

Applied Compositional Thinking for Engineers



Session 7

Life is hard

The story so far

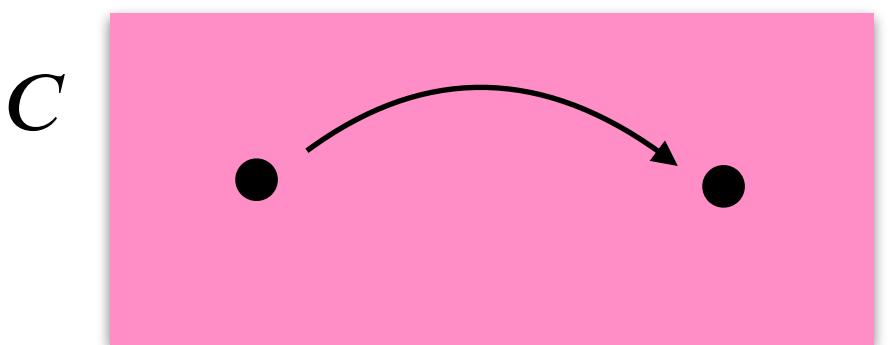
- We have defined **what is a category** and provided several examples.
- We have defined notions of **product**, **coproduct**, **subcategory**, etc.
- We have been looking at **posets** and how they represent **trade-offs**.
- We have seen how **posets can be interpreted as categories**.

- Today:
 - **Monotone functions**
 - **Lattices**; meet, join. Interpretation as product/coproduct.
 - **Functors** as generalization of monotone functions on posets.

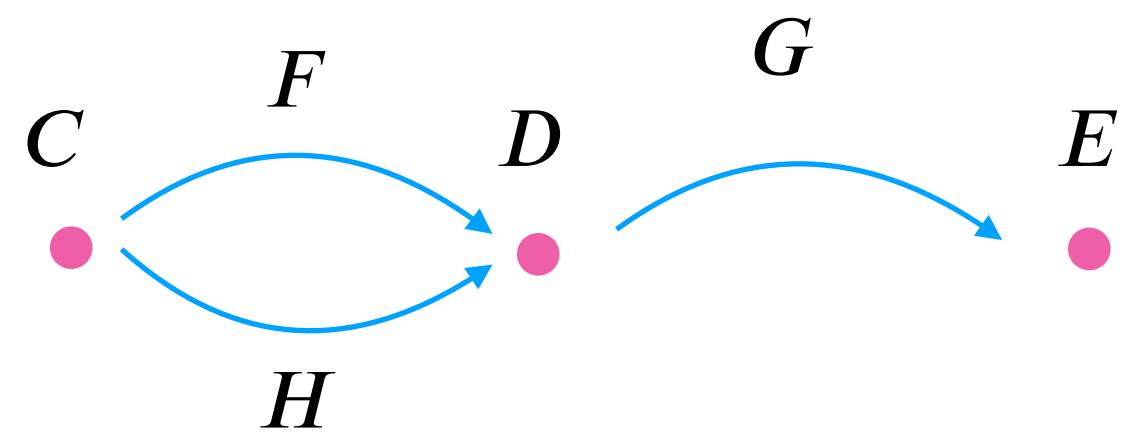


Looking ahead

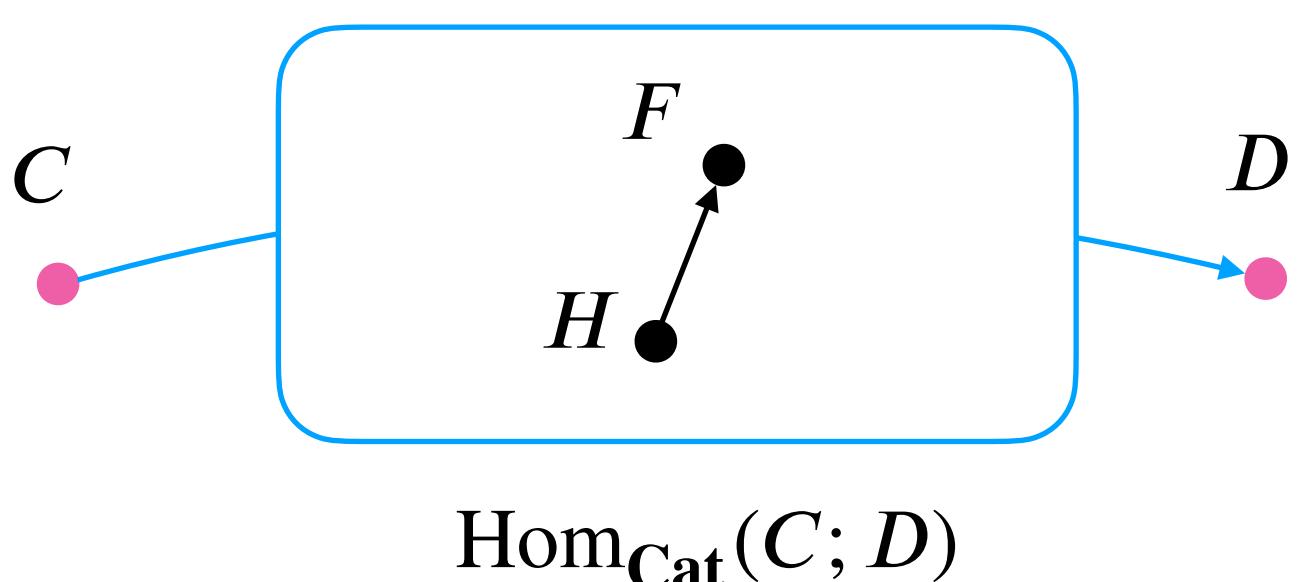
- ▶ In a category, morphisms are arrows between objects.

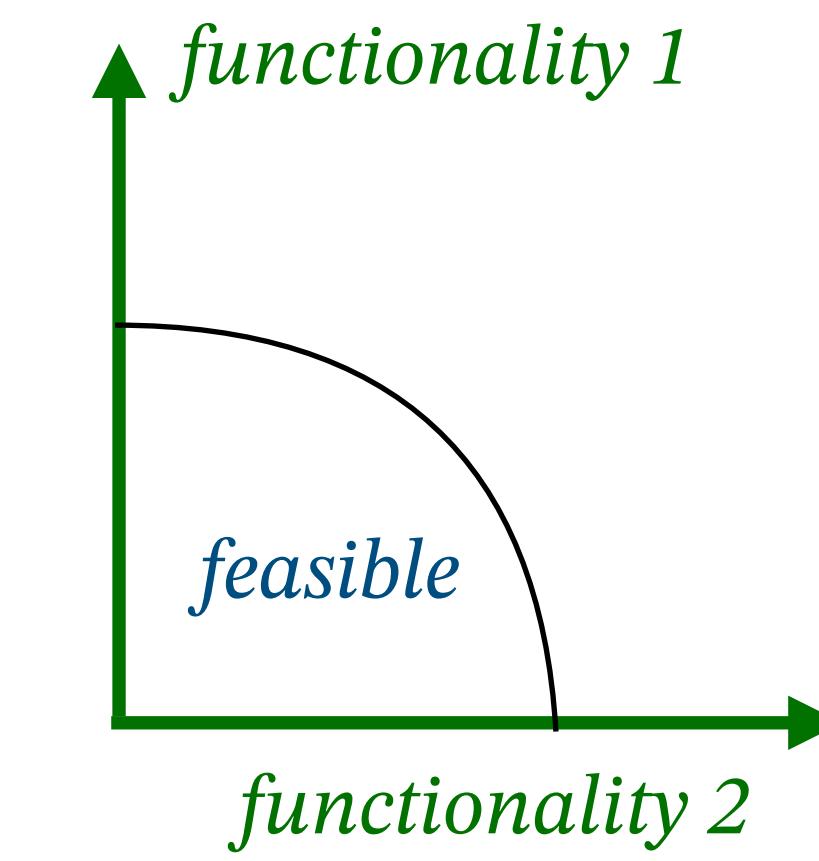
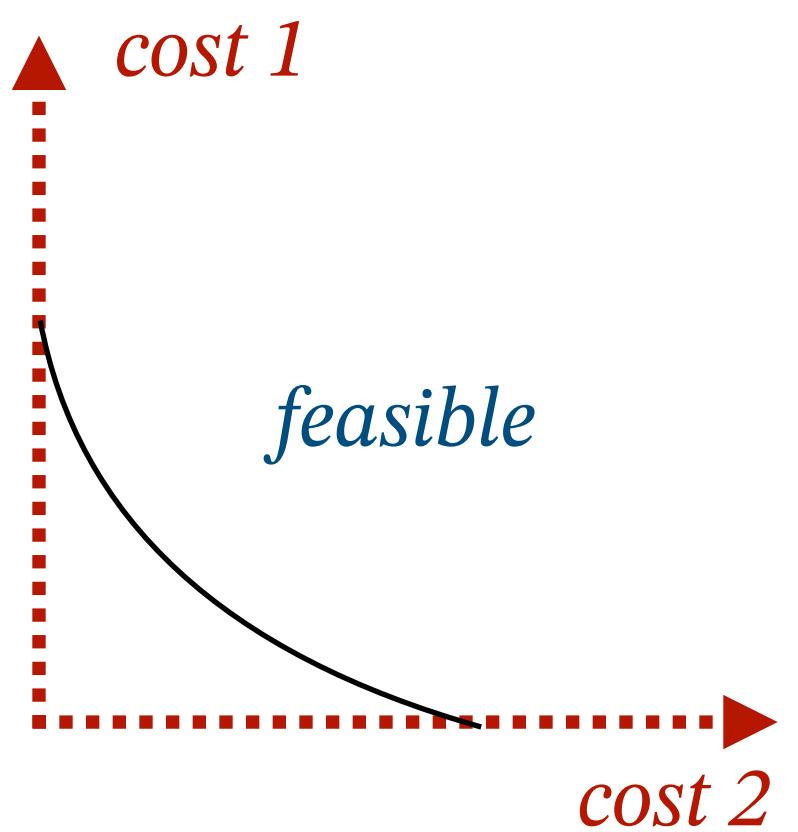
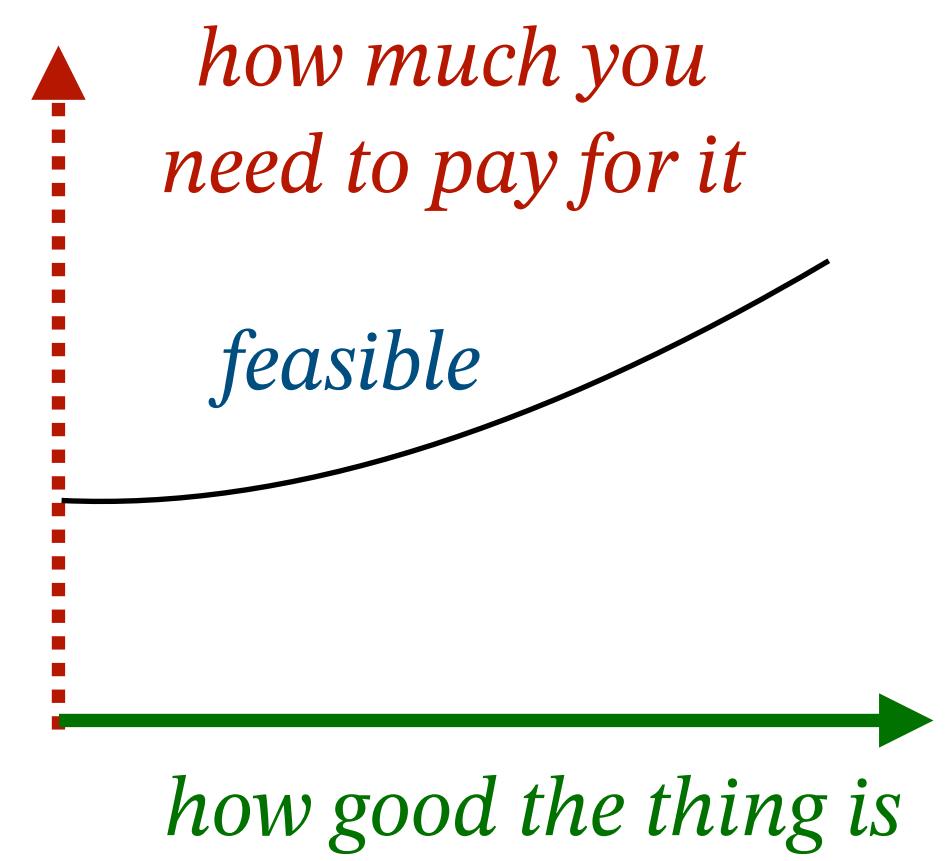
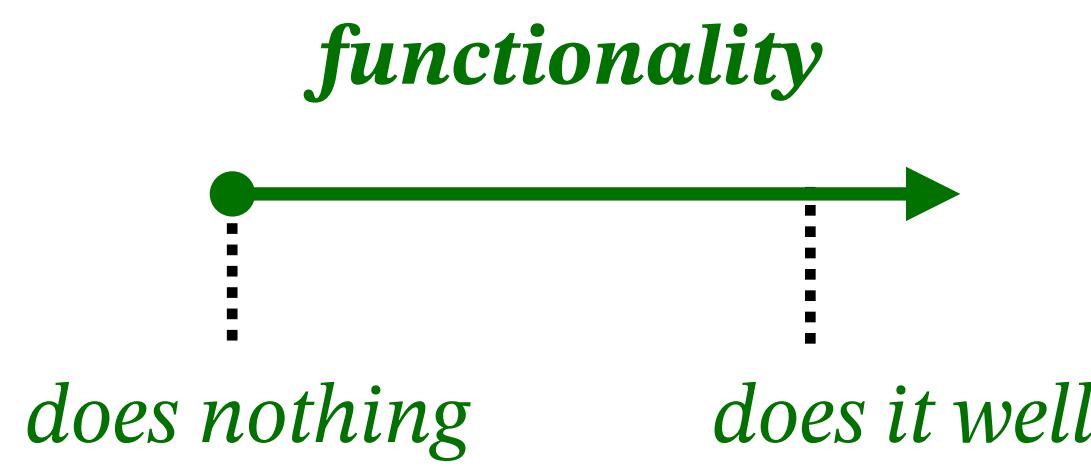
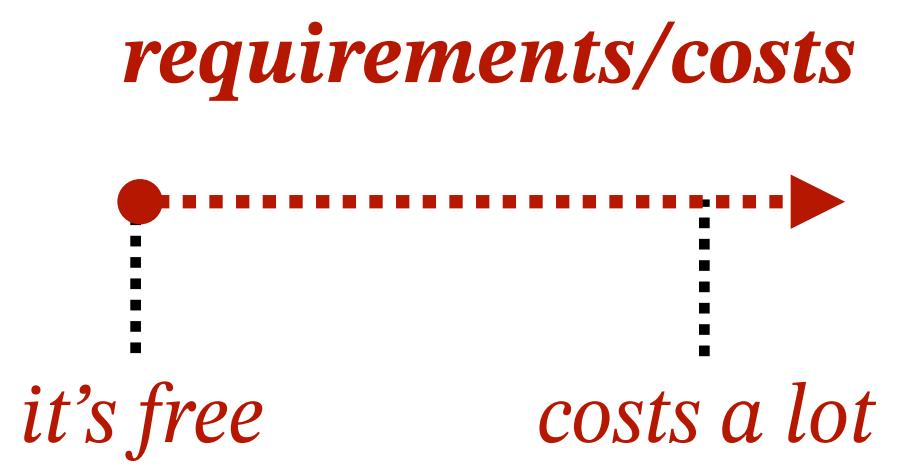


