Advanced Topics in Statistics — PhD program in MAEM

First Semester 2019/20 — Regular Exam — Duration: 2 hours **January 17 2020**

Note: Please fully justify all your answers. Each question is worth 2 points.

1. Let $X_1, \ldots, X_n, \ldots$ denote a sequence of independent and identically distributed random variables with common cumulative distribution function given by

$$F(x) = \left(1 - \frac{1}{x}\right)^{\beta} \quad , \quad x \geq 1$$

where $\beta > 0$. Let $X_{(1)}$ and $X_{(n)}$ denote, respectively, the sample minimum and maximum.

a) Show that $X_{(1)} \xrightarrow{P} 1$.

b) Determine a value of α such that the sequence $n^{\alpha}/X_{(n)}$ converges in distribution to a non-degenerate random variable. Which is the probability distribution of this random variable?

c) Determine the limit in probability of the sequence $1/X_{(n)}$.

2. Consider a population (X, Y) such that, given $\lambda > 0$, X and Y are independent, with $X \mid \lambda \sim \text{Po}(\lambda)$ and $Y \mid \lambda \sim \text{Po}(1/\lambda)$. Let $((X_i, Y_i),\ i = 1, \ldots, n)$ denote a random sample of size n from (X, Y).

a) Show that $T = \sum_{i=1}^{n}(X_i - Y_i)/n$ is a minimal sufficient statistic. Is it complete?

b) Determine the scalar function of λ, $\tau(\lambda)$, such that T is the most efficient estimator of $\tau(\lambda)$.

c) Specify the condition(s) that the function g must satisfy to ensure that

$$\sqrt{n}\,\left(g(T) - g(\mu)\right) \xrightarrow{d} N(0,1)\ .$$

where $\mu = \lambda - 1/\lambda$.

d) Obtain an explicit formula for $\hat{\lambda}$, the maximum likelihood estimator of λ.

3. Consider a random sample from a population Y with probability density function given by

$$f(y \mid \lambda) = \sqrt{\frac{\lambda}{2\pi}}\ \exp\left(-\frac{\lambda y^2}{2}\right) \quad , \quad y \in \mathbb{R}$$

where $\lambda > 0$ is an unknown parameter. Once observed the sample $y_1, \ldots, y_n$, we want to implement a Bayesian analysis using Jeffreys' prior, $\pi^J(\lambda)$.

a) Show that $\pi^J(\lambda) \propto \frac{1}{\lambda}$.

b) The posterior distribution is a member of the gamma family of distributions. Which one?

c) Let W represent a random variable statistically independent of the random sample, and following, given λ, an exponential distribution with mean $1/\lambda$. Determine the posterior predictive distribution of W.

END OF EXAM