Contents

1 Combination

to write

{bmm:1}

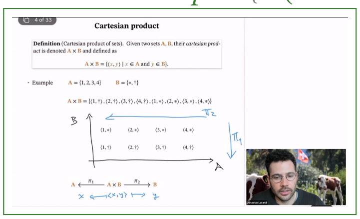
{ch:combination}

1.1 Products

{sec:combination-products}

We'll start off by recalling a familiar way of combining two sets, **A** and **B**.

Watch *Cartesian product* (3 minutes) {def:cartesian-product}



Definition 1.1 (Cartesian product of sets)**.** Given two sets **A**, **B**, their *cartesian product* is denoted $\mathbf{A} \times \mathbf{B}$ and defined as

$$\mathbf{A} \times \mathbf{B} = \{\langle x, y\rangle \mid x \in \mathbf{A} \text{ and } y \in \mathbf{B}\}.$$

Example 1.2. Consider the sets $\mathbf{A} = \{1, 2, 3, 4\}$ and $\mathbf{B} = \{$♋, ♈$\}$. We have

{{eq:cart-prod-example}}

$$\mathbf{A} \times \mathbf{B} = \{\langle 1, ♈\rangle, \langle 2, ♈\rangle, \langle 3, ♈\rangle, \langle 4, ♈\rangle, \langle 1, ♋\rangle, \langle 2, ♋\rangle, \langle 3, ♋\rangle, \langle 4, ♋\rangle\}. \tag{1.1}$$

We can, however, also represent $\mathbf{A} \times \mathbf{B}$ in a way which highlights its structure more (Example 1.2).

{bhfn:2}

$$\begin{array}{ccccc} & \langle 1, ♋\rangle & \langle 2, ♋\rangle & \langle 3, ♋\rangle & \langle 4, ♋\rangle \\ \\ a & \langle 1, ♈\rangle & \langle 2, ♈\rangle & \langle 3, ♈\rangle & \langle 4, ♈\rangle \end{array}$$

{fig:example_cartesian}

In particular, the cartesian product comes naturally equipped with two projection maps π_1 and π_2 which map an element of $\mathbf{A} \times \mathbf{B}$ to its first and second coordinate, respectively:

$$\pi_1(\langle x, y\rangle) = x \text{ and } \pi_2(\langle x, y\rangle) = y. \tag{1.2}$$

We will often depict the situation like this:

{bhfn:3}

$$a \quad \mathbf{A} \xleftarrow{\pi_1} \mathbf{A} \times \mathbf{B} \xrightarrow{\pi_2} \mathbf{B}$$

It turns out that this situation is part of a pattern that unites various different constructions across mathematics. This pattern, of which the cartesian product is a special case, is formalized in the notion of "categorical product". Before introducing the rigorous definition of the categorical product, let us list a number of examples that are part of the pattern.

{fig:diagram_cartesian_2}

$$a \quad x_1 \longleftarrow \min\{x_1, x_2\} \longrightarrow x_2$$

Figure 1.1: Taking the minimum

[a] 050_example_cartesian
[a] 050_example_cartesian_2
[a] 050_example_prod_min

{exa:min-as-prod}

Example 1.3. For any $x_1, x_2 \in \mathbb{R}$ let us draw an arrow $x_1 \to x_2$ iff $x_1 \leq x_2$. Then "taking the minimum" is an example of the categorical product; see Fig. 1.1.

For instance, choosing $x_1 = 3, x_2 = 7$, we have $3 \geq \min\{3, 7\} \leq 7$.

$$a \quad m \longleftarrow \gcd\{m, n\} \longrightarrow n$$

Figure 1.2: Taking the greatest common divisor

{exa:gcd-as-prod}

[a] 050_example_prod_gcd

Example 1.4. For any $m, n \in \mathbb{N}$ let us draw an arrow $m \to n$ iff m divides n, which is written $m|n$. Then "taking the greatest common divisor" is an example of the categorical product; see Fig. 1.2.

For instance, choosing $m = 6, n = 9$, we have $\gcd\{6, 9\} = 3$, and $3|6$ and $3|9$.

$$a \quad \mathbf{S}_1 \longleftarrow \mathbf{S}_1 \cap \mathbf{S}_2 \longrightarrow \mathbf{S}_2$$

{exa:intersection-as-prod}

Figure 1.3: Taking the intersection.

[a] 050_example_prod_intersection

Example 1.5. Let **A** be a set. For any subsets $\mathbf{S}_1, \mathbf{S}_2 \subseteq \mathbf{A}$, let us draw an arrow $\mathbf{S}_1 \to \mathbf{S}_2$ iff $\mathbf{S}_1 \subseteq \mathbf{S}_2$. Then "taking the intersection" is an example of the categorical product; see Fig. 1.3.

For instance, let $\mathbf{A} = \{1,2,3,4\}$, $\mathbf{S}_1 = \{1,2,3\}$, and $\mathbf{S}_2 = \{2,3,4\}$. Then $\mathbf{S}_1 \cap \mathbf{S}_2 = \{2,3\}$, and $\{1,2,3\} \supseteq \{2,3\} \subseteq \{2,3,4\}$.

{exa:conjunction-as-prod}

Example 1.6. Let $\mathbf{A} = \{\top, \bot\}$ be the set of logical propositions consisting of true and false. For any propositions $p_1, p_2 \in \mathbf{A}$, let us draw an arrow $p_1 \to p_2$ iff $p_1 \Rightarrow p_2$. Then "taking the conjunction" (the operation "and", denoted $\wedge$) of two statements is an example of the categorical product; see Fig. 1.4.

For instance, let $p_1 = \top$, $p_2 = \bot$. Then $\top \wedge \bot = \bot$ and $\top \Leftarrow \bot \Rightarrow \bot$ holds.

$$p_1 \longleftarrow p_1 \wedge p_2 \longrightarrow p_2$$ [a]

Figure 1.4: Taking the conjunction

[a] `050_example_prod_conjunction`

{exa:meet-as-prod}

Example 1.7. Let ... be a lattice. For any element $x, y \in$ ..., let us draw an arrow $x \to y$ iff $x \le y$. Then the "meet" (the operation $\wedge$) of two elements is an example of the categorical product; see Fig. 1.5.

$$x \longleftarrow x \wedge y \longrightarrow y$$ [a]

Figure 1.5: Taking the meet

[a] `050_example_prod_meet`

As you can see from the above list of examples, the notion of categorical product involves diagrams of the type in Fig. 1.6.

$$X \longleftarrow \text{"product of } X \text{ and } Y\text{"} \longrightarrow Y$$ [a]

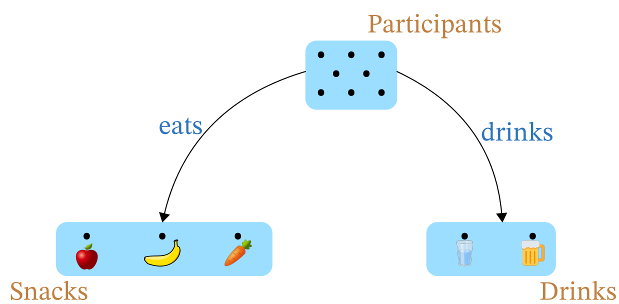
Figure 1.6

[a] `050_prod_generic`

There is more to the story, however. Loosely speaking, the categorical "product of X and Y" is characterized by how it interacts, in a certain way, with all other diagrams which have a similar form. Let us explain using an "applied" example involving the cartesian product of sets.

