

**Definition** (Monoidal category). A *monoidal structure* on a category  $\mathbf{C}$  is specified by:

### Constituents

1. A functor  $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ , called the *monoidal product*.
2. An object  $\mathbf{1} \in \mathbf{Ob}_{\mathbf{C}}$ , called the *monoidal unit*.
3. A natural isomorphism, called the *associator*, whose components are of the type

$$\text{as}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z) \quad X, Y, Z \in \mathbf{Ob}_{\mathbf{C}}.$$

4. A natural isomorphism, called the *left unitor*, whose components are of the type

$$\text{lu}_X : \mathbf{1} \otimes X \xrightarrow{\cong} X \quad X \in \mathbf{Ob}_{\mathbf{C}}.$$

5. A natural isomorphism, called the *right unitor*, whose components are of the type

$$\text{ru}_X : X \otimes \mathbf{1} \xrightarrow{\cong} X \quad X \in \mathbf{Ob}_{\mathbf{C}}.$$

### Conditions

For all  $X, Y, Z, W \in \mathbf{Ob}_{\mathbf{C}}$ , the following diagrams must commute:

1. Triangle identities.

$$\begin{array}{ccc}
 (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{\text{as}_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\
 \searrow \text{ru}_X \otimes \mathbf{1} & & \swarrow \mathbf{1} \otimes \text{lu}_Y \\
 & X \otimes Y &
 \end{array}$$

2. Pentagon identities.

$$\begin{array}{ccc}
 & (X \otimes Y) \otimes (Z \otimes W) & \\
 \text{as}_{X \otimes Y, Z, W} \nearrow & & \searrow \text{as}_{X, Y, Z \otimes W} \\
 ((X \otimes Y) \otimes Z) \otimes W & & (X \otimes (Y \otimes (Z \otimes W))) \\
 \downarrow \text{as}_{X, Y, Z} \otimes \text{Id}_W & & \uparrow \text{Id}_X \otimes \text{as}_{Y, Z, W} \\
 (X \otimes (Y \otimes Z)) \otimes W & \xrightarrow{\text{as}_{X, Y \otimes Z, W}} & X \otimes ((Y \otimes Z) \otimes W)
 \end{array}$$

A category equipped with a monoidal structure is called a *monoidal category*. If the components of the associator, left unitor, and right unitor are all equalities, one calls the category *strict monoidal*.