Advanced Topics in Statistics 2020/2021
Homework set 1

1. Show that convergence in the rth mean implies convergence in the sth mean if $r > s \geq 1$:

   (a) Recall Jensen's inequality and use the fact that $x \mapsto |x|^p$ $(p \geq 1)$ is a convex function in $\mathbb{R}$ (no need to prove these facts) to show that as long as all moments exist,
   $$E[|X|^r] \geq (E[|X|^s])^{r/s} \ .$$

   (b) Conclude the result.

2. Consider the situation where $\Omega = (0,1)$ and P is such that $P(A) = \int_A d\omega$. Let α be a real number and $X_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, be given by
   $$X_n(\omega) = \begin{cases} \alpha^n & \text{if } \omega < 2^{-n} \\ 0 & \text{if } \omega \geq 2^{-n} \ . \end{cases}$$

   (a) Verify using the definition that

      i. $X_n \xrightarrow{as} 0$.

      ii. $X_n \xrightarrow{P} 0$.

      iii. $X_n \xrightarrow{d} 0$.

   (b) Show that $X_n \xrightarrow{r} 0$ iff $|\alpha| < 2^{1/r}$, where $r \geq 1$. Conclude that almost sure convergence does not imply convergence in the rth mean.

   (c) Show that, if $1 \leq r < s$, convergence in the rth mean does not imply convergence in the sth mean.

3. For $n \geq 1$ consider the sample space $\Omega_n = \{1, 2, \ldots, n\}$ with $P_n(A) = \frac{\#A}{n}$ for any $A \subset \Omega_n$. Let $X_n : \Omega_n \to \mathbb{R}$ be given by $X_n(\omega) = \omega/n$.

   (a) Show that $X_n \xrightarrow{d} X$ where $X \sim \mathrm{U}(0,1)$.

   (b) Does it make sense to investigate whether $X_n \xrightarrow{P} X$? Why?

4. Let $\{X_n\}$ be a sequence of random variables defined on the same probability space and c a real number. Show that if $X_n \xrightarrow{d} c$ then $X_n \xrightarrow{P} c$.