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Taller 3

15. Demostrar directamente que la transformación

$$Q = \arctan \frac{\alpha q}{p},$$
$$P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

es canónica, donde α es una constante.

Respuesta a Problema 1

Sean en (p, q) la coordenadas generalizadas de un sistema en el espacion de fase, y sea $H(p, q)$ la funcion de Hamilton, entonces las ecuaciones de Hamilton son:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \quad \text{y} \quad \frac{\partial H}{\partial t} = 0 \quad (15{,}1)$$

Dado que la transformacion puntual de coordenadas generalizadas (p, q) a coordenadas generalizadas (P, Q) dada por:

$$\begin{aligned} Q &= \arctan \frac{\alpha q}{p}, \\ P &= \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right) = \frac{1}{2}\left(\alpha q^2 + \frac{p^2}{\alpha}\right) \end{aligned} \quad (15{,}2)$$

es canónica, entonces deben satisfacer las ecuaciones de Hamilton (15.1) con $K(P, Q) = K$ constante, es decir:

$$\dot{P} = -\frac{\partial K}{\partial Q} \quad \text{y} \quad \dot{Q} = \frac{\partial K}{\partial P} \quad \text{y} \quad \frac{\partial K}{\partial t} = 0 \quad (15{,}3)$$

Para mostrar esto ultimo, usando la regla de la cadena en la ecuaciones (15.2) y reemplazando (15.1)

$$\begin{aligned} \frac{dQ}{dt} &= \frac{\partial Q}{\partial q}\frac{dq}{dt} + \frac{\partial Q}{\partial p}\frac{dp}{dt} \\ &= -\frac{\alpha q}{p^2 + \alpha^2 q^2}\left(-\frac{\partial H}{\partial q}\right) + \frac{\alpha p}{p^2 + \alpha^2 q^2}\left(\frac{\partial H}{\partial p}\right) \\ &= \frac{\alpha}{p^2 + \alpha^2 q^2}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \\ &= \frac{1}{2}\frac{2\alpha}{p^2 + \alpha^2 q^2}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \\ &= \frac{1}{2P}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right) \quad (15{,}4{,}1) \end{aligned}$$

$$\begin{aligned} \frac{dP}{dt} &= \frac{\partial P}{\partial q}\frac{dq}{dt} + \frac{\partial P}{\partial p}\frac{dp}{dt} \\ &= \frac{p}{\alpha}\left(-\frac{\partial H}{\partial q}\right) + q\alpha\left(\frac{\partial H}{\partial p}\right) \quad (15{,}4{,}2) \end{aligned}$$

Aplicando la regla de la cadena en (15.1) tenemos que:

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} - \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}$$
$$= -\frac{\partial H}{\partial Q}\frac{\alpha p}{p^2+\alpha^2 q^2} - \frac{\partial H}{\partial P}\alpha q \quad (15{,}7{,}1)$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}$$
$$= -\frac{\partial H}{\partial Q}\frac{\alpha q}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p}{\alpha} \quad (15{,}7{,}2)$$

Reemplazando (15.5) en (15.7.1) y (15.7.2) tenemos que:

$$\begin{aligned}
\dot{Q} &= \frac{1}{2P}\left(q\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial p}\right)\\
&= \frac{1}{2P}\left(q\left(\frac{\partial H}{\partial Q}\frac{\alpha p}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\alpha q\right) + p\left(-\frac{\partial H}{\partial Q}\frac{\alpha q}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p}{\alpha}\right)\right)\\
&= \frac{1}{2P}\left(\frac{\partial H}{\partial Q}\frac{\alpha pq}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\alpha q^2 - \frac{\partial H}{\partial Q}\frac{\alpha pq}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p^2}{\alpha}\right)\\
&= \frac{1}{2P}\left(\frac{\partial H}{\partial P}\alpha q^2 + \frac{\partial H}{\partial P}\frac{p^2}{\alpha}\right)\\
&= \frac{1}{2P}\frac{\partial H}{\partial P}\left(\alpha q^2 + \frac{p^2}{\alpha}\right)\\
&= \frac{1}{2P}\frac{\partial H}{\partial P}2P = \frac{\partial H}{\partial P}
\end{aligned}$$

$$\begin{aligned}
\dot{P} &= \frac{p}{\alpha}\left(-\frac{\partial H}{\partial q}\right) + q\alpha\left(\frac{\partial H}{\partial p}\right)\\
&= \frac{p}{\alpha}\left(-\frac{\partial H}{\partial Q}\frac{\alpha p}{p^2+\alpha^2 q^2} - \frac{\partial H}{\partial P}\alpha q\right) + q\alpha\left(-\frac{\partial H}{\partial Q}\frac{\alpha q}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}\frac{p}{\alpha}\right)\\
&= -\frac{\partial H}{\partial Q}\frac{p^2}{p^2+\alpha^2 q^2} - \frac{\partial H}{\partial P}pq - \frac{\partial H}{\partial Q}\frac{\alpha^2 q^2}{p^2+\alpha^2 q^2} + \frac{\partial H}{\partial P}pq\\
&= -\frac{\partial H}{\partial Q}\frac{p^2+\alpha^2 q^2}{p^2+\alpha^2 q^2}\\
&= -\frac{\partial H}{\partial Q}
\end{aligned}$$

Por lo que si $K = H(P,Q)$ entonces las anterioes ecuaciones son las ecuaciones de Hamilton (15.3) y por lo tanto la transformacion (15.2) es canónica. Vease que como $H = H(q,p)$ no depende de t entonces $K = H(P,Q)$ tampoco depende de t, es decir $\frac{\partial K}{\partial t} = 0$.

16. Mostrar que una funcion generatriz del segundo tipo cuya forma particular sea $F_2 = q_j P_j$, genera la transformacion identidad

Respuesta a Punto 2

De las ecuaciones de transformacion asociadas a al funcion generatriz $F_2(q_j, P_j) \quad j = 1, \ldots n$ dadas por:

$$Q_j = \frac{\partial F_2}{\partial P_j}, \quad p_j = \frac{\partial F_2}{\partial q_j} \quad \text{y} \quad K = H + \frac{\partial F_2}{\partial t} \quad (16{,}1)$$

Se tiene que:

$$\begin{aligned}
\frac{\partial F_2}{\partial t} &= 0 \quad \Rightarrow \quad K = H(Q_j, P_j) \\
Q_j &= \frac{\partial F_2}{\partial P_j} = \frac{\partial}{\partial P_j}(P_j q_j) = q_j \\
p_j &= \frac{\partial F_2}{\partial q_j} = \frac{\partial}{\partial q_j}(P_j q_j) = P_j
\end{aligned}$$

De esta forma la matriz de transformacion M es tal que:

$$M = \begin{pmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

17. Use una funci´on generatriz para construir una transformaci´on que intercambie cantidades de movimiento y coorde- nadas.

