

# Chapter 1

## Probability

### 1.1 Basic concepts and results

A **random experiment** is when a set of all possible outcomes is known, but it is impossible to predict the actual outcome of the experiment. A **sample space**, denoted as  $\Omega$ , contains all possible outcomes of the experiment. An **event** is a subset of  $\Omega$ . We say that  $A \subset \Omega$  has occurred if and only if the outcome of the experiment is an element of  $A$ . Formally, the family of events forms a  $\sigma$ -algebra of subsets of  $\Omega$  that we denote by  $\mathcal{A}$ .

**Note:**

- $\Omega \in \mathcal{A}$
- $A \in \mathcal{A} \Rightarrow \bar{A} \in \bar{\mathcal{A}}$ , where  $\bar{A}$  indicates the complement of  $A$
- $A_1, A_2, \dots \in \mathcal{A}$
- $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

#### 1.1.1 Probability measures

**Definition 1.1.1: Kolmogorov's axioms**

- $P(A) \geq 0$
- $P(\Omega) = 1$
- If  $A_i \cap A_j = \emptyset, i \neq j$ , then  $P(\bigcup_i A_i) = \sum_i P(A_i)$

Probability measure  $P : \mathcal{A} \rightarrow \mathbb{R}$  satisfying Kolmogorov's axioms has the following properties:

- $P(\emptyset) = 0$
- $A \subset B \Rightarrow P(A) \leq P(B)$
- $0 \leq P(A) \leq 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(\bar{A}) = 1 - P(A)$
- $P(A - B) = P(A \cap \bar{B}) = P(A) - P(A \cap B)$

### Definition 1.1.2: Conditional probability

If  $P(B) > 0$ ,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We are re-evaluating the probability of A given the B space.

Let  $\{A_1, A_2, \dots\}$  denote a partition of  $\Omega : \cup_i A_i = \Omega; A_i \cap A_j = \emptyset, i \neq j$ . Meaning union makes up  $\Omega$  and are mutually exclusive. Then if  $P(A_i) > 0$  for all  $i$

### Theorem 1.1.1 Total probability theorem

$$P(B) = \sum_i P(B|A_i)P(A_i)$$

$$B = B \cap \Omega = B \cap [\cup_i A_i] = \cup_i (B \cap A_i) \text{ and } P(\cup_i B \cap A_i) = \sum_i P(B \cap A_i)$$

### Theorem 1.1.2 Bayes' theorem

If  $P(B) > 0$

$$P(A_i|B) = \frac{P(B|A_j)P(A_j)}{\sum_i P(B|A_i)P(A_i)}$$

$$P(\underbrace{A_j}_{\text{explanation}} \mid \underbrace{B}_{\text{evidence}}) = \frac{P(A_j \cap B)}{P(B)} = \frac{\underbrace{P(B|A_j)P(A_j)}_{\text{substitute with total probability theorem}}}{P(B)}$$

## 1.1.2 Random variables

### Definition 1.1.3: Random variable

Function defined in  $\Omega$  and taking values in  $\mathbb{R}$

$$X : \Omega \rightarrow \mathbb{R}$$

$$\omega \mapsto X(\omega) = x$$

A random variable induces a probability measure in  $\mathbb{R}$  that we denote by  $P_X$ : if  $B \subset \mathbb{R}$ ,  $P_X(B) = P(A)$ , where  $A = X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$ . Formally, there must be a  $\sigma$ -algebra of subsets of  $\mathbb{R}, \mathcal{B}$ , and we have to verify that for every set  $B \in \mathcal{B}$  we have  $X^{-1}(B) \in \mathcal{A}$ . Typically,  $\mathcal{B}$  is the so called Borel  $\sigma$ -algebra and it suffices to make sure that  $X$  satisfies  $X^{-1}((-\infty, x]) \in \mathcal{A}, \forall x \in \mathbb{R}$ .

Basically what it means is that we don't know if  $X^{-1}(B) \in \mathcal{A}$  and for which  $B$  can I compute  $P_X(B)$ . If  $X^{-1}(B) \in \mathcal{A}$  for  $B$  is in the Borel  $\sigma$ -algebra, then  $X$  is measurable.

### Definition 1.1.4: Distribution function of a random variable

$X$ : for all  $x \in \mathbb{R}$

$$F_X(x) = P_X((-\infty, x]) = P(X \leq x)$$

It is suffice to know  $F_X(\cdot)$  to be able to compute  $P_X(B)$  for all  $B \in \mathcal{B}$ .

- For all  $a > b$ ,  $P(a < X \leq b) = F_X(b) - F_X(a)$
- $F_X(-\infty) = 0; F_X(\infty) = 1$

- $F_X$  is right-continuous and non-decreasing
- The set of points at which  $F_X$  is discontinuous is either finite or countable (at most countable)

#### Definition 1.1.5: Discrete random variable

$X$  is a discrete random variable if  $D_X$  is such that  $P_X(D_X) = 1$

The probability mass function of  $X$  is defined as  $f_X(x) = F_X(x) - \lim_{y \rightarrow x} F_X(y) = \begin{cases} P(X = x) & \text{if } x \in D_X \\ 0 & \text{otherwise} \end{cases}$

Any  $f$  satisfying the following is a probability mass function

- $f(x) \geq 0$  for all  $x$
- $f(x) > 0$  iff  $x \in D$ , where  $D \subset \mathbb{R}$  is finite or countable
- $\sum_{x \in D} f(x) = 1$

For any event  $B \subset \mathbb{R}$ ,  $P(X \in B) = \sum_{x \in B \cap D_X} f_X(x)$ .

**Note:**

$$F_X(x) = \sum_{y \leq x} f_X(y)$$

$F_X(x) = P(X \leq x)$  cumulative distribution function

↓

$f_X(x) = P(X = x)$  probability mass function  
where  $0 \leq f_X(x) \leq 1$

Discrete distribution include Bernoulli, binomial, Poisson, geometric, negative binomial, multinomial, hypergeometric, etc.

#### Definition 1.1.6: Continuous random variable

$X$  is continuous if  $P_X(D_X) = 0$ ,  $D_X = \emptyset$  and if additionally there is  $f_X$  such that for all  $x \in \mathbb{R}$

- $f_X(x) \geq 0 \rightarrow$  probability density function
- $F_X(x) = \int_{-\infty}^{+\infty} f(x) dx = 1$

At the points where  $F_X$  is differentiable, we have  $F'_X(x) = f_X(x)$ .

Any  $f$  satisfying the following conditions is a probability density function

- $f(x) \geq 0$  for all  $x$
- $\int_{-\infty}^{+\infty} f(x) dx = 1$

Continuous distributions include uniform, exponential, gamma, chi-squared, normal.  $t$ -student,  $F$ -Snedcor, beta, Pareto, Weibull, log-normal, etc.

### 1.1.3 Functions of a random variable

Let  $X$  be a r.v. and  $Y = h(X)$  where  $h : \mathbb{R} \rightarrow \mathbb{R}$

In general, if  $X = g(Y)$  with  $g$  invertible and differentiable, and  $X$  continuous, we have

$$f_Y(y) = |g'(y)| f_X(g(y))$$

Proof.  $\frac{\partial F_X(x)}{\partial x} = f_X(x)$

Using chain rule:  $(f \circ g)'(x) = [f(g(x))]' = f'(g(x))g'(x)$  ■

### Definition 1.1.7: Expected value

Let  $Y = h(X)$ , a linear function.

The expected value of  $Y$  is defined by  $E[Y] = \begin{cases} \sum_x h(x) f_X(x) & \text{if } X \text{ discrete} \\ \int_{-\infty}^{+\infty} h(x) f_X(x) dx & \text{if } X \text{ continuous} \end{cases}$

Formally, we must additionally verify that the integral or series are absolutely convergent.  $E[Y]$  may not exist.

There are two ways to compute  $E[Y]$  with  $Y = h(X)$ , either use the definition above, or first obtain the distribution of  $Y$  and compute  $E[Y] = \begin{cases} \sum_y y f_Y(y) & \text{if } Y \text{ discrete} \\ \int_{-\infty}^{+\infty} y f_Y(y) dy & \text{if } Y \text{ continuous} \end{cases}$ . The two methods are equivalent.

### Definition 1.1.8: Raw moment of order $k$

$$\mu'_k = E[X^k]$$

### Definition 1.1.9: Central moment of order $k$

$$\mu_k = E[(X - \mu)^k], \mu = E[X]$$

### Definition 1.1.10: Moment generating function

$M_X(s) = E[e^{sX}]$  whenever the expectation exists for  $s$  in a neighborhood of the origin.

- If  $M_X(s)$  exists, then  $X$  has moments of all orders and  $M^{(k)}(0) = E[X^k]$
- The moment generating function, when it exists, identifies the probability distribution

Some useful properties:

- $E[h_1(X) + h_2(X)] = E[h_1(X)] + E[h_2(X)]$
- If  $c \in \mathbb{R}$ , then  $E[cX] = cE[X]; E[c] = c$
- If  $c \in \mathbb{R}$ , then  $Var(cX + b) = c^2 Var(X)$
- $Var(X) = E[X^2] - (E[X])^2$
- $Var(X) \geq 0; Var(X) = 0 \Leftrightarrow P(X = c) = 1$  for some  $c \in \mathbb{R}$

## 1.1.4 Bivariate random variables

$$(X, Y) : \Omega \rightarrow \mathbb{R}^2$$

$$\omega \mapsto (X(\omega), Y(\omega)) = (x, y)$$

If  $(X, Y)$  discrete, we define the joint probability mass function as  $f(x, y) = P(X = x, Y = y)$ . If  $(X, Y)$  continuous, then there exists the joint probability density function,  $f(x, y)$  such that for all  $(x, y) \in \mathbb{R}^2$ ,

- $f(x, y) \geq 0$
- $F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^x \int_{-\infty}^y f(u, v) dv du$

### Example 1.1.1

$X = \text{weight}, Y = \text{height} \Rightarrow Z = \text{BMI}$

### Definition 1.1.11: Marginal distributions

$$f_X(x) = \begin{cases} \sum_y f(x, y) & \text{if } (X, Y) \text{ discrete} \\ \int_{-\infty}^{+\infty} f(x, y) dy & \text{if } (X, Y) \text{ continuous} \end{cases}$$

### Definition 1.1.12: Expectation of $Z = h(X, Y)$

$$E[Z] = \begin{cases} \sum_x \sum_y h(x, y) f(x, y) & \text{if } (X, Y) \text{ discrete} \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(x, y) f(x, y) dy dx & \text{if } (X, Y) \text{ continuous} \end{cases}$$

### Definition 1.1.13: Conditional distributions

$$f_{X|Y=y}(x) = \frac{f(x, y)}{f_Y(y)}, y \text{ fixed: } f_Y(y) > 0$$

function of  $x$  for every  $y$  where  $f_Y(y) > 0$

### Definition 1.1.14: Raw moment of order $(r, s)$

$$\mu'_{(r,s)} = E[X^r Y^s]$$

### Definition 1.1.15: Central moment of order $(r, s)$

$$\mu_{(r,s)} = E[(X - \mu_X)^r (Y - \mu_Y)^s]$$

### Definition 1.1.16: Covariance

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \mu_{(1,1)}$$

If  $x$  and  $y$  are positively associated  $\rightarrow \text{Cov}(x, y) > 0 \rightarrow$  If  $x$  is larger than its mean, then typically  $y$  is larger than its mean.

Some useful properties:

- $\text{Cov}(X, Y) = E[X, Y] - E[X]E[Y]$
- $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
- $\text{Cov}(cX, Y) = c\text{Cov}(X, Y), c \in \mathbb{R}$
- $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$
- $\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y) \pm 2\text{Cov}(X, Y)$

### Example 1.1.2 (Portfolio management)

$$\text{Cov}(x, y) < 0$$

$$\text{Var}(x, y) < \text{Var}(x) + \text{Var}(y)$$

### Theorem 1.1.3 Law of iterated expectation

If  $Z = h(X, Y)$  then  $E[Z] = E_X[E[Z|X]]$

### Theorem 1.1.4 Law of total variance

$$\text{Var}(Y) = \text{Var}_X(E[Y|X]) + E_X[\text{Var}(Y|X)]$$

Other useful tricks:

- $E[h(X) Y | X = x] = h(x) E[Y | X = x]$
- $\text{Cov}(X, Y) = \text{Cov}(X, E[Y|X])$

*Proof.*

$$\begin{aligned}\text{Cov}(X, E[Y|X]) &= E[X E[Y|X]] - E[X] E[E[Y|X]] \\ &= E[E[XY|X]] - E[X] E[Y] \\ &= E[XY] - E[X] E[Y] \\ &= \text{Cov}(X, Y)\end{aligned}$$

■

### 1.1.5 Independence

#### Definition 1.1.17: Stochastic independence

$X$  and  $Y$  are stochastically independent if and only if  $\forall (x, y) \in \mathbb{R}^2, f(x, y) = f_X(x) f_Y(y)$

If  $X$  and  $Y$  are independent, then

- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

*Proof.*  $\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y) \pm 2 \times \underbrace{\text{Cov}(X, Y)}_{\rightarrow 0} \blacksquare$

- $M_{X+Y}(s) = M_X(s) M_Y(s)$

*Proof.*  $M_{X+Y}(s) = E[e^{s(X+Y)}] = E[\underbrace{e^{sx}}_u \underbrace{e^{sy}}_v]$

$x$  and  $y$  independent stochastically  $\Rightarrow u$  and  $v$  independent

$M_{X+Y}(s) = E[e^{sx}] E[e^{sy}] = M_X(s) M_Y(s) \blacksquare$

- $\text{Cov}(X, Y) = 0$

*Proof.*  $\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \underbrace{E[XY]}_{X, Y \text{ uncorrelated}} - E[X]E[Y] = E[X]E[Y] - E[X]E[Y] = 0 \blacksquare$

- $E[X^r Y^s] = E[X^r] E[Y^s]$
- $E[Y | X = x] = E[Y]; E[X | Y = y] = E[X]$
- $f_{X|Y=y}(x) = f_X(x); f_{Y|X=x}(y) = f_Y(y)$

*Proof.*  $f_{X|Y=y}(x) = \frac{f(x, y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x) \blacksquare$

#### Definition 1.1.18: Mean independence

$Y$  is mean independent of  $X$  iff  $E[Y | X = x]$  does not depend on  $x$  for all  $x$ .

*Proof.*  $E[Y | X = x] = c$

$E[Y | X] = c \Rightarrow E[E[Y | X]] = c \Rightarrow E[Y] = c \rightarrow \text{conditional is equal to marginal} \blacksquare$

### Definition 1.1.19: Uncorrelatedness

$X$  and  $Y$  are uncorrelated iff  $\text{Cov}(X, Y) = 0$

Useful results:

- If  $X$  and  $Y$  are stochastically independent, then  $Y$  is mean-independent of  $X$ , and  $X$  is mean independent of  $Y$ .
- If  $Y$  is mean-independent of  $X$ , then  $X$  and  $Y$  are uncorrelated. The converse is not true.

*Proof.*  $Y$  mean independence of  $X \Rightarrow \text{Cov}(X, Y) = \text{Cov}(X, E[Y|X]) = \text{Cov}(X, c) = 0 \Rightarrow$  uncorrelated ■

- If  $Y$  is uncorrelated with  $X$ , then  $E[XY] = E[X]E[Y]$
- If  $Y$  is mean-independent of  $X$ , then  $E[X^kY] = E[X^k]E[Y]$  for all  $k$
- If  $Y$  and  $X$  are stochastically independent, then  $E[X^kY^r] = E[X^k]E[Y^r]$  for all  $k, r$

**Note:**

stochastic independence  $\Rightarrow$  mean independence  $\Rightarrow$  uncorrelatedness

## 1.2 Convergence of sequences of random variables

If  $\{X_n\}_{n=1}^\infty$  is a sequence of random variables and  $X$  is a random variable,

$$\begin{aligned} X_n : \underbrace{\Omega}_{\text{exists probability, } \sigma\text{-algebra}} &\rightarrow \mathbb{R} \\ X_n &\longrightarrow X \quad \text{as } n \rightarrow +\infty \end{aligned}$$

$n$  can be population size, or can be the number of iterations for Monte Carlo simulation.