{bhfn:10}

Suppose you are at an engineering conference in Switzerland, and there will be a hike as a group outing. The organizers have prepared snacks to go. Each participant can choose a food from Snacks = {🍎, 🍌, 🥕} and a drink from Drinks = {🥛, 🍺}. Let Participants denote the set of participants. The choice of snacks could be organized as depicted in Fig. 1.7, in which each participant chooses a food, and chooses a drink. This can be described via functions $\mathrm{eats} : \mathrm{Participants} \to \mathrm{Snacks}$ and $\mathrm{drinks} : \mathrm{Participants} \to \mathrm{Drinks}$.



[a]

Figure 1.7: Each participant chooses a food and a drink.

[a] `50_snacks_1`

{fig:snacks_1}

Alternatively, snacks could be pre-packaged in such a way as to allow all possible combinations of food and drink choices. This corresponds to $\mathrm{Snacks} \times \mathrm{Drinks}$. Then the choice participants make of which lunch package they'd like is described by a single function $\mathrm{meal} : \mathrm{Participants} \to \mathrm{Snacks} \times \mathrm{Drinks}$ (see Fig. 1.8).

Intuitively, the two situations (two choices separately, or one choice of a pre-packaged snack) are "the same" in a certain sense. We can make this "sameness" precise. Specifically, if we start with the functions eats and drinks, we can use them to build the following function:

$$\begin{aligned} \phi_{\mathrm{eats},\mathrm{drinks}} : \mathrm{Participants} &\to \mathrm{Snacks} \times \mathrm{Drinks} \\ p &\mapsto \langle \mathrm{eats}(p), \mathrm{drinks}(p) \rangle . \end{aligned} \tag{1.3}$$

{{eq:snacks-building-univ-map}}

{bhfn:11}

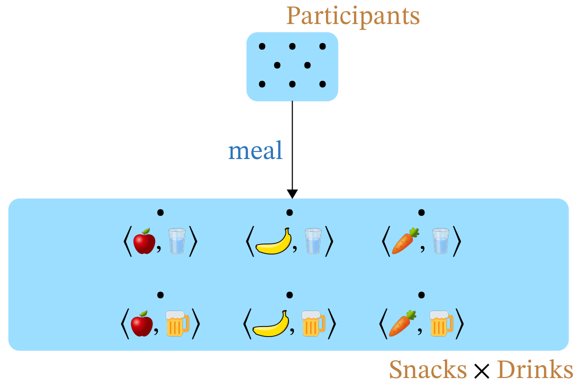


Figure 1.8: Each participant chooses a combination of food and a drink.[a] fig:snacks_2

[a] 50_snacks_2

Furthermore, given $\phi_{\text{eats},\text{drinks}}$, one can recover eats and drinks:

$$\text{eats} = \phi_{\text{eats},\text{drinks}} \,⨟\, \pi_1 \quad \text{and} \quad \text{drinks} = \phi_{\text{eats},\text{drinks}} \,⨟\, \pi_2. \tag{1.4}$$

{{eq:snacks-pro

These two equations say that the diagram in Fig. 1.9 is commutative. The whole situation can be summarized thus: given a set Participants and functions eats : Participants → Snacks and drinks : Participants → Drinks as in Fig. 1.8, there is a unique function

{{eq:snacks-unique-univ-map}}

$$\phi_{\text{eats},\text{drinks}} : \text{Participants} \to \text{Snacks} \times \text{Drinks} \tag{1.5}$$

such that the diagram in Fig. 1.9 commutes.

{bhfn:12}

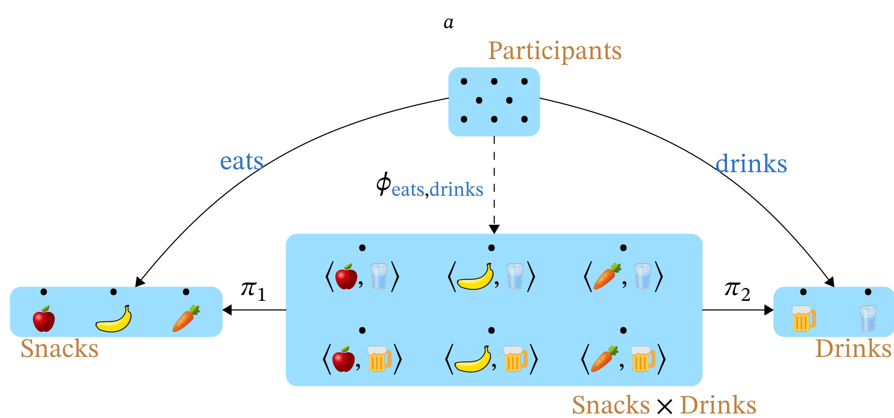


Figure 1.9: Choosing food and drink separately is essentially the same as choosing a combination of the two.[a]

[a] 50_snacks_3

{fig:snacks_3}

It turns out that this state of affairs *characterizes* the cartesian product Snacks × Drinks. We think of the diagram

{{eq:snacks-prod-diagram}}

$$\text{Snacks} \leftarrow \text{Snacks} \times \text{Drinks} \to \text{Drinks} \tag{1.6}$$

as "interacting" with the diagram

{{eq:snacks-prod-test-diagram}}

$$\text{Snacks} \leftarrow \text{Participants} \to \text{Drinks} \tag{1.7}$$

via the fact that such a map $\phi_{\text{eats},\text{drinks}}$ exists which links the two by making the diagram Fig. 1.9 commute.

This describes the general pattern for the definition of the categorical product.

Namely, the categorical product is a diagram of the kind

eric-diagram}}

$$X \leftarrow \text{“product of } X \text{ and } Y\text{”} \rightarrow Y \tag{1.8}$$

which interacts with any other diagram of the form

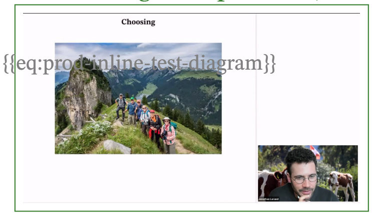
$$X \leftarrow T \rightarrow Y \tag{1.9}$$

{{eq:prod-inline-test-diagram}}

in a way which is analogous to the situation in Fig. 1.9 (here T stands for any set that plays the role of the set Participants) .

Let us now finally state the general definition of categorical product. It is probably helpful to read the definition together with the clarifying remarks that follow it.

Watch *Categorical product* (33 minutes).



