Chapter 1

Probability

1.1 Basic concepts and results

A **random experiment** is when a set of all possible outcomes is known, but it is impossible to predict the actual outcome of the experiment. A **sample space**, denoted as Ω, contains all possible outcomes of the experiment. An **event** is a subset of Ω. We say that $A \subset \Omega$ has occrred if and only if the outcome of the experiment is an element of A. Formally, the family of events forms a σ-algebra of subsets of Ω that we denote by $\mathcal{A}$.

Note:

- $\Omega \in \mathcal{A}$
- $A \in \mathcal{A} \Rightarrow \bar{A} \in \bar{\mathcal{A}}$,where $\bar{A}$ indicates the compliment of A
- $A_1, A_2, \cdots \in \mathcal{A}$
- $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

1.1.1 Probability measures

Definition 1.1.1: Kolmogorov's axioms

- $P(A) \geq 0$
- $P(\Omega) = 1$
- If $A_i \bigcap A_j = \emptyset, i \neq j$, then $P(\cup_i A_i) = \sum_i P(A_i)$

Probability measure $P : \mathcal{A} \to \mathbb{R}$ satisfying Kolmogorov's axioms has the following properties:

- $P(\varnothing) = 0$
- $A \subset B \Rightarrow P(A) \leq P(B)$
- $0 \leq P(A) \leq 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(\bar{A}) = 1 - P(A)$
- $P(A - B) = P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Definition 1.1.2: Conditional probability

If $P(B) > 0$,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We are re-evaluating the probability of A given the B space.

Let $\{A_1, A_2, \cdots\}$ denote a partition of $\Omega : \cup_i A_i = \Omega;\ A_i \cap A_j = \emptyset, i \neq j$. Meaning union makes up Ω and are mutually exclusive. Then if $P(A_i) > 0$ for all i

Theorem 1.1.1 Total probability theorem

$$P(B) = \sum_i P(B|A_i)P(A_i)$$

$B = B \cap \Omega = B \cap [\cup_i A_i] = \cup_i (B \cap A_i)$ and $P(\cup_i B \cap A_i) = \sum_i P(B \cap A_i)$

Theorem 1.1.2 Bayes' theorem

If $P(B) > 0$

$$P(A_i|B) = \frac{P(B|A_j)P(A_j)}{\sum_i P(B|A_i)P(A_i)}$$

$$P(\underbrace{A_j}_{\text{explanation}} | \underbrace{B}_{\text{evidence}}) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\underbrace{P(B)}_{\text{substitute with total probability theorem}}}$$

1.1.2 Random variables

Definition 1.1.3: Random variable

Function defined in Ω and taking values in $\mathbb{R}$

$$X : \Omega \to \mathbb{R}$$

$$\omega \mapsto X(\omega) = x$$

A random variable induces a probability measure in $\mathbb{R}$ that we denote by P_X: if $B \subset \mathbb{R}$, $P_X(B) = P(A)$, where $A = X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$. Formally, there must be a σ-algebra of subsets of $\mathbb{R}$,$\mathcal{B}$, and we have to verify that for every set $B \in \mathcal{B}$ we have $X^{-1}(B) \in \mathcal{A}$. Typically, $\mathcal{B}$ is the so called Borel σ-algebra and it suffices to make sure that X satisfies $X^{-1}((-\infty, x]) \in \mathcal{A}$,$\forall x \in \mathbb{R}$.

Basically what it means is that we don't know if $X^{-1}(B) \in \mathcal{A}$ and for which B can I compute $P_X(B)$. If $X^{-1}(B) \in \mathcal{A}$ for B is in the Borel σ-algebra, then X is measurable.

Definition 1.1.4: Distribution function of a random variable

X: for all $x \in \mathbb{R}$

$$F_X(x) = P_X((-\infty, x]) = P(X \leq x)$$

It is suffice to know $F_X(\cdot)$ to be able to compute $P_X(B)$ for all $B \in \mathcal{B}$.

- For all $a > b$, $P(a < X \leq b) = F_X(b) - F_X(a)$
- $F_X(-\infty) = 0$; $F_X(\infty) = 1$

- F_X is right-continuous and non-decreasing
- The set of points at which F_X is discontinuous is either finite or countable (at most countable)

Definition 1.1.5: Discrete random variable

X is a discrete random variable if D_X is such that $P_X(D_X) = 1$

The probability mass function of X is defined as $f_X(x) = F_X(x) - \lim_{y \to x} F_X(y) = \begin{cases} P(X = x) & \text{if } x \in D_X \\ 0 & \text{otherwise} \end{cases}$

Any f satisfying the following is a probability mass function

- $f(x) \geq 0$ for all x
- $f(x) > 0$ iff $x \in D$, where $D \subset \mathbb{R}$ is finite or countable
- $\sum_{x \in D} f(x) = 1$

For any event $B \subset \mathbb{R}$, $P(X \in B) = \sum_{x \in B \cap D_X} f_X(x)$.

Note:

$F_X(x) = \sum_{y \leq x} f_X(y)$

$F_X(x) = P(X \leq x)$ cumulative distribution function
$\downarrow$
$f_X(x) = P(X = x)$ probability mass function
where $0 \leq f_X(x) \leq 1$

Discrete distribution include Bernoulli, binomial, Poisson, geometric, negative binomial, multinomial, hypergeometric, etc.

Definition 1.1.6: Continuous random variable

X is continuous if $P_X(D_X) = 0, D_X = \varnothing$ and if additionally there is f_X such that for all $x \in \mathbb{R}$

- $f_X(x) \geq 0 \longrightarrow$ probability density function
- $F_X(x) = \int_{-\infty}^{+\infty} f(x)\,dx = 1$

At the points where F_X is differentiable, we have $F'_X(x) = f_X(x)$.

Any f satisfying the following conditions is a probability density function

- $f(x) \geq 0$ for all x
- $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

Continuous distributions include uniform, exponential, gamma, chi-squared, normal. t-student, F-Snedcor, beta, Pareto, Weibull, log-normal, etc.

