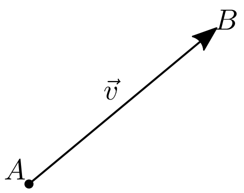
1 Vectors

> Geometric entity characterized by a magnitude and a direction.
>
> A vector's minimal definition, Wikipedia



What is a vector for In its fundamental concept, a vector is the transformation needed to move from one point (usually the origin) in space to another, in physics, it is mainly used to describe quantities that are bound to multiple dimensions, like movement or force. Dimensionless quantities, like temperature or mass, are instead defined through natural numbers, more correctly called *scalars*.

How to represent vectors There are two main ways to represent vectors, the Magnitude-Angle notation and the Component notation: through experience, one may learn when to prefer the use of one over the other.

Magnitude-Angle It describes (in a two dimensional space) the vector as a pair $\langle m, \sigma \rangle$, where $|v| = m$ is the magnitude (or length) of the vector bound as $0 \leq m \leq +\infty$ (a length can't be negative) and σ is its direction (or angle) bound as $0 = 0° \leq \sigma \leq 2\pi = 360°$.

$$\vec{v} = \begin{cases} m \text{ - Magnitude} \\ \sigma \text{ - Angle} \end{cases} \tag{1}$$

Component notation Represents the vector as a linear sum of scalar products between the coordinates' unit vectors.

$$\vec{v} = v_x \hat{\imath} + v_y \hat{\jmath} \equiv \begin{bmatrix} v_x \\ v_y \end{bmatrix} \tag{2}$$

In this case the scalars are unbounded across $\mathbb{R}$, they can hold any value.

Changing notation Being possible to represent a vector in both ways, it is possible to switch from one notation to the other.

$$\begin{cases} m = \sqrt{v_x^2 + v_y^2} \\ \sigma = \tan \frac{v_y}{v_x} \end{cases} \iff \begin{cases} v_x = m \cos \sigma \\ v_y = m \sin \sigma \end{cases} \tag{3}$$

Unit vectors A unit vector is any vector with magnitude $|v| = 1$. In Magnitude-Angle notation, just let the magnitude equal to 1. In Component notation, divide through scalar multiplication on its magnitude.

$$\hat{v} = \begin{cases} |v| = 1 \text{ - Magnitude-Angle notation} \\ \frac{1}{|v|}\vec{v} \text{ - Component notation} \end{cases} \tag{4}$$

Coordinates of a space To represent vectors in any space with n-dimensions at least n coordinate vectors are needed: the most commonly used set of these is $\langle \hat{\imath}, \hat{\jmath}, \hat{k} \rangle$ (for three dimensions, just $\hat{\imath}, \hat{\jmath}$ for two), which are unit vectors holding properties between eachother (orthogonality, etc.) such that they can form a basis (can be used to represent any vector) for the space they are in.

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

Operations on vectors As we've just seen, vectors may be represented in multiple ways: this is why now, when describing each operation, we'll also usually define two processes.

Vector negation The only unary operation, returns a vector holding same magnitude but opposite direction:

$$-\vec{v} = \begin{cases} \langle m, \sigma + \pi \rangle \text{ -Mag/Angl} \\ (-v_x, -v_y) \text{ -Comp.} \end{cases} \tag{6}$$

Vector sum $+$ We've previously said how a vector represents the movement from one point to another: the result of summing vectors is a vector representing movement from a starting point to an ending point if multiple movements happened.

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y) \tag{7}$$

This is one of the few cases in which to compute the operation just one representation is usable, the Component one: indeed to compute through Magnitude-Angle representation we first need to switch into Component notation and back again with the result.
This operation holds both the commutativity and the associativity law.

Scalar multiplication Binary operation taking a scalar value and a vector: the result has its magnitude equal to the product of the original vector's magnitude and the scalar value. Since a magnitude can't be negative if the scalar value is negative the direction of the resulting vector will be opposite to the original one.

$$a\vec{v} = \begin{cases} \langle |am|, \texttt{if } a \geq 0 : \sigma \texttt{ otherwise } \sigma + \pi \rangle \text{ -Mag/Angl} \\ (av_x, av_y) \text{ -Comp.} \end{cases} \tag{8}$$

Dot product $\cdot$ Instinctively speaking, the dot product is an operation which returns a scalar representing the similarity of two vectors: indeed, if two vectors are parallel, their dot product will be equal to the product of their magnitudes, while if they are orthogonal (perpendicular) the result will default to 0.

$$\vec{a} \cdot \vec{b} = \begin{cases} |a||b| \cos(\phi) \text{ -Mag/Angl} \\ a_x b_x + a_y b_y \text{ -Comp.} \end{cases} \tag{9}$$

The angle between the two vectors ϕ can be found by computing:

$$\cos(\phi) = \frac{\vec{a} \cdot \vec{b}}{|a||b|} \tag{10}$$

As we can see, if that angle is not given then the dot product is better found by using Component notation.
The dot product too holds the commutativity and distributivity law, furthermore, if any scalar value is multiplying one of the two vectors, that can be taken out of the operation and multiplied to the result.

$$\vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b}) \tag{11}$$

Cross product $\times$ Maybe the most important operation on vectors, returns a vector orthogonal to the input vectors and with magnitude proportional to the orthogonality of the two inputs: that is, if the two vectors are parallel it is 0, while it will be equal to the product of the two input magnitudes if they are perpendicular.

$$\vec{a} \times \vec{b} = \begin{cases} \langle |a||b| \sin(\phi), \sigma \text{ ortho. to inputs} \rangle \text{ -Mag/Angl} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\hat{i} - (a_x b_z - a_z b_x)\hat{j} + (a_x b_y - a_y b_x)\hat{k} \end{cases} \tag{12}$$

The distributivity law holds for the cross product, but does **not** with the commutativity law: we can see it through the so-called *right hand rule*. Furthermore, here too constants multiplied to our vectors can be brought outside the operation.

Right hand rule The right hand rule is a simple way to imagine the direction of the vector resulting off a cross product, indeed it is not easy to find it through the Magnitude-Angle notation, nor it is so through Component notation (even though by crunching the numbers it is possible to do so). If done well it is possible to visualize how impossible it is to process the same result by switching the arguments: spoiler it would be of the opposite direction.