

Definition (Operad)

An *operad* is defined by:

Constituents

1. *Objects*: A collection $\mathbf{Ob}_{\mathcal{O}}$;
2. *Morphisms*: Let $n \in \mathbb{N}$. For each finite string $[X_1, \dots, X_n]$ of objects and each object Y , one specifies a set $\mathbf{Hom}_{\mathcal{O}}([X_1, \dots, X_n]; Y)$, elements of which are morphisms $[X_1, \dots, X_n] \rightarrow Y$;
3. *Identity morphisms*: For each object X , a morphism $\mathbf{Id}_X \in \mathbf{Hom}_{\mathcal{O}}([X_1, \dots, X_n]; Y)$;
4. *Composition operations*:

$$\mathbf{Hom}_{\mathcal{O}}([X_1^1, \dots, X_{n_1}^1]; Y_1) \times \dots \times \mathbf{Hom}_{\mathcal{O}}([X_1^m, \dots, X_{n_m}^m]; Y_m) \times \mathbf{Hom}_{\mathcal{O}}([Y_1, \dots, Y_m]; Z) \rightarrow \mathbf{Hom}_{\mathcal{O}}([X_1^1, \dots, X_{n_m}^m]; Z)$$
$$\langle f_1, \dots, f_m, g \rangle \mapsto [f_1, \dots, f_m] \circ g.$$

Conditions

1. *Associativity*:

$$[[f_1^1, \dots, f_{n_1}^1] \circ g_1, [f_1^2, \dots, f_{n_2}^2] \circ g_2, \dots, [f_1^m, \dots, f_{n_m}^m] \circ g_m] \circ h = [f_1^1, \dots, f_{n_m}^m] \circ ([g_1, \dots, g_m] \circ h).$$

2. *Unitality*:

$$[\mathbf{Id}_{X_1}, \dots, \mathbf{Id}_{X_n}] \circ f = f = f \circ \mathbf{Id}_Y, \quad \forall f : [X_1, \dots, X_n] \rightarrow Y.$$