### 1.2.1 Notions of convergence of sequences

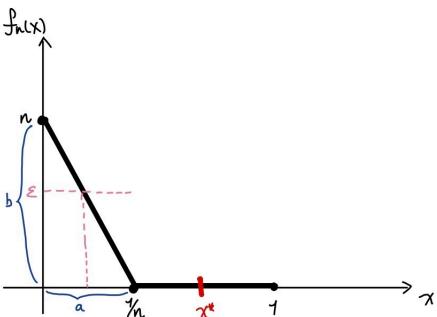
Notions of **convergence of sequences**: let  $f_n, f : [0, 1] \rightarrow \mathbb{R}$

- Point wise convergence:  $f_n(x) \rightarrow f(x)$  for all  $x \in [0, 1]$
- Uniform convergence:  $\sup_{x \in [0,1]} |f_n(x) - f(x)| \rightarrow 0$
- Convergence in  $L^P$ :  $\int_0^1 |f_n(x) - f(x)|^P dx \rightarrow 0$
- Convergence in measure:  $\mu(A_{n,\epsilon}) \rightarrow 0$  for all  $\epsilon > 0$  where  $A_{n,\epsilon} = \{x \in [0, 1] : |f_n(x) - f(x)| > \epsilon\}$

#### Example 1.2.1

$f_n : [0, 1] \rightarrow \mathbb{R}$

$$f_n(x) = \begin{cases} 0 & 1/n \leq x \leq 1 \\ n - n^2x & 0 \leq x < 1/n \end{cases}$$



As  $n \rightarrow \infty$ ,  $a$  becomes smaller,  $b$  becomes bigger.

- Point wise convergence

$$\forall x \in [0, 1]$$

$$\forall x^* > 0, f_n(x^*) = 0 \quad \text{for } n > N \quad \text{except } f_n(0) = 0 \rightarrow \infty$$

$$\Rightarrow f_n(x) \rightarrow \begin{cases} 0 & \text{if } x \in [0, 1] \\ \infty & \text{if } x = 0 \end{cases} \Rightarrow f_n \text{ is not converging pointwise to the null function.}$$

- Uniform convergence

$$\max |f_n(x)| = n \rightarrow +\infty \quad x \in [0, 1] \Rightarrow f_n \text{ does not converge uniformly to the null function.}$$

- Convergence in  $L^1$   $\rightarrow P = 1$

$$\int_0^1 |f_n(x)| dx = \frac{1}{2} = \underbrace{\frac{1}{n} \times n \times \frac{1}{2}}_{\text{area under the triangle}} \Rightarrow f_n \text{ does not converge in } L^1 \text{ to the null function.}$$

- Convergence in measure

$$A_{n,\epsilon} \subset [0, \frac{1}{n}]$$

$$\mu(A_{n,\epsilon}) \leq \mu([0, \frac{1}{n}]) = \frac{1}{n} \rightarrow \text{as } n \rightarrow \infty, \mu \rightarrow 0 \Rightarrow f_n \text{ converges to the null function in measure.}$$

## 1.2.2 Convergence of random variables

Let  $\{X_n\}_{n=1}^\infty$  be a sequence of random variables and  $X$  is a random variable, all defined in the same probability space  $(\Omega, \mathcal{A}, P)$ .

### Definition 1.2.1: Almost surely convergence

$X_n$  converges to  $X$  almost surely, or with probability 1,  $X_n \xrightarrow{\text{a.s.}} X$ , iff

$$P[\{\omega \in \Omega : X_n(\omega) \rightarrow X(\omega)\}] = 1$$

Similar to pointwise convergence, no need for expectation.

#### Note:

$$P(\underbrace{X_n(\omega)}_{\text{set}} \rightarrow \underbrace{x(\omega)}_{\text{point}}) = 1$$

set of which it happens has a probability of 1

### Definition 1.2.2: Convergence in the $r$ th mean

$X_n$  converges to  $X$  in the  $r$ th mean,  $r \geq 1$ ,  $X_n \xrightarrow{r} X$ , iff

$$E[|X_n - X|^r] \rightarrow 0$$

Each point will be weighted with the same probability. Expectation is involved in this case.

**Note:**

When  $r = 2$ , it is the mean square convergence, often used for quality checking.

### Definition 1.2.3: Convergence in probability

$X_n$  converges in probability to  $X$ ,  $X_n \xrightarrow{P} X$ , iff for all  $\epsilon > 0$

$$P(|X_n - X| > \epsilon) \rightarrow 0$$

It is similar to measure in convergence. Often used to check for quality of estimator. Note that this is no longer a Lebesgue measure, it is now a probability measure.  $P\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon\} \rightarrow 0$  as  $n \rightarrow 0$ .

### Definition 1.2.4: Convergence in distribution

$X_n$  converges in distribution to  $X$ ,  $X_n \xrightarrow{d} X$ , iff

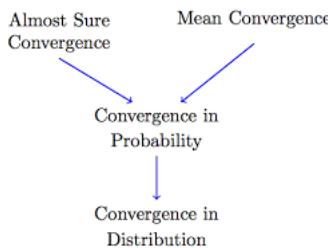
$$F_n(x) \rightarrow F(x)$$

for all  $x$  continuity point of  $F$ , where  $F(x) = P(X \leq x)$  and  $F_n(x) = P(X_n \leq x)$

Has nothing to do with the random variable. Often used for hypothesis testing. It does not need the requirement that all points are defined in the same probability space  $(\Omega, \mathcal{A}, P)$  as there is no  $\omega$  in the density function.

Some useful **remarks**:

- Convergence in distribution is really about the convergence of the sequence of probability functions and not the random variables themselves.
- When defining convergence in the  $r$ th mean, it is assumed that the corresponding expected values exist:  $E[|X_n|^r] < \infty$  and  $E[|X|^r] < \infty$
- When  $X_n \xrightarrow{1} X$ , we say that  $X_n$  converges to  $X$  in mean; when  $X_n \xrightarrow{2} X$ , we say that  $X_n$  converges to  $X$  in quadratic mean.



*Proof.* Convergence in mean implies convergence in probability

$$E[|X_n - X|] \rightarrow 0 \Rightarrow P(|X_n - X| > \epsilon) \rightarrow 0, \forall \epsilon > 0$$

Using **Markov inequality**:  $P(|y| > a) \leq \frac{|E[X_n - X]|}{\epsilon}$

$$0 \leq \lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) \leq \lim_{n \rightarrow \infty} \overbrace{\frac{E[|X_n - X|]}{\epsilon}}^{\rightarrow 0} = 0 \blacksquare$$

*Proof.* **Proof of convergence in probability implies convergence in distribution**

$$X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} \Leftrightarrow P(|X_n - X| > \epsilon) \rightarrow 0 \Rightarrow P(X_n \leq x) \rightarrow P(X \leq x), \forall x$$

let  $\epsilon > 0$ ,

$$F_n(x) = P(X_n \leq x)$$

$$F(x) = P(X \leq x)$$

Using the **total probability theorem**:  $P(A) = P(A \cap B) + P(A \cap \bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$

$$F_n(x) = P(X_n \leq x) = P\left(\underbrace{X_n \leq x}_{A}, \underbrace{X \leq x + \epsilon}_{B}\right) + P\left(\underbrace{X_n \leq x}_{A}, \underbrace{X > x + \epsilon}_{\bar{B}}\right) \leq F(x + \epsilon) * P(|X_n - x| > \epsilon)$$

$$F(x - \epsilon) - P(|X_n - X| > \epsilon) \leq F_n(x) \leq F(x + \epsilon) + \underbrace{P(|X_n - X| < \epsilon)}_{\rightarrow 0}$$

$$\text{as } n \rightarrow \infty, \underbrace{F(x - \epsilon)}_{\substack{\epsilon \rightarrow 0 \\ \longrightarrow F(X)}} \leq \lim_{n \rightarrow \infty} F_n(x) \leq \underbrace{F(x + \epsilon)}_{\substack{\epsilon \rightarrow 0 \\ \longrightarrow F(X)}} \blacksquare$$

Some **converses**:

- If  $X_n \xrightarrow{P} X$ , then there exists  $\{n_k\}_{k=1}^{+\infty}$  such that  $X_{n_k} \xrightarrow{a.s.} X$  when  $k \rightarrow +\infty$
- If  $|X_n|^r$  is uniformly integrable, then  $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{r} X$

### Theorem 1.2.1 Skorokhod representation theorem

If  $X_n \xrightarrow{d} X$  then there exists a probability space  $(\Omega', \mathcal{A}', P')$  and r.v.  $\{Y_n\}$  and  $Y$ , defined in  $\Omega'$  such that

- $P'(Y_n \leq y) = P(X_n \leq y)$  and  $P'(Y \leq y) = P(X \leq y)$  for all  $y \in \mathbb{R}$ . This means that  $X_n$  and  $Y_n$  are marginally equal in distribution, the same for  $X$  and  $Y$ .

$$\bullet Y_n \xrightarrow{a.s.} Y$$

Other useful **results**:

- $X_n \xrightarrow{P} c \Leftrightarrow X_n \xrightarrow{d} c$ , where  $c \in \mathbb{R}$

$$\text{Proof. } X_n \xrightarrow{d} x \Rightarrow X_n \xrightarrow{P} c \Leftrightarrow P(X_n \leq x) \rightarrow \begin{cases} 0 & x < c \\ 1 & x > c \end{cases}, \text{not continuous at } c$$

$P(|X_n - c| > \epsilon) \rightarrow 0, \forall \epsilon > 0$

$$\begin{aligned} P(|X_n - c| > \epsilon) &= P(X_n - c > \epsilon) + P(X_n - c < -\epsilon) \\ &= P(X_n > c + \epsilon) + P(X_n < c - \epsilon) \\ &= 1 - P(X_n \leq c + \epsilon) + P(X_n \leq c - \epsilon) \\ &\leq 1 - P(X_n \leq \underbrace{c + \epsilon}_{>c}) + P(X_n \leq \underbrace{c - \epsilon}_{<c}) \\ &\rightarrow 1 - 1 + 0 = 0 \end{aligned}$$

■

- Since  $E[(X_n - \theta)^2] = \text{Var}(X_n) + (E[X_n] - \theta)^2$  if  $\text{Var}(X_n) \rightarrow 0$  and  $E[X_n] \rightarrow \theta$ . We have convergence in mean square to  $\theta$ , and hence convergence in probability to  $\theta$ .

### Theorem 1.2.2 Continous mapping theorem

Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Then

- $X_n \xrightarrow{a.s.} X \Rightarrow h(X_n) \xrightarrow{a.s.} h(X)$
- $X_n \xrightarrow{d} X \Rightarrow h(X_n) \xrightarrow{d} h(X)$
- $X_n \xrightarrow{P} X \Rightarrow h(X_n) \xrightarrow{P} h(X)$

### Theorem 1.2.3 Slutsky theorem

Let  $\{X_n\}$  and  $\{Y_n\}$  be sequences of random variables,  $X$  a random variable and  $c$  a real number. If  $X_n \xrightarrow{d} X$  and  $Y_n \xrightarrow{P} c$ , then

- $X_n + Y_n \xrightarrow{d} X + c$
- $Y_n X_n \xrightarrow{d} cX$
- $X_n / Y_n \xrightarrow{d} X/c$  as long as  $c \neq 0$

### Wrong Concept 1.1: $X_n + Z_n \neq 2X$

Suppose that  $X_n \xrightarrow{d} X$  where  $X \sim N(0, 1)$ . Then with  $Z_n = -X_n$  we have  $Z_n \xrightarrow{d} X$ . However,  $X_n + Z_n = 0$ , hence  $X_n + Z_n$  does not converge in distribution to  $2X$  as one might expect.

cdf of  $Z_n$  converges to cdf of  $X_n$

$$\begin{aligned} Z_n \xrightarrow{d} X &\Leftrightarrow P(Z_n \leq z_n) \rightarrow \Phi(z_n), \forall z \in \mathbb{R} \\ &\Leftrightarrow P(-X_n \leq z) = P(X_n \geq -z) = 1 - P(X_n \leq -z) \\ &\rightarrow 1 - \Phi(-z) \\ \therefore Z_n \xrightarrow{d} X \end{aligned}$$

This is why the Slutsky theorem is important, it showcases safe procedures.

### Example 1.2.2 ( $X_n \sim t(n) \Rightarrow X_n \xrightarrow{d} N(0, 1)$ using Slutsky)

$$X_n \sim t(n), X_n = \frac{u_n}{\sqrt{\frac{v_n}{n}}}$$

$$\text{Assumptions: } \begin{cases} u_n \text{ independent of } v_n \\ u_n \sim N(0, 1) \\ v_n \sim \chi^2(n) \end{cases}$$

What would be nice is to show that  $\sqrt{\frac{v_n}{n}}$  converges to 1 then we can apply the Slutsky theorem.

Using the **mean square convergence**, we have

$$\begin{aligned} \text{Var}\left(\frac{v_n}{n}\right) &= \frac{\text{Var}(v_n)}{n} = \frac{2n}{n^2} = \frac{2}{n} \rightarrow 0 \\ E\left[\frac{v_n}{n}\right] &= \frac{E[v_n]}{n} = \frac{n}{n} = 1 \rightarrow 1 \end{aligned}$$

We now have mean square convergence to 1.

Using the **Continuous mapping theorem**, we have

$$\begin{aligned} \frac{v_n}{n} &\xrightarrow{P} 1 \Rightarrow \sqrt{\frac{v_n}{n}} \xrightarrow{P} 1 \\ \Rightarrow \frac{v_n}{n} &\xrightarrow{2} 1 \text{ and } \frac{v_n}{n} \xrightarrow{P} 1 \end{aligned}$$

Now using the **Slutsky theorem**, we have

$$X_n = \frac{u_n}{\sqrt{\frac{v_n}{n}}} \xrightarrow{d} u_n \sim N(0, 1)$$

### 1.3 Important asymptotic results

#### Theorem 1.3.1 Weak law of large numbers

Let  $\{X_n\}_{n=1}^{+\infty}$  be a sequence of independent and identically distributed random variables, with  $E[X_n] = \mu$  and  $\text{Var}(X_n) = \sigma^2 < \infty$ . Let also  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ . Then we have that

$$\bar{X}_n \xrightarrow{P} \mu$$

*Proof.* Goal:  $\bar{X}_n \xrightarrow{P} \mu \Rightarrow P(|\bar{X}_n - \mu| > \epsilon) \rightarrow 0$

Checking the validity of Chebychov's inequality,

$$E[\bar{X}_n] = E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n \underbrace{E[X_i]}_{\rightarrow \mu} = \frac{1}{n} n \mu = \mu$$

We can now apply the **Chebychov's inequality**:  $P(\underbrace{|X - \mu|}_{\text{distance of distribution from its mean}} > \epsilon) \leq \frac{\text{Var}(X)}{\epsilon^2}$

$$\begin{aligned} P(|\bar{X}_n - \mu| > \epsilon) &\leq \overbrace{\frac{\text{Var}(\bar{X}_n)}{\epsilon^2}}^{\frac{1}{n^2} \cdot n \sigma^2 = \frac{\sigma^2}{n}} \rightarrow 0 \\ 1: \text{Var}(\bar{X}_n) &= \frac{1}{n} \text{Var}\left(\sum_{i=1}^n X_i\right) = \underbrace{\frac{1}{n^2} n \sigma^2}_{\text{Var}(\Sigma) = \sum \text{Var} + 2\text{Cov}_{\text{iid} \rightarrow 0}} = \frac{\sigma^2}{n} \blacksquare \end{aligned}$$

Intuitively, the WLLN tell us that  $\bar{X}_n$  becomes more and more concentrated around  $\mu$  as  $n$  increases.

#### Theorem 1.3.2 Strong law of large numbers

Under the same conditions as above, we have

$$\bar{X}_n \xrightarrow{a.s.} \mu$$

Actually, it is only necessary to assume that  $E[|X_i|] < +\infty$  for both laws to hold.

**Theorem 1.3.3 Central limit theorem**

Let  $\{X_n\}_{n=1}^{+\infty}$  be a sequence of iid random variables possessing finite variance. Let  $\mu = E[X_n]$  and  $\sigma^2 = \text{Var}(X_n)$ . Let also  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$  and

$$Z_n = \sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} = \frac{\bar{X}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} = \frac{\bar{X}_n - E[\bar{X}]}{\sqrt{\text{Var}(\bar{X})}} \xrightarrow{d} N(0, 1)$$

Then we have

$$Z_n \xrightarrow{d} Z$$

where  $Z \sim N(0, 1)$

*Proof. Proof with assumption of existence of mgf*

Assume

1.  $M_n(S) = E[e^{sX_n}]$  exists
2.  $M_n(s) \rightarrow M(s)$  for  $s \in (-s_0, s_0)$

then  $M(s) = E[e^{sX}] \Rightarrow X_n \xrightarrow{d} X$

Idea:  $X_n$  are iid r.v.

