

Definition (Strong monoidal functor). Let $\langle \mathbf{C}, \otimes_{\mathbf{C}}, \mathbf{1}_{\mathbf{C}} \rangle$ and $\langle \mathbf{D}, \otimes_{\mathbf{D}}, \mathbf{1}_{\mathbf{D}} \rangle$ be two monoidal categories. A *strong monoidal functor* between \mathbf{C} and \mathbf{D} is given by:

1. A functor

$$F : \mathbf{C} \rightarrow \mathbf{D};$$

2. An isomorphism

$$\text{iso} : \mathbf{1}_{\mathbf{D}} \rightarrow F(\mathbf{1}_{\mathbf{C}});$$

3. A natural isomorphism μ

$$\mu_{X,Y} : F(X) \otimes_{\mathbf{D}} F(Y) \rightarrow F(X \otimes_{\mathbf{C}} Y), \quad \forall X, Y \in \mathbf{C},$$

satisfying the following conditions:

- a) *Associativity*: For all objects $X, Y, Z \in \mathbf{C}$, there are *associators* $\text{as}^{\mathbf{C}}$ and $\text{as}^{\mathbf{D}}$ such that the following diagram commutes.

$$\begin{array}{ccc}
 (F(X) \otimes_{\mathbf{D}} F(Y)) \otimes_{\mathbf{D}} F(Z) & \xrightarrow[\text{as}^{\mathbf{D}}]{F(X), F(Y), F(Z)} & F(X) \otimes_{\mathbf{D}} (F(Y) \otimes_{\mathbf{D}} F(Z)) \\
 \downarrow \mu_{X,Y} \otimes_{\mathbf{D}} \text{Id}(F(Z)) & & \downarrow \text{Id}(F(X)) \otimes_{\mathbf{D}} \mu_{Y,Z} \\
 F(X \otimes_{\mathbf{C}} Y) \otimes_{\mathbf{D}} F(Z) & & F(X) \otimes_{\mathbf{D}} F(Y \otimes_{\mathbf{C}} Z) \\
 \downarrow \mu_{X \otimes_{\mathbf{C}} Y, Z} & & \downarrow \mu_{X, Y \otimes_{\mathbf{C}} Z} \\
 F((X \otimes_{\mathbf{C}} Y) \otimes_{\mathbf{C}} Z) & \xrightarrow[F(\text{as}^{\mathbf{C}}_{X,Y,Z})]{} & F(X \otimes_{\mathbf{C}} (Y \otimes_{\mathbf{C}} Z))
 \end{array}$$

- b) *Unitality*: For all $X \in \mathbf{C}$, there exist left and right *unitors* $\text{lu}^{\mathbf{C}}$ and $\text{ru}^{\mathbf{C}}$, the following diagrams commutes

$$\begin{array}{ccc}
 \mathbf{1}_{\mathbf{D}} \otimes_{\mathbf{D}} F(X) & \xrightarrow[\text{iso} \otimes_{\mathbf{D}} \text{Id}_{F(X)}]{} & F(\mathbf{1}_{\mathbf{C}}) \otimes_{\mathbf{D}} F(X) \\
 \downarrow \text{lu}_{\mathbf{D}} & & \downarrow \mu_{\mathbf{1}_{\mathbf{C}}, X} \\
 F(X) & \xleftarrow[F(\text{lu}^{\mathbf{C}}_X)]{} & F(\mathbf{1}_{\mathbf{C}} \otimes_{\mathbf{C}} X)
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(X) \otimes_{\mathbf{D}} \mathbf{1}_{\mathbf{D}} & \xrightarrow[\text{Id}_{F(X)} \otimes_{\mathbf{D}} \text{iso}]{} & F(X) \otimes_{\mathbf{D}} F(\mathbf{1}_{\mathbf{C}}) \\
 \downarrow \text{ru}_{\mathbf{D}} & & \downarrow \mu_{X, \mathbf{1}_{\mathbf{C}}} \\
 F(X) & \xleftarrow[F(\text{ru}^{\mathbf{C}}_X)]{} & F(X \otimes_{\mathbf{C}} \mathbf{1}_{\mathbf{C}})
 \end{array}$$