Part 2:
Imprementing the ACM2 model

Manuel J. Barragán

https://github.com/ACMmodel/MOSFET_model

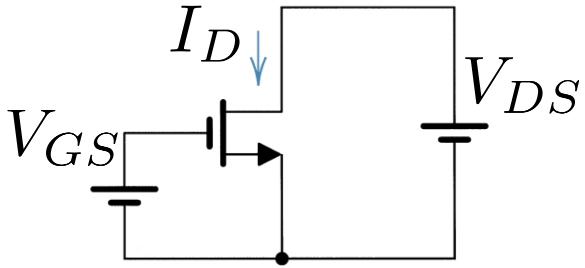
"Scan me"

Outline

- The ACM2 model (a quick recap)
- Numerical solving of the UCCM equations
- acm 433 algorithm and Lambert function
- Analytical approximation of Lambert functions
- Towards a closed-form I-V MOSFET model

The 5-parameter ACM2 model

- Only 5 parameters: n, I_{S0}, V_{T0}, σ and ζ



$$I_D = I_{S0}\frac{q_S + q_D + 2}{1+\zeta|q_S - q_D|}(q_S - q_D)$$

where

$$\frac{V_P - V_{SB}}{U_T} = q_S - 1 + \ln q_S$$

$$\frac{V_{DS}}{U_T} = q_S - q_D + \ln\frac{q_S - q_{Dsat}}{q_D - q_{Dsat}}$$

with

$$V_P = \frac{V_{GB} - V_{T0} + \sigma(V_{DB} + V_{SB})}{n}$$

$$q_{Dsat} = q_S + 1 + \frac{1}{\zeta} - \sqrt{\left(1+\frac{1}{\zeta}\right)^2 + \frac{2q_S}{\zeta}}$$

The ACM2 model (a quick recap)

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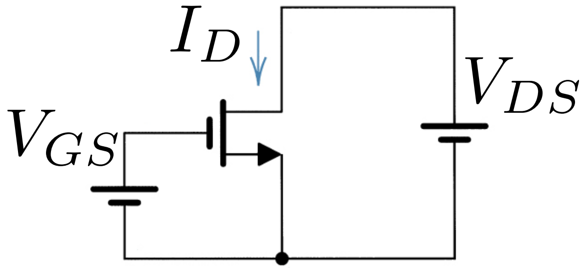

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The ACM2 model (a quick recap)

Computing I_D in practice:

Given the model parameters and polarization voltages

n, I_{S0}, V_{T0}, σ and ζ $\qquad$ V_G, V_D, V_S, V_B

Step 1: Compute V_P: $$V_P = \frac{V_{GB} - V_{T0} + \sigma(V_{DB} + V_{SB})}{n}$$

Step 2: Compute q_S: $$\frac{V_P - V_{SB}}{U_T} = q_S - 1 + \ln q_S$$

Step 3: Compute q_{Dsat}: $$q_{Dsat} = q_S + 1 + \frac{1}{\zeta} - \sqrt{\left(1 + \frac{1}{\zeta}\right)^2 + \frac{2q_S}{\zeta}}$$

Step 4: Compute q_D: $$\frac{V_{DS}}{U_T} = q_S - q_D + \ln \frac{q_S - q_{Dsat}}{q_D - q_{Dsat}}$$

Step 5: Compute I_D: $$I_D = I_{S0} \frac{q_S + q_D + 2}{1 + \zeta|q_S - q_D|} (q_S - q_D)$$

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Transcendental equations

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How to solve $\frac{V_P - V_{SB}}{U_T} = q_S - 1 + \ln q_S$ efficiently?

$$\frac{V_P - V_{SB}}{U_T} = q_S - 1 + \ln q_S \Rightarrow e^{\left(\frac{V_P - V_{SB}}{U_T} - 1\right)} = q_S\, e^{q_S} \Rightarrow x = w\, e^{w}$$

