Definition (Monad). Let $\mathbf{C}$ be a category. A *monad* on $\mathbf{C}$ is specified by: <u>Constituents</u>

1. A functor $M : \mathbf{C} \to \mathbf{C}$;
2. A natural transformation $\mathrm{mu} : M \mathbin{\fatsemi} M \Rightarrow M$, called the *composition* or *multiplication*;
3. A natural transformation $\mathrm{un} : \mathrm{Id}_{\mathbf{C}} \Rightarrow M$, called the *unit*.
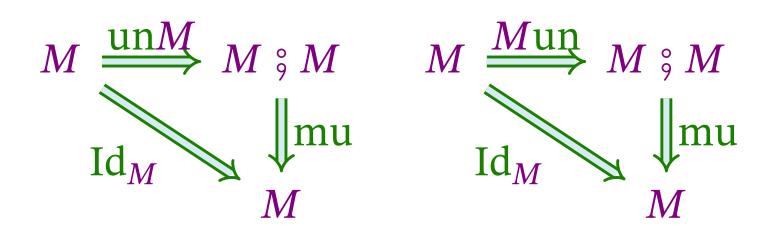
<u>Conditions</u>

1. *Associativity*: the diagram

$$
\begin{array}{ccc}
M \mathbin{\fatsemi} M \mathbin{\fatsemi} M & \xRightarrow{\;M\mathrm{mu}\;} & M \mathbin{\fatsemi} M \\
\Big\Downarrow{\mathrm{mu}M} & & \Big\Downarrow{\mathrm{mu}} \\
M \mathbin{\fatsemi} M & \xRightarrow{\;\mathrm{mu}\;} & M
\end{array}
$$

   must commute.

2. *Left and right unitality*: the diagrams

$$
\begin{array}{ccc}
M & \xRightarrow{\;\mathrm{un}M\;} & M \mathbin{\fatsemi} M \\
& {}_{\mathrm{Id}_M}\!\searrow & \Big\Downarrow{\mathrm{mu}} \\
& & M
\end{array}
\qquad
\begin{array}{ccc}
M & \xRightarrow{\;M\mathrm{un}\;} & M \mathbin{\fatsemi} M \\
& {}_{\mathrm{Id}_M}\!\searrow & \Big\Downarrow{\mathrm{mu}} \\
& & M
\end{array}
$$

   must commute.