Advanced Topics in Statistics – PhD program in MAEM

1.st Semester 2010/2011 – Regular Exam – Duration: 2 hours **January 12 2011**

Note: please fully justify all your answers.

1. Let $X_1, \ldots, X_n$ denote a random sample of size n from an $\mathrm{Ex}(\theta)$ population, where $\theta > 0$. Let $X_{(1)} = \min\{X_1, \ldots, X_n\}$ and $\bar{X} = \sum_{i=1}^n X_i/n$. In this context, show that, as $n \to +\infty$,

a) $a_n X_{(1)} \xrightarrow{d} \mathrm{Ex}(\theta)$ as long as the sequence of real numbers (a_n) is such that $\lim_{n\to+\infty} n/a_n = 1$. (1.5)

b) $b_n X_{(1)} \xrightarrow{P} 0$ as long as the sequence of real numbers (b_n) is such that $\lim_{n\to+\infty} n/b_n = +\infty$. (1.5)

c) $nX_{(1)}(1 - X_{(1)}) \xrightarrow{d} \mathrm{Ex}(\theta)$. (2.0)

d) $2\sqrt{n}(\sqrt{\theta \bar{X}} - 1) \xrightarrow{d} \mathrm{N}(0,1)$. (2.0)

2. Consider a bivariate population (X, Y) whose joint probability distribution is such that, for some $\lambda > 0$,

$$X \mid \lambda \sim \mathrm{Ex}(1/\lambda) \quad \text{and} \quad Y \mid X = x, \lambda \sim \mathrm{N}(0, \lambda x) \ .$$

Denote by $(X_1, Y_1), \ldots, (X_n, Y_n)$ a random sample of size n extracted from this population. In this context,

a) Show that $T = \sum_{i=1}^n X_i + \frac{1}{2} \sum_{i=1}^n Y_i^2/X_i$ is a complete and sufficient statistic for λ. (2.0)

b) Verify that $E[Y^2/X \mid \lambda] = \lambda$. (1.5)

c) Show that $S = 2T/(3n)$ is the UMVU estimator of λ. (1.5)

d) Is S also the most efficient estimator of λ? (1.5)

e) Determine $\mathrm{Var}(S \mid \lambda)$. (1.5)

3. Let $X_1, \ldots X_n$ denote a random sample of size n from a population X with probability density function given by

$$f(x \mid \theta) = \frac{2}{\theta^2}\, x, \quad 0 < x < \theta$$

where $\theta > 0$. Denote by $x_1, \ldots, x_n$ the corresponding observed sample. The goal is to implement a Bayesian analysis of these data.

a) Suppose that, *a priori*,

$$\pi(\theta) \propto \frac{1}{\theta^2}\, I_{(1,+\infty)}(\theta) \ .$$

Does this *a priori* distribution correspond to a proper probability distribution? If so, determine the corresponding normalizing constant. (1.5)

b) Assuming the *a priori* distribution described above, show that

$$\pi(\theta \mid x_1, \ldots, x_n) = (2n+1)\ (x_{(n)} \vee 1)^{2n+1}\ \theta^{-(2n+2)}\ I_{(x_{(n)} \vee 1, +\infty)}(\theta)$$

where $x_{(n)}$ denotes the sample maximum and $a \vee b \equiv \max\{a, b\}$. (2.0)

c) Before observing X_{n+1}, you are given the possibility of betting on the realization of two events: $\{X_{n+1} \le 1\}$ or $\{X_{n+1} > 1\}$. Taking into account the Bayesian analysis herein developed, devise a strategy that, as a function of $x_1, \ldots, x_n$, tells you on which of the events one should bet. (1.5)