Mathematical Logic (IV)

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1 The Semantics of First-order Logic

1.1 Isomorphisms

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two S-structures.

(a) A mapping $\pi : A \to B$ is an **isomorphism from** $\mathcal{A}$ **to** $\mathcal{B}$ (in short $\pi : \mathcal{A} \cong \mathcal{B}$) if the following conditions are satisfied.

   (i) π is a bijection.

   (ii) For any n-ary relation symbol $R \in S$ and $a_0, \ldots, a_{n-1} \in A$

$$(a_0, \ldots, a_{n-1}) \in R^{\mathcal{A}} \iff \big(\pi(a_0), \ldots, \pi(a_{n-1})\big) \in R^{\mathcal{B}}.$$

   (iii) For any n-ary function symbol $f \in S$ and $a_0, \ldots, a_{n-1} \in A$

$$\pi(f^{\mathcal{A}}(a_0, \ldots, a_{n-1})) = f^{\mathcal{B}}(\pi(a_0), \ldots, \pi(a_{n-1})).$$

   (iv) For any constant $c \in S$

$$\pi(c^{\mathcal{A}}) = c^{\mathcal{B}}.$$

(b) $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, written $\mathcal{A} \cong \mathcal{B}$, if there is an isomorphism $\pi : \mathcal{A} \to \mathcal{B}$. ⊣

Observe that the above definition is not symmetric. However we can easily show:

Lemma 1.2. *$\cong$ is an equivalence relation. That is, for all S-structures $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$*

1. *$\mathcal{A} \cong \mathcal{A}$;*
2. *$\mathcal{A} \cong \mathcal{B}$ implies $\mathcal{B} \cong \mathcal{A}$;*
3. *if $\mathcal{A} \cong \mathcal{B}$ and $\mathcal{B} \cong \mathcal{C}$, then $\mathcal{A} \cong \mathcal{C}$.* ⊣

Lemma 1.3 (The Isomorphism Lemma)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two isomorphic S-structures. Then for every S-sentence φ*

$$\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi.$$

⊣

Proof: Let β be an assignment in $\mathcal{A}$. By the coincidence lemma, it suffices to show that there is an assignment β' in $\mathcal{B}$ such that

$$(\mathcal{A}, \beta) \models \varphi \iff (\mathcal{B}, \beta') \models \varphi, \tag{1}$$

where φ is an S-sentence.

Let $\pi : \mathcal{A} \cong \mathcal{B}$ and we define an assignment β^{π} in $\mathcal{B}$ by

$$\beta^{\pi}(x) := \pi(\beta(x))$$

for any variable x. Then we prove for any S-**formula** φ

$$(\mathcal{A},\beta)\models\varphi \quad\Longleftrightarrow\quad (\mathcal{B},\beta^{\pi})\models\varphi, \tag{2}$$

which certainly generalizes (1). To simplify notation, let $\mathfrak{I}:=(\mathcal{A},\beta)$ and $\mathfrak{I}^{\pi}:=\big(\mathcal{B},\beta^{\pi}\big)$. First, it is routine to verify that for every S-term t

$$\pi(\mathfrak{I}(t))=\mathfrak{I}^{\pi}(t). \tag{3}$$

Then we prove (2) by induction on the construction of S-formula φ.

- $\varphi=t_1\equiv t_2$. Then

$$\begin{aligned}\mathfrak{I}\models t_1\equiv t_2 &\Longleftrightarrow \mathfrak{I}(t_1)=\mathfrak{I}(t_2)\\ &\Longleftrightarrow \pi(\mathfrak{I}(t_1))=\pi(\mathfrak{I}(t_2)) && \text{(since π is an injection)}\\ &\Longleftrightarrow \mathfrak{I}^{\pi}(t_1)=\mathfrak{I}^{\pi}(t_2) && \text{(by (3))}\\ &\Longleftrightarrow \mathfrak{I}^{\pi}\models t_1\equiv t_2.\end{aligned}$$

