Physics' notes

by and for the Sapienza's ACSAI 2020/21 students

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1 Measurements

We shall use my largest scales!

Sir Bedivere

Intro Physics is a science based on measurements: in it complex things are to be described and, to do that, we record data and information through units of measure. Every science has to be held by some standards that are shared and comprehended by all of its users; in Math we know that, irrefutably, $1 + 1 = 2$: that is because the properties of *addition* and the entity *one* are widely defined such that there can't be any other outcome but *two*. If, on the other hand, we were processing some Boolean Algebra (those that struggled in Computer Architecture know), we would know that $1 + 1 = 1$, that is because *boolean addition* and the entity *one* in it are defined differently.
In the same fashion Physics needs three different classes of basic entities over which to build everything else on:

- Base quantities: mass, time, length...
- Standards: scales, ticking of a clock, ruler...
- Units of measure: kg, s, m...

An organization which standardized units of measurement is the International System of Units.

1.1 Defining some base quantities

Time Time is measured through the second; which is defined as the amount of time passing every 9.192×10^9 oscillations of a specific radiation emitted by a Cesium-133 atom.

Length Today a meter is defined as the amount of space travelled in a vacuum by light in a time interval of $\frac{1}{299792458}$ seconds.

Mass There exist two different standards for the kilogram:

- A sample of Platinum-Iridium kept in the International Bureau of Measure (near Paris) is regarded as the standard kilogram (there are some problems with it though; its mass has changed during time as it naturally decays).
- Another standard is the amount of atoms contained by 12 atomic mass units of Carbon-12, where an atomic mass unit $1\,\mathrm{u} = 1.66 \times 10^{-27}\,\mathrm{kg}$.

1.2 Handling measurements

Changing units Of course based on where we are in the world or what task we are trying to accomplish there exist different units of measure for the same quantity, a fundamental thing to know is how to switch between them: some changes are fairly trivial, like going from kilometer to meter ($1\,\mathrm{km} = 10^3\,\mathrm{m}$), but others not quite so- an example may be converting minutes to seconds or square kilometers to square miles.
The process is usually the same:

1. Find/know the equivalence between two units of measure.
2. Manipulate the ratio such that the wanted final unit is on top of the fraction.
3. Apply the conversion.

Following on the previous examples, our procedure would look like this:

- $1\,\mathrm{min} = 60\,\mathrm{s} \rightarrow 1 = \frac{60\,\mathrm{s}}{1\,\mathrm{min}}$

$$\begin{aligned} t &= 13\,\mathrm{min} \\ &= 1 \times 13\,\mathrm{min} \\ &= \frac{60\,\mathrm{s}}{1\,\mathrm{min}} \times 13\,\mathrm{min} = \boxed{7.8 \times 10^2\,\mathrm{s}} \end{aligned}$$

- $1.61\,\mathrm{km} = 1\,\mathrm{mi} \rightarrow 1 = \frac{1\,\mathrm{mi}}{1.61\,\mathrm{km}} \rightarrow \frac{1\,\mathrm{mi}^2}{2.59\,\mathrm{km}^2}$

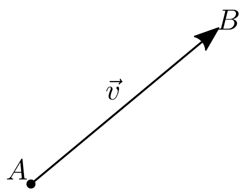
$$\begin{aligned} A &= 27.0\,\mathrm{km}^2 \\ &= \frac{1\,\mathrm{mi}^2}{2.59\,\mathrm{km}^2} \times 27.0\,\mathrm{km}^2 = \boxed{10.4\,\mathrm{mi}^2} \end{aligned}$$

Significant figures The significant figures used to represent a quantity depend on the accuracy of the tool which took the survey: to count the amount of significant figures in a number just count all the digits which are **not** zero, all the zeroes (or groups of) which are in between non-zero figures and all of those zeroes which are deliberately left as decimal digits.
When displaying the result of a calculation, the number of significant figures to be chosen has to be equal to the lower amount of significant figures used by any value of the calculation.

$$1.22357894 \times 2.10 = 2.57$$

2 Vectors

> Geometric entity characterized by a magnitude and a direction.
>
> A vector's minimal definition, Wikipedia



What is a vector In its fundamental concept, a vector is the transformation needed to move from one point (usually the origin) in space to another, in physics, it is mainly used to describe quantities that are bound to multiple dimensions, like movement or force. Dimensionless quantities, like temperature or mass, are instead defined through natural numbers, more correctly called *scalars*.