$E[e^{sX_n}] = M_{X_n}(s)$  exists for  $s \in (-s_0, s_0) \Rightarrow Z_n = \sqrt{n} \frac{\bar{X}_n - \mu}{\sigma}$

Need to show  $M_{Z_n}(s) \rightarrow M_{N(0,1)}(s) = e^{s^2/2} \rightarrow$  mgf of  $Z_n$  goes to  $e^{s^2/2}$ , the mgf of the normal distribution.

$$Z_n = \sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} \underset{\substack{\text{Annex 1} \\ \text{standardized version of the } X_i \text{'s}}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \quad (1.1)$$

Annex 1:

$$\begin{aligned} Y_i &= \frac{X_i - \mu}{\sigma}, \text{ standardized version of the } X_i \text{'s} \\ \sum Y_i &= \frac{\sum(X_i - \mu)}{\sigma} = \frac{\sum X_i - n\mu}{\sigma} = \frac{n\bar{X}_n - n\mu}{\sigma} = n \frac{\bar{X}_n - \mu}{\sigma} \\ \frac{1}{\sqrt{n}} \sum Y_i &= \frac{1}{\sqrt{n}} n \frac{\bar{X}_n - \mu}{\sigma} = \sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} \end{aligned}$$

Using the moment generating function

$$\begin{aligned} M_{Z_n}(s) &= E[e^{sZ_n}] = E[e^{s \frac{1}{\sqrt{n}} \sum Y_i}] \\ &= M_{\sum Y_i} \left( \frac{s}{\sqrt{n}} \right) \\ &= M_{Y_1} \left( \frac{s}{\sqrt{n}} \right) \times M_{Y_2} \left( \frac{s}{\sqrt{n}} \right) \times \cdots \times M_{Y_n} \left( \frac{s}{\sqrt{n}} \right) \rightarrow \text{mgf of the sum of the variable is the product} \\ &= [M_Y \left( \frac{s}{\sqrt{n}} \right)]^n \\ &= \sum_{k=0}^2 M_Y^{(k)}(0) \frac{s^k}{k!} + \underbrace{r(s)}_{\frac{r(s)}{s^2} \rightarrow 0 \text{ as } s \rightarrow 0} \rightarrow \text{Taylor's expansion of 2nd order, Annex 2} \\ &= 1 + \frac{s^2}{2!} + r(s) \end{aligned} \quad (1.2)$$

Annex 2:

$$\begin{aligned}
M_Y^{(k)}(0) &= E[Y^k] \\
M_Y^{(0)}(0) &= E[Y^0] = 1 \\
M_Y^{(1)}(0) &= E[Y^1] = 0 \\
M_Y^{(2)}(0) &= E[Y^2] = \frac{E[(x_i - \mu)^2]}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1
\end{aligned}$$

Back to the moment generating function

$$\begin{aligned}
M_{Z_n}(s) &= [M_Y(\frac{s}{\sqrt{n}})]^n \\
&= [1 + \frac{s^2/2}{n} + r(\frac{s}{\sqrt{n}})]^n \\
&\quad \xrightarrow{\rightarrow 0} \\
&= [1 + \underbrace{\frac{\frac{s^2}{2} + n r(s/\sqrt{n})}{n}}_n]^n \xrightarrow{\text{Annex 3}} e^{s^2/2}
\end{aligned} \tag{1.3}$$

Annex 3:

$$[1 + \frac{u_n}{v_n}]^{v_n} \rightarrow e^c, u_n \rightarrow c, v_n \rightarrow \infty$$

■

The CLT is often used to compute probabilities of the type  $P(\bar{X}_n \leq x)$  approximating them by  $\Phi(\sqrt{n}\frac{(x-\mu)}{\sigma})$  for sufficiently large  $n$ .

$$\begin{aligned}
P(\bar{X}_n \leq x) &= P(\sqrt{n}\frac{\bar{X}_n - \mu}{\sigma} \leq \sqrt{n}\frac{x - \mu}{\sigma}) \\
&\approx \Phi(\sqrt{n}\frac{x - \mu}{\sigma}) \\
P(Z_n \leq z) &\rightarrow \Phi(z)
\end{aligned}$$

Intuitively, the CLT tells us that the distribution of  $\bar{X}_n$  is well approximated by a normal distribution for sufficiently large  $n$  as long as the variance is finite. Additionally, if the distribution of  $X_n$  is close to symmetric, then the rate of convergence is faster. Rate of convergence is related to the coefficient of symmetry,  $\frac{E[(X - \mu)^3]}{(\text{Var}(X))^{3/2}} = \gamma_1$ . If the distribution is symmetric,  $\gamma_1 = 0$ .

#### Theorem 1.3.4 Lévy's continuity theorem

Suppose that  $\{X_n\}_{n=1}^{+\infty}$  is a sequence of random variables and let  $M_n(s)$  denote the mgf of  $X_n$ ,  $n = 1, 2, \dots$ . Additionally assume that

$$\lim_{n \rightarrow +\infty} M_n(s) = M(s)$$

for  $s$  in a neighborhood of the origin, and that  $M(\cdot)$  is the mgf of a random variable  $X$ .

In these circumstances,

$$X_n \xrightarrow{d} X$$

#### Example 1.3.1 (Application : Bernoulli)

$\{X_n\}_{n=1}^{+\infty}$  iid  $B(1, \theta)$  where  $\theta \in (0, 1)$ . By the **central limit theorem**,

$$\sqrt{n}\frac{\bar{X}_n - \theta}{\sqrt{\theta(1-\theta)}} \xrightarrow{d} N(0, 1)$$

On the other hand, the **WLLN** ensures that  $\bar{X}_n \xrightarrow{d} \theta$ .

By the **continuous mapping theorem**,

$$\frac{\sqrt{\theta(1-\theta)}}{\sqrt{\bar{X}_n(1-\bar{X}_n)}} \xrightarrow{P} 1$$

and **Slutsky's theorem** allows us to conclude that

$$\sqrt{n} \frac{\bar{X}_n - \theta}{\sqrt{\theta(1-\theta)}} \frac{\sqrt{\theta(1-\theta)}}{\sqrt{\bar{X}_n(1-\bar{X}_n)}} = \sqrt{n} \frac{\bar{X}_n - \theta}{\sqrt{\bar{X}_n(1-\bar{X}_n)}} \xrightarrow{d} N(1, 0)$$

which in practice means that, for large  $n$

$$P\left(\frac{\bar{X}_n - \theta}{\sqrt{\bar{X}_n(1-\bar{X}_n)}} \leq x\right) \approx \Phi(x)$$

*Proof.*  $X_i \sim B(1, \theta)$ ,  $E[X_i] = \theta$   $\text{Var}(x_i) = \theta(1-\theta)$

By the **CLT**,  $\sqrt{n} \frac{\bar{X}_n - \theta}{\underbrace{\sqrt{\theta(1-\theta)}}_{\text{issue in the denominator}}} \xrightarrow{d} N(0, 1)$

$$\begin{aligned} \sqrt{n} \frac{\bar{X}_n - \theta}{\sqrt{\bar{X}_n(1-\bar{X}_n)}} &\xrightarrow{d} N(0, 1) = \sqrt{n} \underbrace{\frac{\bar{X}_n - \theta}{\sqrt{\theta(1-\theta)}}}_{\xrightarrow{d} N(0,1) \text{ by CLT}} \underbrace{\frac{\sqrt{\theta(1-\theta)}}{\sqrt{\bar{X}_n(1-\bar{X}_n)}}}_{\xrightarrow{P} 1, \text{ Annex 1}} \end{aligned}$$

Annex 1:

- $\bar{X}_n \xrightarrow{P} E[X_i] = \theta$  by **WLLN**
- $\sqrt{\bar{X}_n(1-\bar{X}_n)} \rightarrow \sqrt{\theta(1-\theta)}$  by **continuous mapping theorem**

**Example 1.3.2** (Application :  $P(X \in A)$  using Simple Monte Carlo)

Notice that  $P(X \in A) = E[Y]$  where  $Y = I_A(X) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}$

Let  $X_1, X_2, \dots, X_M$  be iid r.v. with the same distribution as  $X$ , and  $Y_i = I_A(X_i)$ ,  $i = 1, \dots, M$ . Then by **SLLN**,

$$\bar{Y}_M = \frac{1}{M} \sum_{i=1}^M Y_i \xrightarrow{a.s.} E[Y] = P(X \in A)$$

where  $M$  is the simulation length.

For sufficiently large  $M$ ,

$$P(X \in A) \approx \frac{1}{M} \sum_{i=1}^M y_i = \frac{1}{M} \underbrace{\#\{i = 1, \dots, M : x_i \in A\}}_{\text{observed proportions of } x_i \in A}$$

Simple Monte Carlo allows us to replace the analytical knowledge of a probability distribution by a sufficiently large sample of iid draws from the distribution since almost all aspects of that probability distribution can be arbitrarily approximated using that sample.

**Example 1.3.3** (Application :  $f(a)$  for some  $a \in \mathbb{R}$  using Simple Monte Carlo)

For a continuous distribution with density  $f$ ,

$$\begin{aligned} f(a) &= \lim_{\delta \rightarrow 0} \frac{F(a + \delta - F(a))}{\delta} \\ &= \frac{1}{\delta} \frac{1}{M} \#\{i = 1, \dots, M : a < x_i < a + \delta\} \end{aligned}$$

That is, the histogram of  $x_1, \dots, x_M$  is an approximation to the density of  $X$ .

### Theorem 1.3.5 Delta method

Let  $\{X_n\}_{n=1}^{+\infty}$  be a sequence of r.v. such that  $\forall \theta \in \Theta$

$$\sqrt{n}(X_n - \theta) \xrightarrow{d} N(0, \sigma^2)$$

Let  $\underbrace{\theta_0}_{\text{interior point of } \Theta} \in \overbrace{\Theta}^{\text{open set}}$  and  $g$  be a differentiable function such that  $g'(\theta_0) \neq 0$ . Then

$$\sqrt{n} \left( \underbrace{g(X_n)}_{\text{typically non-linear}} - g(\theta_0) \right) \xrightarrow{d} N(0, \sigma^2 [g'(\theta_0)]^2)$$

*Proof.* Using the 1st order Taylor expansion

$$g(x) = g(\theta_0) + g'(\theta_0)(x - \theta_0) + r(x - \theta_0), \quad \frac{r(x - \theta_0)}{x - \theta_0} \rightarrow 0 \text{ as } x \rightarrow \theta_0$$

$$\begin{aligned} g(x_n) - g(\theta_0) &= g'(\theta_0)(x_n - \theta_0) + r(x_n - \theta_0) \\ \sqrt{n}(g(x_n) - g(\theta_0)) &= \underbrace{\sqrt{n}(g(x_n) - g(\theta_0))}_{\text{Annex 1}} + \underbrace{\sqrt{n}r(x_n - \theta_0)}_{\text{Annex 2}} \\ \sqrt{n}(g(x_n) - g(\theta_0)) &= \underbrace{\sqrt{n}(g(x_n) - g(\theta_0))}_{\xrightarrow{d} N(0, [g'(\theta_0)]^2 \sigma^2)} + \underbrace{\sqrt{n}r(x_n - \theta_0)}_{\xrightarrow{P} 0} \end{aligned}$$

Annex 1:

$$\sqrt{n}(x_n - \theta_0) \xrightarrow{d} N(0, \sigma^2)$$

$$\begin{aligned} \text{By Slutsky's theorem, } \underbrace{g'(\theta_0)}_{\text{constant, } \xrightarrow{P} g'(\theta_0)} \underbrace{\sqrt{n}(x_n - \theta_0)}_{\xrightarrow{d} T(\cdot)} &\xrightarrow{d} g'(\theta_0)N(0, \sigma^2) = N(0 \times g'(\theta_0), [g'(\theta_0)]^2 \sigma^2) \\ &\Rightarrow \sqrt{n}(x_n - \theta_0) \xrightarrow{d} N(0, [g'(\theta_0)]^2 \sigma^2) \end{aligned}$$

Annex 2:

Step 1:

$$(T_n - \theta) = \underbrace{\frac{1}{a_n}}_{\xrightarrow{P} 0} \underbrace{a_n(T_n - \theta)}_{\xrightarrow{d} T(\cdot)} \xrightarrow{d} 0 \times T(\cdot) = 0 \Rightarrow T_n \xrightarrow{d} \theta \quad \Leftrightarrow \quad T_n \xrightarrow{P} \theta$$

this applies because  $\theta$  is a constant

$$\therefore \sqrt{n}(T_n - \theta) \xrightarrow{d} T \Rightarrow T_n \xrightarrow{P} \theta$$

Step 2:

$$\sqrt{n}(x_n - \theta_0) \xrightarrow{d} N(0, \sigma^2)$$

By step 1, we can conclude that  $x_n \xrightarrow{P} \theta_0$     $X_n - \theta_0 \xrightarrow{P} 0$

We also know that  $\frac{r(x)}{x} \rightarrow 0$

By **continuous mapping theorem**,  $\frac{r(x_n - \theta_0)}{x_n - \theta_0} \xrightarrow{P} 0$

Step 3:

$$\sqrt{n} r(x_n - \theta_0) \Leftrightarrow \underbrace{\sqrt{n}(x_n - \theta_0)}_{\xrightarrow{d} N(0, \sigma^2)} \underbrace{\frac{r(x_n - \theta_0)}{x_n - \theta_0}}_{\xrightarrow{P} 0}$$

By **Slutsky's theorem**,  $\sqrt{n} r(x_n - \theta_0) \xrightarrow{d} 0 \quad \Rightarrow \quad \sqrt{n} r(x_n - \theta_0) \xrightarrow{P} 0 \blacksquare$   
true for constant

#### Example 1.3.4 (Application : log-odds ratio)

Suppose that  $X_1, \dots, X_n$  are iid  $B(1, \theta)$ . Then the **CLT** ensures that

$$\sqrt{n} \frac{\bar{X}_n - \theta}{\sqrt{\theta(1-\theta)}} \xrightarrow{d} N(0, 1)$$

which is equivalent to  $\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, \theta(1-\theta))$ .

- $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n x_i$  representing the proportion of successes in the random sample
- $\theta$  representing the probability of success in the population

Now we are interested in the asymptotic distribution of  $Y_n = \ln \frac{\bar{X}_n}{1-\bar{X}_n}$  which is the empirical log odds of success, a non-linear function. With  $g(x) = \ln \frac{x}{1-x}$ , following  $g'(x) = \frac{1}{x(1-x)}$ . The **delta method** ensures that

$$\sqrt{n}(Y_n - \ln \frac{\theta}{1-\theta}) \xrightarrow{d} N(0, [\theta(1-\theta)]^{-1})$$

which is often written as

$$Y_n \xrightarrow{a} N\left(\ln \frac{\theta}{1-\theta}, \frac{[\theta(1-\theta)]^{-1}}{n}\right)$$

*Proof.* Asymptotic distribution of  $T_n = \ln \frac{\bar{X}_n}{1-\bar{X}_n}$

$$\sqrt{n} \frac{\bar{X}_n - \theta}{\sqrt{\theta(1-\theta)}} \xrightarrow{d} N(0, 1) \Leftrightarrow \sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, \theta(1-\theta))$$

$$g(x) = \ln \frac{x}{1-x} \rightarrow g'(x) = \frac{\frac{1}{1-x} - \frac{(-1)x}{(1-x)^2}}{\frac{x}{1-x}} = \frac{1-x+x}{(1-x)^2} \frac{1-x}{x} = \frac{1}{x(1-x)}$$

Applying the **delta method**,  $\sqrt{n}(T_n - g(\theta_0)) \xrightarrow{d} N(0, [g'(\theta_0)]^2 \sigma^2)$

$$\Rightarrow \sqrt{n}(T_n - \ln \frac{\theta_0}{1-\theta_0}) \xrightarrow{d} N\left(0, \left(\frac{1}{\theta(1-\theta)}\right)^2 \theta_0(1-\theta_0)\right) \Leftrightarrow \sqrt{n}(T_n - \ln \frac{\theta_0}{1-\theta_0}) \xrightarrow{d} N(0, [\theta_0(1-\theta_0)]^{-1}) \blacksquare$$

#### Example 1.3.5 (Application : variance stabilizing)

Suppose  $X_1, \dots, X_n$  are  $B(0, \theta)$ . Then the **CLT** ensures that

$$\sqrt{n} \frac{\bar{X}_n - \theta}{\sqrt{\theta(1-\theta)}} \xrightarrow{d} N(0, 1)$$