- $\varphi=Rt_1\cdots t_n$.

$$\begin{aligned}\mathfrak{I}\models Rt_1\cdots t_n &\Longleftrightarrow \big(\mathfrak{I}(t_1),\dots,\mathfrak{I}(t_n)\big)\in R^{\mathcal{A}}\\ &\Longleftrightarrow \big(\pi(\mathfrak{I}(t_1)),\dots,\pi(\mathfrak{I}(t_n))\big)\in R^{\mathcal{B}}\\ &\Longleftrightarrow \big(\mathfrak{I}^{\pi}(t_1),\dots,\mathfrak{I}^{\pi}(t_n)\big)\in R^{\mathcal{B}} && \text{(by (3))}\\ &\Longleftrightarrow \mathfrak{I}^{\pi}\models Rt_1\cdots t_n.\end{aligned}$$

- $\varphi=\neg\psi$. It follows that $\mathfrak{I}\models\neg\psi \iff \mathfrak{I}\not\models\psi \iff \mathfrak{I}^{\pi}\not\models \iff \mathfrak{I}^{\pi}\models\neg\psi$.
- $\varphi=\psi\vee\chi$. The inductive argument is similar to the above $\neg\psi$.
- $\varphi=\exists x\psi$. This is again the most complicated case.

$$\begin{aligned}\mathfrak{I}\models\exists x\psi &\Longleftrightarrow \text{there exists an } a\in A \text{ such that } \mathfrak{I}\frac{a}{x}=\left(\mathcal{A},\beta\frac{a}{x}\right)\models\psi\\ &\Longleftrightarrow \text{there exists an } a\in A \text{ such that } \left(\mathfrak{I}\frac{a}{x}\right)^{\pi}=\left(\mathcal{A},\beta\frac{a}{x}\right)^{\pi}\models\psi,\\ &\qquad\qquad \left(\text{by induction hypothesis on } \mathfrak{I}\frac{a}{x},\ \left(\mathfrak{I}\frac{a}{x}\right)^{\pi}\text{, and } \psi\right)\\ &\qquad\qquad \text{that is, there exists an } a\in A \text{ such that } \left(\mathcal{B},\beta^{\pi}\frac{\pi(a)}{x}\right)\models\psi\\ &\Longleftrightarrow \text{there exists a } b\in B \text{ such that } \left(\mathcal{B},\beta^{\pi}\frac{b}{x}\right)\models\psi \qquad \text{(since π is surjective)}\\ &\qquad\qquad \text{i.e., there exists a } b\in B \text{ with } \mathfrak{I}^{\pi}\frac{b}{x}=\big(\mathcal{B},\beta^{\pi}\big)\frac{b}{x}\models\psi\\ &\Longleftrightarrow \mathfrak{I}^{\pi}\models\exists x\psi.\end{aligned}$$

This finishes the proof. □

Corollary 1.4. *Let $\pi:\mathcal{A}\cong\mathcal{B}$ and $\varphi\in L^S_n$. Then for every $a_0,\dots,a_{n-1}$*

$$\mathcal{A}\models\varphi[a_0,\dots,a_{n-1}] \quad\Longleftrightarrow\quad \mathcal{B}\models\varphi\big[\pi(a_0),\dots,\pi(a_{n-1})\big]$$

⊣

1.2 Substitution

In mathematics, when writing $f(y+10)$ we plug the value of $y+10$ into $f(x)$. We will do the same for $\varphi(x)$ where we want to substitute x by a term t. This is not completely trivial, e.g.,

$$\varphi(x) = \exists z\, z+z \equiv x \quad \text{and} \quad t = x+z.$$

It is obviously wrong for

$$\exists z\, z+z \equiv x+z.$$

Definition 1.5. Let t be an S-term, $x_0, \dots, x_r$ variables, and $t_0, \dots, t_r$ S-terms. Then the term

$$t\frac{t_0, \dots, t_r}{x_0, \dots, x_r}$$

$x_i \xrightarrow{\text{换成}} t_i$

is defined inductively as follows.