{def:categorical-product}

Definition 1.8 (Categorical Product). Let $\mathbf{C}$ be a category and let $X, Y \in \mathrm{Ob}_{\mathbf{C}}$ be objects. The *product* of X and Y is:

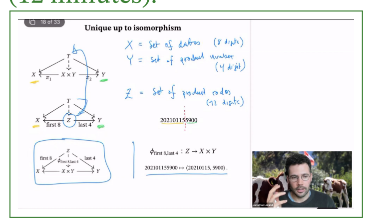
Constituents

1. an object $Z \in \mathrm{Ob}_{\mathbf{C}}$ (this is “the product” of X and Y);
2. *projection morphisms* $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$,

Conditions

1. For any $T \in \mathrm{Ob}_{\mathbf{C}}$ and any morphisms $f : T \to X, g : T \to Y$, there exists a *unique* morphism $\phi_{f,g} : T \to Z$ such that $f = (\phi_{f,g}) ⨟ \pi_1$ and $g = (\phi_{f,g}) ⨟ \pi_2$.

Watch *Definition of categorical product* (12 minutes).



Remark 1.9. Diagrammatically, the condition above states that the diagrams of the form in Fig. 1.10 commute.

$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[d, dashed, "\phi_{f,g}"] \arrow[dr, "g"] & \\ X & Z \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$ [a]

Figure 1.10

[a] 50_defproduct

Remark 1.10. In the above definition, technically both Z *and* the projection morphisms constitute the data of “the product of X and Y”. However, for simplicity, we usually refer only to Z as “the product”. Furthermore, we will usually use the notation $X \times Y$ to denote the product of X and Y, in place of Z. Similarly, we will usually write $f \times g$ in place of $\phi_{f,g}$. Then the diagram in Fig. 1.10 looks as in Fig. 1.11.

The reason we do not do this directly in the definition itself is the following. In general, for fixed X and Y, there may be several different objects Z (together with projection morphisms) that satisfy the definition of being “the product of X and Y”. Thus, there is, technically, no such thing as “*the*” (unique) product of X and Y. However, one can prove that any two candidates which satisfy the definition of being “the product of X and Y” will necessarily be isomorphic in a canonical manner. Thus, for simplicity, we will sometimes be slightly sloppy and speak of “the product of X and Y” as if it were unique. In many categories there is also indeed a choice for “the product of X and Y” that we are used to. For example, in the category **Set**, given sets X and Y, the familiar choice for “the product of X and Y” is the cartesian product $X \times Y$”. However, other representatives of the product of X and Y are possible! Example 1.12 illustrates this.

{prod-unique-up-to-iso}

$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[d, dashed, "f \times g"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$ [a]

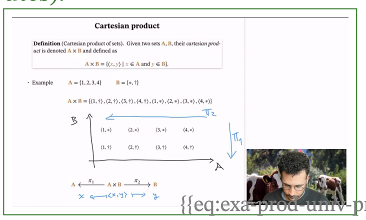
Figure 1.11 fig:prod-def-diagram-generic

[a] 50_defproduct_generic

Remark 1.11. The condition in the definition of the categorical product is know as the “universal property of the product”. We will attempt to explain this naming.

The stated condition involves the product Z of X and Y *interacting* with every possible choice of object T and every possible choice of morphisms $f : T \to X$ and $g : T \to Y$. We think of the ambient category $\mathbf{C}$ as "the universe" (or the "context"), and this condition states how the product must interact "with the whole universe". We choose the letter "T" because we think of this as a "test object" (similar, for instance, to how, in electrodynamics, a "test charge" is used to probe an electromagnetic field).

Watch *Examples of products* (22 minutes).



{ex:univ-prop-prod}

Example 1.12. Suppose that as a manufacturer, you want to label your products with

- ▷ A production date (8-digit code), and
- ▷ a model number (4-digit code).

Instead of two separate labels, you can also make one

{{eq:exa-prod-univ-prop-code}}

$$202101155900 \tag{1.10}$$

where the first 8 digits represent a date, and the last 4 digits are a model number. Let's call this single label the *product code*. Let Z denote the set of all product codes, and consider the maps $\pi_1 : Z \to X$, and $\pi_2 : Z \to Y$ which, respectively, map a 12-digit product code to its first 8 digits and its last 4 digits. One may check that Z, together with the map π_1 and π_2, will satisfy the definition of "the product of X and Y".

{bhfn:15}

{fig:digits-1}

$$\begin{tikzcd} & T \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\ X & Z \arrow[l, "\text{first 8}"'] \arrow[r, "\text{last 4}"'] & Y \end{tikzcd}$$ †

However, Z is not precisely the cartesian product of X and Y (which we will call $X \times Y$). The elements of Z are 12-digit codes, while elements of $X \times Y$ are pairs $\langle x, y\rangle$ where x is a 8-digit code and y is a 4-digit code. Since both Z and $X \times Y$ satisfy the definition of categorical product, they must, by Remark 1.10, be isomorphic.

{bhfn:16}

{fig:digits-3}

$$\begin{tikzcd} & T \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\ X & X \times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$ ‡

To see concretely what this isomorphism between them looks like, note that there is a unique map $\phi_{\text{first 8,last 4}}$ making the following diagram commute:

{bhfn:17}

{fig:digits-2}

$$\begin{tikzcd} & Z \arrow[dl, "\text{first 8}"'] \arrow[d, dashed, "\phi_{\text{first 8,last 4}}"] \arrow[dr, "\text{last 4}"] & \\ X & X \times Y \arrow[l] \arrow[r] & Y \end{tikzcd}$$ §

Concretely,

{{eq:exa-univ-iso}}

$$\phi_{\text{first 8,last 4}} : Z \to X \times Y \tag{1.11}$$

† `050_digits_1`
‡ `050_digits_3`
§ `050_digits_2`

maps for instance

-iso-concrete}}

$$202101155900 \mapsto \langle 20210115, 5900 \rangle \,. \tag{1.12}$$

One can readily show that $\phi_{\text{first } 8, \text{last } 4}$ is an isomorphism.

Now let us revisit the examples that were given earlier, before we stated the definition of categorical product. The idea is that all of these are instantiations of the categorical product, but in each case, we are working the in context of a different category! In each case, arrows denote morphisms in the category in question, specific to the example.

{exa:min-as-prod-cont}

Example 1.13. This is a continuation of Example 1.3. For any $x_1, x_2 \in \mathbb{R}$, we drew an arrow $x_1 \to x_2$ iff $x_1 \leq x_2$. The category in question here is the category whose objects are elements of $\mathbb{R}$, and whose morphisms are inequalities. The product is “taking the minimum”; its universal property is illustrated in Fig. 1.12. It says that if $t \in \mathbb{R}$ is such that $t \leq x_1$ and $t \leq x_2$, then $t \leq \min\{x_1, x_2\}$.

To make things concrete, choose $x_1 = 10$, $x_2 = 18$, and experiment with different choices of t. Verify that everything checks out.

$$\begin{tikzcd} & t \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\ x_1 & \min\{x_1, x_2\} \arrow[l] \arrow[r] & x_2 \end{tikzcd}$$ [a]

Figure 1.12: Taking the minimum

[a] `050_example_prod_min_cont`

{exa:gcd-as-prod-cont}

Example 1.14. This is a continuation of Example 1.4. For any $m, n \in \mathbb{N}$, we drew an arrow $m \to n$ iff m divides n, written $m|n$. The category in questions has natural numbers as its objects, and morphisms are given by the relation "divides". Then product is “taking the greatest common divisor”; its universal property is visualized in Fig. 1.2.

