Advanced Topics in Statistics – PhD program in MAEM

1.$^{\text{st}}$ Semester 2013/2014 – Regular Exam – Duration: 2 hours **January 24 2014**

Note: please fully justify all your answers.

1. Let $X_1, X_2, \ldots, X_n, \ldots$ denote a sequence of independent and identically distributed random variables with common probability distribution function given by

$$F(x) = 1 - \left(\frac{a}{x}\right)^b \ , \quad x > a$$

where $a > 0$ and $b > 0$. Let $X_{(1)} = \min\{X_1, \ldots, X_n\}$, $X_{(n)} = \max\{X_1, \ldots, X_n\}$, and $\bar{X} = \sum_{i=1}^n X_i/n$. You can use the facts $E[X_1] = ab/(b-1)$, if $b > 1$, and $\text{Var}(X_1) = a^2\, b/[(b-1)^2(b-2)]$, if $b > 2$.

a) Show that $X_{(1)} \xrightarrow{P} a$. (1.5)

b) Determine sequences of real numbers, (a_n) and (b_n), such that (2.0)

$$\frac{X_{(n)} - a_n}{b_n} \xrightarrow{d} Z$$

where Z follows a Fréchet distribution: $P(Z \leq z) = e^{-z^{-b}}$, $z > 0$.

c) Consider the case $b = 3$. Determine a transformation $g(\cdot)$ such that (2.0)

$$\sqrt{n}\left(g(\bar{X}) - g(\mu)\right) \xrightarrow{d} \text{N}(0,1) \ ,$$

where $\mu = 3a/2$.

2. Consider a population (X, Y) such that, given an unknown parameter $\theta > 0$, X and Y are independent, $X \mid \theta \sim \text{Ex}(\theta)$ and $Y \mid \theta \sim \text{Ex}(1/\theta)$. Let (X_i, Y_i), $i = 1, \ldots, n$ denote a random of size n from (X, Y), where $n > 1$.

a) Show that $T = (T_1, T_2) = (\sum_{i=1}^n X_i, \sum_{j=1}^n Y_j)$ is a minimal sufficient statistic for the statistical model in question. What does this tells us about the existence of a most efficient estimator for θ? (2.0)

b) Show that T is not complete. Are there any sufficient statistics which are complete? (2.0)

c) Verify that the maximum likelihood estimator of θ^2 is $U = T_2/T_1$. (2.0)

d) Show that U is a biased estimator of θ^2 and determine an unbiased estimator of θ^2. (2.0)

e) Show that U is a (weakly) consistent estimator of θ^2. (1.5)

3. Suppose X follows a geometric distribution with success probability $\theta \in (0,1)$, that is,

$$f(x \mid \theta) = \theta(1-\theta)^{x-1}, \quad x = 1, 2, 3, \ldots$$

You can use the fact $E[X] = 1/\theta$.

Let $x_1, \ldots, x_n$ denote the observed value of a random sample of size $n > 1$ from X. We are interested in implementing a Bayesian analysis of these data, and will adopt the Jeffreys prior for θ.

a) Show that $\theta \mid x_1, \ldots, x_n \sim \text{Be}(n, \sum x_i - n + 1/2)$. (2.0)

b) Suppose we estimate θ using the posterior mean. Do you see any advantages in using this estimate rather than the method of moments estimate? (1.0)

c) Obtain $E[X_{n+1} \mid x_1, \ldots, x_n]$, where X_{n+1} denotes another observation from the model, statistically independent from $X_1, \ldots, X_n$. (2.0)