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Taller 3

15. Demostrar directamente que la transformación

$$Q = \arctan \frac{\alpha q}{p},$$
$$P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

es canónica, donde α es una constante.

Respuesta a Problema 1

Sean en (p, q) la coordenadas generalizadas de un sistema en el espacion de fase, y sea $H(p, q)$ la funcion de Hamilton, entonces las ecuaciones de Hamilton son:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \quad \text{y} \quad \frac{\partial H}{\partial t} = 0 \quad (15{,}1)$$

Dado que la transformacion puntual de coordenadas generalizadas (p, q) a coordenadas generalizadas (P, Q) dada por:

$$\begin{aligned} Q &= \arctan \frac{\alpha q}{p}, \\ P &= \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right) = \frac{1}{2}\left(\alpha q^2 + \frac{p^2}{\alpha}\right) \end{aligned} \quad (15{,}2)$$

es canónica, entonces deben satisfacer las ecuaciones de Hamilton (15.1) con $K(P, Q) = K$ constante, es decir:

$$\dot{P} = -\frac{\partial K}{\partial Q} \quad \text{y} \quad \dot{Q} = \frac{\partial K}{\partial P} \quad \text{y} \quad \frac{\partial K}{\partial t} = 0 \quad (15{,}3)$$

Para mostrar esto ultimo, usando la regla de la cadena en la ecuaciones (15.2) y reemplazando (15.1)

$$\begin{aligned} \frac{dQ}{dt} &= \frac{\partial Q}{\partial q}\frac{dq}{dt} + \frac{\partial Q}{\partial p}\frac{dp}{dt} \\ &= -\frac{\alpha q}{p^2 + \alpha^2 q^2}\left(-\frac{\partial H}{\partial q}\right) + \frac{\alpha p}{p^2 + \alpha^2 q^2}\left(\frac{\partial H}{\partial p}\right) \\ &= \frac{\alpha}{p^2 + \alpha^2 q^2}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \\ &= \frac{1}{2}\frac{2\alpha}{p^2 + \alpha^2 q^2}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \\ &= \frac{1}{2P}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \quad (15{,}4{,}1) \end{aligned}$$

$$\begin{aligned} \frac{dP}{dt} &= \frac{\partial P}{\partial q}\frac{dq}{dt} + \frac{\partial P}{\partial p}\frac{dp}{dt} \\ &= \frac{p}{\alpha}\left(-\frac{\partial H}{\partial q}\right) + q\alpha\left(\frac{\partial H}{\partial p}\right) \quad (15{,}4{,}2) \end{aligned}$$

Aplicando la regla de la cadena en (15.1) tenemos que:

$$\begin{aligned}\dot{p} = -\frac{\partial H}{\partial q} &= -\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} - \frac{\partial H}{\partial P}\frac{\partial P}{\partial q} \\ &= -\frac{\partial H}{\partial Q}\frac{\alpha p}{p^2+\alpha^2 q^2} - \frac{\partial H}{\partial P}\alpha q \quad (15{,}7{,}1)\end{aligned}$$

$$\begin{aligned}\dot{q} = \frac{\partial H}{\partial p} &= \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p} \\ &= -\frac{\partial H}{\partial Q}\frac{\alpha q}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p}{\alpha} \quad (15{,}7{,}2)\end{aligned}$$

Reemplazando (15.5) en (15.7.1) y (15.7.2) tenemos que:

$$\begin{aligned}\dot{Q} &= \frac{1}{2P}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \\ &= \frac{1}{2P}\left(q\left(\frac{\partial H}{\partial Q}\frac{\alpha p}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\alpha q\right) + p\left(-\frac{\partial H}{\partial Q}\frac{\alpha q}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p}{\alpha}\right)\right) \\ &= \frac{1}{2P}\left(\frac{\partial H}{\partial Q}\frac{\alpha pq}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\alpha q^2 - \frac{\partial H}{\partial Q}\frac{\alpha pq}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p^2}{\alpha}\right) \\ &= \frac{1}{2P}\left(\frac{\partial H}{\partial P}\alpha q^2 + \frac{\partial H}{\partial P}\frac{p^2}{\alpha}\right) \\ &= \frac{1}{2P}\frac{\partial H}{\partial P}\left(\alpha q^2 + \frac{p^2}{\alpha}\right) \\ &= \frac{1}{2P}\frac{\partial H}{\partial P}2P = \frac{\partial H}{\partial P}\end{aligned}$$

