Advanced Topics in Statistics – PhD program in MAEM

1.st Semester 2011/2012 – Regular Exam – Duration: 2 hours **january 13 2012**

Note: please fully justify all your answers.

1. Denote by $X_1, X_2, \ldots$ a sequence of random variables which are independent and identically distributed, all following a $\mathrm{N}(0,1)$ distribution. Let $S_n = \sum_{i=1}^n X_i^2$. In this context,

a) Show that $X_1^2 \sim \chi^2(1)$. [NB: $\Gamma(1/2) = \sqrt{\pi}$.] (1.5)

b) Determine θ such that $S_n/n \xrightarrow{P} \theta$. (1.5)

c) Determine the limit in distribution of $\sqrt{n}(S_n/n - 1)$. (2.0)

d) Determine the limit in distribution of $\sqrt{2}(\sqrt{S_n} - \sqrt{n})$. (2.0)

2. The Weibull model with scale parameter equal to 1, $\mathrm{Wei}(\theta, 1)$, $\theta > 0$, is defined by a probability density function given by

$$f(x \mid \theta) = \theta \; x^{\theta - 1} \; \exp(-x^\theta) \;, \quad x > 0 \;.$$

Let $X_1, \ldots, X_n$ denote a random sample of size n from a population $X \sim \mathrm{Wei}(\theta, 1)$, $\theta > 0$.

a) Show that $(X_{(1)}, \ldots, X_{(n)})$ is a minimal sufficient statistic for θ. (1.5)

b) Show that if $X \sim \mathrm{Wei}(\theta, 1)$, then the distribution of $Y = \ln X$ belongs to the scale family with scale parameter $\delta = 1/\theta$. (1.5)

c) Comment on the following statement: the statistic

$$T = (\ln X_{(1)}, \ln X_{(2)} / \ln X_{(1)}, \ldots, \ln X_{(n)} / \ln X_{(n-1)})$$

is minimal sufficient but $n - 1$ of its components are ancillary. It cannot, therefore, be complete. (2.0)

3. Denote by $\mathcal{F}$ the subfamily of the log-normal distribution with log-scale parameter equal to zero, i.e., $\mathcal{F} = \{\mathrm{LN}(0, \sigma^2) : \sigma > 0\}$, whose probability density function is given by

$$f(x \mid \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \; \frac{1}{x} \; \exp\left[-\frac{(\ln x)^2}{2\sigma^2}\right] \;, \quad x > 0 \;.$$

Additionally, we know that if $X \sim \mathrm{LN}(0, \sigma^2)$ then $Y = \ln X \sim \mathrm{N}(0, \sigma^2)$. Let $X_1, \ldots, X_n$ denote a random sample from $\mathcal{F}$.

a) Show that if $X \sim \mathrm{LN}(0, \sigma^2)$ then $E[X] = e^{\sigma^2/2}$. (1.5)

b) Show that the method of moments estimator of σ^2 is a consistent estimator of σ^2. (1.5)

c) Verify that the maximum likelihood estimator of σ^2 is $T = \sum_{i=1}^n (\ln X_i)^2 / n$. (2.0)

d) Show that T is an unbiased estimator of σ^2. (1.5)

e) Can we say that T is the UMVU estimator of σ^2? (1.5)

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