Respuesta a Punto 3

Dado que se busca una funcion generatriz $F' = F'(q_j, p_k, Q_l, P_m, t), \quad j = l, k = m = 1, .., n$ o $j = m, l = m = 1, .., n$, para algun j, k, l, m tal que la transformacion canonica M asociada satisfaga que:

$$\dot{\mathbf{X}} = M\dot{\mathbf{x}} \quad (17{,}1)$$

Donode:

$$\mathbf{X} = \begin{pmatrix} Q_j \\ P_j \end{pmatrix} = \begin{pmatrix} p_j \\ q_j \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} q_j \\ p_j \end{pmatrix} \quad \text{y} \quad M = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i} \\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix} \quad (17{,}2)$$

Por lo que de (17.2):

$$M = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i} \\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial p_j}{\partial q_i} & \frac{\partial p_j}{\partial p_i} \\ \frac{\partial q_j}{\partial q_i} & \frac{\partial q_j}{\partial p_i} \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad (17{,}3)$$

Vease que M de (17.3) no una transformacion canonica, pues:

$$M^T J M = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}^T \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \neq J$$

Por lo que en su lugar se propone:

$$\mathbf{X} = \begin{pmatrix} Q_j \\ P_j \end{pmatrix} = \begin{pmatrix} p_j \\ -q_j \end{pmatrix} \quad \text{y} \quad M = \begin{pmatrix} \frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i} \\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i} \end{pmatrix} \quad (17{,}4)$$

Asi de (17.4):

$$M=\begin{pmatrix}\frac{\partial Q_j}{\partial q_i} & \frac{\partial Q_j}{\partial p_i}\\ \frac{\partial P_j}{\partial q_i} & \frac{\partial P_j}{\partial p_i}\end{pmatrix}=\begin{pmatrix}\frac{\partial p_j}{\partial q_i} & \frac{\partial p_j}{\partial p_i}\\ \frac{\partial(-q_j)}{\partial q_i} & \frac{\partial(-q_j)}{\partial p_i}\end{pmatrix}=\begin{pmatrix}0 & \delta_{ij}\\ -\delta_{ij} & 0\end{pmatrix}=\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}\quad(17{,}5)$$

Vease que M de (17.5) es una transformacion canonica, pues:

$$M^TJM=\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}^T\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}=\begin{pmatrix}0 & -\mathbf{1}\\ \mathbf{1} & 0\end{pmatrix}\begin{pmatrix}-\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}=\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}=J$$

Una una funcion generatriz que me genera este tipo de transformacion canonica es:

$$F(q_j,Q_j,t)=q_jQ_j\quad(17{,}6)$$

Pues de (17.6) se tiene que:

$$p_i=\frac{\partial F}{\partial q_i}=\frac{\partial}{\partial q_i}(q_jQ_j)=Q_i\quad\text{y}\quad P_i=-\frac{\partial F}{\partial Q_i}=-\frac{\partial}{\partial Q_i}(q_jQ_j)=-q_i$$

18. Las ecuaciones de transformación entre dos sistemas de coordenadas son

$$Q=\log\left(1+q^{1/2}\cos p\right)$$
$$P=2\left(1+q^{1/2}\cos p\right)q^{1/2}\sin p$$

a) A partir de estas ecuaciones de transformación, demostrar directamente que Q,P son variables canónicas si lo son q y p. b) Demostrar que la función que genera esta transformación es

$$F_3=-\left(e^Q-1\right)^2\tan p$$

Respuesta a Punto 4

- Si (q,p) son variables canónicas entonces satisfacen las ecuaciones canonicas de Hamilton, es decir:

$$\dot q=\frac{\partial H}{\partial p}\quad\text{y}\quad\dot p=-\frac{\partial H}{\partial q}\quad(18{,}1)$$

Si ademas define la transformacion:

$$\begin{aligned}Q&=\log\left(1+q^{1/2}\cos p\right)\\ P&=2\left(1+q^{1/2}\cos p\right)q^{1/2}\sin p\end{aligned}\qquad(18{,}2)$$

Entonces se tiene que por la regla de la cadena aplicada en (18.1) y (18.2):

$$\begin{aligned}\dot q=\frac{\partial H}{\partial p}&=\frac{\partial H}{\partial P}\frac{\partial P}{\partial p}+\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p}\\ &=\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p+q\cos(2p)\right)-\frac{\partial H}{\partial Q}\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p}\\ &=\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p+q\cos(2p)\right)-\frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\quad(18{,}3{,}1)\end{aligned}$$

$$
\begin{aligned}
\dot{p} &= -\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial P}\frac{\partial P}{\partial q} - \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} \\
&= -\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p}{2(q^{1/2} + q\cos p)} \\
&= -\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p \sin p}{P} \quad (18,3,2)
\end{aligned}
$$

$$
\begin{aligned}
\dot{Q} &= \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} \\
&= \frac{\cos p}{2(q^{1/2} + q\cos p)}\dot{q} - \frac{q^{1/2}\sin p}{1 + q^{1/2}\cos p}\dot{p} \\
&= \frac{\cos p \sin p}{P}\dot{q} - \frac{2q\sin^2 p}{P}\dot{p} \quad (18,4,1)
\end{aligned}
$$

$$
\begin{aligned}
\dot{P} &= \frac{\partial P}{\partial q}\dot{q} + \frac{\partial P}{\partial p}\dot{p} \\
&= \left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p\dot{q} + 2\left(q^{1/2}\cos p + q\cos(2p)\right)\dot{p} \quad (18,4,2)
\end{aligned}
$$

