1 Problem 1

1.1 a

Populations:
I = Infective Population

S = Healthy/Susceptible to infection population
I_p = Infective people who engage in precautionary behavior
I_{np} = Infective people who do not engage in precautionary behavior
S_p = Healthy/Susceptible people who engage in precautionary behavior
S_{np} = Healthy/Susceptible people who do not engage in precautionary behavior
$S = S_{np} + S_p$
$I = I_{np} + I_p$

Types of reactions:
For this problem there are 4 ways to spread the disease: susceptible + invective; susceptible and precautionary+ invective;susceptible + invective and precautionary;susceptible and precautionary + invective and precautionary. These 4 different ways to spread can be modeled by the following equations where α is the reaction rate. Note that for some of the cases thew reaction rate is reduced (divided by a factor) based on if any of the involved parties engage in precautionary behavior.

$$S_{np} + I_{np} \xrightarrow{\alpha} 2I_{np}; \text{ Disease spread reaction}$$

$$S_p + I_p \xrightarrow{\frac{\alpha}{\phi\sigma}} 2I_p; \text{ Disease spread reaction}$$

$$S_p + I_{np} \xrightarrow{\frac{\alpha}{\phi}} I_p + I_{np}; \text{ Disease spread reaction}$$

$$S_{np} + I_p \xrightarrow{\frac{\alpha}{\sigma}} I_p + I_{np}; \text{ Disease spread reaction}$$

For the reaction rate for the mass action model of people adopting precautionary model I think the rate should be proportion to the aggregate infection rate. I think this because the higher the rate of infection is the more likely people are to take it seriously and engage in precautionary measures.Which is the rate at which both S_p and S_{np} get infected. Which is equal to the sum of the 4 previous disease rates: $\alpha + \frac{\alpha}{\phi\sigma} + \frac{\alpha}{\phi} + \frac{\alpha}{\sigma}$. Lets call this aggregate value γ. Now we can model the precautionary adoption reaction as follows:

$$S_{np} \xrightarrow{\gamma} S_p; \text{ Precautionary behavior adoption reaction}$$

$$I_{np} \xrightarrow{\gamma} I_p; \text{ Precautionary behavior adoption reaction}$$

The next two reactions are recovery reactions, which is the reaction where an invective person becomes healthy which puts them back into out susceptible category because there is no immunity from this disease. For both the group taking precautionary action and the group that is not we are assuming that they have the same recovery reaction rate. The logic here is that once you have the disease it doesn't matter if you are being precautious or not so the time it takes you to recover is independent of which group you are in. For the recovery reaction rate I am using the variable μ

$$I_{np} \xrightarrow{\mu} S_{np}; \text{ Recovery reaction}$$
$$I_p \xrightarrow{\mu} S_p; \text{ Recovery reaction}$$

The final two reactions are the fatigue reactions, which are when people get tired of engaging in precautionary behavior and stop doing it. I believe this should be proportional to the recovery reaction rate because if that recovery rate is high people will perceive the disease to not be severe and will not think it is necessary to be precautious.

$$S_p \xrightarrow{\mu} S_{np}; \text{ Fatigue reaction}$$
$$I_p \xrightarrow{\mu} I_{np}; \text{ Fatigue reaction}$$

Change in population:

$$\frac{\mathrm{d}S_p}{\mathrm{d}t} = (-1)\frac{\alpha}{\phi}S_pI_{np}(-1)\frac{\alpha}{\phi\sigma}S_pI_p(+1)\gamma S_{np}I(+1)\mu I_p(-1)\mu S_p$$
$$\frac{\mathrm{d}S_{np}}{\mathrm{d}t} = (-1)\frac{\alpha}{\sigma}S_{np}I_p(-1)\alpha S_{np}I_{np}(-1)\gamma S_{np}I(+1)\mu S_p$$
$$\frac{\mathrm{d}I_{np}}{\mathrm{d}t} = (+1)(2\alpha S_{np}I_{np})(+1)\frac{\alpha}{\phi}S_pI_{np}(+1)\frac{\alpha}{\sigma}S_{np}I_p(-1)\gamma I_{np}I(-1)\mu I_{np}(+1)\mu I_p$$
$$\frac{\mathrm{d}I_p}{\mathrm{d}t} = (+1)(2\alpha S_pI_p)(+1)\frac{\alpha}{\phi}S_pI_{np}(+1)\frac{\alpha}{\sigma}S_{np}I_p(+1)\gamma I_{np}I(-1)\mu I_p(-1)\mu I_p$$