(a) Let $t = x$ be a variable. Then

$$t\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \begin{cases} t_i & \text{if } x = x_i \text{ for some } 0 \leqslant i \leqslant r \\ x & \text{otherwise.} \end{cases}$$

(b) For a constant $t = c$

$$c\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := c.$$

(c) For a function term

$$ft'_1 \dots t'_n \frac{t_0, \dots, t_r}{x_0, \dots, x_r} := ft'_1 \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \dots t'_n \frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

$\dashv$

Definition 1.6. Let φ be an S-formula, $x_0, \dots, x_r$ variables, and $t_0, \dots, t_r$ S-terms. We define

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r}$$

inductively as follow.

(a) Assume $\varphi = t'_1 \equiv t'_2$. Then

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := t'_1\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \equiv t'_2\frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(b) Let $\varphi = Rt'_1 \dots t'_n$. We set

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := Rt'_1\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \dots t'_n\frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(c) For $\varphi = \neg\psi$

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \neg\psi\frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(d) For $\varphi = (\psi_1 \vee \psi_2)$

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \left(\psi_1\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \vee \psi_2\frac{t_0, \dots, t_r}{x_0, \dots, x_r}\right).$$

(e) Assume $\varphi = \exists x \psi$. Let $x_{i_1}, \ldots, x_{i_s}$ $(i_1 < \ldots < i_s)$ be the variables x_i in $x_0, \ldots, x_r$ with $x_i \in \text{free}(\exists x \psi)$ and $x_i \neq t_i$. In particular, $x \neq x_{i_1}, \ldots, x \neq x_{i_s}$. Then

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$$\varphi \frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} := \exists u \left[\psi \frac{t_{i_1}, \ldots, t_{i_s}, u}{x_{i_1}, \ldots, x_{i_s}, x} \right],$$

where $u = x$ if x does not occur in $t_{i_1}, \ldots, t_{t_s}$; otherwise u is the first variable in $\{v_0, v_1, v_2, \ldots\}$ which does not occur in $\psi, t_{i_1}, \ldots, t_{i_s}$. ⊣

$v_0 \notin$ free$(\quad)$

Examples 1.7. 1.

$$[Pv_0 f v_1 v_2] \frac{v_2, v_0, v_1}{v_1, v_2, v_3} = Pv_0 f v_2 v_0.$$

2.

$$[\exists v_0 Pv_0 f v_1 v_2] \frac{v_4, f v_1 v_1}{v_0, v_2} = \exists v_0 \left[Pv_0 f v_1 v_2 \frac{f v_1 v_1, v_0}{v_2, v_0} \right] = \exists v_0 \; Pv_0 f v_1 f v_1 v_1.$$

3.

$$[\exists v_0 \; Pv_0 f v_1 v_2] \frac{v_0, v_2, v_4}{v_1, v_2, v_0} = \exists v_3 \left[Pv_0 f v_1 v_2 \frac{v_0, v_3}{v_1, v_0} \right] = \exists v_3 \; Pv_3 f v_0 v_2.$$ ⊣

Definition 1.8. Let β be an assignment in $\mathcal{A}$ and $a_0, \ldots, a_r \in A$. Then

$$\beta \frac{a_0, \ldots, a_r}{x_0, \ldots, x_r}$$

is an assignment in $\mathcal{A}$ defined by

$$\beta \frac{a_0, \ldots, a_r}{x_0, \ldots, x_r} := \begin{cases} a_i & \text{if } y = x_i \text{ for } 0 \leqslant i \leqslant r \\ \beta(y) & \text{otherwise.} \end{cases}$$

For an S-interpretation $\mathcal{I} = (\mathcal{A}, \beta)$ we let

$$\mathcal{I} \frac{a_0, \ldots, a_r}{x_0, \ldots, x_r} := \left(\mathcal{A}, \beta \frac{a_0, \ldots, a_r}{x_0, \ldots, x_r} \right).$$ ⊣