For a concrete example, let $m = 12$ and $n = 18$, so $\gcd\{12, 18\} = 6$. If we take $t = 3$, which divides both 12 and 18, we see that, indeed, 3 also divides $6 = \gcd\{12, 18\}$. And if we take $t = 2$, which *also* divides both 12 and 18, we see that it is *also* true that 2 also divides $6 = \gcd\{12, 18\}$.

$$\begin{tikzcd} & t \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\ m & \gcd\{m, n\} \arrow[l] \arrow[r] & n \end{tikzcd}$$ [a]

Figure 1.13: Taking the greatest common divisor

[a] `050_example_prod_gcd_cont`

{exa:intersection-as-prod-cont}

Example 1.15. This is a continuation of Example 1.5. Given a set $\mathbf{A}$ and arbitrary subsets $\mathbf{S}_1, \mathbf{S}_2 \subseteq \mathbf{A}$, we drew an arrow $\mathbf{S}_1 \to \mathbf{S}_2$ iff $\mathbf{S}_1 \subseteq \mathbf{S}_2$. The category in question here has as its objects the subsets of $\mathbf{A}$, and its morphisms are inclusions between them. The product is “taking the intersection”; its universal property is visualized in Fig. 1.3.

As a concrete example, consider again $\mathbf{A} = \{1, 2, 3, 4\}$, $\mathbf{S}_1 = \{1, 2, 3\}$, and $\mathbf{S}_2 = \{2, 3, 4\}$. So $\mathbf{S}_1 \cap \mathbf{S}_2 = \{2, 3\}$. If we choose $T = \{2\}$, we see that $T \subseteq \mathbf{S}_1$ and $T \subseteq \mathbf{S}_2$, and that also $T \subseteq \mathbf{S}_1 \cap \mathbf{S}_2$ (as it must, according to the universal property). The situation is similar if we choose $T = \{1\}$ or $T = \emptyset$.

$$\begin{tikzcd} & T \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\ \mathbf{S}_1 & \mathbf{S}_1 \cap \mathbf{S}_2 \arrow[l] \arrow[r] & \mathbf{S}_2 \end{tikzcd}$$ [a]

Figure 1.14: Taking the intersection

[a] `050_example_prod_intersection_cont`

The following Lemma describes a general fact that was illustrated in Example 1.15.

{lem:cat-product-powerset}

Lemma 1.16. Let $\mathbf{A}$ be any set. Its powerset $\mathcal{P}\mathbf{A}$, with the relation of inclusion, is a poset. View this poset as a category (this means there is a single morphism $\mathbf{S}_1 \to \mathbf{S}_2$ if and only if $\mathbf{S}_1 \subseteq \mathbf{S}_2$). For any two objects $\mathbf{S}_1, \mathbf{S}_2 \in \mathcal{P}\mathbf{A}$, their categorical product exists and is given by $\mathbf{S}_1 \cap \mathbf{S}_2 \in \mathcal{P}\mathbf{A}$.

{ex:CatProductPowerset}

Graded exercise 1 (`CatProductPowerset`). Prove Lemma 1.16 by checking that Definition 1.8 is satisfied.

{exa:conjunction}

Example 1.17. This is a continuation of Example 1.6. We considered $\mathbf{A} = \{\top, \bot\}$ as a set of logical propositions and for any $p_1, p_2 \in \mathbf{A}$, we drew an arrow $p_1 \to p_2$ iff $p_1 \Rightarrow p_2$. So, the category at play here has $\mathbf{A}$ as its set of objects, and its morphisms are logical implications. The product is "taking the conjunction" (the logical opertaion "and"); the universal property is visible in Fig. 1.4.

$$\begin{tikzcd} & t \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\ p_1 & p_1 \wedge p_2 \arrow[l] \arrow[r] & p_2 \end{tikzcd}$$ [a]

{exa:meet-as-prod-cont}

Figure 1.15: Taking the conjunction

[a] `050_example_prod_conjunction_cont`

Example 1.18. This is a continuation of Example 1.18. We considered ... and we drew an arrow $x \to y$ iff $x \leq y$. So the category at play here is the one corresponding to the poset underlying ... . The categorical product of two elements is their meet (greatest lower bound); the universal property is illustrated in Fig. 1.5.

$$\begin{tikzcd} & t \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\ x & x \wedge y \arrow[l] \arrow[r] & y \end{tikzcd}$$ [a]

Figure 1.16: Taking the meet

[a] `050_example_prod_meet_cont`

Example 1.19. Suppose that we are designing a vehicle, and we are thinking about choices of engine. Both electric engines and internal combustion engins can produce **motion**, but each from a different source of energy. The electric engine uses **electric energy**; the internal combustion engine uses **gasoline**. The situation is depicted in Fig. 1.17, using the interpretation of the arrows that we have introduced for engineering design components. Namely, the arrow from motion to gasoline represents the internal combustion engine, and its direction is to be read as follows: given the desired functionality **motion**, **internal combustion engine** provides a way of getting it using **gasoline**. The other arrow in the figure represents the component **electric engine**, and is interpreted in a similar way.

{bhfn:23}

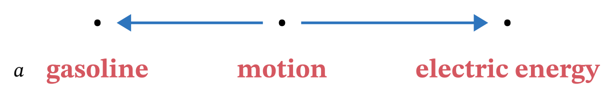


Figure 1.17: Alternative ways to generate **motion**.

[a] `30_dpcatfig_e14`

{fig:e14}

We could also consider building a hybrid vehicle, where we can obtain **motion** from **either** **gasoline** **or** **electric energy** (Fig. 1.18).

{bhfn:24}

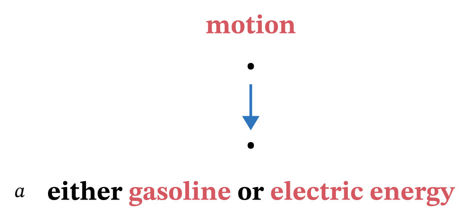


Figure 1.18: We can generate **motion** from either **gasoline** or **electric energy**.

[a] `30_dpcatfig_e15`

{fig:e15}

{def:cartesian-product-category}

Definition 1.20 (Cartesian product of categories)**.** Given two categories $\mathbf{C}$ and $\mathbf{D}$, their *cartesian product* $\mathbf{C} \times \mathbf{D}$ is the category specified as follows:

1. *Objects*: Objects are pairs $\langle X, Y \rangle$, with $X \in \mathrm{Ob}_{\mathbf{C}}$ and $Y \in \mathrm{Ob}_{\mathbf{D}}$.
2. *Morphisms*: Morphisms are pairs of morphisms $\langle f, g \rangle : \langle X, Z \rangle \to \langle Y, W \rangle$, with $f : X \to Y$, $g : Z \to W$.
3. *Identity morphisms*: Given objects $X \in \mathrm{Ob}_{\mathbf{C}}$ and $Y \in \mathrm{Ob}_{\mathbf{D}}$, the identity morphism on $\langle X, Y \rangle$ is the pair $\langle \mathrm{Id}_X, \mathrm{Id}_Y \rangle$.