Reemplazando (18.3.1) y (18.3.2) en (18.4.1) y (18.4.2):

$$
\begin{aligned}
\dot{Q} &= \frac{\cos p \sin p}{P}\left(\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos(2p)\right) - \frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\right) \\
&- \frac{2q\sin^2 p}{P}\left(-\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\right) \\
&= \frac{\cos p \sin p}{P}\left(\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos^2 p - q\sin^2 p\right) - \frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\right) \\
&+ \frac{2q\sin^2 p}{P}\left(\frac{\partial H}{\partial P}\left(\frac{1 + q^{1/2}\cos p}{q^{1/2}} + \cos p\right)\sin p + \frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\right) \\
&= \frac{\cos p \sin p}{P}\left(\frac{\partial H}{\partial P}2\left(q^{1/2} + q\cos p\right)\frac{\sin p}{\sin p}\cos p - \frac{\partial H}{\partial P}2\left(q\sin^2 p\right)\right) - \cancel{\frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P^2}} \\
&+ \frac{2q\sin^2 p}{P}\left(\frac{\partial H}{\partial P}2\left(\frac{q^{1/2} + q\cos p}{2q}\sin p\right) + \frac{\partial H}{\partial P}\cos p\sin p\right) + \cancel{\frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P^2}} \\
&= \frac{\cos p \sin p}{P}\left(\frac{\partial H}{\partial P}P\frac{\cos p}{\sin p} - \frac{\partial H}{\partial P}2\left(q\sin^2 p\right)\right) + \frac{2q\sin^2 p}{P}\left(\frac{\partial H}{\partial P}\left(\frac{P}{2q}\right) + \frac{\partial H}{\partial P}\cos p\sin p\right) \\
&= \frac{\partial H}{\partial P}\cos^2 p - \cancel{2\frac{\partial H}{\partial P}q\sin^3 p\cos p\frac{1}{P}} + \frac{\partial H}{\partial P}\sin^2 p + \cancel{2\frac{\partial H}{\partial P}\cos p\sin^3 p\frac{1}{P}} \\
&= \frac{\partial H}{\partial P} \quad (18,5,1)
\end{aligned}
$$

$$
\begin{aligned}
\dot{P} &= \left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p\left(\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos(2p)\right) - \frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P}\right) \\
&\quad + 2\left(q^{1/2}\cos p + q\cos(2p)\right)\left(-\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p - \frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \cancel{\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p\frac{\partial H}{\partial P}2\left(q^{1/2}\cos p + q\cos(2p)\right)} - \left(\frac{1}{q^{1/2}} + 2\cos p\right)\frac{\partial H}{\partial Q}\frac{2q\sin^3 p}{P} \\
&\quad \cancel{+ 2\left(q^{1/2}\cos p + q\cos(2p)\right)\frac{\partial H}{\partial P}\left(\frac{1}{q^{1/2}} + 2\cos p\right)\sin p} + 2\left(q^{1/2}\cos p + q\cos(2p)\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}
\end{aligned}
$$

$$
= -\left(\frac{1 + q^{1/2}\cos p}{q^{1/2}} + \cos p\right)\sin p\frac{\partial H}{\partial Q}\frac{2q\sin^3 p}{P} - 2\left(q^{1/2}\cos p + q\cos^2 p - q\sin^2 p\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P}
$$

$$
\begin{aligned}
&= -\left(\frac{P}{2q} + \cos p\sin p\right)\frac{\partial H}{\partial Q}\frac{2q\sin^2 p}{P} - \left(2\left(q^{1/2} + q\cos p\right)\frac{\sin p}{\sin p}\cos p - 2\left(q\sin^2 p\right)\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P} \\
&= -\frac{\partial H}{\partial Q}\sin^2 - \frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P} - \left(P\frac{\cos p}{\sin p} - 2\left(q\sin^2 p\right)\right)\frac{\partial H}{\partial Q}\frac{\cos p\sin p}{P} \\
&= -\frac{\partial H}{\partial Q}\sin^2 + \cancel{\frac{\partial H}{\partial Q}\frac{2q\sin^3 p\cos p}{P}} - \frac{\partial H}{\partial Q}\cos^2 p - \cancel{\frac{\partial H}{\partial Q}\frac{2q\cos p\sin^3 p}{P}} \\
&= -\frac{\partial H}{\partial Q} \quad (18{,}5{,}2)
\end{aligned}
$$

Por que si $K = H(Q, P)$ entonces las anterioes ecuaciones son las ecuaciones canonicas de Hamilton y por lo tanto (P, Q) son variables canonicas.

- Dado que $F_3 = F_3(p, Q)$, pues:

$$F_3 = -\left(e^Q - 1\right)^2 \tan p$$

Entonces de la ecuaciones de canonicas de transformacion para F_3:

$$q = -\frac{\partial F_3}{\partial p} \quad \text{y} \quad P = \frac{\partial F_3}{\partial Q} \quad (18{,}6)$$

se sigue que:

$$
\begin{aligned}
q = -\frac{\partial F_3}{\partial p} &= -\frac{\partial}{\partial p}\left(-\left(e^Q - 1\right)^2\tan p\right) = \left(e^Q - 1\right)^2\sec^2 p \quad (18{,}7{,}1) \\
P = -\frac{\partial F_3}{\partial Q} &= -\frac{\partial}{\partial Q}\left(-\left(e^Q - 1\right)^2\tan p\right) = 2\left(e^Q - 1\right)e^Q\tan p \quad (18{,}7{,}2)
\end{aligned}
$$

Despejando Q de (18.7.1):

$$\begin{aligned} q = \left(e^Q - 1\right)^2 \sec^2 p \quad &\Rightarrow \quad \left(e^Q - 1\right)^2 = \frac{q}{\sec^2 p} \\ &\Rightarrow \quad e^Q - 1 = \sqrt{\frac{q}{\sec^2 p}} \\ &\Rightarrow \quad e^Q = 1 + \sqrt{\frac{q}{\sec^2 p}} \\ &\Rightarrow \quad Q = \ln\left(1 + \sqrt{\frac{q}{\sec^2 p}}\right) \\ &\Rightarrow \quad Q = \ln\left(1 + q^{1/2}\cos p\right) \quad (18{,}8{,}1) \end{aligned}$$

Reemplazando (18.8.1) en (18.7.2):

$$\begin{aligned} P = 2\left(e^Q - 1\right) e^Q \tan p \quad &\Rightarrow \quad P = -2\left(1 - q^{1/2}\cos p - 1\right)\left(1 - q^{1/2}\cos p\right)\tan p \\ &\Rightarrow \quad P = -2(1 - 2q^{1/2}\cos p + q\cos^2 p - 1 + q^{1/2}\cos p)\tan p \\ &\Rightarrow \quad P = -2\left(q^{1/2}\cos p + q\cos^2 p\right)\tan p \\ &\Rightarrow \quad P = -2q^{1/2}\cos p\left(1 + q^{1/2}\cos p\right)\tan p \quad (18{,}8{,}2) \end{aligned}$$

De (18.8.1) y (18.8.2) se concluye que F_3 es una funcion generatriz de (18.2)

19. Probar directamente que la transformación

$$\begin{aligned} Q_1 = q_1, \quad P_1 = p_1 - 2p_2 \\ Q_2 = p_2, \quad P_2 = -2q_1 - q_2 \end{aligned}$$

es canónica y hallar una función generatriz.

