Advanced Topics in Statistics 2020/2021
Homework set 7

1. Consider the exponential model with mean $1/\lambda$, $\lambda > 0$, i.e.,

$$f(y \mid \lambda) = \lambda \exp(-\lambda y), \; y > 0 \; .$$

   (a) Determine the Fisher information about λ in an observation of the model.

   (b) Determine the Jeffreys prior for λ. Is it an improper distribution?

   (c) Determine the posterior distribution of λ corresponding to the Jeffreys prior having observed a random sample of size n.

   (d) Suppose that you are interested in the alternative parameterization $\mu = 1/\lambda$. Determine the posterior of μ associated to the Jeffreys prior on μ, again after having observed a random sample of size n. Justify your answer.

2. A random variable Y is said to follow a Pareto distribution if its density function is of the form

$$f(y \mid a, b) = ab^a y^{-(a+1)} I_{(b,+\infty)}(y)$$

   with $a, b > 0$.

   (a) Show that the Pareto family is the natural conjugate family of the statistical model $\{\mathrm{U}(0, \theta) : \; \theta > 0\}$.

   (b) Let $X_1, \ldots, X_n$ denote a random sample of size n from the model $\{\mathrm{U}(0, \theta) : \theta > 0\}$. Assuming that *a priori* θ is distributed according to a member of the natural conjugate family, determine the corresponding posterior distribution of θ.

   (c) The posterior determine in the previous question depends on the sample only through the sample maximum. Could this fact be anticipated? Why?

   (d) Suppose that the *a priori* information about θ allows the statistician to specify the *a priori* mean and variance of θ as m and v, respectively. Determine the member of the natural conjugate family of the model that reflects this information.

3. We say that a random variable X follows an inverse-gamma distribution, $X \sim \mathrm{IG}(\alpha, \beta)$ iff X follows the same distribution as $1/Y$, with $Y \sim \mathrm{G}(\alpha, \beta)$, that is,

$$f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \, y^{\alpha-1} \, \exp(-\beta y) \; , \qquad y > 0$$

   where $\alpha, \beta > 0$.

   (a) Determine the density of X if $X \sim \mathrm{IG}(\alpha, \beta)$.

   (b) Consider the model $\mathcal{F} = \{\mathrm{N}(0, \sigma^2) : \sigma > 0\}$ and let $W_1, \ldots, W_n$ denote a random sample extracted from $\mathcal{F}$.

      i. Determine the Jeffreys prior for σ^2 in these circumstances. Use this prior to solve the following questions.

ii. Show that $\sigma^2 \mid w_1, \ldots, w_n \sim \mathrm{GI}(n/2, \sum w_i^2/2)$.

iii. Show that the posterior mean of σ^2 only exists if $n > 2$, and that neither the posterior mean nor the posterior mode coincide with the maximum likelihood estimate of σ^2.

iv. Determine the posterior predictive distribution of $W_{n+1} \mid \sigma^2 \sim \mathrm{N}(0, \sigma^2)$, with W_{n+1} independent of $W_1, \ldots, W_n$ given σ^2.

v. Suppose that you are interested in predicting the outcome of a random variable $Z \mid \sigma^2 \sim \mathrm{Ex}(\sigma^2)$ that you consider to be statistically independent of $W_1, \ldots, W_n$. Show that $E[Z \mid w_1, \ldots, w_n] = n / \sum_{i=1}^n w_i^2$.