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Taller 3

15. Demostrar directamente que la transformación

$$Q = \arctan \frac{\alpha q}{p},$$
$$P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

es canónica, donde α es una constante.

Respuesta a Problema 1

Sean en (p, q) la coordenadas generalizadas de un sistema en el espacion de fase, y sea $H(p, q)$ la funcion de Hamilton, entonces las ecuaciones de Hamilton son:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \quad \text{y} \quad \frac{\partial H}{\partial t} = 0 \quad (15{,}1)$$

Dado que la transformacion puntual de coordenadas generalizadas (p, q) a coordenadas generalizadas (P, Q) dada por:

$$\begin{aligned} Q &= \arctan \frac{\alpha q}{p}, \\ P &= \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right) = \frac{1}{2}\left(\alpha q^2 + \frac{p^2}{\alpha}\right) \end{aligned} \quad (15{,}2)$$

es canónica, entonces deben satisfacer las ecuaciones de Hamilton (15.1) con $K(P, Q) = K$ constante, es decir:

$$\dot{P} = -\frac{\partial K}{\partial Q} \quad \text{y} \quad \dot{Q} = \frac{\partial K}{\partial P} \quad \text{y} \quad \frac{\partial K}{\partial t} = 0 \quad (15{,}3)$$

Para mostrar esto ultimo, usando la regla de la cadena en la ecuaciones (15.2) y reemplazando (15.1)

$$\begin{aligned} \frac{dQ}{dt} &= \frac{\partial Q}{\partial q}\frac{dq}{dt} + \frac{\partial Q}{\partial p}\frac{dp}{dt} \\ &= -\frac{\alpha q}{p^2 + \alpha^2 q^2}\left(-\frac{\partial H}{\partial q}\right) + \frac{\alpha p}{p^2 + \alpha^2 q^2}\left(\frac{\partial H}{\partial p}\right) \\ &= \frac{\alpha}{p^2 + \alpha^2 q^2}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \\ &= \frac{1}{2}\frac{2\alpha}{p^2 + \alpha^2 q^2}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \\ &= \frac{1}{2P}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \quad (15{,}4{,}1) \end{aligned}$$

$$\begin{aligned} \frac{dP}{dt} &= \frac{\partial P}{\partial q}\frac{dq}{dt} + \frac{\partial P}{\partial p}\frac{dp}{dt} \\ &= \frac{p}{\alpha}\left(-\frac{\partial H}{\partial q}\right) + q\alpha\left(\frac{\partial H}{\partial p}\right) \quad (15{,}4{,}2) \end{aligned}$$

Aplicando la regla de la cadena en (15.1) tenemos que:

$$\begin{aligned}\dot{p} = -\frac{\partial H}{\partial q} &= -\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} - \frac{\partial H}{\partial P}\frac{\partial P}{\partial q} \\ &= -\frac{\partial H}{\partial Q}\frac{\alpha p}{p^2+\alpha^2 q^2} - \frac{\partial H}{\partial P}\alpha q \quad (15{,}7{,}1)\end{aligned}$$

$$\begin{aligned}\dot{q} = \frac{\partial H}{\partial p} &= \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p} \\ &= -\frac{\partial H}{\partial Q}\frac{\alpha q}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p}{\alpha} \quad (15{,}7{,}2)\end{aligned}$$

Reemplazando (15.5) en (15.7.1) y (15.7.2) tenemos que:

$$\begin{aligned}\dot{Q} &= \frac{1}{2P}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \\ &= \frac{1}{2P}\left(q\left(\frac{\partial H}{\partial Q}\frac{\alpha p}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\alpha q\right) + p\left(-\frac{\partial H}{\partial Q}\frac{\alpha q}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p}{\alpha}\right)\right) \\ &= \frac{1}{2P}\left(\frac{\partial H}{\partial Q}\frac{\alpha pq}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\alpha q^2 - \frac{\partial H}{\partial Q}\frac{\alpha pq}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p^2}{\alpha}\right) \\ &= \frac{1}{2P}\left(\frac{\partial H}{\partial P}\alpha q^2 + \frac{\partial H}{\partial P}\frac{p^2}{\alpha}\right) \\ &= \frac{1}{2P}\frac{\partial H}{\partial P}\left(\alpha q^2 + \frac{p^2}{\alpha}\right) \\ &= \frac{1}{2P}\frac{\partial H}{\partial P}2P = \frac{\partial H}{\partial P}\end{aligned}$$

$$\begin{aligned}\dot{P} &= \frac{p}{\alpha}\left(-\frac{\partial H}{\partial q}\right) + q\alpha\left(\frac{\partial H}{\partial p}\right) \\ &= \frac{p}{\alpha}\left(-\frac{\partial H}{\partial Q}\frac{\alpha p}{p^2+\alpha^2 q^2} - \frac{\partial H}{\partial P}\alpha q\right) + q\alpha\left(-\frac{\partial H}{\partial Q}\frac{\alpha q}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p}{\alpha}\right) \\ &= -\frac{\partial H}{\partial Q}\frac{p^2}{p^2+\alpha^2 q^2} - \frac{\partial H}{\partial P}pq - \frac{\partial H}{\partial Q}\frac{\alpha^2 q^2}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}pq \\ &= -\frac{\partial H}{\partial Q}\frac{p^2+\alpha^2 q^2}{p^2+\alpha^2 q^2} \\ &= -\frac{\partial H}{\partial Q}\end{aligned}$$

Por lo que si $K = H(P,Q)$ entonces las anterioes ecuaciones son las ecuaciones de Hamilton (15.3) y por lo tanto la transformacion (15.2) es canónica. Vease que como $H = H(q,p)$ no depende de t entonces $K = H(P,Q)$ tampoco depende de t, es decir $\frac{\partial K}{\partial t} = 0$.

16. Mostrar que una funcion generatriz del segundo tipo cuya forma particular sea $F_2 = q_j P_j$, genera la transformacion identidad

Respuesta a Punto 2

De las ecuaciones de transformacion asociadas a al funcion generatriz $F_2(q_j, P_j) \quad j = 1, \dots n$ dadas por:

$$Q_j = \frac{\partial F_2}{\partial P_j}, \quad p_j = \frac{\partial F_2}{\partial q_j} \quad \text{y} \quad K = H + \frac{\partial F_2}{\partial t} \quad (16,1)$$

Se tiene que:

$$\frac{\partial F_2}{\partial t} = 0 \quad \Rightarrow \quad K = H(Q_j, P_j)$$
$$Q_j = \frac{\partial F_2}{\partial P_j} = \frac{\partial}{\partial P_j}(P_j q_j) = q_j$$
$$p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial}{\partial q_j}(P_j q_j) = P_j$$

De esta forma la matriz de transformacion M es tal que:

$$M = \begin{pmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

17. Use una funci´on generatriz para construir una transformaci´on que intercambie cantidades de movimiento y coorde- nadas.