$$\begin{aligned}\dot{P} &= \frac{p}{\alpha}\left(-\frac{\partial H}{\partial q}\right) + q\alpha\left(\frac{\partial H}{\partial p}\right) \\ &= \frac{p}{\alpha}\left(-\frac{\partial H}{\partial Q}\frac{\alpha p}{p^2+\alpha^2 q^2} - \frac{\partial H}{\partial P}\alpha q\right) + q\alpha\left(-\frac{\partial H}{\partial Q}\frac{\alpha q}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p}{\alpha}\right) \\ &= -\frac{\partial H}{\partial Q}\frac{p^2}{p^2+\alpha^2 q^2} - \frac{\partial H}{\partial P}pq - \frac{\partial H}{\partial Q}\frac{\alpha^2 q^2}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}pq \\ &= -\frac{\partial H}{\partial Q}\frac{p^2+\alpha^2 q^2}{p^2+\alpha^2 q^2} \\ &= -\frac{\partial H}{\partial Q}\end{aligned}$$

Por lo que si $K = H(P,Q)$ entonces las anterioes ecuaciones son las ecuaciones de Hamilton (15.3) y por lo tanto la transformacion (15.2) es canónica. Vease que como $H = H(q,p)$ no depende de t entonces $K = H(P,Q)$ tampoco depende de t, es decir $\frac{\partial K}{\partial t} = 0$.

16. Mostrar que una funcion generatriz del segundo tipo cuya forma particular sea $F_2 = q_j P_j$, genera la transformacion identidad

Respuesta a Punto 2

De las ecuaciones de transformacion asociadas a al funcion generatriz $F_2(q_j, P_j) \quad j = 1, \dots n$ dadas por:

$$Q_j = \frac{\partial F_2}{\partial P_j}, \quad p_j = \frac{\partial F_2}{\partial q_j} \quad \text{y} \quad K = H + \frac{\partial F_2}{\partial t} \quad (16{,}1)$$

Se tiene que:

$$\frac{\partial F_2}{\partial t} = 0 \quad \Rightarrow \quad K = H(Q_j, P_j)$$
$$Q_j = \frac{\partial F_2}{\partial P_j} = \frac{\partial}{\partial P_j}(P_j q_j) = q_j$$
$$p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial}{\partial q_j}(P_j q_j) = P_j$$

De esta forma la matriz de transformacion M es tal que:

$$M = \begin{pmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

17. Use una funci´on generatriz para construir una transformaci´on que intercambie cantidades de movimiento y coorde- nadas.

Respuesta a Punto 3

Dado que se busca una funcion generatriz $F' = F'(q_j, p_k, Q_l, P_m, t), \quad j = l, k = m = 1, .., n$ o $j = m, l = m = 1, .., n$, para algun j, k, l, m tal que la transformacion canonica M asociada satisfaga que:

$$\dot{\mathbf{X}} = M\dot{\mathbf{x}} \quad (17{,}1)$$

Donode:

$$\mathbf{X} = \begin{pmatrix} Q_j \\ P_j \end{pmatrix} = \begin{pmatrix} p_j \\ q_j \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} q_j \\ p_j \end{pmatrix} \quad \text{y} \quad M = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i} \\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix} \quad (17{,}2)$$

Por lo que de (17.2):

$$M = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i} \\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial p_j}{\partial q_i} & \frac{\partial p_j}{\partial p_i} \\ \frac{\partial q_j}{\partial q_i} & \frac{\partial q_j}{\partial p_i} \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad (17{,}3)$$

Vease que M de (17.3) no una transformacion canonica, pues:

$$M^T J M = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}^T \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \neq J$$

Por lo que en su lugar se propone:

$$\mathbf{X} = \begin{pmatrix} Q_j \\ P_j \end{pmatrix} = \begin{pmatrix} p_j \\ -q_j \end{pmatrix} \quad \text{y} \quad M = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i} \\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix} \quad (17{,}4)$$

Asi de (17.4):

$$M=\begin{pmatrix}\frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i}\\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i}\end{pmatrix}=\begin{pmatrix}\frac{\partial p_j}{\partial q_i} & \frac{\partial p_j}{\partial p_i}\\ \frac{\partial(-q_j)}{\partial q_i} & \frac{\partial(-q_j)}{\partial p_i}\end{pmatrix}=\begin{pmatrix}0 & \delta_{ij}\\ -\delta_{ij} & 0\end{pmatrix}=\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}\quad (17{,}5)$$

Vease que M de (17.5) es una transformacion canonica, pues:

$$M^TJM=\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}^T\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}=\begin{pmatrix}0 & -\mathbf{1}\\ \mathbf{1} & 0\end{pmatrix}\begin{pmatrix}-\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}=\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}=J$$

