

Connection

Applied Compositional Thinking for Engineers



Example: "distribution networks"

General situation: something is distributed via a network

Specific example: electric power distributed via a power grid.

- ▶ Toy model:



To model **connectivity**: arrows

Direction of arrows: flow of distribution



Which consumers are connected to which power plants? Look at **paths**:



We also might want to show which high voltage are connected to each other:

Note: there is also a way to make these relationships symmetric.



The information above can also be represented as directed graph:



For comparison, a representation of a power grid taken from Wikipedia:



Binary relations

Definition: A (binary) **relation** from a set X to a set Y is a subset of $X \times Y$

Example: $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$

$R \subseteq X \times Y$ given by

$$R = \{\langle x_1, y_1 \rangle, \langle x_2, y_3 \rangle, \langle x_2, y_4 \rangle\}$$



Notation: If $R \subseteq X \times Y$ is a relation, we write $R : X \rightarrow Y$ or $X \xrightarrow{R} Y$.

Sometimes the notation $R : X \rightrightarrows Y$ is used to emphasize that R is a relation, and not a function.

We will see: we can think of a relation as a type of **morphism**.



Example: In the power grid example, we had

This represents a relation

$$X = \{\text{plant1, plant2, plant3}\} \longrightarrow Y = \{\text{HVN1, HVN2, HVN3, HVN4, HVN5}\}$$



Relations can be composed

Suppose we have relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, i.e. relations

$$X \xrightarrow{R} Y \xrightarrow{S} Z.$$

How might we compose R and S to obtain a relation $X \xrightarrow{R \circ S} Z$?



Example:

What is the composition $R \circ S$?

Look at **paths** from X to Z .



So, $R \circ S$ is this relation:



Definition: Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be relations. Their **composition** is

$$R \circ S := \{ \langle x, z \rangle \in X \times Z \mid \exists y \in Y : \langle x, y \rangle \in R \wedge \langle y, z \rangle \in S \}$$

which is a relation $X \rightarrow Z$.

Definition: The category **Rel** of sets and relations:

- ▶ Objects:
- ▶ Homsets: given sets X and Y ,

$$\text{Hom}_{\mathbf{Rel}}(X, Y) := \mathcal{P}(X \times Y) = \text{all subsets of } X \times Y$$

- ▶ Identity morphisms: given a set X , the identity relation id_X is

$$\text{id}_X := \{ \langle x, x' \rangle \in X \times X \mid x = x' \}.$$

- ▶ Composition: as above.



Example:





(reflexive relations)

Example: In the power grid example, we also had

This represents a relation

$$Y = \{\text{HVN1}, \text{HVN2}, \text{HVN3}, \text{HVN4}, \text{HVN5}\} \longrightarrow Y = \{\text{HVN1}, \text{HVN2}, \text{HVN3}, \text{HVN4}, \text{HVN5}\}$$



...

