Lemma. $\mathbf{Pos}_{\mathcal{U}}$ and $\mathbf{Pos}_{\mathcal{L}}$ are equivalent: there exists a pair of functors

$$\swarrow : \mathbf{Pos}_{\mathcal{U}} \to \mathbf{Pos}_{\mathcal{L}},$$
$$\nearrow : \mathbf{Pos}_{\mathcal{L}} \to \mathbf{Pos}_{\mathcal{U}},$$

such that $\swarrow \fatsemi \nearrow = \mathrm{Id}_{\mathbf{Pos}_{\mathcal{U}}}$ and $\nearrow \fatsemi \swarrow = \mathrm{Id}_{\mathbf{Pos}_{\mathcal{L}}}$, where $\mathrm{Id}_{\mathbf{Pos}_{\mathcal{U}}}$ and $\mathrm{Id}_{\mathbf{Pos}_{\mathcal{L}}}$ are the identity functors on $\mathbf{Pos}_{\mathcal{U}}$ and $\mathbf{Pos}_{\mathcal{L}}$, respectively.