Chapter 1

Typology of $\mathbb{R}^n$ and Fixed Point Theorems

Typology means the "properties of subsets".

1.1 Fundamental concepts

Sets

- $\mathbb{N}$: natural numbers (1, 2, 3, ...)
- $\mathbb{Z}$: integers (..., -2, -1, 0, 1, 2, ...)
- $\mathbb{Q}$: rational numbers (fractions of integers), same as $\mathbb{Z} \cup \{\text{fractionary numbers}\}$
- $\mathbb{R}$: real numbers (all rational and irrational numbers), same as $\mathbb{Q} \cup \{\text{irrational numbers}\}$
- $\mathbb{C}$: complex numbers (numbers with real and imaginary parts), $\{a + bi, a, b \in \mathbb{R}\}$

Symbols

- $\forall$: for all
- $\exists$: there exists

Terminology

- $\in$: belongs to
- $\subset$: is contained
- $\supset$: contains

A, B, 2 subsets of $\mathbb{R}$

- $A \cup B = \{x \in \mathbb{R} : x \in A \text{ or } x \in B\}$: union of A and B, all elements in A or B
- $A \cap B = \{x \in \mathbb{R} : x \in A \text{ and } x \in B\}$: intersection of A and B, all elements in both A and B
- $A^c = \{x \in \mathbb{R} : x \notin A\}$: complement of A, all elements not in A
- $A \setminus B = \{x \in \mathbb{R} : x \in A \text{ and } x \notin B\}$: difference of A and B, all elements in A but not in B
- $\emptyset$: empty set, set with no elements

Definition 1.1.1: Disjoint

$A, B \subset \mathbb{R}$, A and B are disjoint iff $A \cap B = \emptyset$.

Properties

- Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's Laws: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

Terminology

I is a set of indixes, $\forall_j \in I, A_j \subset \mathbb{R}$

- $\bigcap_{j \in I} A_j = \{x \in \mathbb{R} : x \in A_j, \forall_j \in I\}$: intersection of all A_j, all elements in every A_j
- $\bigcup_{j \in I} A_j = \{x \in \mathbb{R} : x \in A_j \text{ for some } j \in I\}$: union of all A_j, all elements in at least one A_j

Example 1.1.1 (Union and Intersection)
$I_n = (-\frac{1}{n}, \frac{1}{n}), n \in \mathbb{N}$
$\bigcap_{j \in \mathbb{N}} I_n = \{0\}$
$\bigcup_{j \in \mathbb{N}} I_n = (-1, 1)$

From now on, extending the domain where the sets live to $\mathbb{R}^n$. $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a, b) : a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = (x_1, x_2, \cdots, x_n)$.

Example 1.1.2 ($\mathbb{R}^2$)
$\mathbb{R} \times \{0\} = \{(x, 0), x \in \mathbb{R}\}$, the order of the pair matters.

1.2 Functions

A, B sets. Correspondence between two sets, A and B, in such a way that to each element of A corresponds one and only one element of B.

Note:
element of A: objects
elements of B that receives arrow: image
$f : \underbrace{A}_{domain} \to \underbrace{B}_{codomain} \Rightarrow x \to \underbrace{f(x)}_{analytical\ expression}$

Example 1.2.1 (Domain-1)
$f : D_f \to \mathbb{R} \Rightarrow x \to \sqrt{x}$
$D_f = [0, +\infty)$ or $D_f = \mathbb{R}_0^+$, also called the maximal domain

Example 1.2.2 (Domain-2)
$g : D_g \to \mathbb{R} \Rightarrow x \to \frac{1}{x}$
$D_g = (-\infty, 0) \cup (0, +\infty)$ or $D_g = \mathbb{R} \setminus \{0\}$

Example 1.2.3 (Domain-3)
$h : D_h \to \mathbb{R} \Rightarrow x \to \ln(x)$
$D_h = (0, +\infty)$ or $D_h = \mathbb{R}^+$

Example 1.2.4 (Domain-4)
$A \subset \mathbb{R}, l : A \to A \Rightarrow x \to x$
$D_l = \mathbb{R}$, also known as the identity map

Definition 1.2.1: Graph

$f : A \to B \Rightarrow x \to f(x)$
Graph of f is defined as $\mathrm{Gr}(f) = \{(x, y) \in A \times B : y = f(x)\}$.

Example 1.2.5 (Graph)
$f : \mathbb{R}^3 \to \mathbb{R} \Rightarrow (x_1, x_2, x_3) \to x_1x_2x_3$
$\mathrm{Gr}(f) = \{(x_1, x_2, x_3, y) \in \mathbb{R}^3 \times \mathbb{R} : y = f(x_1, x_2, x_3) = x_1x_2x_3\}$

$f : A \to B$ where $A, B \subset \mathbb{R}$

Definition 1.2.2: Injective

$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

An injective function, also known as one-to-one function, is a function where distinct elements in the domain map to distinct elements in the codomain. This means that no two different inputs can produce the same output.

Example 1.2.6 (Injective?)
$f : \mathbb{R} \to \mathbb{R} \Rightarrow x \to x^2$
This is not injective because $f(1) = f(-1) = 1$ but $1 \neq -1$.
But, if we restrict the domain to $\mathbb{R}_0^+$, then it is injective.

Definition 1.1: Surjective

$\mathrm{Image}(f) = B$

A surjective function, also known as onto function, is a function where every element in the codomain has at least one element from the domain mapping to it. This means that the function covers the entire codomain.

Example 1.2.7 (Surjective?)
$f : \mathbb{R} \to \mathbb{R} \Rightarrow x \to x^2$
This is not surjective because there is no $x \in \mathbb{R}$ such that $f(x) = -1$.
But, if we restrict the codomain to $\mathbb{R}_0^+$, then it is surjective.

Definition 1.2.3: Bijective

f is bijective iff it is injective and surjective.

