

## Definition (Monoidal category)

A *monoidal structure* on a category  $\mathbf{C}$  is specified by:

### Constituents

1. A functor  $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ , called the *monoidal product*.
2. An object  $\mathbf{1} \in \mathbf{Ob}_{\mathbf{C}}$ , called the *monoidal unit*.
3. A natural isomorphism, called the *associator*, whose components are of the type

$$\text{as}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \quad X, Y, Z \in \mathbf{Ob}_{\mathbf{C}}.$$

4. A natural isomorphism, called the *left unitor*, whose components are of the type

$$\text{lu}_X : \mathbf{1} \otimes X \xrightarrow{\cong} X \quad X \in \mathbf{Ob}_{\mathbf{C}}.$$

5. A natural isomorphism, called the *right unitor*, whose components are of the type

$$\text{ru}_X : X \otimes \mathbf{1} \xrightarrow{\cong} X \quad X \in \mathbf{Ob}_{\mathbf{C}}.$$

### Conditions

For all  $X, Y, Z, U \in \mathbf{Ob}_{\mathbf{C}}$ , the following diagrams must commute:

1. Triangle identities.

$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{\text{as}_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\ & \searrow \text{ru}_X \otimes \text{Id}_Y & \swarrow \text{Id}_X \otimes \text{lu}_Y \\ & X \otimes Y & \end{array}$$

2. Pentagon identities.

$$\begin{array}{ccccc} & & (X \otimes Y) \otimes (Z \otimes U) & & \\ & \nearrow \text{as}_{X \otimes Y, Z, U} & & \searrow \text{as}_{X, Y, Z \otimes U} & \\ ((X \otimes Y) \otimes Z) \otimes U & & & & (X \otimes (Y \otimes (Z \otimes U))) \\ \downarrow \text{as}_{X, Y, Z} \otimes \text{Id}_U & & & & \uparrow \text{Id}_X \otimes \text{as}_{Y, Z, U} \\ (X \otimes (Y \otimes Z)) \otimes U & \xrightarrow{\text{as}_{X, Y \otimes Z, U}} & & & X \otimes ((Y \otimes Z) \otimes U) \end{array}$$

A category equipped with a monoidal structure is called a *monoidal category*. If the components of the associator, left unitor, and right unitor are all equalities, one calls the category *strict monoidal*.