Brief solution to the regular final exam of ATS, PhD program in MAEM, Jan/13/2012.

1. (a) For $x > 0$ we have $P(X_1^2 \le x) = P(-\sqrt{x} \le X_1 \le \sqrt{x}) = \Phi(\sqrt{x}) - \Phi(-\sqrt{x}) = 2\Phi(\sqrt{x}) - 1$. Hence, $f_{X_1^2}(x) = 2\ 1/2\ x^{-1/2}\ \phi(\sqrt{x})$ and noting that $\Gamma(1/2) = \sqrt{\pi}$ it is easy to recognize the formula for the density of a $\chi^2(1)$ distribution.

   (b) S_n/n corresponds to the average of iid random variables, and hence, by the weak law of large numbers it follows that $S_n/n \xrightarrow{P} E[X_1^2] = 1$ due to (a). Hence, $\theta = 1$.

   (c) By the central limit theorem, $\sqrt{n}(S_n/n - E[X_1^2]) \xrightarrow{\mathcal{D}} \mathrm{N}(0, \mathrm{Var}(X_1^2))$. Then, due to (a), $\sqrt{n}(S_n/n - 1) \xrightarrow{\lceil} \mathrm{N}(0, 2)$.

   (d) Apply the delta method with $g(x) = \sqrt{x}$ to the previous result and note that $g'(x) = x^{-1/2}/2$, hence $[g'(1)]^2 = 1/4$, which allows us to conclude that $\sqrt{n}(\sqrt{S_n/n} - 1) \xrightarrow{\lceil} \mathrm{N}(0, 1/2)$. Then, $\sqrt{2}(\sqrt{S_n} - \sqrt{n}) \xrightarrow{\lceil} \mathrm{N}(0, 1)$.

2. (a) We have that $f(\mathbf{x} \mid \theta) = \theta^n (\prod_{i=1}^n x_i)^{\theta - 1}\ \exp(-\sum_{i=1}^n x_i^\theta)$, hence

$$\frac{f(\mathbf{x} \mid \theta)}{f(\mathbf{y} \mid \theta)} = \left(\prod_{i=1}^n x_i / \prod_{i=1}^n y_i \right)^\theta \ \exp\left(\sum_{i=1}^n y_i^\theta - \sum_{i=1}^n x_i^\theta \right)$$

   and this ratio does not depend on θ iff $\forall i\ \exists j : x_i = y_j$, that is, iff the samples $\mathbf{x}$ and $\mathbf{y}$ are the same up to the order. The Lehmman-Scheffe allows us then to conclude that the ordered sample is a minimal sufficient statistic for θ.

   (b) The statement is true if, for instance, we show that the distribution of $W = (\ln X)/\delta = \theta \ln X$ does not depend on unknown parameters. Indeed,

$$P(W \le w) = P(X \le e^{w/\theta})$$

   hence, $f_W(w) = 1/\theta\ e^{w/\theta}\ f_X(e^{w/\theta}) = e^w\ \exp(-e^w)$, $w \in \mathbb{R}$, which clearly does not depend on θ.

   (c) The statistic T is equivalent to the ordered sample, hence it is also minimal sufficient. By the previous result,

$$\frac{\ln X_{(i)}}{\ln X_{(i-1)}} = \frac{\ln X_{(i)}/\delta}{\ln X_{(i-1)}/\delta}\ , \qquad i = 2, \ldots, n$$

   is a statistic with $n-1$ scalar components which are all ancillary. Basu's theorem states that sufficient and complete statistics are independent of any ancillary statistic, hence T clearly cannot be complete.

3. (a) $E[X] = \int_0^{+\infty} x\ f(x \mid \sigma^2)\ dx$ and apply the change of variables $y = \ln x$ to conclude that

$$E[X] = \int_{-\infty}^{+\infty} e^y \frac{1}{\sigma\ \sqrt{2\pi}} e^{-y^2/(2\sigma^2)} dy = E[e^Y] = M_Y(1) = \exp(\sigma^2/2)$$

   where $Y \sim \mathrm{N}(0, \sigma^2)$.

   (b) Because of the previous result, we have that $\sigma^2 = 2 \ln E[X]$, hence the method of moment estimators of σ^2 is $S = 2 \ln \bar{X}$. By the weak law of large numbers, $\bar{X}$ converges in probability to $E[X]$, and by the continuous mapping theorem we can conclude that S converges in probability to $2 \ln E[X] = \sigma^2$. In summary, S is a consistent estimator of σ^2.

   (c) It is easy to see that $L(\sigma^2 \mid x) \propto (\sigma^2)^{-n/2}\ \exp[-\sum (\ln x_i)^2/(2\sigma^2)]$ and that $\partial \ln L / \partial \sigma^2 = 0 \Leftrightarrow -n/2\ 1/\sigma^2 + \sum (\ln x_i)^2/\sigma^4 = 0$. The solution to this equation is $\sigma^2 = t$, and as a consequence the maximum likelihood estimator of σ^2 is T.

   (d) Note that $E[T] = \sigma^2/n\ E[\sum_{i=1}^n (\ln X_i/\sigma)^2]$. Since $\ln X_i/\sigma \sim \mathrm{N}(0, 1)$ it follows that $(\ln X_i/\sigma)^2 \sim \chi^2(1)$, which allows us to conclude that $E[T] = \sigma^2$ as needed.

   (e) It is clear that the model belongs to the one-parameter exponential family with natural parameter $\alpha = -1/(2\sigma^2)$ and natural parameter space $\mathbb{R}_-$, which contains an open subset of $\mathbb{R}$. Thus, the sufficient statistic $\sum (\ln X_i)^2$ is sufficient and complete. Since T is unbiased and it is a function of a sufficient and complete statistic, it follows that it is the UMVU estimator of σ^2.