Lemma. The optimal control law for the LQG problem is $\mathbf{u}_t^\star = -\mathbf{K}\hat{\mathbf{x}}_t = -\mathbf{R}^{-1}\mathbf{B}^*\bar{\mathbf{S}}\hat{\mathbf{x}}_t$, where $\hat{\mathbf{x}}_t$ is the unbiased minimum-variance estimate of $\mathbf{x}_t$ given previous measurements and $\bar{\mathbf{S}} \in \mathcal{P}^+$ solves the Riccati equation

$$\mathbf{SA} + \mathbf{A}^*\mathbf{S} - \mathbf{SBR}^{-1}\mathbf{B}^*\mathbf{S} + \mathbf{Q} = \mathbf{0}.$$

The minimum cost $J^\star$ achieved by the optimal control is:

$$\begin{aligned} J^\star &= \mathrm{Tr}(\bar{\mathbf{S}}\bar{\boldsymbol{\Sigma}}\mathbf{C}^*\mathbf{V}^{-1}\mathbf{C}\bar{\boldsymbol{\Sigma}} + \bar{\boldsymbol{\Sigma}}\mathbf{Q}) \\ &= \mathrm{Tr}(\bar{\boldsymbol{\Sigma}}\bar{\mathbf{S}}\mathbf{BR}^{-1}\mathbf{B}^*\bar{\mathbf{S}} + \bar{\mathbf{S}}\mathbf{W}), \end{aligned}$$

where $\bar{\boldsymbol{\Sigma}} \in \mathcal{P}^+$ is the solution of the Riccati equation

$$\mathbf{A}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\mathbf{A}^* - \boldsymbol{\Sigma}\mathbf{C}^*\mathbf{V}^{-1}\mathbf{C}\boldsymbol{\Sigma} + \mathbf{W} = \mathbf{0}.$$