Mathematical Logic (VI)

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1 Sequent Calculus

1.1 Basic Rules

Antecedent

$$\frac{\Gamma \quad \varphi}{\Gamma' \quad \varphi}\ \Gamma \subseteq \Gamma'$$

The correctness is straightforward. Assume that $\Gamma \models \varphi$ and $\mathfrak{I} \models \Gamma'$. Since $\Gamma \subseteq \Gamma'$, we conclude $\mathfrak{I} \models \Gamma$ and thus $\mathfrak{I} \models \varphi$.

Assumption

$$\frac{}{\Gamma \quad \varphi}\ \varphi \in \Gamma$$

Case Analysis

$$\frac{\begin{array}{lll}\Gamma & \psi & \varphi \\ \Gamma & \neg\psi & \varphi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

Contradiction

$$\frac{\begin{array}{lll}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

$\vee$-introduction in antecedent

$$\frac{\begin{array}{rrr}\Gamma & \varphi & \chi \\ \Gamma & \psi & \chi\end{array}}{\begin{array}{lll}\Gamma & (\varphi \vee \psi) & \chi\end{array}}$$

$\vee$-introduction in succedent

$$\text{(a)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad (\varphi \vee \psi)} \qquad\qquad \text{(b)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad (\psi \vee \varphi)}$$

$\exists$-introduction in succedent

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad \exists x \varphi}$$

$\exists$-introduction in antecedent

$$\frac{\Gamma \quad \varphi\frac{y}{x} \quad \psi}{\Gamma \quad \exists x\varphi \quad \psi} \text{ if } y \notin \text{free}(\Gamma \cup \{\exists x\varphi, \psi\})$$

Equality

$$\frac{}{t \equiv t}$$

Substitution

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad t \equiv t' \quad \varphi\frac{t'}{x}}$$

1.2 Some Derived Rules

Example 1.1 (The law of excluded middle).

$$\begin{array}{lrrr}
1. & \varphi & \varphi & \text{(assumption)} \\
2. & \varphi & (\varphi \vee \neg\varphi) & (\vee\text{-introduction in succedent by 1)} \\
3. & \neg\varphi & \neg\varphi & \text{(assumption)} \\
4. & \neg\varphi & (\varphi \vee \neg\varphi) & (\vee\text{-introduction in succedent by 3)} \\
5. & & (\varphi \vee \neg\varphi) & \text{(case analysis by 2 and 4).}
\end{array}$$

Therefore $\vdash (\varphi \vee \neg\varphi)$. ⊣

Example 1.2 (The modified contradiction).

$$\frac{\begin{array}{ll}\Gamma & \psi \\ \Gamma & \neg\psi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

We argue as follows.

$$\begin{array}{lrrrr}
1. & & \Gamma & \psi & \text{(premise)} \\
2. & & \Gamma & \neg\psi & \text{(premise)} \\
3. & \Gamma & \neg\varphi & \psi & \text{(antecedent by 1)} \\
4. & \Gamma & \neg\varphi & \neg\psi & \text{(antecedent by 2)} \\
5. & & \Gamma & \varphi & \text{(contradiction by 3 and 4).}
\end{array}$$

⊣

Example 1.3 (The chain deduction).

$$\frac{\begin{array}{rrr} & \Gamma & \varphi \\ \Gamma & \varphi & \psi\end{array}}{\begin{array}{rrr} & \Gamma & \psi\end{array}}$$

We have the following deduction.

$$\begin{array}{lrrrr}
1. & & \Gamma & \varphi & \text{(premise)} \\
2. & \Gamma & \varphi & \psi & \text{(premise)} \\
3. & \Gamma & \neg\varphi & \varphi & \text{(antecedent by 1)} \\
4. & \Gamma & \neg\varphi & \neg\varphi & \text{(assumption)} \\
5. & \Gamma & \neg\varphi & \psi & \text{(modified contradiction by 3 and 4)} \\
6. & & \Gamma & \psi & \text{(case analysis by 2 and 5).}
\end{array}$$

⊣

Definition 1.4. Let Φ be a set of S-formulas and φ an S-formula. Then φ is **derivable from** Φ, denoted by $\Phi \vdash \varphi$, if there exists an $n \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_n \in \Phi$ such that

$$\vdash \varphi_1 \dots \varphi_n \ \varphi.$$ ⊣

Let Φ be a set of S-sentences and φ an S-formula.