Respuesta a Punto 19

Si de las ecuaciones:

$$\begin{aligned} Q_1 = q_1, \quad P_1 = p_1 - 2p_2 \\ Q_2 = p_2, \quad P_2 = -2q_1 - q_2 \end{aligned} \qquad (19{,}1)$$

Calculamos la matriz de transformación M:

$$M = \begin{pmatrix} \frac{\partial Q_1}{\partial q_1} & \frac{\partial Q_1}{\partial q_2} & \frac{\partial Q_1}{\partial p_1} & \frac{\partial Q_1}{\partial p_2} \\ \frac{\partial Q_2}{\partial q_1} & \frac{\partial Q_2}{\partial q_2} & \frac{\partial Q_2}{\partial p_1} & \frac{\partial Q_2}{\partial p_2} \\ \frac{\partial P_1}{\partial q_1} & \frac{\partial P_1}{\partial q_2} & \frac{\partial P_1}{\partial p_1} & \frac{\partial P_1}{\partial p_2} \\ \frac{\partial P_2}{\partial q_1} & \frac{\partial P_2}{\partial q_2} & \frac{\partial P_2}{\partial p_1} & \frac{\partial P_2}{\partial p_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix}$$

Entonces la matriz M es una transformación canónica si y solo si:

$$M^T J M = J \quad J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Lo cual podemos comprobar:

$$
\begin{aligned}
M^T J M &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = J
\end{aligned}
$$

Por lo tanto M es una transformación canónica.

Considere la funcion generatriz $F' = F'(q_1, p_2, P_1, P_2, t)$ tal que:

$$F(q_1, q_2, Q_1, Q_2, t) \to F''(q_1, p_2, Q_1, Q_2, t) \quad \Rightarrow$$

$$
\begin{aligned}
F''(q_1, p_1, Q_1, Q_1, t) &= q_2 \frac{\partial F}{\partial q_2} - F \\
&= q_2 p_2 - F
\end{aligned}
$$

$$F''(q_1, p_2, Q_1, Q_2, t) \to F'''(q_1, p_2, P_1, Q_2, t) \quad \Rightarrow$$

$$
\begin{aligned}
F'''(q_1, p_2, P_1, Q_2, t) &= Q_1 \frac{\partial F''}{\partial Q_1} - F'' \\
&= -Q_1 \frac{\partial F}{\partial Q_1} - q_2 p_2 + F \\
&= Q_1 P_1 - q_2 p_2 + F(q_1, q_2, Q_1, Q_2, t) \quad (19{,}2)
\end{aligned}
$$

$$F'''(q_1, p_2, P_1, Q_2, t) \to F'(q_1, p_2, P_1, P_2, t) \quad \Rightarrow$$

$$
\begin{aligned}
F'(q_1, p_2, P_1, Q_2, t) &= Q_2 \frac{\partial F'''}{\partial Q_2} - F''' \\
&= Q_2 \frac{\partial F}{\partial Q_2} - Q_1 P_1 + q_2 p_2 - F \\
&= -Q_2 P_2 - Q_1 P_1 + q_2 p_2 - F(q_1, q_2, Q_1, Q_2, t) \quad (19{,}2)
\end{aligned}
$$

Donde de (19.2) se tiene que:

$$\begin{aligned}
\frac{dF}{dt} &= p_1\dot{q}_1 + p_2\dot{q}_2 - P_1\dot{Q}_1 - P_2\dot{Q}_2 - (H-K) \\
&= \frac{\partial F}{\partial q_1}\dot{q}_1 + \frac{\partial F}{\partial q_2}\dot{q}_2 - \frac{\partial F}{\partial Q_1}\dot{Q}_1 - \frac{\partial F}{\partial Q_2}\dot{Q}_2 \\
&= \frac{d}{dt}\left(-F'(q_1,p_2,P_1,P_2,t) + q_2p_2 - Q_1P_1 - Q_2P_2\right) \\
&= \frac{d}{dt}\left(-F'(q_1,p_2,P_1,P_2,t)\right) + \frac{d}{dt}\left(q_2p_2\right) - \frac{d}{dt}\left(Q_1P_1\right) - \frac{d}{dt}\left(Q_2P_2\right) \\
&= \frac{d}{dt}\left(-F'(q_1,p_2,P_1,P_2,t)\right) + \dot{q}_2p_2 + q_2\dot{p}_2 - \dot{Q}_1P_1 - Q_1\dot{P}_1 - \dot{Q}_2P_2 - Q_2\dot{P}_2 \quad \Rightarrow
\end{aligned}$$

$$\begin{aligned}
\frac{dF'}{dt} &= p_1\dot{q}_1 - q_2\dot{p}_2 + Q_1\dot{P}_1 + Q_2\dot{P}_2 - (H-K) \\
&= \frac{\partial F'}{\partial q_1}\dot{q}_1 + \frac{\partial F'}{\partial p_2}\dot{p}_2 + \frac{\partial F'}{\partial P_1}\dot{P}_1 + \frac{\partial F'}{\partial P_2}\dot{P}_2 + \frac{\partial F'}{\partial t} \quad \Rightarrow
\end{aligned}$$

$$p_1 = \frac{\partial F'}{\partial q_1} \qquad q_2 = -\frac{\partial F'}{\partial p_2}$$

$$Q_1 = \frac{\partial F'}{\partial P_1} \qquad Q_2 = \frac{\partial F'}{\partial P_2} \qquad (19{,}3)$$

$$K = H + \frac{\partial F'}{\partial t}$$

Ahora operando las ecuaciones (19.1) entonces:

$$p_1 = P_1 + 2p_2 \qquad q_2 = -2q_1 - P_2 \quad (19{,}4{,}1)$$
$$Q_1 = q_1 \qquad Q_2 = p_2 \quad (19{,}4{,}2)$$

Comparando (19.3) y (19.4):

$$P_1 + 2p_2 = \frac{\partial F'}{\partial q_1} \qquad 2q_1 + P_2 = \frac{\partial F'}{\partial p_2}$$
$$q_1 = \frac{\partial F'}{\partial P_1} \qquad p_2 = \frac{\partial F'}{\partial P_2}$$

