Advanced Topics in Statistics – PhD program in MAEM

1.st Semester 2012/2013 – Regular Exam – Duration: 2 hours **january 11 2013**

Note: please fully justify all your answers.

1. Let $X_1, X_2, \ldots$ denote a sequence of independent and identically distributed random variables, with probability density function given by

$$f(x \mid \theta) = \frac{2}{\theta^2}\, x \ , \quad 0 < x < \theta$$

where $\theta > 0$. Let $X_{(n)} = \max\{X_1, \ldots, X_n\}$.

a) Verify that $f(x \mid \theta)$ is in fact a probability density function. (1.5)

b) Show that $X_{(n)} \xrightarrow{P} \theta$. (1.5)

c) Show that $-2n\ (X_{(n)} - \theta) \xrightarrow{d} \mathrm{Ex}(1/\theta)$. (2.0)

d) Justify the following statement: (2.0)

$$-2n\, \frac{X_{(n)} - \theta}{X_{(n)}} \xrightarrow{d} \mathrm{Ex}(1)\ .$$

2. Consider a population(X, Y) whose probability distribution is such that, for $\lambda > 0$,

$$X \mid \lambda \sim \mathrm{N}(0, \lambda) \quad \text{amd} \quad Y \mid X = x, \lambda \sim \mathrm{Ex}(x/\lambda)\ .$$

Denote by $(X_1, Y_1), \ldots, (X_n, Y_n)$ and iid random sample of size n from this population. In this context,

a) Show that the maximum likelihood estimator of λ, T_n, can be written in the form (1.5)

$$T_n = \frac{1}{n} \sum_{i=1}^{n} W_i$$

where $W_i = \frac{1}{3}X_i^2 + \frac{2}{3}X_iY_i$, $i = 1, \ldots, n$.

b) Verify that T_n is the most efficient estimator of λ. (2.0)

c) Show that $\mathrm{Var}(W_i \mid \lambda) = 2\lambda^2/3$. (2.0)

d) Justify the following statement: (2.0)

$$\sqrt{n}\, \frac{T_n - \lambda}{T_n} \xrightarrow{d} \mathrm{N}(0, 2/3)\ .$$

3. Consider a random sample of size n from an exponential population of mean δ, which we deonte by $Y_1, \ldots, Y_n$.

a) Show that the sample mean, $\bar{Y}$, is the UMVU estimator of δ. (2.0)

b) Verify that the family of exponential distributions is a member of the scale family, with scale parameter δ. (1.5)

c) Using Basu's theorem, show that $E[Y_{(1)}/\bar{Y}] = 1/n$, where $Y_{(1)} = \min\{Y_1, \ldots, Y_n\}$. (2.0)

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