Compontion of maps

$A, B, C \subset \mathbb{R}$, and $\begin{cases} f : A \to B \\ g : B \to C \end{cases}$ Then, the composition of f and g is defined as $g \circ f : A \to C \Rightarrow x \to g(f(x))$. The map is well defined if $\text{Image}(f) \subset D_g$.

Example 1.2.8 (Composition of maps)

$f : \mathbb{R} \to \mathbb{R} \Rightarrow x \to x^2$
$g : \mathbb{R} \to \mathbb{R} \Rightarrow x \to x + 1$
$g \circ f : \mathbb{R} \to \mathbb{R} \Rightarrow x \to g(f(x)) = x^2 + 1$
$f \circ g : \mathbb{R} \to \mathbb{R} \Rightarrow x \to f(g(x)) = (x+1)^2 = x^2 + 2x + 1$

In general, $g \circ f \neq f \circ g$ unless linear.

Definition 1.2.4: Inverse

f, g are maps. If $f \circ g = g \circ f = I_d$, then f and g are inverses of each other, denoted as $f = g^{-1}$ and $g = f^{-1}$, one with respect to other.

Example 1.2.9 (Inverse?)

$g : \mathbb{R} \to \mathbb{R}^+ \Rightarrow x \to \exp^x$
$f : \mathbb{R}^+ \to \mathbb{R} \Rightarrow x \to \ln(x)$
$f \circ g = f(e^x) = \ln(e^x) = x$
$g \circ f = g(\ln(x)) = e^{\ln(x)} = x$
$f \circ g = g \circ f = I_d \Rightarrow$ f and g are inverses of each other.

Corollary 1.2.1 Invertibility

f is invertible iff f is bijective.
$f : \mathbb{R} \to \mathbb{R}$ is not bijective $\Rightarrow$ f is not invertible.

<u>Cardinal of sets</u>

$\Omega \subseteq \mathbb{R}^n, n \in \mathbb{N}$

Definition 1.2: Finite

Ω is finite if $\#\Omega \in \mathbb{N}$.
There exists a bijection with Ω and $\{1, 2, \cdots, n\}$ for some $n \in \mathbb{N}$.

Example 1.2.10 (Cardinality)

$A = \{a, b, c\}$ where $a \neq b \wedge b \neq c \wedge c \neq a$
$\#A = 3$

And of course, Ω is infinite if it is not finite.

Note:

$\#\emptyset = 0$

Infinite sets can be further classified into countable and uncountable sets.

- Ω is countable if there exists a bijection between Ω and $\mathbb{N}$.
- Ω is uncountable if it is not countable.

Note:

In this course, finite sets are countable sets.

Example 1.2.11 (Countable)

$A = \{2, 4, 8\}$ is finite $\Rightarrow$ A is countable.

Example 1.2.12 (Countable)

$B = \{4, 5, 6, 7, \cdots\} = \{n \in \mathbb{N} : x \geq 4\}$
$f : B \longrightarrow \mathbb{N} \Rightarrow n \longrightarrow n - 3$ is a bijection $\Rightarrow B$ is countable.

Example 1.2.13 (Countable)

$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
$f : \mathbb{N} \longrightarrow \mathbb{Z}$

$$n \longrightarrow \begin{cases} 0, & n = 1 \\ k, & n = 2k \\ -k, & n = 2k + 1 \end{cases}, k \in \mathbb{N}, \text{ is a bijection} \Rightarrow \mathbb{Z} \text{ is countable.}$$

Example 1.2.14 (Uncountable)

$[0, 1]$ is uncountable.
Proof by contradiction: assume $[0, 1]$ is countable. Then, there exists a bijection $f : \mathbb{N} \longrightarrow [0, 1]$. Let $f(n) = 0.a_{n1}a_{n2}a_{n3}\cdots$ be the decimal representation of $f(n)$. Construct a number $b = 0.b_1b_2b_3\cdots$ where $b_n \neq a_{nn}$ and $b_n \in \{0, 1, 2, \cdots, 9\}$. Then, $b \in [0, 1]$ but $b \neq f(n)$ for all $n \in \mathbb{N}$, contradicting the assumption that f is a bijection. Therefore, $[0, 1]$ is uncountable.

Corollary 1.2.2 Countability

$\mathbb{R}$ is uncountable.

1.3 Metric spaces

A metric space is a set equipped with a metric, which is a function that defines a distance between any two elements in the set. $\Omega \neq \emptyset$.

Definition 1.3.1: Metric

A metric or distance is a map $d : X \times X \longrightarrow \mathbb{R}_0^+$ that satisfies the following properties

- (Non-negativity) $d(x, y) \geq 0$
- (Identity of indiscernibles) $d(x, y) = 0 \Leftrightarrow x = y$
- (Symmetry) $d(x, y) = d(y, x)$
- (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in \Omega$

Example 1.3.1 (Metric)

$\Omega = \mathbb{R}$
$d(x, y) = |x - y|$ is a metric.

Example 1.3.2 (Metric)
$\Omega = \mathbb{R}^2$
$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ is a metric.

Example 1.3.3 (Metric)
$\Omega = \mathbb{R}^n$
$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ is a metric.

The metric defined above is called the Euclidean metric or Euclidean distance. There are also other metrics such as the Manhattan metric and the distance along the surface.

Example 1.3.4 (Manhattan distance)
$A \longrightarrow (x_A, y_A), B \longrightarrow (x_B, y_B)$
$d(A, B) = |x_A - x_B| + |y_A - y_B|$

Definition 1.3.2: Bounded map

(X, d) is a metric space, $A \subset X$.
$f : A \longrightarrow \mathbb{R}$ is bounded if there exists $a, b \in \mathbb{R}$ such that $a \leq f(x) \leq b, \forall x \in A$.
$f : A \longrightarrow X$ is bounded iff $\exists a \in X, \forall x \in A, d(f(x), a) \leq M$.

