Applied Compositional Thinking for Engineers

Session 6

Trade-offs

Today’s plan

- **Contents**:
  - Review of notation
  - Trade-offs
  - Definition of pre-order, partial order, total order.
  - Pre-orders as categories.
  - Constructing partial orders

- **Style**:
  - I choose to use notations that is *suggestive* of more advanced topics, but we will stay at very low level, and never actually develop the more powerful formalisms.
  - Let’s not intimidate people!

Logic notation

- **Logic constants:**

true $\top$ *"top"*

false $\bot$ *"bottom"*

- **Operations:**

$a \wedge b$ *"a and b"*

$a \vee b$ *"a or b"*

- **Implication**

$a \Rightarrow b$

"a implies b"

If you give me a proof of *a*,
I can give you a proof of *b*.

If you give me a refutation of *b*,
I can you give you a refutation of *a*.

$a \Leftrightarrow b$

$a \Rightarrow b$

$b \Rightarrow a$

$a \Rightarrow a$

$$(a \Rightarrow b) \wedge (b \Rightarrow c)$$
$$\Downarrow$$
$$(a \Rightarrow c)$$

Mathematical Occultism

- There is always an hidden meaning behind symbols; **the wise reads a deeper story.**

Tarots De Marseille

Mathematical Occultism

- There is always an hidden meaning behind symbols; **the wise reads a deeper story.**

- Resemblance of symbols is not a coincidence:

$$a \wedge b \qquad A \cap B$$

$$a \vee b \qquad A \cup B$$

$$A \subseteq B \qquad A \leq B$$

$$f : X \to Y \qquad \textit{functions}$$

$$f : X \to Y \qquad \textit{morphisms}$$

$$f : (X \to Y) \qquad \textit{types}$$

Sequents notation

- This is a compact way to describe implications.

$$\dfrac{\dfrac{a}{b}}{c} \qquad \begin{array}{l} \quad a \\ \Rightarrow b \\ \Rightarrow c \end{array}$$

$$\begin{array}{c} a \\ \hline\hline b \\ \hline\hline c \end{array} \qquad a \Leftrightarrow b \Leftrightarrow c$$

$$\dfrac{a \qquad b}{c} \qquad (a \wedge b) \Rightarrow c$$

$$\dfrac{f : \dfrac{a}{b} \qquad g : \dfrac{b}{c}}{f ; g : \dfrac{a}{c}}$$

$$\dfrac{\top}{a}$$ *"a is true"*

$$\dfrac{a}{\bot}$$ *"if a is true the world explodes"*

Zoo of relation properties

- Properties of relations

reflexive

$$\frac{\top}{aRa}$$

total

$$\frac{\top}{aRb \quad \vee \quad bRa}$$

transitive

$$\frac{aRb \quad bRc}{aRc}$$

symmetric

$$\frac{aRb}{bRa}$$

irreflexive

$$\frac{aRa}{\bot}$$

asymmetric

$$\frac{aRb \quad bRa}{\bot}$$

antisymmetric

$$\frac{aRb \quad bRa}{a = b}$$

Equivalence relations

- **Equivalence relation:** a **reflexive**, **transitive**, **symmetric** relation**.**
  - Recall: Equivalence relations define partitions of a set.

reflexive	*transitive*	*symmetric*
$\dfrac{\top}{aRa}$	$\dfrac{aRb \quad bRc}{aRc}$	$\dfrac{aRb}{bRa}$ (double line)

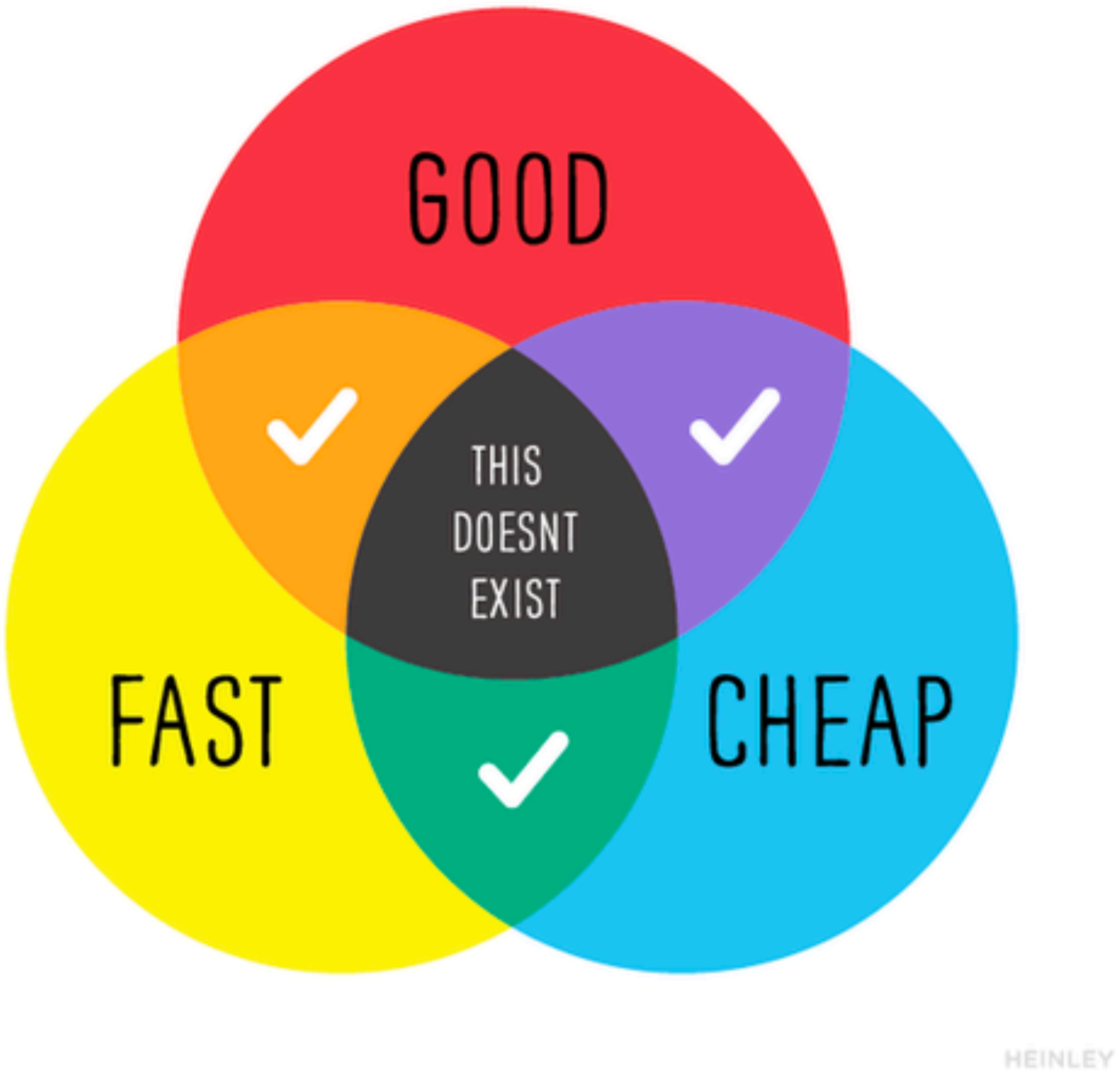
- Examples:
  - "Having the same blood type" is an equivalence relation.
  - For natural numbers: "differing by a multiple of 3" is an equivalence relation.

$$xRy \qquad \Leftrightarrow \qquad (x - y) \mod 3 = 0$$

  - $=$ is an equivalence relation.

Trade-offs

- Every engineer has an intuitive understanding of **trade-offs**.