Simplifications:

$$\frac{\mathrm{d}S_p}{\mathrm{d}t} = -\frac{\alpha}{\phi}S_pI_{np} - \frac{\alpha}{\phi\sigma}S_pI_p + \gamma S_{np}I + \mu I_p - \mu S_p$$
$$\frac{\mathrm{d}S_{np}}{\mathrm{d}t} = -\frac{\alpha}{\sigma}S_{np}I_p - \alpha S_{np}I_{np} - \gamma S_{np}I + \mu I_{np} + \mu S_p$$
$$\frac{\mathrm{d}I_{np}}{\mathrm{d}t} = \alpha S_{np}I_{np} + \frac{\alpha}{\phi}S_pI_{np} + \frac{\alpha}{\sigma}S_{np}I_p - \gamma I_{np}I - \mu I_{np} + \mu I_p$$
$$\frac{\mathrm{d}I_p}{\mathrm{d}t} = \alpha S_pI_p + \frac{\alpha}{\phi}S_pI_{np} + \frac{\alpha}{\sigma}S_{np}I_p + \gamma I_{np}I - \mu I_p - \mu I_p$$

1.2 b

The problem says that the abandoning and adoption of precautionary behavior should be taken as fast. So following the steps to do the QSS:

Step 1: Shut off the slow reactions.

$$\frac{\mathrm{d}S_p}{\mathrm{d}t} = \gamma S_{np} I - \mu S_p$$
$$\frac{\mathrm{d}S_{np}}{\mathrm{d}t} = -\gamma S_{np} I + \mu S_p$$
$$\frac{\mathrm{d}I_{np}}{\mathrm{d}t} = -\gamma I_{np} I + \mu I_p$$
$$\frac{\mathrm{d}I_p}{\mathrm{d}t} = \gamma I_{np} I - \mu I_p$$

Step 2: Find as many independent linear conservation laws of the model with only fast reactions as you can.

Since we only take the fast reactions the slow reactions are the disease spread and the recovery terms. Since they do not have an impact in the fast time, in the fast time S and I are constant.

$$\frac{\mathrm{d}S_p}{\mathrm{d}t} + \frac{\mathrm{d}S_{np}}{\mathrm{d}t} = 0 \Longrightarrow S_p + S_{np} = S = constant$$
$$\frac{\mathrm{d}I_{np}}{\mathrm{d}t} + \frac{\mathrm{d}I_p}{\mathrm{d}t} = 0 \Longrightarrow I_{np} + I_p = P = constant$$

Step 3: Find the quasi-steady states of the model with only fast reactions; these quasi-steady states may depend on the slow variables just defined. The quasi-steady states should express all variables as a function of the slow variables.

We can substitute into our fast reaction equations using the linear combinations.

$$\frac{\mathrm{d}S_p}{\mathrm{d}t} = \gamma(S - S_p)I - \mu S_p$$
$$\frac{\mathrm{d}S_{np}}{\mathrm{d}t} = -\gamma S_{np} I + \mu(S - S_{np)}$$
$$\frac{\mathrm{d}I_{np}}{\mathrm{d}t} = -\gamma I_{np} I + \mu(I - I_{np})$$
$$\frac{\mathrm{d}I_p}{\mathrm{d}t} = \gamma(I - I_p)I - \mu I_p$$