Example 1.3.5 (Unbounded)
$f(x, y) = e^{x^2+y^2}, (x, y) \in \mathbb{R}$
$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$
f is unbounded because $\lim\limits_{x^2+y^2 \to \infty} e^{x^2+y^2} = \infty$.

Example 1.3.6 (Bounded)
$g(x, y) = e^{-x^2-y^2}, (x, y) \in \mathbb{R}$
$g : \mathbb{R}^2 \longrightarrow \mathbb{R}$
$-x^2 - y^2 \in \mathbb{R}_0^-$
$0 < e^{-x^2-y^2} \leq 1$
$\therefore$ g is bounded.

Definition 1.3.3: Diameter

$A, B \subset X$ where (X, d) is a metric space.
The diameter is defined as $\mathrm{diam}(A, B) = \sup\{d(x, y), x \in A, y \in B\}$ which is the maximum distance between A and B.

Example 1.3.7 (Diameter)
$X = \mathbb{R}^2$
$A = \{(x, 0), x \in \mathbb{R}\}$
$B = \{(x, y) \in \mathbb{R}^2 : (x - 4)^2 + (y - 4)^2 \leq 1\}$
$\mathrm{diam}(A, B) = \infty$ because A is unbounded.

Definition 1.3.4: Bounded set

$A \subset X$, A is bounded if $\text{diam}(x, y) \leq M$, for some $M \in \mathbb{R}_0^+$, and for all $x, y \in A$.

Note:

f bounded ≠ A bounded. f being the map and A being the set.

Example 1.3.8 (Bounded?)

$A = \mathbb{R}^2$
$f : \mathbb{R}^2 \to \mathbb{R} \Rightarrow (x, y) \to e^{-x^2-y^2}$
f is bounded, i.e. bounded range, but A is unbounded, i.e. unbounded domain.

Example 1.3.9 (Bounded?)

$A = [-1, 1] \times [2, 3]$
$B = \mathbb{R}^2$
$f : A \to B \Rightarrow (x, y) \to (2x, 2y)$
f is bounded, i.e. bounded range, and A is bounded, i.e. bounded domain.

Example 1.3.10 (Bounded?)

$A = (0, 1)$
$f(x) = \ln(x)$
f is unbounded, i.e. unbounded range, but A is bounded, i.e. bounded domain.

1.4 Definition and typology

$X = \mathbb{R}^n, A \subset X, a \in A, r > 0$

Definition 1.4.1: Open ball

An open ball centered at $a \in X$ and radius $r > 0$ is defined as
$B_r(a) = \{x \in X : d(a, x) < r\}$.

Definition 1.4.2: Closed ball

A closed ball centered at $a \in X$ and radius $r > 0$ is defined as
$D_r(a) = \{x \in X : d(a, x) \leq r\}$. D is for ”disk”. $D_r(a) = \overline{B_r(a)}$.

Definition 1.4.3: Open set

A is an open set of X iff $\forall x \in X, \exists r > 0 : B_r(x) \subset A$.

Definition 1.4.4: Interior point

X_0 is an interior points of A if there exists $r > 0$ such that $B_r(x_0) \subset A$.
We write int$A$.

Definition 1.4.5: Adherent point

X_0 is an adherent point if for all $r > 0$, $B_r(x_0) \cap A \neq \emptyset$.
Same notation as closure: $\overline{A}$ or cl(A). Can also be written as ad(A).

Definition 1.4.6: Frontier point

X_0 is a boundary or frontier point of A if for any $r > 0$, $B_r(x_0) \cap A \neq \emptyset$ and $B_r(x_0) \cap A^c \neq \emptyset$.
We write $\text{fr}(A)$ or ∂A.

Definition 1.4.7: Closed set

$A \subset X$ is a closed set iff A^c is open, $X \to \mathbb{R}^n$.

Example 1.4.1

$A = [0, 1] \subset \mathbb{R}$
$\text{int}(A) = (0, 1)$
$\text{cl}(A) = [0, 1]$
$\text{fr}(A) = \{0, 1\}$
A is closed but not open becasuse $A^c = (-\infty, 0) \cup (1, +\infty)$ which is open.

Definition 1.4.8: Exterior point

$B \subset \mathbb{R}^n$
y is an exterior point of B iff $y \in \text{int}(B^c)$.

Example 1.4.2

$A = [0, 1)$
$A^c = (-\infty, 0) \cup [1, +\infty)$
$\text{int}(A^c) = (-\infty, 0) \cup (1, +\infty)$
$\text{ext}(A) = (-\infty, 0) \cup (1, +\infty)$

Example 1.4.3

$B = [0, 1) \times [0, 1)$
$B^c = [(-\infty, 0) \times \mathbb{R}] \cup [[0, \infty) \times \mathbb{R}^-] \cup [[1, \infty) \times \mathbb{R}] \cup [[0, 1] \times [1, \infty)]$

Proposition: $A \subset \mathbb{R}^n$

1. Any open ball is an open set.
2. $\text{int}(A) \subset A \subset \text{cl}(A)$
3. A is open $\Leftrightarrow$ A = int(A)
4. A is closed $\Leftrightarrow$ A = cl(A)
5. $\text{cl}(A) = \text{int}(A) \cup \partial A$
6. ∂A is closed.
7. A countable union of open sets is open.
8. A countable intersection of closed sets is closed.

