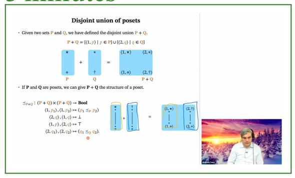
Contents

1 Poset constructions

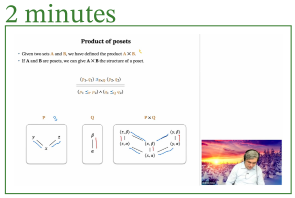
to write

1.1 Product of posets

Watch *Composing posets*.
3 minutes



Watch *Product of posets*.
2 minutes



We can think of the product of posets.

{def:productposet}

Definition 1.1 (Product of posets). Given two posets $\langle \mathbf{P}, \preceq_{\mathbf{P}} \rangle$ and $\langle \mathbf{Q}, \preceq_{\mathbf{Q}} \rangle$, the *product poset* is $\langle \mathbf{P} \times \mathbf{Q}, \preceq_{\mathbf{P}\times\mathbf{Q}} \rangle$, where $\mathbf{P} \times \mathbf{Q}$ is the Cartesian product of two sets (**??**) and the order $\preceq_{\mathbf{P}\times\mathbf{Q}}$ is given by:

$$\frac{\langle p_1, q_1 \rangle \preceq_{\mathbf{P}\times\mathbf{Q}} \langle p_2, q_2 \rangle}{(p_1 \preceq_{\mathbf{P}} p_2) \wedge (q_1 \preceq_{\mathbf{Q}} q_2)} \tag{1.1}$$

Recalling the pizza recipes example, we have the two posets representing time and money. Given that we want to minimize both time and costs, by considering the money poset containing elements 1 ฿, 2 ฿, and 3 ฿, and the time poset containing elements 1 hours, and 2 hours, one can represent the product as in Fig. 1.1.

{bhfn:2}

a

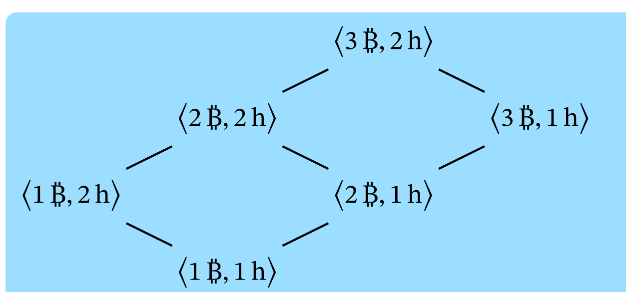


Figure 1.1: Product poset of time and cost for pizza recipes.

[a] `70_hasse_pizza_product`

{fig:productpizza}

Do we really want bullets and labels here? If yes, todo: put labels on the side

Example 1.2. Consider now two posets and their product, given in Fig. 1.2.

[a]

{bhfn:3}

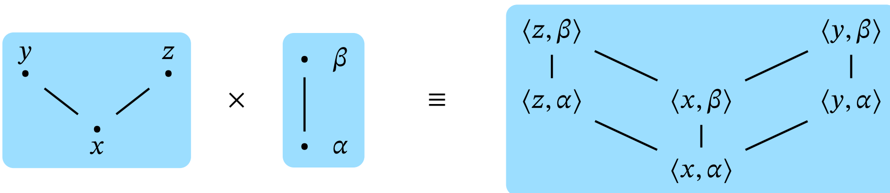
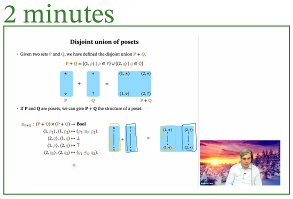


Figure 1.2: Product of two posets. fig:composing_pose

[a] `40_exposet_1_1`

1.2 Disjoint union of posets

Watch *Disjoint union of posets*.
2 minutes



Similarly to what we have done for sets in **??**, we can think of alternatives in the poset case through their disjoint union.