Now we can solve the diff eq's. Notice due to our linear combination we only need to solve one of the diff eq for S and one of the diff eq for I. I decided to solve the diff eq's for S_p and I_p. Funily enough I had the same formula in wolfram for both but the variables mean different things. My wolfram input was this:

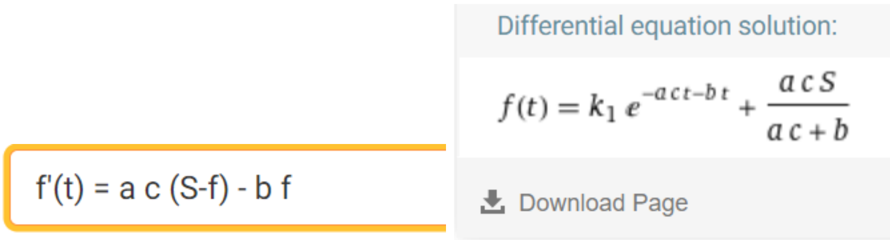

Figure 1: Wolfram Input

For solving S_p the variables mean : f = S_p; a = γ; c = I, S = S ; and b = mu. This makes the solution:

$$S_p(t) = k_1 e^{-\gamma It - \mu t} + \frac{\gamma IS}{\gamma I + \mu}$$

For solving I_p the variables mean : f = I_p; a = γ; c = I, S = I ; and b = mu. This makes the solution:

$$I_p(t) = k_2 e^{-\gamma It - \mu t} + \frac{\gamma I^2}{\gamma I + \mu}$$

Now the other two diff eq's (the ones for S_{np} and I_{np} can be written as $S - S_p$ and $I - I_p$ respectively. So those equations can be written as follows:

$$S_{np}(t) = S - k_1 e^{-\gamma It - \mu t} - \frac{\gamma IS}{\gamma I + \mu}$$
$$I_{np}(t) = I - k_2 e^{-\gamma It - \mu t} - \frac{\gamma I^2}{\gamma I + \mu}$$

As slow time t goes to infinity we are just left with the constant terms and these constants are the quasi steady states.So the quasi steady states are as follows:

$$S_p(t) = \frac{\gamma IS}{\gamma I + \mu}$$
$$S_{np}(t) = S - \frac{\gamma IS}{\gamma I + \mu} = \frac{1}{\gamma I + \mu}(S(\gamma I + \mu) - \gamma IS) = \frac{\mu S}{\gamma I + \mu}$$
$$I_p(t) = \frac{\gamma I^2}{\gamma I + \mu}$$
$$I_{np}(t) = I - \frac{\gamma I^2}{\gamma I + \mu} = \frac{1}{\gamma I + \mu}(I(\gamma I + \mu) - \gamma I^2) = \frac{\mu I}{\gamma I + \mu}$$

Step 4: Verify whether these quasi-steady states are stable under the fast reactions, so that the quasi-steady state approximation is at least plausible.

Under the fast reactions S and I are constant which makes them stable. The variables γ and μ are constant because they are the rates of the fast reaction. Thus the quasi steady state equations are stable.

Step 5: Go back to the full model and derive differential equations for the slow variables. By definition, their rates of change should not involve the fast reactions, but can be nonzero due to the slow reactions.