Entonces:

$$F'(q_1,p_2,P_1,P_2,t) = q_1P_1 + p_2P_2 + 2p_2q_1 \quad (19{,}5)$$

Es la funcion generatriz que genera la transformacion canónica (19.1), pues se puede verificar que:

$$\begin{aligned}
p_1 &= \frac{\partial F'}{\partial q_1} = \frac{\partial}{\partial q_1}\left(q_1P_1 + p_2P_2 + 2p_2q_1\right) = -P_1 - 2p_2 \\
q_2 &= -\frac{\partial F'}{\partial p_2} = -\frac{\partial}{\partial p_2}\left(q_1P_1 + p_2P_2 + 2p_2q_1\right) = -P_2 - 2q_1 \\
Q_1 &= \frac{\partial F'}{\partial P_1} = \frac{\partial}{\partial P_1}\left(q_1P_1 + p_2P_2 + 2p_2q_1\right) = q_1 \\
Q_2 &= \frac{\partial F'}{\partial P_2} = \frac{\partial}{\partial P_2}\left(q_1P_1 + p_2P_2 + 2p_2q_1\right) = p_2
\end{aligned}$$

21. Determine la identidad de Jacobi:

$$\{A,\{B,C\}\} + \{C,\{A,B\}\} + \{B,\{C,A\}\} = 0$$

Respuesta a punto 21

De la definicion de bracket de Lagrange:

$$\{A,B\} \equiv \{A,B\}_{q,p} = \frac{\partial q_i}{\partial A}\frac{\partial p_i}{\partial B} - \frac{\partial p_i}{\partial A}\frac{\partial q_i}{\partial B} \quad (21{,}1)$$

Entonces si definimos $f \equiv f(q_j, p_j) = \{A(q_j, p_j), B(q_j, p_j)\}$ entonces:

$$\begin{aligned}
\frac{\partial f}{\partial q_i} &= \frac{\partial}{\partial q_i}\{A,B\} = \frac{\partial}{\partial q_i}\left(\frac{\partial q_j}{\partial A}\frac{\partial p_j}{\partial B} - \frac{\partial p_j}{\partial A}\frac{\partial q_j}{\partial B}\right) \\
&= \frac{\partial q_j}{\partial A}\frac{\partial^2 p_j}{\partial q_i \partial B} - \frac{\partial p_j}{\partial A}\frac{\partial^2 q_j}{\partial q_i \partial B} - \frac{\partial p_j}{\partial A}\frac{\partial^2 p_j}{\partial q_i \partial B} + \frac{\partial q_j}{\partial A}\frac{\partial^2 q_j}{\partial q_i \partial B} \\
&= \frac{\partial q_j}{\partial A}\frac{\partial^2 p_j}{\partial q_i \partial B} - \frac{\partial p_j}{\partial A}\frac{\partial^2 q_j}{\partial q_i \partial B} \quad (21{,}2{,}1)
\end{aligned}$$

$$\begin{aligned}
\{C,\{A,B\}\} &= \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial f} - \frac{\partial p_i}{\partial C}\frac{\partial q_i}{\partial f} \\
&= \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial A}\frac{\partial A}{\partial f} + \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial B}\frac{\partial B}{\partial f} - \frac{\partial p_i}{\partial C}\frac{\partial q_i}{\partial A}\frac{\partial A}{\partial f} - \frac{\partial p_i}{\partial C}\frac{\partial q_i}{\partial B}\frac{\partial B}{\partial f} \\
&= \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial A}\frac{\partial A}{\partial f} - \frac{\partial p_i}{\partial C}\frac{\partial q_i}{\partial A}\frac{\partial A}{\partial f} + \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial B}\frac{\partial B}{\partial f} - \frac{\partial p_i}{\partial C}\frac{\partial q_i}{\partial B}\frac{\partial B}{\partial f} \\
&= \left[\frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial A} - \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial A}\right]\frac{\partial A}{\partial f} + \left[\frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial B} - \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial B}\right]\frac{\partial B}{\partial f} \\
&= \{C,A\}\frac{\partial A}{\partial f} + \{C,B\}\frac{\partial B}{\partial f}
\end{aligned}$$

$$\begin{aligned}
\{C,\{A,B\}\} &= \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial\{A,B\}} - \frac{\partial p_i}{\partial C}\frac{\partial q_i}{\partial\{A,B\}} \\
&= \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial A}\frac{\partial A}{\partial\{A,B\}} + \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial B}\frac{\partial B}{\partial\{A,B\}} - \frac{\partial p_i}{\partial C}\frac{\partial q_i}{\partial A}\frac{\partial A}{\partial\{A,B\}} - \frac{\partial p_i}{\partial C}\frac{\partial q_i}{\partial B}\frac{\partial B}{\partial\{A,B\}} \\
&= \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial A}\frac{\partial A}{\partial\{A,B\}} - \frac{\partial p_i}{\partial C}\frac{\partial q_i}{\partial A}\frac{\partial A}{\partial\{A,B\}} + \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial B}\frac{\partial B}{\partial\{A,B\}} - \frac{\partial p_i}{\partial C}\frac{\partial q_i}{\partial B}\frac{\partial B}{\partial\{A,B\}} \\
&= \left[\frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial A} - \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial A}\right]\frac{\partial A}{\partial\{A,B\}} + \left[\frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial B} - \frac{\partial q_i}{\partial C}\frac{\partial p_i}{\partial B}\right]\frac{\partial B}{\partial\{A,B\}} \\
&= \{C,A\}\frac{\partial A}{\partial\{A,B\}} + \{C,B\}\frac{\partial B}{\partial\{A,B\}}
\end{aligned}$$

Que es?

$$\begin{aligned}
\frac{\partial n}{\partial f} &= \frac{\partial n}{\partial\{A,B\}} = \frac{\partial n}{\partial A}\frac{\partial A}{\partial\{A,B\}} + \frac{\partial n}{\partial B}\frac{\partial B}{\partial\{A,B\}} = \frac{\partial n}{\partial A}\frac{\partial A}{\partial f} + \frac{\partial n}{\partial B}\frac{\partial B}{\partial f} \\
&= \frac{\partial n}{\partial\{A,B\}} = \frac{\partial n}{\partial q_i}\frac{\partial q_i}{\partial\{A,B\}} + \frac{\partial n}{\partial p_i}\frac{\partial p_i}{\partial\{A,B\}} = \frac{\partial n}{\partial q_i}\frac{\partial q_i}{\partial f} + \frac{\partial n}{\partial p_i}\frac{\partial p_i}{\partial f}
\end{aligned}$$