Note that the asymptotic variance depends on the true value of  $\theta$ , meaning that the variance,  $\sigma^2$  is not fixed, thus giving us the motive to stabilize the variance. Our goal is to find a  $g$  such that  $\sqrt{n}(g(\bar{X}_n) - g(\theta)) \xrightarrow{d} N(0, 1)$ , which is the same as solving for  $g'(x) = \frac{1}{\sqrt{\theta(1-\theta)}}$ .

$$[g'(x)]^2 \theta(1-\theta) = 1 \Leftrightarrow g'(x) = \frac{1}{\theta(1-\theta)} = \theta^{-1/2}(1-\theta)^{-1/2} \Rightarrow g(\theta) = 2 \arcsin \sqrt{\theta}$$

After this, the asymptotic distribution would be normal with a constant variance.

$$\sqrt{n}(2 \arcsin \sqrt{\bar{X}_n} - 2 \arcsin \sqrt{\theta}) \xrightarrow{d} N(0, 1)$$

When can we apply this technique?

$$\sqrt{n}(X_n - \mu) \xrightarrow{d} N(0, \ln(\mu))$$

$$\text{From } \mathbf{delta method}, \sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} N(0, [g'(\mu)]^2 \ln(\mu))$$

$$\text{the variance stabilizing transformation } g \text{ satisfies } [g'(x)]^2 \ln(\mu) = 1 \Leftrightarrow g'(\mu) = \frac{1}{\sqrt{\ln(\mu)}} \Rightarrow g(\mu) = \int_c^\mu \frac{1}{h(t)} dt$$

$c$  being some constant that ensures the integral exists, and with this  $c$ ,

$$\sqrt{n}(g(x_n) - g(\mu)) \xrightarrow{d} N(0, 1)$$

# Chapter 2

## Classical Statistical Model

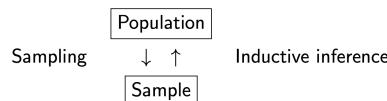
### 2.1 Probability versus statistical inference

**Probability theory** begins with a completely specified model which we assume are correct and we compute the probabilities of certain events. For example,

$$\begin{aligned} X_1, \dots, X_n &\stackrel{iid}{\sim} B(1, \theta) \\ T = \sum_{i=1}^n X_i &\sim B(n, \theta) \\ P(T = t|\theta) &= \binom{n}{t} \theta^t (1-\theta)^{n-t} \quad t = 0, 1, \dots, n \end{aligned}$$

with  $n = 20 \rightarrow P(T = 10|\theta)$  is calculatable if we know  $\theta$ . On the other hand, for **statistical inference**, we observe the realization of certain events, and using that information we try to infer the probabilistic model that governs the corresponding random experiment. For example,  $T = 10 \rightarrow$  observed outcome. I want to use this information to make inference about  $\theta$ .

**Statistical data** result from experiments conducted on a subset of a population, the sample, and we try to extend the conclusions obtained to the whole population.



**Inductive inference** means that there is uncertainty regarding the resulting inference. If we are just drawing finite samples, then we cannot be certain the result is in fact representative of the entire population. The opposite would be **deductive inference** where it is of mathematics. No questions about the validity of the inference. If  $A$  holds  $\rightarrow B$  definitely holds.

### 2.2 Model specification

To formalize the process of statistical inference. The characteristic of interest is modeled as a random variable  $X$  with cumulative distribution function (cdf)  $F$ , the statistical model. The model must be specified either through a **parametric model** where  $F$  is known up to a finite dimensional parameter, e.g.  $X$  as normal with mean  $\mu$  and variance  $\sigma^2$ , both unknown. Or a **nonparametric model** where  $F$  is specified in a nonparametric fashion, e.g.  $F$  is an element of the set of all continuous and symmetric distributions. Focusing on the parametric statistical model:

$$\mathcal{F} = \{F(\cdot|\theta) : \underbrace{\theta}_{\text{parameter}} \in \underbrace{\Theta}_{\text{parameter space}}\}$$

**Example 2.2.1** (Application : daily return of financial asset)

We can propose a normal  $\rightarrow \mathcal{F} = \{N(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma > 0\}$  or a gamma  $\rightarrow \mathcal{F} = \{G(\alpha, \lambda) : \alpha, \lambda > 0\}$

**Example 2.2.2** (Application : insurance policy)

If we are interested in the number of claims per year in an insurance policy, we can propose the Poisson  $\rightarrow \mathcal{F} = \{Po(\lambda) : \lambda > 0\}$

The specification is important and results from many factors, namely based on the knowledge of the problem at hand, knowledge of previous studies, and knowledge of probability theory. The consequence of model misspecification is always negative but is smaller for larger samples.

## 2.2.1 Sampling

**Random sampling** means that the observed data are one of many possible data sets we could have obtained in the same circumstances. The set of  $n$  observations,  $(x_1, \dots, x_n)$  which we have observed is a realization of an  $n$ -dimensional random variables  $(X_1, \dots, X_n)$ .

$$\begin{array}{ll} (X_1, \dots, X_n) & \text{Random sample} \\ (x_1, \dots, x_n) & \text{Observed sample} \end{array}$$

The **sample space** is a subset of  $\mathbb{R}^n$  that contains the set of possible values for  $x_1, \dots, x_n$ . We denote it by  $\mathcal{X}$ .

### Definition 2.2.1: IID random sampling

When the  $n$  random variables that compose the random sample are

- mutually independent  $\rightarrow x_i \perp\!\!\!\perp x_j | \theta$
- identically distributed, with the same distribution as  $X \rightarrow x_i \sim x_j | \theta$

we say that  $(X_1, \dots, X_n)$  constitutes an iid random sample of size  $n$  obtained from the population  $X$ . In notation,  $X_1, \dots, X_n | \theta \stackrel{iid}{\sim} X$ ,  $X$  follows the common distribution of all  $x_i$ 's,  $x_i \sim X$ .

If  $\mathcal{F} = \{f(\cdot | \theta) : \theta \in \Theta\}$  and  $X_1, \dots, X_n | \theta \stackrel{iid}{\sim} X$ , then

$$\begin{aligned} F_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta) &= \prod_{i=1}^n F_{X_i}(x_i | \theta) && \text{by independence} \\ &= \prod_{i=1}^n f(x_i | \theta) && \text{since } X_i \sim X \end{aligned}$$

and similarly for the probability density function

$$f(x_1, \dots, x_n | \theta) = \prod_{i=1}^n f(x_i | \theta)$$

*Proof.* **Poisson distribution**

$$\begin{aligned} f(x_1, \dots, x_n | \lambda) &= \prod_{i=1}^n f(x_i | \lambda) \\ &= \prod_{i=1}^n e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \\ &= P(x_1 = x_1, \dots, x_n = x_n | \lambda) \\ &= e^{-n\lambda} \frac{\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} \rightarrow \text{Annex 1} \end{aligned}$$

Annex 1:

- $e^a e^b = e^{a+b}$
- $a^x a^y = a^{x+y}$

■

## 2.3 Statistics

### Definition 2.3.1: Statistic

A statistic is any function of  $(X_1, \dots, X_n)$  that does not depend on unknown parameters.

#### Example 2.3.1 (Statistic)

In the context of a  $N(\mu, \sigma^2)$ ,  $\mu \in \mathbb{R}$  and  $\sigma > 0$  unknown.

**Uni-dimensional** statistics include

- $T = \sum_{i=1}^n X_i$
- $\bar{X} = \frac{1}{n} T$
- $S^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$

**Bi-dimensional** statistics include

- $(T, \sum_{i=1}^n X_i^2)$
- $(\bar{X}, S^2)$

#### Example 2.3.2 (Not statistic)

$$\sum_{i=1}^n (X_i - \mu)^2 \quad \frac{1}{\sigma^2} \sum_{i=1}^n X_i^2$$

are not statistics because they depend on unknown parameters. If  $\sigma^2$  is known, then  $\frac{1}{\sigma^2} \sum_{i=1}^n X_i^2$  is a statistic.

Statistics operate a data reduction and are summaries of the information contained in the random sample. Statistics are random variables, as usual, it is important to distinguish between the random variable and its observed value.

population $X$	random sample $(X_1, \dots, X_n)$	observed sample $(x_1, \dots, x_n)$
population mean $\mu = E[X]$	sample mean $\bar{X} = \frac{1}{n} \sum_i X_i$	mean of the sample $\bar{x} = \frac{1}{n} \sum_i x_i$
population variance $\sigma^2 = \text{Var}(X)$	sample variance $S^2 = \frac{1}{n} \sum_i (X_i - \bar{X})^2$	variance of the sample $s^2 = \frac{1}{n} \sum_i (x_i - \bar{x})^2$

Probability vs. Statistics vs. Data exploration

## 2.4 Sampling distribution

The sampling distribution of a statistic corresponds to its probability distribution: as  $(X_1, \dots, X_n)$  varies according to its distribution, what is the resulting probabilistic behavior of  $T(X_1, \dots, X_n)$ . In classical inference, it is important to know the sampling distribution of statistics because that is necessary to evaluate the performance of statistical methodologies. The **objective** is to determine aspects of the sampling distribution of a statistic  $T$  knowing aspects of the probability distribution of the population  $X$ .

There are different methods to obtain the sampling distribution of a statistic.

- **Change of variable:** If  $X$  is continuous,

$$F_T(t|\theta) = P(T \leq t|\theta) = \int_{A(t)} \sum_{i=1}^n f(x_i|\theta) dx_1 \dots dx_n$$

where  $A(t) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : T(x_1, \dots, x_n) \leq t\}$ . If  $X$  is discrete, replace integrals with sums.

- Determining the **moment generating function** of  $T$
- Using **well-known properties** of the distribution of  $X$
- **Asymptotic approximations** to the sampling distribution of certain statistics (from CLT and related results)
- Using **simulations**

#### Example 2.4.1 (Change of variable)

Let  $T = \sum_{i=1}^n X_i$

- If  $(X_1, \dots, X_n)$  is an iid random sample from a  $Po(\lambda)$  population, since the sum of independent Poisson is still Poisson, we have  $T \sim Po(n\lambda)$ , hence  $f_T(t|\lambda) = e^{-n\lambda} \frac{(n\lambda)^t}{t!}$ ,  $t \in N_0$ .
- If  $(X_1, \dots, X_n)$  is an iid random sample from a  $N(\mu, \sigma^2)$  population, then  $T \sim N(n\mu, n\sigma^2)$ .
- If  $(X_1, \dots, X_n)$  is an iid random sample from a  $B(1, \theta)$  population, then  $T \sim B(n, \theta)$ .

#### Example 2.4.2 (Monte Carlo simulation)

1. Draw  $N$  independent samples of size  $n$  from the distribution of  $X$
2. For each of those samples, compute the observed values of the statistic  $T$
3. The  $N$  resulting numbers,  $(t_1, \dots, t_n)$  constitute a sample of size  $N$  drawn from the sampling distribution of  $T$

### 2.4.1 Sample distribution of the sample moments

#### Definition 2.4.1: Sample moments

Let  $(X_1, \dots, X_n)$  be an iid random sample of size  $n$  from a population  $X$ . For  $k \in \mathbb{N}$  we define the  $k$ th raw sample moment as

$$M'_k = \frac{1}{n} \sum_{i=1}^n x_i^k$$

and the  $k$ th central sample moment by

$$M_k = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^k$$

$$\begin{cases} \mu'_k = E[x^k] \rightarrow k\text{th raw moment} \\ M'_k = \frac{1}{n} \sum x_i^k \rightarrow k\text{th sample raw moment} \end{cases} \quad \begin{cases} \mu_k = E[(x - \mu)^k] \rightarrow k\text{th central moment} \\ M_k = \frac{1}{n} \sum (x_i - \bar{x})^k \rightarrow k\text{th central sample moment} \end{cases}$$

We want to observe how they behave in relation to each other.

Once again, it is important to distinguish the **sample moments**,  $M'_k$  and  $M_k$ , from the **population moments**,  $\mu'_k = E[X^k]$  and  $\mu_k = E[(X - E[X])^k]$ , and the **observed sample moments**,  $m'_k = \frac{1}{n} \sum_{i=1}^n x_i^k$  and  $m_k = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^k$ .

**Note:**

**Important special cases:**  $\bar{X} = M'_1$  and  $S^2 = M_2$ , the sample mean and the sample variance.

**Theorem 2.4.1 Properties of the sample mean**

If all the moments exist, then

$$\begin{aligned} E[\bar{X}] &= E[X] = \mu \\ Var(\bar{X}) &= \frac{Var(X)}{n} = \frac{\sigma^2}{n} \\ \mu_3 &= \frac{\mu_3}{n^2} \\ \mu_4 &= \frac{3\mu_2^2}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3} \end{aligned}$$

*Proof.*  $E[\bar{X}]$

$$E[\bar{X}] = \frac{1}{n} \sum_{i=1}^n E[x_i] = \frac{1}{n} n\mu = \mu$$

■

*Proof.*  $Var(\bar{X})$

$$\begin{aligned} Var(\bar{X}) &= E[(\bar{X} - \mu)^2] \\ &= E\left[\left(\frac{1}{n} \sum x_i - \mu\right)^2\right] \\ &= E\left[\frac{1}{n^2} \left(\sum (x_i - \mu)\right)^2\right] \\ &\stackrel{\text{Annex 1}}{=} \frac{1}{n^2} E\left[\left(\sum_{\substack{a_i \\ =x_i-\mu}} \underbrace{a_i}_{\text{Annex 1}}\right)^2\right] \\ &= \otimes \end{aligned}$$

Annex 1:  $(\sum a_i)^2 = (\sum_i a_i)(\sum_j a_j) = \sum_i \sum_j a_i a_j = \sum_i a_i^2 + \sum_i \sum_{i \neq j} a_i a_j$  □

$$\begin{aligned} \otimes &= \frac{1}{n^2} [E\left[\sum_i a_i^2\right] + E\left[\sum_i \sum_{i \neq j} a_i a_j\right]] \\ &= \frac{1}{n^2} \left[ \sum_i \underbrace{E[a_i^2]}_{E[(X-\mu)^2]=Var(X)=\sigma^2} + \sum_i \sum_{i \neq j} \underbrace{E[a_i a_j]}_{\text{Cov}(a_i, a_j)=0 \rightarrow \text{Annex 2}} \right] \\ &= \frac{1}{n^2} (n\sigma^2 + 0) \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Annex 2: Since  $a_i \perp\!\!\!\perp a_j$ , expectation of the product is the product of the expectation for independent variables. This implies that  $E[(x_i - \mu)(x_j - \mu)] = E[(x_i - \mu)]E[(x_j - \mu)] = 0 \times 0 = 0$  ■

By the **Weak Law of Large Numbers**,  $\bar{X} \xrightarrow{P} \mu$ . The distribution is more and more concentrated around  $\mu$  as  $n$  increases. The distribution of  $\bar{X}$  is centered around  $\mu$  and  $\lim_{n \rightarrow +\infty} Var(\bar{X}) = 0$ .

*Proof.* Asymmetry  $\mu_3$

$$\mu_3(\bar{X}) = E[(\bar{X} - \mu)^3] = \frac{1}{n^3} E\left[\left(\sum a_i\right)^3\right] = \otimes$$

Annex 1

Annex 1:

$$\begin{aligned}
(\sum a_i)^3 &= (\sum_i a_i)(\sum_j a_j)(\sum_k a_k) = \sum_i \sum_j \sum_k a_i a_j a_k \\
&= (\sum_i a_i)^2 (\sum_k a_k) = (\sum_i a_i^2 + \sum_i \sum_{i \neq j} a_i a_j) (\sum_k a_k) \\
&= \sum_i \sum_k a_i^2 a_k + \sum_i \sum_{i \neq j} \sum_k a_i a_j a_k \\
&= \sum_i a_i^3 + \underbrace{\sum_i \sum_{k \neq i} a_i^2 a_k}_{E[\cdot]=0} + \underbrace{\sum_i \sum_{j \neq i} \sum_{k \neq i} a_i a_j a_k}_{E[\cdot]=0} + \underbrace{\sum_i \sum_{j \neq i} a_i^2 a_j}_{E[\cdot]=0} \quad \square \\
\otimes &= \frac{1}{n^3} \sum_{i=1}^n E[(x_i - \mu)^3] = \frac{1}{n^3} n \mu_3 = \frac{\mu_3}{n^2} \xrightarrow{n \rightarrow \infty} 0
\end{aligned}$$

■ As  $n$  goes to infinity, the distribution becomes symmetric. This evidence is compatible with the **Central Limit Theorem**. The related concept to asymmetry is **skewness**,  $\gamma_1 = \frac{\mu_3}{\sigma^3}$ .  $\sigma^3$  is used to make  $\gamma_1$  dimensionless, as in independent of the unit measurement of the  $x$ .

