Brief solution to the regular final exam of ATS, PhD program in MAEM, Resolução abreviada do exame de época normal de TAE, Jan/11/2013.

1. (a) It is clear that $f(x \mid \theta) \geq 0$ for all x. We must then verify that $\int_{-\infty}^{+\infty} f(x \mid \theta)dx = \int_0^\theta 2x/\theta^2 dx = 1$.

   (b) It is easy to see that $F(x) = (x/\theta)^2$, $0 < x < \theta$, hence $G_n(x) = P(X_{(n)} \leq x) = (x/\theta)^{2n}$, $0 < x < \theta$. We must show that, for all $\varepsilon > 0$, $P(|X_{(n)} - \theta| > \varepsilon) \to 0$. Indeed,

$$\begin{aligned} P(|X_{(n)} - \theta| > \varepsilon) &= 1 - [P(X_{(n)} < \theta + \varepsilon) - P(X_{(n)} < \theta - \varepsilon)] \\ &= [(\theta - \varepsilon)/\theta]^{2n} \to 0 \quad \text{since } 0 < (\theta - \varepsilon)/\theta < 1 \ . \end{aligned}$$

   (c) Note that

$$\begin{aligned} P(-2n(X_{(n)} - \theta) \leq x) &= P(X_{(n)} \geq \theta - x/(2n)) \\ &= 1 - \{[\theta - x/(2n)]/\theta\}^{2n} \ , \quad 0 < x < 2n\theta \\ &= 1 - [1 - (x/\theta)/(2n)]^{2n} \ , \quad 0 < x < 2n\theta \end{aligned}$$

   which shows that, for all $x > 0$, $P(-2n(X_{(n)} - \theta) \leq x) \to 1 - e^{-x/\theta}$, and establishes the result.

   (d) By Slutsky's theorem, and taking into account b) and c), it follows that

$$-2n(X_n - \theta)\frac{1}{X_{(n)}} \xrightarrow{d} \theta^{-1}\mathrm{Ex}(1/\theta) = \mathrm{Ex}(1) \ .$$

2. (a) We have that

$$\begin{aligned} L(\lambda \mid \mathbf{x}, \mathbf{y}) &\propto \prod_{i=1}^{n} f(y_i \mid x_i) \ f(x_i) \\ &\propto \lambda^{-3n/2} \exp[-(\sum x_i y_i + \sum x_i^2/2)/\lambda] \end{aligned}$$

   and hence the score function is

$$\frac{\partial \ln L}{\partial \lambda} = -\frac{3}{2}\ \frac{n}{\lambda} + \frac{1}{\lambda^2}(\sum x_i y_i + \sum x_i^2/2)$$

   and the root of this function is $\lambda = \frac{1}{n}\sum(2x_i y_i + x_i^2/2)$. It is easy to verify that this point is indeed a point of maximum of L, and hence we have proved the result.

   (b) We can write the score function in the form

$$S = \frac{3}{2}\ \frac{n}{\lambda^2}(T_n - \lambda)$$

   which shows the result.

   (c) The previous part allows to conclude that $E[T_n] = E[W_i] = \lambda$ and that $\mathrm{Var}(T_n) = \mathrm{Var}(W_i)/n = 1/I_{X_1,\ldots,X_n}(\lambda)$. We need then to compute the Fisher information. Indeed,

$$I_{X_1,\ldots,X_n}(\lambda) = -E\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{3}{2}\ \frac{n}{\lambda^2} - \frac{3n}{\lambda^3}E[T_n] = \frac{3n}{2\lambda^2} \ ,$$

   and the result follows.

   (d) By the CLT,

$$\sqrt{n}\frac{T_n - \lambda}{\sqrt{2/3\lambda^2}} \xrightarrow{d} N(0,1)$$

   which can be rewritten as

$$\sqrt{n}\frac{T_n - \lambda}{\lambda} \xrightarrow{d} N(0, 2/3) \ .$$

   Since, by the weak law of large numbers, $T_n \xrightarrow{P} \lambda$, one application of Slutky's theorem shows the result. pretendido.

3. (a) It is easy to verify that the exponential model is part of the 1-parameter exponential family with natural parameter $1/\delta$ and sufficient statistic $\sum Y_i$. Since the natural parameter space is $\mathbb{R}^+$, which contains an open subset of $\mathbb{R}$, it follows that $\sum Y_i$ is complete. Since the estimator $\bar{Y}$ is unbiased (since $E[\bar{Y}] = E[Y_i] = \delta$) and it is a function of a complete and sufficient, it follows that it is the UMVU of δ.

(b) $F(y) = 1 - \exp(-y/\delta) = G(y/\delta)$, $y > 0$, where $G(x) = 1 - \exp(x)$, $x > 0$, which does not depend on unknown parameters.

(c) The statistic $Y_{(1)}/\bar{Y}$ is ancillary because we can write it as a function only of $Y_1/\delta, \ldots, Y_n/\delta$ and the model is part of the scale family with scale parameter δ. Basu's theorem guarantees that $Y_{(1)}/\bar{Y}$ is independent of $\bar{Y}$, since this statistic is sufficient and complete. On the other hand, $Y_{(1)} \sim \text{Ex}(n/\delta)$. Hence, $\delta/n = E[Y_{(1)}] = E[\bar{Y}\ Y_{(1)}/\bar{Y}] = E[\bar{Y}]E[Y_{(1)}/\bar{Y}] = \delta E[Y_{(1)}/\bar{Y}]$, which finishes the proof.