

**Definition** (Monoidal poset). A *monoidal structure* on a poset  $\mathbf{P} = \langle \mathbf{P}, \leq_{\mathbf{P}} \rangle$  is specified by:

### Constituents

1. A monotone map  $\otimes : \mathbf{P} \times \mathbf{P} \rightarrow \mathbf{P}$ , called the *monoidal product*.

Note that here we are implicitly assuming  $\mathbf{P} \times \mathbf{P}$  as having the product order.

In detail, monotonicity means that, for all  $x_1, x_2, y_1, y_2 \in \mathbf{P}$ :

$$x_1 \leq_{\mathbf{P}} y_1 \text{ and } x_2 \leq_{\mathbf{P}} y_2 \implies (x_1 \otimes x_2) \leq_{\mathbf{P}} (y_1 \otimes y_2).$$

2. An element  $1 \in \mathbf{P}$ , called the *monoidal unit*.

### Conditions

1. Associativity: for all  $x, y, z \in \mathbf{P}$ :

$$(x \otimes y) \otimes z = x(\otimes y \otimes z).$$

2. Left and right unitality: for all  $x \in \mathbf{P}$ :

$$1 \otimes x = x \quad \text{and} \quad x \otimes 1 = x.$$

A poset equipped with a monoidal structure is called a *monoidal poset*.