$$\frac{\partial n}{\partial\{A,B\}} = \frac{\partial n}{\partial\left(\frac{\partial q_i}{\partial A}\frac{\partial p_i}{\partial B} - \frac{\partial p_i}{\partial A}\frac{\partial q_i}{\partial B}\right)}$$

$$\frac{\partial A}{\partial f} = \frac{\partial A}{\partial\{A,B\}} = \frac{\partial A}{\partial q_i}\frac{\partial q_i}{\partial\{A,B\}} + \frac{\partial A}{\partial p_i}\frac{\partial p_i}{\partial\{A,B\}} = \frac{\partial A}{\partial q_i}\frac{\partial q_i}{\partial f} + \frac{\partial A}{\partial p_i}\frac{\partial p_i}{\partial f}$$

$$A = f(q_i,p_i) \quad A - f(q_i,p_i) = 0$$

Si $F(A,q_i,p_i) = A - f(q_i,p_i) = 0$ entonces:

$$\begin{aligned}
\frac{\partial F}{\partial q_i} &= \frac{\partial A}{\partial q_i} - \frac{\partial f}{\partial q_i} = 0 \\
\frac{\partial F}{\partial p_i} &= \frac{\partial A}{\partial p_i} - \frac{\partial f}{\partial p_i} = 0
\end{aligned}$$

$p_i = p_i(u,v)$ y $q_i = q_i(u,v)$ como tambien $u = u(q_i,p_i)$ y $v = v(q_i,p_i)$, matricialmente se escribe

$$\frac{\partial u}{\partial \boldsymbol{\eta}} = \frac{\partial u}{\partial(q_i,p_i)} = \begin{pmatrix} \frac{\partial u}{\partial q_i} \\ \frac{\partial u}{\partial p_i} \end{pmatrix} \quad \text{y} \quad \frac{\partial v}{\partial \boldsymbol{\eta}} = \frac{\partial v}{\partial(q_i,p_i)} = \begin{pmatrix} \frac{\partial v}{\partial q_i} \\ \frac{\partial v}{\partial p_i} \end{pmatrix}$$

De esta forma

$$
\begin{aligned}
[u,v] &= \frac{\tilde{\partial u}}{\partial \boldsymbol{\eta}} J \frac{\partial v}{\partial \boldsymbol{\eta}} \\
&= \begin{pmatrix} \frac{\partial u}{\partial q_i} & \frac{\partial u}{\partial p_i} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial q_i} \\ \frac{\partial v}{\partial p_i} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial u}{\partial q_i} & \frac{\partial u}{\partial p_i} \end{pmatrix} \begin{pmatrix} \frac{\partial v}{\partial p_i} \\ -\frac{\partial v}{\partial q_i} \end{pmatrix} \\
&= \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}
\end{aligned}
$$

En notacion tensorial $\partial_i u \equiv \frac{\partial u}{\partial \eta_i}$ y $\partial_i v \equiv \frac{\partial v}{\partial \eta_i}$ entonces:

$$[u,v] = \partial_i u J_{ij} \partial_j v$$

Tambien en matricialmente se escribe:

$$\frac{\partial \boldsymbol{\eta}}{\partial u} = \frac{\partial(q_i,p_i)}{\partial u} = \begin{pmatrix} \frac{\partial q_i}{\partial u} \\ \frac{\partial p_i}{\partial u} \end{pmatrix} \quad \text{y} \quad \frac{\partial \boldsymbol{\eta}}{\partial v} = \frac{\partial(q_i,p_i)}{\partial v} = \begin{pmatrix} \frac{\partial q_i}{\partial v} \\ \frac{\partial p_i}{\partial v} \end{pmatrix}$$

Asi tenemos que:

$$
\begin{aligned}
\{u,v\} &= \frac{\tilde{\partial \boldsymbol{\eta}}}{\partial u} J \frac{\partial \boldsymbol{\eta}}{\partial v} \\
&= \begin{pmatrix} \frac{\partial q_i}{\partial u} \\ \frac{\partial p_i}{\partial u} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial q_i}{\partial v} \\ \frac{\partial p_i}{\partial v} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \end{pmatrix} \begin{pmatrix} \frac{\partial q_i}{\partial v} \\ -\frac{\partial p_i}{\partial v} \end{pmatrix} \\
&= \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v}
\end{aligned}
$$

En notacion tensorial $\partial^i u \equiv \frac{\partial \boldsymbol{\eta}_i}{\partial u}$ y $\partial^i v \equiv \frac{\partial \boldsymbol{\eta}_i}{\partial v}$ entonces:

$$\{u,v\} = \partial^i u J_{ij} \partial^j v$$

Por lo tanto:

$$\{A,B\} = \partial^i A J_{ij} \partial^j B \quad \Rightarrow$$

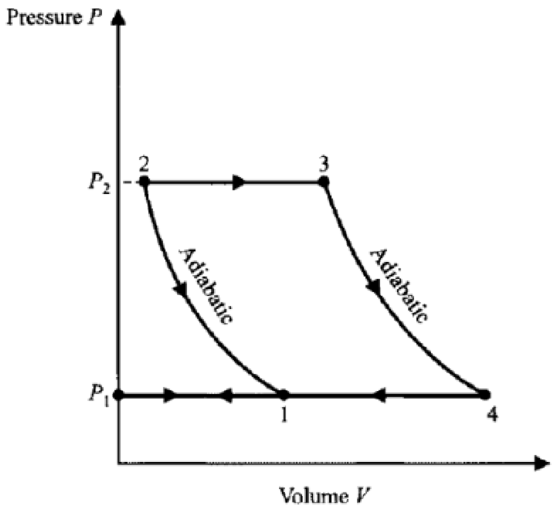
$$
\begin{aligned}
\{C,\{A,B\}\} &= \partial^i C J_{ij} \partial^j \{A,B\} \\
&= \partial^i C J_{ij} \partial^j \left(\partial^k A J_{kl} \partial^l B\right) \\
&= \partial^i C J_{ij} \left(\partial^j \partial^k A J_{kl} \partial^l B + \partial^k A J_{kl} \partial^j \partial^l B\right)
\end{aligned}
$$