The commonly used concept related to  $\mu_4$  is **kurtosis** which is often denoted as  $\gamma_2 = \frac{\mu_4}{\sigma^4}$ . It is the indication of heavy tails. If  $\gamma_2 > 3$ , the distribution has a heavier tail than Gaussian. The excess kurtosis can also be used which is just  $\gamma_2 - 3$ , indicating heavier tail than Gaussian if bigger than 0.

*Proof.* Kurtosis  $\gamma_2$

$$\begin{aligned}
\gamma_2(\bar{X}) &= \frac{\mu_4(\bar{X})}{\underbrace{\sigma^4(\bar{X})}_{\text{s.d. to the power of 4}}} = \frac{\frac{3\mu_2^2}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3}}{(\sqrt{\frac{\sigma^2}{n}})} \\
&= \frac{\frac{3\mu_2^2}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3}}{(\frac{(\sigma^2)^2}{n^2})} \stackrel{\mu_2 = \sigma^2}{=} \frac{\frac{3\mu_2^2}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3}}{(\frac{(\mu_2)^2}{n^2})} \\
&= 3 + \underbrace{\frac{1}{n} \frac{\mu_4 - 3\mu_2^2}{\mu_2^2}}_A \xrightarrow{n \rightarrow \infty} 3
\end{aligned}$$

$$A: \frac{\mu_4}{\mu_2^2} - \frac{3\mu_2^2}{\mu_2^2} = \frac{\mu_4}{\sigma^4} - 3 \rightarrow \gamma_2 - 3 \Rightarrow \text{excess kurtosis.}$$

### Theorem 2.4.2 Properties of the sample variance

If all the moments exist,

$$\begin{aligned}
E[S^2] &= \frac{n-1}{n} \sigma^2 \\
Var(S^2) &= \frac{\mu_4 - \mu_2^2}{n} - 2 \frac{\mu_4 - 2\mu_2^2}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3} \xrightarrow{n \rightarrow \infty} 0, \text{ roughly centered around } S^2
\end{aligned}$$

Since  $E[S^2] = \frac{n-1}{n} \sigma^2 < \sigma^2$ , always strictly smaller than the variance, we define the **bias-corrected sample variance**

$$S'^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 = \frac{n}{n-1} S^2$$

Notice that  $M'_k = \frac{1}{n} \sum X_i^k$  which is the average of iid r.v. but  $M_k = \frac{1}{n} \sum (X_i - \bar{X})^k$ . We cannot use LLN or CLT to study  $M_k$  because  $(X_i - \bar{X})$  and  $(X_j - \bar{X})$  for  $i \neq j$  are not iid as  $\bar{X}$  depends on both  $X_i$  and  $X_j$ .

### Theorem 2.4.3 Properties of the bias-corrected sample variance

If all the moments exist,

$$E[S'^2] = \sigma^2$$

$$\text{Var}(S'^2) = \frac{1}{n} \left( \mu_4 - \frac{n-3}{n-1} \mu_2^2 \right)$$

*Proof.*  $E[S'^2]$

$$S'^2 = \frac{nS^2}{n-1}$$

$$E[S'^2] = \frac{n}{n-1} E[S^2] = \frac{n}{n-1} \frac{n-1}{n} \sigma^2 = \sigma^2$$

■

### Theorem 2.4.4 Properties of central sample moments

If all the moments exist,

$$E[M_k] = \mu_k + O\left(\frac{1}{n}\right)$$

$$\text{Var}(S'^2) = \frac{c}{n} + O\left(\frac{1}{n^2}\right)$$

where  $c$  is a constant which involves central population moments of order  $\leq 2k$ .

The central sample moments have similar behavior as  $S^2$ .

$$a_n = O(b_n) \Leftrightarrow \frac{a_n}{b_n} \text{ is limited}$$

$$a_n = O\left(\frac{1}{n}\right) \Leftrightarrow \frac{a_n}{\frac{1}{n}} \text{ is limited} = na_n$$

$$a_n = \underbrace{\frac{1}{n}}_{\rightarrow 0} \underbrace{na_n}_{\text{limited}} \rightarrow 0$$

$$\Rightarrow O\left(\frac{1}{n}\right) \text{ sequence goes to 0 roughly at the rate of } \frac{1}{n}$$

### Theorem 2.4.5 Asymptotic distribution of $\bar{X}$

As long as  $\text{Var}(X)$  is finite, we have as a direct consequence of the **Central limit theorem** that

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \sqrt{n} \frac{\bar{X} - \mu}{\sigma} \xrightarrow{d} N(0, 1)$$

This result is typically used in the form

$$P(\bar{X} \leq x) \approx \Phi\left(\sqrt{n} \frac{x - \mu}{\sigma}\right)$$

that is,  $\bar{X} \xrightarrow{a} N(\mu, \frac{\sigma^2}{n})$ .

In general, unimodality and symmetry have a positive impact on the speed of convergence. If the distribution is already behaving like normal, the rate of convergence would be faster.

## 2.4.2 Order statistics

### Definition 2.4.2: Order statistics

Let  $(X_1, \dots, X_n)$  be an iid random sample. The  $i$ th order statistic is denoted by  $X_{(i)}$  and satisfies

$$\underbrace{X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}}_{\text{not iid}}$$

Order statistics are not iid because if we know  $X_{(n)}$  is 10, then  $X_{(1)}$  cannot be larger than 10. For notation purposes,  $(Y_1, \dots, Y_n) \equiv (X_{(1)}, \dots, X_{(n)})$ .

The order statistics have a joint pdf given by

$$g(y_1, y_2, \dots, y_n) = n! \prod_{i=1}^n f(y_i) \quad \text{if } y_1 < y_2 < \dots < y_n$$

If  $u > v$ , the joint pdf of  $(Y_u, Y_v)$  is

$$g_{u,v}(y, z) = \frac{n!}{(u-1)!(v-u-1)!(n-v)!} \times [F(y)]^{u-1} [F(z) - F(y)]^{v-u-1} [1 - F(z)]^{n-v} f(y) f(z) \quad \text{if } y < z$$

The cdf of  $Y_v$  is

$$g_v(y) = \frac{n!}{(v-1)!(n-v)!} [F(y)]^{v-1} [1 - F(y)]^{n-v} f(y) = G'_v(y)$$

and the pdf of  $Y_v$  is

$$G_v(y) = \sum_{j=v}^n \binom{n}{j} [F(y)]^j [1 - F(y)]^{n-j} = P(Y_v \leq y)$$

*Proof.* The joint distribution of  $G_{u,v}(y, z)$  follows a **multinomial distribution** with the following characteristics:

- $n$  objects
- $k$  categories
- The probability that each object belongs to category  $i$  is  $p_i$ ,  $i = 1, \dots, k$  independently
- $w_i$  is the number of objects among the  $n$  that belongs to class  $i$ ,  $i = 1, \dots, k$

$$G_{u,v}(y, z) = P(Y_u \leq y, Y_v \leq z)$$

$$g_{u,v}(y, z) = \frac{\partial^2 G(y, z)}{\partial y \partial z} = \lim_{\delta_1 \rightarrow 0^+, \delta_2 \rightarrow 0^+} P(y < y_u \leq y + \delta_1, z < y_v \leq z + \delta_2)$$

This is a multinomial with 5 classes:

$$\begin{cases} P_1 = F(y) \\ P_2 = P(y < x \leq y + \delta_1) = F(y + \delta_1) - F(y) \\ P_3 = F(z) - F(y + \delta_1) \\ P_4 = P(z < x \leq z + \delta_2) = F(z + \delta_2) - F(z) \\ P_5 = 1 - F(z + \delta_2) \end{cases}$$

with

$$\begin{cases} w_1 = u - 1 \\ w_2 = 1 \\ w_3 = v - u - 1 \rightarrow \text{Annex 1} \\ w_4 = 1 \\ w_5 = n - v \end{cases}$$

Annex 1:  $n = u - 1 + 1 + 1 + n - v + w_3 \Leftrightarrow n = u + 1 + n - v + w_3 \Leftrightarrow w_3 = n - u - 1 - n + v = v - u - 1$   
where

$$P(y < Y_u \leq y + \delta_1, z < Y_v \leq z + \delta_2) = \frac{n!}{(u-1)!1!(v-u-1)!1!(n-v)!} P_1^{u-1} P_2^1 P_3^{v-u-1} P_4^1 P_5^{n-v}$$

thus,

$$\begin{aligned} g_{u,v}(y, z) &= \lim_{\delta_1 \rightarrow 0^+, \delta_2 \rightarrow 0^+} \frac{n!}{(u-1)!(v-u-1)!(n-v)!} [F(y)]^{u-1} [F(y + \delta_1) - F(y)] \\ &\quad \times [F(z) - F(y + \delta_1)]^{v-u-1} [F(z + \delta_2) - F(z)] [1 - F(z + \delta_2)]^{n-v} \\ &= \frac{n!}{(u-1)!(v-u-1)!(n-v)!} [F(y)]^{u-1} [F(z) - F(y)]^{v-u-1} [1 - F(z)]^{n-v} \\ &\quad \times \underbrace{\lim_{\delta_1 \rightarrow 0^+} \frac{F(y + \delta_1) - F(y)}{\delta_1}}_{f(y)} \underbrace{\lim_{\delta_2 \rightarrow 0^+} \frac{F(z + \delta_2) - F(z)}{\delta_2}}_{f(z)} \\ &= \frac{n!}{(u-1)!(v-u-1)!(n-v)!} [F(y)]^{u-1} [F(z) - F(y)]^{v-u-1} [1 - F(z)]^{n-v} f(y) f(z) \end{aligned}$$

■

*Proof.* The cdf of  $Y_v$

$$\begin{aligned} G_v(y) &= P(Y_v \leq y) = P(\text{at least } v \text{ of the } X_i \text{'s are } \leq y) \\ &= \underbrace{\sum_{j=v}^n P(\text{exactly } j \text{ of the } X_i \text{'s are } \leq y)}_{N_y = \#\{i : x_i \leq y\}} \\ &= \sum_{j=v}^n \binom{n}{j} [F(y)]^j [1 - F(y)]^{n-j} \quad \text{since } N_y \sim B(n, F(y)) \end{aligned}$$

The pdf is just the derivative of the cdf, thus

$$\begin{aligned} g_v(y) &= G'_v(y) = \sum_{j=v}^n \binom{n}{j} \left[ j[F(y)]^{j-1} f(y) [1 - F(y)]^{n-j} - [F(y)]^j (n-j) [1 - F(y)]^{n-j-1} (-f(y)) \right] \\ &= f(y) \sum_{j=v}^n \binom{n}{j} [F(y)]^{j-1} [1 - F(y)]^{n-j} [j + n - j - nj] \\ &= f(y) \sum_{j=v}^n \binom{n}{j} [F(y)]^{j-1} [1 - F(y)]^{n-j} [n - nj + j] \\ &= f(y) \sum_{j=v}^n \binom{n}{j} [F(y)]^{j-1} [1 - F(y)]^{n-j} [n(1 - j) + j] \\ &= f(y) \sum_{j=v}^n \binom{n}{j} [F(y)]^{j-1} [1 - F(y)]^{n-j} [n - nj + j] \\ &= \frac{n!}{(v-1)!(n-v)!} [F(y)]^{v-1} [1 - F(y)]^{n-v} f(y) \end{aligned}$$

■

**Theorem 2.4.6** Important special cases : the maximum and the minimum

The pdf and the cdf of the minimum and the maximum are

$$\begin{aligned} G_1(y) &= 1 - [1 - F(y)]^n & g_1(y) &= n f(y) [1 - F(y)]^{n-1} \\ G_n(y) &= [F(y)]^n & g_n(y) &= n f(y) [F(y)]^{n-1} \end{aligned}$$

with the joint cdf of

$$g_{1,n}(y, z) = n(n-1)[F(z) - F(y)]^{n-2} f(y) f(z) \quad y < z$$

*Proof.*  $G_n(y)$

$$\begin{aligned} G_n(y) &= P(Y_n \leq y) = P(X_{(n)} \leq y) \\ &= P(X_{(1)} \leq y, \dots, X_{(n)} \leq y) \rightarrow \text{Annex 1} \\ &= P(X_{(1)} \leq y) P(X_{(2)} \leq y) \cdots P(X_{(n)} \leq y) \quad \text{because } x_i \perp\!\!\!\perp x_j, i \neq j \\ &= [F(y)]^n \quad \text{because iid} \end{aligned}$$

Annex 1: If  $X_{(n)} \leq y \Leftrightarrow X_{(1)} \leq y, \dots, X_{(n)} \leq y$

$g_n(y) = G'_n(y)$  if continuous

*Proof.*  $G_1(y)$

The idea is that

$$X_{(1)} \leq y \Leftrightarrow \exists_i X_i \leq y$$

Thus we can write

$$\begin{aligned} G_1(y) &= P(X_{(1)} \leq y) = P(\exists_i X_i \leq y) \\ &= P(\overline{\forall_i X_i > y}) = 1 - P(\forall_i X_i > y) \\ &= 1 - P(X_{(1)} > y, \dots, X_{(n)} > y) = 1 - P(X_{(1)} > y) \cdots P(X_{(n)} > y) \quad \text{because } x_i \perp\!\!\!\perp x_j, i \neq j \\ &= 1 - [1 - F(y)]^n \quad \text{because iid} \end{aligned}$$

*Proof.*  $g_{1,n}(y, z)$

$$\begin{aligned} G_n(y) &= G_{1,n}(x, y) + P(x < X_i \leq y, \forall i) \\ &= G_{1,n}(x, y) + \underbrace{[F(y) - F(x)]^n}_{P(a < X \leq b) = F(b) - F(a)} \quad \text{justified through \textbf{total probability theorem}} \\ &\Leftrightarrow [F(y)]^n = G_{1,n}(x, y) + [F(y) - F(x)]^n \\ &\Leftrightarrow G_{1,n}(x, y) = [F(y)]^n - [F(y) - F(x)]^n, x < y \end{aligned}$$

The cumulative distribution function is just the derivative relative to both variables, so

$$\begin{aligned} g_{1,n}(x, y) &= \frac{\partial^2}{\partial y \partial x} G_{1,n}(x, y) \\ &= \frac{\partial}{\partial y} [0 - n(-f(x)) [F(y) - F(x)]^{n-1}] \\ &= n f(x) (n-1) f(y) [F(y) - F(x)]^{n-2}, n > 2 \end{aligned}$$

**Example 2.4.3 (Pareto)**

Pareto distribution is one of the fat tail distributions.

$$f(x) \propto x^{-(\theta+1)}, x > c \Leftrightarrow f(x) = bx^{-(\theta+1)}$$

To find  $b$ , we impose that the integral of the pdf is equal to 1.

$$\begin{aligned} \int_c^{+\infty} bx^{-(\theta+1)} dx &= 1 \\ b \left[ -\frac{1}{\theta} x^{-\theta} \right]_c^{+\infty} &= 1 \\ b \left( 0 - \frac{c^{-\theta}}{-\theta} \right) &= 1 \\ b \frac{c^{-\theta}}{\theta} &= 1 \\ b &= \theta c^\theta \end{aligned}$$

Polynomial decays slower than exponential decays. This means that the tails are heavier.

$$\begin{aligned} F(x) = P(X \leq x) &= \int_c^x f(u) du \\ &= \int_c^x \theta c^\theta u^{-(\theta+1)} du \\ &= \theta c^\theta \left[ -\frac{u^{-\theta}}{\theta} \right]_c^x \\ &= \theta c^\theta \left( \frac{x^{-\theta} - c^{-\theta}}{-\theta} \right) \\ &= c^\theta (c^{-\theta} - x^{-\theta}) \\ &= 1 - \left( \frac{c}{x} \right)^\theta \\ &= 1 - c^\theta x^{-\theta}, x > c \end{aligned}$$

Now showing the distribution of the minimum order statistic.

$$\begin{aligned} g_1(y) &= nf(y)[1 - F(y)]^{n-1} \\ &= n\theta c^\theta y^{-(\theta+1)} [c^\theta y^{-\theta}]^{n-1} \\ &= n\theta c^\theta y^{-(\theta+1)} c^{\theta(n-1)} y^{-\theta(n-1)} \\ &= n\theta c^{\theta n} y^{-(\theta n+1)}, y > c \end{aligned}$$

This has the same structure as a Pareto distribution with parameter  $\theta n$ . Thus, the minimum of a sample of size  $n$  from a Pareto( $\theta$ ) distribution is still a Pareto distribution with parameter  $\theta n$ .

$$X_{(1)} \sim Pa(c, \theta n)$$

However, the maximum does not have a simple form.

$$g_n(y) = n\theta c^\theta y^{-(\theta+1)} [1 - c^\theta y^{-\theta}]^{n-1}, y > c$$

The maximum does not belong to the Pareto family.

