Connection

Applied Compositional Thinking for Engineers

Example: "distribution networks"

General situation: something is distributed via a network

Specific example: electric power distributed via a power grid.

- Toy model:

To model **connectivity**: arrows
Direction of arrows: flow of distribution

Which consumers are connected to which power plants? Look at **paths**:

We also might want to show which high voltage are connected to each other:

Note: there is also a way to make these relationships symmetric.

The information above can also be represented as directed graph:

For comparison, a representation of a power grid taken from Wikipedia:

Binary relations

Definition: A (binary) **relation** from a set X to a set Y is a subset of $X \times Y$

Example: $X = \{x_1, x_2, x_3\}, \quad Y = \{y_1, y_2, y_3, y_4\}$

$R \subseteq X \times Y$ given by

$$R = \{\langle x_1, y_1 \rangle, \langle x_2, y_3 \rangle, \langle x_2, y_4 \rangle\}$$

Notation: If $R \subseteq X \times Y$ is a relation, we write $R : X \to Y$ or $X \xrightarrow{R} Y$.

Sometimes the notation $R : X \nrightarrow Y$ is used to emphasize that R is a relation, and not a function.

We will see: we can think of a relation as a type of **morphism**.

Example: In the power grid example, we had

This represents a relation

$$X = \{\text{plant1}, \text{plant2}, \text{plant3}\} \quad \longrightarrow \quad Y = \{\text{HVN1}, \text{HVN2}, \text{HVN3}, \text{HVN4}, \text{HVN5}\}$$

Relations can be composed

Suppose we have relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, i.e. relations

$$X \xrightarrow{R} Y \xrightarrow{S} Z.$$

How might we compose R and S to obtain a relation $X \xrightarrow{R ⨾ S} Z$?

Example:

What is the composition $R \fatsemi S$?

Look at **paths** from X to Z.

So, $R \fatsemi S$ is this relation:

Definition: Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be relations. Their **composition** is

$$R \fatsemi S := \{\}$$

which is a relation $X \to Z$.

Definition: The category **Rel** of sets and relations:

- Objects:
- Homsets: given sets X and Y,

$$\hom_{\mathbf{Rel}}(X, Y) := \mathcal{P}(X \times Y) = \text{ all subsets of } X \times Y$$

Example:

(reflexive relations)
Example: In the power grid example, we also had

This represents a relation

$$Y = \{\mathrm{HVN1}, \mathrm{HVN2}, \mathrm{HVN3}, \mathrm{HVN4}, \mathrm{HVN5}\} \quad \longrightarrow \quad Y = \{\mathrm{HVN1}, \mathrm{HVN2}, \mathrm{HVN3}, \mathrm{HVN4}, \mathrm{HVN5}\}$$

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