

Definition (Category $\mathbf{Pos}_{\mathcal{L}}$). The category $\mathbf{Pos}_{\mathcal{L}}$ consists of:

1. *Objects*: objects are posets;
2. *Morphisms*: given objects $X, Y \in \mathbf{Ob}_{\mathbf{Pos}_{\mathcal{L}}}$, morphisms $f : X \rightarrow Y$ are monotone maps of the form $f^{\star} : X \rightarrow_{\mathbf{Pos}} \mathcal{L} Y$.
3. *Composition of morphisms*: Given morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$, their composition $f \circ g : X \rightarrow Z$ is given by

$$(f \circ g)^{\star} : X \rightarrow_{\mathbf{Pos}} \mathcal{L} Z$$

$$x \mapsto \bigcup_{y \in f^{\star}(x)} g^{\star}(y);$$

4. *Identity morphism*: given an object $X \in \mathbf{Ob}_{\mathbf{Pos}_{\mathcal{L}}}$, the identity morphism $\text{Id}_X : X \rightarrow X$ is given by the application of the lower closure operator: $\text{Id}_X^{\star}(x) := \downarrow \{x\}$.