Advanced Topics in Statistics – PhD program in MAEM

1.st Semester 2014/2015 – Regular Exam – Duration: 2 hours **January 13 2015**

Note: please fully justify all your answers.

1. Let $X_1, \ldots, X_n$ denote a random sample of size n extracted from a population X which, for some $\theta > 0$, is uniformly distributed in the interval $[\theta, 2\theta]$.

a) Show that the maximum likelihood estimator of θ is $\hat{\theta} = X_{(n)}/2$. (2.0)

b) Show that $\hat{\theta}$ is a (weakly) consistent estimator of θ. (2.0)

c) Show that $-n(\hat{\theta} - \theta) \xrightarrow{d} Y$ where $Y \sim \text{Ex}(2/\theta)$. (2.0)

d) What does c) imply about the convergence in distribution of $\sqrt{n}(\hat{\theta} - \theta)$? How do you reconcile this result with the result that characterizes the convergence in distribution of maximum likelihood estimators? (1.5)

e) Show that $(X_{(1)}, X_{(n)})$ is a minimal sufficient statistic. What are the implications of this on the estimation of θ? (1.5)

f) Show that $T_1 = \frac{n+1}{n+2}\, X_{(1)}$ and $T_2 = \frac{n+1}{2n+1}\, X_{(n)}$ are unbiased estimators of θ. (3.0)
Hint: $\int x(ax+b)^m dx = \frac{a(m+1)x - b}{a^2(m+1)(m+2)}(ax+b)^{m+1} + C$, $m \notin \{-1, -2\}$.

g) Show that the minimal sufficient statistic in (e) is not complete. (1.0)

2. Consider a population (X, Y) such that, for some $\lambda > 0$,

$$Y \mid X = x, \lambda \sim \text{LN}(0, 1/(\lambda\, x)) \quad \text{and} \quad X \mid \lambda \sim \text{Ex}(\lambda)\ .$$

Let (X_i, Y_i), $i = 1, \ldots, n$, denote a random sample from (X, Y). Recall that $W \sim \text{LN}(\mu, \delta)$ $\Leftrightarrow \ln W \sim \text{N}(\mu, \delta)$. Also,

$$f(w) = \frac{1}{\sqrt{2\pi\delta}}\, \frac{1}{w}\, \exp[-(\ln w)^2/(2\delta)],\ w > 0\ .$$

a) Show that the most efficient estimator of $1/\lambda$ can be written as (3.0)

$$T = \frac{1}{n} \sum_{i=1}^{n} Z_i$$

where $Z_i = (\ln Y_i)^2 X_i/3 + 2X_i/3$, $i = 1, \ldots, n$.

b) Show that $\text{Var}(Z_i \mid \lambda) = 2/(3\lambda^2)$. (2.0)

c) Find a transformation $g(\cdot)$ such that (2.0)

$$\sqrt{n}(g(T) - g(\theta)) \xrightarrow{d} \text{N}(0, 1)$$

where $\theta = 1/\lambda$.