We can combine our four original diff eq's to find the diff eq's for the slow reactions I and S:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}S_p}{\mathrm{d}t} + \frac{\mathrm{d}S_{np}}{\mathrm{d}t} = -\frac{\alpha}{\phi}S_pI_{np} - \frac{\alpha}{\phi\sigma}S_pI_p + \gamma S_{np}I + \mu I_p - \mu S_p - \frac{\alpha}{\sigma}S_{np}I_p$$
$$-\alpha S_{np}I_{np} - \gamma S_{np}I + \mu I_{np} + \mu S_p$$
$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\alpha}{\phi}(S_pI_{np} - S_{np}I_p) - \frac{\alpha}{\sigma\phi}S_pI_p - \alpha S_{np}I_{np} + \mu I_p + \mu I_{np}$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}I_p}{\mathrm{d}t} + \frac{\mathrm{d}I_{np}}{\mathrm{d}t} = \alpha S_{np}I_{np} + \frac{\alpha}{\phi}S_pI_{np} + \frac{\alpha}{\sigma}S_{np}I_p - \gamma I_{np}I - \mu I_{np} + \mu I_p$$
$$+\alpha S_pI_p + \frac{\alpha}{\phi}S_pI_{np} + \frac{\alpha}{\sigma}S_{np}I_p + \gamma I_{np}I - \mu I_p - \mu I_p$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \alpha(S_pI_p + S_{np}I_{np}) + \frac{\alpha}{\phi}S_pI_{np} + \frac{\alpha}{\sigma}S_{np}I_p - \mu I_{np} - \mu I_p$$

Step 6: The quasi-steady state assumption amounts to assuming the fast reactions always cause the variables appearing in the differential equations of the slow variables to be expressible in terms of their quasi-steady relationship to the slow variables, at least after some rapid adjustment from the initial conditions. This will result in a closed system of equations for only the slow variables, which should be fewer in number than the number of variables in the original model. Now we can plug in our qss into these two diff eq's.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\alpha}{\phi}((\frac{\gamma IS}{\gamma I+\mu})(\frac{\mu I}{\gamma I+\mu}) - (\frac{\mu S}{\gamma I+\mu})(\frac{\gamma I^2}{\gamma I+\mu})) - \frac{\alpha}{\sigma\phi}(\frac{\gamma IS}{\gamma I+\mu})(\frac{\gamma I^2}{\gamma I+\mu})$$
$$-\alpha(\frac{\mu S}{\gamma I+\mu})(\frac{\mu I}{\gamma I+\mu}) + \mu(\frac{\gamma I^2}{\gamma I+\mu} + \frac{\mu I}{\gamma I+\mu})$$
$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\alpha SI}{\gamma I+\mu}(-\frac{1}{\sigma\phi}(\gamma^2I^2) - (\mu^2)) + \mu(\frac{\gamma I^2}{\gamma I+\mu} + \frac{\mu I}{\gamma I+\mu})$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \alpha((\frac{\gamma IS}{\gamma I+\mu})(\frac{\gamma I^2}{\gamma I+\mu}) + (\frac{\mu S}{\gamma I+\mu})(\frac{\mu I}{\gamma I+\mu})) + \frac{\alpha}{\phi}(\frac{\gamma IS}{\gamma I+\mu})(\frac{\mu I}{\gamma I+\mu})$$
$$+\frac{\alpha}{\sigma}(\frac{\mu S}{\gamma I+\mu})(\frac{\gamma I^2}{\gamma I+\mu}) - \mu(\frac{\mu I}{\gamma I+\mu} + \frac{\gamma I^2}{\gamma I+\mu})$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\alpha S}{\gamma I+\mu}((\gamma^2I^2 + \mu^2) + \frac{1}{\phi}(\gamma I\mu) + \frac{1}{\sigma}(\mu\gamma I)) - \mu(\frac{\mu I}{\gamma I+\mu} + \frac{\gamma I^2}{\gamma I+\mu})$$

1.3 c

The relationship we acquired in part 1b indicates that the rate of change in the two larger populations (Total infectives and Total Susceptibles) are so slow that these changes can be considered constants when modeling the system.

