Definition (Monoidal category)

A monoidal structure on a category **C** is specified by:

Constituents

- 1. A functor \otimes : $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, called the *monoidal product*.
- 2. An object $1 \in Ob_{\mathbb{C}}$, called the *monoidal unit*.
- 3. A natural isomorphism, called the *associator*, whose components are of the type

$$\operatorname{as}_{X,Y,Z}: (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \qquad X,Y,Z \in \operatorname{Ob}_{\mathbf{C}}.$$

4. A natural isomorphism, called the *left unitor*, whose components are of the type

$$lu_X: \mathbf{1} \otimes X \xrightarrow{\cong} X \qquad X \in Ob_{\mathbf{C}}.$$

5. A natural isomorphism, called the *right unitor*, whose components are of the type

$$\operatorname{ru}_X: X \otimes \mathbf{1} \xrightarrow{\cong} X \qquad X \in \operatorname{Ob}_{\mathbf{C}}.$$

Conditions

For all $X, Y, Z, U \in Ob_{\mathbb{C}}$, the following diagrams must commute:

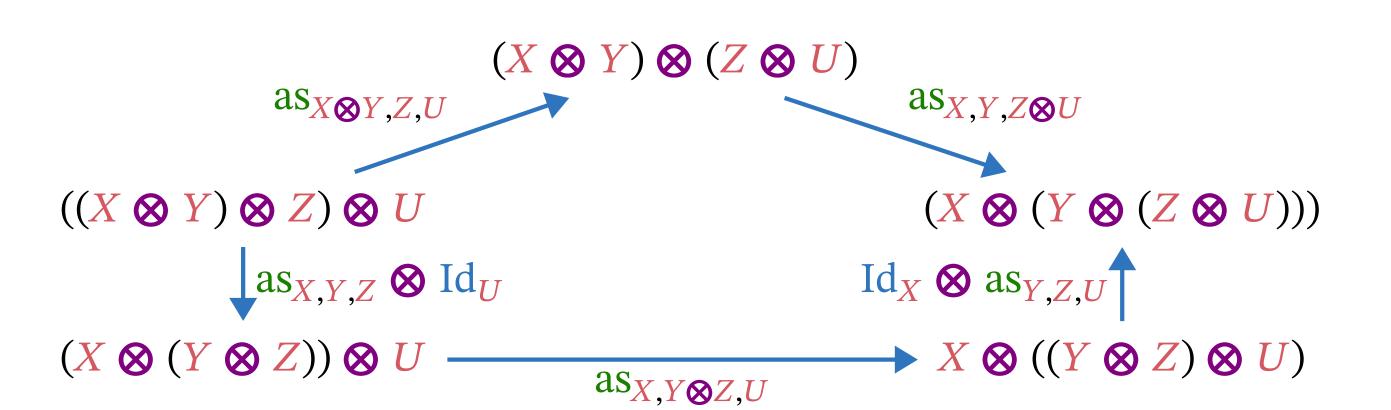
1. Triangle identities.

$$(X \otimes \mathbf{1}) \otimes Y \longrightarrow X \otimes (\mathbf{1} \otimes Y)$$

$$\operatorname{ru}_X \otimes \operatorname{Id}_Y \longrightarrow X \otimes \operatorname{lu}_Y$$

$$X \otimes Y \longrightarrow X \otimes \operatorname{Id}_Y \otimes \operatorname{Id$$

2. Pentagon identities.



A category equipped with a monoidal structure is called a *monoidal category*. If the components of the associator, left unitor, and right unitor are all equalities, one calls the category *strict* monoidal.