ary relation symbol* $R \in S$ *and* $2 \cdot n$ *S-terms* $t_1 \sim t_1', \ldots, t_n \sim t_n'$*, we have*

$$\Phi \vdash Rt_1 \cdots t_n \iff \Phi \vdash Rt_1' \cdots t_n'. \quad \dashv$$

Proof: By the equality rule and the substitution rule. $\square$

Now for every $t \in T^S$ we define

$$\bar{t} := \{t' \in T^S \mid t' \sim t\},$$

i.e., the equivalence class of t.

\mathcal{T}

Definition 1.3. The **term structure for** Φ, denoted by $\mathcal{T}^\Phi$, is defined as below.

(i) The universe is $T^\Phi := \{\bar{t} \mid t \in T^S\}$.

$t_1, t_2 \in T^S$

$-$either $\bar{t}_1 = \bar{t}_2$

$-$or $\bar{t}_1 \cap \bar{t}_2 = \phi$

(ii) For every n-ary relation symbol $R \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$(\bar{t}_1, \ldots, \bar{t}_n) \in R^{\mathcal{T}^\Phi} \quad \text{if} \quad \Phi \vdash Rt_1 \ldots t_n.$$

(iii) For every n-ary function symbol $f \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$f^{\mathcal{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) := \overline{ft_1 \cdots t_n}.$$

(iv) For every constant $c \in S$

$$c^{\mathcal{T}^\Phi} := \bar{c}.$$

This finishes the construction of $\mathcal{T}^\Phi$. $\dashv$

Using Lemma 1.2, in particular (ii), it is easy to verify that:

Lemma 1.4. $\mathcal{T}^\Phi$ *is well-defined.* $\dashv$

To complete the definition of an S-interpretation, we still need to provide an assignment of the variables $v_0, v_1, \ldots$ in $\mathcal{T}^\Phi$.

Definition 1.5. For every variable v_i we let

$$\beta^\Phi(v_i) := \bar{v}_i. \qquad \dashv$$

Finally we let

$$\mathfrak{I}^\Phi := \left(\mathcal{T}^\Phi, \beta^\Phi\right).$$

Lemma 1.6. *(i) For any* $t \in T^S$

$$\mathfrak{I}^\Phi(t) = \bar{t}.$$

(ii) For every ***atomic*** φ

$$\mathfrak{I}^\Phi \models \varphi \iff \Phi \vdash \varphi. \qquad \dashv$$

Proof: (i) We proceed by structural induction on t.

- $t = v_i$ is a variable. Then
$$\mathfrak{I}^\Phi(v_i) = \beta^\Phi(v_i) = \bar{v}_i.$$
- $t = c$ is a constant. Then
$$\mathfrak{I}^\Phi(c) = c^{\mathcal{T}^\Phi} = \bar{c}$$
- $t = ft_1 \cdots t_n$. Then
$$\begin{aligned} \mathfrak{I}^\Phi(ft_1 \cdots t_n) &= f^{\mathcal{T}^\Phi}(\mathfrak{I}^\Phi(t_1), \ldots, \mathfrak{I}^\Phi(t_n)) \\ &= f^{\mathcal{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) && \text{(by induction hypothesis)} \\ &= \overline{ft_1 \cdots t_n}. \end{aligned}$$

(ii) Recall that there are two types of atomic formulas. For the first type, let $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned} \mathfrak{I}^\Phi \models t_1 \equiv t_2 &\overset{\text{def}}{\iff} \mathfrak{I}^\Phi(t_1) = \mathfrak{I}^\Phi(t_2) \\ &\iff \bar{t}_1 = \bar{t}_2 && \text{(by (i))} \\ &\iff t_1 \sim t_2 \\ &\overset{\text{def}}{\iff} \Phi \vdash t_1 \equiv t_2. \end{aligned}$$

Second, let $\varphi = Rt_1 \cdots t_n$. We deduce

$$\begin{aligned} \mathfrak{I}^\Phi \models Rt_1 \cdots t_n &\iff \left(\mathfrak{I}^\Phi(t_1), \ldots, \mathfrak{I}^\Phi(t_n)\right) \in R^{\mathcal{T}^\Phi} \\ &\iff \left(\bar{t}_1, \ldots, \bar{t}_n\right) \in R^{\mathcal{T}^\Phi} && \text{(by (i))} \\ &\iff \Phi \vdash Rt_1 \cdots t_n. \end{aligned}$$

□

Lemma 1.7. *Let* φ *be an S-formula and* $x_1, \ldots, x_n$ *pairwise distinct variables. Then*

(i) $\mathfrak{I}^\Phi \models \exists x_1 \ldots \exists x_n \varphi$ *if and only if there are S-terms* $t_1, \ldots, t_n$ *such that*

$$\mathfrak{I}^\Phi \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n}.$$

(ii) $\mathfrak{I}^{\Phi} \models \forall x_1 \ldots \forall x_n \varphi$ *if and only if for all S-terms* $t_1, \ldots, t_n$ *we have*

$$\mathfrak{I}^{\Phi} \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n}.$$

