Advanced Topics in Statistics 2020/2021
Homework set 6

1. Let $X_1, \ldots, X_n$ $(n \geq 3)$ denote a random sample from a $\mathrm{Ex}(\lambda)$ population, and let $T = \sum_i X_i$.

   (a) Show that T is sufficient and complete for λ.

   (b) Show that $E[1/T \mid \lambda] = \lambda/(n-1)$.

   (c) Determine the UMVU estimator of λ, S.

   (d) Compute the Cramer-Rao lower bound for the estimation of λ.

   (e) Verify that $E[1/T^2 \mid \lambda] = \lambda^2/[(n-1)(n-2)]$ and compute the efficiency of S.

   (f) What can we conclude about the existence of a most efficient estimator for λ?

2. Let $X_1, \ldots, X_n$ denote a random sample from a Pareto distribution with scale parameter equal to 1:
$$f(x \mid \beta) = \beta \; x^{-(\beta+1)} \; , \quad x \geq 1 \; .$$
   where $\beta > 0$.

   (a) Show that we can only apply the method of moments if $\beta > 1$. Verify that, in that case, the method of moments estimator of β is $\tilde{\beta} = \bar{X}/(\bar{X} - 1)$.

   (b) Show that if $\beta > 1$ then $\tilde{\beta}$ is a consistent estimator of β.

   (c) Show that the maximum likelihood estimator of β is $\hat{\beta} = n/[\sum \ln X_i]$.

   (d) Verify that $\hat{\beta}$ is not an unbiased estimator of β. [Hint: determine the distribution of $\ln X_i$ and recall part b) of question 1.]

   (e) Determine the UMVU estimator of β and compare it with $\hat{\beta}$ in terms of mean square error.

3. Consider a population (X, Y) where, given an unknown parameter $\theta > 0$, X and Y are independent, $X \mid \theta \sim \mathrm{N}(0, \theta)$, and $Y \mid \theta \sim \mathrm{N}(0, \theta^{-1})$.

   Let (X_i, Y_i), $i = 1, \ldots, n$ denote a random sample of size $n > 4$ from (X, Y).

   (a) Show that $T = (T_1, T_2) = (\sum_{i=1}^n X_i^2, \sum_{i=1}^n Y_i^2)$ is a minimal sufficient statistic for θ.

   (b) Show that T is not complete.

   (c) Show that the maximum likelihood estimator of θ is $U = \sqrt{T_1/T_2}$.

   (d) Show that $V = (n-2)U^2/n$ is an unbiased estimator of θ^2.

   (e) Is it possible that V is the most efficient estimator of θ^2? Justify your answer.