UNIVERSITY OF LISBON | ISEG
2025/2026 | PAULO M.D.C. PARENTE

Advanced Econometrics (PhD in Economics) and Advanced Topics in Econometrics (PhD in Mathematics Applied to Economics and Management) 2025/2026
Exercise Sheet 3 - Ordered Data and Count Data Models
(version 28/9/2025)

1. Consider a count data model that satisfies the conditions

$$\begin{aligned} \mathrm{E}(Y|X) &= \lambda = \exp\left(X\beta_0\right), \\ \mathrm{Var}(Y|X) &= f\left(X\right), \end{aligned}$$

where X is a univariate exogenous random variable, β_0 is a single parameter and $f\left(X\right)$ is a positive unknown function of X. The probability density function of Y conditional on X is also unknown. Additionally, suppose a random sample $\{(Y_i, X_i)\}_{i=1}^n$ is available.

(a) Obtain the asymptotic variance of the pseudo-maximum likelihood estimator of β_0 based on a likelihood function obtained using:

i. The Poisson distribution. [**Hint:** Recall that

$$f\left(y|\lambda\right) = \frac{\lambda^y \exp\left(-\lambda\right)}{y!}$$

is the probability function of the Poisson random variable with mean λ]

ii. The normal distribution with fixed known variance σ^2 (the non-linear least squares estimator). [**Hint:** Recall that

$$f\left(y|\lambda, \sigma\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\left(y-\lambda\right)^2}{2\sigma^2}\right)$$

is the probability function of the normal random variable with mean λ]

(b) Suppose now that σ^2 is unknown in case (a-ii) and β_0 and σ^2 are estimated jointly by maximum likelihood. Is the resulting estimator for β_0 consistent? Justify your answer.

(c) Suppose that the log-likelihood function is contructed using the probability function of the negative binomial distribution with parameter γ^2. [**Hint:** Recall that

$$f\left(y|\lambda, \gamma\right) = \frac{\Gamma\left(y+\gamma^{-2}\right)\left[1+\gamma^{-2}/\lambda\right]^{-y}}{\Gamma\left(\gamma^{-2}\right)\Gamma\left(y+1\right)\left(1+\gamma^2\lambda\right)^{\gamma^{-2}}}.$$

is the probability function of the negative binomial random variable with mean λ, where $\Gamma\left(\cdot\right)$ is the gamma function].

i. If γ^2 is a fixed known parameter is the resultant pseudo-maximum likelihood estimator consistent for β_0? Justify your answer.

ii. Suppose now that γ^2 is unknown and β_0 and γ^2 are estimated jointly by maximum likelihood. Is the resulting estimator for β_0 consistent? Justify your answer.

2. Consider the random variable Y, defined as the number of times that an individual went to a restaurant in the previous week. Assume that for part of the population the distribution of Y conditional on X is Poisson with parameter $\lambda = \exp(X'\beta_0)$, where X is a vector of exogenous regressors. For the remaining individuals of the population, for professional reasons, the variable Y is always greater than 4. The probability that an individual belongs to this latter part of the population conditional on X is given by π_0, where π_0 does not vary with X. Unfortunately, the variable Y is observed with errors so large that cannot be used for estimation purposes. However, we can observe without errors the binary variable $D = \mathbf{1}(Y > 0)$, where $\mathbf{1}(\cdot)$ is the indicator function. Assume that a random sample $\left\{(D_i, X_i')'\right\}_{i=1}^{n}$ drawn from the distribution of $(D, X')'$ is available. We would like to estimate π_0 and β_0. Obtain $P(D = 1|X)$ and write down the log-likelihood function of this problem.