Definition 1.3 (Disjoint union of posets). Given posets $\langle \mathbf{P}, \preceq_{\mathbf{P}} \rangle$ and $\langle \mathbf{Q}, \preceq_{\mathbf{Q}} \rangle$, we can define their *disjoint union* $\langle \mathbf{P}+\mathbf{Q}, \preceq_{\mathbf{P}+\mathbf{Q}} \rangle$, where $\mathbf{P}+\mathbf{Q}$ is the disjoint union of the sets $\mathbf{P}$ and $\mathbf{Q}$ (**??**), and the order $\preceq_{\mathbf{P}+\mathbf{Q}}$ is given by:

$$p \preceq_{\mathbf{P}+\mathbf{Q}} q \quad \equiv \quad \begin{cases} p \preceq_{\mathbf{P}} q, & p, q \in \mathbf{P}, \\ p \preceq_{\mathbf{Q}} q, & p, q \in \mathbf{Q}, \end{cases} \tag{1.2}$$

with

$$\begin{aligned} \preceq_{\mathbf{P}+\mathbf{Q}} : (\mathbf{P}+\mathbf{Q}) \times (\mathbf{P}+\mathbf{Q}) &\to \mathbf{Bool} \\ \langle 1, p_1 \rangle, \langle 1, p_2 \rangle &\mapsto (p_1 \preceq_{\mathbf{P}} p_2) \\ \langle 2, q \rangle, \langle 1, q \rangle &\mapsto \bot \\ \langle 1, p \rangle, \langle 2, q \rangle &\mapsto \bot \\ \langle 2, q_1 \rangle, \langle 2, q_2 \rangle &\mapsto (q_1 \preceq_{\mathbf{Q}} q_2). \end{aligned} \tag{1.3}$$

Example 1.4. Consider the posets $\mathbf{P} = \langle \diamond, \star \rangle$ with $\diamond \preceq_{\mathbf{P}} \star$, and $\mathbf{Q} = \langle \dagger, * \rangle$, with $* \preceq_{\mathbf{Q}} \dagger$. Their disjoint union can be represented as in Fig. 1.3.

{bhfn:4}

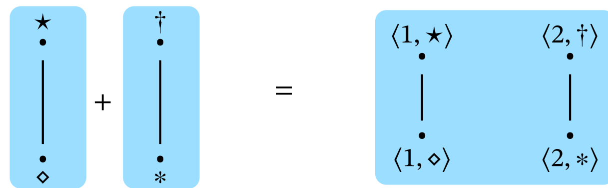


Figure 1.3: Disjoint union of posets.

[a] 40_disjoint_union

{fig:poset-copr

1.3 Opposite of a poset

{sec:{def:poset-opposite}}

Definition 1.5. The *opposite* of a poset $\langle \mathbf{P}, \preceq_{\mathbf{P}} \rangle$ is the poset denoted as $\langle \mathbf{P}^{\mathrm{op}}, {\preceq_{\mathbf{P}}}^{\mathrm{op}} \rangle$ that has the same elements as $\mathbf{P}$ and the reverse ordering (Fig. 1.4). For a given $p \in \mathbf{P}$, we use p^* to represent its corresponding copy in $\mathbf{P}^{\mathrm{op}}$; note that p and p^* belong to distinct posets. Reversing the order means that, for all $p, q \in \mathbf{P}$,

$$\frac{p \preceq_{\mathbf{P}} q}{q^* \ {\preceq_{\mathbf{P}}}^{\mathrm{op}} \ p^*} \tag{1.4}$$

{bhfn:5}

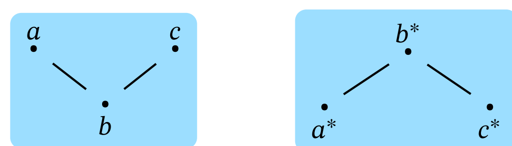


Figure 1.4: Opposite of a poset.

[a] 40_dpcatfig_opposite

{fig:poset-opposite}

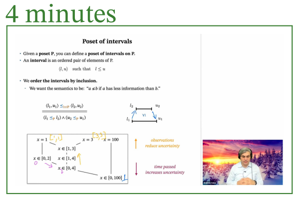
Example 1.6 (Credit and debt). Let us define the set

$$\mathbf{P} = \mathbb{R} \times \{\text{₿}\} = \{0.00, 0.01, 0.02, \dots\}$$

of all ₿ monetary quantities approximated to the cent. From this set we can define two posets: $\mathbf{P}^+ = \langle \mathbf{P}, \preceq_{\mathbf{P}} \rangle$ and $\mathbf{P}^- = \langle \mathbf{P}, \succeq_{\mathbf{P}} \rangle$, that are the opposite of each other. If the context is that, given two quantities 1 ₿ and 2 ₿, we prefer 1 ₿ to 2 ₿ (for example because it is a cost to pay to acquire a component), then we are working in $\mathbf{P}^+$, otherwise we are working in $\mathbf{P}^-$ (for example because it represents the price at which we are selling our product). Traditionally, in double-entry ledger systems, the numbers were not written with negative signs, but rather in color: red and black. From this convention we get the idioms “being in the black” and “being in the red”.