**Example 2.4.4 (Exponential)**

Let  $(X_1, \dots, X_n)$  be an iid random sample from an  $\text{Exp}(\lambda)$  population. The pdf and cdf of the exponential distribution are

$$f(x) = \lambda e^{-\lambda x}, x > 0 \quad F(x) = 1 - e^{-\lambda x}, x > 0$$

The pdf of the minimum order statistic is

$$\begin{aligned} g_1(y) &= nf(y)[1 - F(y)]^{n-1} \\ &= n\lambda e^{-\lambda y} [e^{-\lambda y}]^{n-1} \\ &= n\lambda e^{-\lambda y} e^{-\lambda ny + \lambda y} \\ &= n\lambda e^{-\lambda ny}, y > 0 \end{aligned}$$

This has the same structure as an exponential distribution with parameter  $\lambda n$ . Thus, the minimum of a sample of size  $n$  from an  $\text{Exp}(\lambda)$  distribution is still an exponential distribution with parameter  $\lambda n$ .

$$X_{(1)} \sim \text{Exp}(\lambda n)$$

Also notice that

$$E[X] = \frac{1}{\lambda} \Rightarrow E[X_{(1)}] = \frac{1}{\lambda n} < \frac{1}{\lambda}$$

This makes sense as it is often used to model duration.

The pdf of the maximum order statistic is

$$\begin{aligned} g_n(y) &= nf(y)[F(y)]^{n-1} \\ &= n\lambda e^{-\lambda y} [1 - e^{-\lambda y}]^{n-1}, y > 0 \end{aligned}$$

The maximum does not belong to the exponential family.

**Definition 2.4.3: Sample quantile**

Let  $p \in (0, 1)$  and  $k = np$ . Then the sample quantile of order  $p$  is  $Z_p$  such that

$$Z_p = \begin{cases} \frac{Y_k + Y_{k+1}}{2} & \text{if } k \text{ is an integer} \\ Y_{(\lceil k \rceil)} & \text{if } k \text{ is not an integer} \end{cases}$$

where  $\lceil k \rceil$  is the integer part of  $k$ .

**Theorem 2.4.7 Asymptotic distribution of the sample quantile of order  $p$**

Let  $Z_p$  be the sample quantile of order  $p$  of an iid random sample of size  $n$ , obtained from a continuous population with density  $f$ . Denoted by  $\xi_p$  the population quantile of order  $p$ . If  $f$  is continuous and positive at  $\xi_p$ , then

$$\sqrt{n} f(\xi_p) \frac{Z_p - \xi_p}{\sqrt{p(1-p)}} \xrightarrow{d} N(0, 1)$$

$$\xi_p : P(X \leq \xi_p) \geq p \quad \text{and} \quad P(X \geq \xi_p) \geq 1 - p$$

If  $X$  is continuous,  $F(\xi_p) = p$ . It is actually quite difficult to prove the relationship between the population quantile  $\xi_p$  and the sample quantile  $Z_p$ .

**Example 2.4.5 (Normal)**

If  $X \sim N(\mu, \sigma^2)$ , then  $\xi_{\frac{1}{2}} = \mu$ . Then  $f(\xi_{\frac{1}{2}}) = (2\pi\sigma^2)^{-\frac{1}{2}}$ . Hence,

$$\sqrt{n} \frac{Z_{\frac{1}{2}} - \mu}{\sigma} \xrightarrow{d} N(0, \frac{\pi}{2}) \Leftrightarrow \sqrt{\frac{2n}{\pi\sigma^2}} (Z_{\frac{1}{2}} - \mu) \xrightarrow{d} N(0, 1)$$

### Theorem 2.4.8 Fisher-Tippett-Gnedenko Theorem

If  $\exists a_n, b_n, b_n > 0$  then

$$\lim_{n \rightarrow \infty} P\left(\frac{X_{(n)} - a_n}{b_n} \leq x\right) = F(x)$$

and  $F$  is the cdf of a non-degenerate r.v., then  $F$  belongs to one of the following three families:

- **Gumbel** :  $F(x) = e^{-e^{-(x-\mu)/\beta}}$ ,  $x \in \mathbb{R}$ , with  $\mu \in \mathbb{R}$  and  $\beta > 0$
- **Fréchet** :  $F(x) = 0, x \leq \mu$  and  $F(x) = e^{-((x-\mu)/\beta)^{-\alpha}}$ ,  $x > \mu$  with  $\mu \in \mathbb{R}$ ,  $\beta > 0$  and  $\alpha > 0$
- **Weibull** :  $F(x) = e^{-((\mu-x)/\beta)^\alpha}$ ,  $x < \mu$  and  $F(x) = 1, x \geq \mu$  with  $\mu \in \mathbb{R}$ ,  $\beta > 0$  and  $\alpha > 0$

One of the three extreme value distributions.

### 2.4.3 A few sampling distributions

#### Normal population

Let  $(X_1, \dots, X_n)$  be an iid random sample from a  $N(\mu, \sigma^2)$  distribution.

- Distribution of the sample mean  $\bar{X}$

$\bar{X}$  is a linear combination of independent normals, hence it follows a normal distribution. We know that  $E[\bar{X}] = \mu$  and  $Var(\bar{X}) = \frac{\sigma^2}{n}$ . Thus,

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{or} \quad \sqrt{n} \frac{\bar{X} - \mu}{\sigma} \sim N(0, 1)$$

Unlike the CLT which is approximately  $N(0, 1)$ , this is exact.

- Distribution of the bias-corrected sample variance  $S'^2$

Because  $\frac{X_i - \mu}{\sigma} \sim N(0, 1)$ , we have  $\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_1^2$ . Thus,  $\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_n^2$ .

Also  $\sum_{i=1}^n (X_i - \mu)^2 = \sum_{i=1}^n [(X_i - \bar{X}) + (\bar{X} - \mu)]^2 = \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$ . Hence,

$$\sum_{i=1}^n (X_i - \mu)^2 = (n-1) \frac{S'^2}{\sigma^2} + n \frac{(\bar{X} - \mu)^2}{\sigma^2}$$

Since  $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ , we have  $n \frac{(\bar{X} - \mu)^2}{\sigma^2} \sim \chi_1^2$ . Also,  $\bar{X}$  and  $S'^2$  are independent in the context of the normal distribution. Thus,

$$(n-1) \frac{S'^2}{\sigma^2} = \sum_{i=1}^n (X_i - \mu)^2 - n \frac{(\bar{X} - \mu)^2}{\sigma^2} \sim \chi_{n-1}^2$$

Therefore,

$$\frac{(n-1)S'^2}{\sigma^2} \sim \chi_{n-1}^2$$

- Student ratio

When the population variance is unknown, we have the student ratio:

$$\frac{\bar{X} - \mu}{S'/\sqrt{n}} = \frac{\bar{X} - \mu}{S/\sqrt{n-1}} \sim t(n-1)$$

*Proof.*  $\frac{N(0,1)}{\sqrt{\chi^2(n)/n}} \sim t(n), N(0,1) \perp\!\!\!\perp \chi^2(n)$

We know that  $\bar{X}$  and  $S'^2$  are independent, thus

$$\frac{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S'^2}{\sigma^2}/(n-1)}} = \frac{\bar{X} - \mu}{S'/\sqrt{n}} \sim t(n-1)$$

and because  $t(n) \xrightarrow{d} N(0, 1)$ , we have

$$\frac{\bar{X} - \mu}{S'/\sqrt{n}} \xrightarrow{d} N(0, 1)$$

■

## Two normal distributions

$$\begin{aligned} X_1 &\sim N(\mu_1, \sigma_1^2) \\ X_2 &\sim N(\mu_2, \sigma_2^2) \end{aligned}$$

Two random samples, mutually independent, of size  $m$  and  $n$  respectively:  $(X_{11}, \dots, X_{1m})$  and  $(X_{21}, \dots, X_{2n})$ .

- Difference of the sample means

$$\bar{X}_1 - \bar{X}_2 \sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}) \Leftrightarrow \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \sim N(0, 1)$$

However, the previous result is limited if the population variances are not known. We can assume both variances are equal, i.e.  $\sigma_1^2 = \sigma_2^2 = \sigma^2$ . Then,

$$T = \frac{\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{1}{m} + \frac{1}{n}}}}{\sqrt{\frac{(m-1)S_1'^2 + (n-1)S_2'^2}{m+n-2}}} \sim t(m+n-2)$$

Considering the equation as  $T = \frac{U}{\sqrt{V/(m+n-2)}}$ ,  $U \frac{1}{\sigma} \sim N(0, 1)$  and  $V \frac{1}{\sigma^2} \sim \chi^2(m+n-2)$ .

When the variances are unknown and different, if the sample sizes are large, Slutsky's theorem allows us to replace the population variances by the sample variances and obtain the same distribution in the limit. However, for small sample sizes, we can use the Welch-Satterthwaite approximation

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1'^2}{m} + \frac{S_2'^2}{n}}} \stackrel{a}{\sim} t(v)$$

where  $v$  is the largest integer that does not exceed

$$v = \frac{\left(\frac{S_1'^2}{m} + \frac{S_2'^2}{n}\right)^2}{\frac{(S_1'^2/m)^2}{m-1} + \frac{(S_2'^2/n)^2}{n-1}}$$

- Two sample variances

$$U = \frac{(m-1)S_1'^2}{\sigma_1^2} \sim \chi_{m-1}^2$$

$$V = \frac{(n-1)S_2'^2}{\sigma_2^2} \sim \chi_{n-1}^2$$

Since  $U \perp\!\!\!\perp V$ , we have

$$F = \frac{U/(m-1)}{V/(n-1)} = \underbrace{\frac{S_1'^2}{S_2'^2}}_{\text{ratio of the corrected sample variance}} \times \underbrace{\frac{\sigma_2^2}{\sigma_1^2}}_{\text{inverted ratio of the variances}} \sim F(m-1, n-1)$$

### Bernoulli population

Consider there are two types of individuals in the population: the ones who posses a certain attribute and the one who do not. Let  $(X_1, \dots, X_n)$  be an iid random sample of size  $n$  from a  $B(1, \theta)$  population, where  $\theta$  is the proportion of successes in the population (known). It is useful to establish the sampling distribution of two statistics:

- The number of individuals in the sample who posses the attribute:  $T = \sum_{i=1}^n X_i$
- The proportion of individuals in the sample who posses the attribute:  $\bar{X} = \frac{T}{n}$

Clearly,  $T \sim B(n, \theta)$ , hence

- $P(T = t) = \binom{n}{t} \theta^t (1-\theta)^{n-t}, t = 0, 1, \dots, n$
- $P(\bar{X} = z) = P(T = nz) = \binom{n}{nz} \theta^{nz} (1-\theta)^{n-nz}, z = \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}$

### Theorem 2.4.9 De Moivre-Laplace theorem

For large sample sizes,

$$\frac{T - n\theta}{\sqrt{n\theta(1-\theta)}} \xrightarrow{d} N(0, 1) \Leftrightarrow \frac{\bar{X} - \theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \xrightarrow{d} N(0, 1)$$

use when  $n > 20, n\theta \geq 5, n\theta(1-\theta) \geq 5, 0.1 < \theta < 0.9$ .

Combined with the continuity correction, we have

$$P(a \leq T \leq b) \approx P\left(\frac{a - 0.5 - n\theta}{\sqrt{n\theta(1-\theta)}} \leq Z \leq \frac{b + 0.5 - n\theta}{\sqrt{n\theta(1-\theta)}}\right)$$

$$= \Phi\left(\frac{b + 0.5 - n\theta}{\sqrt{n\theta(1-\theta)}}\right) - \Phi\left(\frac{a - 0.5 - n\theta}{\sqrt{n\theta(1-\theta)}}\right)$$

with  $a < b, a, b = 0, 1, \dots, n$ .

*Proof.* From the CLT, we get

$$\sqrt{n} \frac{\bar{X} - \theta}{\sqrt{\theta(1-\theta)}} \xrightarrow{a} N(0, 1)$$

Now going through some algebraic manipulations and applying the CLT again, we have

$$\sqrt{n} \frac{T - n\theta}{n\sqrt{\theta(1-\theta)}} = \frac{T - n\theta}{\sqrt{n}\sqrt{\theta(1-\theta)}} = \frac{T - n\theta}{\sqrt{n\theta(1-\theta)}} \xrightarrow{a} N(0, 1)$$

■

The motivation for the continuity correction is that the binomial distribution is discrete while the normal distribution is continuous. When approximating a discrete distribution with a continuous one, we need to adjust for this difference.

$$p(T = x) = P\left(\frac{T - n\theta}{\sqrt{n\theta(1 - \theta)}} = \frac{x - n\theta}{\sqrt{n\theta(1 - \theta)}}\right) \stackrel{a}{\sim} N(0, 1)$$

$$\approx 0$$

Thus we need to consider the interval around  $x$  that captures the probability mass:  $P(T = x) = P(x - 0.5 < T < x + 0.5)$ .

### Theorem 2.4.10 Law of rare events

Technically,  $n \rightarrow \infty$  and  $\theta \rightarrow 0$  such that  $n\theta = \lambda$  is constant. Then,

$$T \stackrel{a}{\sim} Po(\lambda)$$

use for  $n > 20$  and  $\theta \notin (0.1, 0.9)$  and  $n\theta < 5$ .

### Two Bernoulli populations

Consider two Bernoulli populations with success probabilities  $\theta_1$  and  $\theta_2$  respectively. We want to compare  $\theta_1$  and  $\theta_2$ .  $\theta_1 - \theta_2$  will be unknown and we want to make inference about this quantity through the statistics  $\bar{X}_1 - \bar{X}_2$ , the difference between the sample proportions in two independent samples:

- $(X_{11}, \dots, X_{1m}) \Rightarrow \bar{X}_1 = \sum_{i=1}^n \frac{X_{1i}}{m}$
- $(X_{21}, \dots, X_{2n}) \Rightarrow \bar{X}_2 = \sum_{j=1}^n \frac{X_{2j}}{n}$

By the **De Moivre-Laplace theorem**, we have

$$\frac{\bar{X}_1 - \theta_1}{\sqrt{\theta_1(1 - \theta_1)/m}} \xrightarrow{d} N(0, 1) \quad \frac{\bar{X}_2 - \theta_2}{\sqrt{\theta_2(1 - \theta_2)/n}} \xrightarrow{d} N(0, 1)$$

and using the independence between the two samples, we get

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\theta_1 - \theta_2)}{\sqrt{\frac{\theta_1(1 - \theta_1)}{m} + \frac{\theta_2(1 - \theta_2)}{n}}} \xrightarrow{d} N(0, 1)$$

*Proof.*

$$X_m^* \rightarrow N(0, \sigma^2)$$

$$Y_n^* \rightarrow N(0, \delta^2)$$

Then,

$$\begin{aligned} Z_{m,n} &= \frac{(X_m^* + Y_n^*) - (\mu - \theta)}{\sqrt{\underbrace{\frac{\sigma^2}{m} + \frac{\delta^2}{n}}_{\sqrt{A}}}} \\ &= \frac{(X_m - \mu) - (Y_n - \theta)}{\sqrt{A}} \\ &= \frac{X_m - \mu}{\sqrt{\sigma^2/m}} \times \underbrace{\frac{\sqrt{\sigma^2/m}}{\sqrt{A}}}_{\rightarrow \sqrt{c}} - \frac{Y_n - \theta}{\sqrt{\delta^2/n}} \times \underbrace{\frac{\sqrt{\delta^2/n}}{\sqrt{A}}}_{= \sqrt{1 - \frac{\sigma^2/m}{A}}} \rightarrow \sqrt{1 - c} \end{aligned}$$

Using the moment generating functions, we have

$$\begin{aligned}
M_{Z_{m,n}}(S) &= E[e^{SZ_{m,n}}] \\
&= E\left[e^{S \frac{X_m - \mu}{\sqrt{\sigma^2/m}} \frac{\sqrt{\sigma^2/m}}{\sqrt{A}} - S \frac{Y_n - \theta}{\sqrt{\delta^2/n}} \frac{\sqrt{\delta^2/n}}{\sqrt{A}}}\right] \\
&= M_{\frac{X_m - \mu}{\sqrt{\sigma^2/m}}} \left(S \frac{\sqrt{\sigma^2/m}}{\sqrt{A}}\right) M_{\frac{Y_n - \theta}{\sqrt{\delta^2/n}}} \left(S \frac{\sqrt{\delta^2/n}}{\sqrt{A}}\right) \\
&= \underbrace{M_{\sqrt{n} \frac{X_m - \mu}{\sqrt{\sigma^2/m}}} \left(S \frac{\sqrt{\sigma^2/m}}{\sqrt{A}}\right)}_{\rightarrow e^{S^2 c/2}} \underbrace{M_{\sqrt{n} \frac{Y_n - \theta}{\sqrt{\delta^2/n}}} \left(S \frac{\sqrt{\delta^2/n}}{\sqrt{A}}\right)}_{\rightarrow e^{S^2(1-c)/2}} \\
&\rightarrow e^{S^2 c/2 + S^2/2 - S^2 c/2} = e^{S^2/2}
\end{aligned}$$

which is the mgf of a standard normal distribution.