Example 1.4.4

$A = [0, 1) \cup [2, 3]$

- $\text{int}(A) = (0, 1) \cup (2, 3) \to$ interior is always open
- $\text{cl}(A) = [0, 1] \cup [2, 3] \to$ closure is the interior and the walls

- $\partial A = \{0, 1, 2, 3\}$
- A is open? No. Because $A \neq \text{int}(A)$
- A is closed? No. Because $A \neq \text{cl}(A)$

Remarks:

1. There are sets that are neither open nor closed.
2. $\mathbb{R}^n, \emptyset$ are either open or closed.

Example 1.4.5

$B = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 2)^2 \leq 4\} \backslash \{(0, 0)\}$

- $\text{int}(B) = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 2)^2 < 4\}$
- $\text{ext}(B) = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 2)^2 > 4\}$
- $\text{cl}(B) = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 2)^2 \leq 4\}$
- $\partial B = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 2)^2 = 4\}$
- B is open? No. Because $B \neq \text{int}(B)$
- B is closed? No. Because $B \neq \text{cl}(B)$

Definition 1.4.9: Bounded

$A \subset \mathbb{R}^n$

A is bounded if $A \subset B_r(x_0)$, for some $r > 0$ and some $x_0 \in \mathbb{R}^n$.

Example 1.4.6

$\mathbb{R} \times \{0\} \subset \mathbb{R}^n$

Not bounded because we cannot find a x large enough to contain the entire set.

Definition 1.4.10: Compact

$A \subset \mathbb{R}^n$

A is compact iff A is closed and bounded.

Definition 1.4.11: Association point

$A \subset \mathbb{R}^n$

x_0 is an association point of A if for any $r > 0$, $B_r(x_0) \cap [A \backslash \{x_0\}] \neq \emptyset$

Example 1.4.7

$A = \{\frac{1}{n}, n \in \mathbb{N}\} \subset \mathbb{R}$

$0 \in A$? No. But there are points from A that accumulates in the ball of 0.

$\therefore$ 0 is an accumulation point of A.

Definition 1.4.12: Isolated point

$A \subset \mathbb{R}^n$

x_0 is an isolated point of A if $x_0 \in A$ and x_0 is not an accumulation point of A.

Proposition:
$\mathrm{cl}(A) = \mathrm{int}(A) \cup \partial A = \{\text{accumulation points}\} \cup \{\text{isolated points}\}$, accumulation points can also be written as derivative of A, A'.

Remarks:
$x^2 = 9 \Leftrightarrow x = -3 \vee x = 3$
$\{x \in \mathbb{R} : x^2 = 9\} \rightarrow$ two points
$\{(x, y) \in \mathbb{R}^2 : x^2 = 9\} \rightarrow$ two lines
$\{(x, y, z) \in \mathbb{R}^3 : x^2 = 9\} \rightarrow$ two planes

Definition 1.4.13: Neighborhood

$\mathbb{R}^n, x_0 \in \mathbb{R}^n$
v is a neighborhood of x_0, if there exists $r > 0$ such that $B_r(x_0) \subset v$.
In general, we consider open neighborhood.

Remarks:

1. $g = \frac{1}{f}, D_g = \{x \in \mathbb{R}^n : f(x) \neq 0\}$
2. $g = \log(f), D_g = \{x \in \mathbb{R}^n : f(x) > 0\}$
3. $g = \sqrt{f}, D_g = \{x \in \mathbb{R}^n : f(x) \geq 0\}$

Definition 1.4.14: Convex

$A \subset \mathbb{R}^n$
A is convex if for any two points X and Y in A, then the segment $[X, Y]$ is contained in A.

1.5 Sequences in $\mathbb{R}^n$

Map from a subset of $\mathbb{N}$ into $\mathbb{R}^n$.

Example 1.5.1 (Sequence)
$f(n) = n^2, n \in \mathbb{N}$
$f : \mathbb{N} \rightarrow \mathbb{R} \Rightarrow n \rightarrow n^2$
$u_n = f(n) = n^2 \rightarrow$ general term

Example 1.5.2 (Sequence)
$V_n = (n^2, n^2 + 1)$ is a sequence in $\mathbb{R}^2$.
$V_5 = (25, 26)$

Sometimes, it is important to see the convergence of sequences.
$u_n = \frac{1}{n}$
$\lim_{n\to\infty} u_n = 0 \Rightarrow u_n \rightarrow 0$

Lemma 1.5.1
If a sequnece u_n in $\mathbb{R}$ converges, the limit is unique.

$u_n = (a_{1n}, a_{2n}, \cdots, a_{kn}) \in \mathbb{R}^k \rightarrow$ the sequence converges if each component converges.

Example 1.5.3 (Convergence)

$u_n = (\frac{1}{n}, \frac{n^2-2}{n^2})$ is a sequence in $\mathbb{R}^2$

- $\frac{1}{n} \longrightarrow 0$
- $\frac{n^2-2}{n^2} = 1 - \frac{2}{n^2} \longrightarrow 1$

$\therefore u_n \longrightarrow (0, 1)$, u_n converges.

When one of the components does not converge, we say that the sequence diverges.

Example 1.5.4 (Divergence)

$u_n = (n^3 + n; \sqrt{n}; \frac{\sqrt{n}+3}{\sqrt{n}})$ in $\mathbb{R}^3$

- $\lim_{n\to\infty} n^3 + n = \infty$ (diverges)
- $\lim_{n\to\infty} \sqrt{n} = \infty$ (diverges)
- $\lim_{n\to\infty} \frac{\sqrt{n}+3}{\sqrt{n}} = \lim_{n\to\infty} 1 + \frac{3}{\sqrt{n}} = 1$

$\therefore u_n$ diverges.

Remarks:

- $(k \in \mathbb{R})$ $\frac{k}{n} \longrightarrow 0$
- $(k \in \mathbb{R})$ $\frac{p(n)}{q(n)}$, then
    1. $\pm\infty$ if $deg(p) > deg(q)$
    2. quotient of the coefficient associated to the highest degree if $deg(p) = deg(q)$
    3. 0 if $deg(p) < deg(q)$

Example 1.5.5 (Sequences and subsequences)

$u_n = (-1)^n$

- $(u_n)_n$ diverges.
- $(u_{2n})_n \longrightarrow 1$
- $(u_{2n+1})_n \longrightarrow -1$, it is a subsequence (infinite sequence of terms of the original sequence).