1.1.3 Functions of a random variable

Let X be a r.v. and $Y = h(X)$ where $h : \mathbb{R} \longrightarrow \mathbb{R}$
In general, if $X = g(Y)$ with g invertible and differentiable, and X continuous, we have

$$f_Y(y) = |g'(y)|\, f_x(g(y))$$

Proof. $\dfrac{\partial F_X(x)}{\partial x} = f_X(x)$
Using chain rule: $(f \circ g)'(x) = [f(g(x))]' = f'(g(x))g'(x)$ ■

Definition 1.1.7: Expected value

Let $Y = h(X)$, a linear function.

The expected value of Y is defined by $E[Y] = \begin{cases} \sum_x h(x)\, f_X(x) & \text{if } X \text{ discrete} \\ \int_{-\infty}^{+\infty} h(x)\, f_X(x)\, dx & \text{if } X \text{ continuous} \end{cases}$

Formally, we must addtionally verify that the integral or series are absolutely convergent. $E[Y]$ may not exist.

There are two ways to compute $E[Y]$ with $Y = h(X)$, either use the definition above, or first obtain the distribution of Y and compute $E[Y] = \begin{cases} \sum_y y\, f_Y(y) & \text{if } Y \text{ discrete} \\ \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy & \text{if } Y \text{ continuous} \end{cases}$. The two methods are equivalent.

Definition 1.1.8: Raw moment of oder k

$\mu'_k = E[X^k]$

Definition 1.1.9: Central moment of order k

$\mu_k = E[(X-\mu)^k]$, $\mu = E[X]$

Definition 1.1.10: Moment generating function

$M_X(s) = E[e^{sX}]$ whenever the expectation exists for s in a neighborhood of the origin.

- If $M_X(s)$ exists, then X has moments of all orders and $M^{(k)}(0) = E[X^k]$
- The moment generating function, when it exists, identifies the probability distribution

Some useful **properties**:

- $E[h_1(X) + h_2(X)] = E[h_1(X)] + E[h_2(X)]$
- If $c \in \mathbb{R}$, then $E[cX] = cE[X]$; $E[c] = c$
- If $c \in \mathbb{R}$, then $Var(cX + b) = c^2 Var(X)$
- $Var(X) = E[X^2] - (E[X])^2$
- $Var(X) \geq 0$; $Var(X) = 0 \Leftrightarrow P(X = c) = 1$ for some $c \in \mathbb{R}$

1.1.4 Bivariate random variables

$$\begin{aligned} (X, Y) : \Omega &\to \mathbb{R}^2 \\ \omega &\mapsto (X(\omega), Y(\omega)) = (x, y) \end{aligned}$$

If (X, Y) discrete, we define the joint probability mass function as $f(x, y) = P(X = x, Y = y)$. If $(X; Y)$ continuous, then there exists the joint probability density function, $f(x, y)$ such that for all $(x, y) \in \mathbb{R}^2$,

- $f(x, y) \geq 0$
- $F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, dv\, du$

Example 1.1.1

X = weight, Y = height $\Rightarrow$ Z = BMI

Definition 1.1.11: Marginal distributions

$$f_X(x) = \begin{cases} \sum_y f(x,y) & \text{if } (X,Y) \text{ discrete} \\ \int_{-\infty}^{+\infty} f(x,y)\,dy & \text{if } (X,Y) \text{ continuous} \end{cases}$$

Definition 1.1.12: Expectation of $Z = h(X,Y)$

$$E[Z] = \begin{cases} \sum_x \sum_y h(x,y)\,f(x,y) & \text{if } (X,Y) \text{ discrete} \\ \int_{-\infty}^{+\infty} h(x,y)\,f(x,y)\,dy\,dx & \text{if } (X,Y) \text{ continuous} \end{cases}$$

Definition 1.1.13: Conditional didstributions

$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$, y fixed: $f_Y(y) > 0$

function of x for every y where $f_Y(y) > 0$

Definition 1.1.14: Raw moment of order (r,s)

$\mu'_{(r,s)} = E[X^r\,Y^s]$

Definition 1.1.15: Central moment of order (r,s)

$\mu_{(r,s)} = E[(X-\mu_X)^r\,(Y-\mu_Y)^s]$

Definition 1.1.16: Covariance

$\text{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \mu_{(1,1)}$

If x and y are positively associated $\longrightarrow \text{Cov}(x,y) > 0 \longrightarrow$ If x is larger than its mean, then typically y is larger than its mean.

Some useful **properties**:

- $\text{Cov}(X,Y) = E[X,Y] - E[X]E[Y]$
- $\text{Cov}(X,Y) = \text{Cov}(Y,X)$
- $\text{Cov}(cX,Y) = c\text{Cov}(X,Y), c \in \mathbb{R}$
- $\text{Cov}(X+Y,Z) = \text{Cov}(X,Z) + \text{Cov}(Y,Z)$
- $\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y) \pm 2\,\text{Cov}(X,Y)$

Example 1.1.2 (Portfolio management)

$\text{Cov}(x,y) < 0$
$\text{Var}(x,y) < \text{Var}(x) + \text{Var}(y)$

Theorem 1.1.3 Law of iterated expectation

If $Z = h(X,Y)$ then $E[Z] = E_X[E[Z|X]]$

Theorem 1.1.4 Law of total variance

$\text{Var}(Y) = \text{Var}_X(E[Y|X]) + E_X[\text{Var}(Y|X)]$

Other useful tricks:

- $E[h(X)\, Y \mid X = x] = h(x)\, E[Y \mid X = x]$
- $\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, E[Y|X])$

  Proof.
  $$\begin{aligned}\mathrm{Cov}(X, E[Y|X]) &= E[X\, E[Y|X]] - E[X]\, E[E[Y|X]] \\ &= E[E[XY|X]] - E[X]\, E[Y] \\ &= E[XY] - E[X]\, E[Y] \\ &= \mathrm{Cov}(X, Y)\end{aligned}$$
  ■

1.1.5 Independence

Definition 1.1.17: Stochastic independence

X and Y are stochastically independent if and only if $\forall (x, y) \in \mathbb{R}^2$, $f(x, y) = f_X(x)\, f_Y(y)$

If X and Y are independent, then

- $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

  Proof. $\mathrm{Var}(X \pm Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \pm 2 \times \underbrace{\mathrm{Cov}(X, Y)}_{\to 0}$ ■

