Definition (Monoidal category)

A *monoidal structure* on a category $\mathbf{C}$ is specified by:

Constituents

1. A functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, called the *monoidal product*.
2. An object $\mathbf{1} \in \mathrm{Ob}_{\mathbf{C}}$, called the *monoidal unit*.
3. A natural isomorphism, called the *associator*, whose components are of the type

$$\mathrm{as}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad X, Y, Z \in \mathrm{Ob}_{\mathbf{C}}.$$

4. A natural isomorphism, called the *left unitor*, whose components are of the type

$$\mathrm{lu}_X : \mathbf{1} \otimes X \xrightarrow{\cong} X \qquad X \in \mathrm{Ob}_{\mathbf{C}}.$$

5. A natural isomorphism, called the *right unitor*, whose components are of the type

$$\mathrm{ru}_X : X \otimes \mathbf{1} \xrightarrow{\cong} X \qquad X \in \mathrm{Ob}_{\mathbf{C}}.$$

Conditions

For all $X, Y, Z, U \in \mathrm{Ob}_{\mathbf{C}}$, the following diagrams must commute:

1. Triangle identities.

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "\mathrm{as}_{X,\mathbf{1},Y}"] \arrow[dr, "\mathrm{ru}_X \otimes \mathrm{Id}_Y"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "\mathrm{Id}_X \otimes \mathrm{lu}_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

2. Pentagon identities.

$$\begin{tikzcd}
& (X \otimes Y) \otimes (Z \otimes U) \arrow[dr, "\mathrm{as}_{X,Y,Z\otimes U}"] & \\
((X \otimes Y) \otimes Z) \otimes U \arrow[ur, "\mathrm{as}_{X\otimes Y,Z,U}"] \arrow[d, "\mathrm{as}_{X,Y,Z} \otimes \mathrm{Id}_U"] & & X \otimes (Y \otimes (Z \otimes U)) \\
(X \otimes (Y \otimes Z)) \otimes U \arrow[rr, "\mathrm{as}_{X,Y\otimes Z,U}"'] & & X \otimes ((Y \otimes Z) \otimes U) \arrow[u, "\mathrm{Id}_X \otimes \mathrm{as}_{Y,Z,U}"']
\end{tikzcd}$$

A category equipped with a monoidal structure is called a *monoidal category*. If the components of the associator, left unitor, and right unitor are all equalities, one calls the category *strict* monoidal.