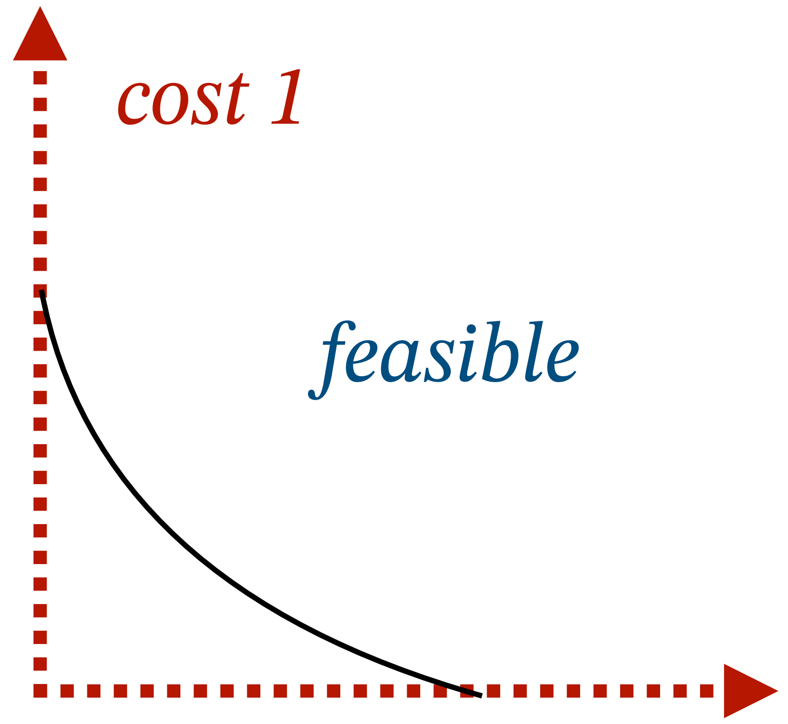
- **Partial orders** are the mathematical structure used to describe them.
- Partial orders will be the objects in the category of design problems.

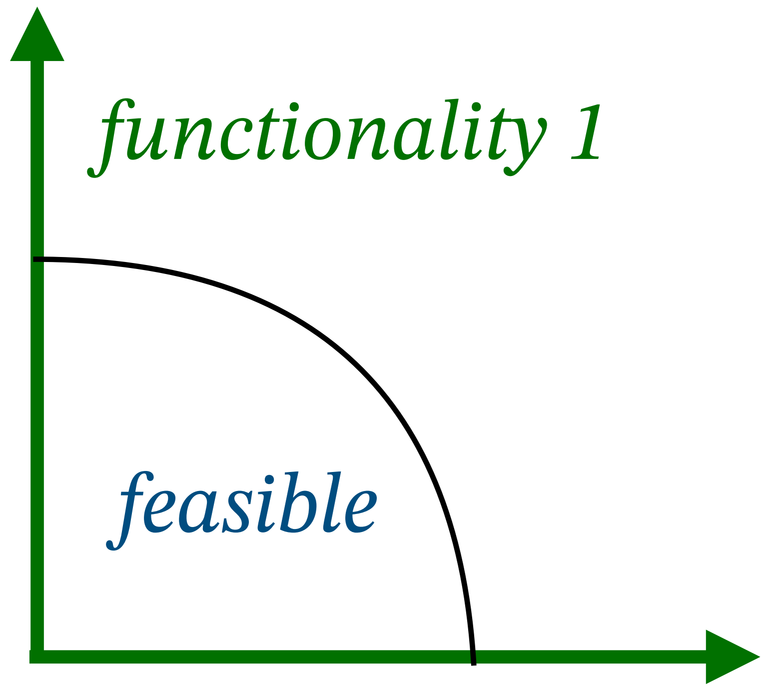
Trade-offs

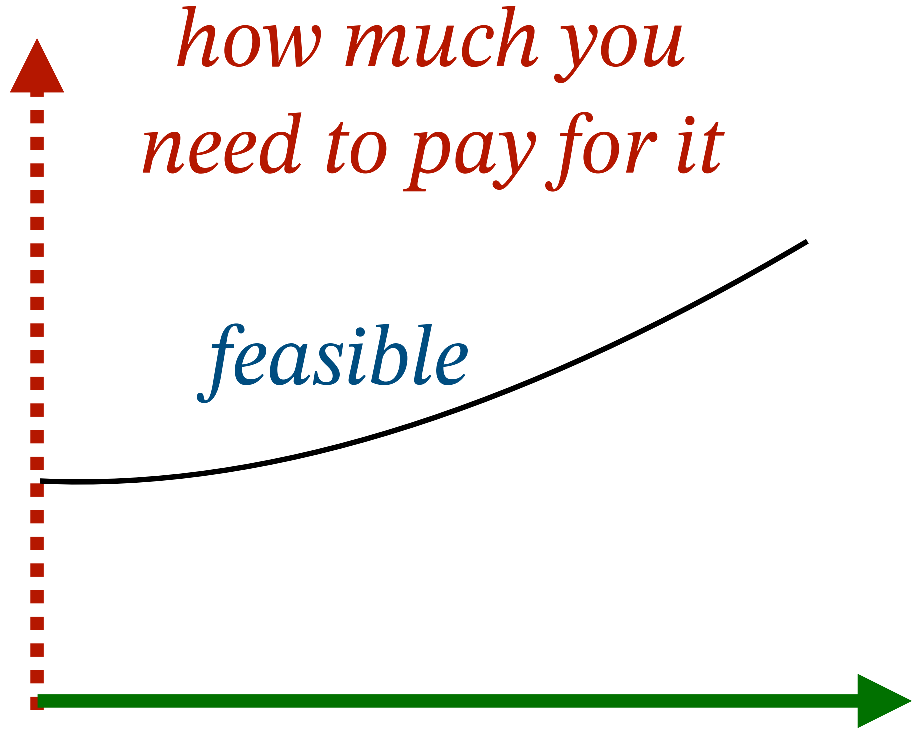
- We distinguish (semantically) between **functionality** and **requirements/costs.**