- $M_{X+Y}(s) = M_X(s)\, M_Y(s)$

  Proof. $M_{X+Y}(s) = E[e^{s(X+Y)}] = E[\underbrace{e^{sx}}_{u}\ \underbrace{e^{sy}}_{v}]$

  x and y independent stochastically $\Rightarrow$ u and v independent

  $M_{X+Y}(s) = E[e^{sx}]\, E[e^{sy}] = M_X(s)\, M_Y(s)$ ■

- $\mathrm{Cov}(X, Y) = 0$

  Proof. $\mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \underbrace{E[XY]}_{X, Y \text{uncorrelated}} - E[X]E[Y] = E[X]E[Y] - E[x]E[Y] = 0$ ■

- $E[X^r Y^s] = E[X^r]\, E[Y^s]$
- $E[Y \mid X = x] = E[Y]$; $E[X \mid Y = y] = E[X]$
- $f_{X|Y=y}(x) = f_X(x)$; $f_{Y|X=x}(y) = f_Y(y)$

  Proof. $f_{X|Y=y}(x) = \dfrac{f(x, y)}{f_Y(y)} = \dfrac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x)$ ■

Definition 1.1.18: Mean independence

Y is mean independent of X iff $E[Y \mid X = x]$ does not depend on x for all x.

Proof. $E[Y|X = x] = c$

$E[Y|X] = c \Rightarrow E[E[Y|X]] = c \Rightarrow E[Y] = c \longrightarrow$ conditional is equal to marginal ■

Definition 1.1.19: Uncorrelatedness

X and Y are uncorrelated iff $\text{Cov}(X, Y) = 0$

Useful **results**:

- If X and Y are stochastically independent, then Y is mean-independent of X, and X is mean independent of Y.
- If Y is mean-independet of X, then X and Y are uncorrelated. The converse is not true.

  Proof. Y mean independence of $X \Rightarrow \text{Cov}(X, Y) = \text{Cov}(X, E[Y|X]) = \text{Cov}(X, c) = 0 \Rightarrow$ uncorrelated ■

- If Y is uncorrelated with X, then $E[XY] = E[X]E[Y]$
- If Y is mean-independent of X, then $E[X^kY] = E[X^k]E[Y]$ for all k
- If Y and X are stochastically independent, then $E[X^kY^r] = E[X^k]E[Y^r]$ for all k, r

Note:
stochastic independence $\Rightarrow$ mean independence $\Rightarrow$ uncorrelatedness

1.2 Convergence of sequences of random variables

If $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables and X is a random variable,

$$X_n : \underbrace{\Omega}_{\text{exists probability, } \sigma\text{-algebra}} \longrightarrow \mathbb{R}$$

$$X_n \longrightarrow X \qquad \text{as } n \longrightarrow +\infty$$

n can be population size, or can be the number of iterations for Monte Carlo simulation.

1.2.1 Notions of convergence of sequences

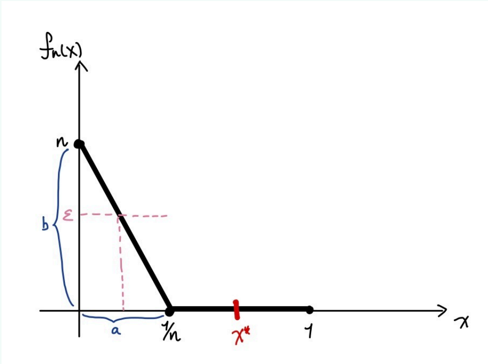
Notions of **convergence of sequences**: let $f_n, f : [0, 1] \longrightarrow \mathbb{R}$

- Point wise convergence: $f_n(x) \longrightarrow f(x)$ for all $x \in [0, 1]$
- Uniform convergence: $\sup\limits_{x\in[0,1]} |f_n(x) - f(x)| \longrightarrow 0$
- Convergence in L^P: $\int_0^1 |f_n(x) - f(x)|^P \, dx \longrightarrow 0$
- Convergence in measure: $\mu(A_{n,\epsilon}) \longrightarrow 0$ for all $\epsilon > 0$ where $A_{n,\epsilon} = \{x \in [0, 1] : |f_n(x) - f(x)| > \epsilon\}$

Example 1.2.1

$f_n : [0, 1] \longrightarrow \mathbb{R}$

$$f_n(x) = \begin{cases} 0 & 1/n \le x \le 1 \\ n - n^2 x & 0 \le x < 1/n \end{cases}$$



As $n \longrightarrow \infty$, a becomes smaller, b becomes bigger.

- Point wise convergence
  $\forall x \in [0,1]$
  $\forall x^* > 0, f_n(x^*) = 0 \quad \text{for } n > N \quad \text{except } f_n(0) = 0 \longrightarrow \infty$

  $$\Rightarrow f_n(x) \longrightarrow \begin{cases} 0 & \text{if } x \in [0,1] \\ \infty & \text{if } x = 0 \end{cases} \Rightarrow f_n \text{ is not converging pointwise to the null function.}$$

- Uniform convergence
  $\max |f_n(x)| = n \longrightarrow +\infty \quad x \in [0,1] \Rightarrow f_n$ does not converge uniformly to the null function.

- Convergence in $L^1 \longrightarrow P = 1$

  $$\int_0^1 |f_n(x)|\, dx = \frac{1}{2} = \underbrace{\frac{1}{n} \times n \times \frac{1}{2}}_{\text{area under the triangle}} \quad \Rightarrow f_n \text{ does not converge in } L^1 \text{ to the null function.}$$

- Convergence in measure
  $A_{n,\epsilon} \subset [0, \frac{1}{n}]$
  $\mu(A_{n,\epsilon}) \leq \mu([0, \frac{1}{n}]) = \frac{1}{n} \longrightarrow$ as $n \longrightarrow \infty, \mu \longrightarrow \infty \Rightarrow f_n$ converges to the null function in measure.

1.2.2 Convergence of random variables

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables and X is a random variable, all defined in the same probability space $(\Omega, \mathcal{A}, P)$.

Definition 1.2.1: Almost surely convergence

X_n converges to X almost surely, or with probability 1, $X_n \xrightarrow{\text{a.s.}} X$, iff

$$P[\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}] = 1$$

Similar to pointwise convergence, no need for expectation.