Lemma 1.5. $\Phi \vdash \varphi$ *if and only if there exists a* ***finite*** $\Phi_0 \subseteq \Phi$ *such that* $\Phi_0 \vdash \varphi$. ⊣

Theorem 1.6 (Soundness). *If* $\Phi \vdash \varphi$*, then* $\Phi \models \varphi$. ⊣

2 Consistency

Definition 2.1. Φ is **consistent**, written $\mathrm{cons}(\Phi)$, if there is no φ such that both $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. Otherwise, Φ is **inconsistent**.

Lemma 2.2. Φ *is inconsistent if and only if* $\Phi \vdash \varphi$ *for any formula* φ.

Proof: The direction from right to left is by Definition 2.1. For the converse direction, assume that there is a ψ such that $\Phi \vdash \psi$ and $\Phi \vdash \neg\psi$. Then there exist two finite sequences of formulas, Γ_1 and Γ_2, such that we have derivation

$$\begin{array}{cc} \vdots & \\ \Gamma_1 & \psi \end{array} \qquad \text{and} \qquad \begin{array}{cc} \vdots & \\ \Gamma_2 & \neg\psi. \end{array}$$

Then for every φ we can obtain the derivation of $\Gamma_1\ \Gamma_2\ \varphi$ as below.

$$\begin{array}{lccrr}
 & & \vdots & & \\
m. & & \Gamma_1 & \psi & \\
 & & \vdots & & \\
n. & & \Gamma_2 & \neg\psi & \\
(n+1). & \Gamma_1 & \Gamma_2 & \psi & \text{(antecedent by } m) \\
(n+2). & \Gamma_1 & \Gamma_2 & \neg\psi & \text{(antecedent by } n) \\
(n+3). & \Gamma_1 & \Gamma_2 & \varphi & \text{(modified contradiction by } n+1 \text{ and } n+2).
\end{array}$$

□

Corollary 2.3. Φ *is consistent if and only if there is a* φ *such that* $\Phi \nvdash \varphi$. $\Rightarrow$ trivial

Lemma 2.4. Φ *is consistent if and only if every finite* $\Phi_0 \subseteq \Phi$ *is consistent.* $\Leftarrow$ trivial

Lemma 2.5. *Every satisfiable* Φ *is consistent.* sequence calculus

semantic

Proof: Assume that Φ is inconsistent. Then there is a φ such that $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. By the Soundness Theorem, i.e., Theorem 1.6, we conclude $\Phi \models \varphi$ and $\Phi \models \neg\varphi$. Thus, Φ cannot be satisfiable. □

"$\Leftarrow$": $\Phi \cup \{\neg\varphi\} \vdash \varphi$

Lemma 2.6. *(a)* $\Phi \vdash \varphi$ *if and only if* $\Phi \cup \{\neg\varphi\}$ *is inconsistent.*

$\Phi \cup \{\neg\varphi\} \vdash \neg\varphi$

(b) $\Phi \vdash \neg\varphi$ *if and only if* $\Phi \cup \{\varphi\}$ *is inconsistent.*

contradiction $\Rightarrow$ $\Phi \vdash \varphi$

(c) If $\mathrm{cons}(\Phi)$*, then* ~~either~~ $\mathrm{cons}(\Phi \cup \{\varphi\})$ *or* $\mathrm{cons}(\Phi \cup \{\neg\varphi\})$.

↳ $\Phi \cup \{\neg\varphi\}$ inconsistent $\overset{(a)}{\Rightarrow}$ $\Phi \vdash \varphi \Rightarrow \mathrm{cons}(\Phi \cup \{\varphi\})$

vise versa.