Respuesta a Punto 3

Dado que se busca una funcion generatriz $F' = F'(q_j, p_k, Q_l, P_m, t), \quad j = l, k = m = 1, .., n$ o $j = m, l = m = 1, .., n$, para algun j, k, l, m tal que la transformacion canonica M asociada satisfaga que:

$$\dot{\mathbf{X}} = M\dot{\mathbf{x}} \quad (17,1)$$

Donode:

$$\mathbf{X} = \begin{pmatrix} Q_j \\ P_j \end{pmatrix} = \begin{pmatrix} p_j \\ q_j \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} q_j \\ p_j \end{pmatrix} \quad \text{y} \quad M = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i} \\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix} \quad (17,2)$$

Por lo que de (17.2):

$$M = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i} \\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial p_j}{\partial q_i} & \frac{\partial p_j}{\partial p_i} \\ \frac{\partial q_j}{\partial q_i} & \frac{\partial q_j}{\partial p_i} \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad (17,3)$$

Vease que M de (17.3) no una transformacion canonica, pues:

$$M^T J M = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}^T \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \neq J$$

Por lo que en su lugar se propone:

$$\mathbf{X} = \begin{pmatrix} Q_j \\ P_j \end{pmatrix} = \begin{pmatrix} p_j \\ -q_j \end{pmatrix} \quad \text{y} \quad M = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i} \\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix} \quad (17,4)$$

Asi de (17.4):

$$M = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i} \\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial p_j}{\partial q_i} & \frac{\partial p_j}{\partial p_i} \\ \frac{\partial(-q_j)}{\partial q_i} & \frac{\partial(-q_j)}{\partial p_i} \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \quad (17{,}5)$$

Vease que M de (17.5) es una transformacion canonica, pues:

$$M^T J M = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}^T \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} = J$$

Una una funcion generatriz que me genera este tipo de transformacion canonica es:

$$F(q_j, Q_j, t) = q_j Q_j \quad (17{,}6)$$

Pues de (17.6) se tiene que:

$$p_i = \frac{\partial F}{\partial q_i} = \frac{\partial}{\partial q_i}(q_j Q_j) = Q_i \quad \text{y} \quad P_i = -\frac{\partial F}{\partial Q_i} = -\frac{\partial}{\partial Q_i}(q_j Q_j) = -q_i$$

18. Las ecuaciones de transformación entre dos sistemas de coordenadas son

$$Q = \log\left(1 + q^{1/2}\cos p\right)$$
$$P = 2\left(1 + q^{1/2}\cos p\right) q^{1/2}\sin p$$

a) A partir de estas ecuaciones de transformación, demostrar directamente que Q, P son variables canónicas si lo son q y p. b) Demostrar que la función que genera esta transformación es

$$F_3 = -\left(e^Q - 1\right)^2 \tan p$$

Respuesta a Punto 4

- Si (q, p) son variables canónicas entonces satisfacen las ecuaciones canonicas de Hamilton, es decir:

$$\dot{q} = \frac{\partial H}{\partial p} \quad \text{y} \quad \dot{p} = -\frac{\partial H}{\partial q} \quad (18{,}1)$$

Si ademas define la transformacion:

$$\begin{aligned} Q &= \log\left(1 + q^{1/2}\cos p\right) \\ P &= 2\left(1 + q^{1/2}\cos p\right) q^{1/2}\sin p \end{aligned} \quad (18{,}2)$$

Entonces se tiene que por la regla de la cadena aplicada en (18.1) y (18.2):

$$\begin{aligned} \dot{q} = \frac{\partial H}{\partial p} &= \frac{\partial H}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} \\ &= \frac{\partial H}{\partial P} 2\left(q^{1/2}\cos p + q\cos(2p)\right) - \frac{\partial H}{\partial Q}\frac{q^{1/2}\sin p}{1 + q^{1/2}\cos p} \\ &= \frac{\partial H}{\partial P} 2\left(q^{1/2}\cos p + q\cos(2p)\right) - \frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P} \quad (18{,}3{,}1) \end{aligned}$$

$$
\begin{aligned}
\dot{p} &= -\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial P}\frac{\partial P}{\partial q} - \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} \\
&= -\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p}{2(q^{1/2}+q\cos p)} \\
&= -\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P} \quad (18{,}3{,}2)
\end{aligned}
$$

$$
\begin{aligned}
\dot{Q} &= \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} \\
&= \frac{\cos p}{2(q^{1/2}+q\cos p)}\dot{q} - \frac{q^{1/2}\sin p}{1+q^{1/2}\cos p}\dot{p} \\
&= \frac{\cos p\sin p}{P}\dot{q} - \frac{2q\sin^2 p}{P}\dot{p} \quad (18{,}4{,}1)
\end{aligned}
$$

$$
\begin{aligned}
\dot{P} &= \frac{\partial P}{\partial q}\dot{q} + \frac{\partial P}{\partial p}\dot{p} \\
&= \left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p\dot{q} + 2\left(q^{1/2}\cos p + q\cos(2p)\right)\dot{p} \quad (18{,}4{,}2)
\end{aligned}
$$

Reemplazando (18.3.1) y (18.3.2) en (18.4.1) y (18.4.2):

