Mathematical Logic (III)

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1 The Semantics of First-order Logic

1.1 Structures and interpretations

We fix a symbol set S.

Definition 1.1. An S-**structure** is a pair $\mathcal{A} = (A, \mathfrak{a})$ which satisfies the following conditions.

1. $A \neq \emptyset$ is the **universe** of $\mathcal{A}$.
2. $\mathfrak{a}$ is a function defined on S such that:
   (a) Let $R \in S$ be an n-ary relation symbol. Then $\mathfrak{a}(R) \subseteq A^n$.
   (b) Let $f \in S$ be an n-ary function symbol. Then $\mathfrak{a}(f) : A^n \to A$.
   (c) $\mathfrak{a}(c) \in A$ for every constant $c \in S$.

For better readability, we write $R^{\mathcal{A}}$, $f^{\mathcal{A}}$, and $c^{\mathcal{A}}$, or even R^A, f^A, and c^A, instead of $\mathfrak{a}(R)$, $\mathfrak{a}(f)$, and $\mathfrak{a}(c)$. Thus for $S = \{R, f, c\}$ we might write an S-structure as

$$\mathcal{A} = \left(A, R^{\mathcal{A}}, f^{\mathcal{A}}, c^{\mathcal{A}}\right) = \left(A, R^A, f^A, c^A\right). \qquad \dashv$$

Examples 1.2.

1. For $S_{\mathrm{Ar}} := \{+, \cdot, 0, 1\}$ the S_{Ar}-structure

$$\mathcal{N} = \left(\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}\right)$$

   is the standard model of natural numbers with addition, multiplication, and constants 0 and 1.

2. For $S_{\mathrm{Ar}}^{<} := \{+, \cdot, 0, 1, <\}$ we have an $S_{\mathrm{Ar}}^{<}$-structure

$$\mathcal{N}^{<} = \left(\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}, <^{\mathbb{N}}\right),$$

   i.e., the standard model of $\mathbb{N}$ with the natural ordering $<$. $\dashv$

Definition 1.3. An **assignment** in an S-structure $\mathcal{A}$ is a mapping

$$\beta : \left\{v_i \mid i \in \mathbb{N}\right\} \to A. \qquad \dashv$$

Definition 1.4. An S-**interpretation** $\mathfrak{I}$ is a pair $(\mathcal{A}, \beta)$ where $\mathcal{A}$ is an S-structure and β is an assignment in $\mathcal{A}$. $\dashv$

Definition 1.5. Let β be an assignment in $\mathcal{A}$, $a \in A$, and x a variable. Then $\beta\frac{a}{x}$ is the assignment defined by

$$\beta\frac{a}{x}(y) := \begin{cases} a, & \text{if } y = x, \\ \beta(y), & \text{otherwise.} \end{cases}$$

Then, for the S-interpretation $\mathfrak{I} = (\mathcal{A}, \beta)$ we use $\mathfrak{I}\frac{a}{x}$ to denote the S-interpretation $\left(\mathcal{A}, \beta\frac{a}{x}\right)$. $\dashv$

Definition 1.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two S-structures. Their **direct product** $\mathcal{A} \times \mathcal{B}$ is the S-structure defined as follows.

- The universe of $\mathcal{A} \times \mathcal{B}$ is $A \times B$.
- For every n-ary relation symbol $R \in S$

$$R^{\mathcal{A}\times\mathcal{B}} := \big\{\big((a_1, b_1), \dots, (a_n, b_n)\big) \;\big|\; (a_1, \dots, a_n) \in R^{\mathcal{A}} \text{ and } (b_1, \dots, b_n) \in R^{\mathcal{B}}\big\}.$$

- For every n-ary function symbol $f \in S$

$$f^{\mathcal{A}\times\mathcal{B}}\big((a_1, b_1), \dots, (a_n, b_n)\big) := \big(f^{\mathcal{A}}(a_1, \dots, a_n), f^{\mathcal{B}}(b_1, \dots, b_n)\big).$$

- For every constant $c \in S$

$$c^{\mathcal{A}\times\mathcal{B}} := \big(c^{\mathcal{A}}, c^{\mathcal{B}}\big).$$ ⊣

1.2 The satisfaction relation $\mathfrak{I} \models \varphi$

We fix an S-interpretation $\mathfrak{I} = (\mathcal{A}, \beta)$.

Definition 1.7. For every S-term t we define its **interpretation** $\mathfrak{I}(t)$ by induction on the construction of t.

(a) $\mathfrak{I}(x) = \beta(x)$ for a variable x.

(b) $\mathfrak{I}(c) = c^{\mathcal{A}}$ for a constant $c \in S$.

