ATS Jan 13 2015 <u>ABBREVIATED</u> SOLUTION 1.

1.

(a) $f(x|\theta) = \frac{1}{2\theta-\theta} I_{[\theta,2\theta]}(x) = \frac{1}{\theta} I_{[\theta,2\theta]}(x)$

$\Rightarrow f(\underset{\sim}{x}|\theta) = \theta^{-n} I_{[\theta,+\infty[}(x_{(1)}) \, I_{(0,2\theta]}(x_{(n)})$

$= \theta^{-n} I_{[x_{(n)}/2;\, x_{(1)}]}(\theta)$

$x_{(1)} \geq \theta \, , \; x_{(n)} \leq 2\theta \iff \frac{x_{(n)}}{2} \leq \theta \leq x_{(1)}$

$\therefore \; L(\theta|\underset{\sim}{x}) \propto \theta^{-n} I_{[x_{(n)}/2;\, x_{(1)}]}(\theta)$

θ^{-n} is a decreasing function of θ; hence the mle is $x_{(n)}/2$

(b) $P(|\hat{\theta}-\theta|>\varepsilon) = P(\hat{\theta}-\theta>\varepsilon) + P(\hat{\theta}-\theta<-\varepsilon)$

$= P(\hat{\theta} > \theta+\varepsilon) + P(\hat{\theta} < \theta-\varepsilon)$

$= P(X_{(n)} > 2(\theta+\varepsilon)) + P(X_{(n)} < 2(\theta-\varepsilon))$

$= 0 + \left[F(2(\theta-\varepsilon))\right]^n$

$F(x) = \frac{x-\theta}{\theta} \, , \; \theta < x < 2\theta$

$= \left(\frac{2(\theta-\varepsilon)-\theta}{\theta}\right)^n \qquad \theta \leq 2(\theta-\varepsilon) \leq 2\theta$

$= \left(\frac{2\theta-2\varepsilon-\theta}{\theta}\right)^n$

$= \left(\frac{\theta-2\varepsilon}{\theta}\right)^n \longrightarrow 0$

$\therefore \; \hat{\theta} \xrightarrow{P} \theta$

(c) $P(-n(\hat\theta-\theta) \le x) = P(\hat\theta - \theta \ge -x/n)$

$$= 1 - P\left(X_{(n)} \le 2(\theta - x/n)\right)$$

$$= 1 - \left[F\left(2(\theta - x/n)\right)\right]^n$$

$$= 1 - \left[\frac{2(\theta - x/n) - \theta}{\theta}\right]^n \qquad \theta < 2(\theta - x/n) < 2\theta$$

$$= 1 - \left(\frac{\theta - 2x/n}{\theta}\right)^n$$

$$= 1 - \left(1 - \frac{2x/\theta}{n}\right)^n \qquad \frac{\theta}{2} < \theta - x/n < \theta \Leftrightarrow 0 < x < n\theta$$

$$\cancel{-\theta/2 < -x/n < 0}$$

$$\longrightarrow 1 - e^{-2x/\theta}, \quad x > 0$$

$$= F_{Ex(2/\theta)}(x) \qquad x \in \mathbb{R}$$

(d) $\sqrt{n}(\hat\theta - \theta) = n(\hat\theta - \theta)\frac{1}{\sqrt{n}} \xrightarrow{d} 0$ by Slutsky etc ...

the result that states that $\sqrt{n}(\hat\theta - \theta) \xrightarrow{d} N(0, [I_X(\theta)]^{-1})$ is valid only under regularity conditions. This model is not regular as the support depends on θ.

(e)

$$\frac{f(\underset{\sim}{y}|\theta)}{f(\underset{\sim}{x}|\theta)} = \frac{I^{(\theta)}_{[y_{(n)}/2\ ;\ y_{(1)}]}}{I^{(\theta)}_{[x_{(n)}/2\ ;\ x_{(1)}]}}$$ which does not depend on θ iff

$$\begin{cases} x_{(n)}/2 = y_{(n)}/2 \\ x_{(1)} = y_{(1)} \end{cases}$$

And we can hence conclude that a minimal sufficient statistic is

$$(X_{(1)}, X_{(n)})$$

there are no estimators of θ which are sufficient.

(f)

$$g_n(x) = \frac{d}{dx}\left[\frac{x-\theta}{\theta}\right]^n = n\,\theta^{-n}(x-\theta)^{n-1}, \theta < x < 2\theta$$

$$g_1(x) = \frac{d}{dx}\left[1-\left(1-\frac{x-\theta}{\theta}\right)^n\right] = \frac{d}{dx}\left[1-\left(\frac{2\theta-x}{\theta}\right)^n\right]$$

$$= n\,\theta^{-n}\,(2\theta-x)^{n-1} \quad \theta < x < 2\theta$$

$$E[X_{(n)}] = \int_\theta^{2\theta} n\,\theta^{-n}\,x\,(x-\theta)^{n-1}\,dx =$$

$$= n\theta^{-n}\int_\theta^{2\theta} x\,(x-\theta)^{n-1}\,dx$$

$$a = 1 \quad b = -\theta \quad m = n-1$$

$$= n\,\theta^{-n}\,\frac{n\,x+\theta}{n(n+1)}(x-\theta)^n\Big|_\theta^{2\theta}$$