$*$ when $\Phi = \emptyset$, $\varphi = x \equiv y$

then both $\mathrm{cons}(\Phi \cup \{\varphi\})$ and $\mathrm{cons}(\Phi \cup \{\neg\varphi\})$

3 Completeness

The goal of this section is to show:

mathematical truth

formal proof

Theorem 3.1 (Completeness). *If* $\Phi \models \varphi$, *then* $\Phi \vdash \varphi$. $\dashv$

(exists finite $\Phi_0 \subseteq \Phi$, $\Phi_0 \vdash \varphi$)

We observe that the contrapositive of Theorem 3.1 is:

$$\Phi \nvdash \varphi \text{ implies } \Phi \not\models \varphi$$

$$\iff \text{if } \Phi \cup \{\neg\varphi\} \text{ is consistent, then } \Phi \cup \{\neg\varphi\} \text{ is satisfiable.}$$

As a matter of fact, we actually will prove the following general statement.

Theorem 3.2. $\mathrm{cons}(\Phi)$ *implies that* Φ *is satisfiable.* $\dashv$

3.1 Henkin's Theorem

We fix a set Φ of S-formulas and will construct an S-interpretation out of Φ. To that end, we first define a binary relation over the set T^S of S-terms.

Definition 3.3. Let $t_1, t_2 \in T^S$. Then $t_1 \sim t_2$ if $\Phi \vdash t_1 \equiv t_2$. $\dashv$

Lemma 3.4. *(i)* $\sim$ *is an* ***equivalence*** *relation.*

① $t \sim t$ trivial

② $t_1 \sim t_2 \Rightarrow t_2 \sim t_1$:

by $\dfrac{\Gamma\ \varphi\frac{t}{x}}{\Gamma\ t \equiv t'\ \varphi\frac{t'}{x}}$ ← $t_1 \equiv t_1$, → $t_2 \equiv t_1$, $t_1 \equiv t_2$

$\varphi : x \equiv t_1$

$t : t_1$, $t' : t_2$

③ $t_1 \sim t_2, t_2 \sim t_3 \Rightarrow t_1 \sim t_3$:

$\varphi : t_1 \equiv x$

$\dfrac{\Gamma\ \varphi\frac{t_2}{x}}{\Gamma\ t_2 \equiv t_3\ \varphi\frac{t_3}{x}}$

(ii) $\sim$ *is a* ***congruence*** *relation. That is:*

- *For every* n*-ary function symbol* $f \in S$ *and* $2 \cdot n$ *S-terms* $t_1 \sim t_1', \ldots, t_n \sim t_n'$, *we have*

$$ft_1 \cdots t_n \sim ft_1' \cdots t_n'.$$

- *For every* n*-ary relation symbol* $R \in S$ *and* $2 \cdot n$ *S-terms* $t_1 \sim t_1', \ldots, t_n \sim t_n'$, *we have*

$$\Phi \vdash Rt_1 \cdots t_n \quad \iff \quad \Phi \vdash Rt_1' \cdots t_n'.$$

$\dashv$

Proof: By the equality rule and the substitution rule. $\square$

Now for every $t \in T^S$ we define

$$\bar{t} := \{t' \in T^S \mid t' \sim t\},$$

i.e., the equivalence class of t.

Definition 3.5. The **term structure for** Φ, denoted by $\mathfrak{T}^\Phi$, is defined as follows.

(i) The universe is $T^\Phi := \{\bar{t} \mid t \in T^S\}$.

