Advanced Econometrics (PhD in Economics) and Advanced Topics in Econometrics (PhD in Mathematics Applied to Economics and Management) 2025/2026
Exercise Sheet 1 - Maximum Likelihood
(version 6/9/2025)

1. Consider a random variable y and a scalar regressor x. The density function of y conditional of on the regressor x is given by

$$f(y|x) = \lambda^2 y \exp\left(-\lambda y\right), y > 0,$$

and $\lambda = 2\exp(-x\beta_0)$, where β_0 is a scalar parameter. Note that $E(y|x) = \exp(x\beta_0)$ and $Var(y|x) = [\exp(x\beta_0)]^2 \Big/ 2$. Our objective is to estimate the true parameter β_0. Suppose that a random sample $\left\{(y_i, x_i)'\right\}_{i=1}^n$ is available and that the regressor x is exogenous.

   (a) Write-down the log-likelihood function of the problem.

   (b) Show that the score-vector is given by

$$S\left(\beta\right) = \sum_{i=1}^{n} s_i\left(\beta\right), \text{ where } s_i\left(\beta\right) = 2 \times \frac{y_i - \exp(x_i\beta)}{\exp\left(x_i\beta\right)} x_i,$$

   (c) Show that $E\left(S\left(\beta_0\right)\right) = 0$.

   (d) Let $\hat{\beta}$ be the maximum likelihood estimator of β_0. Obtain the asymptotic variance of $\sqrt{n}\left(\hat{\beta} - \beta_0\right)$, assuming that the density function is correctly specified.

   (e) Prove that the information matrix identity holds in this model.

2. Consider the simple linear model

$$y_i = \alpha + \beta x_i + u_i, \qquad i = 1, \ldots, n$$

where α and β are scalar parameters, x_i is exogenous, and the error terms u_i are independently identically normally distributed with mean zero and variance σ^2 and $\text{Cov}(x, u) = 0$.

   (a) Write the likelihood function for this model and derive $s_i(\theta) = \partial \log L_i(\theta)/\partial\theta$, where $\theta = (\alpha, \beta, \sigma^2)'$. Show that each element of this vector has expected value equal to zero when evaluated at the true values of the parameters.

   (b) Derive the Maximum Likelihood estimators of α, β and σ^2, using the first order conditions and compare these with the corresponding Ordinary Least Squares estimators.

   (c) Explain and discuss the general properties of Maximum Likelihood estimators

(d) Suppose you want to test the null hypothesis $H_0 : \beta = 0$ in the above model. Discuss how you would test this hypothesis by conducting a Likelihood-Ratio test. Derive the expression for the test statistic in terms of the ratio of the sum of squared residuals from the restricted and unrestricted model.

(e) Explain and contrast the general principles that underlie the Likelihood-Ratio, Wald and Lagrange Multiplier tests. Briefly discuss how you would construct a test for omitted variables in the linear model

$$y_i = \alpha + \beta x_i + \gamma z_i + u_i, \qquad i = 1, \ldots, n$$

where α, β and γ are scalar parameters, and where the error terms u_i are independently identically normally distributed with mean zero and variance σ^2.