Proof: We prove (i), then (ii) follows immediately.

$$\begin{aligned}
&\mathfrak{I}^{\Phi} \models \exists x_1 \ldots \exists x_n \varphi \\
&\quad \overset{\text{def}}{\iff} \mathfrak{I}^{\Phi} \frac{a_1 \ldots a_n}{x_1 \ldots x_n} \models \varphi \text{ for some } a_1, \ldots, a_n \in T^{\Phi}, \\
&\qquad \text{i.e., } \mathfrak{I}^{\Phi} \frac{\bar{t}_1 \ldots \bar{t}_n}{x_1 \ldots x_n} \models \varphi \text{ for some } t_1, \ldots, t_n \in T^{S}, \\
&\quad \iff \mathfrak{I}^{\Phi} \frac{\mathfrak{I}^{\Phi}(t_1) \ldots \mathfrak{I}^{\Phi}(t_n)}{x_1 \ldots x_n} \models \varphi \text{ for some } t_1, \ldots, t_n \in T^{S}, && \text{(by Lemma 1.6 (i))} \\
&\quad \iff \mathfrak{I}^{\Phi} \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n} \text{ for some } t_1, \ldots, t_n \in T^{S}, && \text{(by the Substitution Lemma).}
\end{aligned}$$

e.g. ① $\Phi = \{\neg x \equiv x\}$ ② $\Phi(\mathcal{I}) = \{\varphi \in L^S \mid \mathcal{I} \models \varphi\}$

$\mathcal{I}$ is a S-interpretation

Definition 1.8. (i) Φ is **negation complete** if for every S-formula φ

$$\Phi \vdash \varphi \quad \text{or} \quad \Phi \vdash \neg\varphi.$$

(ii) Φ contains witnesses if for every S-formula φ and every variable x there is a term $t \in T^S$ with

$$\Phi \vdash \left(\exists x \varphi \to \varphi \frac{t}{x}\right).$$

☆ $\exists x$ will traverse universe

$\mathcal{I}(t)$ might not cover all universe.

Lemma 1.9. *Assume that Φ is consistent, negation complete, and contains witnesses. Then for all S-formulas φ and ψ:*

(i) $\Phi \vdash \varphi$ *if and only if* $\Phi \nvdash \neg\varphi$.

(ii) $\Phi \vdash (\varphi \vee \psi)$ *if and only if* $\Phi \vdash \varphi$ *or* $\Phi \vdash \psi$.

(iii) $\Phi \vdash \exists x \varphi$ *if and only if there is a term* $t \in T^s$ *such that* $\Phi \vdash \varphi \frac{t}{x}$.

Proof: (i) Assume that $\Phi \vdash \varphi$. Since Φ is consistent, we conclude that $\Phi \nvdash \neg\varphi$. Conversely, if $\Phi \nvdash \neg\varphi$, then $\Phi \vdash \varphi$ by the negation completeness.

(ii) The direction from right to left is trivial by $\vee$-introduction in succedent. For the other direction, assume that $\Phi \vdash (\varphi \vee \psi)$ and $\Phi \nvdash \varphi$. By the negation completeness, $\Phi \vdash \neg\varphi$. Then for some finite $\Gamma \subseteq \Phi$ we have the following sequent proof.

$$\begin{array}{llrl}
 & \vdots & & \\
m. & \Gamma_1 & (\varphi \vee \psi) & \\
 & \vdots & & \\
n. & \Gamma_2 & \neg\varphi & \\
(n+1). & \Gamma_1 \ \Gamma_2 \ \varphi & \neg\varphi & \text{(antecedent by } n\text{)} \\
(n+2). & \Gamma_1 \ \Gamma_2 \ \varphi & \varphi & \text{(assumption)} \\
(n+3). & \Gamma_1 \ \Gamma_2 \ \varphi & \psi & \text{(modified contradiction by } n+1 \text{ and } n+2\text{)} \\
(n+4). & \Gamma_1 \ \Gamma_2 \ \psi & \psi & \text{(assumption)} \\
(n+5). & \Gamma_1 \ \Gamma_2 \ (\varphi \vee \psi) & \psi & (\vee\text{-introduction in antecedent}) \\
(n+6). & \Gamma_1 \ \Gamma_2 & \psi & \text{(chain rule by } m \text{ and } n+5\text{)}
\end{array}$$

(iii) Let $\Phi \vdash \exists x\varphi$ and Φ contain witnesses. Thus there is a term $t \in T^S$ such that

$$\Phi \vdash \left(\exists x\varphi \to \varphi\frac{t}{x}\right).$$

By Modus ponens[1], we conclude $\Phi \vdash \varphi\frac{t}{x}$. The converse is by the rule of the $\exists$-introduction in succedent. □

Theorem 1.10 (Henkin's Theorem). *Let $\Phi \subseteq L^S$ be consistent, negation complete, and contain witnesses. Then for every S-formula φ*

$$\mathfrak{I}^\Phi \models \varphi \quad \Longleftrightarrow \quad \Phi \vdash \varphi.$$

Proof: We proceed by induction on φ.