$$
\begin{aligned}
\dot{Q} &= \frac{\cos p\sin p}{P}\left(\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos(2p)\right) - \frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\right) \\
&\quad - \frac{2q\sin^2 p}{P}\left(-\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\right) \\
&= \frac{\cos p\sin p}{P}\left(\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos^2 p - q\sin^2 p\right) - \frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\right) \\
&\quad + \frac{2q\sin^2 p}{P}\left(\frac{\partial H}{\partial P}\left(\frac{1+q^{1/2}\cos p}{q^{1/2}} + \cos p\right)\sin p + \frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\right) \\
&= \frac{\cos p\sin p}{P}\left(\frac{\partial H}{\partial P}2\left(q^{1/2} + q\cos p\right)\frac{\sin p}{\sin p}\cos p - \frac{\partial H}{\partial P}2\left(q\sin^2 p\right)\right) - \cancel{\frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P^2}} \\
&\quad + \frac{2q\sin^2 p}{P}\left(\frac{\partial H}{\partial P}2\left(\frac{q^{1/2}+q\cos p}{2q}\sin p\right) + \frac{\partial H}{\partial P}\cos p\sin p\right) + \cancel{\frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P^2}} \\
&= \frac{\cos p\sin p}{P}\left(\frac{\partial H}{\partial P}P\frac{\cos p}{\sin p} - \frac{\partial H}{\partial P}2\left(q\sin^2 p\right)\right) + \frac{2q\sin^2 p}{P}\left(\frac{\partial H}{\partial P}\left(\frac{P}{2q}\right) + \frac{\partial H}{\partial P}\cos p\sin p\right) \\
&= \frac{\partial H}{\partial P}\cos^2 p - \cancel{2\frac{\partial H}{\partial P}q\sin^3 p\cos p\frac{1}{P}} + \frac{\partial H}{\partial P}\sin^2 p + \cancel{2\frac{\partial H}{\partial P}\cos p\sin^3 p\frac{1}{P}} \\
&= \frac{\partial H}{\partial P} \quad (18{,}5{,}1)
\end{aligned}
$$

$$
\begin{aligned}
\dot{P} &= \left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p\left(\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos(2p)\right) - \frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\right)\\
&+ 2\left(q^{1/2}\cos p + q\cos(2p)\right)\left(-\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \cancel{\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos(2p)\right)} - \left(\frac{1}{q^{1/2}} + 2\cos p\right)\frac{\partial H}{\partial Q}\frac{2q\sin^3 p}{P}\\
&+ \cancel{2\left(q^{1/2}\cos p + q\cos(2p)\right)\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p} + 2\left(q^{1/2}\cos p + q\cos(2p)\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}
\end{aligned}
$$

$$
= -\left(\frac{1 + q^{1/2}\cos p}{q^{1/2}} + \cos p\right)\sin p\frac{\partial H}{\partial Q}\frac{2q\sin^3 p}{P} - 2\left(q^{1/2}\cos p + q\cos^2 p - q\sin^2 p\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}
$$

$$
\begin{aligned}
&= -\left(\frac{P}{2q} + \cos p\sin p\right)\frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P} - \left(2\left(q^{1/2} + q\cos p\right)\frac{\sin p}{\sin p}\cos p - 2\left(q\sin^2 p\right)\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\\
&= -\frac{\partial H}{\partial Q}\sin^2 - \frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P} - \left(P\frac{\cos p}{\sin p} - 2\left(q\sin^2 p\right)\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\\
&= -\frac{\partial H}{\partial Q}\sin^2 + \cancel{\frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P}} - \frac{\partial H}{\partial Q}\cos^2 p - \cancel{\frac{\partial H}{\partial Q}\frac{2q\cos p\sin^3 p}{P}}\\
&= -\frac{\partial H}{\partial Q} \quad (18{,}5{,}2)
\end{aligned}
$$

Por que si $K = H(Q, P)$ entonces las anterioes ecuaciones son las ecuaciones canonicas de Hamilton y por lo tanto (P, Q) son variables canonicas.

- Dado que $F_3 = F_3(p, Q)$, pues:

$$
F_3 = -\left(e^Q - 1\right)^2 \tan p
$$

Entonces de la ecuaciones de canonicas de transformacion para F_3:

$$
q = -\frac{\partial F_3}{\partial p} \quad \text{y} \quad P = \frac{\partial F_3}{\partial Q} \quad (18{,}6)
$$

se sigue que:

$$
\begin{aligned}
q = -\frac{\partial F_3}{\partial p} = -\frac{\partial}{\partial p}\left(-\left(e^Q - 1\right)^2\tan p\right) = \left(e^Q - 1\right)^2\sec^2 p \quad (18{,}7{,}1)\\
P = -\frac{\partial F_3}{\partial Q} = -\frac{\partial}{\partial Q}\left(-\left(e^Q - 1\right)^2\tan p\right) = 2\left(e^Q - 1\right)e^Q\tan p \quad (18{,}7{,}2)
\end{aligned}
$$

Despejando Q de (18.7.1):

$$\begin{aligned} q = \left(e^Q - 1\right)^2 \sec^2 p \quad &\Rightarrow \quad \left(e^Q - 1\right)^2 = \frac{q}{\sec^2 p} \\ &\Rightarrow \quad e^Q - 1 = \sqrt{\frac{q}{\sec^2 p}} \\ &\Rightarrow \quad e^Q = 1 + \sqrt{\frac{q}{\sec^2 p}} \\ &\Rightarrow \quad Q = \ln\left(1 + \sqrt{\frac{q}{\sec^2 p}}\right) \\ &\Rightarrow \quad Q = \ln\left(1 + q^{1/2}\cos p\right) \quad (18{,}8{,}1) \end{aligned}$$

Reemplazando (18.8.1) en (18.7.2):

$$\begin{aligned} P = 2\left(e^Q - 1\right)e^Q \tan p \quad &\Rightarrow \quad P = -2\left(1 - q^{1/2}\cos p - 1\right)\left(1 - q^{1/2}\cos p\right)\tan p \\ &\Rightarrow \quad P = -2(1 - 2q^{1/2}\cos p + q\cos^2 p - 1 + q^{1/2}\cos p)\tan p \\ &\Rightarrow \quad P = -2\left(q^{1/2}\cos p + q\cos^2 p\right)\tan p \\ &\Rightarrow \quad P = -2q^{1/2}\cos p\left(1 + q^{1/2}\cos p\right)\tan p \quad (18{,}8{,}2) \end{aligned}$$

De (18.8.1) y (18.8.2) se concluye que F_3 es una funcion generatriz de (18.2)

19. Probar directamente que la transformación

$$\begin{aligned} Q_1 = q_1, \quad P_1 = p_1 - 2p_2 \\ Q_2 = p_2, \quad P_2 = -2q_1 - q_2 \end{aligned}$$

es canónica y hallar una función generatriz.

