Definition (Disjoint union category)**.** Given two categories $\mathbf{C}$ and $\mathbf{D}$, their *disjoint union* $\mathbf{C} + \mathbf{D}$ is the category specified as follows:

1. *Objects*: Objects are elements of $\mathrm{Ob}_{\mathbf{C}} + \mathrm{Ob}_{\mathbf{D}}$; that is, objects are tuples of the form $\langle X, i\rangle$, with $i = 1$ or $i = 2$, depending on whether $X \in \mathrm{Ob}_{\mathbf{C}}$ or $X \in \mathrm{Ob}_{\mathbf{D}}$.
2. *Morphisms*: Given objects $\langle X, i\rangle, \langle Y, j\rangle \in \mathrm{Ob}_{\mathbf{C}+\mathbf{D}}$,

$$\mathrm{Hom}_{\mathbf{C}+\mathbf{D}}(\langle X, i\rangle; \langle Y, j\rangle) := \begin{cases} \mathrm{Hom}_{\mathbf{C}}(X; Y) & \text{if } i = j = 1, \\ \mathrm{Hom}_{\mathbf{D}}(X; Y) & \text{if } i = j = 2, \\ \emptyset & \text{else.} \end{cases}$$

3. *Identity morphisms*: Given a morphism $f : X \to Y \in \mathrm{Hom}_{\mathbf{C}+\mathbf{D}}(\langle X, i\rangle; \langle Y, i\rangle)$, one has

$$\mathrm{Id}_{\mathbf{C}+\mathbf{D}} := \begin{cases} \mathrm{Id}_{\mathbf{C}} & \text{if } i = 1, \\ \mathrm{Id}_{\mathbf{D}} & \text{if } i = 2. \end{cases}$$

4. *Composition of morphisms*: Given morphisms $f : X \to Y \in \mathrm{Hom}_{\mathbf{C}+\mathbf{D}}(\langle X, i\rangle; \langle Y, i\rangle)$ and $g : Y \to Z \in \mathrm{Hom}_{\mathbf{C}+\mathbf{D}}(\langle Y, j\rangle; \langle Z, j\rangle)$, one has

$$f \mathbin{⨾}_{\mathbf{C}+\mathbf{D}} g := \begin{cases} f \mathbin{⨾}_{\mathbf{C}} g & \text{if } i = j = 1, \\ f \mathbin{⨾}_{\mathbf{D}} g & \text{if } i = j = 2, \\ \text{does not exist} & \text{else.} \end{cases}$$