- ▶ In the **category of categories** \mathbf{Cat} , functors are **arrows between categories**.



- ▶ **Natural transformations** are arrows between functors with same domain/codomain.

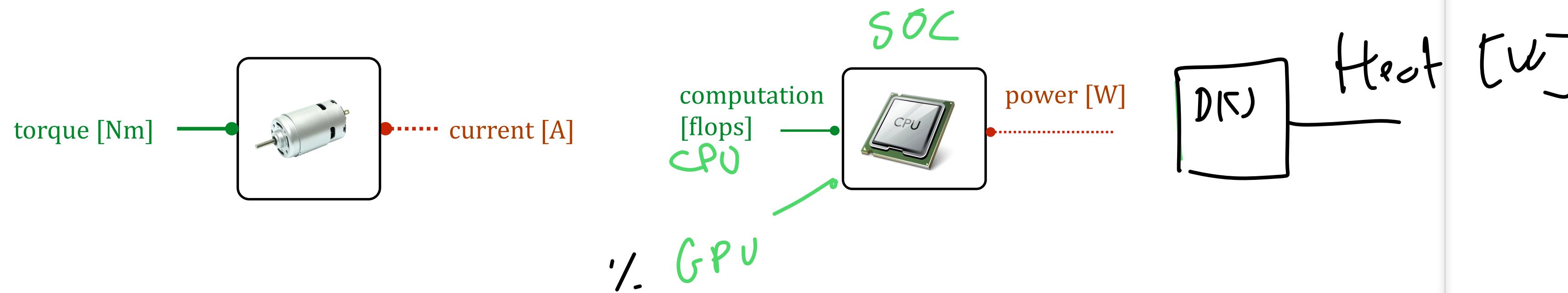




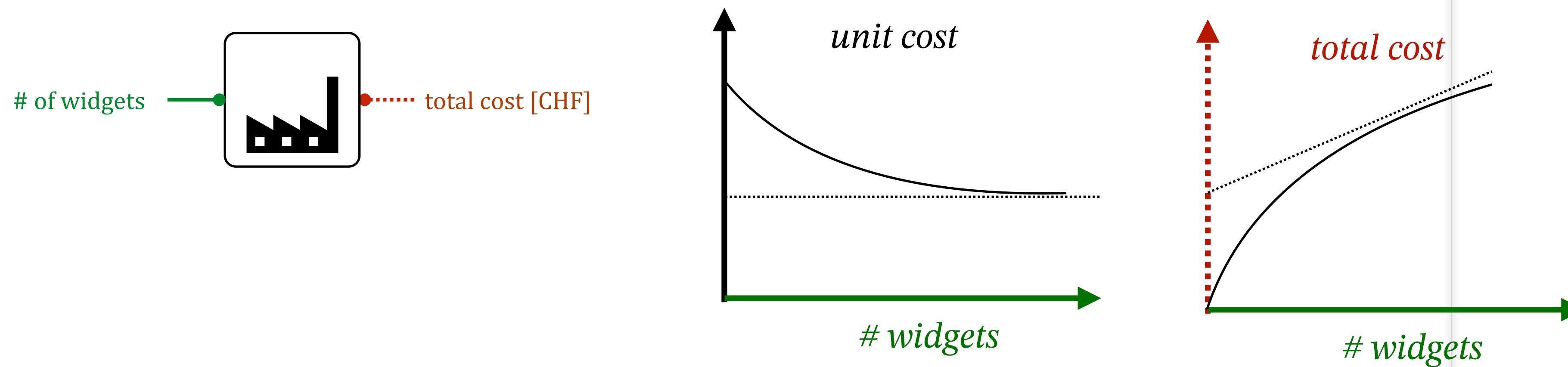
Monotone functions

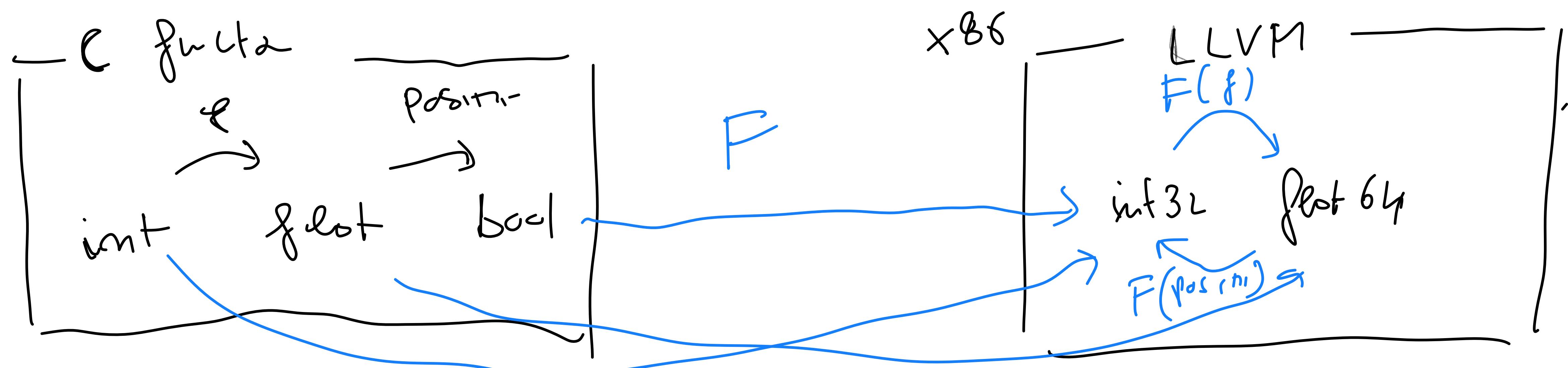
- Monotone functions on \mathbb{R} = “**non-decreasing** functions”.

- Examples: all dimensioning relations in engineering:



- Example: Manufacturing cost as a function of # of widgets.

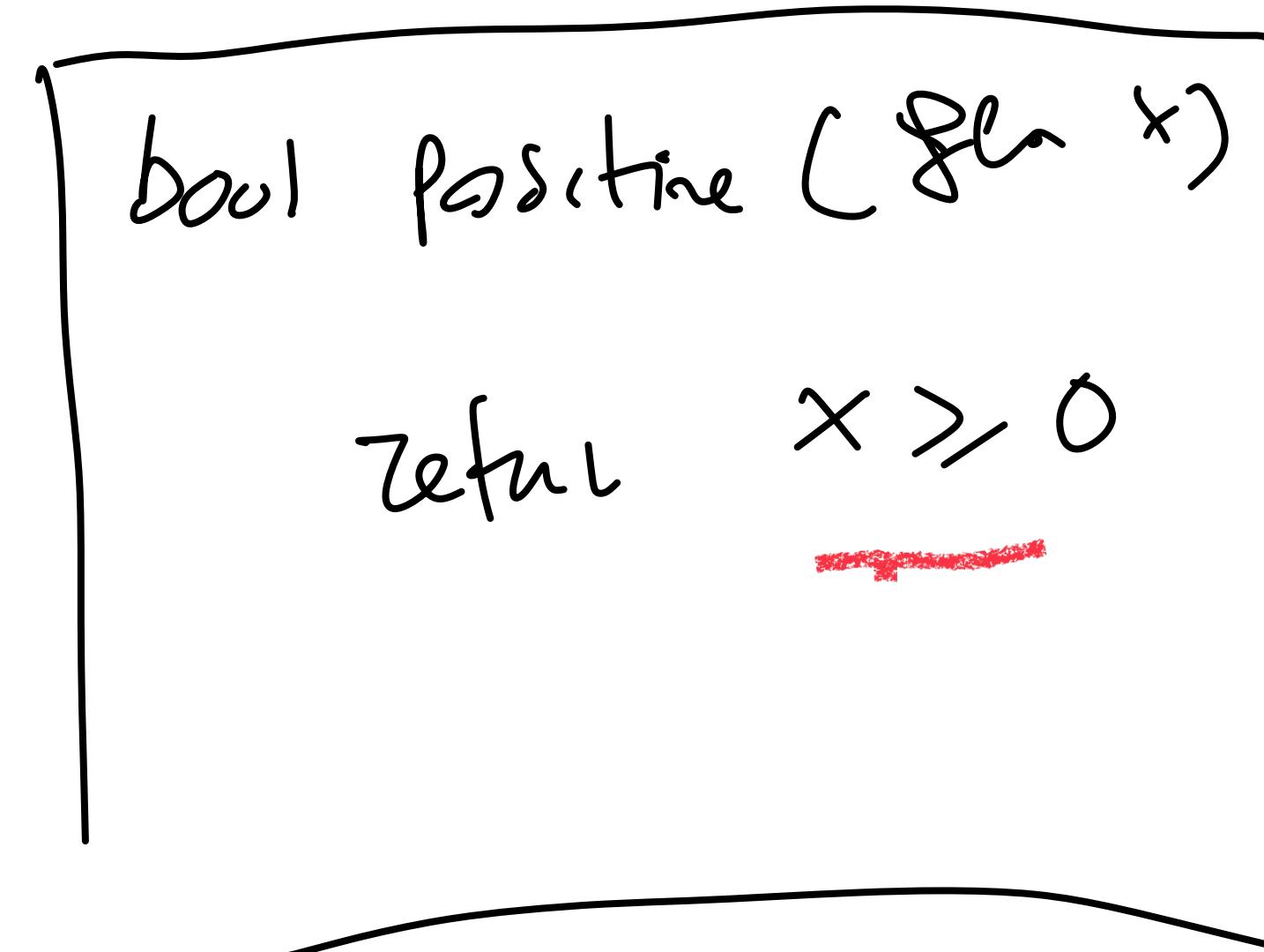
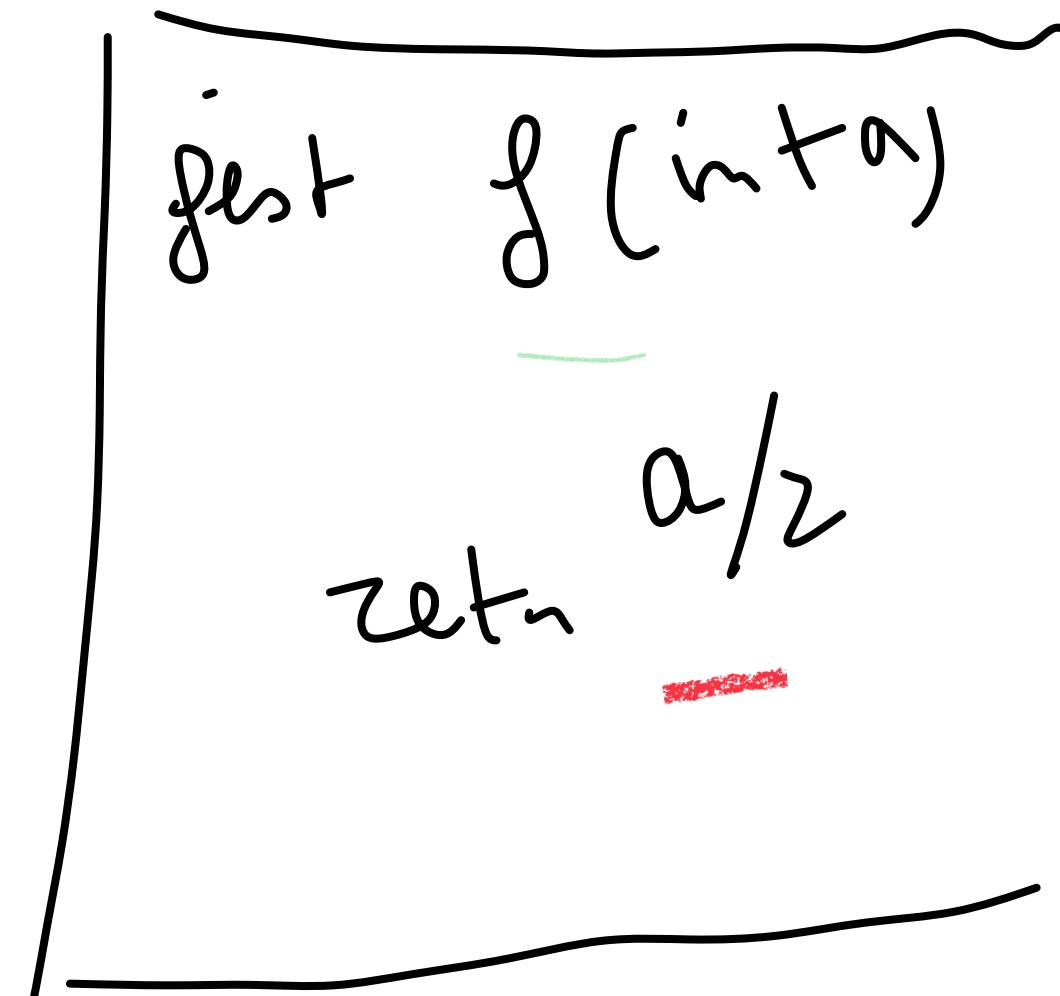