(c) Let $f \in S$ be an n-ary function symbol and $t_1, \dots, t_n$ S-terms. Then

$$\mathfrak{I}\big(ft_1 \cdots t_n\big) = f^{\mathcal{A}}\big(\mathfrak{I}(t_1), \dots, \mathfrak{I}(t_n)\big).$$ ⊣

Example 1.8. Let $S := S_{\mathrm{Gr}} = \{\circ, e\}$ and $\mathfrak{I} := (\mathcal{A}, \beta)$ with $\mathcal{A} = (\mathbb{R}, +, 0)$, $\beta(v_0) = 2$, and $\beta(v_2) = 6$. Then

$$\begin{aligned}\mathfrak{I}\big(v_0 \circ (e \circ v_2)\big) &= \mathfrak{I}(v_0) + \mathfrak{I}(e \circ v_2)\\ &= 2 + \big(\mathfrak{I}(e) + \mathfrak{I}(v_2)\big) = 2 + (0 + 6) = 2 + 6 = 8.\end{aligned}$$ ⊣

Definition 1.9. Let φ be an S-formula. We define $\mathfrak{I} \models \varphi$ by induction on the construction of φ.

(a) $\mathfrak{I} \models t_1 \equiv t_2$ if $\mathfrak{I}(t_1) = \mathfrak{I}(t_2)$.

(b) $\mathfrak{I} \models Rt_1 \cdots t_n$ if $\big(\mathfrak{I}(t_1), \dots, \mathfrak{I}(t_n)\big) \in R^{\mathcal{A}}$.

(c) $\mathfrak{I} \models \neg\varphi$ if $\mathfrak{I} \not\models \varphi$ (i.e., it is **not** the case that $\mathfrak{I} \models \varphi$).

(d) $\mathfrak{I} \models (\varphi \wedge \psi)$ if $\mathfrak{I} \models \varphi$ and $\mathfrak{I} \models \psi$.

(e) $\mathfrak{I} \models (\varphi \vee \psi)$ if $\mathfrak{I} \models \varphi$ or $\mathfrak{I} \models \psi$.

(f) $\mathfrak{I} \models (\varphi \to \psi)$ if $\mathfrak{I} \models \varphi$ implies $\mathfrak{I} \models \psi$.

(g) $\mathfrak{I} \models (\varphi \leftrightarrow \psi)$ if $\big(\mathfrak{I} \models \varphi$ if and only if $\mathfrak{I} \models \psi\big)$.

(h) $\mathfrak{I} \models \forall x\varphi$ if for all $a \in A$ we have $\mathfrak{I}\frac{a}{x} \models \varphi$.

(i) $\mathfrak{I} \models \exists x\varphi$ if for some $a \in A$ we have $\mathfrak{I}\frac{a}{x} \models \varphi$.

If $\mathfrak{I} \models \varphi$, then $\mathfrak{I}$ is a **model** of φ, of $\mathfrak{I}$ **satisfies** φ.

Let Φ be a set of S-formulas. Then $\mathfrak{I} \models \Phi$ if $\mathfrak{I} \models \varphi$ for all $\varphi \in \Phi$. Similarly as above, we say that $\mathfrak{I}$ is a model of Φ, or $\mathfrak{I}$ satisfies Φ. ⊣

Example 1.10. Let $S := S_{\mathrm{Gr}}$ and $\mathfrak{I} := (\mathcal{A}, \beta)$ with $\mathcal{A} = (\mathbb{R}, +, 0)$ and $\beta(x) = 9$ for all variables x. Then

$$\begin{aligned}
\mathfrak{I} \models \forall v_0\, v_0 \circ e \equiv v_0 &\iff \text{for all } r \in \mathbb{R} \text{ we have } \mathfrak{I}\frac{r}{v_0} \models v_0 \circ e \equiv v_0, \\
&\iff \text{for all } r \in \mathbb{R} \text{ we have } r + 0 = r.
\end{aligned}$$ ⊣

Definition 1.11. Let Φ be a set of S-formulas and φ an S-formula. Then φ is a **consequence of** Φ, written $\Phi \models \varphi$, if for any interpretation $\mathfrak{I}$ it holds that $\mathfrak{I} \models \Phi$ implies $\mathfrak{I} \models \varphi$.