Una una funcion generatriz que me genera este tipo de transformacion canonica es:

$$F(q_j,Q_j,t)=q_jQ_j\quad (17{,}6)$$

Pues de (17.6) se tiene que:

$$p_i=\frac{\partial F}{\partial q_i}=\frac{\partial}{\partial q_i}(q_jQ_j)=Q_i\quad \text{y}\quad P_i=-\frac{\partial F}{\partial Q_i}=-\frac{\partial}{\partial Q_i}(q_jQ_j)=-q_i$$

18. Las ecuaciones de transformación entre dos sistemas de coordenadas son

$$Q=\log\left(1+q^{1/2}\cos p\right)$$
$$P=2\left(1+q^{1/2}\cos p\right)q^{1/2}\sin p$$

a) A partir de estas ecuaciones de transformación, demostrar directamente que Q,P son variables canónicas si lo son q y p. b) Demostrar que la función que genera esta transformación es

$$F_3=-\left(e^Q-1\right)^2\tan p$$

Respuesta a Punto 4

- Si (q,p) son variables canónicas entonces satisfacen las ecuaciones canonicas de Hamilton, es decir:

$$\dot q=\frac{\partial H}{\partial p}\quad \text{y}\quad \dot p=-\frac{\partial H}{\partial q}\quad (18{,}1)$$

Si ademas define la transformacion:

$$\begin{aligned}Q&=\log\left(1+q^{1/2}\cos p\right)\\ P&=2\left(1+q^{1/2}\cos p\right)q^{1/2}\sin p\end{aligned}\quad (18{,}2)$$

Entonces se tiene que por la regla de la cadena aplicada en (18.1) y (18.2):

$$\begin{aligned}\dot q=\frac{\partial H}{\partial p}&=\frac{\partial H}{\partial P}\frac{\partial P}{\partial p}+\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p}\\ &=\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p+q\cos(2p)\right)-\frac{\partial H}{\partial Q}\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p}\\ &=\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p+q\cos(2p)\right)-\frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\quad (18{,}3{,}1)\end{aligned}$$

$$
\begin{aligned}
\dot{p} &= -\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial P}\frac{\partial P}{\partial q} - \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} \\
&= -\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p}{2(q^{1/2}+q\cos p)} \\
&= -\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p \sin p}{P} \quad (18{,}3{,}2)
\end{aligned}
$$

$$
\begin{aligned}
\dot{Q} &= \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} \\
&= \frac{\cos p}{2(q^{1/2}+q\cos p)}\dot{q} - \frac{q^{1/2}\sin p}{1+q^{1/2}\cos p}\dot{p} \\
&= \frac{\cos p \sin p}{P}\dot{q} - \frac{2q\sin^2 p}{P}\dot{p} \quad (18{,}4{,}1)
\end{aligned}
$$

$$
\begin{aligned}
\dot{P} &= \frac{\partial P}{\partial q}\dot{q} + \frac{\partial P}{\partial p}\dot{p} \\
&= \left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p\dot{q} + 2\left(q^{1/2}\cos p + q\cos(2p)\right)\dot{p} \quad (18{,}4{,}2)
\end{aligned}
$$

Reemplazando (18.3.1) y (18.3.2) en (18.4.1) y (18.4.2):

$$
\begin{aligned}
\dot{Q} &= \frac{\cos p \sin p}{P}\left(\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos(2p)\right) - \frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\right) \\
&\quad - \frac{2q\sin^2 p}{P}\left(-\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\right) \\
&= \frac{\cos p \sin p}{P}\left(\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos^2 p - q\sin^2 p\right) - \frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\right) \\
&\quad + \frac{2q\sin^2 p}{P}\left(\frac{\partial H}{\partial P}\left(\frac{1+q^{1/2}\cos p}{q^{1/2}} + \cos p\right)\sin p + \frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\right) \\
&= \frac{\cos p \sin p}{P}\left(\frac{\partial H}{\partial P}2\left(q^{1/2} + q\cos p\right)\frac{\sin p}{\sin p}\cos p - \frac{\partial H}{\partial P}2\left(q\sin^2 p\right)\right) - \cancel{\frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P^2}} \\
&\quad + \frac{2q\sin^2 p}{P}\left(\frac{\partial H}{\partial P}2\left(\frac{q^{1/2}+q\cos p}{2q}\sin p\right) + \frac{\partial H}{\partial P}\cos p\sin p\right) + \cancel{\frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P^2}} \\
&= \frac{\cos p \sin p}{P}\left(\frac{\partial H}{\partial P}P\frac{\cos p}{\sin p} - \frac{\partial H}{\partial P}2\left(q\sin^2 p\right)\right) + \frac{2q\sin^2 p}{P}\left(\frac{\partial H}{\partial P}\left(\frac{P}{2q}\right) + \frac{\partial H}{\partial P}\cos p\sin p\right) \\
&= \frac{\partial H}{\partial P}\cos^2 p - \cancel{2\frac{\partial H}{\partial P}q\sin^3 p\cos p\frac{1}{P}} + \frac{\partial H}{\partial P}\sin^2 p + \cancel{2\frac{\partial H}{\partial P}\cos p\sin^3 p\frac{1}{P}} \\
&= \frac{\partial H}{\partial P} \quad (18{,}5{,}1)
\end{aligned}
$$

