Advanced Topics of Statistics 2020/2021
Homework set 0

1. Let A, B and C events of positive probability of a sample space Ω.
   (a) If $P(B) = 1$ then $P(A \mid B) = P(A)$
   (b) If A and B are incompatible, $P(A \mid A \cup B) = P(A)/[P(A) + P(B)]$
   (c) $P(A \cap B \cap C) = P(A \mid B \cap C)P(B \mid C)P(C)$

2. Consider a random variable Y with cumulative distribution function

$$F_Y(y) = 1 - \frac{1}{y^2}, \; y \geq 1 \; .$$

   (a) Verify that F_Y is indeed a distribution function.
   (b) Determine the density function of Y.
   (c) Compute $P(Y > 10 \mid Y < 20)$.
   (d) Determine $E[Y]$ e verify that $\mathrm{Var}(Y)$ does not exist.
   (e) Consider the random variable $Z = 10(Y - 1)$. Determine the density of Z.

3. Let $X_i \mid \lambda \underset{iid}{\sim} \mathrm{Po}(\lambda)$, $i = 1, 2$. Determine the distribution of $X_1 \mid X_1 + X_2 = x$, where $x \in \mathbb{N}$.

4. Show that the moment generating function of $Z \sim \mathrm{N}(0, 1)$ is given by $M_Z(z) = \exp(z^2/2)$, $z \in \mathbb{R}$. Using the McLaurin series of M_Z, show that the raw moments of odd order of Z are zero, and the raw moments of even order are given by $\mu'_{2r} = (2r)!/(r!\ 2^r)$.

5. Supose that $X = Y + Z$, where Y and Z are independent random variables and, with $0 < n < m$, $X \sim \chi^2(m)$, $Z \sim \chi^2(n)$. Determine the distribution of Y.

6. Let $X \mid \lambda \sim \mathrm{U}(0, \lambda)$, where $\lambda > 0$.
   (a) Show that $-\ln(X/\lambda) \mid \lambda \sim \mathrm{Ex}(1)$.
   (b) Consider random variables $X_i \mid \lambda \underset{iid}{\sim} X$, $i = 1, \ldots, n$, where $n \in \mathbb{N}$. Determine the probability distribution of $Y = -2\sum_{i=1}^{n} \ln(X_i/\lambda)$.

7. Let (X, Y) be a bivariate random variable with joint probability density given by $f(x, y) = \lambda^2\ e^{-\lambda y}$, $0 < x < y$, where $\lambda > 0$. Show that
   (a) $X \mid \lambda \sim \mathrm{Ex}(\lambda)$.
   (b) $Y \mid \lambda \sim \mathrm{Ga}(2, \lambda)$
   (c) $X \mid Y = y, \lambda \sim \mathrm{U}(0, y)$, $y > 0$.

8. Consider a discrete pair (X, Y) with joint probability mass function given by

$x \backslash y$	-1	0	1
0	0	1/3	0
1	1/3	0	1/3

   Use this example to verify that mean independence is not symmetric, and that uncorrelatedness does not imply mean-independence.

9. Supppose that $Y = Z - X$ and that Y is stochastically independent of both Z and X. Show that $P(Y = c) = 1$ for some constant c.