Advanced Topics in Statistics — PhD program in MAEM

First Semester 2019/20 — Repeat Exam — Duration: 2 hours **February 6 2020**

Note: Please fully justify all your answers. Each question is worth 2 points.

1. Consider a population X with probability density function given by

$$f(x \mid \theta) = \frac{2}{\theta^2}\, x \ , \quad 0 \le x \le \theta$$

where $\theta > 0$ is an unknown parameter. Let $X_1, \ldots, X_n$ denote a random sample from X. Denote by $X_{(n)}$ the sample maximum.

a) Determine the limit in distribution of $-2n(X_{(n)} - \theta)$.

b) Show that

$$\sqrt{2n}\left(\frac{3\bar{X}}{\theta} - 2\right) \xrightarrow{d} \mathrm{N}(0,1)$$

c) Find transformations g and h such that

$$\sqrt{n}\,\left(g(\bar{X}) - h(\theta)\right) \xrightarrow{d} \mathrm{N}(0,1) \ .$$

2. We say that X follows a Lévy distribution, and write $X \sim \mathrm{L}(\mu, \lambda)$, $\lambda > 0$, $\mu \in \mathbb{R}$, if its density function is given by

$$f(x \mid \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi}}\,(x-\mu)^{-3/2}\ \exp\left[-\frac{\lambda}{2(x-\mu)}\right], \quad x \ge \mu \ .$$

Assume that $\mu = \mu_0$ is known, and let $X_1, \ldots, X_n$ denote a random sample from $X \mid \lambda \sim \mathrm{L}(\mu_0, \lambda)$.

a) Show that $T = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_0)^{-1}$ is the most efficient estimator of $\frac{1}{\lambda}$.

b) Show that $\mathrm{Var}((X_i - \mu_0)^{-1}) = 2/\lambda^2$

c) Is T also the UMVU estimator of $1/\lambda$?

d) What can we say about the joint distribution of T and $T \times (X_1 - \mu_0)$?

3. Consider a population (X, Y) such that

$$X \mid \theta \sim \mathrm{Ra}(\theta)$$
$$Y \mid X = x, \theta \sim \mathrm{N}(0, \theta x) \ .$$

Recall that $X \mid \theta \sim \mathrm{Ra}(\theta)$, that is, that X follows a Rayleigh distribution with parameter θ if, for some $\theta > 0$,

$$f_X(x \mid \theta) = \frac{x}{\theta}\ \exp\left(-\frac{x^2}{2\theta}\right), \quad x > 0 \ .$$

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ denote a random sample from (X, Y). To implement a Bayesian analysis of the observed data, denoted by $(\mathbf{x}, \mathbf{y})$, we decide to parametrize the model in terms of $\psi = 1/\theta$ and to use Jeffreys prior, $\pi^J(\psi)$.

a) Show that $\pi^J(\psi) \propto 1/\psi$.

b) Obtain the posterior mean of ψ.

c) Compute $E[X_{n+1}^2 \mid \mathbf{x}, \mathbf{y}]$.

END OF THE EXAM