Using the **Lévy theorem**, we conclude that  $Z_{m,n} \xrightarrow{d} N(0, 1)$ . ■

#### Example 2.4.6 (Bernoulli : observation of large proportion of bad clients)

$\theta_1 = 0.05, \theta_2 = 0.06, m = 400, n = 500$

$$\begin{aligned}
P(\bar{X}_1 - \bar{X}_2 > 0) &= P\left(\frac{(\bar{X}_1 - \bar{X}_2) - (\theta_1 - \theta_2)}{\sqrt{\frac{\theta_1(1-\theta_1)}{m} + \frac{\theta_2(1-\theta_2)}{n}}} > \frac{0 - (-0.01)}{\sqrt{\frac{0.05 \times 0.95}{400} + \frac{0.06 \times 0.94}{500}}}\right) \\
&= 1 - \Phi(0.66) \approx 0.2546
\end{aligned}$$

**Note:**

Care must be taken when extrapolating conclusions from the samples to the whole universe.

#### Gamma population

If  $X \sim G(\alpha, \lambda)$ , then for  $\alpha, \lambda > 0$ ,

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, x > 0$$

where  $\alpha$  is the shape parameter and  $\lambda$  is the rate parameter. Sometimes, the scale parameter  $\beta = \frac{1}{\lambda}$  is used instead of the rate parameter.

Couple of special cases regarding the Gamma distribution:

- If  $\alpha \in \mathbb{N}$ , it is also known as the Erlang distribution.
- If  $\alpha = 1$ , it reduces to the exponential distribution with parameter  $\lambda$ .
- If  $\alpha = \frac{n}{2}$  and  $\lambda = \frac{1}{2}$ , then  $X \sim \chi_n^2$

Other special properties of the Gamma distribution:

- If  $X_1 \sim G(\alpha_1, \lambda)$  is independent of  $X_2 \sim G(\alpha_2, \lambda)$ , then  $X_1 + X_2 \sim G(\alpha_1 + \alpha_2, \lambda)$
- If  $c > 0$  and  $X \sim G(\alpha, \lambda)$ , then  $cX \sim G(\alpha, \frac{\lambda}{c})$
- If  $X \sim G(\alpha, \lambda)$ , then  $2\lambda X \sim G(\alpha, \frac{1}{2}) = \chi_{2\alpha}^2$

*Proof.*

$$X \sim G(\alpha, \lambda)$$

Using the property  $cX \sim G(\alpha, \frac{\lambda}{c})$  with  $c = 2\lambda$ , we have

$$2\lambda X \sim G\left(\alpha, \frac{\lambda}{2\lambda}\right) = G\left(\alpha, \frac{1}{2}\right)$$

With some algebraic manipulation and using the property  $G(\frac{n}{2}, \frac{1}{2}) = \chi_n^2$ , we get

$$G\left(\frac{2\alpha}{2}, \frac{1}{2}\right) = \chi_{2\alpha}^2$$

■

Another useful result is often used:

$$X \sim \chi_n^2 \Rightarrow \sqrt{2X} - \sqrt{2n} \xrightarrow{d} N(0, 1)$$

where the square root is used to make the  $\chi_1^2$  distribution more symmetric.

*Proof.*  $\chi^2$  are additive if independent,  $X \sim \sum_{i=1}^n X_i$  where  $X_i \sim \chi_1^2$  with  $E[X_i] = 1$  and  $Var(X_i) = 2$ .

Using the **CLT**, we have

$$\frac{X - nE[X_i]}{\sqrt{nVar(X_i)}} \xrightarrow{d} N(0, 1) \Leftrightarrow \frac{X - n}{\sqrt{2n}} \xrightarrow{d} N(0, 1)$$

Now doing some algebraic manipulation,

$$\frac{\frac{X}{n} - 1}{\sqrt{\frac{2}{n}}} = \frac{\frac{X}{n} - 1}{\frac{\sqrt{2}}{\sqrt{n}}} = \sqrt{n} \frac{\frac{X}{n} - 1}{\sqrt{2}} \xrightarrow{d} N(0, 1)$$

Using **Slutsky's**, we have

$$\sqrt{n} \left( \frac{X}{n} - 1 \right) \xrightarrow{d} N(0, 2)$$

Now applying the **Delta method** with  $g(x) = \sqrt{x}$  and  $g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ ,

$$\sqrt{n} \left( \sqrt{\frac{X}{n}} - 1 \right) \xrightarrow{d} N \left( 0, \underbrace{\left( \frac{1}{2} \right)^2}_{g'(1)} \underbrace{2}_{\sigma^2} \right) \Leftrightarrow \sqrt{X} - \sqrt{n} \xrightarrow{d} N \left( 0, \frac{1}{2} \right)$$

Now dividing by  $\sqrt{\frac{1}{2}}$  on both sides and using **Slutsky's** again, we get

$$\frac{\sqrt{X} - \sqrt{n}}{\sqrt{\frac{1}{2}}} = \sqrt{2X} - \sqrt{2n} \xrightarrow{d} N(0, 1)$$

■  
Another useful property of the Gamma distribution. Let  $X_1, \dots, X_n$  be a random sample of size  $n$  from a  $G(\alpha, \lambda)$  population. Then,

$$\sum_{i=1}^n X_i \sim G(n\alpha, \lambda) \Leftrightarrow \bar{X} \sim G(n\alpha, n\lambda) \Leftrightarrow 2n\lambda\bar{X} \sim \chi_{2n\alpha}^2$$

# Chapter 3

## Sufficiency and Information

### 3.1 Sufficiency

Consider a parametric model  $\mathcal{F} = \{f(\cdot|\theta) : \theta \in \Theta\}$  with  $X_1, \dots, X_n$  iid random sample of size  $n$  extracted from  $X$ . The goal is to use the information contained in  $X_1, \dots, X_n$  to produce inferential statements about the unknown parameter  $\theta$ . When we compute statistics, i.e. functions of the random sample, we are summarizing the information contained in the random sample.

**Open Question 3.1.1:** When can we be assured that in the process we are not losing any relevant information about the parameter?

When all information that is lost is spurious, i.e. irrelevant for  $\theta$ .

#### Example 3.1.1 (Poisson)

$X|\lambda \sim Po(\lambda)$  where  $\lambda > 0$ . We observe a random sample of size  $n = 2$  and we know that the observed value of the statistic  $T = \sum_{i=1}^n X_i = 31$ . What can we say about the random sample  $X_1, X_2$ ?

We want the possibility of the sample given all the information that I have.

$$\begin{aligned} \frac{P(X_1 = x_1, X_2 = x_2 | \lambda)}{P(T = 31 | \lambda)} &= \frac{\frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \frac{e^{-\lambda} \lambda^{x_2}}{x_2!}}{\frac{e^{-2\lambda} (2\lambda)^{31}}{31!}} \\ &= \frac{31!}{x_1! x_2!} \times \frac{\lambda^{x_1+x_2}}{\lambda^{31}} \times \left(\frac{1}{2}\right)^{31} \\ &= \frac{31!}{(31-x_2)! x_2!} \times \left(\frac{1}{2}\right)^{x_2} \left(1 - \frac{1}{2}\right)^{31-x_2} \\ \Rightarrow X_2 | T = 31 &\sim B(31, \frac{1}{2}) \end{aligned}$$

We don't need to know  $\lambda$ .

#### Definition 3.1.1: Sufficient statistics

We say that a statistic  $T$  is sufficient for  $\mathcal{F}$  or for  $\theta$ , if the conditional distribution of the random sample given the observed value of  $T$  does not depend on the unknown parameter  $\theta$  for all  $\theta$ .

What is essentially means is that  $f(x_1, \dots, x_n | \theta, t)$  does not depend on  $\theta$ .

$$\begin{aligned} f(x_1, \dots, x_n | \theta, t) &= \frac{f(x_1, \dots, x_n, t | \theta)}{f_T(t | \theta)} \\ &= \begin{cases} \frac{f(x_1, \dots, x_n | \theta)}{f_T(t | \theta)} & \text{if } T(x_1, \dots, x_n) = t \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

**Example 3.1.2** (Definition : Bernoulli)

$X | \theta \sim B(1, \theta)$ , and  $X_1, \dots, X_n | \theta \stackrel{iid}{\sim} B(1, \theta)$

Intuitively, the sequence of success and failure should not matter, only the proportion is important. Thus,  $T = \sum_{i=1}^n X_i$  should be sufficient for  $\theta$ .

$$\begin{aligned} f(x_1, \dots, x_n | \theta, t) &= \frac{f(x_1, \dots, x_n | \theta)}{f_T(t | \theta)} \quad \text{if } T(x_1, \dots, x_n) = t, \sum x_i = t \\ &= \frac{\prod_{i=1}^n f(x_i | \theta)}{f_T(t | \theta)} \end{aligned}$$

Because  $T | \theta \sim B(n, \theta)$

$$\begin{aligned} f(x_1, \dots, x_n | \theta, t) &= \frac{\prod_{i=1}^n \theta^{x_i} (1-\theta)^{1-x_i}}{\binom{n}{t} \theta^t (1-\theta)^{n-t}} \quad \text{if } \sum x_i = t \\ &= \frac{\theta^{\sum x_i} (1-\theta)^{n-\sum x_i}}{\binom{n}{t} \theta^t (1-\theta)^{n-t}} \\ &= \frac{1}{\binom{n}{t}} \end{aligned}$$

The definition of sufficient statistic is not very useful to discover sufficient statistics. Thus we need to rely on another theorem.

**Theorem 3.1.1 Halmos-Savage Factorization Criterion**

A statistic  $T$  is sufficient for  $\theta$  if and only if there are non-negative functions  $g$  and  $h$  such that

- $g$  depends on  $\theta$  and on the random sample exclusively through the observed value of  $T$ . Meaning it depends on parameter and statistics.
- $h$  depends exclusively on the random sample.
- $f(x_1, \dots, x_n | \theta) = g(T(x_1, \dots, x_n); \theta) \times h(x_1, \dots, x_n)$

*Proof.*  $T$  is sufficient for  $\theta \Leftrightarrow$  There are non-negative function of  $g$  and  $h$ .

We first start from left to right.

If  $T$  is sufficient for  $\theta$ , then

$$f(x_1, \dots, x_n | t, \theta) = \frac{f(x_1, \dots, x_n | \theta)}{f_T(t | \theta)} \quad \text{does not depend on } \theta$$

meaning that

$$f(x_1, \dots, x_n | t) = \frac{f(x_1, \dots, x_n | \theta)}{f_T(t | \theta)} \quad \text{if } T(x_1, \dots, x_n) = t$$

so we get

$$f(x_1, \dots, x_n | \theta) = \underbrace{f_T(t | \theta)}_{g(t, \theta)} \underbrace{f(x_1, \dots, x_n | t)}_{h(x_1, \dots, x_n)}$$

Now going from right to left.

For  $X$  discrete,  $T$  is discrete. And for simplification,  $\tilde{x} = (x_1, \dots, x_n)$ .

$$\begin{aligned} f_T(t | \theta) &= P(T = t | \theta) = \sum_{\tilde{x}: T(\tilde{x})=t} f(\tilde{x} | \theta) \\ &= \sum_{\tilde{x}: T(\tilde{x})=t} g(t; \theta) h(\tilde{x}) \\ &= g(t; \theta) \sum_{\tilde{x}: T(\tilde{x})=t} h(\tilde{x}) \end{aligned}$$

we now have

$$\begin{aligned} f(x_1, \dots, x_n | t, \theta) &= \frac{f(x_1, \dots, x_n | \theta)}{f_T(t | \theta)} \quad \text{if } T(\tilde{x}) = t \\ &= \frac{g(t, \theta) h(\tilde{x})}{g(t, \theta) \sum_{\tilde{x}: T(\tilde{x})=t} h(\tilde{x})} \\ &= \frac{h(\tilde{x})}{\sum_{\tilde{x}: T(\tilde{x})=t} h(\tilde{x})} \quad \text{does not depend on } \theta \end{aligned}$$

■

### Example 3.1.3 (Factorization criterion : Poisson)

Let  $X|\lambda \sim Po(\lambda)$

$$\begin{aligned} f(x_1, \dots, x_n | \lambda) &= \prod_{i=1}^n f(x_i | \lambda) = \prod_{i=1}^n e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \\ &= e^{-n\lambda} \frac{\lambda^{\sum x_i}}{\prod x_i!} \\ &= \underbrace{e^{-n\lambda} \lambda^{\sum x_i}}_{g(\sum x_i; \lambda)} \underbrace{\frac{1}{\prod x_i!}}_{h(\tilde{x})} \end{aligned}$$

### Example 3.1.4 (Factorization criterion : Uniform)

Let  $X|\theta \sim U(0, \theta)$ ,  $\theta > 0$ .

$$\begin{aligned} f(x | \theta) &= \frac{1}{\theta}, \quad 0 < x < \theta \\ &= \frac{1}{\theta} I_{(0, \theta)}(x) \end{aligned}$$

with  $I_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , \text{otherwise} \end{cases}$

$$\begin{aligned} f(\tilde{x} | \theta) &= \prod_{i=1}^n f(x_i | \theta) = \prod_{i=1}^n \frac{1}{\theta} I_{(0, \theta)}(x_i) \\ &= \theta^{-n} I_{(0, \infty)}(x_{(1)}) I_{(0, \theta)}(x_{(n)}) \\ &= \underbrace{\theta^{-n} I_{(0, \theta)}(x_{(n)})}_{g(x_{(n)})} \underbrace{I_{(0, \infty)}(x_{(1)})}_{h(\tilde{x})} \end{aligned}$$

**Example 3.1.5 (Factorization criterion : Shifted exponential)**

Consider the shifted exponential distribution  $f(x|\lambda, \delta) = \lambda e^{-\lambda(x-\delta)} I_{[\delta, \infty)}(x)$

$$\begin{aligned} f(\tilde{x}|\lambda, \delta) &= \prod_{i=1}^n f(x_i|\lambda, \delta) = \prod_{i=1}^n \lambda e^{-\lambda(x_i-\delta)} I_{[\delta, \infty)}(x_i) \\ &= \underbrace{\lambda^n e^{-\lambda \sum x_i} e^{n\lambda\delta}}_{g(\sum x_i, x_{(1)}; \lambda, \delta)} \underbrace{I_{[\delta, \infty)}(x_{(1)})}_{h(\tilde{x})} \times \underbrace{1}_{h(\tilde{x})} \end{aligned}$$

By the factorization criterion,  $(\sum x_i, x_{(1)})$  is sufficient for  $(\lambda, \delta)$ .

Two of things to notice regarding sufficient statistics.

- There is always a sufficient statistic, it can be all the samples.
- Sufficient statistics are not unique.

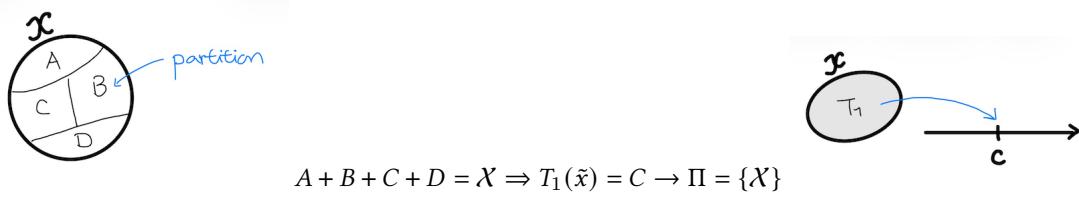
**Example 3.1.6 (Un-uniqueness of sufficient statistics : Poisson)**

$X \sim Po(\lambda)$  where  $t = \sum x_i$

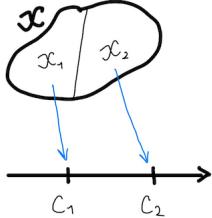
- $f(\tilde{x}|\lambda) = e^{-n\lambda} \lambda^t \frac{1}{\prod x_i!} \Rightarrow T = \sum x_i$  is sufficient.
- $f(\tilde{x}|\lambda) = e^{-n\lambda} \lambda^{\bar{x}} \frac{1}{\prod x_i!} \Rightarrow T = \bar{x}$  is sufficient.
- $f(\tilde{x}|\lambda) = e^{-n\lambda} \lambda^{T_1} \lambda^{T_2} \frac{1}{\prod x_i!} \Rightarrow (T_1, T_2)$  is sufficient where  $T_1 = \sum_{i=1}^{n_1} x_i$  and  $T_2 = \sum_{i=n_1+1}^n x_i$ .