you prefer these to be small ***you prefer these to be large***

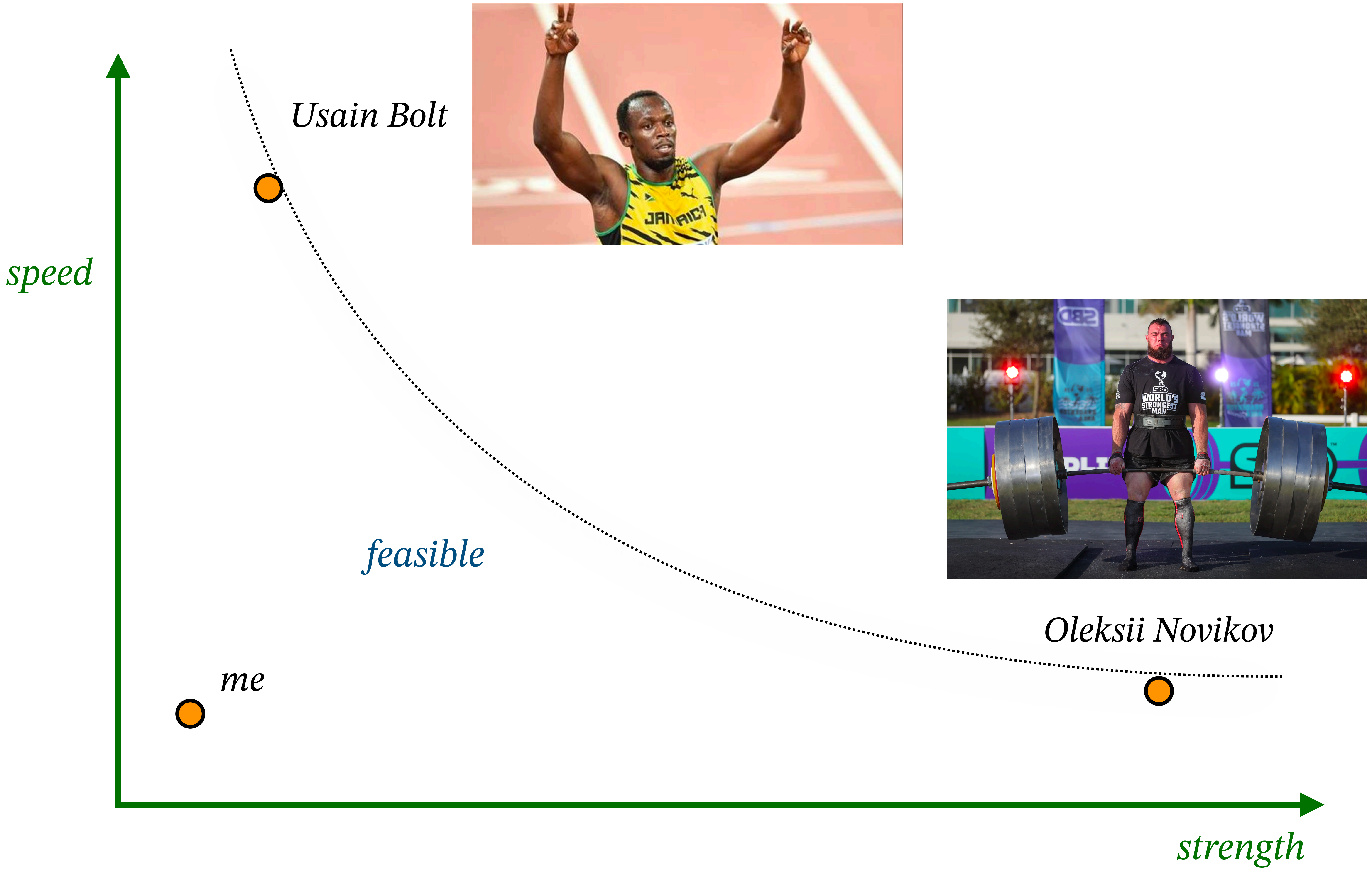
- Exercise: Open an engineering book. Find the graphs talking about “achievable performance”. What colors should the axes be? Classify into one of these:



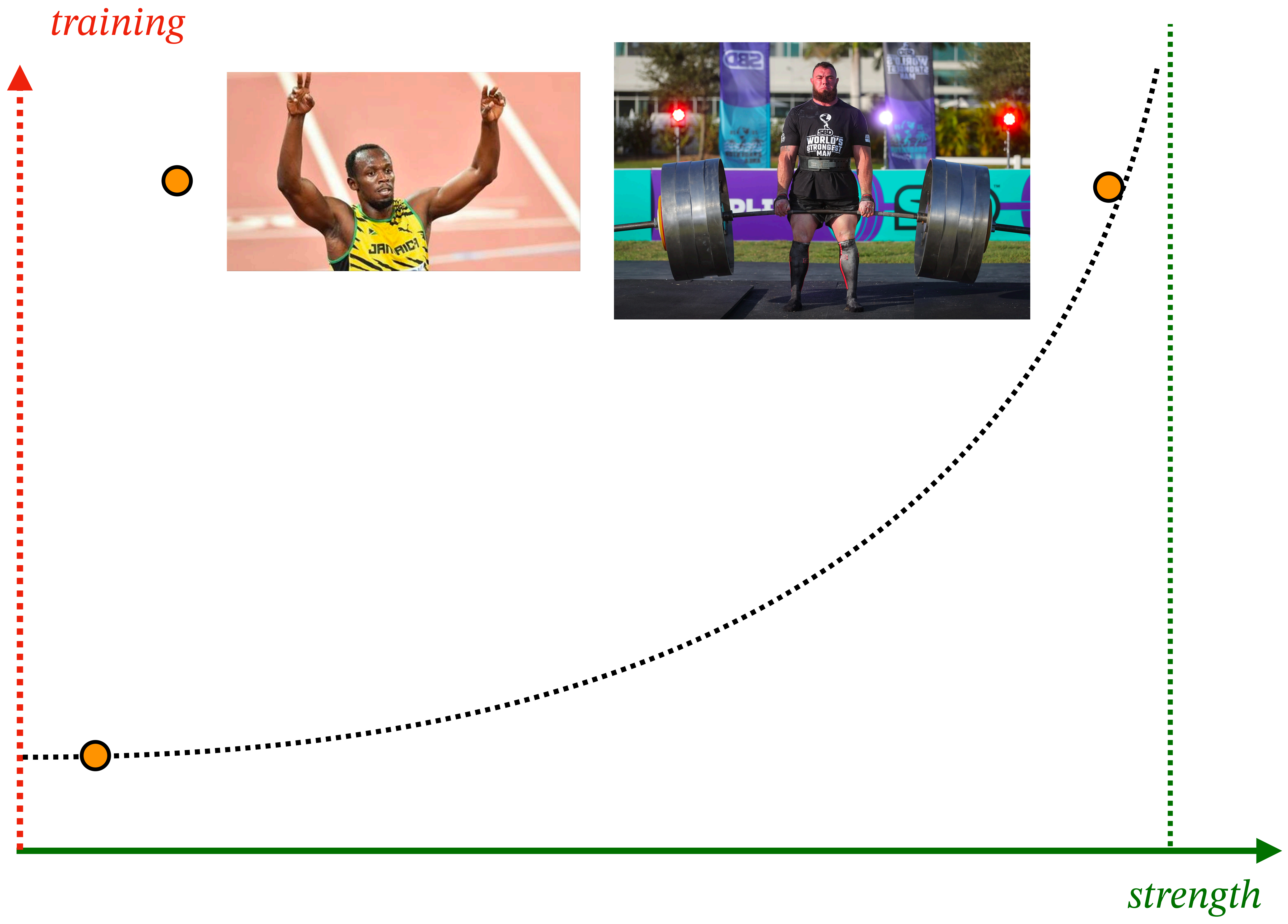




Trade-offs for the human body



Trade-offs for the human body



Ordering by functionality

protects the wearer from COVID-19

FPP3

Class[5]	Filter penetration limit (at 95 L/min air flow)	Inward leakage
FFP1	Filters at least 80% of airborne particles	<22%
FFP2	Filters at least 94% of airborne particles	<8%
FFP3	Filters at least 99% of airborne particles	<2%