Note:

$$\underbrace{P(\underbrace{X_n(\omega)}_{\text{set}} \longrightarrow \underbrace{x(\omega)}_{\text{point}})}_{\text{set of which it happens has a probability of 1}} = 1$$

Definition 1.2.2: Convergence in the rth mean

X_n converges to X in the rth mean, $r \geq 1$, $X_n \xrightarrow{r} X$, iff

$$E[|X_n - X|^r] \to 0$$

Each point will be weighted with the same probability. Expectation is involved in this case.

Note:

When $r = 2$, it is the mean square convergence, often used for quality checking.

Definition 1.2.3: Convergence in probability

X_n converges in probability to X, $X_n \xrightarrow{P} X$, iff for all $\epsilon > 0$

$$P(|X_n - X| > \epsilon) \to 0$$

It is similar to measure in convergence. Often used to check for quality of estimator. Note that this is no longer a Lebesque measure, it is now a probability measure. $P\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon\} \to 0$ as $n \to 0$.

Definition 1.2.4: Convergence in distribution

X_n converges in distribution to X, $X_n \xrightarrow{d} X$, iff

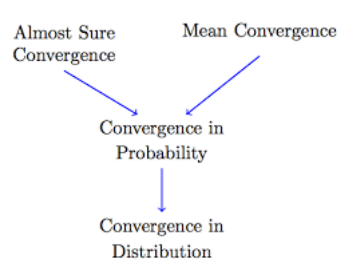
$$F_n(x) \to F_n$$

for all x continuity point of F, where $F(x) = P(X \leq x)$ and $F_n(x) = P(X_n \leq x)$

Has nothing to do with the random variable. Often used for hypothesis testing. It does not need the requirement that all points are defined in the same probability space $(\Omega, \mathcal{A}, P)$ as there is no ω in the density function.

Some useful **remarks**:

- Convergence in distribution is really about the convergence of the sequence of probability functions and not the random variables themselves.
- When defining convergence in the rth mean, it is assumed that the corresponding expected values exist: $E[|X_n|^r] < \infty$ and $E[|X|^r] < \infty$
- When $X_n \xrightarrow{1} X$, we say that X_n converges to X in mean; when $X_n \xrightarrow{2} X$, we say that X_n converges to X in quadratic mean.



Proof. **Convergence in mean implies convergence in probability**

$E[|X_n - X|] \to 0 \Rightarrow P(|X_n - X| > \epsilon) \to 0, \forall \epsilon > 0$

Using **Markov inequality**: $P(|y| > a) \leq \dfrac{|E[X_n - X|]}{\epsilon}$

$$0 \le \lim_{n\to\infty} P(|X_n - X| > \epsilon) \le \lim_{n\to\infty} \frac{\overbrace{E[|X_n - X|]}^{\to 0}}{\epsilon} = 0 \blacksquare$$

Proof. **Proof of convergence in probability implies convergence in distribution**

$$X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} \Leftrightarrow P(|X_n - X| > \epsilon) \to 0 \Rightarrow P(X_n \le x) \to P(X \le x), \forall x$$

let $\epsilon > 0$,

$$F_n(x) = P(X_n \le x)$$

$$F(x) = P(X \le x)$$

Using the **total probability theorem**: $P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$

$$F_n(x) = P(\underbrace{X_n \le x}_{A}) = P(\underbrace{X_n \le x}_{A}, \underbrace{X \le x + \epsilon}_{B}) + P(\underbrace{X_n \le x}_{A}, \underbrace{X > x + \epsilon}_{\bar{B}}) \le F(x + \epsilon) * P(|X_n - x| > \epsilon)$$

$$F(x - \epsilon) - P(|X_n - X| > \epsilon) \le F_n(x) \le F(x + \epsilon) + \underbrace{P(|X_n - X| < \epsilon)}_{\to 0}$$

as $n \to \infty$, $\underbrace{F(x - \epsilon)}_{\xrightarrow{\epsilon \to 0} F(X)} \le \lim_{n\to\infty} F_n(x) \le \underbrace{F(x + \epsilon)}_{\xrightarrow{\epsilon \to 0} F(X)}$ $\blacksquare$

Some **converses**:

- If $X_n \xrightarrow{P} X$, then there exists $\{n_k\}_{k=1}^{+\infty}$ such that $X_{n_k} \xrightarrow{a.s.} X$ when $k \to +\infty$
- If $|X_n|^r$ is uniformly integratable, then $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{r} X$

Theorem 1.2.1 Skorokhod representation theorem

If $X_n \xrightarrow{d} X$ then there exists a probability space $(\Omega', \mathcal{A}', P')$ and r.v. $\{Y_n\}$ and Y, defined in Ω' such that

- $P'(Y_n \le y) = P(X_n \le y)$ and $P'(Y \le y) = P(X \le y)$ for all $y \in \mathbb{R}$. This means that X_n and Y_n are marginally equal in distribution, the same for X and Y.
- $Y_n \xrightarrow{a.s.} Y$

Other useful **results**:

- $X_n \xrightarrow{P} c \Leftrightarrow X_n \xrightarrow{d} c$, where $c \in \mathbb{R}$

  Proof. $X_n \xrightarrow{d} x \Rightarrow X_n \xrightarrow{P} c \Leftrightarrow P(X_n \le x) \to \begin{cases} 0 & x < c \\ 1 & x < c \end{cases}$, not continuous at c

  $P(|X_n - c| > \epsilon) \to 0$, $\forall \epsilon > 0$

$$\begin{aligned} P(|X_n - c| > \epsilon) &= P(X_n - c > \epsilon) + P(X_n - c < -\epsilon) \\ &= P(X_n > \epsilon + c) + P(X_n < c - \epsilon) \\ &= 1 - P(X_n \le \epsilon + c) + P(X_n < c - \epsilon) \\ &\le 1 - P(X_n \le \underbrace{\epsilon + c}_{>c}) + P(X_n \le \underbrace{c - \epsilon}_{<c}) \\ &\to 1 - 1 + 0 = 0 \end{aligned}$$

$\blacksquare$

- Since $E[(X_n - \theta)^2] = \text{Var}(X_n) + (E[X_n] - \theta)^2$ if $\text{Var}(X_n) \to 0$ and $E[X_n] \to \theta$. We have convergence in mean square to θ, and hence convergence in probability to θ.

