

Definition (Monoidal category)

A *monoidal structure* on a category \mathbf{C} is specified by:

Constituents

1. A functor $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$, called the *monoidal product*.
2. An object $\mathbf{1} \in \mathbf{Ob}_{\mathbf{C}}$, called the *monoidal unit*.
3. A natural isomorphism, called the *associator*, whose components are of the type

$$as_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \quad X, Y, Z \in \mathbf{Ob}_{\mathbf{C}}.$$

4. A natural isomorphism, called the *left unitor*, whose components are of the type

$$lu_X : \mathbf{1} \otimes X \xrightarrow{\cong} X \quad X \in \mathbf{Ob}_{\mathbf{C}}.$$

5. A natural isomorphism, called the *right unitor*, whose components are of the type

$$ru_X : X \otimes \mathbf{1} \xrightarrow{\cong} X \quad X \in \mathbf{Ob}_{\mathbf{C}}.$$

Conditions For all $X, Y, Z, U \in \mathbf{Ob}_{\mathbf{C}}$, the following diagrams must commute:

1. Triangle identities.

$$\begin{array}{ccc}
 (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{as_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\
 & \searrow ru_X \otimes Id_Y & \swarrow Id_X \otimes lu_Y \\
 & X \otimes Y &
 \end{array}$$

2. Pentagon identities.

$$\begin{array}{ccc}
 & (X \otimes Y) \otimes (Z \otimes U) & \\
 as_{X \otimes Y, Z, U} \nearrow & & \searrow as_{X, Y, Z \otimes U} \\
 ((X \otimes Y) \otimes Z) \otimes U & & (X \otimes (Y \otimes (Z \otimes U))) \\
 \downarrow as_{X,Y,Z} \otimes Id_U & & \uparrow Id_X \otimes as_{Y,Z,U} \\
 (X \otimes (Y \otimes Z)) \otimes U & \xrightarrow{as_{X,Y \otimes Z, U}} & X \otimes ((Y \otimes Z) \otimes U)
 \end{array}$$

A category equipped with a monoidal structure is called a *monoidal category*. If the components of the associator, left unitor, and right unitor are all equalities, one calls the category *strict monoidal*.