Definition (Disjoint union category)

Given two categories $\mathbf{C}$ and $\mathbf{D}$, their *disjoint union* $\mathbf{C}+\mathbf{D}$ is the category specified as follows:

1. *Objects*: Objects are elements of $\mathrm{Ob}_{\mathbf{C}} + \mathrm{Ob}_{\mathbf{D}}$; that is, objects are tuples of the form $\langle X, i \rangle$, with $i = 1$ or $i = 2$, depending on whether $X \in \mathrm{Ob}_{\mathbf{C}}$ or $X \in \mathrm{Ob}_{\mathbf{D}}$.

2. *Morphisms*: Given objects $\langle X, i \rangle, \langle Y, j \rangle \in \mathrm{Ob}_{\mathbf{C}+\mathbf{D}}$,

$$\mathrm{Hom}_{\mathbf{C}+\mathbf{D}}\left(\langle X, i \rangle; \langle Y, j \rangle\right) := \begin{cases} \mathrm{Hom}_{\mathbf{C}}\left(X; Y\right) & \text{if } i = j = 1, \\ \mathrm{Hom}_{\mathbf{D}}\left(X; Y\right) & \text{if } i = j = 2, \\ \varnothing & \text{else.} \end{cases}$$

3. *Identity morphisms*: Given $f : X \to Y \in \mathrm{Hom}_{\mathbf{C}+\mathbf{D}}\left(\langle X, i \rangle; \langle Y, i \rangle\right)$, one has

$$\mathrm{Id}_{\mathbf{C}+\mathbf{D}} := \begin{cases} \mathrm{Id}_{\mathbf{C}} & \text{if } i = 1, \\ \mathrm{Id}_{\mathbf{D}} & \text{if } i = 2. \end{cases}$$

4. *Composition*: Given $f : X \to Y \in \mathrm{Hom}_{\mathbf{C}+\mathbf{D}}\left(\langle X, i \rangle; \langle Y, i \rangle\right)$ and $g : Y \to Z \in \mathrm{Hom}_{\mathbf{C}+\mathbf{D}}\left(\langle Y, j \rangle; \langle Z, j \rangle\right)$, one has

$$f \mathbin{\fatsemi}_{\mathbf{C}+\mathbf{D}} g := \begin{cases} f \mathbin{\fatsemi}_{\mathbf{C}} g & \text{if } i = j = 1, \\ f \mathbin{\fatsemi}_{\mathbf{D}} g & \text{if } i = j = 2, \\ \text{does not exist} & \text{else.} \end{cases}$$