For simplicity, in case $\Phi = \{\psi\}$ we write $\psi \models \varphi$ instead of $\{\psi\} \models \varphi$. ⊣

Example 1.12. Let

$$\begin{aligned}
\Phi_{\mathrm{Gr}} :=\big\{&\forall v_0 \forall v_1 \forall v_2\, (v_0 \circ v_1) \circ v_2 \equiv v_0 \circ (v_1 \circ v_2), \\
&\forall v_0\, v_0 \circ e \equiv v_0, \forall v_0 \exists v_1\, v_0 \circ v_1 \equiv e\big\}.
\end{aligned}$$

Then it can be shown that

$$\Phi_{\mathrm{Gr}} \models \forall v_0\, e \circ v_0 \equiv v_0.$$

and

$$\Phi_{\mathrm{Gr}} \models \forall v_0 \exists v_1\, v_1 \circ v_0 \equiv e.$$ ⊣

Definition 1.13. An S-formula φ is **valid**, written $\models \varphi$, if $\emptyset \models \varphi$. Or equivalently, $\mathfrak{I} \models \varphi$ for any $\mathfrak{I}$. ⊣

Definition 1.14. An S-formula φ is **satisfiable**, if there exists an S-interpretation $\mathfrak{I}$ with $\mathfrak{I} \models \varphi$. A set Φ of S-formulas is satisfiable if there exists an S-interpretation $\mathfrak{I}$ such that $\mathfrak{I} \models \varphi$ for every $\varphi \in \Phi$. ⊣

The next lemma is essentially the method of **proof by contradiction**.

Lemma 1.15. *Let Φ be a set of S-formulas and φ an S-formula. Then $\Phi \models \varphi$ if and only if $\Phi \cup \{\neg\varphi\}$ is not satisfiable.* ⊣

Proof:

$$\begin{aligned}
\Phi \models \varphi &\iff \text{Every model of } \Phi \text{ is a model of } \varphi, \\
&\iff \text{there is no model } \mathfrak{I} \text{ with } \mathfrak{I} \models \Phi \text{ and } \mathfrak{I} \not\models \varphi, \\
&\iff \text{there is no model } \mathfrak{I} \text{ with } \mathfrak{I} \models \Phi \cup \{\neg\varphi\}, \\
&\iff \Phi \cup \{\neg\varphi\} \text{ is not satisfiable.}
\end{aligned}$$ □

Definition 1.16. Two S-formulas φ and ψ are **logic equivalent** if $\varphi \models \psi$ and $\psi \models \varphi$. ⊣

Example 1.17. Let φ be an S-formula. We define a logic equivalent φ^* which does not contain the logic symbols $\wedge, \rightarrow, \leftrightarrow, \forall$.

$$\begin{aligned}
\varphi^* &:= \varphi \qquad \text{if } \varphi \text{ is atomic,} \\
(\neg\varphi)^* &:= \neg\varphi^*, \\
(\varphi \wedge \psi)^* &:= \neg(\neg\varphi^* \vee \neg\psi^*), \\
(\varphi \vee \psi)^* &:= (\varphi^* \vee \psi^*), \\
(\varphi \rightarrow \psi)^* &:= (\neg\varphi^* \vee \psi^*), \\
(\varphi \leftrightarrow \psi)^* &:= \neg(\varphi^* \vee \psi^*) \vee \neg(\neg\varphi^* \vee \neg\psi^*), \\
(\forall x \varphi)^* &:= \neg\exists x \neg\varphi^*, \\
(\exists x \varphi)^* &:= \exists x \varphi^*.
\end{aligned}$$

Thus, it suffices to consider $\neg, \vee, \exists$ as the only logic symbols in any given φ. ⊣

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Lemma 1.18 (The Coincidence Lemma). *For* $i \in \{1,2\}$ *let* $\mathfrak{I}_i = (\mathcal{A}_i, \beta_i)$ *be an* S_i*-interpretation such that* $A_1 = A_2$ *and every symbol in* $S := S_1 \cap S_2$ *has the same interpretation in* $\mathcal{A}_1$ *and* $\mathcal{A}_2$.

(a) *Let* t *be an* S*-term (thus also an* S_1*-term and an* S_2*-term). Assume further that* $\beta_1(x) = \beta_2(x)$ *for every variable* $x \in \mathrm{var}(t)$. *Then* $\mathfrak{I}_1(t) = \mathfrak{I}_2(t)$.

(b) *Let* φ *be an* S*-formula where* $\beta_1(x) = \beta_2(x)$ *for every* $x \in \mathrm{free}(\varphi)$. *Then*

$$\mathfrak{I}_1 \models \varphi \quad \Longleftrightarrow \quad \mathfrak{I}_2 \models \varphi.$$

$\dashv$

Proof: (a) We prove by induction on t.

- $t = x$. Then $\mathfrak{I}_1(x) = \beta_1(x) = \beta_2(x) = \mathfrak{I}_2(x)$.
- $t = c$. We deduce $\mathfrak{I}_1(c) = c^{\mathcal{A}_1} = c^{\mathcal{A}_2} = \mathfrak{I}_2(x)$.
- $t = ft_1 \cdots t_n$. It holds that

$$\begin{aligned}
\mathfrak{I}_1(ft_1 \cdots t_n) &= f^{\mathcal{A}_1}(\mathfrak{I}_1(t_1), \dots, \mathfrak{I}_2(t_n)) \\
&= f^{\mathcal{A}_2}(\mathfrak{I}_1(t_1), \dots, \mathfrak{I}_1(t_n)) \\
&= f^{\mathcal{A}_2}(\mathfrak{I}_2(t_1), \dots, \mathfrak{I}_2(t_n)) \\
&= \mathfrak{I}_2(ft_1 \cdots t_n).
\end{aligned}$$

(b) The induction proof is on the structure of φ.

