

# Definition (Enriched functor)

Given two categories  $\mathbf{C}$  and  $\mathbf{D}$  enriched in the same monoidal category  $\mathbf{V}$ , an enriched functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  consists of:

1. A map  $F : \mathbf{Ob}_{\mathbf{C}} \rightarrow \mathbf{Ob}_{\mathbf{D}}$  that maps objects of  $\mathbf{C}$  to objects of  $\mathbf{D}$ .
2. For each  $X, Y$  in  $\mathbf{Ob}_{\mathbf{C}}$ , there exists a morphism in  $\mathbf{V}$  given by

$$F_{X,Y} : \mathbf{Hom}_{\mathbf{C}}(X; Y) \rightarrow \mathbf{Hom}_{\mathbf{D}}(F(X); F(Y)),$$

such that composing maps “across  $F$ ” respects the composition in  $\mathbf{C}$  and the unit in  $\mathbf{V}$  in the obvious ways:

$$\begin{array}{ccc} \mathbf{Hom}_{\mathbf{C}}(X; Y) \otimes \mathbf{Hom}_{\mathbf{C}}(Y; Z) & \xrightarrow{m_{X,Y,Z}} & \mathbf{Hom}_{\mathbf{C}}(X; Z) \\ F_{X,Y} \otimes F_{Y,Z} \downarrow & & \downarrow F_{X,Z} \\ \mathbf{Hom}_{\mathbf{D}}(F(Y); F(Z)) \otimes \mathbf{Hom}_{\mathbf{D}}(F(X); F(Y)) & \xrightarrow{m_{F(X),F(Y),F(Z)}} & \mathbf{Hom}_{\mathbf{D}}(F(X); F(Z)) \end{array}$$

and

$$\begin{array}{ccc} \mathbf{1} & & \\ \downarrow j_X & \searrow j_{F(X)} & \\ \mathbf{Hom}_{\mathbf{C}}(X; X) & \xrightarrow{F_{X,X}} & \mathbf{Hom}_{\mathbf{D}}(F(X); F(X)) \end{array}$$

where  $\otimes$  and  $\mathbf{1}$  are the monoidal product and monoidal unit in  $\mathbf{V}$ .