**Definition** (Monoid isomorphism). A morphism of semigroups $f : \mathbf{M} \to \mathbf{N}$ is called a *monoid isomorphism* if there exists a homomorphism of monoids $g : \mathbf{N} \to \mathbf{M}$ such that

$$f \fatsemi g = \mathrm{id}_{\mathbf{M}} \qquad \text{and} \qquad g \fatsemi f = \mathrm{id}_{\mathbf{N}}\,.$$