

Definition (Braided monoidal category). A *braided monoidal category* is a monoidal category $\langle \mathbf{C}, \otimes, \mathbf{1} \rangle$, cf. ??, equipped with a *braiding*, which is specified by

Constituents

1. A natural isomorphism br , called the braiding, whose components are of the type

$$\text{br}_{X,Y} : (X \otimes Y) \xrightarrow{\cong} (Y \otimes X), \quad X, Y \in \text{Ob}_{\mathbf{C}}.$$

Explicitly, this means that for any morphisms $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$, the following diagram

$$\begin{array}{ccc} X_1 \otimes X_2 & \xrightarrow{f_1 \otimes f_2} & Y_1 \otimes Y_2 \\ \text{br}_{X_1, X_2} \downarrow & & \downarrow \text{br}_{Y_1, Y_2} \\ X_2 \otimes X_1 & \xrightarrow{f_2 \otimes f_1} & Y_2 \otimes Y_1 \end{array}$$

commutes.

Conditions

1. *Hexagon identities*: Given any objects $X, Y, Z \in \text{Ob}_{\mathbf{C}}$, the following diagrams must commute.

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{\text{br}_{X,Y} \otimes \text{Id}_Z} & (Y \otimes X) \otimes Z & \xrightarrow{\text{as}_{Y,X,Z}} & Y \otimes (X \otimes Z) \\ \text{as}_{X,Y,Z} \downarrow & & & & \downarrow \text{Id}_Y \otimes \text{br}_{X,Z} \\ X \otimes (Y \otimes Z) & \xrightarrow{\text{br}_{X,Y \otimes Z}} & (Y \otimes Z) \otimes X & \xrightarrow{\text{as}_{Y,Z,X}} & Y \otimes (Z \otimes X) \\ & & & & \\ X \otimes (Y \otimes Z) & \xrightarrow{\text{Id}_X \otimes \text{br}_{Y,Z}} & X \otimes (Z \otimes Y) & \xrightarrow{\text{as}_{Y,X,Z}^{-1}} & (X \otimes Z) \otimes Y \\ \text{as}_{X,Y,Z}^{-1} \downarrow & & & & \downarrow \text{br}_{X,Z} \otimes \text{Id}_Y \\ (X \otimes Y) \otimes Z & \xrightarrow{\text{br}_{X \otimes Y, Z}} & Z \otimes (X \otimes Y) & \xrightarrow{\text{as}_{Y,Z,X}^{-1}} & (Z \otimes X) \otimes Y \end{array}$$