1.4 e

To characterize the differential equations you need to set the two diff eq to 0 and solve for values of I and S.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0 = \frac{1}{\gamma I + \mu}(\alpha SI(-\frac{1}{\phi}(\gamma I\mu - \gamma I\mu) - \frac{1}{\sigma\phi}(\gamma^2 I^2) - (\mu^2)) + \mu(\gamma I^2 + \mu I))$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0 = \frac{I}{\gamma I + \mu}(2I\alpha S((\gamma^2 I^2 + \mu^2) + \frac{1}{\phi}(\gamma I\mu) + \frac{1}{\sigma}(\mu\gamma I)) - \mu(\mu I + \gamma I^2))$$

As you may notice I is multiplied by every term inside the outermost parenthesis for both $\frac{\mathrm{d}S}{\mathrm{d}t}and\frac{\mathrm{d}I}{\mathrm{d}t}$ so I needs to equal 0 and S can be any value. Note there are no other solutions besides these. Now we need to find the Jacobian of these two diff eq so we can determine stability.

$$\frac{\partial}{\partial I}\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial}{\partial I}[\frac{1}{\gamma I + \mu}(\alpha SI(-\frac{1}{\phi}(\gamma I\mu - \gamma I\mu) - \frac{1}{\sigma\phi}(\gamma^2 I^2) - (\mu^2)) + \mu(\gamma I^2 + \mu I))]$$
$$\frac{\partial}{\partial I}\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\frac{3I^2\alpha S\gamma^2}{\sigma\phi} - \mu^2\alpha S + \mu\gamma\alpha S3I^2 + \mu^2 2I\alpha S}{(\gamma I + \mu)^2}$$
$$\frac{\partial}{\partial I}\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial}{\partial I}[\frac{I}{\gamma I + \mu}(I\alpha S((\gamma^2 I^2 + \mu^2) + \frac{1}{\phi}(\gamma I\mu) + \frac{1}{\sigma}(\mu\gamma I)) - \mu(\mu I + \gamma I^2))]$$
$$\frac{\partial}{\partial I}\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{(\gamma I + \mu)[4I^3\alpha S\gamma + 2I\alpha S\mu^2 + \frac{\gamma\mu S\alpha 3I^2}{\phi} + \frac{\mu\gamma 3I^2\alpha S}{\sigma} - mu^2 2I + \gamma\mu 3I^2]}{(\gamma I + \mu)^2}$$
$$\frac{\partial}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial}{\partial S}[\frac{1}{\gamma I + \mu}(\alpha SI(-\frac{1}{\phi}(\gamma I\mu - \gamma I\mu) - \frac{1}{\sigma\phi}(\gamma^2 I^2) - (\mu^2)) + \mu(\gamma I^2 + \mu I))]$$
$$\frac{\partial}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{(\alpha I(-\frac{1}{\phi}(\gamma I\mu - \gamma I\mu) - \frac{1}{\sigma\phi}(\gamma^2 I^2) - (\mu^2))}{(\gamma I + \mu)^2}$$
$$\frac{\partial}{\partial S}\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial}{\partial S}[\frac{I}{\gamma I + \mu}(I\alpha S((\gamma^2 I^2 + \mu^2) + \frac{1}{\phi}(\gamma I\mu) + \frac{1}{\sigma}(\mu\gamma I)) - \mu(\mu I + \gamma I^2))]$$
$$\frac{\partial}{\partial S}\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{(I\alpha((\gamma^2 I^2 + \mu^2) + \frac{1}{\phi}(\gamma I\mu) + \frac{1}{\sigma}(\mu\gamma I))}{(\gamma I + \mu)^2}$$

Now we evaluate these variables at the steady state which is $I = 0 \text{and} S = S$.

$$\frac{\partial}{\partial I}\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\frac{3I^2\alpha S\gamma^2}{\sigma\phi} - \mu^2\alpha S + \mu\gamma\alpha S 3I^2 + \mu^2 2I\alpha S}{(\gamma I+\mu)^2} = -\alpha S$$

$$\frac{\partial}{\partial I}\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{(\gamma I+\mu)[4I^3\alpha S\gamma + 2I\alpha S\mu^2 + \frac{\gamma\mu S\alpha 3I^2}{\phi} + \frac{\mu\gamma 3I^2\alpha S}{\sigma} - mu^2 2I + \gamma\mu 3I^2]}{(\gamma I+\mu)^2} = 0$$