$$\dot{P} = \left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p\left(\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos(2p)\right) - \frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\right)$$
$$+ 2\left(q^{1/2}\cos p + q\cos(2p)\right)\left(-\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\right)$$

$$= \cancel{\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos(2p)\right)} - \left(\frac{1}{q^{1/2}} + 2\cos p\right)\frac{\partial H}{\partial Q}\frac{2q\sin^3 p}{P}$$
$$+ \cancel{2\left(q^{1/2}\cos p + q\cos(2p)\right)\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)}\sin p + 2\left(q^{1/2}\cos p + q\cos(2p)\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}$$

$$= -\left(\frac{1 + q^{1/2}\cos p}{q^{1/2}} + \cos p\right)\sin p\frac{\partial H}{\partial Q}\frac{2q\sin^3 p}{P} - 2\left(q^{1/2}\cos p + q\cos^2 p - q\sin^2 p\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}$$

$$= -\left(\frac{P}{2q} + \cos p\sin p\right)\frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P} - \left(2\left(q^{1/2} + q\cos p\right)\frac{\sin p}{\sin p}\cos p - 2\left(q\sin^2 p\right)\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}$$
$$= -\frac{\partial H}{\partial Q}\sin^2 - \frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P} - \left(P\frac{\cos p}{\sin p} - 2\left(q\sin^2 p\right)\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}$$
$$= -\frac{\partial H}{\partial Q}\sin^2 + \cancel{\frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P}} - \frac{\partial H}{\partial Q}\cos^2 p - \cancel{\frac{\partial H}{\partial Q}\frac{2q\cos p\sin^3 p}{P}}$$
$$= -\frac{\partial H}{\partial Q} \quad (18,5,2)$$

Por que si $K = H(Q,P)$ entonces las anterioes ecuaciones son las ecuaciones canonicas de Hamilton y por lo tanto (P,Q) son variables canonicas.

- Dado que $F_3 = F_3(p,Q)$, pues:

$$F_3 = -\left(e^Q - 1\right)^2\tan p$$

Entonces de la ecuaciones de canonicas de transformacion para F_3:

$$q = -\frac{\partial F_3}{\partial p} \quad \text{y} \quad P = \frac{\partial F_3}{\partial Q} \quad (18,6)$$

se sigue que:

$$q = -\frac{\partial F_3}{\partial p} = -\frac{\partial}{\partial p}\left(-\left(e^Q - 1\right)^2\tan p\right) = \left(e^Q - 1\right)^2\sec^2 p \quad (18,7,1)$$
$$P = -\frac{\partial F_3}{\partial Q} = -\frac{\partial}{\partial Q}\left(-\left(e^Q - 1\right)^2\tan p\right) = 2\left(e^Q - 1\right)e^Q\tan p \quad (18,7,2)$$

Despejando Q de (18.7.1):

$$\begin{aligned}
q = \left(e^Q - 1\right)^2 \sec^2 p \quad &\Rightarrow \quad \left(e^Q - 1\right)^2 = \frac{q}{\sec^2 p} \\
&\Rightarrow \quad e^Q - 1 = \sqrt{\frac{q}{\sec^2 p}} \\
&\Rightarrow \quad e^Q = 1 + \sqrt{\frac{q}{\sec^2 p}} \\
&\Rightarrow \quad Q = \ln\left(1 + \sqrt{\frac{q}{\sec^2 p}}\right) \\
&\Rightarrow \quad Q = \ln\left(1 + q^{1/2}\cos p\right) \quad (18{,}8{,}1)
\end{aligned}$$