Lemma 1.9 (The Substitution Lemma). *(a) For every S-term* t

$$\mathcal{I} \left(t \frac{t_0, \ldots, t_r}{x_0, \ldots x_r} \right) = \mathcal{I} \frac{\mathcal{I}(t_0), \ldots, \mathcal{I}(t_r)}{x_0, \ldots x_r} (t).$$

syntax sub — semantic sub

(b) For every S-formula φ

T&L. $f(x)=x^2$ $y=7$

$$\mathcal{I} \models \varphi \frac{t_0, \ldots, t_r}{x_0, \ldots x_r} \iff \mathcal{I} \frac{\mathcal{I}(t_0), \ldots, \mathcal{I}(t_r)}{x_0, \ldots x_r} \models \varphi.$$

if $f(y+3)$

① $f(y+3) = f(10) = 100$

② $f(y+3) = (y+3)^2 = 100$

Proof: (a) Assume $t = x$. If $x \neq x_i$ for all $0 \leqslant i \leqslant r$, then

$$t \frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} = x.$$

Therefore,

$$\mathcal{I} \left(t \frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} \right) = \mathcal{I}(x) = \mathcal{I} \frac{\mathcal{I}(t_0), \ldots, \mathcal{I}(t_r)}{x_0, \ldots, x_r} (x) = \mathcal{I} \frac{\mathcal{I}(t_0), \ldots, \mathcal{I}(t_r)}{x_0, \ldots, x_r} (t).$$

Otherwise, $x = x_i$ for some $0 \leqslant i \leqslant r$. Then $t\frac{t_0,\dots,t_r}{x_0,\dots,x_r} = t_i$. It follows that

$$\mathcal{I}\left(t\frac{t_0,\dots,t_r}{x_0,\dots,x_r}\right) = \mathcal{I}(t_i) = \mathcal{I}\frac{\mathcal{I}(t_0),\dots,\mathcal{I}(t_r)}{x_0,\dots,x_r}(x_i) = \mathcal{I}\frac{\mathcal{I}(t_0),\dots,\mathcal{I}(t_r)}{x_0,\dots,x_r}(t).$$

The other cases of t can be shown similarly.

(b) Assume that $\varphi = Rt'_1 \dots t'_n$. Then

$$\begin{aligned}
\mathcal{I} \models \varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r} &\iff \left(\mathcal{I}\left(t'_1\frac{t_0,\dots,t_r}{x_0,\dots,x_r}\right),\dots,\mathcal{I}\left(t'_n\frac{t_0,\dots,t_r}{x_0,\dots,x_r}\right)\right) \in R^{\mathcal{A}} \\
&\iff \left(\mathcal{I}\frac{\mathcal{I}(t_0),\dots,\mathcal{I}(t_r)}{x_0,\dots,x_r}(t'_1),\dots,\mathcal{I}\frac{\mathcal{I}(t_0),\dots,\mathcal{I}(t_r)}{x_0,\dots,x_r}(t'_n)\right) \in R^{\mathcal{A}} \qquad \text{(by (a))} \\
&\iff \mathcal{I}\frac{\mathcal{I}(t_0),\dots,\mathcal{I}(t_r)}{x_0,\dots,x_r} \models Rt'_1\dots t'_n \\
&\qquad \text{i.e., } \mathcal{I}\frac{\mathcal{I}(t_0),\dots,\mathcal{I}(t_r)}{x_0,\dots,x_r} \models \varphi.
\end{aligned}$$