8. La figura 1 , se representa un diagrama PV simplificado del ciclo de gas ideal de Joule. Todos los procesos son cuasi-estáticos y C_P es constante. Demuestre que la eficiencia térmica de un motor que

realiza este ciclo es

$$\eta = 1 - \left(\frac{P_1}{P_2}\right)^{(\gamma-1)/\gamma}$$

figura 1. Ciclo de gas ideal Joule



Respuesta a Punto 8

Dado que durante todo el ciclo se tiene que durante proceso isobarico se cumple

$$C_P = \left(\frac{dQ}{dT}\right)_P \quad \Rightarrow \quad Q_{12} = C_P \int_{T_1}^{T_2} dT = C_P(T_2 - T_1) \quad (8{,}1{,}1) \quad \text{si el calor es absorbido}$$

$$\Rightarrow \quad Q_{12} = -C_P \int_{T_1}^{T_2} dT = -C_P(T_2 - T_1) \quad (8{,}1{,}2) \quad \text{si el calor es cedido}$$

Como de $1 \to 2$ es un proceso adiabatico entonces $dQ = 0$ y por lo tanto

$$P_1 V_1^\gamma = P_2 V_2^\gamma \quad (8{,}2)$$

De $2 \to 3$ es un proceso isobarico el calor absorbido por el sistema es:

$$Q_{2\to3} = C_P \Delta T = C_P(T_3 - T_2) \quad \text{Aplicando (8.1.1)}$$

De $3 \to 4$ es un proceso adiabatico:

$$P_1 V_4^\gamma = P_2 V_3^\gamma \quad (8{,}3)$$

Por ultimo de $4 \to 1$ es un proceso isocorico por lo que el calor absorbido es

$$Q_{4\to1} = C_V \Delta T = C_P(T_4 - T_1) \quad \text{Aplicando (8.1.2)}$$

Dividiendo ahora las expresiones (8.2) y (8.3) tenemos que:

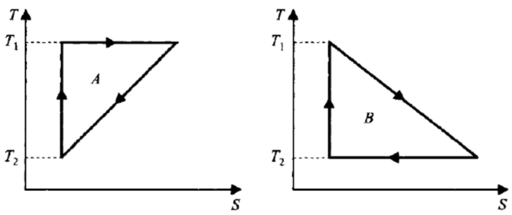
$$\frac{V_1^\gamma}{V_4^\gamma} = \frac{V_2^\gamma}{V_3^\gamma} \quad \Rightarrow \quad \frac{V_1}{V_4} = \frac{V_2}{V_3}$$

$$\Rightarrow \quad V_1 V_3 = V_2 V_4 \quad (8{,}4)$$

Por la definicion de eficiencia:

$$\begin{aligned}
\eta &= 1 - \frac{Q_{4\to 1}}{Q_{2\to 3}} = 1 - \frac{C_P(T_4 - T_1)}{C_P(T_3 - T_2)} \\
&= 1 - \frac{T_3 - T_2}{T_4 - T_1} = 1 - \frac{\frac{P_1 V_4}{nR} - \frac{P_1 V_1}{nR}}{\frac{P_2 V_3}{nR} - \frac{P_2 V_2}{nR}} \quad \text{Aplicando (6.1)} \\
&= 1 - \frac{P_1 V_4 - P_1 V_1}{P_2 V_3 - P_2 V_2} = 1 - \frac{P_1}{P_2}\frac{V_3 - V_2}{V_1 - V_4} \\
&= 1 - \left(\frac{V_2}{V_1}\right)^{\gamma} \frac{V_4 - V_1}{V_3 - V_2} = \left(\frac{V_2}{V_1}\right)^{\gamma - 1} \frac{V_4 V_2 - V_1 V_2}{V_3 V_1 - V_2 V_1} \quad \text{Aplicando (8.2)} \\
&= 1 - \left(\frac{V_2}{V_1}\right)^{\gamma - 1} \frac{V_3 V_1 - V_2 V_1}{V_3 V_1 - V_2 V_1} = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma - 1} \quad \text{Aplicando (8.4)} \\
&= 1 - \left(\frac{V_2^{\gamma}}{V_1^{\gamma}}\right)^{\frac{\gamma - 1}{\gamma}} = 1 - \left(\frac{P_1}{P_2}\right)^{\frac{\gamma - 1}{\gamma}} \quad \text{Aplicando (8.2)}
\end{aligned}$$

9. (a) Deduzca la expresión para la eficiencia de un motor de Carnot directamente de un diagrama TS (Temperatura vs Entropía). (b) Compare las eficiencias de los ciclos A y B de la Figura 2. figura 2.



Respuesta a Punto 9

Dado que el calor y la entropia estan relacionada por la ecuacion:

$$dQ = TdS \quad (9,1)$$

a. En un diagrama PV del un ciclo de Carnot consiste de dos curvas adiabaticas, la cuales en un digrama TS consisten de dos lineas rectas que representa la entropia constante para distintos valores de la temperatura, pero a difrencia del diagrama anterior, este diagrama tambien consistira de dos curvas isotermicas representadas por dos lineas horizontales conectando las dos lineas adiabaticas, como se ve en la figura 1 De esta manera la eficiencia va estar dada por

$$\begin{aligned}
\eta &= 1 - \frac{Q_{4\to 1}}{Q_{2\to 3}} = 1 - \frac{|Q_L|}{|Q_H|} \\
&= 1 - \frac{T_L \Delta S_L}{T_H \Delta S_H} \quad \text{Aplicando (9.1)} \\
&= 1 - \frac{T_L}{T_H}
\end{aligned}$$

Esto ultimo debido a que $2 \to 3$ y $4 \to 1$ son isoentropicos es decir $S_3 = S_2$ y $S_4 = S_1 \Rightarrow \Delta S_L = \Delta S_H$

b. Del diagrama de la izquierda obtenemos que:

$$
\begin{aligned}
|Q_H| &= T_1(S_1 - S_2) = T_1 \Delta S_H \\
|Q_L| &= -\int_{S_1}^{S_2} T(S) dS = -\int_{S_1}^{S_2} \left(\frac{T_1 - T_2}{S_1 - S_2} S - \frac{T_1 S_2 - T_2 S_1}{S_1 - S_2} \right) dS \\
&= -\left(\frac{T_1 - T_2}{S_1 - S_2} \frac{S_2^2 - S_1^2}{2} - \frac{T_1 S_2 - T_2 S_1}{S_1 - S_2} (S_2 - S_1) \right) \\
&= -\left(T_2 - T_1 \frac{S_2 + S_1}{2} + T_1 S_2 - T_2 S_1 \right) \\
&= -\left(\frac{1}{2} T_2 S_1 + \frac{1}{2} T_2 S_2 - \frac{1}{2} T_1 S_1 - \frac{1}{2} T_1 S_2 + T_1 S_2 - T_2 S_1 \right) \\
&= -\left(-\frac{1}{2} T_1 S_1 + \frac{1}{2} T_1 S_2 - \frac{1}{2} T_2 S_1 + \frac{1}{2} T_2 S_2 \right) \\
&= -\left(\frac{1}{2} T_1 (S_2 - S_1) + \frac{1}{2} T_2 (S_2 - S_1) \right) \\
&= \frac{1}{2} (T_1 + T_2)(S_1 - S_2)
\end{aligned}
$$