Theorem 1.2.2 Continous mapping theorem

Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then

- $X_n \xrightarrow{a.s.} X \Rightarrow h(X_n) \xrightarrow{a.s.} h(X)$
- $X_n \xrightarrow{d} X \Rightarrow h(X_n) \xrightarrow{d} h(X)$
- $X_n \xrightarrow{P} X \Rightarrow h(X_n) \xrightarrow{P} h(X)$

Theorem 1.2.3 Slutsky theorem

Let $\{X_n\}$ and $\{Y_n\}$ be sequences of random variables, X a random variable and c a real number. If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$, then

- $X_n + Y_n \xrightarrow{d} X + c$
- $Y_n X_n \xrightarrow{d} cX$
- $X_n / Y_n \xrightarrow{d} X/c$ as long as $c \neq 0$

Wrong Concept 1.1: $X_n + Z_n \neq 2X$

Suppose that $X_n \xrightarrow{d} X$ where $X \sim N(0,1)$. Then with $Z_n = -X_n$ we have $Z_n \xrightarrow{d} X$. However, $X_n + Z_n = 0$, hence $X_n + Z_n$ does not converge in distribution to $2X$ as one might expect.

cdf of Z_n converges to cdf of X_n

$$\begin{aligned}
Z_n \xrightarrow{d} X &\Leftrightarrow P(Z_n \leq z_n) \to \Phi(z_n), \forall z \in \mathbb{R} \\
&\Leftrightarrow P(-X_n \leq z) = P(X_n \geq -z) = 1 - P(X_n \leq -z) \\
&\to 1 - \Phi(-z) \\
\therefore Z_n \xrightarrow{d} X &
\end{aligned}$$

This is why the Slutsky theorem is important, it showcases safe procedures.

Example 1.2.2 ($X_n \sim t(n) \Rightarrow X_n \xrightarrow{d} N(0,1)$ using Slutsky)

$X_n \sim t(n), X_n = \dfrac{u_n}{\sqrt{\frac{v_n}{n}}}$

$$\text{Assumptions: } \begin{cases} u_n \text{independent of } v_n \\ u_n \sim N(0,1) \\ v_n \sim \chi^2(n) \end{cases}$$

What would be nice is to show that $\sqrt{\frac{v_n}{n}}$ converges to 1 then we can apply the Slutsky theorem.

Using the **mean square convergence**, we have

$$\begin{aligned}
\text{Var}(\frac{v_n}{n}) &= \frac{\text{Var}(v_n)}{n} = \frac{2n}{n^2} = \frac{2}{n} \to 0 \\
E[\frac{v_n}{n}] &= \frac{E[v_n]}{n} = \frac{n}{n} = 1 \to 1
\end{aligned}$$

We now have mean square convergence to 1.

Using the **Continuous mapping theorem**, we have

$$\frac{v_n}{n} \xrightarrow{P} 1 \Rightarrow \sqrt{\frac{v_n}{n}} \xrightarrow{P} 1$$

$\Rightarrow \frac{v_n}{n} \xrightarrow{2} 1$ and $\frac{v_n}{n} \xrightarrow{P} 1$

Now using the **Slutsky theorem**, we have

$$X_n = \frac{u_n}{\sqrt{\frac{v_n}{n}}} \xrightarrow{d} u_n \sim N(0,1)$$

1.3 Important asymptotic results

Theorem 1.3.1 Weak law of large numbers
Let $\{X_n\}_{n=1}^{+\infty}$ be a sequence of independent and identically distributed random variables, with $E[X_n] = \mu$ and $\mathrm{Var}(X_n) = \sigma^2 < \infty$. Let also $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then we have that

$$\overline{X}_n \xrightarrow{P} \mu$$

Proof. Goal: $\overline{X}_n \xrightarrow{P} \mu \Rightarrow P(|\overline{X}_n - \mu| > \epsilon) \to 0$

Checking the validity of Chebychov's inequality,

$$E[\overline{X}_n] = E[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n}\sum_{i=1}^{n} \underbrace{E[X_i]}_{\to\mu} = \frac{1}{n}\, n\, \mu = \mu$$

We can now apply the **Chebychov's inequality**: $P(\underbrace{|X-\mu|}_{\text{distance of distribution from its mean}} > \epsilon) \leq \frac{\mathrm{Var}(X)}{\epsilon^2}$

$$P(|\overline{X}_n - \mu| > \epsilon) \leq \frac{\overbrace{\mathrm{Var}(\overline{X}_n)}^{1}}{\epsilon^2} = \frac{\sigma^2}{n\,\epsilon^2} \to 0$$

$$1\text{: } \mathrm{Var}(\overline{X}_n) = \frac{1}{n}\mathrm{Var}(\sum_{i=1}^{n} X_i) \underbrace{=}_{\mathrm{Var}(\sum)=\sum \mathrm{Var}+\underbrace{2\mathrm{Cov}}_{\text{iid}\to 0}} \frac{1}{n^2}\, n\, \sigma^2 = \frac{\sigma^2}{n} \blacksquare$$

Intuitively, the WLLN tell us that $\overline{X}_n$ becomes more and more concentrated around μ as n increases.

Theorem 1.3.2 Strong law of large numbers
Under the same conditions as above, we have

$$\overline{X}_n \xrightarrow{a.s.} \mu$$

Actually, it is only necessary to assume that $E[|X_i|] < +\infty$ for both laws to hold.

Theorem 1.3.3 Central limit theorem

Let $\{X_n\}_{n=1}^{+\infty}$ be a sequence of iid random variables possessing finite variance. Let $\mu = E[X_n]$ and $\sigma^2 = \text{Var}(X_n)$. Let also $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and

$$Z_n = \sqrt{n}\,\frac{\overline{X}_n - \mu}{\sigma} = \frac{\overline{X}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} = \frac{\overline{X}_n - E[\overline{X}]}{\sqrt{\text{Var}(\overline{X})}} \xrightarrow{d} N(0,1)$$

Then we have

$$Z_n \xrightarrow{d} Z$$

where $Z \sim N(0,1)$

Proof. **Proof with assumption of existance of mgf**

Assume

1. $M_n(S) = E[e^{sX_n}]$ exists
2. $M_n(s) \longrightarrow M(s)$ for $s \in (-s_0, s_0)$

then $M(s) = E[e^{sX}] \Rightarrow X_n \xrightarrow{d} X$

Idea: X_n are iid r.v.
$E[e^{sX_n}] = M_{X_n}(s)$ exists for $s \in (-s_0, s_0) \Rightarrow Z_n = \sqrt{n}\frac{\bar{X}_n - \mu}{\sigma}$
Need to show $M_{Z_n}(s) \longrightarrow M_{N(0,1)}(s) = e^{s^2/2} \longrightarrow$ mgf of Z_n goes to $e^{s^2/2}$, the mgf of the normal distribution.