Respuesta a Punto 19

Si de las ecuaciones:

$$\begin{aligned} Q_1 = q_1, \quad P_1 = p_1 - 2p_2 \\ Q_2 = p_2, \quad P_2 = -2q_1 - q_2 \end{aligned} \qquad (19{,}1)$$

Calculamos la matriz de transformación M:

$$M = \begin{pmatrix} \frac{\partial Q_1}{\partial q_1} & \frac{\partial Q_1}{\partial q_2} & \frac{\partial Q_1}{\partial p_1} & \frac{\partial Q_1}{\partial p_2} \\ \frac{\partial Q_2}{\partial q_1} & \frac{\partial Q_2}{\partial q_2} & \frac{\partial Q_2}{\partial p_1} & \frac{\partial Q_2}{\partial p_2} \\ \frac{\partial P_1}{\partial q_1} & \frac{\partial P_1}{\partial q_2} & \frac{\partial P_1}{\partial p_1} & \frac{\partial P_1}{\partial p_2} \\ \frac{\partial P_2}{\partial q_1} & \frac{\partial P_2}{\partial q_2} & \frac{\partial P_2}{\partial p_1} & \frac{\partial P_2}{\partial p_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix}$$

Entonces la matriz M es una transformación canónica si y solo si:

$$M^T J M = J \quad J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Lo cual podemos comprobar:

$$
\begin{aligned}
M^T J M &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = J
\end{aligned}
$$

Por lo tanto M es una transformación canónica.

Considere la funcion generatriz $F' = F'(q_1, p_2, P_1, P_2, t)$ tal que:

$$F(q_1, q_2, Q_1, Q_2, t) \to F''(q_1, p_2, Q_1, Q_2, t) \quad \Rightarrow$$

$$
\begin{aligned}
F''(q_1, p_1, Q_1, Q_1, t) &= q_2 \frac{\partial F}{\partial q_2} - F \\
&= q_2 p_2 - F
\end{aligned}
$$

$$F''(q_1, p_2, Q_1, Q_2, t) \to F'''(q_1, p_2, P_1, Q_2, t) \quad \Rightarrow$$

$$
\begin{aligned}
F'''(q_1, p_2, P_1, Q_2, t) &= Q_1 \frac{\partial F''}{\partial Q_1} - F'' \\
&= -Q_1 \frac{\partial F}{\partial Q_1} - q_2 p_2 + F \\
&= Q_1 P_1 - q_2 p_2 + F(q_1, q_2, Q_1, Q_2, t) \quad (19{,}2)
\end{aligned}
$$

$$F'''(q_1, p_2, P_1, Q_2, t) \to F'(q_1, p_2, P_1, P_2, t) \quad \Rightarrow$$

$$
\begin{aligned}
F'(q_1, p_2, P_1, Q_2, t) &= Q_2 \frac{\partial F'''}{\partial Q_2} - F''' \\
&= Q_2 \frac{\partial F}{\partial Q_2} - Q_1 P_1 + q_2 p_2 - F \\
&= -Q_2 P_2 - Q_1 P_1 + q_2 p_2 - F(q_1, q_2, Q_1, Q_2, t) \quad (19{,}2)
\end{aligned}
$$

Donde de (19.2) se tiene que:

$$
\begin{aligned}
\frac{dF}{dt} &= p_1\dot{q}_1 + p_2\dot{q}_2 - P_1\dot{Q}_1 - P_2\dot{Q}_2 - (H-K)\\
&= \frac{\partial F}{\partial q_1}\dot{q}_1 + \frac{\partial F}{\partial q_2}\dot{q}_2 - \frac{\partial F}{\partial Q_1}\dot{Q}_1 - \frac{\partial F}{\partial Q_2}\dot{Q}_2\\
&= \frac{d}{dt}\left(-F'(q_1,p_2,P_1,P_2,t) + q_2p_2 - Q_1P_1 - Q_2P_2\right)\\
&= \frac{d}{dt}\left(-F'(q_1,p_2,P_1,P_2,t)\right) + \frac{d}{dt}\left(q_2p_2\right) - \frac{d}{dt}\left(Q_1P_1\right) - \frac{d}{dt}\left(Q_2P_2\right)\\
&= \frac{d}{dt}\left(-F'(q_1,p_2,P_1,P_2,t)\right) + \dot{q}_2p_2 + q_2\dot{p}_2 - \dot{Q}_1P_1 - Q_1\dot{P}_1 - \dot{Q}_2P_2 - Q_2\dot{P}_2 \quad \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
\frac{dF'}{dt} &= p_1\dot{q}_1 - q_2\dot{p}_2 + Q_1\dot{P}_1 + Q_2\dot{P}_2 - (H-K)\\
&= \frac{\partial F'}{\partial q_1}\dot{q}_1 + \frac{\partial F'}{\partial p_2}\dot{p}_2 + \frac{\partial F'}{\partial P_1}\dot{P}_1 + \frac{\partial F'}{\partial P_2}\dot{P}_2 + \frac{\partial F'}{\partial t} \quad \Rightarrow
\end{aligned}
$$

$$
p_1 = \frac{\partial F'}{\partial q_1} \qquad q_2 = -\frac{\partial F'}{\partial p_2}
$$

$$
Q_1 = \frac{\partial F'}{\partial P_1} \qquad Q_2 = \frac{\partial F'}{\partial P_2} \qquad (19{,}3)
$$

$$
K = H + \frac{\partial F'}{\partial t}
$$

Ahora operando las ecuaciones (19.1) entonces:

$$
p_1 = P_1 + 2p_2 \qquad q_2 = -2q_1 - P_2 \quad (19{,}4{,}1)
$$
$$
Q_1 = q_1 \qquad Q_2 = p_2 \quad (19{,}4{,}2)
$$

Comparando (19.3) y (19.4):

$$
P_1 + 2p_2 = \frac{\partial F'}{\partial q_1} \qquad 2q_1 + P_2 = \frac{\partial F'}{\partial p_2}
$$
$$
q_1 = \frac{\partial F'}{\partial P_1} \qquad p_2 = \frac{\partial F'}{\partial P_2}
$$

Entonces:

$$
F'(q_1,p_2,P_1,P_2,t) = q_1P_1 + p_2P_2 + 2p_2q_1 \quad (19{,}5)
$$

Es la funcion generatriz que genera la transformacion canónica (19.1), pues se puede verificar que:

$$\begin{aligned}
p_1 &= \frac{\partial F'}{\partial q_1} = \frac{\partial}{\partial q_1}\left(q_1 P_1 + p_2 P_2 + 2p_2 q_1\right) = -P_1 - 2p_2 \\
q_2 &= -\frac{\partial F'}{\partial p_2} = -\frac{\partial}{\partial p_2}\left(q_1 P_1 + p_2 P_2 + 2p_2 q_1\right) = -P_2 - 2q_1 \\
Q_1 &= \frac{\partial F'}{\partial P_1} = \frac{\partial}{\partial P_1}\left(q_1 P_1 + p_2 P_2 + 2p_2 q_1\right) = q_1 \\
Q_2 &= \frac{\partial F'}{\partial P_2} = \frac{\partial}{\partial P_2}\left(q_1 P_1 + p_2 P_2 + 2p_2 q_1\right) = p_2
\end{aligned}$$