2.1 How to represent vectors

There are two main ways to represent vectors, the Magnitude-Angle notation and the Component notation: through experience, one may learn when to prefer the use of one over the other.

Magnitude-Angle It describes (in a two-dimensional space) the vector as a pair $\langle m, \sigma \rangle$, where $|v| = m$ is the magnitude (or length) of the vector bound as $0 \leq m \leq +\infty$ (a length can't be negative) and σ is its direction (or angle) bound as $0 = 0° \leq \sigma \leq 2\pi = 360°$.

$$\vec{v} = \begin{cases} m \text{ - Magnitude} \\ \sigma \text{ - Angle} \end{cases} \tag{1}$$

Component notation Represents the vector as a linear sum of scalar products between the coordinates' unit vectors.

$$\vec{v} = v_x \hat{\imath} + v_y \hat{\jmath} \equiv \begin{bmatrix} v_x \\ v_y \end{bmatrix} \tag{2}$$

In this case the scalars are unbounded across $\mathbb{R}$, they can hold any value.

Changing notation Being possible to represent a vector in both ways, it is possible to switch from one notation to the other.

$$\begin{cases} m = \sqrt{v_x^2 + v_y^2} \\ \sigma = \tan \frac{v_y}{v_x} \end{cases} \iff \begin{cases} v_x = m \cos \sigma \\ v_y = m \sin \sigma \end{cases} \tag{3}$$

Unit vectors A unit vector is any vector with magnitude $|v| = 1$. In Magnitude-Angle notation, just let the magnitude equal to 1. In Component notation, divide through scalar multiplication on its magnitude.

$$\hat{v} = \begin{cases} |v| = 1 \text{ - Magnitude-Angle notation} \\ \frac{1}{|v|}\vec{v} \text{ - Component notation} \end{cases} \tag{4}$$

Coordinates of a space To represent vectors in any space with n-dimensions at least n coordinate vectors are needed: the most commonly used set of these is $\langle \hat{\imath}, \hat{\jmath}, \hat{k} \rangle$ (for three dimensions, just $\hat{\imath}, \hat{\jmath}$ for two), which are unit vectors holding properties between each other (orthogonality, etc.) such that they can form a basis (can be used to represent any vector) for the space they are in.

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

2.2 Operations on vectors

As we've just seen, vectors may be represented in multiple ways: this is why, when describing operations, we'll sometimes define two processes.

Vector negation The only unary operation, returns a vector holding same magnitude but opposite direction:

$$-\vec{v} = \begin{cases} \langle m, \sigma + \pi \rangle \text{ -Mag/Angl} \\ (-v_x, -v_y) \text{ -Comp.} \end{cases} \tag{6}$$

Vector sum $+$ We've previously said how a vector represents the movement from one point to another: the result of summing vectors is a vector representing movement from a starting point to an ending point if multiple movements happened.

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y) \tag{7}$$

This is one of the few cases in which to compute the operation just one representation is usable, the Component one: indeed to compute through Magnitude-Angle representation we first need to switch into Component notation and back again with the result.
This operation holds both the commutativity and the associativity law.

Scalar multiplication Binary operation taking a scalar value and a vector: the result has its magnitude equal to the product of the original vector's magnitude and the scalar value. Since a magnitude can't be negative if the scalar value is negative the direction of the resulting vector will be opposite to the original one.

$$a\vec{v} = \begin{cases} \langle |am|, \texttt{if } a \geq 0 : \sigma \texttt{ otherwise } \sigma + \pi \rangle \text{ -Mag/Angl} \\ (av_x, av_y) \text{ -Comp.} \end{cases} \tag{8}$$

Dot product $\cdot$ Instinctively speaking, the dot product is an operation which returns a scalar representing the similarity of two vectors: indeed, if two vectors are parallel, their dot product will be equal to the product of their magnitudes, while if they are orthogonal (perpendicular) the result will default to 0.

$$\vec{a} \cdot \vec{b} = \begin{cases} |a||b|\cos(\phi) \text{ -Mag/Angl} \\ a_x b_x + a_y b_y \text{ -Comp.} \end{cases} \tag{9}$$

The angle between the two vectors ϕ can be found by computing:

$$\cos(\phi) = \frac{\vec{a} \cdot \vec{b}}{|a||b|} \tag{10}$$

As we can see, if that angle is not given then the dot product is better found by using Component notation.
The dot product too holds the commutativity and distributivity law, furthermore, if any scalar value is multiplying one of the two vectors, that can be taken out of the operation and multiplied to the result.

