

Definition (Profunctor composition)

Given two profunctors $F : \mathbf{C} \multimap \mathbf{D}$ and $G : \mathbf{D} \multimap \mathbf{E}$ we can define their composition $(F \circ G) : \mathbf{C} \multimap \mathbf{E}$ as follows:

$$(F \circ G)_{\text{ob}} : \text{Ob}(\mathbf{C}^{\text{op}} \times \mathbf{E}) \rightarrow \text{Ob Set},$$

$$\langle C^*, E \rangle \mapsto \coprod_{D \in \text{Ob} \mathbf{D}} F_{\text{ob}}(C^*, D) \times G_{\text{ob}}(D^*, E) / \sim$$

$$(F \circ G)_{\text{mor}} : \text{Hom}_{(\mathbf{C}^{\text{op}} \times \mathbf{E})}(\langle C_1^*, E_1 \rangle; \langle C_2^*, E_2 \rangle) \rightarrow \text{Hom}_{\text{Set}}((F \circ G)_{\text{ob}}(C_1^*, E_1); (F \circ G)_{\text{ob}}(C_2^*, E_2))$$

$$\langle \alpha^*, \beta \rangle \mapsto \begin{cases} (F \circ G)_{\text{ob}}(C_1^*, E_1) \rightarrow (F \circ G)_{\text{ob}}(C_2^*, E_2) \\ \langle s, t \rangle \mapsto \langle F_{\text{mor}}(\langle \alpha, \text{Id}_D \rangle)(s), G_{\text{mor}}(\langle \text{Id}_{D^*}, \beta \rangle)(t) \rangle \end{cases}$$

In the formulas:

$$\alpha : C_2 \rightarrow C_1, \quad \beta : E_1 \rightarrow E_2,$$

and $\langle s, t \rangle$ is a pair of elements for which there exists a $D \in \text{Ob} \mathbf{D}$ such that

$$s \in F_{\text{ob}}(C_1^*, D), \quad t \in G_{\text{ob}}(D^*, E).$$