$$Z_n = \sqrt{n}\frac{\overline{X}_n - \mu}{\sigma} \underbrace{=}_{\text{Annex 1}} \frac{1}{\sqrt{n}}\sum_{i=1}^{n} y_i \tag{1.1}$$

Annex 1:

$$\begin{aligned}
Y_i &= \frac{X_i - \mu}{\sigma}, \text{standardized version of the } X_i\text{'s} \\
\sum Y_i &= \frac{\sum(X_i - \mu)}{\sigma} = \frac{\sum X_i - n\mu}{\sigma} = \frac{n\overline{X}_n - n\mu}{\sigma} = n\frac{\overline{X}_n - mu}{\sigma} \\
\frac{1}{\sqrt{n}}\sum Y_i &= \frac{1}{\sqrt{n}} n\frac{\overline{X}_n - mu}{\sigma} = \sqrt{n}\frac{\overline{X}_n - \mu}{\sigma}
\end{aligned}$$

Using the moment generating function

$$\begin{aligned}
M_{Z_n}(s) &= E[e^{sZ_n}] = E[e^{s\frac{1}{\sqrt{n}}\sum Y_i}] \\
&= M_{\sum Y_i}(\frac{s}{\sqrt{n}}) \\
&= M_{Y_1}(\frac{s}{\sqrt{n}}) \times M_{Y_2}(\frac{s}{\sqrt{n}}) \times \cdots \times M_{Y_n}(\frac{s}{\sqrt{n}}) \longrightarrow \text{mgf of the sum of the variable is the product} \\
&= [M_Y(\frac{s}{\sqrt{n}})]^n \\
&= \sum_{k=0}^{2} M_Y^{(k)}(0)\frac{s^k}{k!} + \underbrace{r(s)}_{\frac{r(s)}{s^2}\to 0 \text{ as } s\to 0} \qquad \longrightarrow \text{Taylor's expansion of 2nd order, Annex 2} \\
&= 1 + \frac{s^2}{2!} + r(s)
\end{aligned} \tag{1.2}$$

Annex 2:

$$M_Y^{(k)}(0) = E[Y^k]$$
$$M_Y^{(0)}(0) = E[Y^0] = 1$$
$$M_Y^{(1)}(0) = E[Y^1] = 0$$
$$M_Y^{(2)}(0) = E[Y^2] = \frac{E[(x_i-\mu)^2]}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1$$

Back to the moment generating function

$$\begin{aligned} M_{Z_n}(s) &= [M_Y(\frac{s}{\sqrt{n}})]^n \\ &= [1 + \frac{s^2/2}{n} + r(\frac{s}{\sqrt{n}})]^n \\ &= [1 + \frac{\frac{s^2}{2} + \overbrace{n\, r(s/\sqrt{n})}^{\to 0}}{n}]^n \underbrace{\to}_{\text{Annex 3}} e^{s^2/2} \end{aligned} \tag{1.3}$$

Annex 3:

$$[1 + \frac{u_n}{v_n}]^{v_n} \to e^c, u_n \to c, v_n \to \infty$$

■

The CLT is often used to compute probabilities of the type $P(\overline{X}_n \leq x)$ approximating them by $\Phi(\sqrt{n}\frac{(x-\mu)}{\sigma})$ for sufficiently large n.

$$\begin{aligned} P(\overline{X}_n \leq x) &= P(\sqrt{n}\frac{\bar{X}-\mu}{\sigma} \leq \sqrt{n}\frac{x-\mu}{\sigma}) \\ &\approx \Phi(\sqrt{n}\frac{x-\mu}{\sigma}) \\ P(Z_n \leq z) &\to \Phi(z) \end{aligned}$$

Intuitively, the CLT tells us that the distribution of $\bar{X}_n$ is well approximated by a normal distribution for sufficiently large n as long as the variance is finite. Additionally, if the distribution of X_n is close to symmetric, then the rate of convergence is faster. Rate of convergence is related to the coefficient of symmetry, $\frac{E[(X-\mu)^3]}{(\text{Var}(X))^{3/2}} = \gamma_1$. If the distribution is symmetric, $\gamma_1 = 0$.

Theorem 1.3.4 Lévy's continuity theorem

Suppose that $\{X_n\}_{n=1}^{+\infty}$ is a sequence of random variables and let $M_n(s)$ denote the mgf of X_n, $n = 1, 2, \cdots$. Additionally assume that

$$\lim_{n\to+\infty} M_n(s) = M(s)$$

for s in a neighborhood of the origin, and that $M(\cdot)$ is the mgf of a random variable X.

In these circumstances,

$$X_n \xrightarrow{d} X$$

Example 1.3.1 (Application : Bernoulli)

$\{X_n\}_{n=1}^{+\infty}$ iid $B(1,\theta)$ where $\theta \in (0,1)$. By the **central limit theorem**,

$$\sqrt{n}\frac{\overline{X}_n - \theta}{\sqrt{\theta(1-\theta)}} \xrightarrow{d} N(0,1)$$

On the other hand, the **WLLN** ensures that $\overline{X}_n \xrightarrow{d} \theta$.
By the **continuous mapping theorem**,

$$\frac{\sqrt{\theta(1-\theta)}}{\sqrt{\overline{X}_n(1-\overline{X}_n)}} \xrightarrow{P} 1$$

and **Slutsky's theorem** allows us to conclude that

$$\sqrt{n}\,\frac{\overline{X}_n-\theta}{\sqrt{\theta(1-\theta)}}\,\frac{\sqrt{\theta(1-\theta)}}{\sqrt{\overline{X}_n(1-\overline{X}_n)}} = \sqrt{n}\,\frac{\overline{X}_n-\theta}{\sqrt{\overline{X}_n(1-\overline{X}_n)}} \xrightarrow{d} N(1,0)$$

which in practice means that, for large n

$$P\left(\frac{\overline{X}_n-\theta}{\sqrt{\overline{X}_n(1-\overline{X}_n)}} \leq x\right) \approx \Phi(x)$$