$f: \text{int} \rightarrow \text{float}$

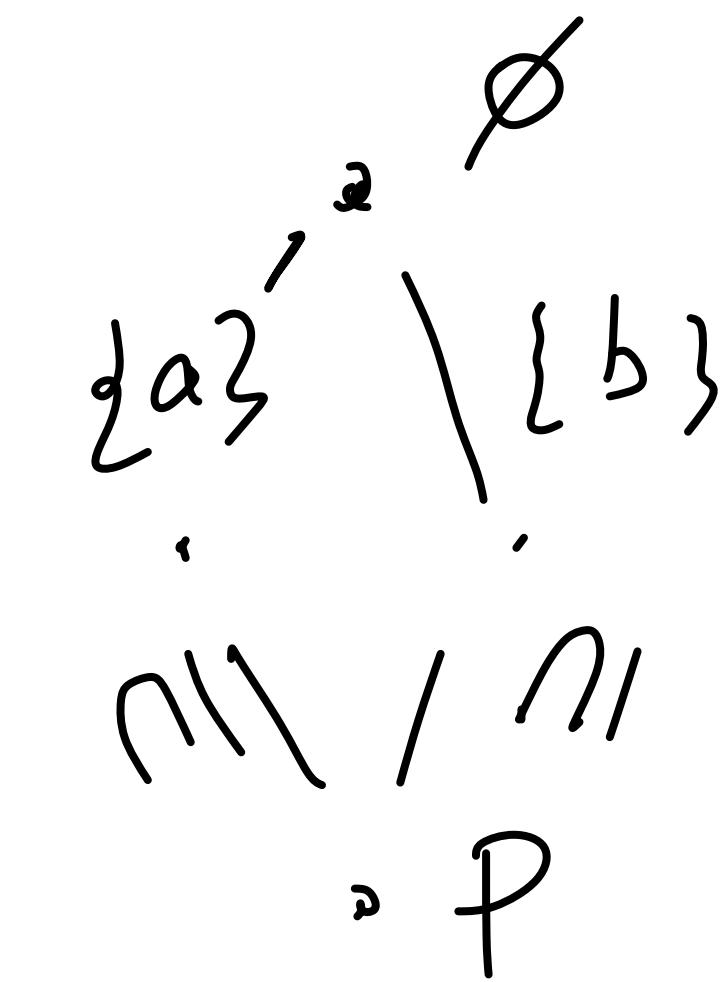
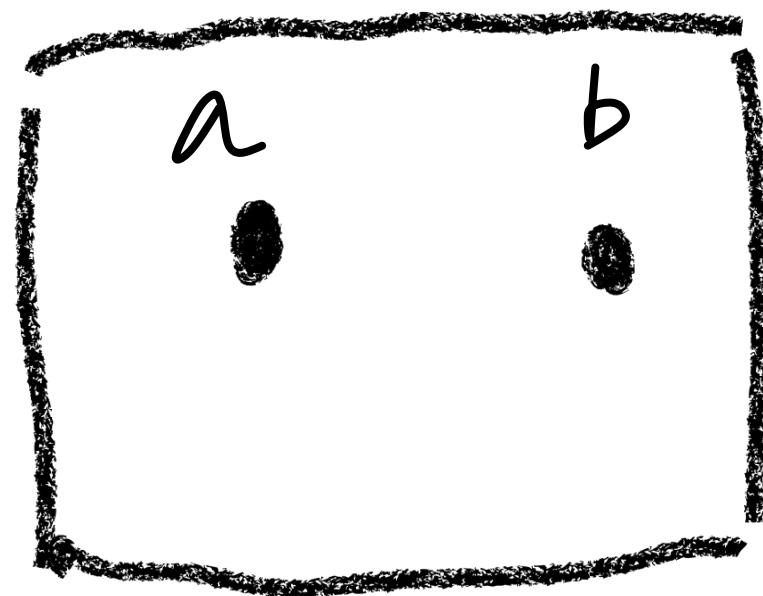
POSITIVE : $\text{float} \rightarrow \text{bool}$

$F(\underline{f}; \underline{\text{POSITIVE}})$



? $= F(\underline{f}) ; F(\underline{\text{POSITIVE}})$

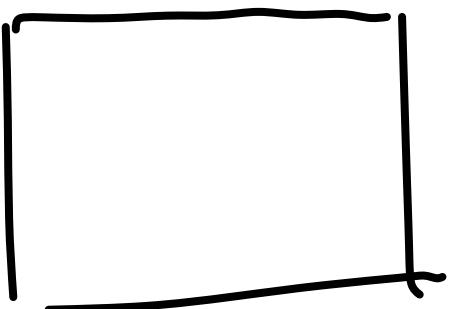
$$\langle P, \leq \rangle \longrightarrow \langle UP, \supseteq \rangle$$



THM . THE UPPER SETS OF
A POSSET FORM A LATTICE.

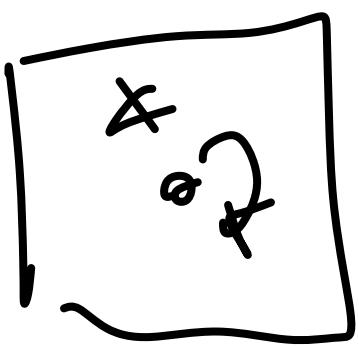
DAVES - PRESENTLY (SEXIST)

0



$$\text{ob } 0 = \emptyset$$

-



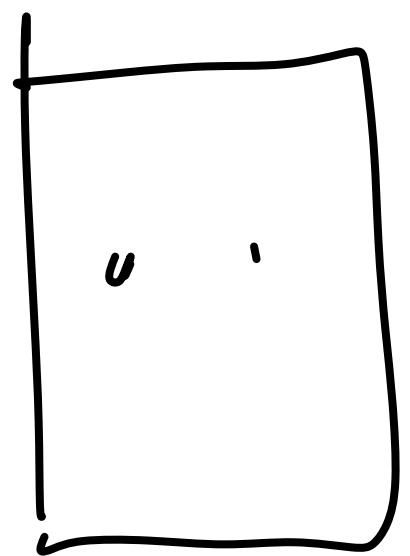
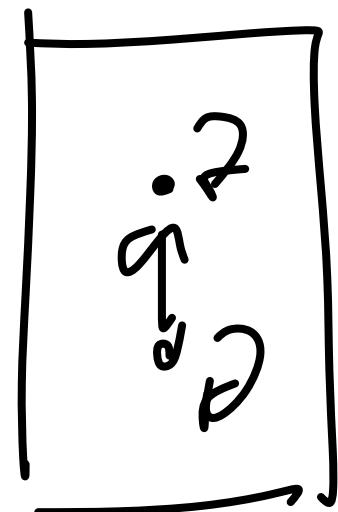
$$\text{ob } 1 = \{*\}$$

$$\text{Hcm}(x; *) = \{ \text{Id}_x \}$$

Bool

2

$$2 = 1 + 1$$



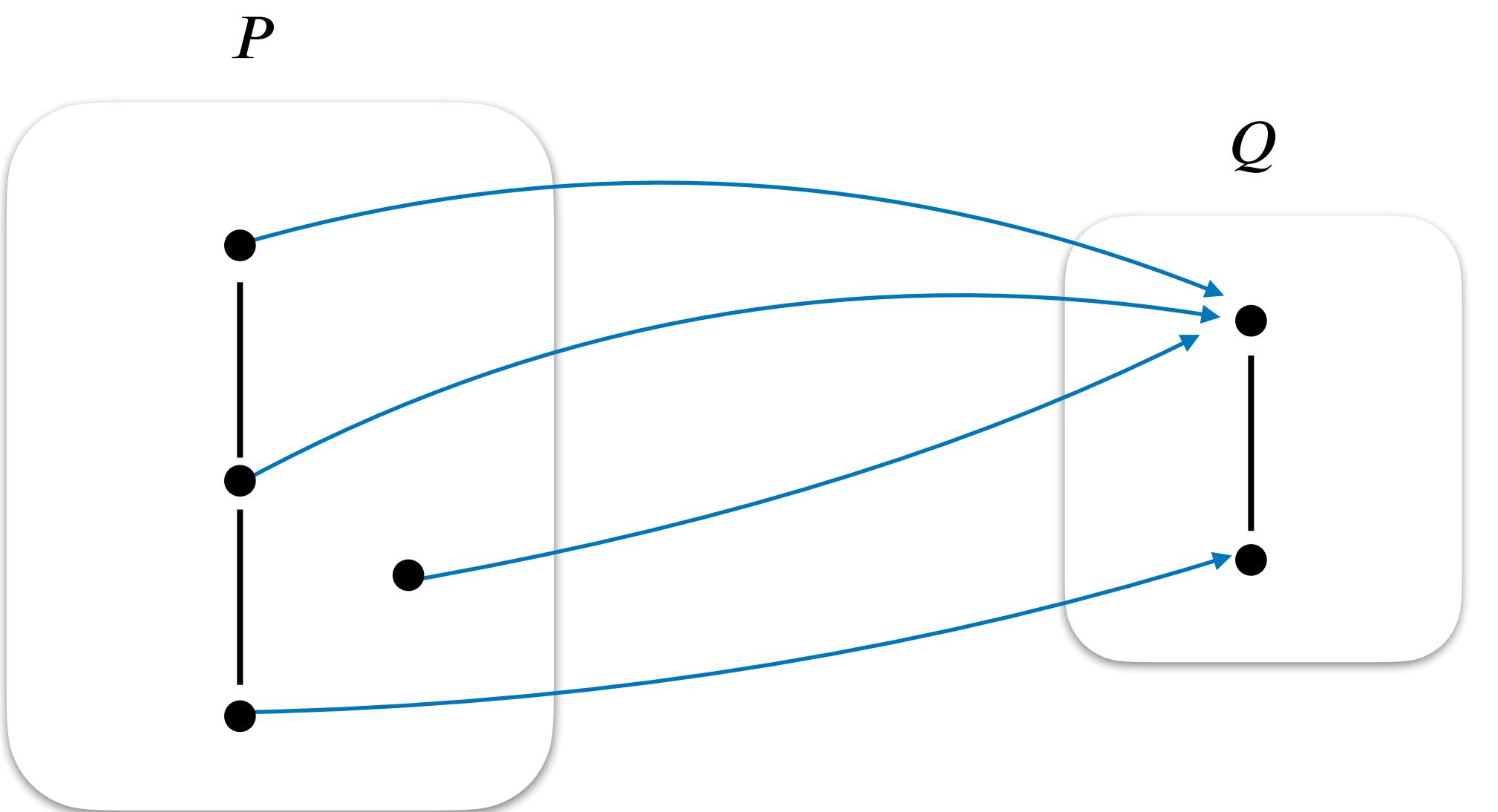
$$(X + Y) \times A \underset{\sim}{=} X \times A + X \times Y$$

Monotone functions on posets

- ▶ A **function** $f : \langle P, \leq_P \rangle \rightarrow \langle Q, \leq_Q \rangle$ is **monotone** iff

$$a \leq_P b$$

$$\frac{}{f(a) \leq_Q f(b)}$$



- ▶ It is an **order isomorphism** if it is **injective** and

$$a \leq_P b$$

$$\frac{}{f(a) \leq_Q f(b)}$$



Rounding functions

- There are a few **rounding functions**:

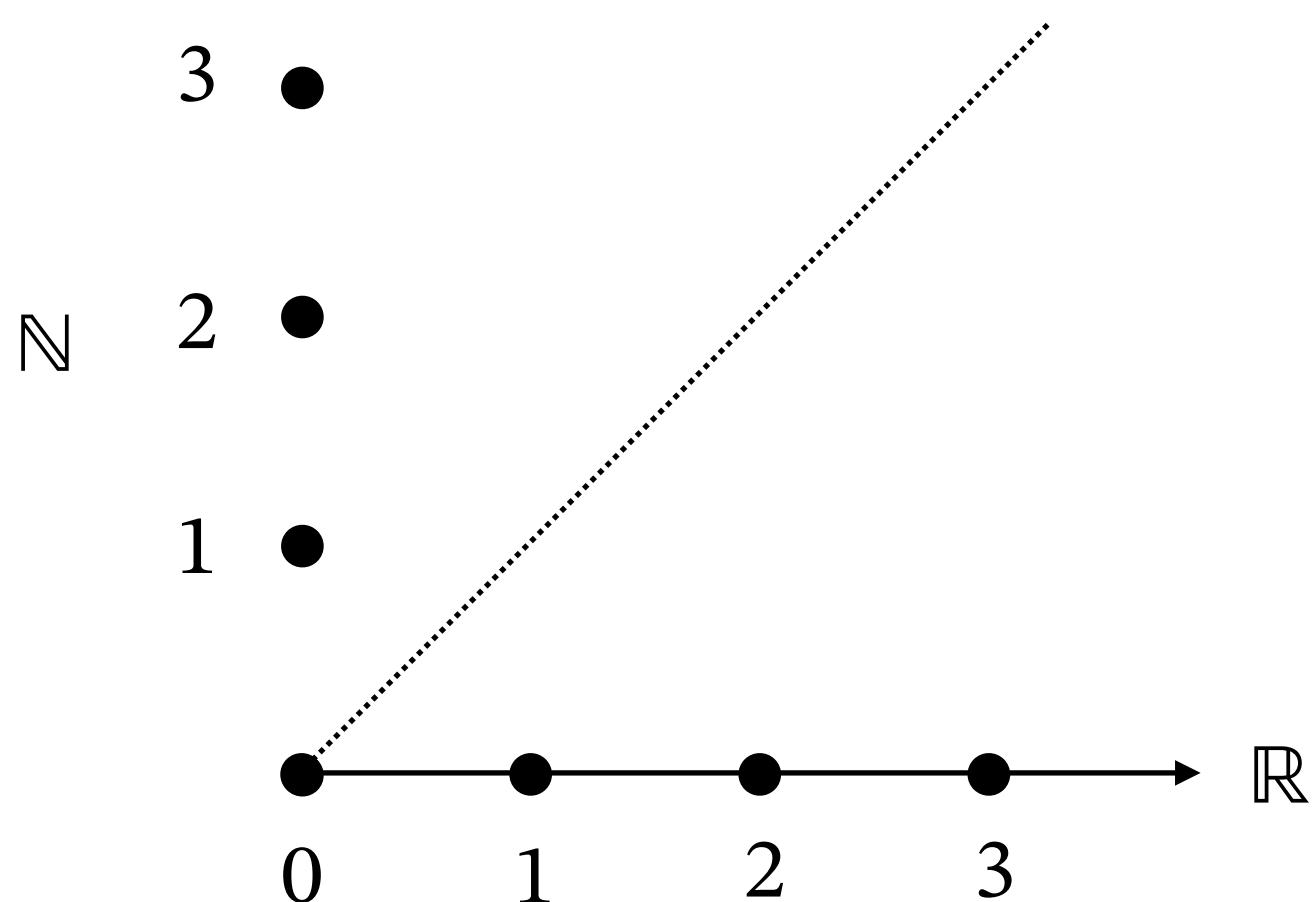
$\text{ceil} : \langle \mathbb{R}, \leq \rangle \rightarrow \langle \mathbb{N}, \leq \rangle$

$\text{floor} : \langle \mathbb{R}, \leq \rangle \rightarrow \langle \mathbb{N}, \leq \rangle$

$\text{rntte} : \langle \mathbb{R}, \leq \rangle \rightarrow \langle \mathbb{N}, \leq \rangle$

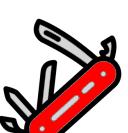
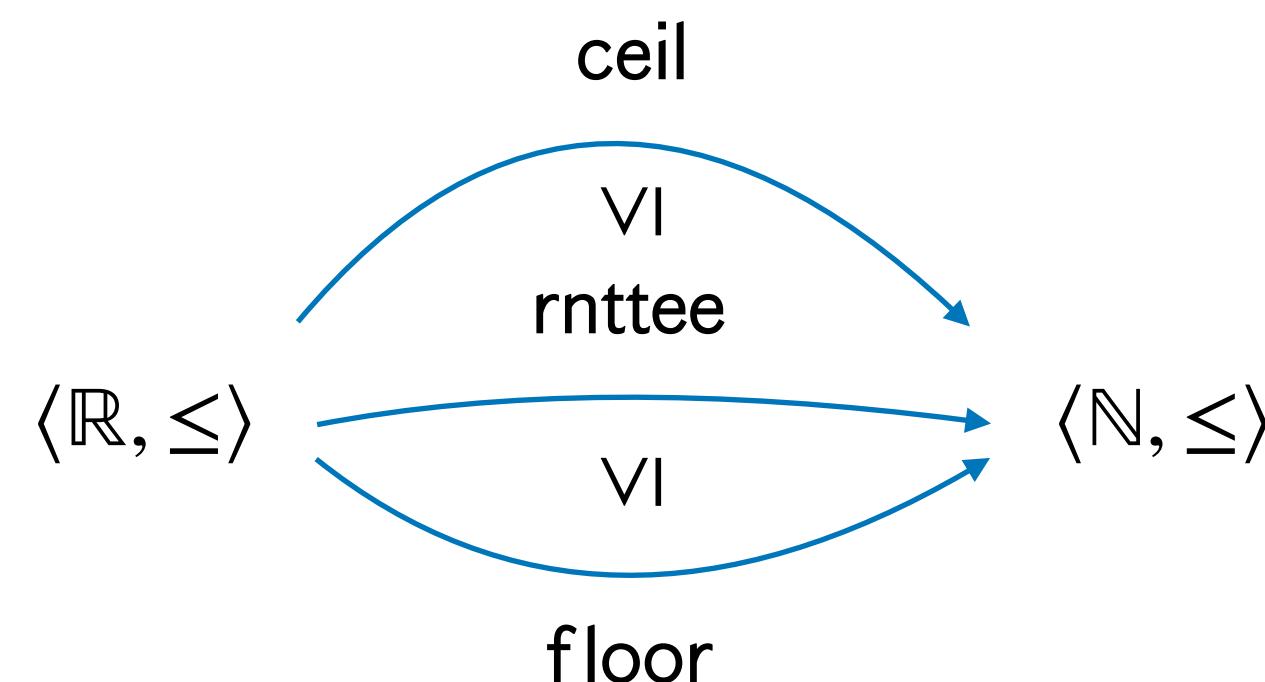
“round to nearest, ties to even”

(default IEEE-754)



- Also note:

$$\forall x \in \mathbb{R} : \text{floor}(x) \leq \text{rntte}(x) \leq \text{ceil}(x)$$



Upper and lower bounds

- ▶ The **upper bounds** of a subset S of a poset P are, if they exist, the elements that dominate all elements in S .



- ▶ The **least upper bound**, if it exists, is the least among the upper bounds of S .
We also call it the *join*, *supremum*, $\vee S$.
- ▶ The least upper bound need not exist.



- ▶ The least upper bound need not exist even in a total order.

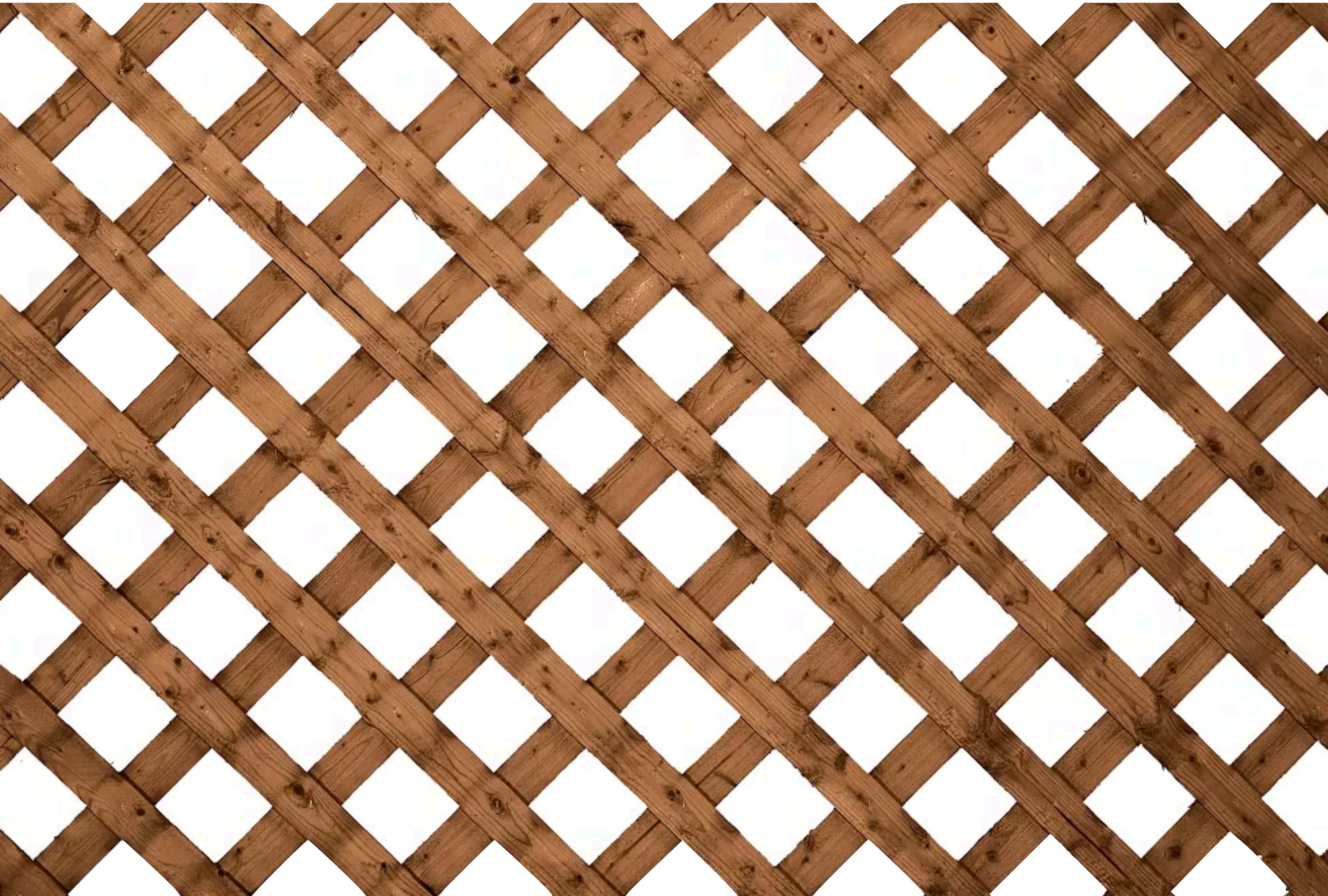
$$\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$$

- ▶ Dually: define **lower bound**, also called **meet**, **minimum**, $\wedge S$.



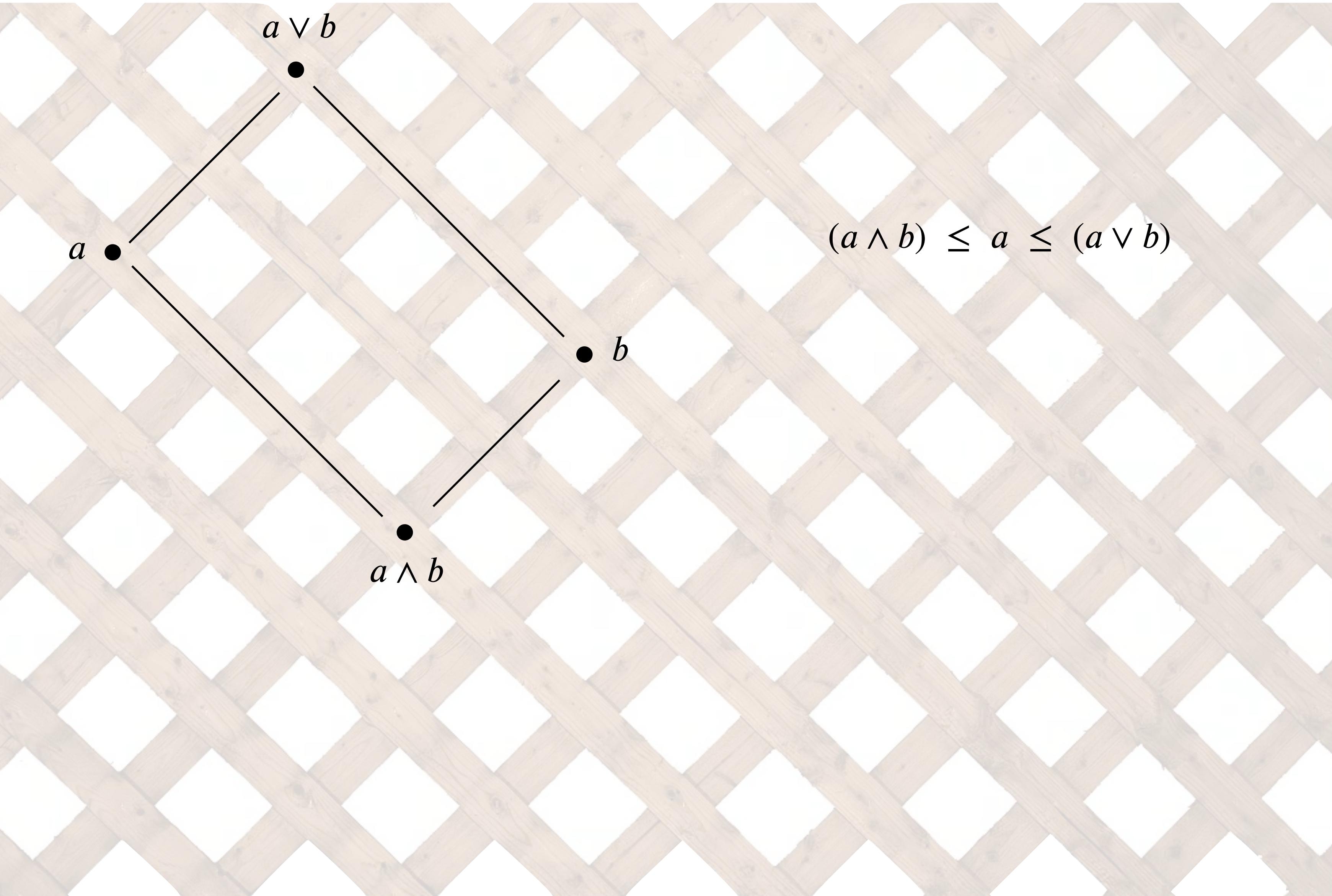
Lattices

- ▶ **Lattices** are posets where the **meet** \wedge and the **join** \vee always exist.