Proposition:

1. If (u_n) is defined in a compact set, it admits a convergent sequence.
2. The accumulation point of a set A can be seen as a limit of a sequence in cl(A).

Example 1.5.6 (Accumulation point)

$A = \{\frac{1}{n}, n \in \mathbb{R}\}$
$A' = \{0\}$

Example 1.5.7 (Accumulation point)

$u_n = \frac{(-1)^n}{n}$
$A' = \{0\}$

Example 1.5.8 (Accumulation point)
$u_n = (-1)^n$
A' of the sequence: $\{-1, 1\}$

Definition 1.5.1: Cauchy sequence

$(u_n)_n$ is a sequence in $\mathbb{R}$. We say that $(u_n)_n$ is a Cauchy sequence if $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$: $\forall m, n \geq n_0, d(x_n, x_m) = |u_n - u_m| < \epsilon$.
The difference between two terms is as small as I want.

Informally, $(u_n)_n$ convergent but when the limit is not in the space.

Example 1.5.9 (Cauchy sequence)
$X = (0, 1)$
$u_n = \frac{1}{n}$
$(u_n)_n$ is a Cauchy sequence that is not (or could be) convergent.

Sequence is very close each order but could converge or not.

Proposition: (X,d) metric space
If X is compact, then any Cauchy sequence converges.

Remark:
X is compact $\Rightarrow$ It "includes" all possible limits of subsequences of $(u_n)_n$.

Definition 1.5.2: Complete

(X,d) metric space
We say that X is complete if any Cauchy sequence in X converges.

Remark:

1. X is compact $\Rightarrow$ X is complete.
2. $\mathbb{R}$ is complete, however it is not compact.

1.6 Continuity

$f : D_f \to \mathbb{R}, D_f \subset \mathbb{R}$
a is accumulation point of f. $a \in D_f \to$ f is continuous if and only if $\lim_{x \to a} f(x) = f(a)$.

Definition 1.6.1: Continuity (Heine)

$(X, d_1), (Y, d_2)$ metric spaces

$f : X \to Y, a \in X$
f is continuous $\Leftrightarrow \begin{cases} (x_n)_n \text{ is a squence in X} \\ x_n \to a \end{cases} \Leftrightarrow f(x_n)$ is a sequence in Y and $f(x_n) \to f(a)$.

Example 1.6.1 (Continuity)

$$f : \begin{cases} \mathbb{R}^2 \to \mathbb{R} \\ (x, y) \to e^x + y \end{cases}$$

f is continuous because e^x and y are continuous. It is the sum of continuous maps.

Example 1.6.2 (Continuity)

$$f : \begin{cases} \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R} \\ x \longrightarrow \frac{1}{x} \end{cases}$$

f is continous because $\frac{1}{x}$ is continuous in its domain.

Example 1.6.3 (Continuity)

$$f : \begin{cases} \mathbb{R}^2\backslash\{(x,0), x \in \mathbb{R}\} \longrightarrow \mathbb{R} \\ (x,y) \longrightarrow \frac{x^2}{y} \end{cases}$$

$(u_n)_n = (\frac{1}{n}, \frac{1}{n}) \longrightarrow (0,0)$
$(v_n)_n = (\frac{1}{n}, \frac{1}{n^2}) \longrightarrow (0,0)$

$$f(u_n) = \frac{(\frac{1}{n})^2}{\frac{1}{n}} = \frac{1}{n} \longrightarrow 0$$
$$f(v_n) = \frac{(\frac{1}{n})^2}{\frac{1}{n^2}} = 1 \longrightarrow 1$$

f is not continuous at (0,0) because the limit depends on the path.

Proposition: $(X, d_1), (Y, d_2)$ metric spaces
$f : X \longrightarrow Y$ is continous. If $k \subset X$ is compact, then $f(k)$ is a compact set of Y.

Theorem 1.6.1 Weierstrass Theorem

$(\mathbb{R}^n, d)$ metric space

$$f : \begin{cases} \mathbb{R}^n \longrightarrow \mathbb{R} \text{ continuous} \\ k \text{ is a compact set} \end{cases}$$
$\Downarrow$
$f(k)$ has a maximum and a minimum. $f(k)$ is closed and bounded.

Theorem 1.6.2 Intermediate Value Theorem

$f : [a,b] \longrightarrow [c,d]$ is continuous
$\mathrm{Im}(f) = [c,d]$

1.7 Fixed point theorems

(X, d) metric spaace.
$f : X \longrightarrow X$

Definition 1.7.1: Fixed point

$x_0 \in X$ is a fixed point if $f(x_0) = x_0$.
$x_0 \in X$ is a K-periodic point if $f^K(x_0) = x_0$ and $f'(x_0) \neq x_0, \forall j = 1, \cdots, k-1$.

Note: $f^K = \underbrace{f \circ f \circ \cdots \circ f}_{K \text{ times}}$

Example 1.7.1 (Fixed point)

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \to x^2 \end{cases}$$

$x_0 = 0$ is a fixed point because $f(0) = 0^2 = 0$.
$x_0 = 1$ is a fixed point because $f(1) = 1^2 = 1$.

Example 1.7.2 (Fixed point)

$$f : \begin{cases} [0,1] \to [0,1] \\ x \to 1 - x \end{cases}$$

$x_0 = \frac{1}{2}$ is a fixed point because $f(\frac{1}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$.