$$\frac{\partial}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{(\alpha I(-\frac{1}{\phi}(\gamma I\mu - \gamma I\mu) - \frac{1}{\sigma\phi}(\gamma^2 I^2) - (\mu^2))}{(\gamma I+\mu)^2} = 0$$

$$\frac{\partial}{\partial S}\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{(I\alpha((\gamma^2 I^2 + \mu^2) + \frac{1}{\phi}(\gamma I\mu) + \frac{1}{\sigma}(\mu\gamma I))}{(\gamma I+\mu)^2} = 0$$

Then we can make the Jacobian with these derivatives : $\begin{bmatrix} 0 & -\alpha S \\ 0 & 0 \end{bmatrix}$ The trace of this matrix is $-\alpha$ S and the determinant is 0 so we are not able to determine the stability of this matrix.

1.5 f

Since there is a steady state at $I = 0 \text{and} S = S$(any value). The way I interpret this model is that when our population of infective's are 0 it doesn't matter how many susceptible people are left because there are no more infective's to spread the disease.

1.6 g

The arrows point to up and to the right at the points where $\frac{\mathrm{d}I}{\mathrm{d}t}$ and $\frac{\mathrm{d}S}{\mathrm{d}t}$ are positive and down to the left when $\frac{\mathrm{d}I}{\mathrm{d}t}$ and $\frac{\mathrm{d}S}{\mathrm{d}t}$ are negative.There is a steady state at the origin where I = 0. The horizontal axis is for values of I and the vertical axis is for value of S. There is a second steady state when the derivative of this line equals 0, all the arrows after this point down and to the left whereas all the arrows before point up and to the right. This phase plane sketch indicates that as the infective population is low the susceptible population will grow but once we pass the steady state the susceptible population will decline until it reaches 0 which would mean all the susceptible people have been infected.

1.7 j

In my current model the adoption of precautionary behavior is only linked to the total number of infectives. The logic was that if the infective population is large people are more likely to adopt precautionary behavior. If the precautionary behavior was visible like wearing a mask this also has an effect on the rate of adoption. I believe that the larger the population of precautionary behavior is the more likely people are to adopt precautionary behavior. If you live in a town where everyone wears a mask you are more likely to start wearing a mass than if you live in a town where no one wears a mask.

This is what I previously had for my modeling of precautionary behavior.

$$S_{np} \overset{\gamma}{\to} S_p\text{; Precautionary behavior adoption reaction}$$
$$I_{np} \overset{\gamma}{\to} I_p\text{; Precautionary behavior adoption reaction}$$

This lead to the terms:

$$-S_{np}I\gamma + S_{np}I\gamma - I_{np}I\gamma + I_{np}I\gamma$$

These terms only take into account total number of infective people and the number of non precautionary people. To Take into account total population of precautionary each term should be multiplied by $(I_p + S_p)$. So the terms should look like:

$$-S_{np}I\gamma(I_p + S_p) + S_{np}I\gamma(I_p + S_p) - I_{np}I\gamma(I_p + S_p) + I_{np}I\gamma(I_p + S_p)$$

This changes the diff eq to look like :

$$\frac{\mathrm{d}S_p}{\mathrm{d}t} = -\frac{\alpha}{\phi}S_pI_{np} - \frac{\alpha}{\phi\sigma}S_pI_p + S_{np}I\gamma(I_p + S_p) + \mu I_p - \mu S_p$$
$$\frac{\mathrm{d}S_{np}}{\mathrm{d}t} = -\frac{\alpha}{\sigma}S_{np}I_p - \alpha S_{np}I_{np} - S_{np}I\gamma(I_p + S_p) + \mu I_{np} + \mu S_p$$
$$\frac{\mathrm{d}I_{np}}{\mathrm{d}t} = \alpha S_{np}I_{np} + \frac{\alpha}{\phi}S_pI_{np} + \frac{\alpha}{\sigma}S_{np}I_p - I_{np}I\gamma(I_p + S_p) - \mu I_{np} + \mu I_p$$
$$\frac{\mathrm{d}I_p}{\mathrm{d}t} = \alpha S_pI_p + \frac{\alpha}{\phi}S_pI_{np} + \frac{\alpha}{\sigma}S_{np}I_p + I_{np}I\gamma(I_p + S_p) - \mu I_p - \mu I_p$$