$$= \theta^{-n}\,\frac{2n\theta+\theta}{n+1}(2\theta-\theta)^n = \frac{2n+1}{n+1}\theta$$

$$\therefore \frac{n+1}{2n+1}X_{(n)} \text{ is unbiased}$$

$$E[X_{(1)}] = \int_\theta^{2\theta} x \cdot n\theta^{-n}(2\theta - x)^{n-1}\,dx$$

$$= n\theta^{-n}\int_\theta^{2\theta} x(-x+2\theta)^{n-1}\,dx$$

$$a = -1 \; ; \; b = 2\theta \; ; \; m = n-1$$

$$= n\theta^{-n} \; \frac{-nx - 2\theta}{n(n+1)} \cdot \left[-x+2\theta\right]^n \Big|_\theta^{2\theta}$$

$$= n\theta^{-n} - \frac{-n\theta - 2\theta}{n(n+1)} \; \theta^n$$

$$= \frac{n+2}{n+1}\,\theta$$

$$\therefore \frac{n+1}{n+2} X_{(1)} \text{ is unbiased for } \theta$$

(g) $T_1 - T_2$ is a function of the minimal sufficient statistic which has zero expectation but is clearly not zero. Hence, $(X_{(1)} ; X_{(n)})$ is not complete.

Since the m.s. statistic is bi-dimensional, there are no 1-dimensional sufficient statistics; hence, there are no estimators of θ which are sufficient.

2.

(a) $$f(x,y|\lambda) = f(y|x,\lambda)\, f(x|\lambda)$$

$$= \frac{\sqrt{\lambda x}}{\sqrt{2\pi}} \cdot \frac{1}{y} \cdot \exp\left\{ \frac{-(\ln y)^2}{2} \lambda x \right\} \lambda e^{-\lambda x},$$

$y>0, x>0$

$$= \frac{\sqrt{x}}{\sqrt{2\pi}\, y} \lambda^{3/2} \exp\left\{ -\lambda \left(\frac{(\ln(y))^2}{2} x + x \right) \right\}$$

$$\Rightarrow \ln L = c + \frac{3}{2}\ln\lambda - \lambda\left(\frac{(\ln y)^2}{2} x + x\right)$$

$$\frac{d\ln L}{d\lambda} = \frac{3}{2}\frac{1}{\lambda} - \left[\frac{(\ln(y))^2}{2} x + x\right] = S(\lambda|x,y)$$

$$\Rightarrow S(\lambda|\underset{\sim}{x},\underset{\sim}{y}) = \frac{3}{2}\frac{n}{\lambda} - \sum_{i=1}^{n}\left[\frac{(\ln y_i)^2}{2} x_i + x_i\right]$$

$$= -\frac{3}{2} n \left(\frac{2}{3} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{(\ln(y_i))^2}{2} x_i + x_i \right) - \frac{1}{\lambda} \right)$$

$$= -\frac{3}{2} n \left(\frac{1}{n} \sum_{i=1}^{n} \left[\frac{(\ln y_i)^2}{3} x_i + \frac{2}{3} x_i \right] - \frac{1}{\lambda} \right)$$

$\therefore$ the most efficient stimator of $\frac{1}{\lambda}$ is

$$T = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{(\ln(y_i))^2}{3} X_i + \frac{2}{3} X_i \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} Z_i$$

b) Since T is the most efficient estimator of $\frac{1}{\lambda}$,

$$Var(T) = \frac{(1/\lambda^2)^2}{I_{\underset{\sim}{X}}(\lambda)} = \frac{Var(Z_i)}{n}$$

$$\frac{d^2 \ln L}{d\lambda^2} = -\frac{3}{2}\frac{n}{\lambda^2} \Rightarrow I_{\underset{\sim}{X}}(\lambda) = \frac{3}{2}\frac{n}{\lambda^2}$$

$$\therefore \frac{1/\lambda^4}{\frac{3}{2} n/\lambda^2} = \frac{Var(Z_i)}{n} \Leftrightarrow \frac{1}{\lambda^4}\lambda^2 \frac{2}{3}\frac{1}{n} = \frac{Var(Z_i)}{n}$$

$$\Leftrightarrow Var(Z_i) = \frac{2}{3\lambda^2}$$

(c) by the CLT, since the $\{Z_i\}$ are iid,

$$\sqrt{n}\,(T - 1/\lambda) \xrightarrow{d} N\left(0, \frac{2}{3\lambda^2}\right)$$

$$\Leftrightarrow \sqrt{n}\,(T - \theta) \xrightarrow{d} N\left(0, \frac{2}{3}\theta^2\right)$$

by the Delta method we want $g(\cdot)$ such that $g'(\theta) = \sqrt{\frac{3}{2}}\,\frac{1}{\theta} \Rightarrow g(\theta) = \sqrt{\frac{3}{2}}\ln\theta$

$$\therefore g(x) = \sqrt{3/2}\,\ln x$$