Algorithm 443: Solution of the transcendental equation $we^w=x$

$$w_n = \begin{cases} \dfrac{x + \frac{4}{3}x^2}{1 + \frac{7}{3}x + \frac{5}{6}x^2}, & x < 0.7385 \\ \ln(x) - \dfrac{24(\ln(x)\ln(x) + 2\ln(x) - 3)}{7\ln(x)\ln(x) + 58\ln(x) + 127}, & x \geq 0.7385 \end{cases}$$

$$z_n = \ln(x) - w_n - \ln(w_n)$$

$$e_n = \frac{z_n}{1 + w_n} \frac{2(1 + w_n)\left(1 + w_n + \frac{2}{3}z_n\right) - z_n}{2(1 + w_n)\left(1 + w_n + \frac{2}{3}z_n\right) - 2z_n}$$

$$w = w_n(1 + e_n)$$

Fritsch et al. Communications of the ACM, 1973

acm 443 Algorithm and Lambert W Functions

How to solve $\frac{V_{DS}}{U_T} = q_S - q_D + \ln \frac{q_S - q_{Dsat}}{q_D - q_{Dsat}}$ efficiently?

$$\frac{V_{DS}}{U_T} = q_S - q_{Dsat} - (q_D - q_{Dsat}) + ln\frac{q_S - q_{Dsat}}{q_D - q_{Dsat}} \Rightarrow$$

$$\Rightarrow (q_S - q_{Dsat})\frac{e^{(q_S - q_{Dsat})}}{e^{V_{DS}/U_T}} = (q_D - q_{Dsat})e^{(q_D - q_{Dsat})} \Rightarrow x = w\,e^w$$

Algorithm 443

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$$w = w_n(1 + e_n)$$

$$q_D = w + q_{Dsat}$$

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Can we consider the problem analytically?

- The Lambert W function $z = w\, e^{w}$ for w any complex number was studied by Johann Lambert in 1758, then properly defined by L. Euler in 1783

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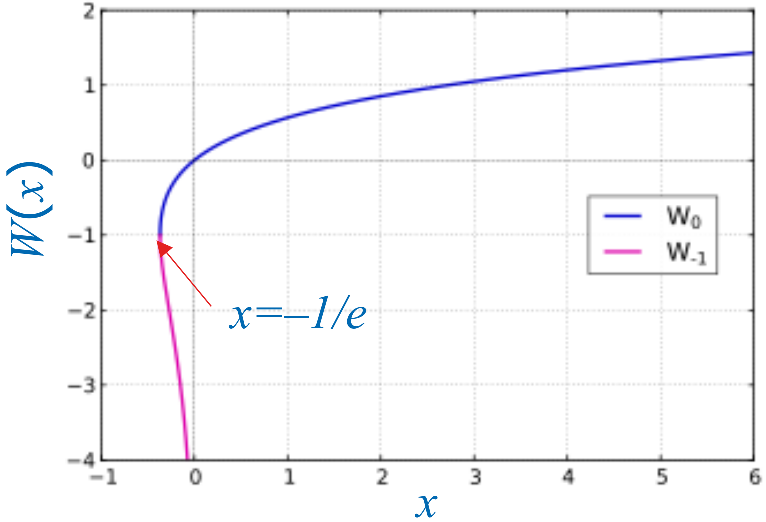
- For real numbers x and $W(x)$, equation $x = W(x)\, e^{W(x)}$ has solutions W_0 and W_{-1}:



q_S, q_D and q_{Dsat}, and I_D can be expressed as a function of V_G, V_D, V_s using W_0

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Expressing I_D using Lamber W function

$$I_D = I_{S0}\frac{q_S + q_D + 2}{1+\zeta|q_S - q_D|}\left(q_S - q_D\right)$$

$$q_S = W_0\left(e^{\left(\frac{V_P - V_{SB}}{U_T}-1\right)}\right)$$

$$q_D = q_{Dsat} + W_0\left(\left(W_0\left(e^{\frac{V_P - V_{SB}}{U_T}-1}\right) - q_{Dsat}\right)e^{\left(W_0\left(e^{\frac{V_P - V_{SB}}{U_T}-1}\right)-q_{Dsat}\right)-\frac{V_{DS}}{U_T}}\right)$$

