Brief solution to the regular final exam of ATS, PhD program in MAEM, Jan/12/2011.

1. Recall that $X_{(1)} \sim \text{Ex}(n\theta)$.

   (a) It is necessary to show that $P(a_n X_{(1)} \le x) \to 1 - \exp(-\theta x)$ for $x > 0$. Since $n/a_n \to 1$, there exists N such that, $\forall n \ge N$, $a_n > 0$. For $n \ge N$ and $x > 0$, we have that
   $$P(a_n X_{(1)} \le x) = 1 - \exp(-n\theta x/a_n) \to 1 - \exp(-\theta x)$$
   as we wanted to show.

   (b) We need to show that $P(|b_n X_{(1)}| > \varepsilon) \to 0$ for all $\varepsilon > 0$. Since $n/b_n \to +\infty$, there exists N such that, $\forall n \ge N$, $b_n > 0$. For $n \ge N$, we have that
   $$P(|b_n X_{(1)}| > \varepsilon) = P(b_n X_{(1)} > \varepsilon) = \exp(-n\theta\varepsilon/b_n) \to 0$$
   as we wanted to show.

   (c) By part a), $nX_{(1)} \xrightarrow{d} \text{Ex}(\theta)$. By part b), $X_{(1)} \xrightarrow{P} 0$, hence, by the continuous mapping theorem, $1 - X_{(1)} \xrightarrow{P} 1$. Slutsky's theorem guarantees the result.

   (d) The central limit theorem allows us to conclude that $\sqrt{n}(\theta\bar{X} - 1)$ converges in distribution to a $\text{N}(0,1)$ distribution. Let $X_n = \theta\bar{X}$ and $g(X_n) = \sqrt{X_n}$. Apply the delta method to obtain the result.

2. (a) Note that
   $$f(x,y) = f(y \mid x) f(x) = \frac{1}{\sqrt{2\pi\lambda x}} \exp(-\frac{1}{2\lambda x} y^2) \times \frac{1}{\lambda} \exp(-x/\lambda)$$
   and this allows us to conclude that the model belongs to the one-parameter exponential family with natural parameter $-1/\lambda$ and natural parameter space $\mathbb{R}_-$, which obviously contains an open subset of $\mathbb{R}$. The sufficient statistic T is as a consequence also complete.

   (b) By the law of the iterated expectation,
   $$E[Y^2/X] = E[E[Y^2/X] \mid X] = E[1/X \; E[Y^2 \mid X]] = E[1/X \; \lambda X] = \lambda \; .$$
   Alternatively, we could also explicitely compute the integral.

   (c) $E[S] = 2E[T]/(3n)$; $E[T] = nE[X] + nE[Y^2/X]/2 = n\lambda + n\lambda/2$ by part (b). We then conclude that S is an unbiased estimator of λ. Since S is a function of the sufficient and complete, it follows that S is the UMVU estimator of λ.

   (d) $L(\lambda \mid x, y) \propto \lambda^{-3/2} \exp(-t/\lambda)$, where t represents the observed value of the statistic T. Hence,
   $$\frac{d \ln L}{d\lambda} = \frac{3n}{2} \frac{1}{\lambda^2} (s - \lambda)$$
   where s is the observed value of S. We then conclude that S is also the most efficient estimator of λ.

   (e) Since S is the most efficient estimator of λ, its variance coincide with the Cramer-Rao lower bound. If we compute the Fisher information about λ, it's easy to conclude that $\text{Var}(S) = 2\lambda^2/(3n)$.

3. (a) $\int \theta^{-2} I_{(1,+\infty)}(\theta) d\theta = 1$, and hence this distribution is proper and the normalizing constant is one.

   (b) $L(\theta \mid x) \propto \prod_{i=1}^n \theta^{-2} I_{(0,\theta)}(x_i) \propto \theta^{-2n} I_{(x_{(n)},+\infty)}(\theta)$, hence,
   $$\pi(\theta \mid x) \propto \pi(\theta) L(\theta \mid x) \propto \theta^{-2(n+1)} I_{(x_{(n)} \vee 1, +\infty)}(\theta) \; .$$
   To find the normalizing constant, it suffices to see that
   $$\int_{x_{(n)} \vee 1}^{+\infty} \theta^{-2(n+1)} d\theta = \frac{1}{(2n+1)(x_{(n)} \vee 1)^{2n+1}} \; .$$

   (c) One possible strategy is based on computing $P(X_{n+1} \le 1 \mid x_1, \ldots, x_n)$ and on betting on the realization of $\{X_{n+1} \le 1\}$ iff that probability exceedes $1/2$. In order to compute that probability, note that
   $$\begin{aligned} P(X_{n+1} \le 1 \mid x_1, \ldots, x_n) &= \int_{-\infty}^{+\infty} P(X_{n+1} \le 1 \mid \theta) \pi(\theta \mid x_1, \ldots, x_n) d\theta \\ &= \int_{x_{(n)} \vee 1}^{+\infty} \frac{1}{\theta^2} \; (2n+1) \; (x_{(n)} \vee 1)^{2n+1} \; \theta^{-(2n+2)} d\theta \end{aligned}$$
   and this allows us to conclude that we should bet on the realization of $\{X_{n+1} \le 1\}$ iff
   $$(2n+1)/[(2n+3) \; (x_{(n)} \vee 1)^2] > 1/2 \; .$$