21. Determine la identidad de Jacobi:

$$\{A,\{B,C\}\} + \{C,\{A,B\}\} + \{B,\{C,A\}\} = 0$$

Respuesta a punto 21

De la definicion de bracket de Lagrange:

$$\{A,B\} \equiv \{A,B\}_{q,p} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \qquad (21,1)$$

Considerando $u = u(q_i, p_i)$ y $v = v(q_i, p_i)$, y usando la notacion matricial:

$$\frac{\partial u}{\partial \boldsymbol{\eta}} = \frac{\partial u}{\partial (q_i, p_i)} = \begin{pmatrix} \frac{\partial u}{\partial q_i} \\ \frac{\partial u}{\partial p_i} \end{pmatrix} \quad \text{y} \quad \frac{\partial v}{\partial \boldsymbol{\eta}} = \frac{\partial v}{\partial (q_i, p_i)} = \begin{pmatrix} \frac{\partial v}{\partial q_i} \\ \frac{\partial v}{\partial p_i} \end{pmatrix}$$

De esta forma

$$\begin{aligned}
[u,v] &= \tilde{\frac{\partial u}{\partial \boldsymbol{\eta}}} J \frac{\partial v}{\partial \boldsymbol{\eta}} \\
&= \begin{pmatrix} \frac{\partial u}{\partial q_i} & \frac{\partial u}{\partial p_i} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial q_i} \\ \frac{\partial v}{\partial p_i} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial u}{\partial q_i} & \frac{\partial u}{\partial p_i} \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial p_i} \\ -\frac{\partial v}{\partial q_i} \end{pmatrix} \\
&= \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}
\end{aligned}$$

En notacion tensorial $\partial_i u \equiv \frac{\partial u}{\partial \eta_i}$ y $\partial_i v \equiv \frac{\partial v}{\partial \eta_i}$ entonces:

$$[u,v] = \partial_i u J_{ij} \partial_j v$$

Por lo que para $A = A(q_i, p_i)$ y $B = B(q_i, p_i)$ y $C = C(q_i, p_i)$ funciones analiticas, se tiene que:

$$\begin{aligned}
\{A, \{B, C\}\} &= \{A, \partial_i B J_{ij} \partial_j C\} \\
&= \partial_i A J_{ij} \partial_j \left(\partial_k B J_{kl} \partial_l C\right) \\
&= \partial_i A J_{ij} \left(\partial_j \partial_k B J_{kl} \partial_l C + \partial_k B J_{kl} \partial_j \partial_l C\right) \\
&= \partial_i A J_{ij} \partial_j \partial_k B J_{kl} \partial_l C + \partial_i A J_{ij} \partial_k B J_{kl} \partial_j \partial_l C \\
&= \partial_l C J_{kl} \partial_i A J_{ij} \partial_k \partial_j B + \partial_k B J_{kl} \partial_j \partial_l C \partial_i A J_{ij} \\
&= -\partial_l C J_{lk} \partial_k \partial_j B \partial_i A J_{ij} + \partial_k B J_{kl} \partial_i A J_{ij} \partial_j \partial_l C \\
&= -\partial_l C J_{lk} \partial_k \partial_j B J_{ij} \partial_i A + \partial_k B J_{kl} \partial_i A J_{ij} \partial_j \partial_l C \\
&= \partial_l C J_{lk} \partial_k \partial_j B J_{ji} \partial_i A + \partial_l C J_{lk} \partial_j B J_{ji} \partial_k \partial_i A_i + \partial_l C J_{lk} \partial_j B J_{ij} \partial_k \partial_i A_i + \partial_k B J_{kl} \partial_i A J_{ij} \partial_l \partial_j C \\
&= \partial_l C J_{lk} \left(\partial_k \partial_j B J_{ji} \partial_i A + \partial_j B J_{ji} \partial_k \partial_i A\right) + \partial_j B J_{ji} \partial_l \partial_i A J_{lk} \partial_k C + \partial_k B J_{kl} \partial_i A J_{ij} \partial_l \partial_j C \\
&= \partial_l C J_{lk} \partial_k \left(\partial_j B J_{ji} \partial_i A\right) + \partial_k B J_{kl} \partial_i \partial_l A J_{ij} \partial_j C + \partial_k B J_{kl} \partial_i A J_{ij} \partial_l \partial_j C \\
&= -\partial_l C J_{lk} \partial_k \left(\partial_i A J_{ij} \partial_j B\right) + \partial_k B J_{kl} \partial_l (\partial_i A J_{ij} \partial_j C) \\
&= -\partial_l C J_{lk} \partial_k \{A, B\} - \partial_k B J_{kl} \partial_l \{C, A\} \\
&= -\{C, \{A, B\}\} - \{B, \{C, A\}\} \quad \Rightarrow
\end{aligned}$$

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 \quad (21{,}2)$$

$$J_{ij} J_{kl}$$

22. Reformule las ecuaciones de Hamilton en terminos de los Corchetes de Poisson.

Respuesta a Punto 22

Partiendo de las ecuaciones canonicas de Hamilton:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{y} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad (22{,}1)$$

Y sea $u = u(q_i, p_i, t)$, entonces:

$$\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial q_i} \dot{q}_i + \frac{\partial u}{\partial p_i} \dot{p}_i + \frac{\partial u}{\partial t} \\
&= \frac{\partial u}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t} \\
&= \{u, H\} + \frac{\partial u}{\partial t} \quad (22{,}2)
\end{aligned}$$

Si definimos: $\boldsymbol{\eta} \equiv (q_i(t), p_i(t))$, tal que $\dot{\boldsymbol{\eta}} = (\dot{q}_i, \dot{p}_i)$, entonces:

$$\begin{aligned}
\dot{\boldsymbol{\eta}} &= \begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q_i} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = J \frac{\partial H}{\partial \boldsymbol{\eta}} = \frac{\partial \tilde{\boldsymbol{\eta}}}{\partial \boldsymbol{\eta}} J \frac{\partial H}{\partial \boldsymbol{\eta}} \\
&= \{\boldsymbol{\eta}, H\} \quad \Rightarrow
\end{aligned}$$

$$\dot{\boldsymbol{\eta}} = \{\boldsymbol{\eta}, H\} \qquad (22{,}2)$$

Son las ecuaciones canonicas de Hamilton en corchetes de Poisson, con $\frac{\partial \boldsymbol{\eta}}{\partial t} = 0$, pues podemos notar que:

$$\{q_i, H\} = \frac{\partial \tilde{q_i}}{\partial \boldsymbol{\eta}} J \frac{\partial H}{\partial \boldsymbol{\eta}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = \frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$\{p_i, H\} = \frac{\partial \tilde{p_i}}{\partial \boldsymbol{\eta}} J \frac{\partial H}{\partial \boldsymbol{\eta}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = -\frac{\partial H}{\partial q_i} = \dot{p}_i$$

$$\{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial \tilde{H}}{\partial \boldsymbol{\eta}} J \frac{\partial H}{\partial \boldsymbol{\eta}} = \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = \frac{dH}{dt}$$