$f[f(0.25)] = f[0.75] = 0.25 \to f^2(0.25) = 0.25$
$f[f(0.1)] = f[0.9] = 0.1 \to f^2(0.1) = 0.1$
$\forall x \in [0,1] \backslash \{\frac{1}{2}\}, f^2(x) = x \to$ every point except $\frac{1}{2}$ is a 2-periodic point.
$\mathrm{Per}_2(f) = [0,1] \backslash \{\frac{1}{2}\}$

Remark: $\frac{1}{2}$ is not 2-periodic because it is a fixed point.9

Example 1.7.3 (Fixed point)

$$R_\theta : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ (x,y) \to R_\theta(x,y) \\ \theta \in (0, 2\pi) \end{cases}$$

R_θ is a rotation of angle θ around the origin.
R_θ has a unique fixed point which is the origin $\Rightarrow \mathrm{Fix}(R_\theta) = \{(0,0)\}$.

Example 1.7.4 (Fixed point)

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \to \begin{cases} 3x, x < \frac{1}{2} \\ 3 - 3x, x \geq \frac{1}{2} \end{cases} \end{cases}$$

$f(x) = 3x \Rightarrow x = 3x \Rightarrow x = 0$
$f(x) = 3 - 3x \Rightarrow x = 3 - 3x \Rightarrow x = \frac{3}{4}$
0 and $\frac{3}{4}$ are the only fixed points of f $\Rightarrow \mathrm{Fix}(f) = \{0, \frac{3}{4}\}$

Definition 1.7.2: Lipschitz contraction

$f : \begin{cases} X \longrightarrow X \text{ is a map} \\ (X, d) \text{ metric space} \end{cases}$

f is a contraction if there exists $0 \leq k < 1$ such that $d(f(x), f(y)) < K \cdot d(x, y), \forall x, y \in X$.

Remark: K is called the Lipschitz constant, it is the ratio of contraction.

Example 1.7.5 (Contraction)

$f : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longrightarrow \frac{1}{2}x \end{cases}$

$|f(x) - f(y)| = |\frac{1}{2}x - \frac{1}{2}y| = \frac{1}{2}|x - y|$
$\therefore f$ is a contraction with $K = \frac{1}{2}$.

In general, to prove that f is a contraction is a difficult task. That is why the next result will be useful in the sequel.

Lemma 1.7.1 Contraction

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable map.
If the eigenvalues of the Jacobian matrix $Df(x)$ have modules less than1, then f is a contraction.

Reminder: The Jacobian matrix is the matrix of partial derivatives.

$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$

Example 1.7.6 (Contraction)

$f : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x, y) \longrightarrow (-\frac{1}{2}x, \frac{y}{3}) \end{cases}$

$Df(x, y) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

Since the Jacobian matrix is diagonal, the eigenvalues are $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{3}$.
And since both have modules less than 1.
$\therefore f$ is a contraction.

Note:
Eigenvalues of $A \in M_{n\times n}(\mathbb{R})$ are the roots of the characteristic polynomial $P_A(\lambda) = \det(A - \lambda I)$.

Theorem 1.7.2 Banach Theorem

If

- (X, d) is a complete metric space.
- $f : X \longrightarrow X$ is a contraction.

then f has a **unique** fixed point.

Proof. **Unicity** of the fixed point
Suppose that we have two fixed points $A \neq B$.

Since f is a contraction, then $d(A, B) = d(f(A), f(B)) < K \cdot d(A, B)$ with $0 \leq K < 1$.
$\therefore\ d(A, B) < d(A, B)$ which is a contradiction. ■

Proof. **Existence** of the fixed point
$x_0 \in X$ define the sequence $x_n = f^n(x_0)$
$(x_n)_n$ is a Cauchy sequence $\underbrace{\Rightarrow}_{x \text{ is complete}}$ $(x_n)_n$ converges to the fixed point. ■

Note:
The distance decreases each time you composite with f.

Example 1.7.7 (Banach)

$$f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \to x^2 \end{cases}$$

$|f'(x)| = |2x| \not< 1 \Rightarrow$ not a contraction.
$\therefore$ Banach theorem does not apply.

Example 1.7.8 (Banach)

$$f : \begin{cases} [-\frac{1}{3}, \frac{1}{3}] \to \mathbb{R} \\ x \to x^2 \end{cases}$$

$|f'(x)| = |2x| \leq \frac{2}{3} < 1 \Rightarrow$ f is a contraction.
$\therefore$ Banach theorem applies and f has a unique fixed point.

$x = x^2 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$
Since $1 \notin [-\frac{1}{3}, \frac{1}{3}]$, the unique fixed point is 0.

Example 1.7.9 (Banach)

$$f : \begin{cases} B_r(0,0) \to \mathbb{R}^2 \\ x \to \lambda x, \lambda \in (0,1) \end{cases}$$

- $B_r(0,0)$ is complete because it is compact (closed and bounded).
- f is a contraction because $|f'(x)| = |\lambda| < 1$.

$\therefore$ Banach theorem applies and f has a unique fixed point at the origin. $\Rightarrow \text{Fix}(f) = \{(0,0)\}$

Note:
$f : x \to x$ is a contraction of ratio K. Then f is a contraction of ratio $\tilde{K} \in (K, 1)$.

Remark:
$(x_0, y_0) \in \mathbb{R}^2$. If I take any point Q, then $\lim_{n\to\infty} f^n(Q) = P$ where P is the unique fixed point of f. This means that the fixed point is an attractor of the map.

Theorem 1.7.3 Brouwer fixed point theorem
If

- $f : D_f \to D_f$ is a continuous map
- D_f is compact and convex

then, f has a fixed point, but not necessarily unique.

Example 1.7.10 (Brouwer)

$$f : \begin{cases} [0,1] \to [0,1] \\ x \to x^2 \end{cases}$$

- $[0,1]$ is compact and convex.
- f is continuous (polynomial).

$\therefore$ Brouwer theorem applies and f has at least one fixed point.

$x = x^2 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$
$\therefore \mathrm{Fix}(f) = \{0, 1\}$

Sidenote: Interval is always convex.