4. *Composition of morphisms*: The composition of morphisms is given by composing each component of the pair separately:

$$\langle f, g\rangle \fatsemi_{\mathbf{C}\times\mathbf{D}} \langle h, i\rangle = \langle f \fatsemi_{\mathbf{C}} h,\, g \fatsemi_{\mathbf{D}} i\rangle. \tag{1.13}$$

Example 1.21. Consider two posets $\mathbf{P}$, $\mathbf{Q}$ as categories. The product poset $\mathbf{P} \times \mathbf{Q}$ (**??**) is the product category of the two posetal categories.

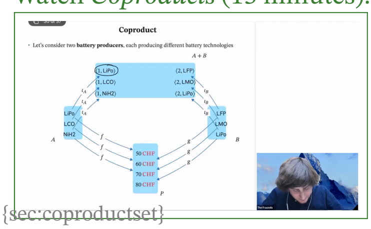
Graded exercise 2. Prove that the product poset $\mathbf{P} \times \mathbf{Q}$ "is" the categorical product of $\mathbf{P}$ and $\mathbf{Q}$ within the category of posets.

Write the solution

Graded exercise 3. Prove that the product category $\mathbf{C} \times \mathbf{D}$ of two small categories "is" the categorical product of $\mathbf{C}$ and $\mathbf{D}$ within the category of small categories.

Write the solution

Watch *Coproducts* (13 minutes).



1.2 Coproducts

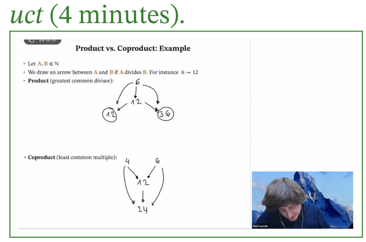
{sec:coproductset}

There exists a "dual" notion to "product" that is called "coproduct". Just like the notion of categorical product generalized the definition of the cartesian product of two sets, the categorical coproduct generalizes the definition of the *disjoint union* of two sets.

Given sets $\mathbf{A}$ and $\mathbf{B}$, their disjoint union $\mathbf{A} + \mathbf{B}$ is a set that contains a distinct copy of $\mathbf{A}$ and $\mathbf{B}$ each. If an element is contained in both $\mathbf{A}$ and $\mathbf{B}$, then there will be two distinct copies of it in the disjoint union $\mathbf{A} + \mathbf{B}$.

{def:disjoint union}

Watch *Introductory examples of coproduct* (4 minutes).



Definition 1.22 (Disjoint union of sets). The *disjoint union* (or sum) of sets $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} + \mathbf{B} = \{\langle 1, x\rangle \mid x \in \mathbf{A}\} \cup \{\langle 2, y\rangle \mid y \in \mathbf{B}\}. \tag{1.14}$$

Example 1.23. Consider the sets $\mathbf{A} = \{\star, \diamond\}$ and $\mathbf{B} = \{*, \dagger\}$. Their disjoint union can be represented as in Fig. 1.19.

We can define the disjoint union of a set with itself; this corresponds to having two distinct copies of the set (Fig. 1.20).

In the case of the cartesian product of two sets we had projection maps, as in [REF]. For the disjoint union of sets, we have instead *inclusion maps*. Thus we have a diagram of this form:

{bhfn:25}

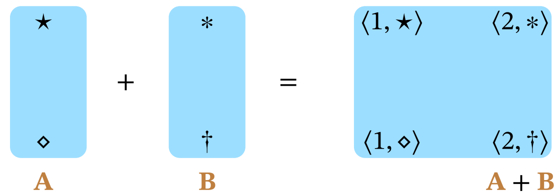
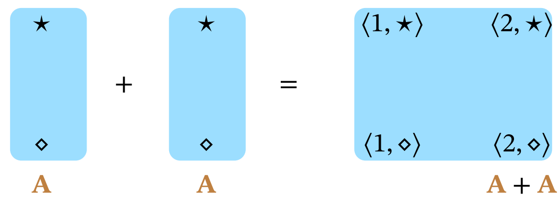
[a] 

Figure 1.19: Example of a disjoint union of sets.

[a] `30_disjoint_union`

{fig:disjoint}
{bhfn:26}

[a] 

{fig:disjointself}
{bhfn:27}

[a]
$$\begin{tikzcd} \mathbf{A} \arrow[r, "\iota_1"'] & \mathbf{A}+\mathbf{B} & \mathbf{B} \arrow[l, "\iota_2"] \end{tikzcd}$$

{fig:coprod_dis

{exa:min-as-pr

[a]
$$\begin{tikzcd} & t & \\ x_1 \arrow[ur] \arrow[r] & \max\{x_1,x_2\} \arrow[u, dashed] & x_2 \arrow[ul] \arrow[l] \end{tikzcd}$$

{exa:lcm-as-coprod-cont}

Figure 1.21: Taking the minimum

{exa:union-as-coprod-cont}

[a] `30_disjoint_union_self`
[a] `050_coprod_disunion_diagram`
[a] `050_example_coprod_max_cont`

[a]
$$\begin{tikzcd} & t & \\ m \arrow[ur] \arrow[r] & \operatorname{lcm}\{m,n\} \arrow[u, dashed] & n \arrow[ul] \arrow[l] \end{tikzcd}$$

{exa:disjunction-as-coprod-cont}

Figure 1.22: Taking the least common multiple

[a] `050_example_coprod_lcm_cont`

[a]
$$\begin{tikzcd} & T & \\ \mathbf{S}_1 \arrow[ur] \arrow[r] & \mathbf{S}_1 \cup \mathbf{S}_2 \arrow[u, dashed] & \mathbf{S}_1 \arrow[ul] \arrow[l] \end{tikzcd}$$

{exa:join-as-coprod-cont}

Figure 1.23: Taking the union

[a] `050_example_coprod_union_cont`

Example 1.24. This example is "dual" to Example 1.13. The category in question in the same one: objects are elements of $\mathbb{R}$ and morphisms are inequalities. The coproduct is "taking the maximum"; its universal property is illustrated in Fig. 1.12. It says that if $t \in \mathbb{R}$ is such that $t \geq x_1$ and $t \geq x_2$, then $t \geq \min\{x_1, x_2\}$.

Example 1.25. This example is "dual" to Example 1.14. The category we're working in has natural numbers as its objects, and morphisms are given by the relation "divides". The cproduct is "taking the least common multiple"; its universal property is visualized in Fig. 1.2.

Example 1.26. This example is "dual" to Example 1.15. Given a set $\mathbf{A}$ and arbitrary subsets $\mathbf{S}_1, \mathbf{S}_2 \subseteq \mathbf{A}$, we drew an arrow $\mathbf{S}_1 \to \mathbf{S}_2$ iff $\mathbf{S}_1 \subseteq \mathbf{S}_2$. The category in question here has, as its objects, the subsets of $\mathbf{A}$, and its morphisms are inclusions between them. The product is "taking the intersection"; its universal property is visualized in Fig. 1.3.