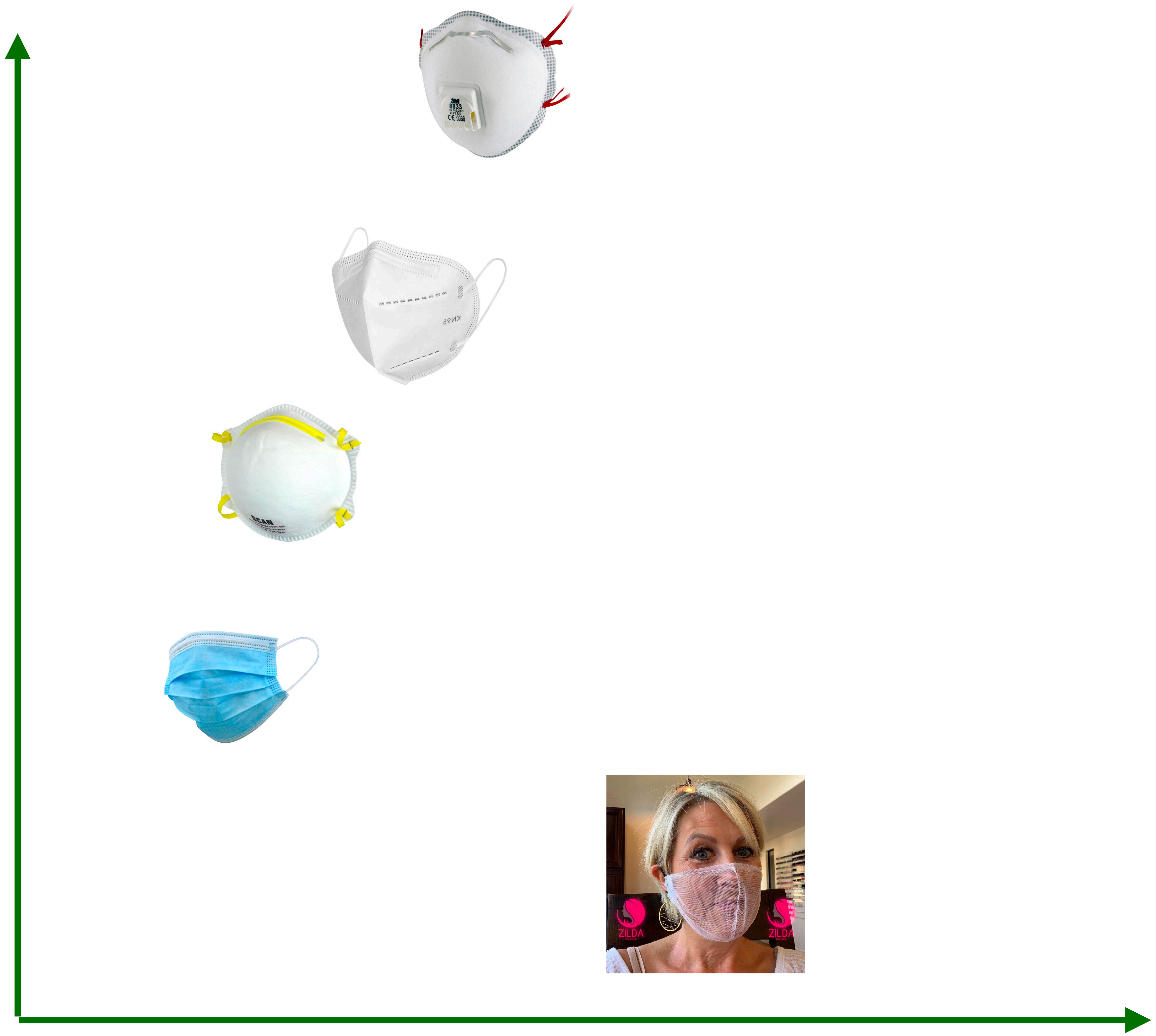
FPP2

FPP1

Surgical mask

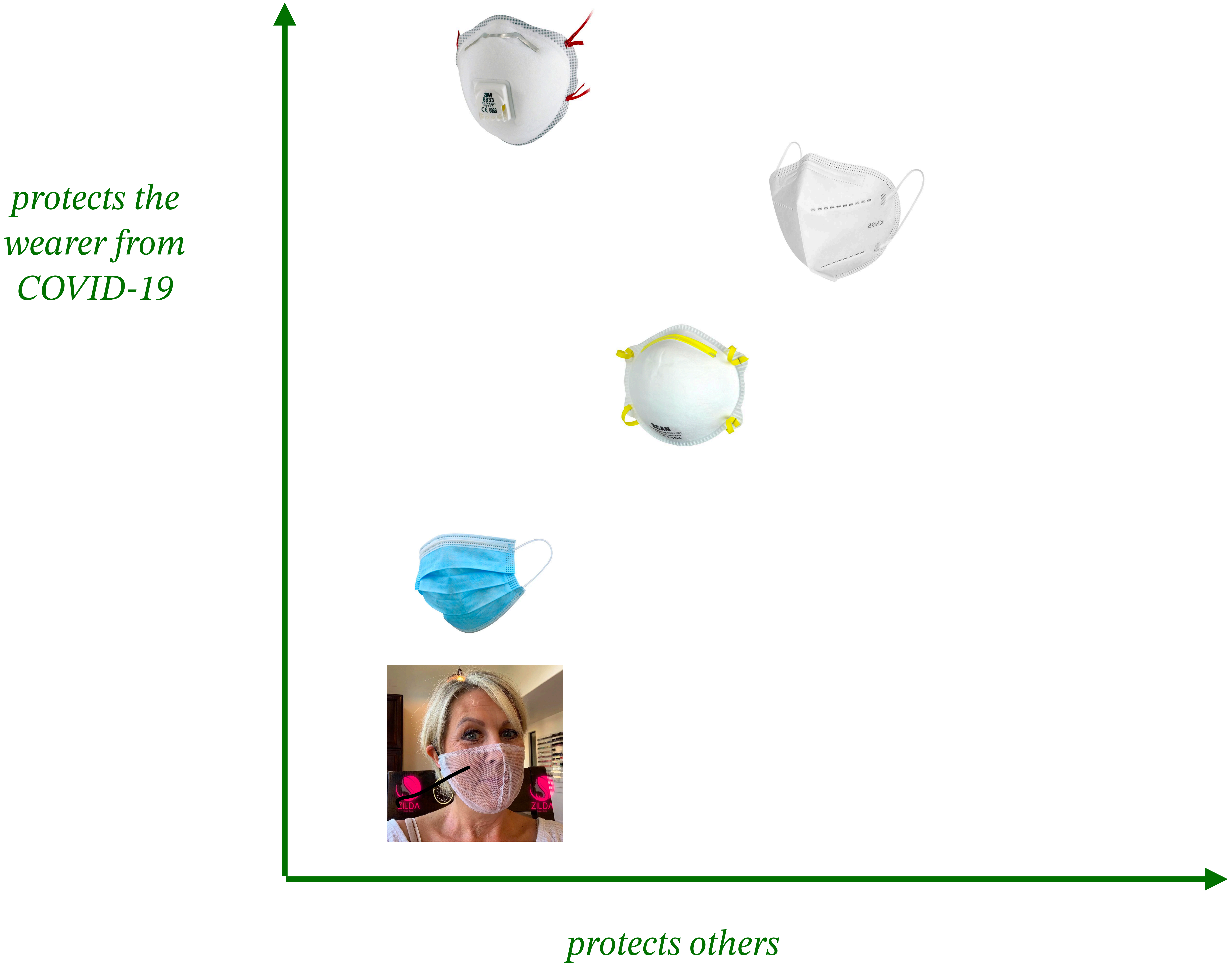
Ordering by functionality

protects the wearer from COVID-19

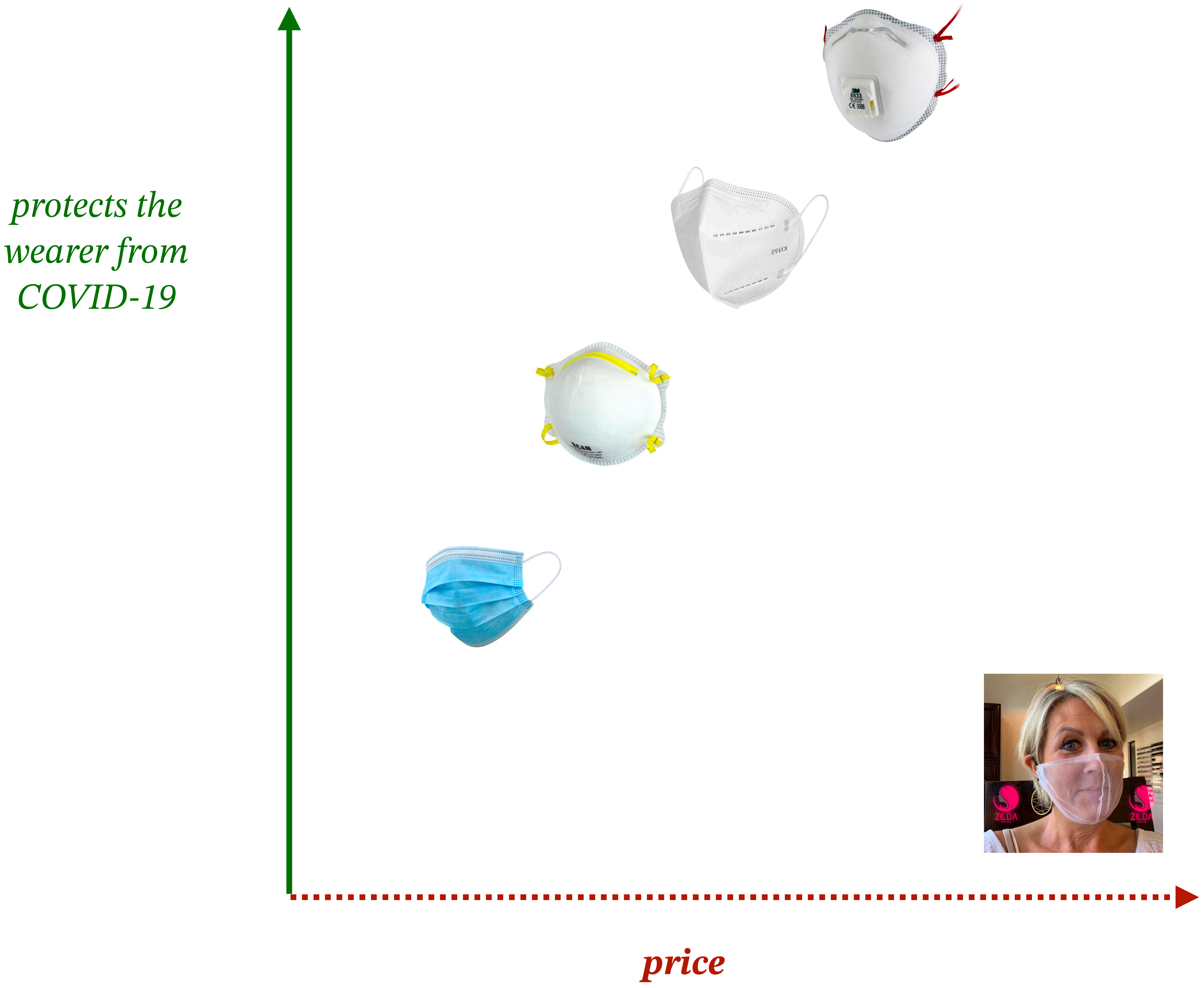


says something about the wearer

Ordering by functionality



Ordering by functionality + cost



Multiple functionalities and costs

Multiple functionalities and costs

Multiple functionalities and costs

Multiple functionalities and costs

Multiple functionalities and costs

- If we consider all characteristics:

(keeps warm × music fidelity) × *(price × frequency of charging × wires hassle)*

- No product dominates another:

- **Law of successful products**: at equilibrium, in an efficient and free market, no product completely dominates another by both functionality and costs.
(Otherwise, the dominated product wouldn't sell.)

- Only when we **specify the design purpose** and constraints we can (partially) order things.

Definition of pre-order, partial order, total order

- **Pre-order:** a relation that is **transitive** and **reflexive**.
- **Partial order**: a relation that is **transitive**, **reflexive**, and **antisymmetric**.
- **Total order**: a relation that is **transitive**, **reflexive**, **antisymmetric** and **total**.

more specific

total order

partial order

pre-order

reflexive

$$\frac{\top}{aRa}$$

transitive

$$\frac{aRb \quad bRc}{aRc}$$

antisymmetric

$$\frac{aRb \quad bRa}{a = b}$$

total

$$\frac{\top}{aRb \quad \vee \quad bRa}$$

- Customary symbols for a pre-order relation:

$$\leq \qquad\qquad \preceq$$

Drawing ordered sets as graphs

- **Pre-order:** a relation that is **transitive** and **reflexive**.
- **Partial order**: a relation that is **transitive**, **reflexive**, and **antisymmetric**.
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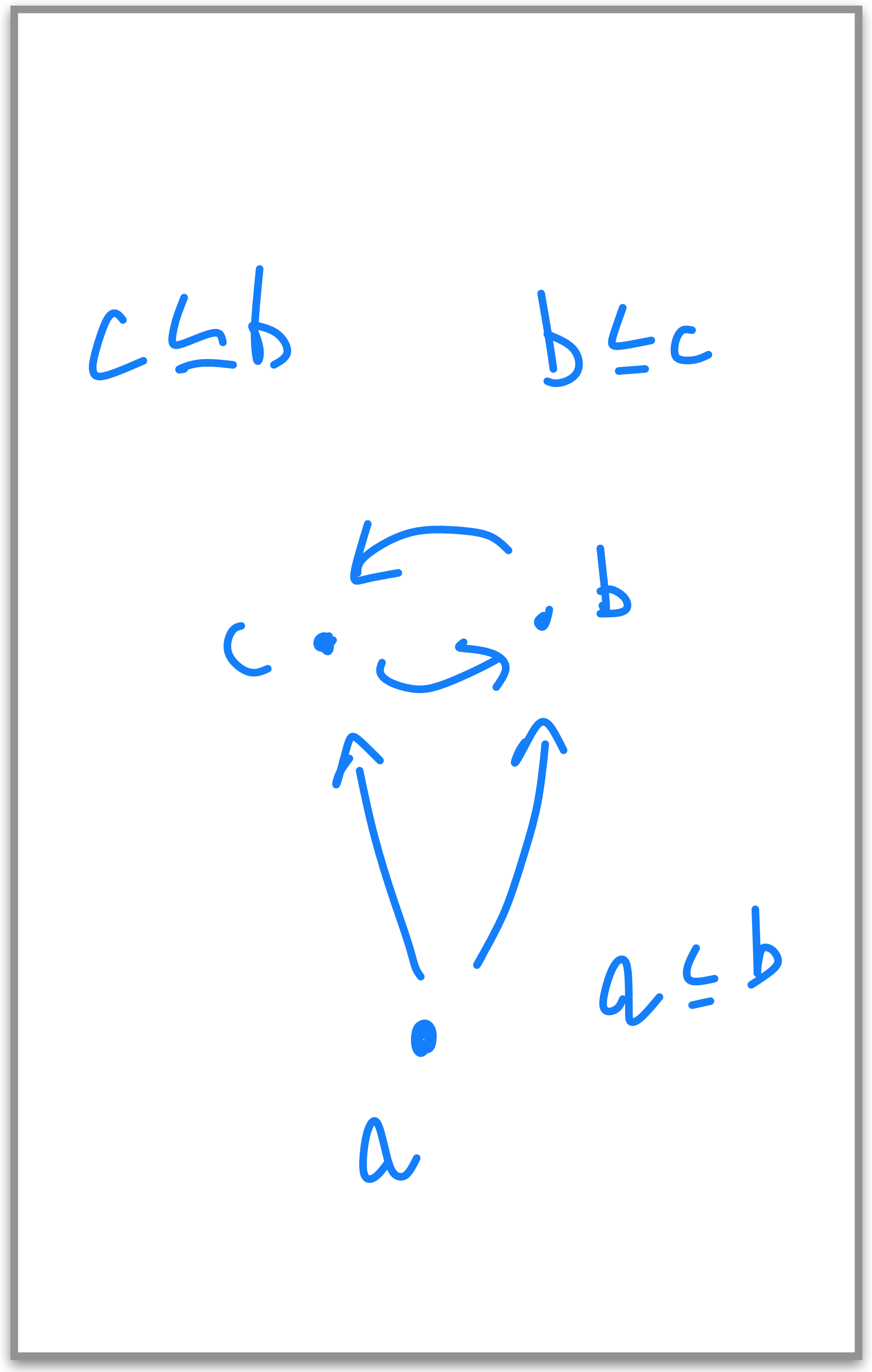
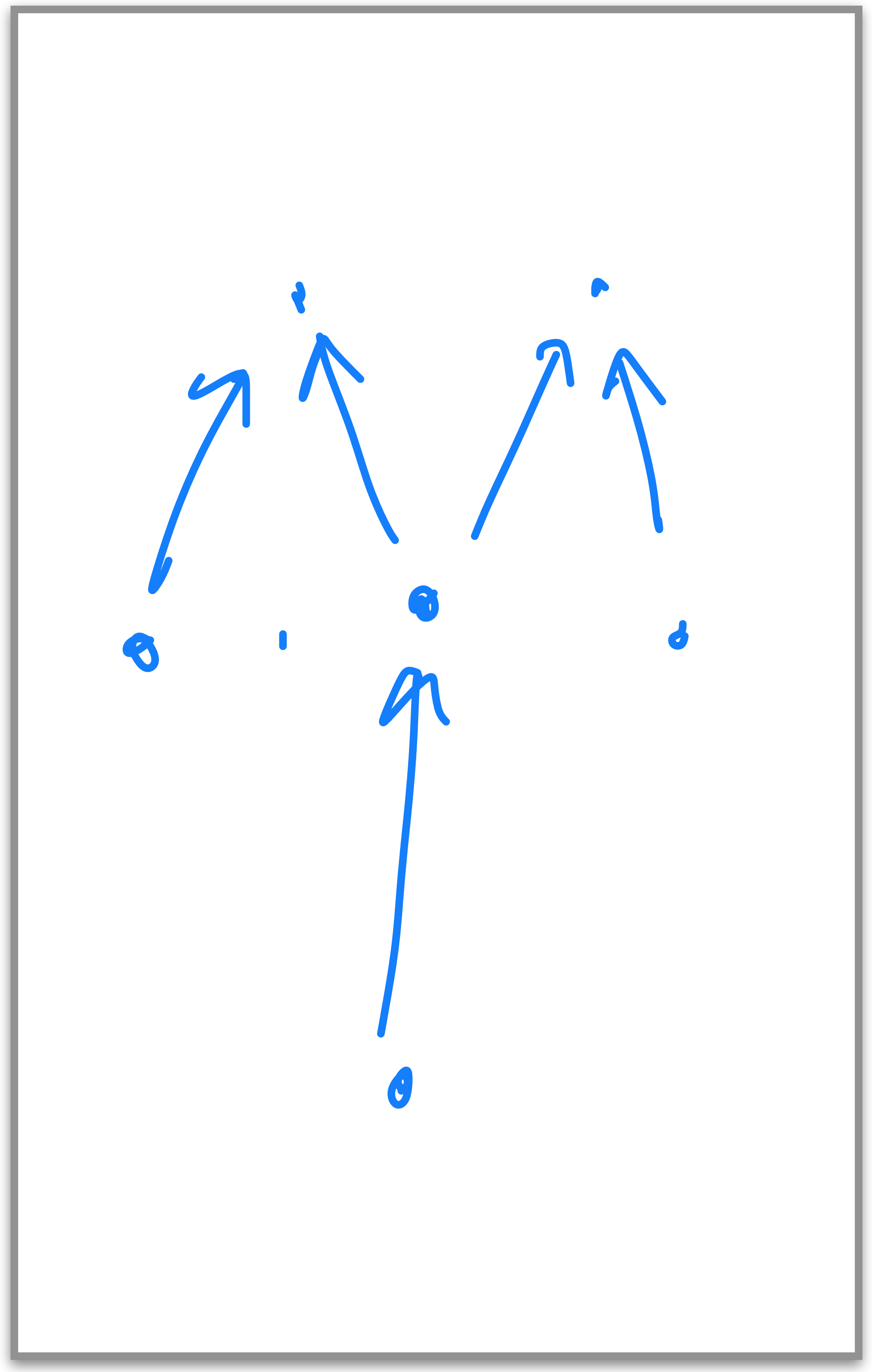
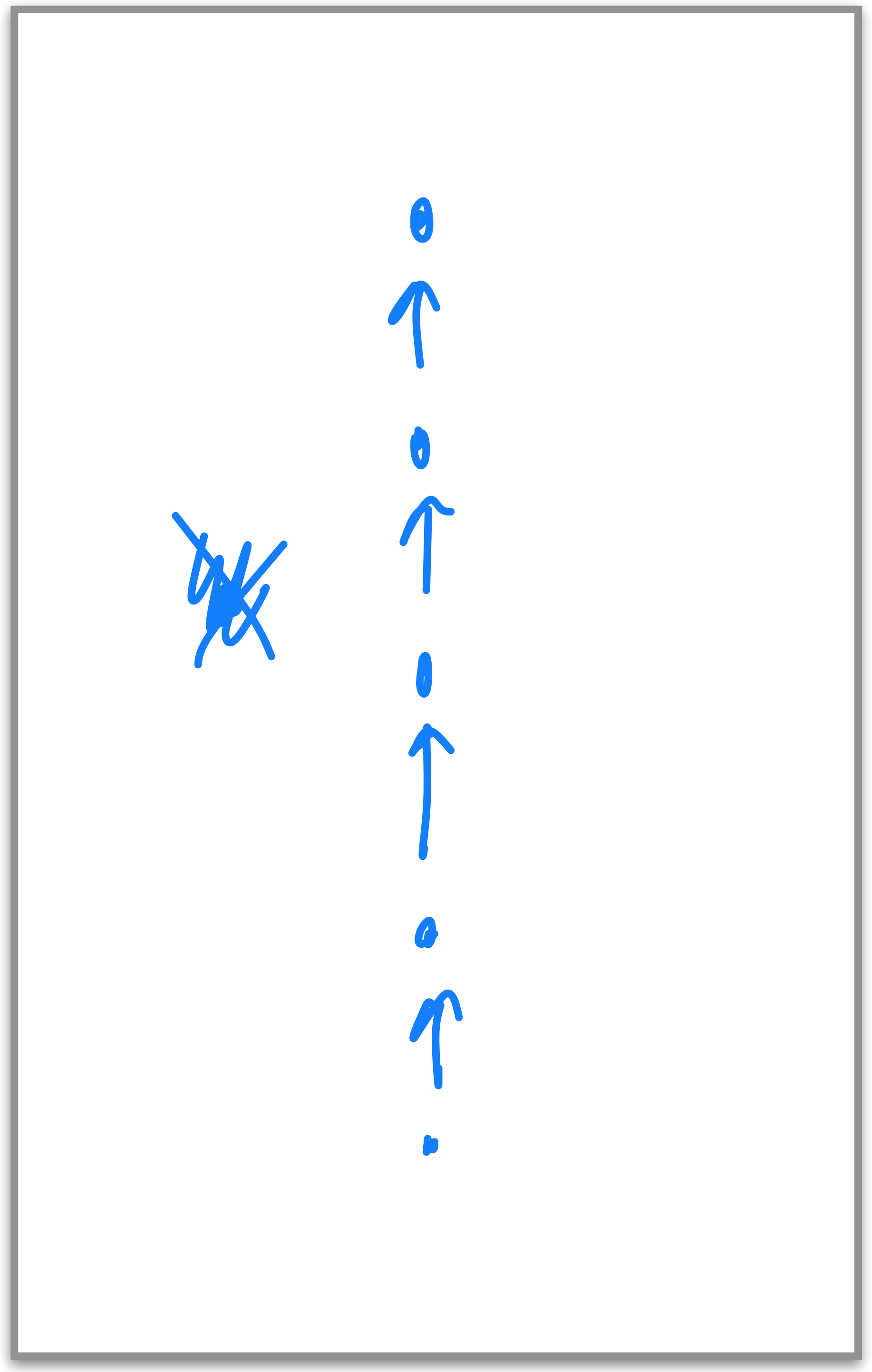
pre-order



partial order



total order



There are more general ways to describe "preferences"

- Hierarchy of order types:
  - **Quasitransitive** relation
  - **Interval order**
  - **Semi-order**
  - Pre-order
  - Partial order
  - Total order

more specific

Example of semi-order preferences:

I am indifferent between 10 and 11,
and indifferent between 11 and 12,
but I prefer 10 to 12.

