Connection

Applied Compositional Thinking for Engineers

Example: "distribution networks"

General situation: something is distributed via a network

Specific example: electric power distributed via a power grid.

- Toy model:

To model **connectivity**: arrows
Direction of arrows: flow of distribution

Which consumers are connected to which power plants? Look at **paths**:

We also might want to show which high voltage are connected to each other:

Note: there is also a way to make these relationships symmetric.

The information above can also be represented as directed graph:

For comparison, a representation of a power grid taken from Wikipedia:

Binary relations

Definition: A (binary) **relation** from a set X to a set Y is a subset of $X \times Y$

Example: $X = \{x_1, x_2, x_3\}, \quad Y = \{y_1, y_2, y_3, y_4\}$

$R \subseteq X \times Y$ given by

$$R = \{\langle x_1, y_1 \rangle, \langle x_2, y_3 \rangle, \langle x_2, y_4 \rangle\}$$

Notation: If $R \subseteq X \times Y$ is a relation, we write $R : X \to Y$ or $X \xrightarrow{R} Y$.

Sometimes the notation $R : X \relbar\joinrel\mapstochar\rightarrow Y$ is used to emphasize that R is a relation, and not a function.

We will see: we can think of a relation as a type of **morphism**.

Example: In the power grid example, we had

This represents a relation

$$X = \{\text{plant1}, \text{plant2}, \text{plant3}\} \quad \longrightarrow \quad Y = \{\text{HVN1}, \text{HVN2}, \text{HVN3}, \text{HVN4}, \text{HVN5}\}$$

Relations can be composed

Suppose we have relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, i.e. relations

$$X \xrightarrow{R} Y \xrightarrow{S} Z.$$

How might we compose R and S to obtain a relation $X \xrightarrow{R \fatsemi S} Z$?

Example:

What is the composition $R \fatsemi S$?

Look at **paths** from X to Z.

So, $R \fatsemi S$ is this relation:

Definition: Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be relations. Their **composition** is

$$R \mathbin{;} S := \{\langle x, z\rangle \in X \times Z \mid \exists\, y \in Y : \langle x, y\rangle \in R \ \wedge \ \langle y, z\rangle \in S\}$$

which is a relation $X \to Z$.

Definition: The category **Rel** of sets and relations:

- Objects:
- Homsets: given sets X and Y,

$$\mathrm{Hom}_{\mathbf{Rel}}(X, Y) := \mathcal{P}(X \times Y) = \text{ all subsets of } X \times Y$$

- Identity morphisms: given a set X, the identity relation id_X is

$$\mathrm{id}_X := \{\langle x, x'\rangle \in X \times X \mid x = x'\}.$$

- Composition: as above.

Note: Graphically, identity morphisms looks like this:

Relations and functions

Functions are **special types** of relations. Given a function $f : X \to Y$, we can turn it into a relation by considering its **graph**,

$$R_f := \{\langle x, y \rangle \in X \times Y \mid y = f(x)\}.$$

Example:

Definition: Let X and Y be sets. A **function** $f : X \to Y$ is a relation $R_f \subseteq X \times Y$ such that

1. $\forall\, x \in X \quad \exists\, y \in Y \;:\; \langle x, y\rangle \in R_f$

$$\forall\, x \in X\ \exists y \in Y \;:\; y = f(x)$$

"every element of the source X gets mapped by f to some element of the target Y"

2. $\forall\, \langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in R_f$ holds : $\quad x_1 = x_2 \;\Rightarrow\; y_1 = y_2$

$$\forall\, x_1, x_2 \in X \text{ holds :} \quad x_1 = x_2 \;\Rightarrow\; f(x_1) = f(x_2)$$

"f is single-valued"

Example: This relation is *not* a function.

Example: This relation *is* a function.

Example: Can we have a function (or relation?) whose source is the empty set?

Example: Can we have a function (or relation?) whose target is the empty set?

Lemma: Composition of relations generalizes the "usual" composition of functions. That is, if we have functions

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

then

$$R_f \fatsemi R_g = R_{f \fatsemi g}.$$

Proof:

$$\begin{aligned} R_f \fatsemi R_g &= \{\langle x, z\rangle \in X \times Z \mid \exists y \in Y : \langle x, y\rangle \in R_f \wedge \langle y, z\rangle \in R_g\} \\ &= \{\langle x, z\rangle \in X \times Z \mid \exists y \in Y : y = f(x) \wedge z = g(y) \in R_g\} \end{aligned}$$

$$R_{f \fatsemi g} = \{\langle x, z\rangle \in X \times Z \mid z = g(f(x)) \in R_g\}$$

The opposite of a relation

Definition: Let $R \subseteq X \times Y$ be a relation.
Possible properties R might have:

1. **surjective:**
$$\forall y \in Y \ \exists x \in X : \langle x, y \rangle \in R$$
2. **injective:**
$$\forall \langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in R \text{ holds} : y_1 = y_2 \ \Rightarrow \ x_1 = x_2$$
3. **defined-everywhere:**
$$\forall x \in X \ \exists y \in Y : \langle x, y \rangle \in R$$
4. **single-valued:**
$$\forall \langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in R \text{ holds} : x_1 = x_2 \ \Rightarrow \ y_1 = y_2$$

Note a certain "duality" here!

Example:

Example:

Example:

(reflexive relations)
Example: In the power grid example, we also had

This represents a relation

$$Y = \{\text{HVN1}, \text{HVN2}, \text{HVN3}, \text{HVN4}, \text{HVN5}\} \quad \longrightarrow \quad Y = \{\text{HVN1}, \text{HVN2}, \text{HVN3}, \text{HVN4}, \text{HVN5}\}$$

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