

**Lemma.** Given any  $X, Y \in \mathbf{Ob}_{\mathbf{Pos}_{\mathcal{U}}}$ ,  $\mathbf{Hom}_{\mathbf{Pos}_{\mathcal{U}}}(X; Y)$  is a bounded lattice with union  $\vee$  of morphisms in  $\mathbf{Pos}_{\mathcal{U}}$  as join, intersection  $\wedge$  of morphisms in  $\mathbf{Pos}_{\mathcal{U}}$  as meet, least upper bound  $\top_{\mathbf{Hom}_{\mathbf{Pos}_{\mathcal{U}}}(X; Y)} : X \rightarrow Y$  given by

$$\begin{aligned} \top_{\mathbf{Hom}_{\mathbf{Pos}_{\mathcal{U}}}(X; Y)}^{\star} : X &\rightarrow \mathcal{U}Y \\ x &\mapsto \emptyset, \end{aligned}$$

and greatest lower bound  $\perp_{\mathbf{Hom}_{\mathbf{Pos}_{\mathcal{U}}}(X; Y)} : X \rightarrow Y$  given by

$$\begin{aligned} \perp_{\mathbf{Hom}_{\mathbf{Pos}_{\mathcal{U}}}(X; Y)}^{\star} : X &\rightarrow \mathcal{U}Y \\ x &\mapsto Y. \end{aligned}$$