Proof. $X_i \sim B(1,\theta), \quad E[x_i]=\theta \quad \mathrm{Var}(x_i)=\theta(1-\theta)$

By the **CLT**, $\sqrt{n}\,\dfrac{\overline{X}_n-\theta}{\underbrace{\sqrt{\theta(1-\theta)}}_{\text{issue in the denominator}}} \xrightarrow{d} N(0,1)$

$$\sqrt{n}\,\frac{\overline{X}_n-\theta}{\sqrt{\overline{X}_n(1-\overline{X}_n)}} \xrightarrow{d} N(0,1) = \underbrace{\sqrt{n}\,\frac{\overline{X}_n-\theta}{\sqrt{\theta(1-\theta)}}}_{\xrightarrow{d} N(0,1) \text{ by CLT}}\;\underbrace{\frac{\sqrt{\theta(1-\theta)}}{\sqrt{\overline{X}_n(1-\overline{X}_n)}}}_{\xrightarrow{P} 1,\text{ Annex 1}}$$

Annex 1:

- $\overline{X}_n \xrightarrow{P} E[X_i]=\theta$ by **WLLN**
- $\sqrt{\overline{X}_n(1-\overline{X}_n)} \to \sqrt{\theta(1-\theta)}$ by **continuous mapping theorem**

■

Example 1.3.2 (Application : $P(X\in A)$ using Simple Monte Carlo)

Notice that $P(X\in A)=E[Y]$ where $Y=I_A(X)=\begin{cases}1 & ,x\in A\\ 0 & ,x\notin A\end{cases}$

Let $X_1, X_2, \cdots, X_M$ be iid r.v. with the same distribution as X, and $Y_i = I_A(X_i), i=1,\dots,M$. Then by **SLLN**,

$$\bar{Y}_M = \frac{1}{M}\sum_{i=1}^{M} Y_i \xrightarrow{a.s.} E[Y] = P(X\in A)$$

where M is the simulation length.

For sufficiently large M,

$$P(X\in A) \approx \frac{1}{M}\sum_{i=1}^{M} y_i = \frac{1}{M}\underbrace{\#\{i=1,\dots,M : x_i\in A\}}_{\text{observed proportions of } x_i\in A}$$

Simple Monte Carlo allows us to replace the analytical knowledge of a probability distribution by a sufficiently large sample of iid draws from the distribution since almost all aspects of that probability distribution can be arbitrarilly approximated using that sample.

Example 1.3.3 (Application : $f(a)$ for some $a \in \mathbb{R}$ using Simple Monte Carlo)
For a continous distribution with density f,

$$\begin{aligned} f(a) &= \lim_{\delta \to 0} \frac{F(a+\delta - F(a))}{\delta} \\ &= \frac{1}{\delta}\frac{1}{M}\#\{i = 1, \ldots, M : a < x_i < a+\delta\} \end{aligned}$$

That is, the histogram of $x_1, \ldots, x_M$ is an approximation to the density of X.

Theorem 1.3.5 Delta method
Let $\{X_n\}_{n=1}^{+\infty}$ be a sequence of r.v. such that $\forall \theta \in \Theta$

$$\sqrt{n}(X_n - \theta) \xrightarrow{d} N(0, \sigma^2)$$

Let $\underbrace{\theta_0}_{\text{interior point of } \Theta} \in \overbrace{\Theta}^{\text{open set}}$ and g be a differentiable function such that $g'(\theta_0) \neq 0$. Then

$$\sqrt{n}(\underbrace{g(X_n)}_{\text{typically non-linear}} - g(\theta_0)) \xrightarrow{d} N(0, \sigma^2 [g'(\theta_0)]^2)$$

Proof. Using the 1st order Taylor expansion
$g(x) = g(\theta_0) + g'(\theta_0)(x-\theta_0) + r(x-\theta_0), \quad \dfrac{r(x-\theta_0)}{x-\theta_0} \longrightarrow 0$ as $x \longrightarrow \theta_0$

$$\begin{aligned} g(x_n) - g(\theta_0) &= g'(\theta_0)(x_n - \theta_0) + r(x_n - \theta_0) \\ \sqrt{n}\,(g(x_n) - g(\theta_0)) &= \underbrace{\sqrt{n}\,(g(x_n) - g(\theta_0))}_{\text{Annex 1}} + \underbrace{\sqrt{n}\, r(x_n - \theta_0)}_{\text{Annex 2}} \\ \sqrt{n}\,(g(x_n) - g(\theta_0)) &= \underbrace{\sqrt{n}\,(g(x_n) - g(\theta_0))}_{\xrightarrow{d} N(0, [g'(\theta_0)]^2 \sigma^2)} + \underbrace{\sqrt{n}\, r(x_n - \theta_0)}_{\xrightarrow{P} 0} \end{aligned}$$

Annex 1:
$\sqrt{n}(x_n - \theta_0) \xrightarrow{d} N(0, \sigma^2)$
By **Slutsky's theorem**, $\underbrace{g'(\theta_o)}_{\text{constant}, \xrightarrow{P} g'(\theta_0)} \underbrace{\sqrt{n}(x_n - \theta_0)}_{\xrightarrow{d} T(\cdot)} \xrightarrow{d} g'(\theta_0) N(0, \sigma^2) = N(0 \times g'(\theta_0), [g'(\theta_0)]^2 \sigma^2)$
$\Rightarrow \sqrt{n}(x_n - \theta_0) \xrightarrow{d} N(0, [g'(\theta_0)]^2 \sigma^2)$

Annex 2:
Step 1:
$(T_n - \theta) = \underbrace{\frac{1}{a_n}}_{\xrightarrow{P} 0} \underbrace{a_n(T_n - \theta)}_{\xrightarrow{d} T(\cdot)} \xrightarrow{d} 0 \times T(\cdot) = 0 \Rightarrow T_n \xrightarrow{d} \theta \quad \underbrace{\Leftrightarrow}_{\text{this applies because } \theta \text{ is a constant}} \quad T_n \xrightarrow{P} \theta$
$\therefore \sqrt{n}(T_n - \theta) \xrightarrow{d} T \Rightarrow T_n \xrightarrow{P} \theta$
Step 2:
$\sqrt{n}(x_n - \theta_0) \xrightarrow{d} N(0, \sigma^2)$
By step 1, we can conclude that $x_n \xrightarrow{P} \theta_0 \quad X_n - \theta_0 \xrightarrow{P} 0$
We also know that $\dfrac{r(x)}{x} \longrightarrow 0$