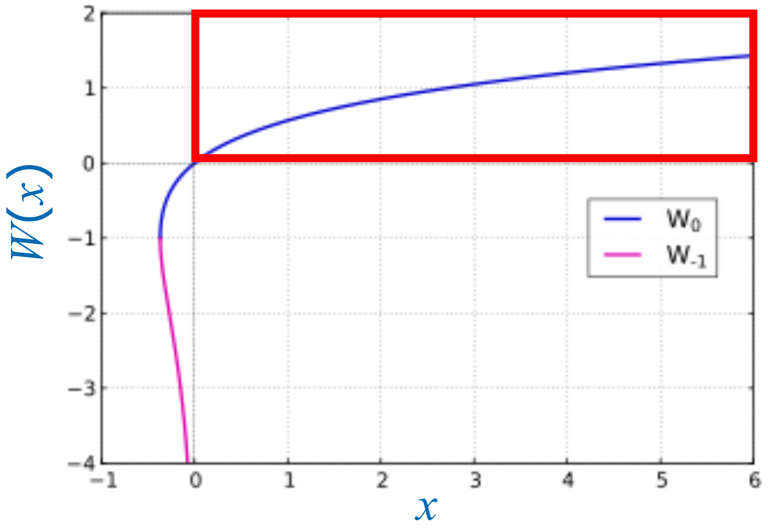
$$q_{Dsat} = W_0\left(e^{\frac{V_P - V_{SB}}{U_T}-1}\right) + 1 + \frac{1}{\zeta} - \sqrt{\left(1+\frac{1}{\zeta}\right)^2 + \frac{2W_0\left(e^{\frac{V_P - V_{SB}}{U_T}-1}\right)}{\zeta}}$$

W_0 is available in main programming languages: Matlab (lambertw), Octave (lambertw), Python (lambertw in scipy lib), R (lambertW0), …

…but can we approximate W_0 in terms of elementary functions?

Approximating the Lambert W function

For real numbers x and $W(x)$, $x = W(x)\, e^{W(x)}$



Approximation of $W_0(x)$ for $x>0$

$$W_{0,0}(x) = \ln\left[1+\alpha(x)x\right]$$

$$W_{0,n}(x) = \frac{W_{n-1}(x)}{1+W_{n-1}(x)}\left[1+\ln\left(\frac{x}{W_{n-1}(x)}\right)\right]$$

For n=1,2,3,…

where $\alpha(x) = \dfrac{1}{1+\dfrac{\ln(1+x)}{2}}$

R. Iacono et al. Advances in Computational Mathematics, 2017

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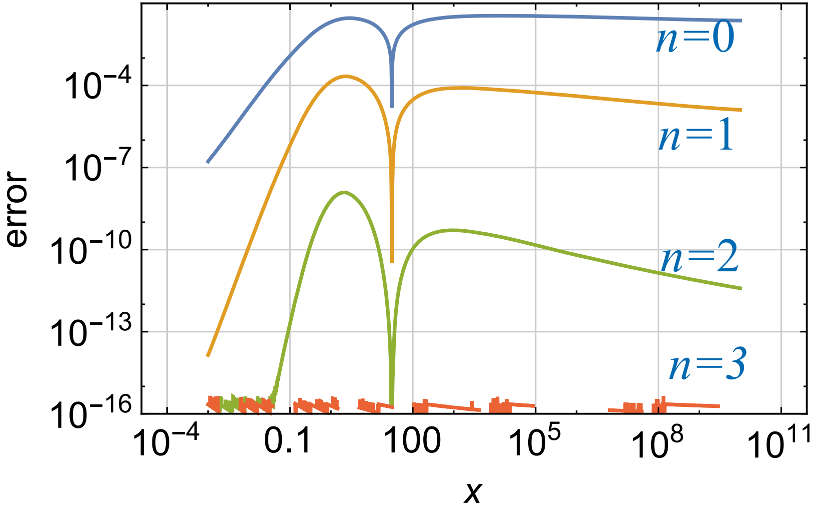