Reemplazando (18.8.1) en (18.7.2):

$$\begin{aligned}
P = 2\left(e^Q - 1\right)e^Q \tan p \quad &\Rightarrow \quad P = -2\left(1 - q^{1/2}\cos p - 1\right)\left(1 - q^{1/2}\cos p\right)\tan p \\
&\Rightarrow \quad P = -2(1 - 2q^{1/2}\cos p + q\cos^2 p - 1 + q^{1/2}\cos p)\tan p \\
&\Rightarrow \quad P = -2\left(q^{1/2}\cos p + q\cos^2 p\right)\tan p \\
&\Rightarrow \quad P = -2q^{1/2}\cos p\left(1 + q^{1/2}\cos p\right)\tan p \quad (18{,}8{,}2)
\end{aligned}$$

De (18.8.1) y (18.8.2) se concluye que F_3 es una funcion generatriz de (18.2)

19. Probar directamente que la transformación

$$\begin{aligned}
Q_1 = q_1, \quad P_1 = p_1 - 2p_2 \\
Q_2 = p_2, \quad P_2 = -2q_1 - q_2
\end{aligned}$$

es canónica y hallar una función generatriz.

Respuesta a Punto 19

Si de las ecuaciones:

$$\begin{aligned}
Q_1 = q_1, \quad P_1 = p_1 - 2p_2 \\
Q_2 = p_2, \quad P_2 = -2q_1 - q_2
\end{aligned} \qquad (19{,}1)$$

Calculamos la matriz de transformación M:

$$M = \begin{pmatrix} \frac{\partial Q_1}{\partial q_1} & \frac{\partial Q_1}{\partial q_2} & \frac{\partial Q_1}{\partial p_1} & \frac{\partial Q_1}{\partial p_2} \\ \frac{\partial Q_2}{\partial q_1} & \frac{\partial Q_2}{\partial q_2} & \frac{\partial Q_2}{\partial p_1} & \frac{\partial Q_2}{\partial p_2} \\ \frac{\partial P_1}{\partial q_1} & \frac{\partial P_1}{\partial q_2} & \frac{\partial P_1}{\partial p_1} & \frac{\partial P_1}{\partial p_2} \\ \frac{\partial P_2}{\partial q_1} & \frac{\partial P_2}{\partial q_2} & \frac{\partial P_2}{\partial p_1} & \frac{\partial P_2}{\partial p_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix}$$

Entonces la matriz M es una transformación canónica si y solo si:

$$M^T J M = J \quad J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Lo cual podemos comprobar:

$$
\begin{aligned}
M^T J M &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = J
\end{aligned}
$$

Por lo tanto M es una transformación canónica.

Considere la funcion generatriz $F' = F'(q_1, p_2, P_1, P_2, t)$ tal que:

$$F(q_1, q_2, Q_1, Q_2, t) \rightarrow F''(q_1, p_2, Q_1, Q_2, t) \quad \Rightarrow$$

$$
\begin{aligned}
F''(q_1, p_1, Q_1, Q_1, t) &= q_2 \frac{\partial F}{\partial q_2} - F \\
&= q_2 p_2 - F
\end{aligned}
$$

$$F''(q_1, p_2, Q_1, Q_2, t) \rightarrow F'''(q_1, p_2, P_1, Q_2, t) \quad \Rightarrow$$

$$
\begin{aligned}
F'''(q_1, p_2, P_1, Q_2, t) &= Q_1 \frac{\partial F''}{\partial Q_1} - F'' \\
&= -Q_1 \frac{\partial F}{\partial Q_1} - q_2 p_2 + F \\
&= Q_1 P_1 - q_2 p_2 + F(q_1, q_2, Q_1, Q_2, t) \quad (19{,}2)
\end{aligned}
$$

$$F'''(q_1, p_2, P_1, Q_2, t) \rightarrow F'(q_1, p_2, P_1, P_2, t) \quad \Rightarrow$$

$$
\begin{aligned}
F'(q_1, p_2, P_1, Q_2, t) &= Q_2 \frac{\partial F'''}{\partial Q_2} - F''' \\
&= Q_2 \frac{\partial F}{\partial Q_2} - Q_1 P_1 + q_2 p_2 - F \\
&= -Q_2 P_2 - Q_1 P_1 + q_2 p_2 - F(q_1, q_2, Q_1, Q_2, t) \quad (19{,}2)
\end{aligned}
$$

