Advanced Topics in Statistics 2020/2021
Homework set 3

1. Let $X_1, \ldots, X_n$ be an iid random sample extracted from a $\mathrm{U}(0,\theta)$ population, where $\theta > 0$. Show that, as $n \to +\infty$,

   (a) $X_{(n)} \xrightarrow{P} \theta$

   (b) $n(1 - X_{(n)}/\theta) \xrightarrow{d} \mathrm{Ex}(1)$

   (c) $\sqrt{3n}\ (2\bar{X}/\theta - 1) \xrightarrow{d} \mathrm{N}(0,1)$

2. Denote by $X_1, \ldots, X_n$ an iid random sample of size $n > 2$ extracted from a continuous population with cdf F and pdf f.

   (a) Justify the following identity
   $$P(X_{(n)} \le y) = P(X_{(n)} \le y, X_{(1)} \le x) + P(X_{(n)} \le y, X_{(1)} > x)$$
   and use it to show that the joint pdf of $(X_{(1)}, X_{(n)})$ is such that
   $$G_{1,n}(x,y) = [F(y)]^n - [F(y) - F(x)]^n \ , \quad x < y \ .$$

   (b) Conclude that the joint pdf of the pair $(X_{(1)}, X_{(n)})$ is given by
   $$g_{1,n}(x,y) = n(n-1)f(x)f(y)[F(y) - F(x)]^{n-2} \ , \quad x < y \ .$$

3. Let $X_1, \ldots, X_n$ be an iid random sample extracted from a population X with mean μ and variance σ^2. Let $S_\star^2 = \sum_{i=1}^n (X_i - \mu)^2/n$, $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/n$, and assume that $\mu_4 = E[(X-\mu)^4]$ is finite.

   (a) Show that $S_\star^2 \xrightarrow{P} \sigma^2$.

   (b) Show that $\sqrt{n}(S_\star^2 - \sigma^2) \xrightarrow{d} \mathrm{N}(0, \mu_4 - \sigma^4)$.

   (c) Prove that $S^2 = S_\star^2 - (\bar{X} - \mu)^2$.

   (d) Show that $S^2 \xrightarrow{P} \sigma^2$.

   (e) Justify the following statement: $\sqrt{n}(\bar{X} - \mu)/S \xrightarrow{d} \mathrm{N}(0,1)$.

   (f) Show that $\sqrt{n}(S^2 - \sigma^2) \xrightarrow{d} \mathrm{N}(0, \mu_4 - \sigma^4)$, i.e., that S^2 and $S_\star^2$ have the same asymptotic distribution.

4. Let $\{X_m\}_{m=1}^{+\infty}$ and $\{Y_n\}_{n=1}^{+\infty}$ be two sequences of independent random variables possessing moment generating functions and
   $$\begin{aligned} X_m^\star &= \sqrt{m}(X_m - \mu) \\ Y_n^\star &= \sqrt{n}(Y_n - \theta) \ , \end{aligned}$$
   where $\sigma > 0$ and $\delta > 0$. Suppose $M_{X_m^\star}(s) \to \exp(\sigma^2 s^2/2)$ and $M_{Y_n^\star}(s) \to \exp(\delta^2 s^2/2)$ and $n, m \to +\infty$ in such a way that $\frac{\sigma^2/m}{\sigma^2/m + \delta^2/n} \to c$, where $0 < c < 1$. Show that, in these circumstances,
   $$Z_{m,n} = \frac{(X_m - Y_n) - (\mu - \theta)}{\sqrt{\sigma^2/m + \delta^2/n}} \xrightarrow{d} \mathrm{N}(0,1) \ .$$