

Lemma. The optimal control law for the LQG problem is $\mathbf{u}_t^\star = -\mathbf{K}\hat{\mathbf{x}}_t = -\mathbf{R}^{-1}\mathbf{B}^*\bar{\mathbf{S}}\hat{\mathbf{x}}_t$, where $\hat{\mathbf{x}}_t$ is the unbiased minimum-variance estimate of \mathbf{x}_t given previous measurements and $\bar{\mathbf{S}} \in \mathcal{P}^+$ solves the Riccati equation

$$\mathbf{S}\mathbf{A} + \mathbf{A}^*\mathbf{S} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{S} + \mathbf{Q} = \mathbf{0}.$$

The minimum cost J^\star achieved by the optimal control is:

$$\begin{aligned} J^\star &= \text{Tr}(\bar{\mathbf{S}}\bar{\Sigma}\mathbf{C}^*\mathbf{V}^{-1}\mathbf{C}\bar{\Sigma} + \bar{\Sigma}\mathbf{Q}) \\ &= \text{Tr}(\bar{\Sigma}\bar{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\bar{\mathbf{S}} + \bar{\Sigma}\mathbf{W}), \end{aligned}$$

where $\bar{\Sigma} \in \mathcal{P}^+$ is the solution of the Riccati equation

$$\mathbf{A}\Sigma + \Sigma\mathbf{A}^* - \Sigma\mathbf{C}^*\mathbf{V}^{-1}\mathbf{C}\Sigma + \mathbf{W} = \mathbf{0}.$$