23. / PROBLEMA OPCIONAL. Determine las siguientes propiedades de los Corchetes de Poisson y el momentum angular:

$$\{q_i, L_j\} = \varepsilon_{ijk} q_k,$$
$$\{p_i, L_j\} = \varepsilon_{ijk} p_k,$$
$$\{L_i, L_j\} = \varepsilon_{ijk} L_k,$$

donde ε_{ijk} es el tensor de Levi-Civita.

Respuesta a Punto 23

Sean q_{ji} y p_{ji} coordenadas represntando las posiciones y momentos usuales, tales que $p_{ji} = m_i \dot{q}_{ji}$, con $i = 1, 2, 3$ entonces:

$$\{\mathbf{p}, \mathbf{L}\} = \frac{\partial \tilde{\mathbf{p}}}{\partial \boldsymbol{\eta}} J \frac{\partial \mathbf{L}}{\partial \boldsymbol{\eta}} = \frac{\tilde{\mathbf{p}}}{\partial \boldsymbol{\eta}} J \frac{\partial}{\partial \boldsymbol{\eta}} (\mathbf{x} \times \mathbf{p}) \qquad (23{,}1)$$

donde

$$
\frac{\partial \mathbf{C}}{\partial \boldsymbol{\eta}} = \begin{pmatrix} \frac{\partial C_x}{\partial q_i} & \frac{\partial C_y}{\partial q_i} & \frac{\partial C_z}{\partial q_i} \\ \frac{\partial C_x}{\partial p_i} & \frac{\partial C_y}{\partial p_i} & \frac{\partial C_z}{\partial p_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial (A_y B_z - B_y A_z)}{\partial q_i} & \frac{\partial (A_z B_x - B_x A_z)}{\partial q_i} & \frac{\partial (A_x B_y - A_y B_x)}{\partial q_i} \\ \frac{\partial (A_y B_z - B_y A_z)}{\partial p_i} & \frac{\partial (A_z B_x - B_x A_z)}{\partial p_i} & \frac{\partial (A_x B_y - A_y B_x)}{\partial p_i} \end{pmatrix}
$$

$$
\begin{aligned}
&= \begin{pmatrix} \frac{\partial A_y B_z}{\partial q_i} - \frac{\partial A_z B_y}{\partial q_i} & \frac{\partial A_z B_x}{\partial q_i} - \frac{\partial B_x A_z}{\partial q_i} & \frac{\partial A_x B_y}{\partial q_i} - \frac{\partial A_y B_x}{\partial q_i} \\ \frac{\partial A_y B_z}{\partial p_i} - \frac{\partial A_z B_y}{\partial p_i} & \frac{\partial A_z B_x}{\partial p_i} - \frac{\partial B_x A_z}{\partial p_i} & \frac{\partial A_x B_y}{\partial p_i} - \frac{\partial A_y B_x}{\partial p_i} \end{pmatrix} \\
&= \begin{pmatrix} A_y \frac{\partial B_z}{\partial q_i} + B_z \frac{\partial A_y}{\partial q_i} & A_z \frac{\partial B_x}{\partial q_i} + B_x \frac{\partial A_z}{\partial q_i} & A_x \frac{\partial B_y}{\partial q_i} + B_y \frac{\partial A_x}{\partial q_i} \\ A_y \frac{\partial B_z}{\partial p_i} + B_z \frac{\partial A_y}{\partial p_i} & A_z \frac{\partial B_x}{\partial p_i} + B_x \frac{\partial A_z}{\partial p_i} & A_x \frac{\partial B_y}{\partial p_i} + B_y \frac{\partial A_x}{\partial p_i} \end{pmatrix} \\
&- \begin{pmatrix} A_z \frac{\partial B_y}{\partial q_i} + B_y \frac{\partial A_z}{\partial q_i} & A_x \frac{\partial B_z}{\partial q_i} + B_z \frac{\partial A_x}{\partial q_i} & A_y \frac{\partial B_x}{\partial q_i} + B_x \frac{\partial A_y}{\partial q_i} \\ A_z \frac{\partial B_y}{\partial p_i} + B_y \frac{\partial A_z}{\partial p_i} & A_x \frac{\partial B_z}{\partial p_i} + B_z \frac{\partial A_x}{\partial p_i} & A_y \frac{\partial B_x}{\partial p_i} + B_x \frac{\partial A_y}{\partial p_i} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{pmatrix} A_y \frac{\partial B_z}{\partial q_i} & A_z \frac{\partial B_x}{\partial q_i} & A_x \frac{\partial B_y}{\partial q_i} \\ A_y \frac{\partial B_z}{\partial p_i} & A_z \frac{\partial B_x}{\partial p_i} & A_x \frac{\partial B_y}{\partial p_i} \end{pmatrix} + \begin{pmatrix} B_z \frac{\partial A_y}{\partial q_i} & B_x \frac{\partial A_z}{\partial q_i} & B_y \frac{\partial A_x}{\partial q_i} \\ B_z \frac{\partial A_y}{\partial p_i} & B_x \frac{\partial A_z}{\partial p_i} & B_y \frac{\partial A_x}{\partial p_i} \end{pmatrix} \\
&- \begin{pmatrix} A_z \frac{\partial B_y}{\partial q_i} & A_x \frac{\partial B_z}{\partial q_i} & A_y \frac{\partial B_x}{\partial q_i} \\ A_z \frac{\partial B_y}{\partial p_i} & A_x \frac{\partial B_z}{\partial p_i} & A_y \frac{\partial B_x}{\partial p_i} \end{pmatrix} - \begin{pmatrix} B_y \frac{\partial A_z}{\partial q_i} & B_z \frac{\partial A_x}{\partial q_i} & B_x \frac{\partial A_y}{\partial q_i} \\ B_y \frac{\partial A_z}{\partial p_i} & B_z \frac{\partial A_x}{\partial p_i} & B_x \frac{\partial A_y}{\partial p_i} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_y & 0 & 0 \\ 0 & A_z & 0 \\ 0 & 0 & A_x \end{pmatrix} + \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} B_z & 0 & 0 \\ 0 & B_x & 0 \\ 0 & 0 & B_y \end{pmatrix} \\
&- \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_z & 0 & 0 \\ 0 & A_x & 0 \\ 0 & 0 & A_y \end{pmatrix} - \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} B_y & 0 & 0 \\ 0 & B_z & 0 \\ 0 & 0 & B_x \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}} \begin{pmatrix} 0 & A_z & 0 \\ 0 & 0 & A_x \\ A_y & 0 & 0 \end{pmatrix} + \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}} \begin{pmatrix} 0 & 0 & B_y \\ B_z & 0 & 0 \\ 0 & B_x & 0 \end{pmatrix} \\
&- \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}} \begin{pmatrix} 0 & 0 & A_y \\ A_z & 0 & 0 \\ 0 & A_x & 0 \end{pmatrix} - \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}} \begin{pmatrix} 0 & B_z & 0 \\ 0 & 0 & B_x \\ B_y & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
= \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}} \begin{pmatrix} 0 & A_z & -A_y \\ -A_z & 0 & A_x \\ A_y & -A_x & 0 \end{pmatrix} + \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}} \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix}
$$

