called a *monoid isomorphism* if there is a morphism of monoids $g: \mathbb{N} \to \mathbb{M}$ such

Definition (Monoid isomorphism). A morphism of semigroups $f: \mathbf{M} \to \mathbf{N}$ is

that $f \circ g = \mathrm{id}_{\mathbf{M}}$ and $g \circ f = \mathrm{id}_{\mathbf{N}}$.