1.4 Poset of intervals

Watch *Poset of intervals.*
4 minutes



oset_intervals}

Definition 1.7 (Poset of intervals)**.** An interval is an ordered pair of elements $\langle p, q\rangle$ of $\mathbf{P}$, such that $p \preceq_{\mathbf{P}} q$. Given a poset $\mathbf{P}$, one can define a *poset of intervals* on $\mathbf{P}$. Intervals can be ordered by inclusion:

$$\frac{\langle p_1, q_1\rangle \preceq_{\mathbf{Int}(\mathbf{P})} \langle p_2, q_2\rangle}{(p_1 \preceq_{\mathbf{P}} p_2) \wedge (q_2 \preceq_{\mathbf{P}} q_1)} \tag{1.5}$$

{dvvv_1_label}

Exercise 1. Check that the relation defined in Def. 1.7 is indeed a poset.

See solution on page 5.

1.5 A different poset of intervals

{def:second_interval_poset}

Definition 1.8 (Another poset of intervals)**.** Given a partially ordered set $\mathbf{P}$, an interval is an ordered pair of elements $\langle l, u\rangle$ of $\mathbf{P}$, such that $l \preceq_{\mathbf{P}} u$. One can define a *poset of intervals* on $\mathbf{P}$, denoted $\mathbf{Int}'(P)$. Intervals can be ordered using the following rule:

$$\frac{\langle p_1, p_2\rangle \preceq_{\mathbf{Int}'(\mathbf{P})} \langle q_1, q_2\rangle}{(p_1 \preceq_{\mathbf{P}} q_1) \wedge (p_2 \preceq_{\mathbf{P}} q_2)} \tag{1.6}$$

This partially ordered set will be instrumental when we define uncertainty in design problems.

{dvvv_2_label}

Exercise 2. Check that the relation defined in Def. 1.8 is indeed a poset.

See solution on page 5.

Solutions to exercises

Solution of Exercise 1. We prove the three conditions. {dvvv_1_reverse_label}

- First, we know that $\langle p_1, q_1\rangle \preceq_{\mathbf{Int}(\mathbf{P})} \langle p_1, q_1\rangle$, since $p_1 \preceq_{\mathbf{P}} p_1$ and $q_1 \preceq_{\mathbf{P}} q_1$.
- Second, $\langle p_1, q_1\rangle \preceq_{\mathbf{Int}(\mathbf{P})} \langle p_2, q_2\rangle$ and $\langle p_2, q_2\rangle \preceq_{\mathbf{Int}(\mathbf{P})} \langle p_3, q_3\rangle$ imply $\langle p_1, q_1\rangle \preceq_{\mathbf{Int}(\mathbf{P})} \langle p_3, q_3\rangle$.
- Third, if $\langle p_1, q_1\rangle \preceq_{\mathbf{Int}(\mathbf{P})} \langle p_2, q_2\rangle$ and $\langle p_2, q_2\rangle \preceq_{\mathbf{Int}(\mathbf{P})} \langle p_1, q_1\rangle$, then $p_1 = p_2$ and $q_1 = q_2$.

Solution of Exercise 2. We check the three conditions. {dvvv_2_reverse_label}

- First, we know that $\langle p_1, q_1\rangle \preceq_{\mathbf{Int}'(\mathbf{P})} \langle p_1, q_1\rangle$, since $p_1 \preceq_{\mathbf{P}} p_1$ and $q_1 \preceq_{\mathbf{P}} q_1$.
- Second, $\langle p_1, q_1\rangle \preceq_{\mathbf{Int}'(\mathbf{P})} \langle p_2, q_2\rangle$ and $\langle p_2, q_2\rangle \preceq_{\mathbf{Int}'(\mathbf{P})} \langle p_3, q_3\rangle$ imply $\langle p_1, q_1\rangle \preceq_{\mathbf{Int}'(\mathbf{P})} \langle p_3, q_3\rangle$.
- Third, if $\langle p_1, q_1\rangle \preceq_{\mathbf{Int}'(\mathbf{P})} \langle p_2, q_2\rangle$ and $\langle p_2, q_2\rangle \preceq_{\mathbf{Int}'(\mathbf{P})} \langle p_1, q_1\rangle$, then $p_1 = p_2$ and $q_1 = q_2$.