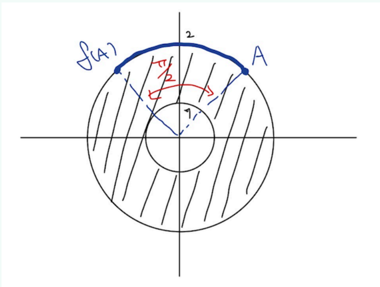
Example 1.7.11 (Brouwer)

$$f : \begin{cases} A \to A \\ (x,y) \to R_{\frac{\pi}{2}}(x,y) \\ A = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4\} \end{cases}$$

- A is compact but not convex.
- f is continuous (rotation).

$\therefore$ Brouwer theorem does not apply and f has no fixed point.

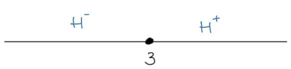
$R_{\frac{\pi}{2}}(x,y) = (x,y)$ has no solution $\Rightarrow \mathrm{Fix}(f) = \emptyset$



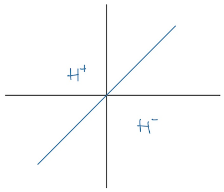
Definition 1.7.3: Hyperplane

Hyperplane in $\mathbb{R}^n$ is a plane of dimension $n-1$ that divides $\mathbb{R}^n$ into semi planes.

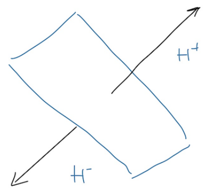
- Hyperplane in $\mathbb{R}^1$ is a point.



- Hyperplane in $\mathbb{R}^2$ is a line that divides the plane into two half-planes.



- Hyperplane in $\mathbb{R}^3$ is a plane that divides the space into two half-spaces.



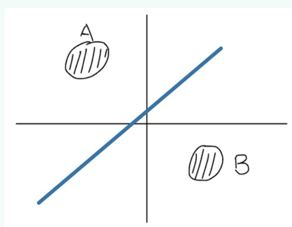
Theorem 1.7.4 Hyperplane theorem

If

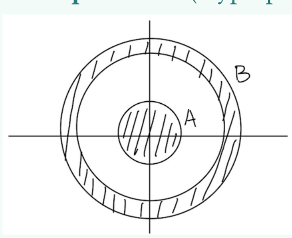
- $A, B \subset \mathbb{R}^n$
- A, B disjoint and convex

then A, B can be separated by a hyperplane.

Example 1.7.12 (Hyperplane)

Separated by a line. 

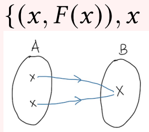
Example 1.7.13 (Hyperplane)



Separation theorem cannot be applied because B is not convex.

Definition 1.7.4: Correspondence

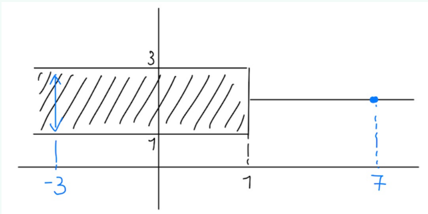
A, B sets $\subseteq \mathbb{R}$
$f : A \longrightarrow B$ map
$F : A \rightrightarrows B$
$\{(x, F(x)), x \in A, F(x) \subset B\}$



Example 1.7.14 (Corrrespondence)

$F : \mathbb{R} \rightrightarrows \mathbb{R}$

$$x \longrightarrow \begin{cases} [1,3] & x \leq 1 \\ \{2\} & x > 1 \end{cases}$$

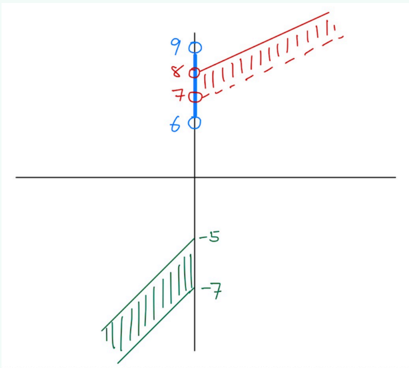


$F(\{7\}) = \{2\}$
$F(\{-3\}) = [1,3]$

Example 1.7.15 (Correspondence)

$G : \mathbb{R} \rightrightarrows \mathbb{R}$

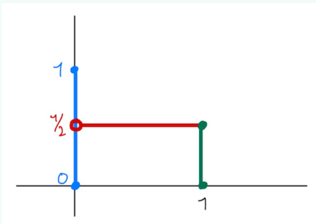
$$G(x) = \begin{cases} [x-7, x-5] & x < 0 \\ (6,9) & x = 0 \\ (x+7, x+8] & x > 0 \end{cases}$$



Example 1.7.16 (Correspondence)

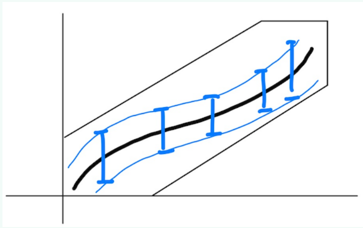
$f : [0,1] \rightrightarrows [0,1]$

$$x \longrightarrow \begin{cases} [0,1] & x = 0 \\ \{\frac{1}{2}\} & 0 < x < 1 \\ [0, \frac{1}{2}] & x = 1 \end{cases}$$



Example 1.7.17 (Correspondence)

Important for confidence interval construction of financial data.

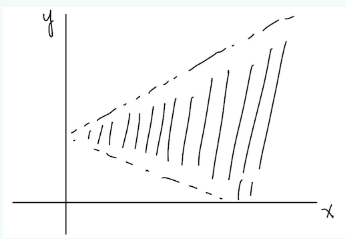


Definition 1.7.5: Closed graph property

$F : A \rightrightarrows B$ a correspondence, $x \in A$.

We say that F has the closed graph property at x if for any converging sequence $(x_n, y_n)_n$ of element in the graph F, its limit belongs to the graph of F.
We say that F has the closed graph property if it has the property above for all $x \in A$.