- $\varphi = t_1 \equiv t_2$. We have

$$\begin{aligned}
\mathfrak{I}_1 \models t_1 \equiv t_2 &\Longleftrightarrow \mathfrak{I}_1(t_1) = \mathfrak{I}_1(t_2) \\
&\Longleftrightarrow \mathfrak{I}_2(t_1) = \mathfrak{I}_2(t_2) && \text{(by (a))} \\
&\Longleftrightarrow \mathfrak{I}_2 \models t_1 \equiv t_2.
\end{aligned}$$

- $\varphi = Rt_1 \cdots t_n$. Then

$$\begin{aligned}
\mathfrak{I}_1 \models Rt_1 \cdots t_n &\Longleftrightarrow (\mathfrak{I}_1(t_1), \dots, \mathfrak{I}_1(t_n)) \in R^{\mathcal{A}_1} \\
&\Longleftrightarrow (\mathfrak{I}_1(t_1), \dots, \mathfrak{I}_1(t_n)) \in R^{\mathcal{A}_2} \\
&\Longleftrightarrow (\mathfrak{I}_2(t_1), \dots, \mathfrak{I}_2(t_n)) \in R^{\mathcal{A}_2} \\
&\Longleftrightarrow \mathfrak{I}_2 \models Rt_1 \cdots t_n.
\end{aligned}$$

- $\varphi = \neg\psi$. We conclude

$$\mathfrak{I}_1 \models \neg\psi \Longleftrightarrow \mathfrak{I}_1 \not\models \psi \Longleftrightarrow \mathfrak{I}_2 \not\models \psi \Longleftrightarrow \mathfrak{I}_2 \models \neg\psi.$$

- $\varphi = (\psi \vee \chi)$.

$$\begin{aligned}
\mathfrak{I}_1 \models (\psi \vee \chi) &\Longleftrightarrow \mathfrak{I}_1 \models \psi \text{ or } \mathfrak{I}_1 \models \chi \\
&\Longleftrightarrow \mathfrak{I}_2 \models \psi \text{ or } \mathfrak{I}_2 \models \chi \\
&\Longleftrightarrow \mathfrak{I}_2 \models (\psi \vee \chi).
\end{aligned}$$

- $\varphi = \exists x\psi$.

$$\begin{aligned}
\mathfrak{I}_1 \models \exists x\psi &\Longleftrightarrow \text{for some } a \in A_1 \text{ we have } \mathfrak{I}_1\frac{a}{x} \models \psi \\
&\Longleftrightarrow \text{for some } a \in A_1 \text{ we have } \mathfrak{I}_2\frac{a}{x} \models \psi \\
&\qquad\qquad \left(\text{by induction hypothesis on } \mathfrak{I}_1\frac{a}{x}, \mathfrak{I}_2\frac{a}{x}, \text{ and } \psi\right) \\
&\Longleftrightarrow \mathfrak{I}_2 \models \exists x\psi.
\end{aligned}$$

□

Remark 1.19. Let $\varphi \in L^S_n$, i.e., φ is an S-formula with $\mathrm{free}(\varphi) \subseteq \{v_0, \dots, v_{n-1}\}$. By the coincidence lemma whether $\mathfrak{I} = (\mathcal{A}, \beta) \models \varphi$ is completely determined by $\mathcal{A}$ and $\beta(v_0), \dots, \beta(v_{n-1})$. So in case $\mathfrak{I} \models \varphi$ we can write

$$\mathcal{A} \models \varphi[a_0, \dots, a_{n-1}]$$

where $a_i := \beta(v_i)$ for $0 \leqslant i < n$. In particular, if φ is an S-sentence, i.e., $\varphi \in L^S_0$, then $\mathcal{A} \models \varphi$ is well-defined.

Similarly, we write

$$t^{\mathcal{A}}[a_0, \dots, a_{n-1}]$$

instead of $\mathfrak{I}(t)$. ⊣

2 Exercises

Example 2.1. Using induction, prove that in every formula we have the same numbers of symbols (and).

Exercise 2.2. Prove that for every φ we have $\varphi \models \varphi^*$ and $\varphi^* \models \varphi$.

Exercise 2.3. Let φ, ψ, and χ be S-formulas. Prove that:

(a) $(\varphi \vee \psi) \models \chi$ if and only if $\varphi \models \chi$ and $\psi \models \chi$.

(b) $\models \varphi \to \psi$ if and only if $\varphi \models \psi$.