2 Problem 2

2.1 a

Using the equation from class we can solve this problem. The equation is :

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} + S(x,t) - D(x,t)$$

Since there is no leaks the source $S(x,t)$ and the sink $-D(x,t)$ equal 0.
To find the cross sectional mass density you need to imagine the pipe as slices of cubic density, $p_3(x)$, and then multiply that by the area at that point x. So we have:

$$p_m(x,t) = A_x p_3(x) = A_x * (\frac{1g}{m^3})$$

If you take the partial with respect to time you get:

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial t} = 0\ ;\ \text{while} x \leq x_2 \text{ and } x \geq x_1$$

The mass flux at a cross section x is $J_{mass}(x,t) =$ the number of agents in the pipe $*$ speed at which the agents move This ends up being:

$$J_{mass}(x,t) = \rho_{mass}(x,t) * v(x,t) = \rho_3 A_x v_x$$

Now lets plug these values into the first equation.

$$\begin{aligned}
&\frac{\partial\rho(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} + S(x,t) - D(x,t); x_1 \le x \le x_2 \\
&0 = -\frac{\partial J(x,t)}{\partial x} + 0 - 0 \\
&0 = \int_{x_1}^{x^2} \left(-\frac{\partial J(x,t)}{\partial x} dx \right) = [-J(x,t)]_{x_1}^{x_2} = A_{x_1} v_{x_1} \rho_3 - A_{x_2} v_{x_2} \rho_3 \\
&0 = A_{x_1} v_{x_1} \rho_3 - A_{x_2} v_{x_2} \rho_3 \to A_{x_1} v_{x_1} \rho_3 = A_{x_2} v_{x_2} \rho_3 \to A_{x_1} v_{x_1} = A_{x_2} v_{x_2}
\end{aligned}$$

The relationship that we are able to derive from this problem is that $A_{x_1} v_{x_1} = A_{x_2} v_{x_2}$.

2.2 b

Since there is a new destruction our $D(x,t)$ is no longer 0. So our equations is now:

$$0 = \int_{x_1}^{x^2} \left(-\frac{\partial J(x,t)}{\partial x} - D(x,t)dx \right)$$

Next we need to find what $D(x,t)$ is and then evaluate the integral. If we take our F_{Leak} and dot it with a vector that is perpendicular to the surface of the pipe we can capture the amount of flux that leaves through the destruction. So if we have a vector V we can write the destruction as $D(x,t) = F_{Leak} \cdot V$. Then if we take the integral of the destruction we are left with $\int_{x_1}^{x^2} D(x,t)dx = F_{Leak} A_L$. This makes the above integral equal to:

$$\begin{aligned}
&0 = \int_{x_1}^{x^2} \left(-\frac{\partial J(x,t)}{\partial x} - D(x,t)dx \right) = [-J(x,t)]_{x_1}^{x_2} - F_{Leak} A_L \\
&0 = A_{x_1} v_{x_1} \rho_3 - A_{x_2} v_{x_2} \rho_3 - F_{Leak} A_L \\
&A_{x_1} v_{x_1} \rho_3 = A_{x_2} v_{x_2} \rho_3 + F_{Leak} A_L
\end{aligned}$$

This means leaves with, change in our relationship is $A_{x_1} v_{x_1} = A_{x_2} v_{x_2} + \frac{F_{Leak} A_L}{\rho_3}$

3 Collaborators

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