MAT1002 Lecture 5, Tuesday, Jan/23/2024

Outline

- Power series (10.7)
  - ↳ Operations
- Taylor series (10.8)

Operations on Power Series

Power series can be operated essentially like polynomials (within the radii of convergence).

Addition/Subtraction

This is done in the same way as series of numbers. (Term-by-term addition/subtraction)

Multiplication

Let $\sum a_n(x-c)^n$ and $\sum b_n(x-c)^n$ be power series with radii of convergence R_a and R_b, respectively. Let $R := \min\{R_a, R_b\}$. Define

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k} \quad \text{for } n \geq 0.$$

Then

$$\left(\sum_{n=0}^{\infty} a_n(x-c)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-c)^n\right) = \sum_{n=0}^{\infty} c_n(x-c)^n$$

for all x with $|x-c| < R$.

Substitution

10.7.20

THEOREM ~~20~~ If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)| < R$.

e.g., Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for all $x \in (-1, 1)$, we have

$\sum_{n=0}^{\infty} (3x)^n = \frac{1}{1-3x}$ valid for $|3x| < 1$, i.e., $x \in (\frac{-1}{3}, \frac{1}{3})$.

Differentiation and Integration

Theorem (Term-by-Term Differentiation and Integration) (10.7.22)

Suppose that $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has a radius of convergence* R*, with* $R>0$*. Define* f *on* $(a-R, a+R)$ *by*

$$f(x) := \sum_{n=0}^{\infty} c_n(x-a)^n.$$

Then on $(a-R, a+R)$*, the function* f *is differentiable and has an antiderivative, with*

(i) $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$.

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The theorem above may be useful in converting a function between its elementary form and power series form.

e.g. Can you express $\ln(1+x)$ in a power series form $\sum_{n=0}^{\infty} c_n x^n$?

Sol: Using the theorem 10.7.22,

$$\ln(1+x) = \underbrace{\sum_{n=0}^{\infty} \frac{1}{n+1} (-1)^n x^{n+1}}_{(*)} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \text{ for } x \in (-1,1).$$

Remarks

* "Abel's theorem"

- It can be shown (using a theorem which is not within the scope of the course) that the series (*) also converges to $\ln(1+x)$ at $x=1$ —— this will give

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

The alternating harmonic series

We omit the proof.

- See e.g. 10.7.6. for another approach of the example above.

e.g. Let $f(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

(a) Find its radius R of convergence.

(b) Find an elementary expression of f on $(-R, R)$.

Sol: (a)

$R = 1$.

(b)

$f(x) = \arctan x$, $\forall x \in (-1, 1)$.

Remarks

- It can be shown that $f(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ also converges to $\arctan x$ at $x = \pm 1$ (we omit the proof now) —— This will imply that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \arctan 1 = \frac{\pi}{4}.$$

- In general, term-by-term differentiation / integration may fail at endpoints $a \pm R$; e.g., $f(x) := \sum_{n=1}^{\infty}\frac{x^n}{n^2}$ converges at $x=1$ (so $f(1)$ is defined), but $\sum_{n=1}^{\infty}\left(\frac{d}{dx}\frac{x^n}{n^2}\right)$ diverges at $x=1$.

  (So in the theorem, you cannot replace $(a-R, a+R)$ with $[a-R, a+R]$.)

  - $\frac{|x|^{n+1}}{(n+1)^2}\cdot\frac{n^2}{|x|^n} = \left(\frac{n}{n+1}\right)^2|x| \to |x| \quad \Rightarrow \quad R=1.$
  - At $x=1$, $\sum_{n=0}^{\infty}\frac{x^n}{n^2}$ converges.
  - But $\sum_{n=1}^{\infty} n\frac{x^{n-1}}{n^2} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n}$ does not converge at $x=1$.

- Term-by-term differentiation / integration does not work for other types of series (that is not a power series); e.g., $f(x) := \sum_{n=1}^{\infty}\frac{\sin(n!x)}{n^2}$

  - Series converges absolutely $\forall x \in (-\infty, \infty)$.
  - $\sum_{n=1}^{\infty}\frac{d}{dx}\frac{\sin(n!x)}{n^2} = \sum_{n=1}^{\infty}\frac{n!}{n^2}\cos(n!x)$ diverges $\forall x \in \mathbb{R}$.

    $\Rightarrow f'(x) = \sum_{n=1}^{\infty}\frac{d}{dx}\frac{\sin(n!x)}{n^2}$ is false.

Taylor Series

- Can every function represented by some power series?
- What is the form of such a series?

Suppose $\sum_n C_n(x-a)^n$ converges to $f(x)$ on some open interval I. i.e., $f(x)=\sum_n C_n(x-a)^n$ $\forall x\in I$. By the previous theorem,

$f'(x)=\sum_{n=1}^{\infty} nC_n(x-a)^{n-1}$, $f''(x)=\sum_{n=2}^{\infty} n(n-1)C_n(x-a)^{n-2}$, ...

i.e., if f has a power series representation on I, it must be infinitely differentiable on I ($f^{(n)}$ exists for all $n\geq 0$).

Conversely, given an infinitely differentiable function f on I:

Q1: Can f be represented as a power series on I?

Q2: If so, what is the form of this power series?

We look at Q2 first. Suppose $f(x)=\sum_{n=0}^{\infty} C_n(x-a)^n$ on some open interval I containing a. What are the values of C_n?

- $f(a)=\sum_{n=0}^{\infty} C_n(a-a)^n = C_0$.
- $f'(a)=\sum_{n=1}^{\infty} nC_n(a-a)^{n-1} \implies f'(a)=C_1$.

$\cdot\ f''(a) = \sum_{n=2}^{\infty} n(n-1)C_n(a-a)^{n-2} \Rightarrow f''(a) = 2\cdot 1\cdot C_2$

$\Rightarrow C_2 = \dfrac{f''(a)}{2!}$

$\cdot\ f^{(k)}(a) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\,C_n(x-a)^{n-k}\Big|_{x=a} = k(k-1)\cdots 1\cdot C_k$

$\Rightarrow C_k = \dfrac{f^{(k)}(a)}{k!}$

Hence, if $f(x)$ has a power series representation $\sum_{n=0}^{\infty} C_n(x-a)^n$ on I, then it must be

C_n

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Definition

Let f be a function such that for all $n \in \mathbb{N}$, $f^{(n)}$ exists on some open interval containing a. The Taylor series of f centered at a is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

or "Taylor series generated by f"

The Maclaurin series of f is the Taylor series of f centered at 0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Example

(a) The Maclaurin series of $f(x) := e^x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(b) The Maclaurin series of $f(x) := \cos x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

Partial sums of a Taylor series are Taylor polynomials. More generally:

Definition

Let f be a function such that for all $n \in \{0, 1, \ldots, N\}$, $f^{(n)}$ exists on some open interval containing a. The Taylor polynomial of f (of order n) centered at a is the polynomial

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$