Lattices

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Lattices

- Lattices are posets where the **meet** \wedge and the **join** \vee always exist.

- Example: Consider a poset **Propositions** where:

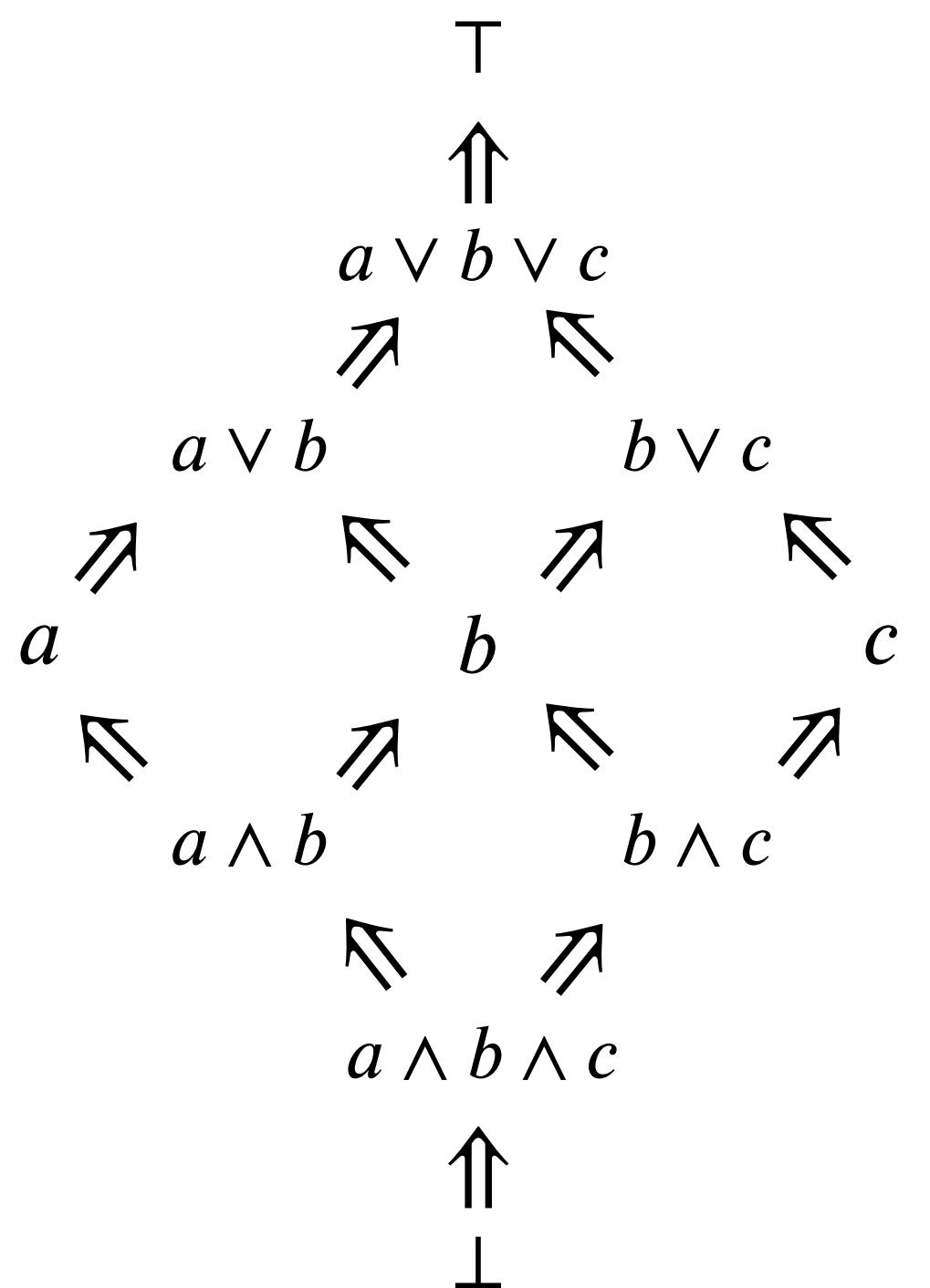
- the elements are propositions (equivalence classes of propositions)
- the partial order is given by

$$\frac{a \leq b}{a \Rightarrow b}$$

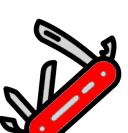
$$\frac{a \Leftrightarrow b}{a = b}$$

$$\frac{\top}{a \Rightarrow a}$$

- This is a lattice with the logical operations \wedge and \vee .



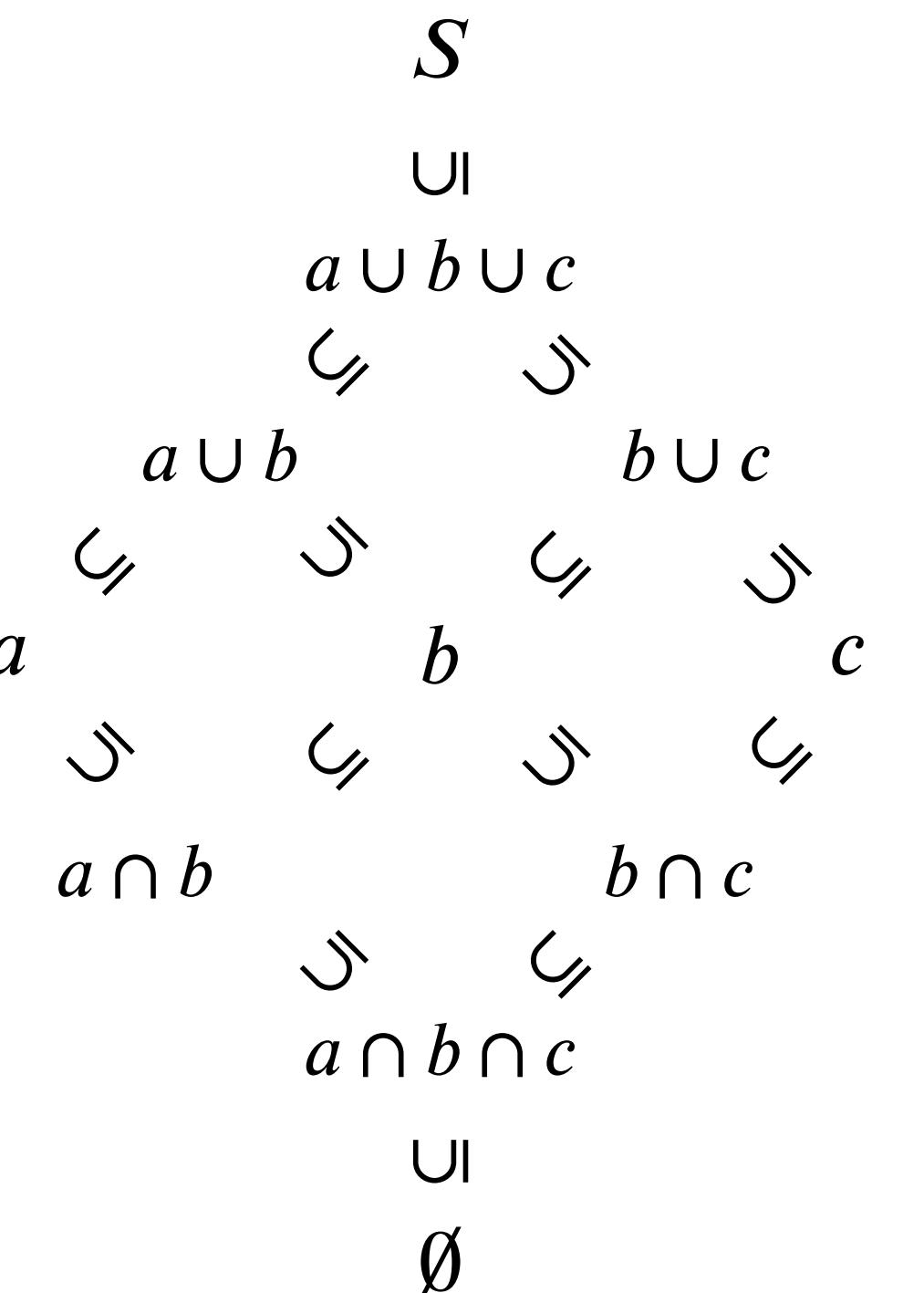
$$\frac{(a \Rightarrow b) \wedge (b \Rightarrow c)}{a \Rightarrow c}$$



Powersets

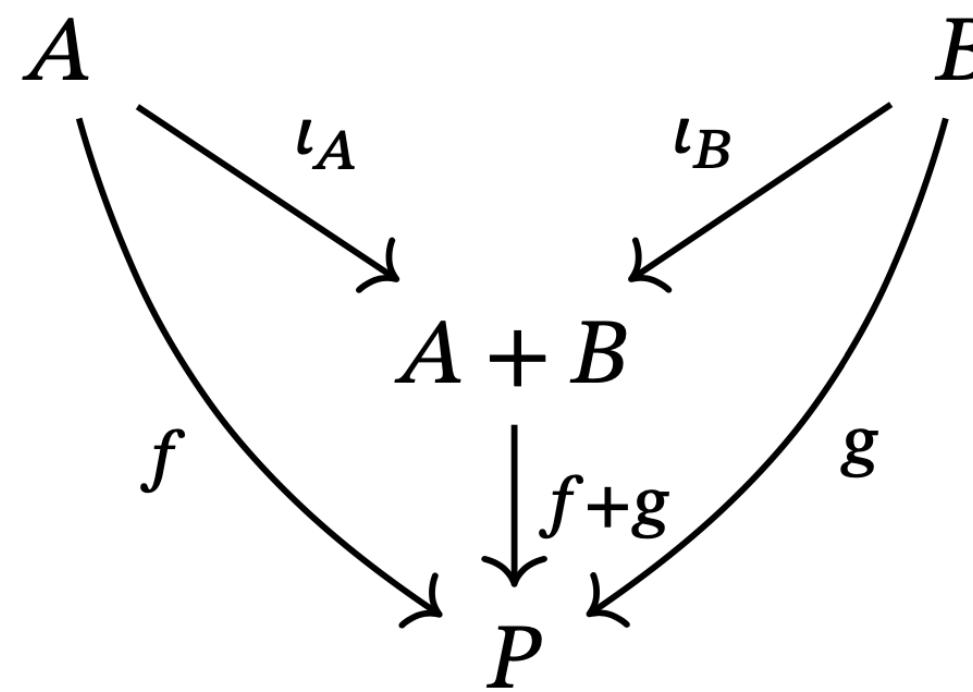
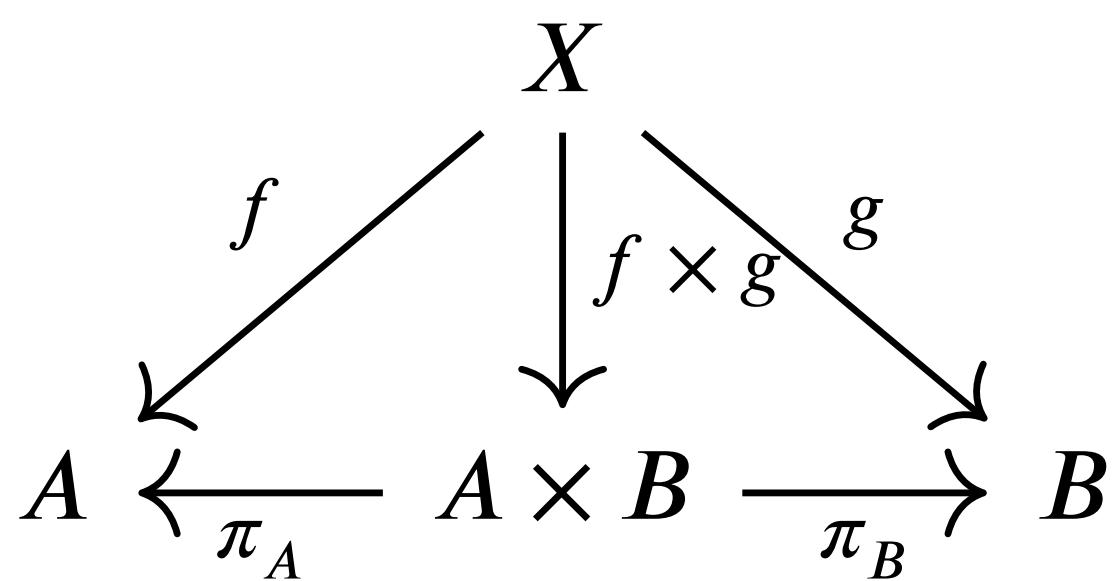
- ▶ **Lattices** are posets where the **meet** \wedge and the **join** \vee always exist.
- ▶ Example: The **power set** $\mathcal{P}(S)$ (set of all subsets) of a set S is a lattice with:

- $a \leq b \doteq a \subseteq b$
- $a \vee b \doteq a \cup b$
- $a \wedge b \doteq a \cap b$
- $\top = S$
- $\perp = \emptyset$



In a lattice, meet and join are product and co-product

- If you see a lattice as a category:
 - the **meet** \wedge is the **product** in the category.
 - the **join** \vee is the **co-product** in the category.



$$f : \frac{X}{A} \quad g : \frac{X}{B}$$

$$f : \frac{A}{P} \quad g : \frac{B}{P}$$

$$f \times g : \frac{X}{A \wedge B}$$

$$f + g : \frac{A \vee B}{P}$$

$$\pi_A : \frac{A \wedge B}{A} \quad \pi_B : \frac{A \wedge B}{B}$$

$$i_A : \frac{A}{A \vee B} \quad i_B : \frac{B}{A \vee B}$$



Upper/lower sets

- An **upper set** is a subset of a poset such that, if x is in the poset, all “higher” elements are.
- A **lower set** is a subset of a poset such that, if x is in the poset, all “lower” elements are.

$$\frac{x \in U \quad x \leq y}{y \in U}$$

$$\frac{x \in L \quad y \leq x}{y \in U}$$



- Let \mathbf{UP} and \mathbf{LP} be the sets of all upper/lower sets. These are posets choosing:

$$\langle \mathbf{UP}, \supseteq \rangle \quad \top = \emptyset \quad \perp = P$$

$$\langle \mathbf{LP}, \subseteq \rangle \quad \top = P \quad \perp = \emptyset$$



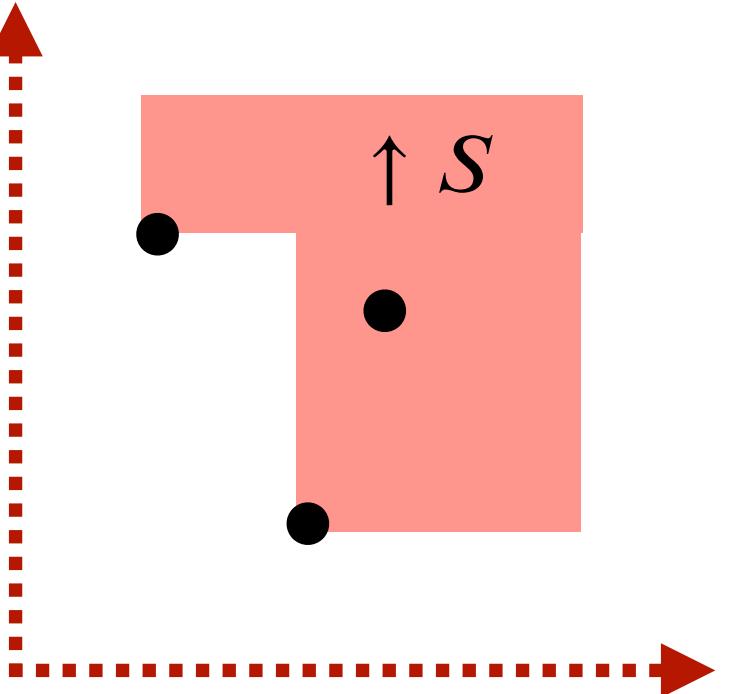
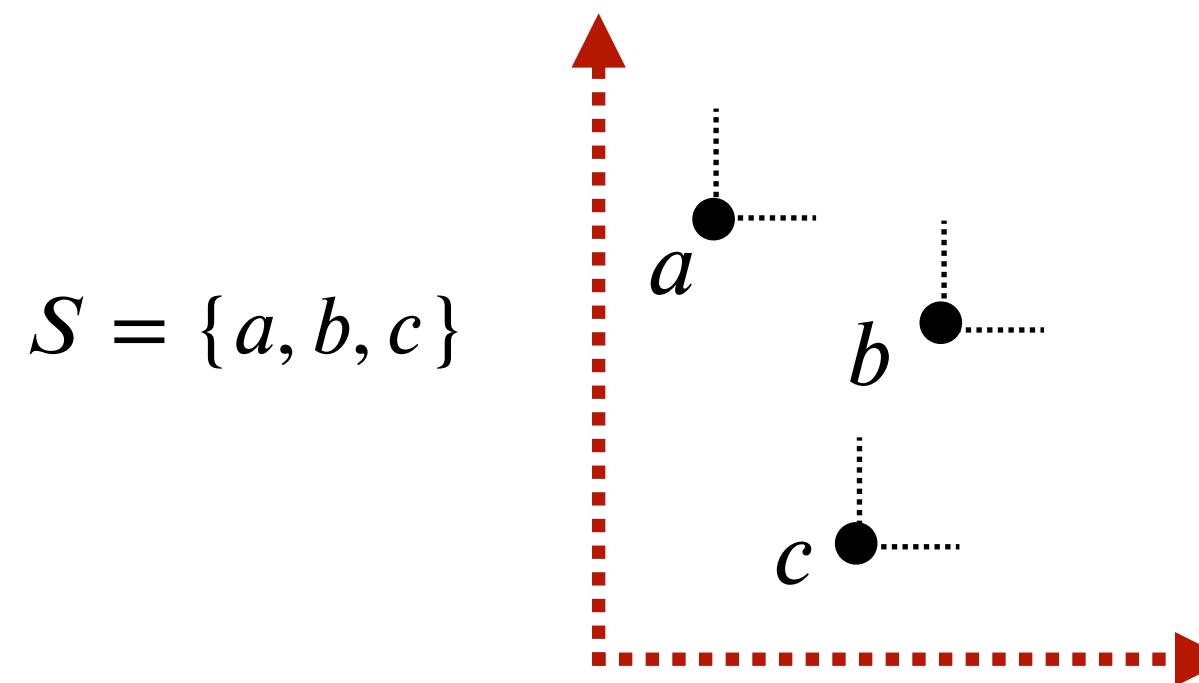
Upper / lower closure

- ▶ The **upper closure** of an element is the set of elements that dominate it:

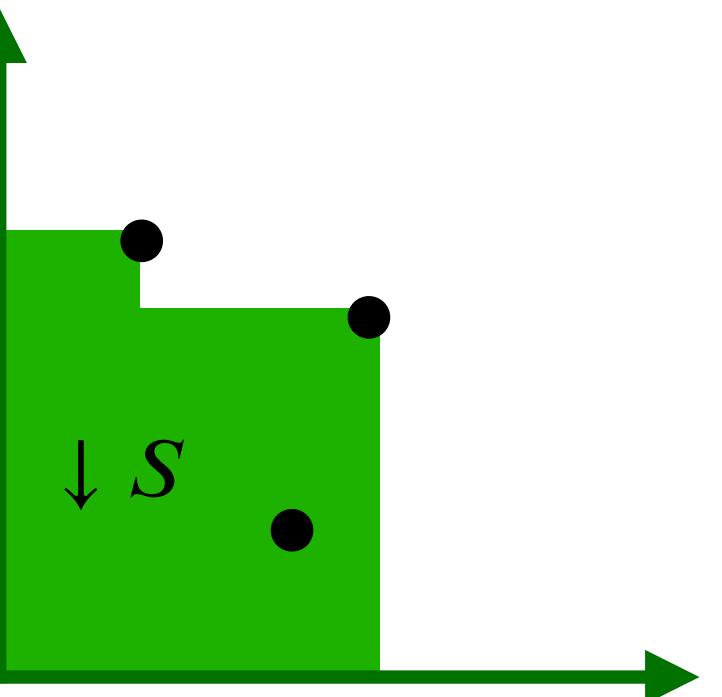
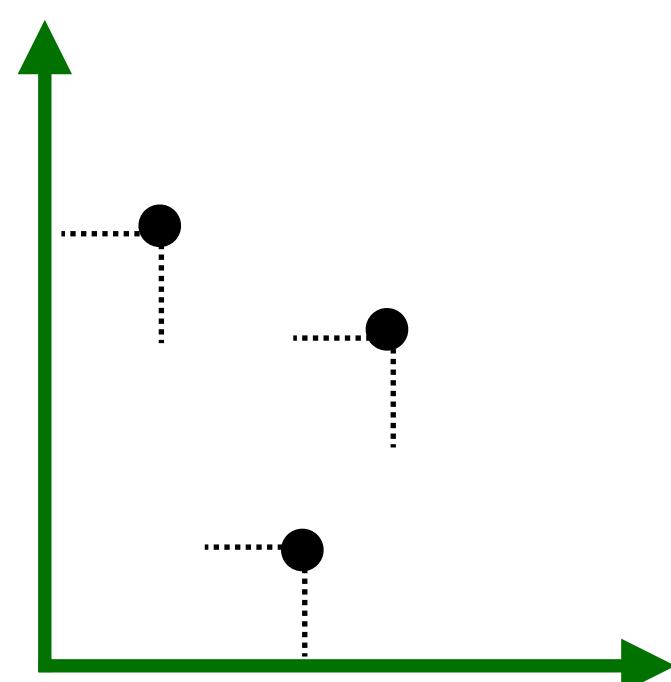
$$\uparrow a \doteq \{x \in P : a \leq x\}$$

- ▶ Upper closure of a set:

$$\uparrow S \doteq \cup_{a \in S} \uparrow a$$



- ▶ Lower closure is defined symmetrically:



Upper / lower closure

- The **upper closure** of an element is the set of elements that dominate it:

$$\uparrow a \doteq \{x \in P : a \leq x\}$$

- Upper closure is a **monotone function**:

$$\langle P, \leq_P \rangle \xrightarrow{\quad} \langle UP, \supseteq \rangle$$

$$\frac{a \leq_P b}{\uparrow a \supseteq \uparrow b}$$



Functors

- ▶ A **functor** $F : C \rightarrow D$ between two categories is defined by two maps:
 - a map from objects to objects $f_0 : \text{Ob}_C \rightarrow \text{Ob}_D$
 - maps from hom-sets to hom-sets $f_{ab} : \text{Hom}(a; b) \rightarrow \text{Hom}(f_0(a); f_0(b))$
- ▶ (We overload the notation and write F for both functions.)
- ▶ These maps need satisfy two conditions:
 - Identities map to identities: $F(\text{Id}_a) = \text{Id}_{F(a)}$
 - Composition is respected: $F(g ; h) = F(g) ; F(h)$

C

D



Monotone functions as functors

- ▶ A **functor** $F : C \rightarrow D$ need satisfy two conditions:

- Identities map to identities: $F(\text{Id}_a) = \text{Id}_{F(a)}$
- Composition is respected: $F(g ; h) = F(g) ; F(h)$

- ▶ A **function** $f : \langle P, \leq_P \rangle \rightarrow \langle Q, \leq_Q \rangle$ is **monotone** iff