- φ is atomic. This is Lemma 1.6 (ii).
- $\varphi = \neg\psi$. Then

$$\begin{aligned} \mathfrak{I}^\Phi \models \neg\psi &\Longleftrightarrow \mathfrak{I}^\Phi \not\models \psi \\ &\Longleftrightarrow \Phi \nvdash \psi && \text{(by induction hypothesis)} \\ &\Longleftrightarrow \Phi \vdash \neg\psi && \text{(by Lemma 1.9 (i)).} \end{aligned}$$

- $\varphi = (\psi_1 \vee \psi_2)$. We deduce

$$\begin{aligned} \mathfrak{I}^\Phi \models (\psi_1 \vee \psi_2) &\Longleftrightarrow \mathfrak{I}^\Phi \models \psi_1 \text{ or } \mathfrak{I}^\Phi \models \psi_2 \\ &\Longleftrightarrow \Phi \vdash \psi_1 \text{ or } \Phi \vdash \psi_2 && \text{(by induction hypothesis)} \\ &\Longleftrightarrow \Phi \vdash (\psi_1 \vee \psi_2) && \text{(by Lemma 1.9 (ii)).} \end{aligned}$$

- $\varphi = \exists x\psi$.

$$\begin{aligned} \mathfrak{I}^\Phi \models \exists x\psi &\Longleftrightarrow \mathfrak{I}^\Phi \models \psi\frac{t}{x} \text{ for some } t \in T^S && \text{(by Lemma 1.7)} \\ &\Longleftrightarrow \Phi \vdash \psi\frac{t}{x} \text{ for some } t \in T^S && \text{(by induction hypothesis)} \\ &\Longleftrightarrow \Phi \vdash \exists x\psi && \text{(by Lemma 1.9 (iii)).} \end{aligned}$$

  Here, note that the length of $\psi\frac{t}{x}$ could be well larger than that $\exists x\psi$. Thus, our induction is on the so-called **connective rank** of ψ, denoted by $\mathrm{rk}(\varphi)$, which is defined as follows:

$$\mathrm{rk}(\varphi) := \begin{cases} 0 & \text{if } \varphi \text{ is atomic,} \\ 1 + \mathrm{rk}(\psi) & \text{if } \varphi = \neg\psi, \\ 1 + \mathrm{rk}(\psi_1) + \mathrm{rk}(\psi_2) & \text{if } \varphi = (\psi_1 \vee \psi_2), \\ 1 + \mathrm{rk}(\psi) & \text{if } \varphi = \exists x\psi. \end{cases}$$

□

Corollary 1.11. *Let $\Phi \subseteq L^S$ be consistent, negation complete, and contain witnesses. Then*

$$\mathfrak{I}^\Phi \models \Phi.$$

In particular, Φ is satisfiable.

[1] That is, if $\Phi \vdash \varphi$ and $\Phi \vdash \varphi \to \psi$, then $\Phi \vdash \psi$.

2 Exercises

Exercise 2.1. Assume that Φ is inconsistent. Please describe the structure $\mathfrak{T}^{\Phi}$.

Exercise 2.2. Again let $S := \{R\}$ with unary relation symbol R, and

$$\Phi := \{Rx \vee Ry\}.$$

Prove that:

- Φ is consistent.
- $\Phi \nvdash Rx$ and $\Phi \nvdash Ry$.
- $\mathfrak{I}^{\Phi} \not\models \Phi$.

Exercise 2.3. Let

$$\Phi := \left\{ v_0 \equiv t \;\middle|\; t \in T^S \right\} \cup \left\{ \exists v_0 \exists v_1 \neg v_0 \equiv v_1 \right\}.$$

Prove that Φ is consistent, but there is no consistent Ψ with $\Phi \subseteq \Psi \subseteq L^S$ which contains witnesses. $\dashv$

Exercise 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two S-structures. A mapping $h : A \to B$ is a **homomorphism** from $\mathcal{A}$ to $\mathcal{B}$ if the following properties hold.

1. For every n-ary relation symbol $R \in S$ and $a_1, \ldots, a_n \in A$ we have
$$(a_1, \ldots, a_n) \in R^{\mathcal{A}} \text{ implies } (h(a_1), \ldots, h(a_n)) \in R^{\mathcal{B}}.$$
2. For every n-ary function symbol $f \in S$ and $a_1, \ldots, a_n \in A$ we have
$$h(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(h(a_1), \ldots, h(a_n)).$$
3. For every constant $c \in S$
$$h(c^{\mathcal{A}}) = c^{\mathcal{B}}.$$

Now let $\Phi \subseteq L^S$ and $\mathcal{A}$ be an S-structure with $\mathcal{A} \models \Phi$. Prove that there is a homomorphism from the term model $\mathfrak{T}^{\Phi}$ to $\mathcal{A}$. $\dashv$