As a concrete example, consider again $\mathbf{A} = \{1,2,3,4\}$, $\mathbf{S}_1 = \{1,2,3\}$, and $\mathbf{S}_2 = \{2,3,4\}$. So $\mathbf{S}_1 \cap \mathbf{S}_2 = \{2,3\}$. If we choose $T = \{2\}$, we see that $T \subseteq \mathbf{S}_1$ and $T \subseteq \mathbf{S}_2$, and that also $T \subseteq \mathbf{S}_1 \cap \mathbf{S}_2$ (as it must, according to the universal property). The situation is similar if we choose $T = \{1\}$ or $T = \emptyset$.

Example 1.27. This example is "dual" to Example 1.15. Again consider the set $\mathbf{A} = \{\top, \bot\}$ of logical propositions and for any $p_1, p_2 \in \mathbf{A}$, we drew an arrow $p_1 \to p_2$ iff $p_1 \Rightarrow p_2$. The category we are working with has $\mathbf{A}$ as its set of objects, and its morphisms are logical implications. The coproduct is "taking the disjunction" (the logical operation "or"); the universal property is shown in Fig. 1.4.

Example 1.28. This example is "dual" to Example 1.18. We considered ... and we drew an arrow $x \to y$ iff $x \leq y$. The category at play here is the one corresponding to the poset underlying ... . The categorical product of two elements is their join (least upper bound); the universal property is illustrated in Fig. 1.5.

As you can see from the above list of examples, the notion of coproduct involves diagrams of the type in Fig. 1.26.

{bhfn:33}
oprod_generic}

$$a \quad X \longrightarrow \text{“coproduct of } X \text{ and } Y\text{”} \longleftarrow Y$$

Figure 1.26

[a] 050_coprod_generic

{bhfn:31}

As mentioned above, the disjoint union is a particular instance – in the category **Set**– of the notion of “coproduct”. We will now give the definition of a coproduct in an arbitrary category. Note that it is very similar to the definition that we gave, in the previous section, for the product – but with a few twists. Analogous remarks to those we gave following the definition of the product apply here!

{def:catcoproduct}

Definition 1.29 (Coproduct). Let $\mathbf{C}$ be a category and let $X, Y \in \mathrm{Ob}_{\mathbf{C}}$ be objects. The *coproduct* of X and Y is:

Constituents

1. an object $Z \in \mathrm{Ob}_{\mathbf{C}}$ (“the coproduct” of X and Y)
2. *injection morphisms* $\iota_1 : X \to Z$ and $\iota_2 : Y \to Z$

Conditions

1. For any $T \in \mathrm{Ob}_{\mathbf{C}}$ and any morphisms $f : X \to T, g : Y \to T$, there exists a *unique* morphism $\psi_{f,g} : Z \to T$ such that $f = \iota_1 ⨟ \psi_{f,g}$ and $g = \iota_2 ⨟ \psi_{f,g}$.

Remark 1.30. Diagrammatically, the condition above states that diagrams of this form commute: Similarly as was the case with the categorical product, “the

$$a \quad \begin{tikzcd} & T & \\ X \arrow[ur, "f"] \arrow[r, "\iota_1"'] & Z \arrow[u, dashed, "\psi_{f,g}"] & Y \arrow[ul, "g"'] \arrow[l, "\iota_2"] \end{tikzcd}$$

Figure 1.27 fig:coprod_general_1

[a] 60_defcoproduct

{fig:def-coproduct-diagram}

coproduct” of X and Y is unique only “up to isomorphism”. Nevertheless, we will usually simply write $X+Y$ for “the” coproduct (in place of Z above), and we will usually write $f+g$ in place of $\psi_{f,g}$. The diagram in Fig. 1.27 then looks as in Fig. 1.28

{bhfn:35}

$$a \quad \begin{tikzcd} & T & \\ X \arrow[ur, "f"] \arrow[r, "\iota_1"'] & X+Y \arrow[u, dashed, "f+g"] & Y \arrow[ul, "g"'] \arrow[l, "\iota_2"] \end{tikzcd}$$

Figure 1.28 fig:def-coproduct-diagram-generic

[a] 60_defcoproduct_generic

$$a \quad \begin{tikzcd} & t & \\ p_1 \arrow[ur] \arrow[r] & p_1 \vee p_2 \arrow[u, dashed] & p_2 \arrow[ul] \arrow[l] \end{tikzcd}$$

Figure 1.24: Taking the disjunction

[a] 050_example_coprod_disjunction_cont

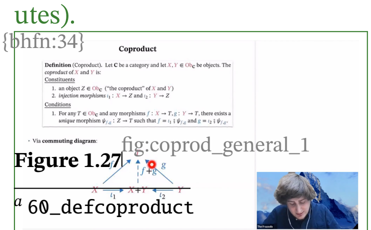
$$a \quad \begin{tikzcd} & t & \\ x \arrow[ur] \arrow[r] & x \vee y \arrow[u, dashed] & y \arrow[ul] \arrow[l] \end{tikzcd}$$

Figure 1.25: Taking the join

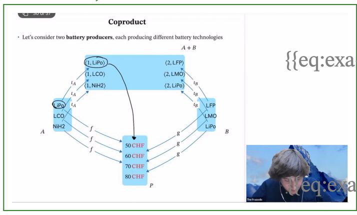
[a] 050_example_coprod_join_cont

Watch *Categorical coproduct* (1 minutes).

{bhfn:34}



Watch *Extended coproduct example* (8 minutes).



Example 1.31. Let's consider two battery producers, each producing specific battery technologies. The first company produces a set

{{eq:exa-batt-setA}}

$$\mathbf{A} = \{\mathsf{LiPo}, \mathsf{LCO}, \mathsf{NiH2}\} \tag{1.15}$$

of technologies, and the second one a set

{{eq:exa-batt-setB}}

$$\mathbf{B} = \{\mathsf{LFP}, \mathsf{LMO}, \mathsf{LiPo}\}. \tag{1.16}$$

Each technology has, for a specific desired battery mass, a specific price, belonging to a set of prices

{{eq:exa-batt-prices}}

$$\mathbf{P} = \{50\,, 60\,, 70\,, 80\,\} \times \{CHF\}. \tag{1.17}$$

We specify the price mappings for different technologies via the functions $f : \mathbf{A} \to \mathbf{P}$ and $g : \mathbf{B} \to \mathbf{P}$. A battery vendor wants to sell batteries from both producers and wants to create a battery catalogue, which needs to take into account which technology comes from which producer, to be able to distribute the earnings from the sales fairly. To this end, the disjoint union of the sets of technology is considered:

{{eq:exa-batt-disunion}}

$$\mathbf{A} + \mathbf{B} = \{\langle 1, \mathsf{LiPo}\rangle, \langle 1, \mathsf{LCO}\rangle, \langle 1, \mathsf{NiH2}\rangle, \langle 2, \mathsf{LFP}\rangle, \langle 2, \mathsf{LMO}\rangle, \langle 2, \mathsf{LiPo}\rangle\}. \tag{1.18}$$

It is possible to map each technology in $\mathbf{A}$, $\mathbf{B}$ to its own representative in $\mathbf{A} + \mathbf{B}$ via the so-called injection maps:

{{eq:exa-batt-inclusion-1}}

$$\begin{aligned} \iota_{\mathbf{A}} : \mathbf{A} &\to \mathbf{A} + \mathbf{B} \\ a &\mapsto \langle 1, a\rangle \end{aligned} \tag{1.19}$$

{{eq:exa-batt-inclusion-2}}

$$\begin{aligned} \iota_{\mathbf{B}} : \mathbf{B} &\to \mathbf{A} + \mathbf{B} \\ b &\mapsto \langle 2, b\rangle\,. \end{aligned} \tag{1.20}$$

This situation is graphically represented in Fig. 1.29, and mimics the coproduct diagram presented in Definition 1.29.