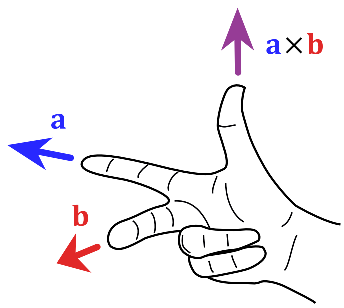
$$\vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b}) \tag{11}$$

Cross product $\times$ Maybe the most important operation on vectors, returns a vector orthogonal to the input vectors and with magnitude proportional to the orthogonality of the two inputs: that is, if the two vectors are parallel it is 0, while it will be equal to the product of the two input magnitudes if they are perpendicular.

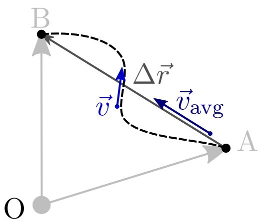
$$\vec{a} \times \vec{b} = \begin{cases} \langle |a||b|\sin(\phi), \sigma \text{ ortho. to inputs} \rangle \text{ -Mag/Angl} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\hat{i} - (a_x b_z - a_z b_x)\hat{j} + (a_x b_y - a_y b_x)\hat{k} \end{cases} \tag{12}$$

The distributivity law holds for the cross product, but does **not** with the commutativity law: we can see it through the so-called *right-hand rule*. Furthermore, here too constants multiplied to our vectors can be brought outside the operation.

Right-hand rule The right-hand rule is a simple way to imagine the direction of the vector resulting off a cross product, indeed it is not easy to find it through the Magnitude-Angle notation, nor it is so through Component notation (even though by crunching the numbers it is possible to do so). If done well it is possible to visualize how impossible it is to process the same result by switching the arguments: spoiler it would be of the opposite direction.



3 Motion in Two and Three Dimensions



As previously stated in the Vector section, the base definition of vector is the movement from one point to another: it is only natural that *real* movement may be described through vectors.

3.1 Basic definitions

Position It can be described through a vector that extends from a reference point to a particle, letting the user know its position; in unit vector notation:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{13}$$

Displacement If we let the particle move, its position vector will change. This difference can be reflected through the displacement $\Delta\vec{r}$, which is:

$$\Delta\vec{r} = \vec{r_2} - \vec{r_1} \tag{14}$$

Or, in unit vector notation:

$$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \tag{15}$$

Velocity When a particle moves through a displacement-Δr in a time interval Δt, its average velocity is:

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{r}}{\Delta t} \tag{16}$$

The equation clarifies that the direction of the velocity will be the same as the direction of the displacement, in vector notation:

$$\vec{v}_{\text{avg}} = \frac{\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k} \tag{17}$$

Instantaneous Velocity To find the velocity of a particle at instant t we take the value that $\vec{v}_{\text{avg}}$ assumes as the interval Δt approaches 0.

$$\vec{x} = \frac{d\vec{r}}{dt} \tag{18}$$

Or, in unit vector notation:

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \quad \leftarrow \quad v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt} \tag{19}$$

Acceleration When a particle's velocity changes, its average acceleration $\vec{a}_{avg}$ is:

$$\vec{a}_{\text{avg}} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t} \tag{20}$$

Instantaneous Acceleration As for the velocity, to know the acceleration in an instant t, we take the value $\vec{a}_{\text{avg}}$ assumes as the interval Δt approaches 0.

$$\vec{a} = \frac{d\vec{v}}{dt} \tag{21}$$

In unit vector notation:

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k} \quad \leftarrow \quad a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad a_z = \frac{dv_z}{dt} \tag{22}$$

The direction of an acceleration vector does not extend from one position to another, it just shows the direction for a particle located at its tail.