$$a \leq_P b$$

$$f(a) \leq_Q f(b)$$

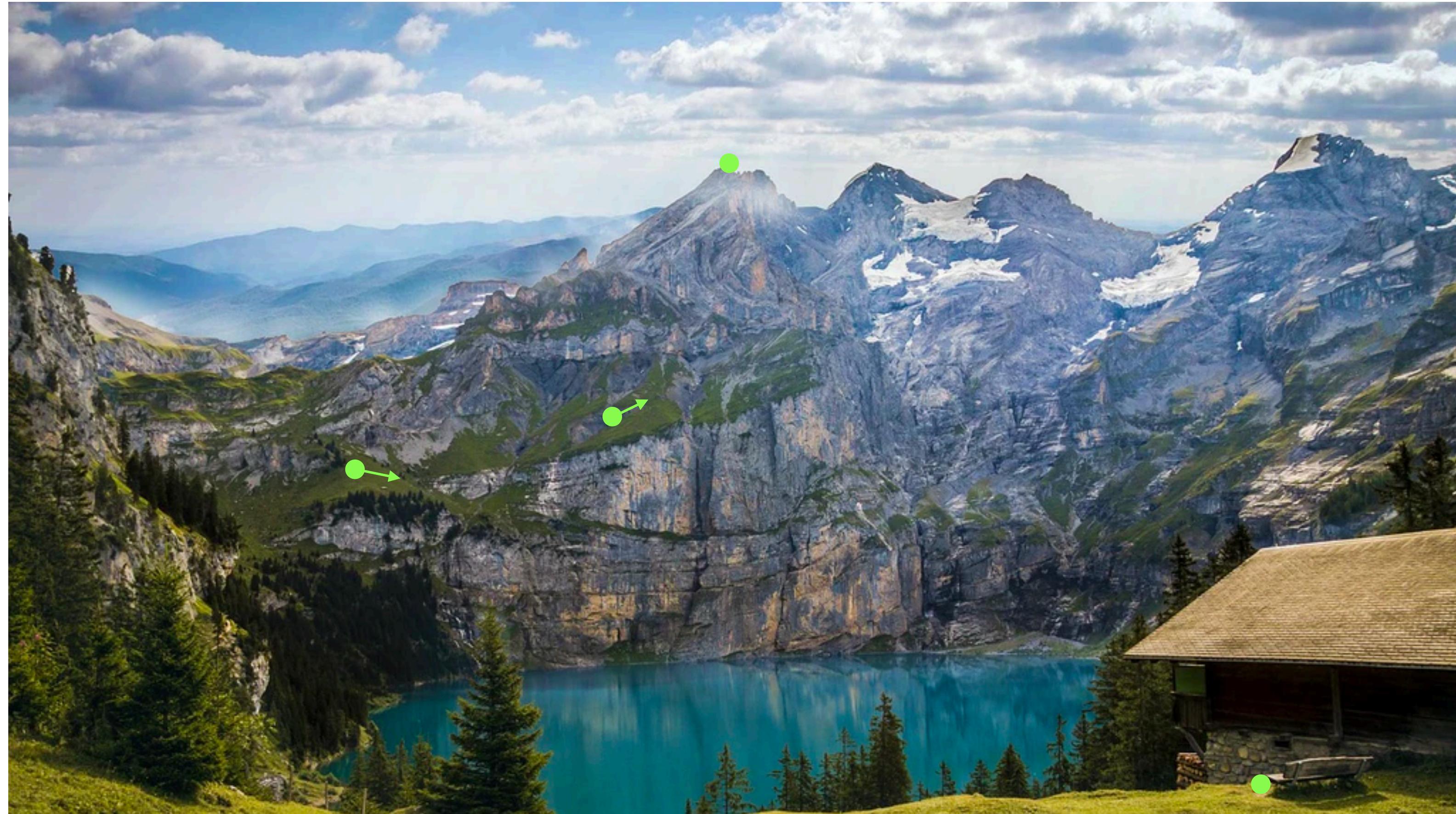
$\langle P, \leq_P \rangle$

$\langle Q, \leq_Q \rangle$



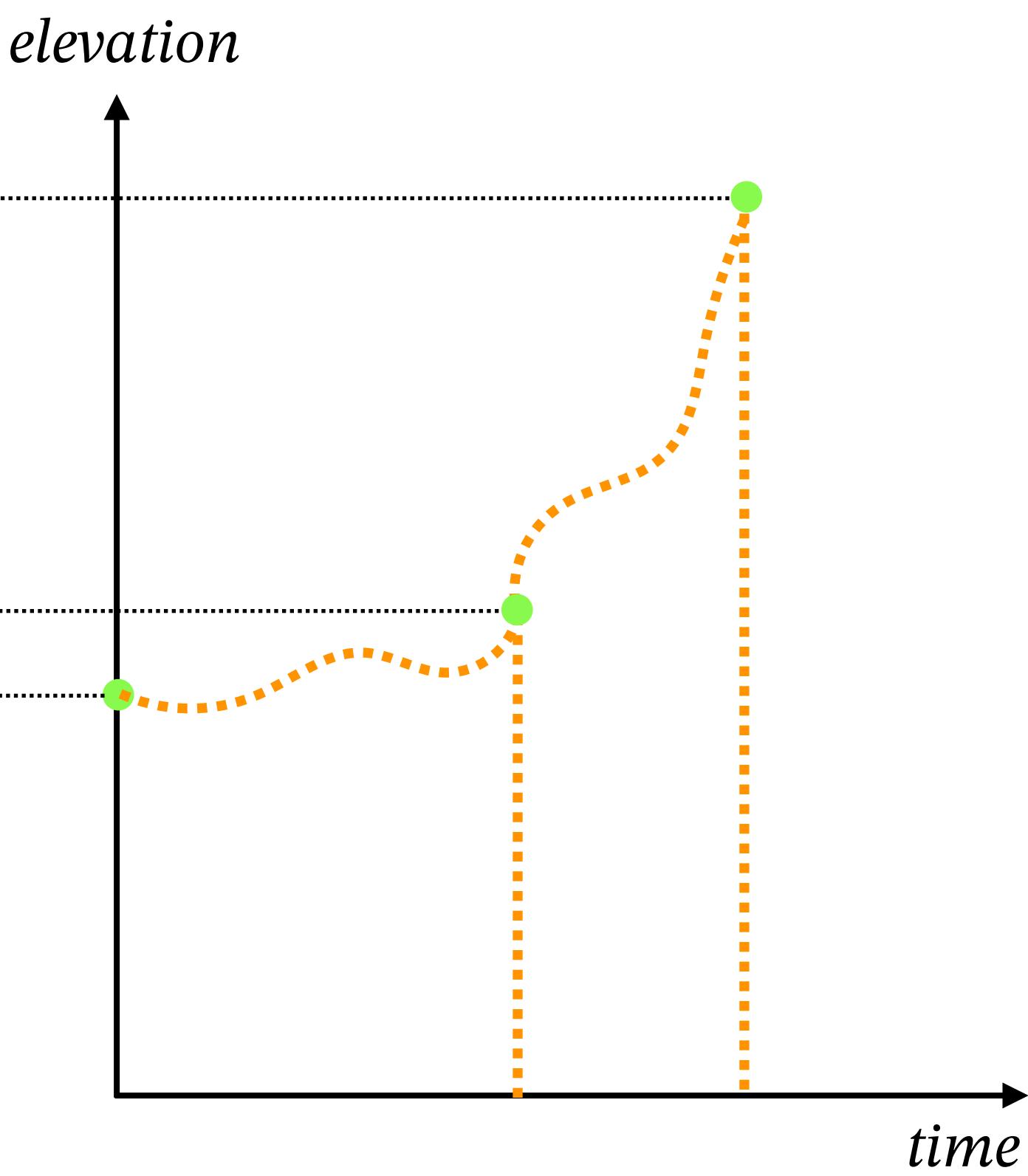
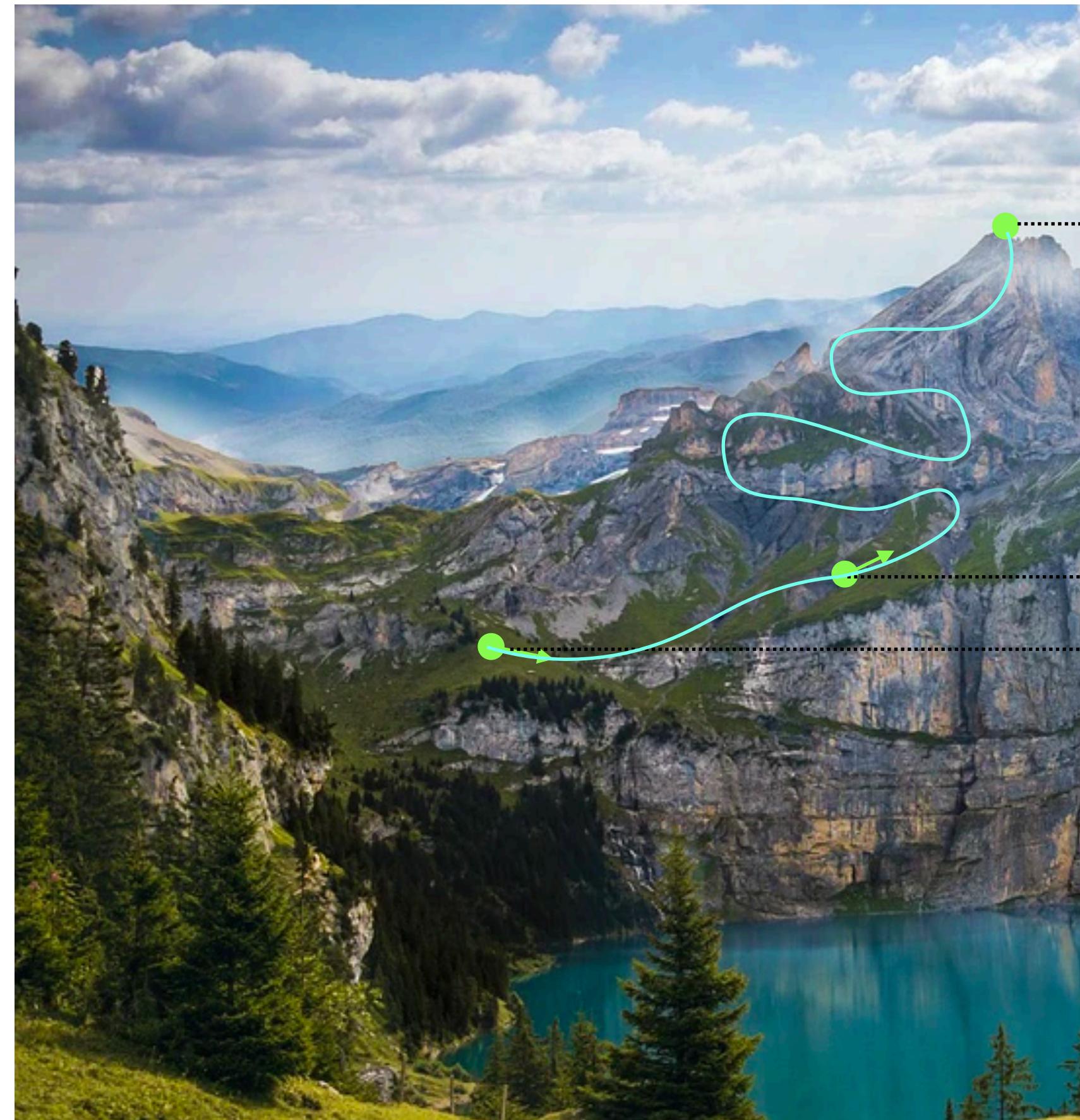
Back to the Swiss Mountains

- **Berg**: objects are (position, velocities) tuples; morphisms are continuous paths.
- **BergAma**: subcategory of Berg where inclination $\leq 1/2$.



Elevation as a functor

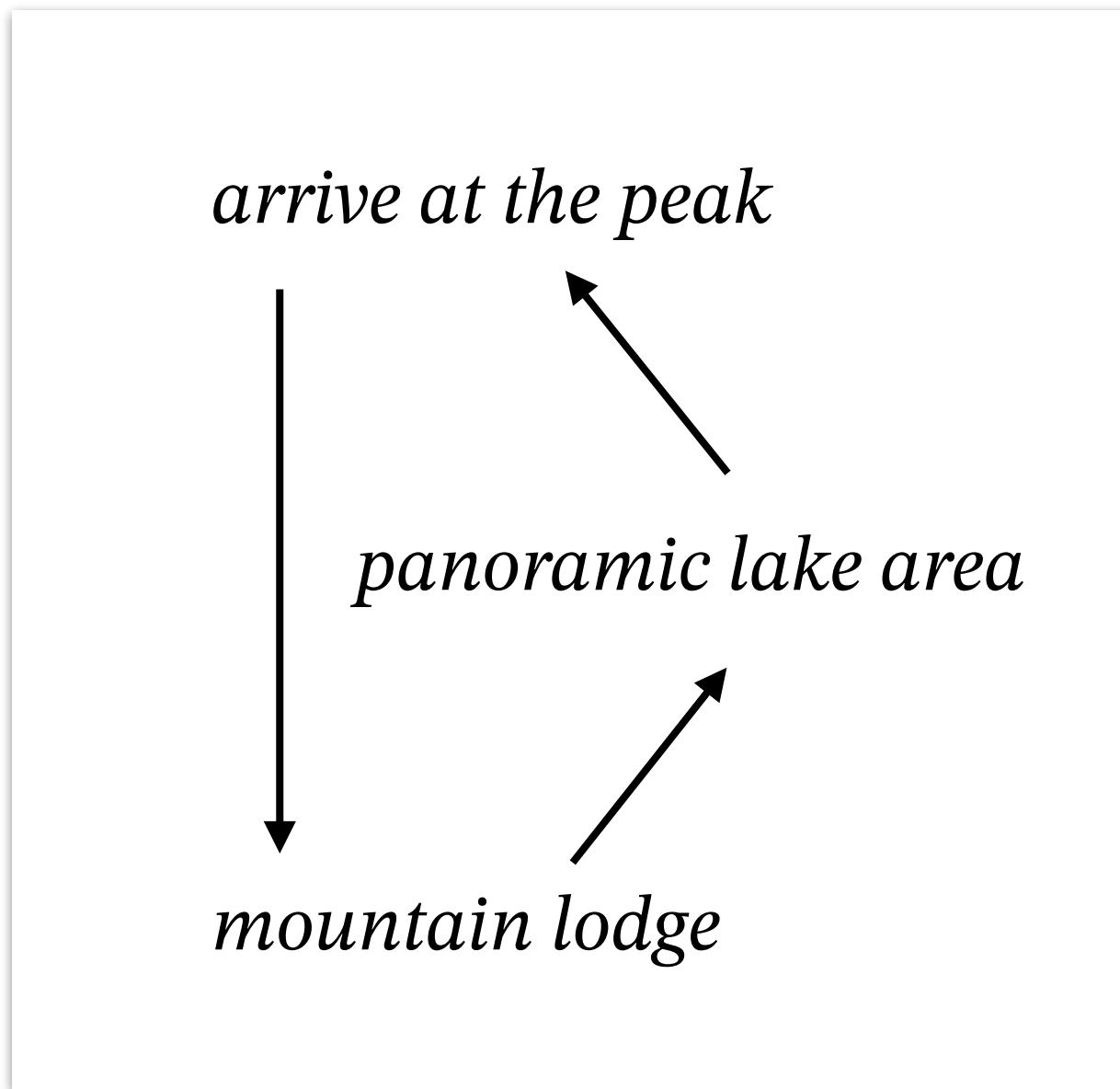
- **Berg**: objects are (position, velocities) tuples; morphisms are continuous paths.
- **BergAma**: subcategory of Berg where inclination $\leq 1/2$.
- There is a functor elevation from Berg to elevation profiles.