En terminos del algebra de Dyadicas se pude escribir:

$$\begin{aligned}
\frac{\partial \mathbf{C}}{\partial \boldsymbol{\eta}} &= \frac{\partial}{\partial \boldsymbol{\eta}}(\mathbf{A}\times\mathbf{B}) \\
&= \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}}\left(A_z\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 - A_y\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 + A_x\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 - A_z\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + A_y\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 - A_x\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2\right) \\
&+ \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}}\left(-B_z\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + B_y\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 - B_x\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 + B_z\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 - B_y\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + B_x\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2\right) \\
&= \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}}\left(A_z\left(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1\right) + A_y\left(\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_1\hat{\mathbf{e}}_3\right) + A_x\left(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 - \hat{\mathbf{e}}_3\hat{\mathbf{e}}_2\right)\right) \\
&+ \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}}\left(-B_z\left(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1\right) - B_y\left(\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_1\hat{\mathbf{e}}_3\right) - B_x\left(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 - \hat{\mathbf{e}}_3\hat{\mathbf{e}}_2\right)\right) \\
&= \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}}\left(A_i\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k\epsilon_{ijk}\right) - \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}}\left(B_i\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k\epsilon_{ijk}\right) \\
&= \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}}\left(A_i\hat{\mathbf{e}}_i\cdot\left(\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right)\epsilon_{jkl}\right) - \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}}\left(B_i\hat{\mathbf{e}}_i\cdot\left(\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k\hat{\mathbf{e}}_l\right)\epsilon_{jkl}\right) \\
&= \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}}\left(\mathbf{A}\cdot\begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\end{vmatrix}\right) - \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}}\left(\mathbf{B}\cdot\begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\end{vmatrix}\right) \\
&= \frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}}\left(\mathbf{A}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right) - \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}}\left(\mathbf{B}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right) \quad \Rightarrow
\end{aligned}$$

$$\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{A}\times\mathbf{B}\right) = \frac{\partial \mathbf{B}}{\partial \mathbf{x}}\left(\mathbf{A}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right) - \frac{\partial \mathbf{A}}{\partial \mathbf{x}}\left(\mathbf{B}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right)$$

Es claro que

$$\bar{\bar{\boldsymbol{\epsilon}}} = \begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\end{vmatrix} = \hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k\epsilon_{ijk}$$

Es el pseudo tensor de rango 3 de Levi-Civita. Ademas:

$$\begin{aligned}
\frac{\tilde{\partial}}{\partial \boldsymbol{\eta}}\left(\mathbf{A}\times\mathbf{B}\right) &= \left[\frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}}\left(\mathbf{A}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right) - \frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}}\left(\mathbf{B}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right)\right]^T \\
&= \left[\frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}}\left(\mathbf{A}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right)\right]^T - \left[\frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}}\left(\mathbf{B}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right)\right]^T \\
&= \left(\mathbf{A}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right)^T\left(\frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}}\right)^T - \left(\mathbf{B}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right)^T\left(\frac{\partial \mathbf{A}}{\partial \boldsymbol{\eta}}\right)^T \\
&= \left(\mathbf{B}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right)\frac{\partial \tilde{\mathbf{A}}}{\partial \boldsymbol{\eta}} - \left(\mathbf{A}\cdot\bar{\bar{\boldsymbol{\epsilon}}}\right)\frac{\partial \tilde{\mathbf{B}}}{\partial \boldsymbol{\eta}} \quad (23{,}2)
\end{aligned}$$

De este modo (23.1) se puede escribir como:

$$
\begin{aligned}
\{\mathbf{p},\mathbf{L}\} &= \frac{\partial\tilde{\mathbf{p}}}{\partial\boldsymbol{\eta}} J \frac{\partial\mathbf{L}}{\partial\boldsymbol{\eta}} = \frac{\partial\tilde{\mathbf{p}}}{\partial\boldsymbol{\eta}} J \frac{\partial}{\partial\boldsymbol{\eta}}(\mathbf{q}\times\mathbf{p}) \\
&= \frac{\partial\tilde{\mathbf{p}}}{\partial\boldsymbol{\eta}} J \left[\frac{\partial\mathbf{p}}{\partial\boldsymbol{\eta}}(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) - \frac{\partial\mathbf{q}}{\partial\boldsymbol{\eta}}(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}})\right] \\
&= \frac{\partial\tilde{\mathbf{p}}}{\partial\boldsymbol{\eta}} J \frac{\partial\mathbf{p}}{\partial\boldsymbol{\eta}}(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) - \frac{\partial\tilde{\mathbf{p}}}{\partial\boldsymbol{\eta}} J \frac{\partial\mathbf{q}}{\partial\boldsymbol{\eta}}(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) \\
&= \cancelto{0}{\{\mathbf{p},\mathbf{p}\}}(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) - \{\mathbf{p},\mathbf{q}\}(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) \\
&= \cancelto{\mathbb{I}}{\{\mathbf{q},\mathbf{p}\}}(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) = \mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}}
\end{aligned}
$$

