

Definition (Natural transformation)

Let \mathbf{C} and \mathbf{D} be categories, and let $F, G : \mathbf{C} \rightarrow \mathbf{D}$ be functors. A *natural transformation* $\alpha : F \Rightarrow G$ is specified by:

Constituents

1. For each object $X \in \mathbf{C}$, a morphism $\alpha_X : F(X) \rightarrow G(X)$ in \mathbf{D} , called the *X -component* of α .

Conditions

1. For every morphism $f : X \rightarrow Y$ in \mathbf{C} , the components of α must satisfy the *naturality condition*

$$F(f) \circ \alpha_Y = \alpha_X \circ G(f).$$

In other words, the following diagram must commute:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

A natural transformation $\alpha : F \Rightarrow G$ is denoted visually as follows:

$$\begin{array}{ccc} & F & \\ \mathbf{C} & \xrightarrow{\quad} & \mathbf{D} \\ & G & \\ & \Downarrow \alpha & \end{array}$$