{bhfn:36}

a

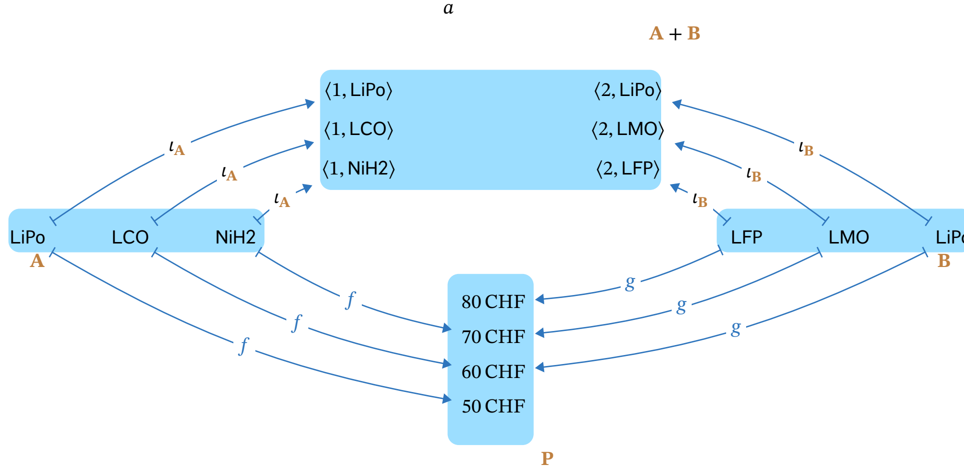


Figure 1.29: Battery technologies, companies, prices, and a catalogue.

[a] 60_coprod_batt_bis

{fig:coprod_batteries_1}

Here, the universal property says that there is a **unique** function $f+g : \mathbf{A}+\mathbf{B} \to \mathbf{P}$ such that

{{eq:exa-batt-factoring-maps}}

$$\iota_{\mathbf{A}} \mathbin{⨾} (f + g) = f \text{ and } \iota_{\mathbf{B}} \mathbin{⨾} (f + g) = g. \tag{1.21}$$

{bhfn:37}

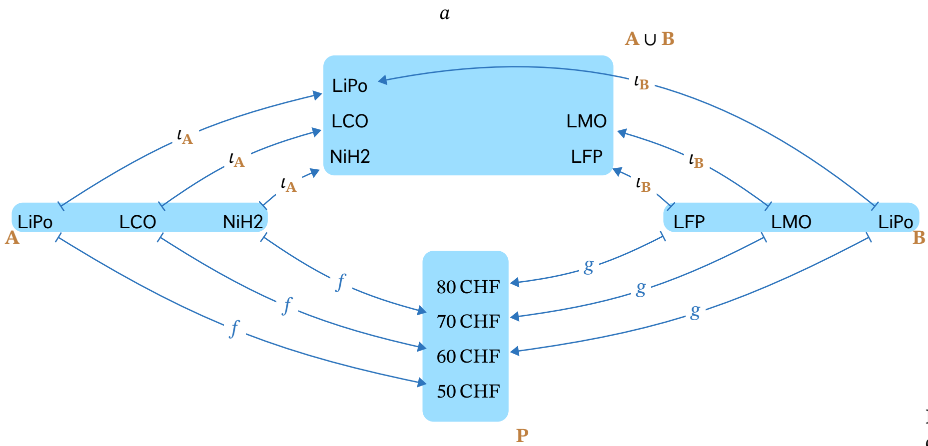


Figure 1.30: Example: why the union is not the coproduct in **Set**.

[a] 60_coprod_batt_2_bis

od_batteries_2}

If we take a $x \in \mathbf{A} + \mathbf{B}$ is either "from $\mathbf{A}$ or from $\mathbf{B}$":

ain-sum-map}}

$$\text{either } \exists a \in \mathbf{A} : \ x = \iota_{\mathbf{A}}(a) \text{ or } \exists b \in \mathbf{B} : \ x = \iota_{\mathbf{B}}(b). \tag{1.22}$$

From this, we can deduce that the desired map $f+g$ is:

$$\begin{aligned} f+g : \ & \mathbf{A} + \mathbf{B} \to \mathbf{P} \\ & x \mapsto \begin{cases} f(x), & \text{if } x = \iota_{\mathbf{A}}(a), \quad a \in \mathbf{A}, \\ g(x), & \text{if } x = \iota_{\mathbf{B}}(b), \quad b \in \mathbf{B}. \end{cases} \end{aligned} \tag{1.23}$$

{{eq:exa-batt-def-sum-map}}

This is a specific example of **Set**/**FinSet**, in which the coproduct is a generalization of the concept of disjoint union. Now, we could spontaneously ask ourselves: why does the union not "suffice" for the coproduct definition in **Set**? To see this, let's consider the same situation as before, but now having the catalogue of technologies given by $\mathbf{A} \cup \mathbf{B}$ (Fig. 1.30). The interpretation of maps f, g does not change, and injections work as depicted. Note, however, that when asked for a map from the technology LiPo $\in \mathbf{A} \cup \mathbf{B}$, we have no notion of the company which produces it, and we are therefore unsure whether to assign it to $f(\text{LiPo}) = 50$ CHF or $g(\text{LiPo}) = 60$ CHF. Indeed, the unique map $f+g$ required by the universal property of the coproduct cannot exist, since in case $\mathbf{A} \cap \mathbf{B} \neq \emptyset$, any element $x \in \mathbf{A} \cap \mathbf{B}$ should be simultaneously sent to $f(x)$ and $g(x)$.

Example 1.32. Given $\mathbf{A}, \mathbf{B} \in \mathrm{Ob}_{\mathbf{Rel}}$ (so $\mathbf{A}$ and $\mathbf{B}$ are sets) their coproduct is the disjoint union $\mathbf{A} + \mathbf{B}$. The disjoint union of sets comes equipped with inclusion functions $\iota_{\mathbf{A}} : \mathbf{A} \to \mathbf{A} + \mathbf{B}$ and $\iota_{\mathbf{B}} : \mathbf{B} \to \mathbf{A} + \mathbf{B}$. If we turn these functions into relations

$$\begin{aligned} R_{\iota_{\mathbf{A}}} &\subseteq \mathbf{A} \times (\mathbf{A} + \mathbf{B}) \\ R_{\iota_{\mathbf{B}}} &\subseteq \mathbf{B} \times (\mathbf{A} + \mathbf{B}). \end{aligned}$$

then these are the injection morphisms for the coproduct in **Rel**. As an aside, we note that in **Rel** products and coproducts are *both* given by the disjoint union of sets. We will see later why this is might be expected.