3.2 Applications

Projectile motion A special case of two-dimensional motion, the particle moves with a constant acceleration directed downwards: the free fall acceleration $\vec{g}$. When the projectile is launched, its initial velocity $\vec{v}_0$ is writable as:

$$\vec{v}_0 = v_{0x}\hat{\imath} + v_{0y}\hat{\jmath} \tag{23}$$

The two components v_{0x} and v_{0y} can be found if we know the angle θ_0 between $\vec{v}_0$ and the positive x direction:

$$v_{0x} = v_0 \cos\theta_0, \quad v_{0y} = v_0 \sin\theta_0 \tag{24}$$

During the motion both position vector $\vec{r}$ and velocity vector $\vec{v}$ change continuously, though the acceleration remain constant. (There's no horizontal acceleration!) The two motions are independent of each other. The horizontal motion has no effect on the vertical one.

Horizontal motion Since there is no acceleration, the horizontal component v_x remains unchanged: therefore at any time t the horizontal displacement $x - x_0$ is:

$$x - x_0 = v_{0x}t \tag{25}$$

Vertical motion Since the acceleration is constant we can apply the equation we have seen in the one dimensional motion chapter. Thus, we have the displacement $y - y_0$

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2 \tag{26}$$

Similarly, for the final velocity v_y

$$v_y = v_{0y} - gt \qquad v_y^2 = v_{0y}^2 - 2g(y - y_0) \tag{27}$$

The path's equation We can find the equation of the path of the projectile (called *trajectory*) by solving equation (25) for t and substituting into equation (26).

$$y = (\tan\theta_0)x - \frac{gx^2}{2v_{0x}^2} \tag{28}$$

The horizontal range The horizontal distance the projectile has traveled when it returns to its initial height is called horizontal range R:

$$R = \frac{v_0^2}{g}\sin(2\theta_0) \tag{29}$$

The maximum horizontal range R can be reached at angle $\theta = 45° = \frac{\pi}{4}$.

Calculation done with the formula seen in this paragraph may differ a lot with the actual motion of the projectile as we assume the air has no effect on the projectile.

Uniform Circular Motion A particle in uniform circular motion travels around a circle or a circular arc at constant speed. However, since the particle is moving in circles the direction of the velocity changes, thus the particle is accelerating. Velocity is always directed tangent to the circle in the direction of motion, while the acceleration is directed radially inward. Because of this, the acceleration is called centripetal acceleration. The magnitude of this acceleration $\vec{a}$ is:

$$a = \frac{v^2}{r} \tag{30}$$

where r is the radius of the circle and v is the speed.

During the motion, the particle travels the circumference of the circle in time T:

$$T = \frac{2\pi r}{v} \tag{31}$$

T is called *period of revolution* or simply *period.*

Relative motion in one dimension The velocity of a particle depends on the reference frame of whoever is observing it. Let's have two reference frames A and B, both observing a particle P, then the following equations hold:

- Position:

$$x_{\mathrm{PA}} = x_{\mathrm{PB}} + x_{\mathrm{BA}} \tag{32}$$

- Velocity, the time derivative of equation (32):

$$v_{\mathrm{PA}} = v_{\mathrm{PB}} + v_{\mathrm{BA}} \tag{33}$$

- Acceleration, the time derivative of equation (33), since v_{BA} is constant the derivative will be 0, thus:

$$a_{\mathrm{PA}} = a_{\mathrm{PB}} \tag{34}$$

  Observers on different frames of reference that move at constant velocity relative to each other will measure the same acceleration for a moving particle.

Relative motion in multiple dimensions Similarly as in one dimension: by having two reference frames A and B, both observing a particle P, the following equations hold:

- Position:
$$\vec{r}_{\mathrm{PA}} = \vec{r}_{\mathrm{PB}} + \vec{r}_{\mathrm{BA}} \tag{35}$$
- Velocity, the time derivative of equation (35):
$$\vec{v}_{\mathrm{PA}} = \vec{v}_{\mathrm{PB}} + \vec{v}_{\mathrm{BA}} \tag{36}$$
- Acceleration, the time derivative of equation (36), since v_{BA} is constant the derivative will be 0, thus:
$$\vec{a}_{\mathrm{PA}} = \vec{a}_{\mathrm{PB}} \tag{37}$$

  The same rule of one dimensional motion holds, observers on different frames of reference that move at constant velocity relative to each other will measure the same acceleration for a moving particle.