Donde de (19.2) se tiene que:

$$
\begin{aligned}
\frac{dF}{dt} &= p_1\dot{q}_1 + p_2\dot{q}_2 - P_1\dot{Q}_1 - P_2\dot{Q}_2 - (H-K)\\
&= \frac{\partial F}{\partial q_1}\dot{q}_1 + \frac{\partial F}{\partial q_2}\dot{q}_2 - \frac{\partial F}{\partial Q_1}\dot{Q}_1 - \frac{\partial F}{\partial Q_2}\dot{Q}_2\\
&= \frac{d}{dt}\left(-F'(q_1,p_2,P_1,P_2,t) + q_2p_2 - Q_1P_1 - Q_2P_2\right)\\
&= \frac{d}{dt}\left(-F'(q_1,p_2,P_1,P_2,t)\right) + \frac{d}{dt}\left(q_2p_2\right) - \frac{d}{dt}\left(Q_1P_1\right) - \frac{d}{dt}\left(Q_2P_2\right)\\
&= \frac{d}{dt}\left(-F'(q_1,p_2,P_1,P_2,t)\right) + \dot{q}_2p_2 + q_2\dot{p}_2 - \dot{Q}_1P_1 - Q_1\dot{P}_1 - \dot{Q}_2P_2 - Q_2\dot{P}_2 \quad \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
\frac{dF'}{dt} &= p_1\dot{q}_1 - q_2\dot{p}_2 + Q_1\dot{P}_1 + Q_2\dot{P}_2 - (H-K)\\
&= \frac{\partial F'}{\partial q_1}\dot{q}_1 + \frac{\partial F'}{\partial p_2}\dot{p}_2 + \frac{\partial F'}{\partial P_1}\dot{P}_1 + \frac{\partial F'}{\partial P_2}\dot{P}_2 + \frac{\partial F'}{\partial t} \quad \Rightarrow
\end{aligned}
$$

$$
p_1 = \frac{\partial F'}{\partial q_1} \qquad q_2 = -\frac{\partial F'}{\partial p_2}
$$

$$
Q_1 = \frac{\partial F'}{\partial P_1} \qquad Q_2 = \frac{\partial F'}{\partial P_2} \qquad (19{,}3)
$$

$$
K = H + \frac{\partial F'}{\partial t}
$$

Ahora operando las ecuaciones (19.1) entonces:

$$
\begin{gathered}
p_1 = P_1 + 2p_2 \qquad q_2 = -2q_1 - P_2 \quad (19{,}4{,}1)\\
Q_1 = q_1 \qquad Q_2 = p_2 \quad (19{,}4{,}2)
\end{gathered}
$$

Comparando (19.3) y (19.4):

$$
\begin{aligned}
P_1 + 2p_2 &= \frac{\partial F'}{\partial q_1} \qquad 2q_1 + P_2 = \frac{\partial F'}{\partial p_2}\\
q_1 &= \frac{\partial F'}{\partial P_1} \qquad p_2 = \frac{\partial F'}{\partial P_2}
\end{aligned}
$$

Entonces:

$$
F'(q_1,p_2,P_1,P_2,t) = q_1P_1 + p_2P_2 + 2p_2q_1 \quad (19{,}5)
$$

Es la funcion generatriz que genera la transformacion canónica (19.1), pues se puede verificar que:

$$p_1 = \frac{\partial F'}{\partial q_1} = \frac{\partial}{\partial q_1}(q_1 P_1 + p_2 P_2 + 2p_2 q_1) = -P_1 - 2p_2$$
$$q_2 = -\frac{\partial F'}{\partial p_2} = -\frac{\partial}{\partial p_2}(q_1 P_1 + p_2 P_2 + 2p_2 q_1) = -P_2 - 2q_1$$
$$Q_1 = \frac{\partial F'}{\partial P_1} = \frac{\partial}{\partial P_1}(q_1 P_1 + p_2 P_2 + 2p_2 q_1) = q_1$$
$$Q_2 = \frac{\partial F'}{\partial P_2} = \frac{\partial}{\partial P_2}(q_1 P_1 + p_2 P_2 + 2p_2 q_1) = p_2$$

21. Determine la identidad de Jacobi:

$$\{A,\{B,C\}\} + \{C,\{A,B\}\} + \{B,\{C,A\}\} = 0$$

Respuesta a punto 21

De la definicion de bracket de Lagrange:

$$\{A,B\} \equiv \{A,B\}_{q,p} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \qquad (21{,}1)$$

Considerando $u = u(q_i, p_i)$ y $v = v(q_i, p_i)$, y usando la notacion matricial:

$$\frac{\partial u}{\partial \boldsymbol{\eta}} = \frac{\partial u}{\partial (q_i, p_i)} = \begin{pmatrix} \frac{\partial u}{\partial q_i} \\ \frac{\partial u}{\partial p_i} \end{pmatrix} \quad \text{y} \quad \frac{\partial v}{\partial \boldsymbol{\eta}} = \frac{\partial v}{\partial (q_i, p_i)} = \begin{pmatrix} \frac{\partial v}{\partial q_i} \\ \frac{\partial v}{\partial p_i} \end{pmatrix}$$