Another important concept is the **partition** induced in the sample space by a statistic.

- Any statistics  $T : \mathcal{X} \rightarrow \mathbb{R}^q$  induces a partition in the sample space  $\mathcal{X}$  and the partitions do not intercept.



If a statistic takes 2 values :  $T_1(\tilde{x}) = \begin{cases} c_1 & , \tilde{x} \in X_1 \\ c_2 & , \tilde{x} \in X_2 \end{cases} \rightarrow \begin{cases} X_1 \cup X_2 = X \\ X_1 \cap X_2 = \emptyset \\ \Pi = \{X_1, X_2\} \end{cases}$



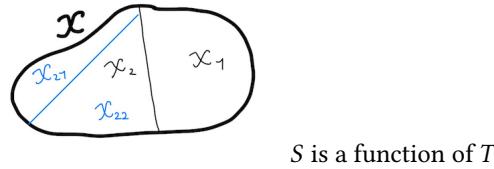
- The finer the partition induced by  $T$  in  $\mathcal{X}$ , the less information is lost; the smaller is the data reduction operated by  $T$ .
- A sufficient statistic operates a data reduction that does not involve loss of relevant information about the parameter; the partition it induces is also said to be sufficient.

**Note:**

When the partition is finer as it can be,  $T(\tilde{x}) = \tilde{x}$ ,  $\Pi = \cup_{\tilde{x} \in \mathcal{X}} \{\tilde{x}\}$ , because no information will be lost.

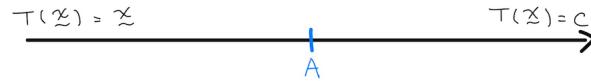
- The notion of partition is more general than that of statistic; different statistics can induce the same partition, in which case they are said to be equivalent, i.e. they are one-to-one.
- If the partition induced by  $T$  is finer than the partition induced by  $S$ , then  $S$  is a function of  $T$ . In that case, if  $S$  is sufficient, so is  $T$ , that is

$$\left. \begin{array}{l} S = h(T) \\ S \text{ sufficient} \end{array} \right\} \Rightarrow T \text{ sufficient}$$



$$\left. \begin{array}{l} S \rightarrow \{X_1, X_2\} \\ T \rightarrow \{X_1, X_{21}, X_{22}\} \end{array} \right\} \text{If } S \text{ is sufficient, then } T \text{ is sufficient, because } T \text{ is finer than } S \text{ partition wise.}$$

- However, if  $T$  is sufficient and  $S = h(T)$ , it is not a given that  $S$  is also sufficient unless  $S$  is injective, in which case,  $S$  and  $T$  are equivalent.
- We are interested in finding statistics which are sufficient but operates the least data reduction. In other words, statistics that induce the coarsest partition that is still sufficient.



**Point A** is what we want to find, the minimum sufficient statistic. We lose information as we are moving to the right hand side.

**Definition 3.1.2: Minimal sufficient statistic**

A statistic is said to be minimal sufficient for  $\mathcal{F}$  if it is sufficient, and if  $S$  is any other sufficient statistic, then  $T = h(S)$  for some  $h$ .

Consider a binary relation in  $\mathcal{X}$  defined by  $\left\{ \begin{array}{l} \tilde{y} \in \mathcal{X} \\ \tilde{x} \in \mathcal{X} \end{array} \right. , \tilde{y} R \tilde{x}$ .

$$f(\tilde{y}|\theta) = c(\tilde{x}, \tilde{y}) f(\tilde{x}|\theta) \Leftrightarrow \underbrace{\frac{f(\tilde{y}|\theta)}{f(\tilde{x}|\theta)}}_{\text{likelihood ratio}} = c(\tilde{x}, \tilde{y})$$

where  $c(\tilde{x}, \tilde{y}) > 0$  and does not depend on  $\theta$ .

This binary relation is an equivalence relation, that is, it is

- symmetric:  $\tilde{x} R \tilde{y} \Leftrightarrow \tilde{y} R \tilde{x}$
- reflexive:  $\tilde{x} R \tilde{x}$
- transitive:  $\tilde{x} R \tilde{y}$  and  $\tilde{y} R \tilde{z} \Rightarrow \tilde{x} R \tilde{z}$

Hence, it induces a partition in  $X$  with parts

$$\begin{aligned}\Pi_x &= \{y \in X : y R x\} \text{ for } x : f(x|\theta) > 0 \text{ for some } \theta \in \Theta \\ \Pi_0 &= \{y \in X : f(y|\theta) = 0 \quad \forall \theta \in \Theta\}\end{aligned}$$

*Proof.* Transitive

Consider the case where

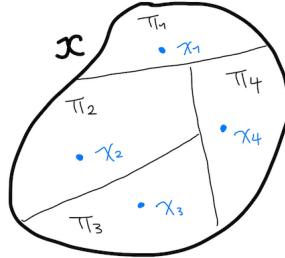
$$\begin{aligned}\tilde{x} R \tilde{y} &\Rightarrow f(\tilde{y}|\theta) = c_1 f(\tilde{x}|\theta) \\ \tilde{y} R \tilde{z} &\Rightarrow f(\tilde{y}|\theta) = c_2 f(\tilde{z}|\theta)\end{aligned}$$

then we have

$$c_1 f(\tilde{x}|\theta) = c_2 f(\tilde{z}|\theta) \Rightarrow f(\tilde{x}|\theta) = \frac{c_2}{c_1} f(\tilde{z}|\theta) \Rightarrow \tilde{x} R \tilde{z}$$

### Theorem 3.1.2 Lehmann-Scheffe

The partition with parts  $\Pi_0$  and  $\{\Pi_x\}$  described above is minimal sufficient, and any statistic which induces it is minimal sufficient.



$$\Pi = \{\Pi_1, \Pi_2, \Pi_3, \Pi_4\}, G(\tilde{x}) = \tilde{x}_i \text{ if } \tilde{x} \in \Pi_i$$

*Proof.* Consider the statistic  $G$  that to each  $x \in X$  associates a representative of the element of the partition to which it belongs

$$x \in X \mapsto x_{\Pi} \in \Pi_x = G(x)$$

#### Proof that $G$ is sufficient

$$\begin{aligned}\tilde{x} \in \Pi_i &\Rightarrow \tilde{x} R \tilde{x}_i \\ f(\tilde{x}|\theta) &= c(\tilde{x}_i, \tilde{x}) f(\tilde{x}|\theta) \underset{G(\tilde{x})=\tilde{x}_i}{=} \underbrace{c(\tilde{x}_i, \tilde{x})}_{g(\tilde{x})} \underbrace{f(G(\tilde{x}|\theta))}_{g(G; \theta)} \\ &\Rightarrow G \text{ is sufficient}\end{aligned}$$

Now proof that  $G$  is a "grosser" partition

$U$  is sufficient

$$\Pi_U = \{\Pi_x^*\}, \quad \left. \begin{array}{l} \tilde{x} \in X \\ \tilde{y} \in \Pi_x^* \end{array} \right\} \rightarrow \tilde{x} \in \Pi_i$$

since  $U$  is sufficient,

$$f(\tilde{x}|\theta) = g(U(\tilde{x}); \theta) h(\tilde{x})$$

$$f(\tilde{y}|\theta) = g(U(\tilde{y}); \theta) h(\tilde{y}) = g(U(\tilde{x}); \theta) h(\tilde{y}) \Leftrightarrow$$

$$h(\tilde{x}) f(\tilde{y}|\theta) = \underbrace{h(\tilde{x}) g(U(\tilde{x}); \theta)}_{f(\tilde{x}|\theta)} h(\tilde{y}) \Leftrightarrow$$

$$f(\tilde{y}|\theta) = f(\tilde{x}|\theta) \frac{h(\tilde{y}|\theta)}{h(\tilde{x}|\theta)} \Rightarrow \tilde{y} R \tilde{x} \Rightarrow U \text{ is a finer partition than } G$$

$\therefore G$  is the minimal sufficient statistic

**Example 3.1.7** (Minimal sufficient statistic : Normal)

Let  $X_1, \dots, X_n$  be a random sample from a  $N(\mu, \sigma^2)$  population and let  $\theta = (\mu, \sigma^2)$

$$\begin{aligned} f(\tilde{x}|\theta) &= \prod_{i=1}^n f(x_i|\mu\sigma^2) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \mu)^2\right\} \\ &= (2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left(\sum x_i^2 - 2n\bar{x}\mu + n\mu^2\right)\right\} \\ &= (2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum x_i^2\right\} \exp\left\{\frac{n\bar{x}\mu}{\sigma^2}\right\} \exp\left\{-\frac{1}{2\sigma^2}n\mu^2\right\} \end{aligned}$$

Now using the **Lehmann-Scheffe theorem**,

$$\begin{aligned} \frac{f(\tilde{y}|\theta)}{f(\tilde{x}|\theta)} &= \frac{(2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum y_i^2\right\} \exp\left\{\frac{n\bar{x}\mu}{\sigma^2}\right\} \exp\left\{-\frac{1}{2\sigma^2}n\mu^2\right\}}{(2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum x_i^2\right\} \exp\left\{\frac{n\bar{x}\mu}{\sigma^2}\right\} \exp\left\{-\frac{1}{2\sigma^2}n\mu^2\right\}} \\ &= \exp\left\{-\frac{1}{2\sigma^2}(\sum y_i^2 - \sum x_i^2)\right\} \exp\left\{\frac{n\mu}{\sigma^2}(\bar{y} - \bar{x})\right\} \quad \text{does not depend on } \theta \\ &\Leftrightarrow \begin{cases} \sum y_i^2 = \sum x_i^2 \\ \bar{y} = \bar{x} \end{cases} \end{aligned}$$

A minimal sufficient statistic is  $(\bar{x}, \sum x_i^2)$  or  $(\bar{x}, s^2)$

**Example 3.1.8** (Minimal sufficient statistic : Exponential)

Let  $X_1, \dots, X_n$  be a random sample from a  $Ex(\lambda)$  population and let  $\theta = \lambda$

$$\begin{aligned} f(x|\lambda) &= \lambda e^{-\lambda n}, n > 0 \\ f(\tilde{x}|\lambda) &= \prod_{i=1}^n \lambda e^{-\lambda x_i} = \lambda^n \exp\{-\lambda \sum x_i\} \\ \frac{f(\tilde{y}|\theta)}{f(\tilde{x}|\theta)} &= \frac{\lambda^n \exp\{-\lambda \sum y_i\}}{\lambda^n \exp\{-\lambda \sum x_i\}} = \exp\{-\lambda(\sum y_i - \sum x_i)\} \end{aligned}$$

Does not depend on  $\lambda$  if and only if  $\sum y_i = \sum x_i \Leftrightarrow \bar{x} = \bar{y} \Rightarrow \bar{x}$  is minimal sufficient

## 3.2 Ancillarity and completeness

**Definition 3.2.1: Ancillary statistic**

A statistic  $T$  is said to be ancillary if its sampling distribution does not depend on the unknown parameter  $\theta$ .

$T$  is ancillary if  $f_T(t|\theta)$  does not depend on  $\theta$ .

### Definition 3.2.2: Location-scale family of distributions

The location-scale family of distribution is composed by all the probability distributions such that the associated cumulative distribution function is of the form

$$G(x|\delta, \lambda) = G\left(\frac{x-\lambda}{\delta}\right)$$

where  $G$  is a function that does not involve unknown parameters,  $\lambda \in \mathbb{R}$  is the location parameter, and  $\delta > 0$  is the scale parameter.

This family includes the location family ( $\delta$  is known) and the scale family ( $\lambda$  is known).

#### Example 3.2.1 (Location-scale family : Normal)

Suppose that  $X \sim N(\mu, \sigma^2)$

$$\frac{x-\mu}{\sigma} \sim N(0, 1)$$

$$\begin{aligned} f(x|\mu, \sigma^2) &= P(X \leq x|\mu, \sigma^2) \\ &= P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma} | \mu, \sigma^2\right) \\ &= \Phi\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

where  $G = \Phi, \lambda = \mu, \delta = \sigma$ .

The normal family of distributions is a member of the location-scale family with location parameter  $\mu$  and scale parameter  $\delta$ .

#### Example 3.2.2 (Location-scale family : Uniform)

Suppose that  $X \sim U(0, \theta), \theta > 0$ , then we have

$$f(x|\theta) = \frac{1}{\theta} I_{(0,\theta)}(x)$$

$$\begin{aligned} F(x|\theta) &= \frac{x}{\theta} \underbrace{I_{(0,\theta)}(x)}_{0 < x < \theta \Leftrightarrow 0 < \frac{x}{\theta} < 1} \\ &= \frac{x}{\theta} I_{(0,1)}\left(\frac{x}{\theta}\right) \\ &= G\left(\frac{x}{\theta}\right) \end{aligned}$$

where  $G(y) = y I_{(0,1)}(y)$

$U(0, \theta)$  belongs to the scale family with scale parameter  $\delta = \theta$  and  $\lambda = 0$ .

Couple **remarks** regarding the location-scale family

- The distribution of  $X$  is part of the location-scale family with location parameter  $\lambda$  and scale parameter  $\delta$  if and only if the distribution of  $\frac{(x-\lambda)}{\delta}$  does not depend on unknown parameters.
- If the distribution of  $X$  is a member of the location-scale family with location parameter  $\lambda$  and scale parameter  $\delta$  then

any statistic which is a function of  $X_1, \dots, X_n$  only through the vector  $\left(\frac{X_i - \lambda}{\delta}, i = 1, \dots, n\right)$  is ancillary.

$$T = T(X_1, \dots, X_n) = H\left(\frac{X_1 - \lambda}{\delta}, \dots, \frac{X_n - \lambda}{\delta}\right), \quad T \text{ is ancillary}$$

**Insert proof**

**Example 3.2.3 (Location-scale and statistic : Uniform)**

Let  $X|\theta \sim U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$ ,  $\theta \in \mathbb{R}$

$$f(x|\theta) = \frac{1}{1} \underbrace{I_{(\theta - \frac{1}{2}, \theta + \frac{1}{2})}(x)}_{\theta - \frac{1}{2} < x < \theta + \frac{1}{2} \Leftrightarrow -\frac{1}{2} < x - \theta < \frac{1}{2}} = I_{(-\frac{1}{2}, \frac{1}{2})}(x - \theta) = G(x - \theta)$$

where  $G(y) = I_{(-\frac{1}{2}, \frac{1}{2})}(y)$

The distribution of  $X$  belongs to the location family with  $\lambda = \theta$  and  $\delta = 1$ . The distribution of  $x - \theta$  does not depend on unknown parameters.

$$R = X_{(n)} - X_{(1)} = (X_{(n)} - \theta) - (X_{(1)} - \theta) = H(X_1 - \theta_1, \dots, X_n - \theta_n)$$

- It would be natural to expect a minimal sufficient statistic and an ancillary statistic to be independent. However, that is not the case in general.

**Example 3.2.4 (Minimal sufficient statistic and ancillary statistic : Uniform)**

Let  $X|\theta \sim U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$$\frac{f(\tilde{y}|\theta)}{f(\tilde{x}|\theta)} = \frac{I_{(y_{(n)} - \frac{1}{2}, \infty)}(\theta)I_{(-\infty, y_{(1)} + \frac{1}{2})}(\theta)}{I_{(x_{(n)} - \frac{1}{2}, \infty)}(\theta)I_{(-\infty, x_{(1)} + \frac{1}{2})}(\theta)}$$

does not depend on  $\theta$  if and only if  $\begin{cases} x_{(n)} = y_{(n)} \\ x_{(1)} = y_{(1)} \end{cases} \Rightarrow (x_{(1)}, x_{(n)})$  is minimal sufficient,  $(x_{(n)} - x_{(1)}, x_{(1)})$  is minimal sufficient.

$(R, X_{(1)})$  is minimal sufficient but contains an element that is ancillary.