Pre-order as a category

- **Pre-order:** a relation that is **transitive** and **reflexive**.

reflexive

$$\frac{\top}{aRa}$$

transitive

$$\frac{aRb \quad bRc}{aRc}$$

- Axioms for a category:

1. Existence of identities

$$\frac{\top}{\mathrm{Id}_A : A \to A}$$

2. Composition operation

$$\frac{f : A \to B \qquad g : B \to C}{f ; g : A \to C}$$

3. Unitality

4. Associativity of composition

Definition of pre-order as a category

- Given a preordered set, we can define a category as follows:
  - The objects are the elements of the sets
  - There is a morphism between a and b iff $a \leq b$:

$$\frac{* : a \to b}{a \leq b}$$

$$\mathrm{Hom}(a, b) = \begin{cases} \{*\} & \text{iff} \quad x \leq y \\ \emptyset & \text{otherwise} \end{cases}$$

$$*;* = *$$

$$a, b, c$$

$$a \leq b$$

$$c \leq b$$

$$\begin{tikzcd} & b & \\ a \arrow[ur, "*"] & & c \arrow[ul, "*"'] \end{tikzcd}$$

The skeleton of a pre-order is a poset

- Let's look at the following equivalence relation:

$$\frac{x \sim y}{(x \leq y) \wedge (y \leq x)}$$

- The **skeleton of the pre-order** is a new pre-order where the elements are the equivalence classes of ~:

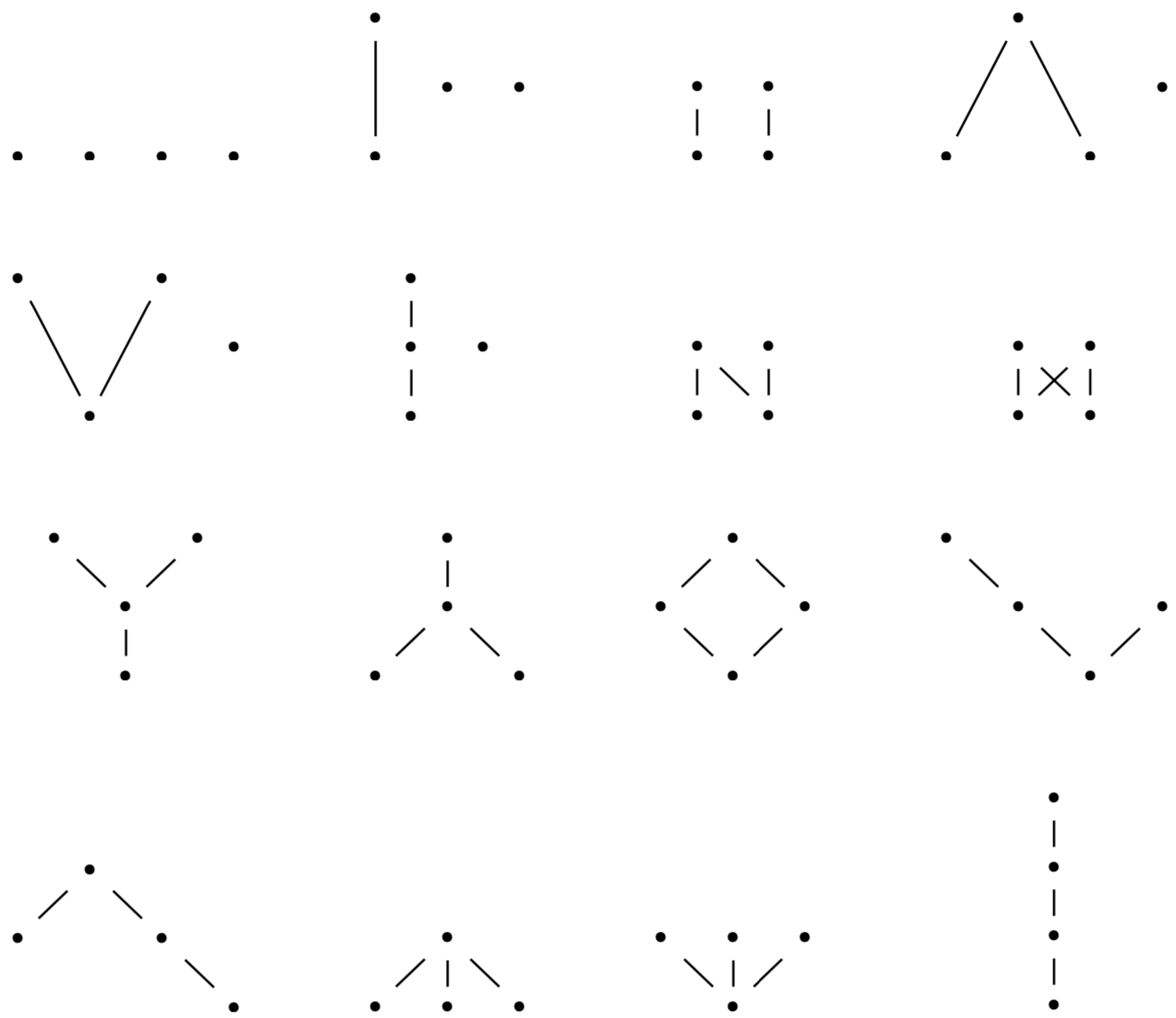
- Exercise: **the skeleton of a pre-order is a partial order.**

Hasse Diagrams

- **Hasse Diagrams** are an **economical way to draw partial orders.**

Posets

- All posets on a set with 4 elements (up to relabeling of the points).



Top and bottom

- The **Top** of a partial order is the element that dominates all others.
- The **Bottom** of a partial order is an element that is dominated by all others.

Example: set-based filtering

- Here's an example of **set-based filtering** ("filtering": online inference).

- **Scenario**:
  - Suppose we want to track the value of a quantity x in the interval [0, 100].
  - We don't know anything about x a priori.
  - We have sensors that periodically measure the quantity with some variable precision.
    $$x_t \in [l_t, u_t]$$
  - The quantity fluctuates randomly:
    $\dot{x}_t \in [-1, +1]$ *(except at boundaries)*

- When dealing with inference, these are the basic questions:
  - What are the possible **information states**?
  - Is it **possible to represent** the information states exactly?
    Do I need an approximation?
  - What is the **prior information**?
  - What are the **update rules**?