Y de forma analoga

$$
\begin{aligned}
\{\mathbf{q},\mathbf{L}\} &= \frac{\partial\tilde{\mathbf{q}}}{\partial\boldsymbol{\eta}} J \frac{\partial\mathbf{L}}{\partial\boldsymbol{\eta}} = \frac{\partial\tilde{\mathbf{q}}}{\partial\boldsymbol{\eta}} J \frac{\partial}{\partial\boldsymbol{\eta}}(\mathbf{q}\times\mathbf{p}) \\
&= \frac{\partial\tilde{\mathbf{q}}}{\partial\boldsymbol{\eta}} J \left[\frac{\partial\mathbf{p}}{\partial\boldsymbol{\eta}}(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) - \frac{\partial\mathbf{q}}{\partial\boldsymbol{\eta}}(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}})\right] \\
&= \frac{\partial\tilde{\mathbf{q}}}{\partial\boldsymbol{\eta}} J \frac{\partial\mathbf{p}}{\partial\boldsymbol{\eta}}(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) - \frac{\partial\tilde{\mathbf{q}}}{\partial\boldsymbol{\eta}} J \frac{\partial\mathbf{q}}{\partial\boldsymbol{\eta}}(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) \\
&= -\cancelto{0}{\{\mathbf{q},\mathbf{q}\}}(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) + \{\mathbf{q},\mathbf{p}\}(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) \\
&= \cancelto{\mathbb{I}}{\{\mathbf{q},\mathbf{p}\}}(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) = (\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}})
\end{aligned}
$$

$$
\begin{aligned}
\{\mathbf{L},\mathbf{L}\} &= \frac{\partial\tilde{\mathbf{L}}}{\partial\boldsymbol{\eta}} J \frac{\partial\mathbf{L}}{\partial\boldsymbol{\eta}} = \frac{\partial\tilde{\mathbf{L}}}{\partial\boldsymbol{\eta}} J \frac{\partial}{\partial\boldsymbol{\eta}}(\mathbf{q}\times\mathbf{p}) \\
&= \frac{\partial\tilde{\mathbf{L}}}{\partial\boldsymbol{\eta}} J \left[\frac{\partial\mathbf{p}}{\partial\boldsymbol{\eta}}(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) - \frac{\partial\mathbf{q}}{\partial\boldsymbol{\eta}}(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}})\right] \\
&= \frac{\partial\tilde{\mathbf{L}}}{\partial\boldsymbol{\eta}} J \frac{\partial\mathbf{p}}{\partial\boldsymbol{\eta}}(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) - \frac{\partial\tilde{\mathbf{L}}}{\partial\boldsymbol{\eta}} J \frac{\partial\mathbf{q}}{\partial\boldsymbol{\eta}}(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) \\
&= \{\mathbf{L},\mathbf{p}\}(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) - \{\mathbf{L},\mathbf{q}\}(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) \\
&= -(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}})(\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) + (\mathbf{q}\cdot\bar{\bar{\boldsymbol{\epsilon}}})(\mathbf{p}\cdot\bar{\bar{\boldsymbol{\epsilon}}}) \\
&= (\mathbf{q}\mathbf{p} - \mathbf{p}\mathbf{q})\cdot\bar{\bar{\boldsymbol{\epsilon}}} \\
&= \mathbf{L}\cdot\bar{\bar{\boldsymbol{\epsilon}}}
\end{aligned}
$$

Conclusion: Todos aquellos tensores de rango 2 tales que:

$$\mathbf{T} = \mathbf{t}\cdot\bar{\bar{\boldsymbol{\epsilon}}} \tag{1}$$

para algun vector $\mathbf{t}$ son pseudotensores antisimetricos y ¿son invariantes bajo transformaciones canonicas?.

¿Es valida el resultado (23.2) para cualquier tensores de rango mayor que 1? permitiendo de esta forma definir el producto cruz para tensores de rango superior a 1, utilizando el tensor de Levi-Civita de rango correspondiente

24. Resolver el problema del movimiento de un proyectil puntiforme en un plano vertical, utilizando el método de Hamilton-Jacobi. Hallar la ecuación de la trayectoria y la dependencia del tiempo de las

coordenadas, suponiendo que el proyectil se ha disparado en el instante $t = 0$ desde el origen con una velocidad v_0 que forma un ángulo α con la horizontal.

Respuesta a Punto 10

La entropia esta definida por:

$$
\begin{aligned}
dQ = TdS \quad & \Rightarrow \quad \left(\frac{dQ}{dT}\right)_{\mathscr{H}} = T\left(\frac{dS}{dT}\right)_{\mathscr{H}} \\
& \Rightarrow \quad C_{\mathscr{H}} = T\left(\frac{dS}{dT}\right)_{\mathscr{H}} \\
& \Rightarrow \quad dS = \frac{C_{\mathscr{H}}}{T}dT \\
& \Rightarrow \quad dS = \left(\frac{B + C\mathscr{H}_0^2}{T^2} + DT^2\right)\frac{dT}{T} \\
& \Rightarrow \quad S = \int_{T_i}^{T_f}\left(\frac{B + C\mathscr{H}_0^2}{T^2} + DT^2\right)\frac{dT}{T} \\
& \Rightarrow \quad S = \int_{T_i}^{T_f}\left(\frac{B + C\mathscr{H}_0^2}{T^3} + DT\right)dT \\
& \Rightarrow \quad S = \left[-\frac{B + C\mathscr{H}_0^2}{2T^2} + D\frac{T^2}{2}\right]_{T_i}^{T_f} \\
& \Rightarrow \quad S = -\frac{B + C\mathscr{H}_0^2}{2T_f^2} + D\frac{T_f^2}{2} + \frac{B + C\mathscr{H}_0^2}{2T_i^2} - D\frac{T_i^2}{2} \\
& \Rightarrow \quad S = \frac{B + C\mathscr{H}_0^2}{2}\left(\frac{1}{T_i^2} - \frac{1}{T_f^2}\right) + D\left(\frac{T_f^2}{2} - \frac{T_i^2}{2}\right) \\
& \Rightarrow \quad S = \frac{B + C\mathscr{H}_0^2}{2}\left(\frac{T_f^2 - T_i^2}{T_i^2 T_f^2}\right) + D\left(\frac{T_f^2 - T_i^2}{2}\right) \\
& \Rightarrow \quad S = \frac{T_f^2 - T_i^2}{2}\left[\left(\frac{B + C\mathscr{H}_0^2}{T_i^2 T_f^2}\right) + D\left(T_f^2 - T_i^2\right)\right]
\end{aligned}
$$