De esta forma

$$\begin{aligned}
[u,v] &= \tilde{\frac{\partial u}{\partial \boldsymbol{\eta}}} J \frac{\partial v}{\partial \boldsymbol{\eta}} \\
&= \begin{pmatrix} \frac{\partial u}{\partial q_i} & \frac{\partial u}{\partial p_i} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial q_i} \\ \frac{\partial v}{\partial p_i} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial u}{\partial q_i} & \frac{\partial u}{\partial p_i} \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial p_i} \\ -\frac{\partial v}{\partial q_i} \end{pmatrix} \\
&= \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}
\end{aligned}$$

En notacion tensorial $\partial_i u \equiv \frac{\partial u}{\partial \eta_i}$ y $\partial_i v \equiv \frac{\partial v}{\partial \eta_i}$ entonces:

$$[u,v] = \partial_i u J_{ij} \partial_j v$$

Por lo que para $A = A(q_i, p_i)$ y $B = B(q_i, p_i)$ y $C = C(q_i, p_i)$ funciones analiticas, se tiene que:

$$
\begin{aligned}
\{A, \{B, C\}\} &= \{A, \partial_i B J_{ij} \partial_j C\} \\
&= \partial_i A J_{ij} \partial_j \left(\partial_k B J_{kl} \partial_l C\right) \\
&= \partial_i A J_{ij} \left(\partial_j \partial_k B J_{kl} \partial_l C + \partial_k B J_{kl} \partial_j \partial_l C\right) \\
&= \partial_i A J_{ij} \partial_j \partial_k B J_{kl} \partial_l C + \partial_i A J_{ij} \partial_k B J_{kl} \partial_j \partial_l C \\
&= \partial_l C J_{kl} \partial_i A J_{ij} \partial_k \partial_j B + \partial_k B J_{kl} \partial_j \partial_l C \partial_i A J_{ij} \\
&= -\partial_l C J_{lk} \partial_k \partial_j B \partial_i A J_{ij} + \partial_k B J_{kl} \partial_i A J_{ij} \partial_j \partial_l C \\
&= -\partial_l C J_{lk} \partial_k \partial_j B J_{ij} \partial_i A + \partial_k B J_{kl} \partial_i A J_{ij} \partial_j \partial_l C \\
&= \partial_l C J_{lk} \partial_k \partial_j B J_{ji} \partial_i A + \partial_l C J_{lk} \partial_j B J_{ji} \partial_k \partial_i A_i + \partial_l C J_{lk} \partial_j B J_{ij} \partial_k \partial_i A_i + \partial_k B J_{kl} \partial_i A J_{ij} \partial_l \partial_j C \\
&= \partial_l C J_{lk} \left(\partial_k \partial_j B J_{ji} \partial_i A + \partial_j B J_{ji} \partial_k \partial_i A\right) + \partial_j B J_{ji} \partial_l \partial_i A J_{lk} \partial_k C + \partial_k B J_{kl} \partial_i A J_{ij} \partial_l \partial_j C \\
&= \partial_l C J_{lk} \partial_k \left(\partial_j B J_{ji} \partial_i A\right) + \partial_k B J_{kl} \partial_i \partial_l A J_{ij} \partial_j C + \partial_k B J_{kl} \partial_i A J_{ij} \partial_l \partial_j C \\
&= -\partial_l C J_{lk} \partial_k \left(\partial_i A J_{ij} \partial_j B\right) + \partial_k B J_{kl} \partial_l (\partial_i A J_{ij} \partial_j C) \\
&= -\partial_l C J_{lk} \partial_k \{A, B\} - \partial_k B J_{kl} \partial_l \{C, A\} \\
&= -\{C, \{A, B\}\} - \{B, \{C, A\}\} \quad \Rightarrow
\end{aligned}
$$

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 \quad (21{,}2)$$

$$J_{ij} J_{kl}$$

22. Reformule las ecuaciones de Hamilton en terminos de los Corchetes de Poisson.

Respuesta a Punto 22

Partiendo de las ecuaciones canonicas de Hamilton:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{y} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad (22{,}1)$$

Y sea $u = u(q_i, p_i, t)$, entonces:

$$
\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t} \\
&= \frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t} \\
&= \{u, H\} + \frac{\partial u}{\partial t} \quad (22{,}2)
\end{aligned}
$$

