

**Lemma.** **UPos** is a monoidal category with the following additional structure:

1. *Tensor product*  $\circledast$ : On objects, the tensor product corresponds to the product of posets. Given two morphisms  $f : X \rightarrow \mathcal{U}Y$  and  $g : Z \rightarrow \mathcal{U}W$ , we have:

$$f \circledast g : X \times Z \rightarrow \mathcal{U}(Y \times W)$$

$$\langle x, z \rangle \mapsto f(x) \times g(z).$$

Note that the Cartesian product of upper sets is an upper set.

2. *Unit*: The unit is the identity poset ( $\{\bullet\}$ ).
3. *Left unitor*: The left unitor is given by the pair of morphisms

$$\text{lu}_{\mathbf{P}} : \{\bullet\} \circledast X \rightarrow \mathcal{U}X$$

$$\langle \bullet, x \rangle \mapsto \uparrow\{x\},$$

and

$$\text{lu}_{\mathbf{P}}^{-1} : X \rightarrow \mathcal{U}(\{\bullet\} \circledast X)$$

$$x \mapsto \{\bullet\} \times \uparrow\{x\}.$$

4. *Right unitor*: The right unitor is given by the pair of morphisms

$$\text{ru}_{\mathbf{P}} : X \circledast \{\bullet\} \rightarrow \mathcal{U}X$$

$$\langle x, \bullet \rangle \mapsto \uparrow\{x\},$$

and

$$\text{ru}_{\mathbf{P}}^{-1} : X \rightarrow \mathcal{U}(X \circledast \{\bullet\})$$

$$x \mapsto \uparrow\{x\} \times \{\bullet\}.$$

5. *Associator*: The associator is given by the pair of morphisms:

$$\text{as}_{XY,Z} : (X \circledast Y) \circledast Z \rightarrow \mathcal{U}X \times (\mathcal{U}Y \times \mathcal{U}Z)$$

$$\langle \langle x, y \rangle, z \rangle \mapsto \uparrow\{x\} \times (\uparrow\{y\} \times \uparrow\{z\}),$$

and

$$\text{as}_{X,YZ} : X \circledast (Y \circledast Z) \rightarrow (\mathcal{U}X \times \mathcal{U}Y) \times \mathcal{U}Z$$

$$\langle x, \langle y, z \rangle \rangle \mapsto (\uparrow\{x\} \times \uparrow\{y\}) \times \uparrow\{z\}.$$