By **continuous mapping theorem**, $\dfrac{r(x_n-\theta_0)}{x_n-\theta_0} \xrightarrow{P} 0$

Step 3:

$$\sqrt{n}\,r(x_n-\theta_0) \Leftrightarrow \underbrace{\sqrt{n}(x_n-\theta_0)}_{\xrightarrow{d} N(0,\sigma^2)} \underbrace{\frac{r(x_n-\theta_0)}{x_n-\theta_0}}_{\xrightarrow{P} 0}$$

By **Slutsky's theorem**, $\sqrt{n}\,r(x_n-\theta_0) \xrightarrow{d} 0 \underbrace{\Rightarrow}_{\text{true for constant}} \sqrt{n}r(x_n-\theta_0) \xrightarrow{P} 0$ ■

Example 1.3.4 (Application : log-odds ratio)

Suppose that $X_1,\dots,X_n$ are iid $B(1,\theta)$. Then the **CLT** ensures that

$$\sqrt{n}\frac{\bar{X}_n-\theta}{\sqrt{\theta(1-\theta)}} \xrightarrow{d} N(0,1)$$

which is equivalent to $\sqrt{n}(\bar{X}_n-\theta) \xrightarrow{d} N(0,\theta(1-\theta))$.

- $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n x_i$ representing the proportion of successes in the random sample
- θ representing the probability of success in the population

Now we are interested in the asymptotic distrbution of $Y_n = \ln\frac{\bar{X}_n}{1-\bar{X}_n}$ which is the empirical log odds of success, a non-linear function. With $g(x) = \ln\frac{x}{1-x}$, following $g'(x) = \frac{1}{x(1-x)}$. The **delta method** ensures that

$$\sqrt{n}(Y_n - \ln\frac{\theta}{1-\theta}) \xrightarrow{d} N(0, [\theta(1-\theta)]^{-1})$$

which is often written as

$$Y_n \overset{a}{\sim} N(\ln\frac{\theta}{1-\theta}, \frac{[\theta(1-\theta)]^{-1}}{n})$$

Proof. Asymptotic distribution of $T_n = \ln\frac{\bar{X}_n}{1-\bar{X}_n}$

$$\sqrt{n}\tfrac{\bar{X}_n-\theta}{\sqrt{\theta(1-\theta)}} \xrightarrow{d} N(0,1) \Leftrightarrow \sqrt{n}(\bar{X}_n-\theta) \xrightarrow{d} N(0,\theta(1-\theta))$$

$$g(x) = \ln\frac{x}{1-x} \to g'(x) = \frac{(\frac{x}{1-x})'}{(\frac{x}{1-x})} = \frac{\frac{1-x-(-1)x}{(1-x)^2}}{\frac{x}{1-x}} = \frac{1-x+x}{(1-x)^2}\frac{1-x}{x} = \frac{1}{x(1-x)}$$

Applying the **delta method**, $\sqrt{n}(T_n - g(\theta_0)) \xrightarrow{d} N(0, [g'(\theta_0)]^2\sigma^2)$

$$\Rightarrow \sqrt{n}(T_n - \ln\frac{\theta_0}{1-\theta_0}) \xrightarrow{d} N\left(0, \left(\frac{1}{\theta(1-\theta)}\right)^2\theta_0(1-\theta_0)\right) \Leftrightarrow \sqrt{n}(T_n - \ln\frac{\theta_0}{1-\theta_0}) \xrightarrow{d} N(0, [\theta_0(1-\theta_0)]^{-1})$$ ■

Example 1.3.5 (Application : variance stabalizing)

Suppose $X_1,\dots,X_n$ are $B(0,\theta)$. Then the **CLT** ensures that

$$\sqrt{n}\frac{\bar{X}_n-\theta}{\sqrt{\theta(1-\theta)}} \xrightarrow{d} N(0,1)$$

Note that the asymptotic variance depends on the true value of θ, meaning that the variance, σ^2 is not fixed, thus giving us the motive to stabalize the variance. Our goal is to find a g such that $\sqrt{n}(g(\bar{X}_n) - g(\theta)) \xrightarrow{d} N(0,1)$, which is the same as solving for $g'(x) = \frac{1}{\sqrt{\theta(1-\theta)}}$.

$[g'(x)]^2\theta(1-\theta) = 1 \Leftrightarrow g'(x) = \frac{1}{\theta(1-\theta)} = \theta^{-1/2}(1-\theta)^{-1/2} \Rightarrow g(\theta) = 2 \arcsin\sqrt{\theta}$

After this, the asymptotic distribution would be normal with a constant variance.

$$\sqrt{n}(2 \arcsin\sqrt{\bar{X}_n} - 2 \arcsin\sqrt{\theta}) \xrightarrow{d} N(0,1)$$

When can we apply this technique?
$\sqrt{n}(X_n - \mu) \xrightarrow{d} N(0, \ln(\mu))$
From **delta method**, $\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} N(0, [g'(\mu)]^2 ln(\mu))$
the variance stabalizing transformation g satisfies $[g'(x)]^2\ln(\mu) = 1 \Leftrightarrow g'(\mu) = \dfrac{1}{\sqrt{\ln(\mu)}} \Rightarrow g(\mu) = \displaystyle\int_c^\mu \frac{1}{h(t)}\,dt$
c being some constant that ensures the integral exists, and with this c,
$\sqrt{n}(g(x_n) - g(\mu)) \xrightarrow{d} N(0,1)$

Chapter 2

Classical Statistical Model

2.1 Probability versus statistical inference

- Probability theory
- Statistical inference

2.2 Model specification

- Random sample
- Sampling
- IID random sampling

2.3 Statistics

- Statistic definition

2.4 Sampling distribution

- Definition
- Methods to obtain the sampling distribution of a statistic
  - Monte Carlo simulation
- Sample distribution of the sample moments
  - Sample moments
  - Properties of the sample mean
  - Properties of the sample variance
  - Properties of the bias-corrected sample variance
  - Properties of central sample moments
  - Asymptotic distribution of $\bar{X}$
- Order statistics