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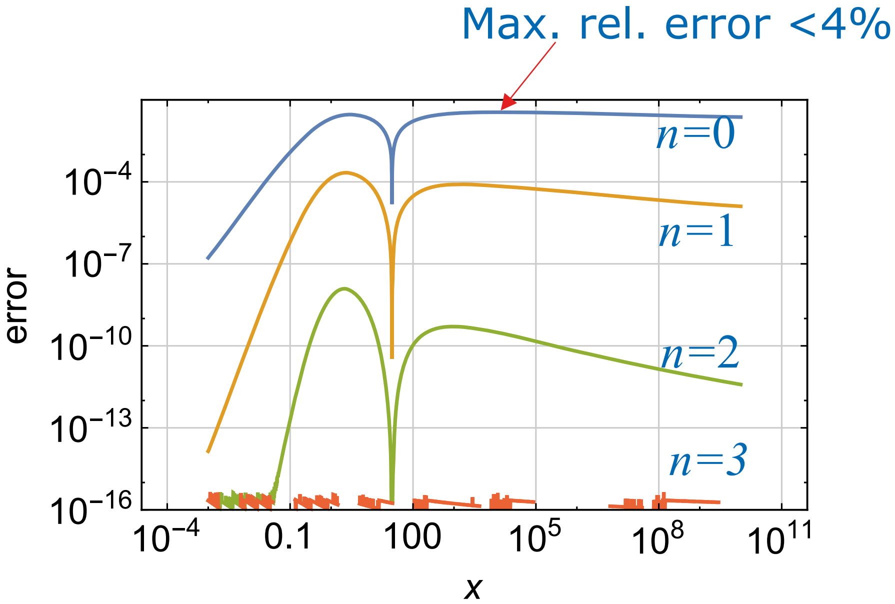

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Approximating the Lambert W function

Approximating q_S

$$q_S = W_0\left(e^{\left(\frac{V_P - V_{SB}}{UT} - 1\right)}\right) \Longrightarrow q_S \approx W_{0,0}(x) = \ln\left[1 + \frac{1}{1 + \frac{\ln\left[1+e^{\left(\frac{V_P - V_{SB}}{U_T} - 1\right)}\right]}{2}} e^{\left(\frac{V_P - V_{SB}}{U_T} - 1\right)}\right]$$

Approximating q_D is similar from:

$$q_D = q_{Dsat} + W_0\left(\left(W_0\left(e^{\frac{V_P - V_{SB}}{UT} - 1}\right) - q_{Dsat}\right) e^{\left(W_0\left(e^{\frac{V_P - V_{SB}}{UT} - 1}\right) - q_{Dsat}\right) - \frac{V_{DS}}{UT}}\right)$$

$$q_{Dsat} = W_0\left(e^{\frac{V_P - V_{SB}}{UT} - 1}\right) + 1 + \frac{1}{\zeta} - \sqrt{\left(1 + \frac{1}{\zeta}\right)^2 + \frac{2W_0\left(e^{\frac{V_P - V_{SB}}{UT} - 1}\right)}{\zeta}}$$

Towards an analytical I-V MOSFET model

Replacing $q_S(V_G, V_S, V_D,)$ and $q_D(V_G, V_S, V_D)$ in I_D will give a closed-form I-V model… but the expression is complex

Proposal: modified empirical model targeting closed-form I-V

$$i_D = \frac{I_D}{I_{S0}} = \frac{(q_S - q_D)(q_S + q_D + 2)}{[1 + \zeta(q_S - q_D)][1 + \theta(q_S + q_D)]}$$

where

$$\frac{V_P - V_{S(D)B}}{U_T} = q_{S(D)} - q_{sat} - 1 + \ln\left(q_{S(D)} - q_{sat}\right)$$

J. Poupon et al., IEEE Access 2024

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where

$$q_{S(D)} = \ln\left[1 + \frac{e^{v_P - v_{S(D)B}} + 1}{1 + \frac{1}{2}\ln\left(1 + e^{v_P - v_{S(D)B}} + 1\right)}\right]\left[1 + \frac{\zeta}{2}\ln\left[1 + \frac{e^{v_P - v_{S(D)B}} + 1}{1 + \frac{1}{2}\ln\left(1 + e^{v_P - v_{S(D)B}} + 1\right)}\right] + \zeta\right]$$

J. Poupon et al., IEEE Access 2024

Take aways

- The ACM2 model contains transcendental equations
- Numerical solving and/or approximations are needed
- acm 433 algorithm and Lambert W function are two useful mathematical tools in this context
- Happy coding!!


Thank you!