Example 1.7.18 (Closed graph property)

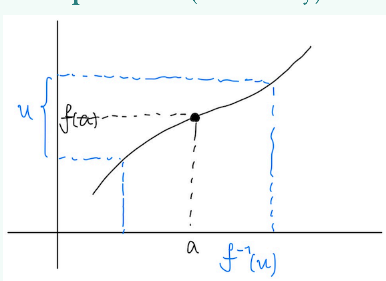


F does not have the closed graph property because the graph of F is not closed.

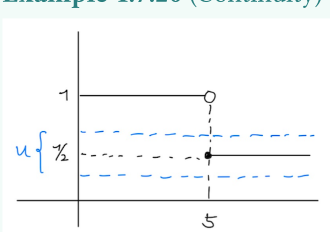
Proposition: $F : A \rightrightarrows B$ correspondence. If graph F is **compact**, then F has the closed graph property.

$f : \mathbb{R} \to \mathbb{R}, x \to f(x)$. f is continuous at $x = a \in \mathbb{R}$, if and only if $\lim_{x \to a} f(x) = f(a)$. Equivalently, f is continuous at $x = a$ if and only if for any open set u containing $f(a)$, $f^{-1}(u)$ is an open set containing a.

Example 1.7.19 (Continuity)



Example 1.7.20 (Continuity)



u open set containing $f(5) = \frac{1}{2}$, $f^{-1}(u) = [5, \infty)$ is not an open set containing 5.

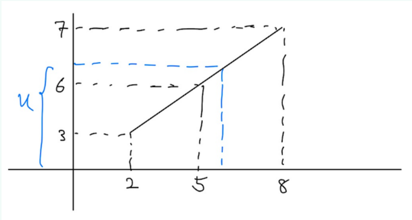
We are now going to generalize the notion of continuity for correspondence.

Definition 1.7.6: Hemi-continuity

$A, B \subseteq \mathbb{R}, a \in A$
$F : A \rightrightarrows B$ is a correspondence

We say that F is upper hemi-continuous at $x = a$ if for all open set u containing $f(a)$, its pre-image $F^{-1}(u)$ is an open set containing a. The correspondence F is upper hemi-continuous if the above property holds for every $a \in A$.

Example 1.7.21 (Hemi-continuity)

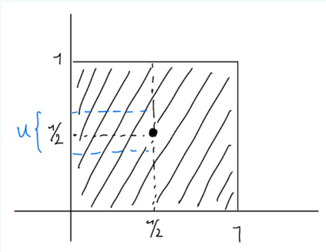


$f(\{5\}) = [0, 6]$
f is upper hemi-continuous.

Example 1.7.22 (Hemi-continuity)

$f : [0, 1] \rightrightarrows [0, 1]$

$$x \longrightarrow \begin{cases} [0,1] & x \neq \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \end{cases}$$



u open set containing $f(\frac{1}{2}) = \frac{1}{2}$
$f^{-1}(u) = \{\frac{1}{2}\}$ is not an open set. $\Rightarrow f$ is not hemi-continuous.

Theorem 1.7.5 Kakutani fixed point theorem

- $A, B \subseteq \mathbb{R}^n$ convex and compact
- $F : A \rightrightarrows B$

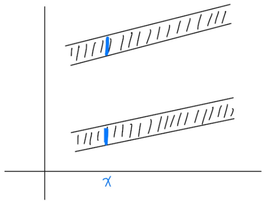
If

1. F is upper hemi-continuous
2. $F(x)$ is convex, $\forall x \in A$

then F has at least one fixed point.

Remark:

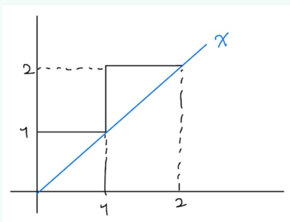
- This theorem is a generalization of the Brouwer fixed point theorem.
- The second property of Kakutani fixed point theorem means that correpondence cannot have the follwing behavior



Example 1.7.23 (Kakutani)

$F : [0,2] \rightrightarrows [0,2]$

$$x \longrightarrow \begin{cases} \{1\} & 0 \le x < 1 \\ [1,2] & x = 1 \\ \{2\} & 1 < x \le 2 \end{cases}$$

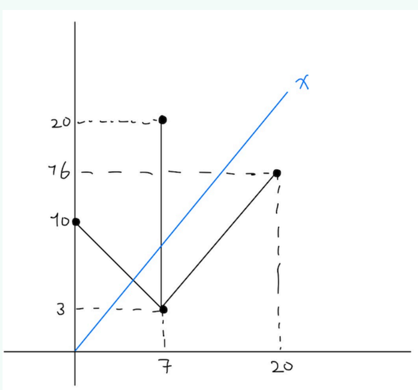


Fix $F = \{1, 2\}$

Example 1.7.24 (Kakutani)

$F : [0,20] \rightrightarrows [0,20]$

$$x \longrightarrow \begin{cases} 10 - x & 0 \le x < 7 \\ [3,20] & x = 7 \\ x - 4 & 7 < x \le 20 \end{cases}$$



$10 - x = x \leftrightarrow x = 5$

F is a fixed point because $\{F\} \subset F(\{7\})$

Fix $F = \{5, 7\}$

Hemi-continuous is hard to prove for the Kakutani theorem, thus we have the following **proposition**: $F : A \rightrightarrows B$ correspondence. If F has the closed graph property, then F is upper hemi-continuous. The scheme is as the following: Graph F is compact $\Rightarrow$ Graph F has the closed graph property $\Rightarrow$ F is upper hemi-continous.

1.8 What is necessary to know in this chapter?

1. Functions, sequences, cardinality, continuity
2. Typology in $\mathbb{R}^n$
3. Hyperplane separation theorem
4. Weierstrass theorem
5. Intermediate value theorem
6. Banach fixed point theorem
7. Brouwer fixed point theorem
8. Kakutani fixed point theorem