For another case, let $\varphi = \exists x\psi$. Again, let $x_{i_1},\dots,x_{i_s}$ be the variables x_i with $x_i \in \mathrm{free}(\exists x\psi)$ and $x_i \neq t_i$. Choose u according to Definition 1.6 (e). In particular, u does not occur in $t_{i_1},\dots,t_{i_s}$. Then

$$\begin{aligned}
\mathcal{I} \models \varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r} &\iff \mathcal{I} \models \exists u\left[\psi\frac{t_{i_1},\dots,t_{i_s},u}{x_{i_1},\dots,x_{i_s},x}\right] \\
&\iff \text{there exists an } a \in A \text{ such that } \mathcal{I}\frac{a}{u} \models \psi\frac{t_{i_1},\dots,t_{i_s},u}{x_{i_1},\dots,x_{i_s},x} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathcal{I}\frac{a}{u}\right]\frac{\mathcal{I}\frac{a}{u}(t_{i_1}),\dots,\mathcal{I}\frac{a}{u}(t_{i_s}),\mathcal{I}\frac{a}{u}(u)}{x_{i_1},\dots,x_{i_s},x} \models \psi \\
&\qquad \text{(by induction hypothesis)} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathcal{I}\frac{a}{u}\right]\frac{\mathcal{I}(t_{i_1}),\dots,\mathcal{I}(t_{i_s}),a}{x_{i_1},\dots,x_{i_s},x} \models \psi \\
&\qquad \text{(by the coincidence lemma and that } u \text{ does not occur in } t_{i_1},\dots t_{i_s}) \\
&\iff \text{there exists an } a \in A \text{ such that } \mathcal{I}\frac{\mathcal{I}(t_{i_1}),\dots,\mathcal{I}(t_{i_s}),a}{x_{i_1},\dots,x_{i_s},x} \models \psi \\
&\qquad \text{(by (either } u = x \text{ or } u \text{ does not occur in } \psi) \text{ and the coincidence lemma)} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathcal{I}\frac{\mathcal{I}(t_{i_1}),\dots,\mathcal{I}(t_{i_s})}{x_{i_1},\dots,x_{i_s}}\right]\frac{a}{x} \models \psi \\
&\qquad \text{(since } x \neq x_{i_1},\dots,x \neq x_{i_s}) \\
&\iff \mathcal{I}\frac{\mathcal{I}(t_{i_1}),\dots,\mathcal{I}(t_{i_s})}{x_{i_1},\dots,x_{i_s}} \models \exists x\psi \\
&\iff \mathcal{I}\frac{\mathcal{I}(t_0),\dots,\mathcal{I}(t_r)}{x_0,\dots,x_r} \models \exists x\psi \\
&\qquad \text{(by } x_i \notin \mathrm{free}(\exists x\psi) \text{ or } x_i = t_i \text{ for } i \neq i_1,\dots,i \neq i_s). \qquad \square
\end{aligned}$$

coincidence between $\mathcal{I}$ and $\mathcal{I}\frac{a}{u}$

2 Exercises

Exercise 2.1. Prove Lemma 1.2.

Exercise 2.2. Let S be finite, i.e., containing finitely many relation symbols, function symbols, and constants. Prove that two **finite** structures $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if for any S-sentence φ

$$\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi.$$

Exercise 2.3. Let P be a binary relation symbol and f a binary function symbol. Set $x := v_0$, $y := v_1$, $u := v_2$, $v := v_3$, and $w := v_4$. Show that:

(a)

$$\exists x \exists y \big(Pxu \wedge Pyv\big) \frac{u, u, u}{x, y, v} = \exists x \exists y \big(Pxu \wedge Pyu\big).$$

(b)

$$\exists x \exists y \big(Pxu \wedge Pyv\big) \frac{v, fuv}{u, v} = \exists x \exists y \big(Pxv \wedge Pyfuv\big).$$

(c)

$$\exists x \exists y \big(Pxu \wedge Pyv\big) \frac{u, x, fuv}{x, u, v} = \exists w \exists y \big(Pwx \wedge Pyfuv\big).$$

(c)

$$\big[\forall x \exists y (Pxy \wedge Pxu) vee \exists u\ fuu \equiv x\big] \frac{x, fxy}{x, u} = \forall v \exists w (Pvw \wedge Pvfxy) \vee \exists ufuu \equiv x.$$