Advanced Topics in Statistics 2020/2021
Homework set 2

1. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common density given by

$$f(x \mid \theta) = \theta \; x^{-(\theta+1)} \; , \quad x > 1$$

where $\theta > 0$. Denote by G_n the geometric mean of the first n variables, that is,

$$G_n = \left(\prod_{i=1}^{n} X_i \right)^{1/n} \; .$$

(a) Verify that $\ln X_i \sim \mathrm{Ex}(\theta)$.

(b) Show that $G_n \xrightarrow{P} \exp(1/\theta)$.

(c) Prove that

$$\sqrt{n} \; (\ln G_n - 1/\theta) \xrightarrow{d} \mathrm{N}(0, \theta^{-2}) \; .$$

(d) Verify that

$$\sqrt{n} \; [G_n - \exp(1/\theta)] \xrightarrow{d} \mathrm{N}(0, \theta^{-2} \exp(2/\theta)) \; .$$

(e) Show that

$$\sqrt{n} \; \frac{G_n - \exp(1/\theta)}{G_n} \xrightarrow{d} \mathrm{N}(0, \theta^{-2}) \; .$$

(Finalizing the proof of the Delta Method)

2. Show that if $a_n(T_n - \theta) \xrightarrow{d} T$ then $T_n \xrightarrow{P} \theta$ as long as the sequence of real numbers (a_n) is such that $a_n \to +\infty$.

3. Recall that in class we were able to conclude that

$$\sqrt{n} \; (g(X_n) - g(\theta_0)) = \sqrt{n} \; g'(\theta_0)(X_n - \theta_0) + \sqrt{n} \; r(X_n - \theta_0)$$

where $r(x)/x \to 0$ as $x \to 0$, and that we only need to show that $\sqrt{n} \, r(X_n - \theta_0) \xrightarrow{P} 0$ to finish the proof.

(a) Justify the statement $r(X_n - \theta_0)/(X_n - \theta_0) \xrightarrow{P} 0$.

(b) Finally show that $\sqrt{n} \; r(X_n - \theta_0) \xrightarrow{P} 0$.