Si definimos: $\boldsymbol{\eta} \equiv (q_i(t), p_i(t))$, tal que $\dot{\boldsymbol{\eta}} = (\dot{q}_i, \dot{p}_i)$, entonces:

$$
\begin{aligned}
\dot{\boldsymbol{\eta}} &= \begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q_i} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = J\frac{\partial H}{\partial \boldsymbol{\eta}} = \frac{\partial \tilde{\boldsymbol{\eta}}}{\partial \boldsymbol{\eta}} J \frac{\partial H}{\partial \boldsymbol{\eta}} \\
&= \{\boldsymbol{\eta}, H\} \quad \Rightarrow
\end{aligned}
$$

$$\dot{\boldsymbol{\eta}} = \{\boldsymbol{\eta}, H\} \qquad (22,2)$$

Son las ecuaciones canonicas de Hamilton en corchetes de Poisson, con $\frac{\partial \boldsymbol{\eta}}{\partial t} = 0$, pues podemos notar que:

$$\{q_i, H\} = \frac{\tilde{\partial q_i}}{\partial \boldsymbol{\eta}} J \frac{\partial H}{\partial \boldsymbol{\eta}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = \frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$\{p_i, H\} = \frac{\tilde{\partial p_i}}{\partial \boldsymbol{\eta}} J \frac{\partial H}{\partial \boldsymbol{\eta}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = -\frac{\partial H}{\partial q_i} = \dot{p}_i$$

$$\{H, H\} + \frac{\partial H}{\partial t} = \frac{\tilde{\partial H}}{\partial \boldsymbol{\eta}} J \frac{\partial H}{\partial \boldsymbol{\eta}} = \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = \frac{dH}{dt}$$

23. / PROBLEMA OPCIONAL. Determine las siguientes propiedades de los Corchetes de Poisson y el momentum angular:

$$\{q_i, L_j\} = \varepsilon_{ijk} q_k,$$
$$\{p_i, L_j\} = \varepsilon_{ijk} p_k,$$
$$\{L_i, L_j\} = \varepsilon_{ijk} L_k,$$

donde ε_{ijk} es el tensor de Levi-Civita.

Respuesta a Punto 23

10. La capacidad calorífica molar a campo magnético constante de un sólido paramagnético a bajas temperaturas varía con la temperatura y el campo según la relación

$$C_{\mathscr{H}} = \frac{B + C\mathscr{H}}{T^2} + DT^2$$

donde B, C y D son constantes. ¿Cuál es el cambio de entropía de n moles de material cuando la temperatura cambia de T_i a T_f mientras que $\mathscr{H}_0$ permanece constante en el valor $\mathscr{H}$

Respuesta a Punto 10

La entropia esta definida por:

$$
\begin{aligned}
dQ = TdS \quad & \Rightarrow \quad \left(\frac{dQ}{dT}\right)_{\mathscr{H}} = T\left(\frac{dS}{dT}\right)_{\mathscr{H}} \\
& \Rightarrow \quad C_{\mathscr{H}} = T\left(\frac{dS}{dT}\right)_{\mathscr{H}} \\
& \Rightarrow \quad dS = \frac{C_{\mathscr{H}}}{T}dT \\
& \Rightarrow \quad dS = \left(\frac{B + C\mathscr{H}_0^2}{T^2} + DT^2\right)\frac{dT}{T} \\
& \Rightarrow \quad S = \int_{T_i}^{T_f}\left(\frac{B + C\mathscr{H}_0^2}{T^2} + DT^2\right)\frac{dT}{T} \\
& \Rightarrow \quad S = \int_{T_i}^{T_f}\left(\frac{B + C\mathscr{H}_0^2}{T^3} + DT\right)dT \\
& \Rightarrow \quad S = \left[-\frac{B + C\mathscr{H}_0^2}{2T^2} + D\frac{T^2}{2}\right]_{T_i}^{T_f} \\
& \Rightarrow \quad S = -\frac{B + C\mathscr{H}_0^2}{2T_f^2} + D\frac{T_f^2}{2} + \frac{B + C\mathscr{H}_0^2}{2T_i^2} - D\frac{T_i^2}{2} \\
& \Rightarrow \quad S = \frac{B + C\mathscr{H}_0^2}{2}\left(\frac{1}{T_i^2} - \frac{1}{T_f^2}\right) + D\left(\frac{T_f^2}{2} - \frac{T_i^2}{2}\right) \\
& \Rightarrow \quad S = \frac{B + C\mathscr{H}_0^2}{2}\left(\frac{T_f^2 - T_i^2}{T_i^2 T_f^2}\right) + D\left(\frac{T_f^2 - T_i^2}{2}\right) \\
& \Rightarrow \quad S = \frac{T_f^2 - T_i^2}{2}\left[\left(\frac{B + C\mathscr{H}_0^2}{T_i^2 T_f^2}\right) + D\left(T_f^2 - T_i^2\right)\right]
\end{aligned}
$$