

**Definition** (Strong monoidal functor). Let  $\langle \mathbf{C}, \otimes_{\mathbf{C}}, \mathbf{1}_{\mathbf{C}} \rangle$  and  $\langle \mathbf{D}, \otimes_{\mathbf{D}}, \mathbf{1}_{\mathbf{D}} \rangle$  be two monoidal categories. A *strong monoidal functor* between  $\mathbf{C}$  and  $\mathbf{D}$  is given by:

1. A functor

$$F : \mathbf{C} \rightarrow \mathbf{D};$$

2. An isomorphism

$$\text{iso} : \mathbf{1}_{\mathbf{D}} \rightarrow F(\mathbf{1}_{\mathbf{C}});$$

3. A natural isomorphism  $\mu$

$$\mu_{X,Y} : F(X) \otimes_{\mathbf{D}} F(Y) \rightarrow F(X \otimes_{\mathbf{C}} Y), \quad \forall X, Y \in \mathbf{C},$$

satisfying the following conditions:

- a) *Associativity*: For all objects  $X, Y, Z \in \mathbf{C}$ , the following diagram commutes.

$$\begin{array}{ccc}
 (F(X) \otimes_{\mathbf{D}} F(Y)) \otimes_{\mathbf{D}} F(Z) & \xrightarrow{\text{as}_{F(X),F(Y),F(Z)}^{\mathbf{D}}} & F(X) \otimes_{\mathbf{D}} (F(Y) \otimes_{\mathbf{D}} F(Z)) \\
 \downarrow & & \downarrow \\
 F(X \otimes_{\mathbf{C}} Y) \otimes_{\mathbf{D}} F(Z) & & F(X) \otimes_{\mathbf{D}} F(Y \otimes_{\mathbf{C}} Z) \\
 \downarrow & & \downarrow \\
 F((X \otimes_{\mathbf{C}} Y) \otimes_{\mathbf{C}} Z) & \xrightarrow{F(\text{as}_{X,Y,Z}^{\mathbf{C}})} & F(X \otimes_{\mathbf{C}} (Y \otimes_{\mathbf{C}} Z))
 \end{array}$$

where  $\text{as}^{\mathbf{C}}$  and  $\text{as}^{\mathbf{D}}$  are called *associators*.

- b) *Unitality*: For all  $X \in \mathbf{C}$ , the following diagrams commute:

$$\begin{array}{ccc}
 \mathbf{1}_{\mathbf{D}} \otimes_{\mathbf{D}} F(X) & \xrightarrow{\text{iso} \otimes_{\mathbf{D}} \text{Id}_{F(X)}} & F(\mathbf{1}_{\mathbf{C}}) \otimes_{\mathbf{D}} F(X) \\
 \downarrow & & \downarrow \\
 F(X) & \xleftarrow{F(\text{lu}_X^{\mathbf{C}})} & F(\mathbf{1}_{\mathbf{C}} \otimes_{\mathbf{C}} X) \\
 \\ 
 F(X) \otimes_{\mathbf{D}} \mathbf{1}_{\mathbf{D}} & \xrightarrow{\text{Id}_{F(X)} \otimes_{\mathbf{D}} \text{iso}} & F(X) \otimes_{\mathbf{D}} F(\mathbf{1}_{\mathbf{C}}) \\
 \downarrow & & \downarrow \\
 F(X) & \xleftarrow{F(\text{ru}_X^{\mathbf{C}})} & F(X \otimes_{\mathbf{C}} \mathbf{1}_{\mathbf{C}})
 \end{array}$$

where  $\text{lu}^{\mathbf{C}}$  and  $\text{ru}^{\mathbf{C}}$  represent the left and right *unitors*.