Advanced Topics in Statistics 2020/2021
Homework set 4

1. Using the definition, show that $(\bar{X}, S^2)$ is sufficient for (μ, σ^2) in the context of a random sample of size n from a $\mathrm{N}(\mu, \sigma^2)$ population, where $\mu \in \mathbb{R}$ and $\sigma > 0$. Show the same result using the factorization criterion.

2. Consider a random sample of size 3 from a $B(1, \theta)$ population. Define the sample space and show that the following statistics are all equivalent:

$$\begin{aligned}
\boldsymbol{T} &= (X_{(1)}, X_{(2)}, X_{(3)}) \\
\boldsymbol{U} &= (X_1 + X_2 + X_3, X_1X_2 + X_1X_3 + X_2X_3, X_1X_2X_3) \\
\boldsymbol{V} &= (X_1 + X_2 + X_3, X_1^2 + X_2^2 + X_3^2, X_1^3 + X_2^3 + X_3^3)
\end{aligned}$$

3. Let $\Pi^1 = \{\Pi^1_\alpha : \alpha \in A\}$ e $\Pi^2 = \{\Pi^2_\beta : \beta \in B\}$ denote two partitions of the sample space $\mathcal{X}$. We say that Π^1 is finer than Π^2 iff for all $\alpha \in A$ there exists $\beta \in B$ such that $\Pi^1_\alpha \subset \Pi^2_\beta$. Show that if Π^1 is induced by the statistics T_1 and Π^2 is induced by the statistic T_2, than to say that Π_1 is finer that Π_2 is equivalent to state that $T_2(x) = g(T({}_1(x)))$ for all $x \in \mathcal{X}$, that is, that T_2 is a function of T_1.

4. Determine minimal sufficient statistics in the context of random samples of size n extracted from the following models:

   (a) Shifted exponential: $f(x \mid \lambda, \delta) = \delta e^{-\delta(x-\lambda)} I_{[\lambda,+\infty)}(x)$, $\lambda, \delta > 0$

   (b) Logistic: $f(x \mid \lambda) = e^{-(x-\lambda)}/[1 + e^{-(x-\lambda)}]^2$, $\lambda \in \mathbb{R}$

   (c) Double exponential, or Laplace: $f(x \mid \lambda) = \frac{1}{2} e^{-|x-\lambda|}$, $\lambda \in \mathbb{R}$

   (d) Cauchy: $f(x \mid \lambda) = \{\pi[1 + (x-\lambda)^2]\}^{-1}$, $\lambda \in \mathbb{R}$

5. Show that the family of distributions of X is part of the location-scale family with location parameter λ and scale parameter δ if and only if the distribution of the random variable $(X - \lambda)/\delta$ does not depend on the unknown parameters (λ, δ).

6. Show that in the context of a statistical model $\mathcal{F} = \{f(\cdot \mid \theta) : \theta \in \Theta\}$ the class of sufficient and complete statistics is either empty or it coincides with the class of minimal sufficient statistics.

7. Show that in the context of the Cauchy model defined by

$$f(x \mid \lambda) = \{\pi[1 + (x-\lambda)^2]\}^{-1}, \ \lambda \in \mathbb{R}$$

   the statistic $\boldsymbol{T} = (T_1, \ldots, T_n)$ with $T_i = X_{(n)} - X_{(i)}$, $i = 1, \ldots, n-1$ and $T_n = X_{(n)}$ is minimal sufficient and $n - 1$ of its components are ancillary. What can we say about the completeness of T?