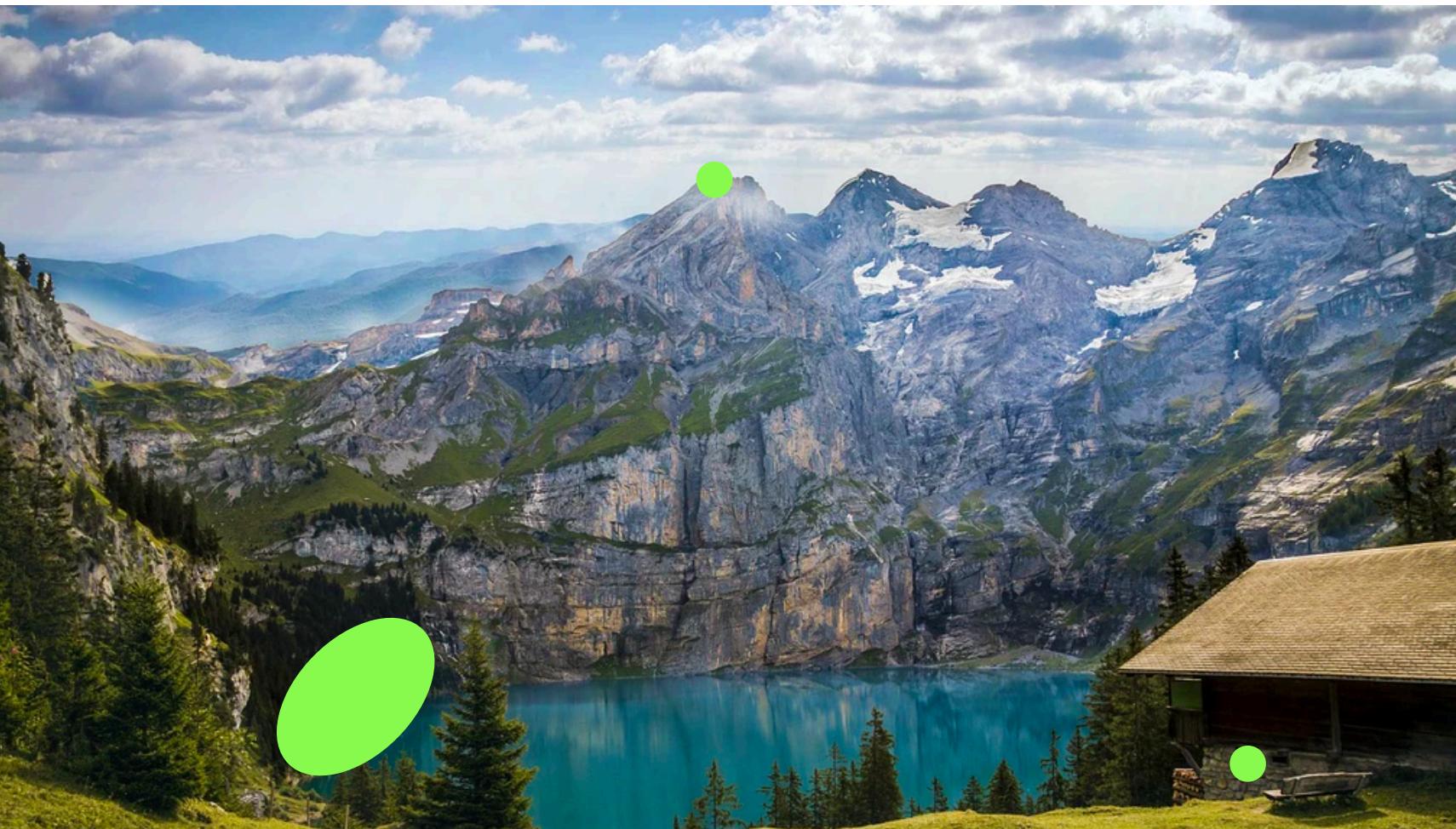
Path planning as the search for a functor

- **Berg**: objects are (position, velocities) tuples; morphisms are continuous paths.
- **BergAma**: subcategory of Berg where inclination $\leq 1/2$.
- Define a category **Plans** where objects are areas of the mountain and morphisms describe visiting order constraints.

Plans

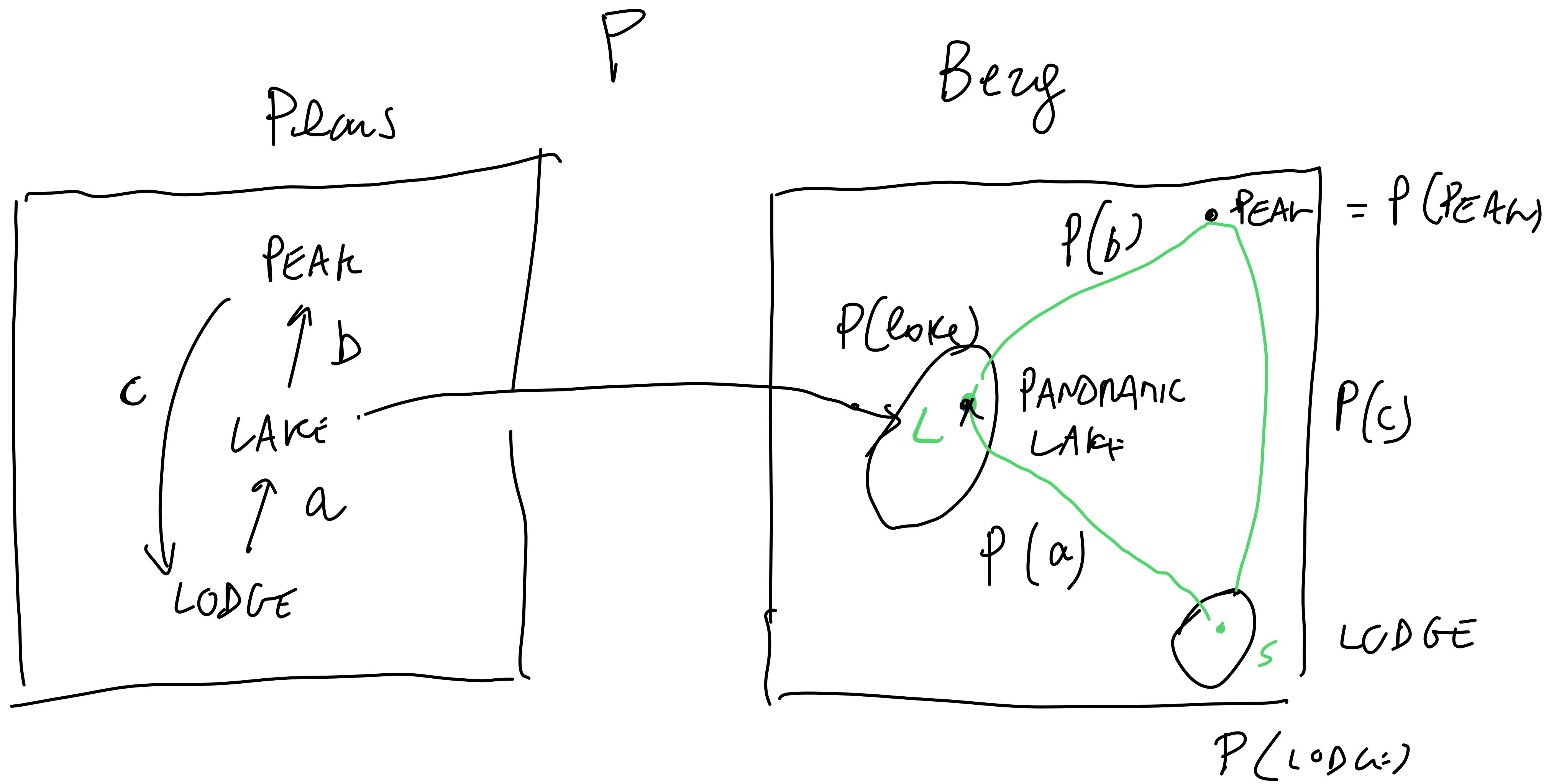


BergAma



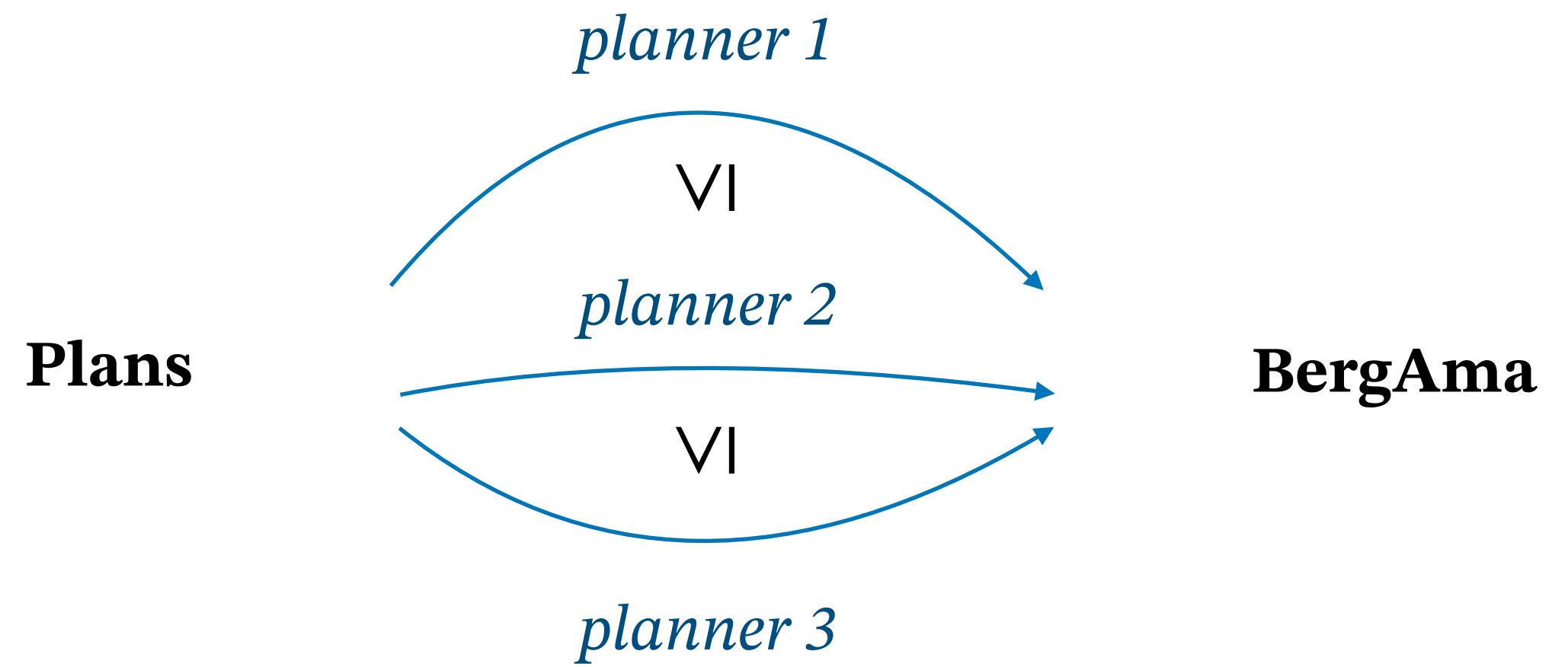
- A **Plan** is a morphism in **Plans**.
- **Planning** means finding a functor from **Plans** to **BergAma**.





Path planning as the search for a functor

- **Berg**: objects are (position, velocities) tuples; morphisms are continuous paths.
- **BergAma**: subcategory of Berg where inclination $\leq 1/2$.
- Define a category **Plans** where objects are areas of the mountains and morphisms describe order constraints.



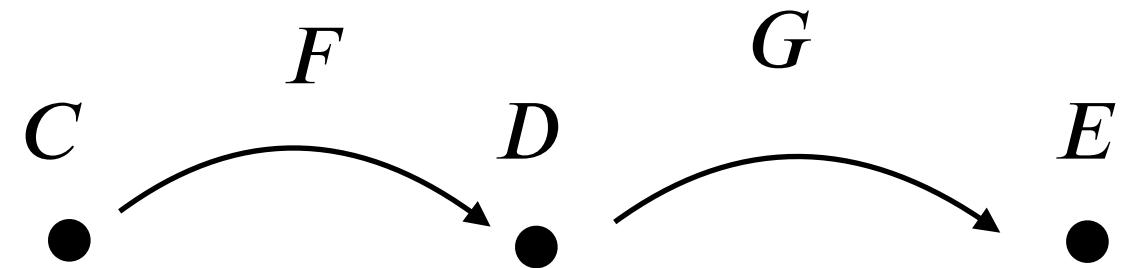
- A **Plan** is a morphism in **Plans**.
- **Planning** means finding a functor from **Plans** to **BergAma**.
- **An optimal planner is one that chooses the shortest paths.**



A category of categories

- There exists a **category of (small) categories** called **Cat**:

- objects are categories
- morphisms are functors
- the identities are identity functors



- Need to prove that functors compose and that composition is associative:

$$F : C \rightarrow D$$

$$G : D \rightarrow E$$

$$\underline{F(\text{Id}_a) = \text{Id}_{F(a)}}$$

$$F ; G : C \rightarrow E$$

$$F(g ; h) = F(g) ; F(h)$$

