

## Definition (Natural transformation)

Let  $\mathbf{C}$  and  $\mathbf{D}$  be categories, and let  $F, G : \mathbf{C} \rightarrow \mathbf{D}$  be functors. A *natural transformation*  $\alpha : F \Rightarrow G$  is specified by:

### Constituents

1. For each object  $X \in \mathbf{C}$ , a morphism  $\alpha_X : F(X) \rightarrow G(X)$  in  $\mathbf{D}$ , called the  *$X$ -component* of  $\alpha$ .

### Conditions

1. For every morphism  $f : X \rightarrow Y$  in  $\mathbf{C}$ , the components of  $\alpha$  must satisfy the *naturality condition*

$$F(f) \circ \alpha_Y = \alpha_X \circ G(f).$$

In other words, the following diagram must commute:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

A natural transformation  $\alpha : F \Rightarrow G$  is denoted visually as follows:

$$\begin{array}{ccc} & F & \\ \mathbf{C} & \xrightarrow{\quad} & \mathbf{D} \\ & G & \\ & \Downarrow \alpha & \end{array}$$