De esta manera la eficiencia queda:

$$\eta_L = 1 - \frac{|Q_L|}{|Q_H|} = 1 - \frac{T_1 + T_2}{2T_1} = \frac{T_1 - T_2}{2T_1}$$

Ahora del diagrama de la derecha obtenemos que:

$$
\begin{aligned}
|Q_L| &= T_2(S_2 - S_1) \\
|Q_H| &= \int_{S_1}^{S_2} T(S) dS = \int_{S_1}^{S_2} \left(\frac{T_2 - T_1}{S_2 - S_1} S - \frac{T_2 S_1 - T_1 S_2}{S_2 - S_1} \right) dS \\
&= \left(\frac{T_2 - T_1}{S_2 - S_1} \frac{S_2^2 - S_1^2}{2} - \frac{T_2 S_1 - T_1 S_2}{S_2 - S_1} (S_2 - S_1) \right) \\
&= \left(T_2 - T_1 \frac{S_2 + S_1}{2} - T_2 S_1 + T_1 S_2 \right) \\
&= \left(-\frac{1}{2} T_1 S_1 - \frac{1}{2} T_1 S_2 + \frac{1}{2} T_2 S_1 + \frac{1}{2} T_2 S_2 - T_2 S_1 + T_1 S_2 \right) \\
&= \left(-\frac{1}{2} T_1 S_1 + \frac{1}{2} T_1 S_2 - \frac{1}{2} T_2 S_1 + \frac{1}{2} T_2 S_2 \right) \\
&= \left(\frac{1}{2} T_1 (S_2 - S_1) + \frac{1}{2} T_2 (S_2 - S_1) \right) \\
&= \frac{1}{2} (T_1 + T_2)(S_2 - S_1)
\end{aligned}
$$

Por lo que ahora la eficiencia queda:

$$\eta_R = 1 - \frac{|Q_L|}{|Q_H|} = 1 - \frac{2T_2}{T_1 + T_2} = \frac{T_1 - T_2}{T_1 + T_2}$$

Si realizamos la difrencia de estas dos eficiencias tenemos que:

$$\begin{aligned}\eta_L - \eta_R &= \frac{T_1 - T_2}{2T_1} - \frac{T_1 - T_2}{T_1 + T_2} = \frac{(T_1 - T_2)(T_1 + T_2) - 2T_1(T_1 - T_2)}{2T_1(T_1 + T_2)} \\ &= \frac{T_1^2 - T_2^2 - 2T_1^2 + 2T_1T_2}{2T_1(T_1 + T_2)} = \frac{-T_1^2 + 2T_1T_2 - T_2^2}{2T_1(T_1 + T_2)} \\ &= \frac{-(T_1 - T_2)^2}{2T_1(T_1 + T_2)} < 0\end{aligned}$$

De esta forma el ciclo de la derecha es mas eficiente que el de la izquierda.

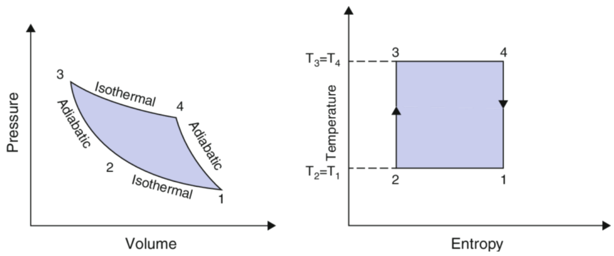


Figura 1: Diagrama TS de un ciclo de Carnot

10. La capacidad calorífica molar a campo magnético constante de un sólido paramagnético a bajas temperaturas varía con la temperatura y el campo según la relación

$$C_{\mathscr{H}} = \frac{B + C\mathscr{H}}{T^2} + DT^2$$

donde B, C y D son constantes. ¿Cuál es el cambio de entropía de n moles de material cuando la temperatura cambia de T_i a T_f mientras que $\mathscr{H}_0$ permanece constante en el valor $\mathscr{H}$

Respuesta a Punto 10

La entropia esta definida por:

$$
\begin{aligned}
dQ = TdS \quad &\Rightarrow \quad \left(\frac{dQ}{dT}\right)_{\mathscr{H}} = T\left(\frac{dS}{dT}\right)_{\mathscr{H}} \\
&\Rightarrow \quad C_{\mathscr{H}} = T\left(\frac{dS}{dT}\right)_{\mathscr{H}} \\
&\Rightarrow \quad dS = \frac{C_{\mathscr{H}}}{T}dT \\
&\Rightarrow \quad dS = \left(\frac{B + C\mathscr{H}_0^2}{T^2} + DT^2\right)\frac{dT}{T} \\
&\Rightarrow \quad S = \int_{T_i}^{T_f}\left(\frac{B + C\mathscr{H}_0^2}{T^2} + DT^2\right)\frac{dT}{T} \\
&\Rightarrow \quad S = \int_{T_i}^{T_f}\left(\frac{B + C\mathscr{H}_0^2}{T^3} + DT\right)dT \\
&\Rightarrow \quad S = \left[-\frac{B + C\mathscr{H}_0^2}{2T^2} + D\frac{T^2}{2}\right]_{T_i}^{T_f} \\
&\Rightarrow \quad S = -\frac{B + C\mathscr{H}_0^2}{2T_f^2} + D\frac{T_f^2}{2} + \frac{B + C\mathscr{H}_0^2}{2T_i^2} - D\frac{T_i^2}{2} \\
&\Rightarrow \quad S = \frac{B + C\mathscr{H}_0^2}{2}\left(\frac{1}{T_i^2} - \frac{1}{T_f^2}\right) + D\left(\frac{T_f^2}{2} - \frac{T_i^2}{2}\right) \\
&\Rightarrow \quad S = \frac{B + C\mathscr{H}_0^2}{2}\left(\frac{T_f^2 - T_i^2}{T_i^2 T_f^2}\right) + D\left(\frac{T_f^2 - T_i^2}{2}\right) \\
&\Rightarrow \quad S = \frac{T_f^2 - T_i^2}{2}\left[\left(\frac{B + C\mathscr{H}_0^2}{T_i^2 T_f^2}\right) + D\left(T_f^2 - T_i^2\right)\right]
\end{aligned}
$$