(ii) For every n-ary relation symbol $R \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$(\bar{t}_1, \ldots, \bar{t}_n) \in R^{\mathfrak{T}^\Phi} \quad \text{if} \quad \Phi \vdash Rt_1 \ldots t_n.$$

(iii) For every n-ary function symbol $f \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$f^{\mathfrak{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) := \overline{ft_1 \cdots t_n}.$$

(iv) For every constant $c \in S$

$$c^{\mathfrak{T}^\Phi} := \bar{c}.$$

This finishes the construction of $\mathfrak{T}^\Phi$. $\dashv$

Using Lemma 3.4, in particular (ii), it is easy to verify that:

Lemma 3.6. *$\mathfrak{T}^\Phi$ is well-defined.* $\dashv$

To complete the definition of an S-interpretation, we still need to provide an assignment of the variables $v_0, v_1, \ldots$ in $\mathfrak{T}^\Phi$.

Definition 3.7. For every variable v_i we let

$$\beta^\Phi(v_i) := \bar{v}_i.$$ $\dashv$

Finally we let

$$\mathfrak{I}^\Phi := \left(\mathfrak{T}^\Phi, \beta^\Phi\right).$$

Lemma 3.8. *(i) For any* $t \in T^S$

$$\mathfrak{I}^\Phi(t) = \bar{t}.$$

(ii) For every ***atomic*** φ

$$\mathfrak{I}^\Phi \models \varphi \quad \Longleftrightarrow \quad \Phi \vdash \varphi.$$

Proof: (i) We proceed by induction on t.

- $t = v_i$ is a variable. Then
$$\mathfrak{I}^\Phi(v_i) = \beta^\Phi(v_i) = \bar{v}_i.$$
- $t = c$ is a constant. Then
$$\mathfrak{I}^\Phi(c) = c^{\mathfrak{T}^\Phi} = \bar{c}$$
- $t = ft_1 \cdots t_n$. Then
$$\begin{aligned} \mathfrak{I}^\Phi(ft_1 \cdots t_n) &= f^{\mathfrak{T}^\Phi}(\mathfrak{I}^\Phi(t_1), \ldots, \mathfrak{I}^\Phi(t_n)) \\ &= f^{\mathfrak{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) && \text{(by induction hypothesis)} \\ &= \overline{ft_1 \cdots t_n}. \end{aligned}$$

(ii) Recall that there are two types of atomic formulas. For the first type, let $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned} \mathfrak{I}^\Phi \models t_1 \equiv t_2 &\Longleftrightarrow \mathfrak{I}^\Phi(t_1) = \mathfrak{I}^\Phi(t_2) \\ &\Longleftrightarrow \bar{t}_1 = \bar{t}_2 && \text{(by (i))} \\ &\Longleftrightarrow t_1 \sim t_2 \\ &\Longleftrightarrow \Phi \vdash t_1 \equiv t_2. \end{aligned}$$

Second, let $\varphi = Rt_1 \cdots t_n$. We deduce

$$\begin{aligned} \mathfrak{I}^\Phi \models Rt_1 \cdots t_n &\Longleftrightarrow \left(\mathfrak{I}^\Phi(t_1), \ldots, \mathfrak{I}^\Phi(t_n)\right) \in R^{\mathfrak{T}^\Phi} \\ &\Longleftrightarrow \left(\bar{t}_1, \ldots, \bar{t}_n\right) \in R^{\mathfrak{T}^\Phi} && \text{(by (i))} \\ &\Longleftrightarrow \Phi \vdash Rt_1 \cdots t_n. \end{aligned}$$

$\square$

4 Exercises

Exercise 4.1. Prove Lemma 3.4

Exercise 4.2. Let

$$\Phi := \big\{\forall x \neg Rxx, \forall x \forall y \forall z (Rxy \wedge Ryz) \to Rxz), \forall x \forall y \big(x \equiv y \vee Rxy \vee Ryx\big), \forall x \exists y Rxy\big\}.$$

Prove that Φ is consistent.

Exercise 4.3. Let $S := \{R\}$ with unary relation symbol R. Moreover we define

$$\Phi := \{\exists x Rx\} \cup \{\neg Ry \mid \text{for every variable } y\}.$$

Prove that:

- Φ is consistent.
- There is no term $t \in T^S$ with $\Phi \vdash Rt$.