Example 1.33. Let $m, n \in \mathbb{N}$, and draw an arrow $m \to n$ if m divides n. For instance, 6 divides 12 and hence there is an arrow $6 \to 12$. The coproduct between any two $m, n \in \mathbb{N}$ in this category is given by the least common multiple.

Example 1.34. Let's consider the ordered set $\langle \mathbb{R}, \leq \rangle$, where given $x_1, x_2 \in \mathbb{R}$ we can draw an arrow $x_1 \to x_2$ if $x_1 \leq x_2$. By following the coproduct's commutative diagram, we know that the coproduct of x_1 and x_2 is a $z \in \mathbb{R}$ such that

- $x_1 \leq z$;
- $x_2 \leq z$;
- For all $x \in \mathbb{R}$ with $x_1 \leq x$ and $x_2 \leq x$, we have $z \leq x$.

In other words, the coproduct of $x_1, x_2 \in \mathbb{R}$ is given by $\max\{x_1, x_2\}$, and is also called *join*.

{ex:subset_coprod}

Example 1.35. Let $\mathbf{S}$ be a set, and $\mathbf{A}, \mathbf{B} \subseteq \mathbf{S}$ subsets. We can draw an arrow $\mathbf{A} \to \mathbf{B}$ if $\mathbf{A} \subseteq \mathbf{B}$. By following the coproduct's commutative diagram, it is easy to see that the coproduct of $\mathbf{A}$ and $\mathbf{B}$ is given by $\mathbf{A} \cup \mathbf{B}$: the "smallest" set containing both $\mathbf{A}$ and $\mathbf{B}$.

finish this defintion

{def:disjoint-union-category}

Definition 1.36 (Disjoint union category). Given two categories $\mathbf{C}$ and $\mathbf{D}$, their *disjoint union* $\mathbf{C} + \mathbf{D}$ is the category specified as follows:

1. *Objects*: Objects are elements of $\mathrm{Ob}_{\mathbf{C}} + \mathrm{Ob}_{\mathbf{D}}$; that is, objects are tuples of the form $\langle X, i \rangle$, with $i = 1$ or $i = 2$, depending on whether $X \in \mathrm{Ob}_{\mathbf{C}}$ or $X \in \mathrm{Ob}_{\mathbf{D}}$.
2. *Morphisms*: Given objects $\langle X, i \rangle, \langle Y, j \rangle \in \mathrm{Ob}_{\mathbf{C}+\mathbf{D}}$,

$$\mathrm{Hom}_{\mathbf{C}+\mathbf{D}}(\langle X, i \rangle, \langle Y, j \rangle) := \begin{cases} \mathrm{Hom}_{\mathbf{C}}(X, Y) & \text{if } i = j = 1, \\ \mathrm{Hom}_{\mathbf{D}}(X, Y) & \text{if } i = j = 2, \\ \emptyset & \text{else.} \end{cases} \tag{1.24}$$

3. *Identity morphisms*:
4. *Composition of morphisms*:

1.3 Other examples

Product and coproduct for power set

Example 1.5 and Example 1.35 are specific instances of the power set lattice.

{def:power-set-as-lattice}

Definition 1.37 (Power set as lattice). Given a set $\mathbf{S}$, its power set $\mathscr{P}\mathbf{S}$ (the set of all subsets) is a lattice where, given $\mathbf{A}, \mathbf{B} \in \mathscr{P}\mathbf{S}$:

- Order is given by inclusion:

$$\mathbf{A} \preceq \mathbf{B} := \mathbf{A} \subseteq \mathbf{B};$$

- The join is given by the union of sets:

$$\mathbf{A} \vee \mathbf{B} := \mathbf{A} \cup \mathbf{B};$$

▷ The meet is given by the intersection of sets:

$$\mathbf{A} \wedge \mathbf{B} := \mathbf{A} \cap \mathbf{B};$$

▷ The top element is the set $\mathbf{S}$ itself:

$$\top = \mathbf{S};$$

▷ The bottom element is the empty set:

$$\bot = \emptyset.$$

The Hasse diagram reported in Fig. 1.31 illustrates the structure of the power set lattice for three sets $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathscr{P}\mathbf{S}$.

{bhfn:38}

$$\begin{tikzcd}
 & & \mathbf{S} \arrow[d, no head] & & \\
 & & \mathbf{A} \cup \mathbf{B} \cup \mathbf{C} \arrow[dl, no head] \arrow[dr, no head] & & \\
 & \mathbf{A} \cup \mathbf{B} \arrow[dl, no head] \arrow[dr, no head] & & \mathbf{B} \cup \mathbf{C} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbf{A} \arrow[dr, no head] & & \mathbf{B} \arrow[dl, no head] \arrow[dr, no head] & & \mathbf{C} \arrow[dl, no head] \\
 & \mathbf{A} \cap \mathbf{B} \arrow[dr, no head] & & \mathbf{B} \cap \mathbf{C} \arrow[dl, no head] & \\
 & & \mathbf{A} \cap \mathbf{B} \cap \mathbf{C} \arrow[d, no head] & & \\
 & & \emptyset & &
\end{tikzcd}$$

[a]

Figure 1.31

[a] 060_powerset_coprod

{fig:prod_coprod_power}

As previously discovered, the lattice can be seen as a category. In this category, the meet $\wedge$ is the product, and the join $\vee$ is the coproduct. Specifically, for $\mathbf{A}, \mathbf{B} \subseteq \mathbf{S}$ the product corresponds to $\mathbf{A} \cap \mathbf{B}$, and the projection maps $\pi_{\mathbf{A}} : \mathbf{A} \cap \mathbf{B} \to \mathbf{A}$ and $\pi_{\mathbf{B}} : \mathbf{A} \cap \mathbf{B} \to \mathbf{B}$ simply state the inclusions of $\mathbf{A} \cap \mathbf{B}$ in $\mathbf{A}$ and $\mathbf{B}$. Similarly, the coproduct corresponds to $\mathbf{A} \cup \mathbf{B}$, and the injection maps $\iota_1 : \mathbf{A} \to \mathbf{A} \cup \mathbf{B}$ and $\iota_2 : \mathbf{B} \to \mathbf{A} \cup \mathbf{B}$ simply state the inclusion of $\mathbf{A}, \mathbf{B}$ in $\mathbf{A} \cup \mathbf{B}$.

Product and coproduct for logical sequents

We are working on this section, and more content will appear.

Add here content of Andrea's slides

1.4 Biproducts

We are working on this section, and more content will appear.

1.5 Occultism

We are working on this section, and more content will appear.

Solutions to exercises