Example: set-based filtering

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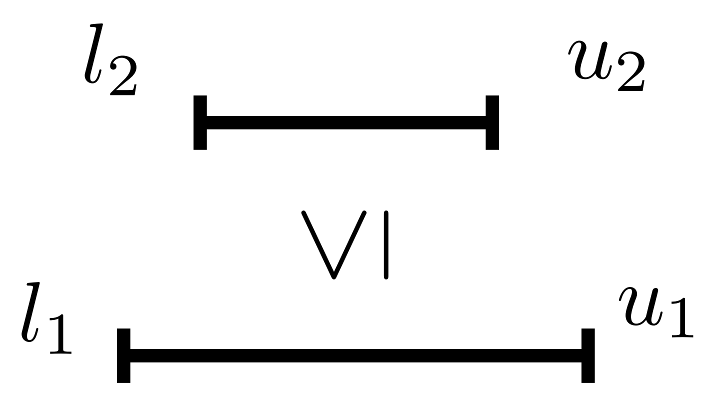
Poset of intervals

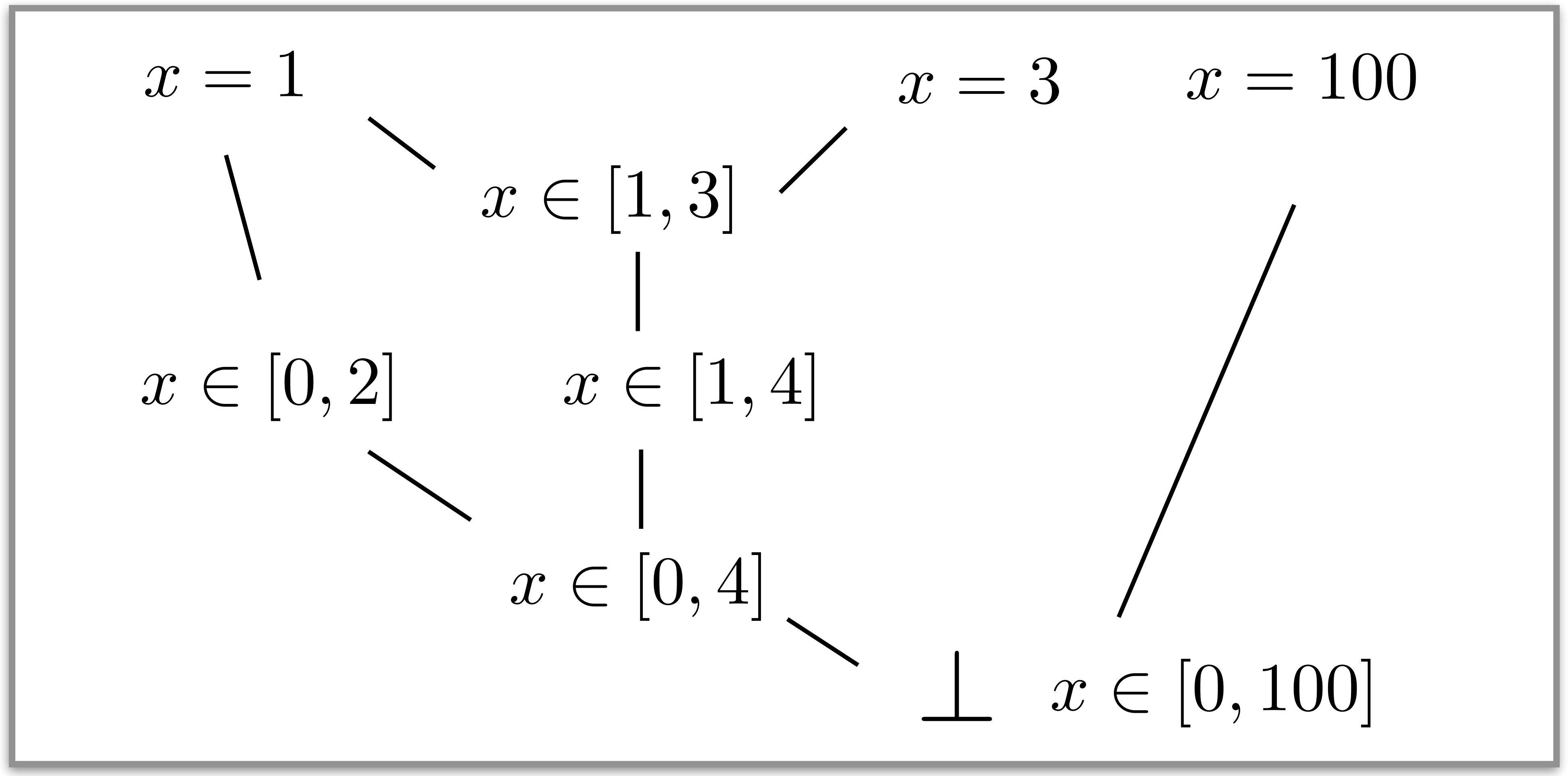
- Given a **poset P**, you can define a **poset of intervals on P.**
- An **interval** is an ordered pair of elements of P.

$$\langle l, u \rangle \quad \text{such that} \quad l \leq u$$

- We order the intervals by inclusion.
  - You could choose either directions
  - Here we want the semantics as: "*a* ≤ *b* if *a* has less information than *b*."

$$\frac{\langle l_1, u_1 \rangle \leq_{\text{Int } P} \langle l_2, u_2 \rangle}{(l_1 \leq_P l_2) \wedge (u_2 \leq_P u_1)}$$





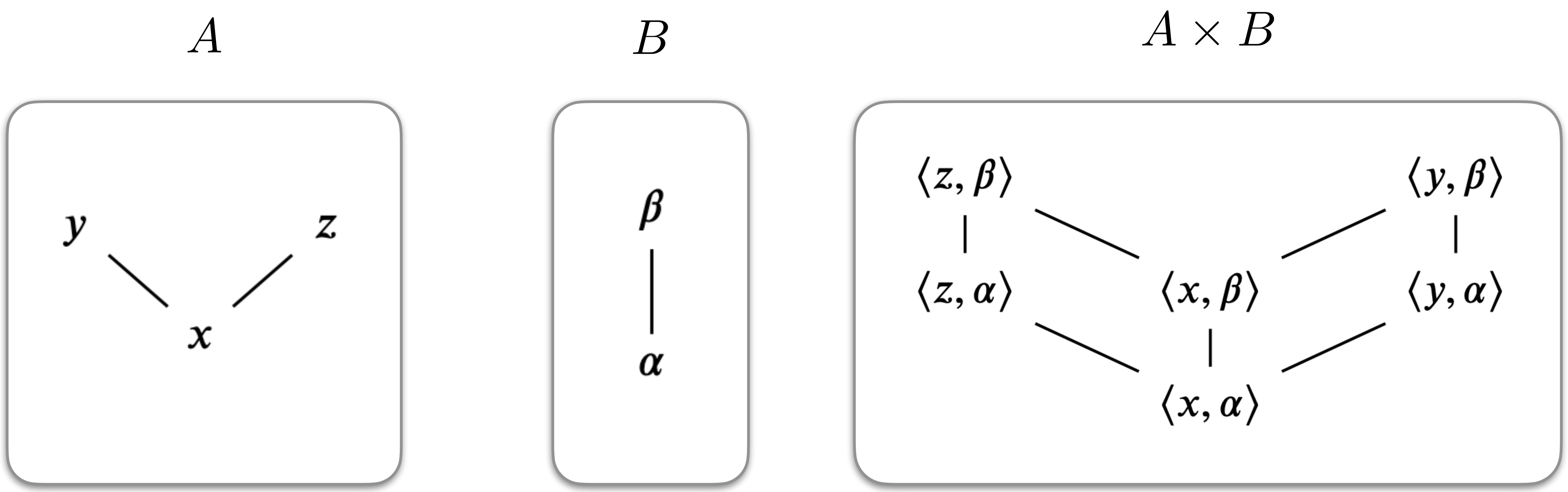
observations reduce uncertainty

time passed increases uncertainty

Product of posets

- Given two sets A and B, we have defined the product $A \times B$.
- If A and B are posets, we can give $A \times B$ the structure of a poset.

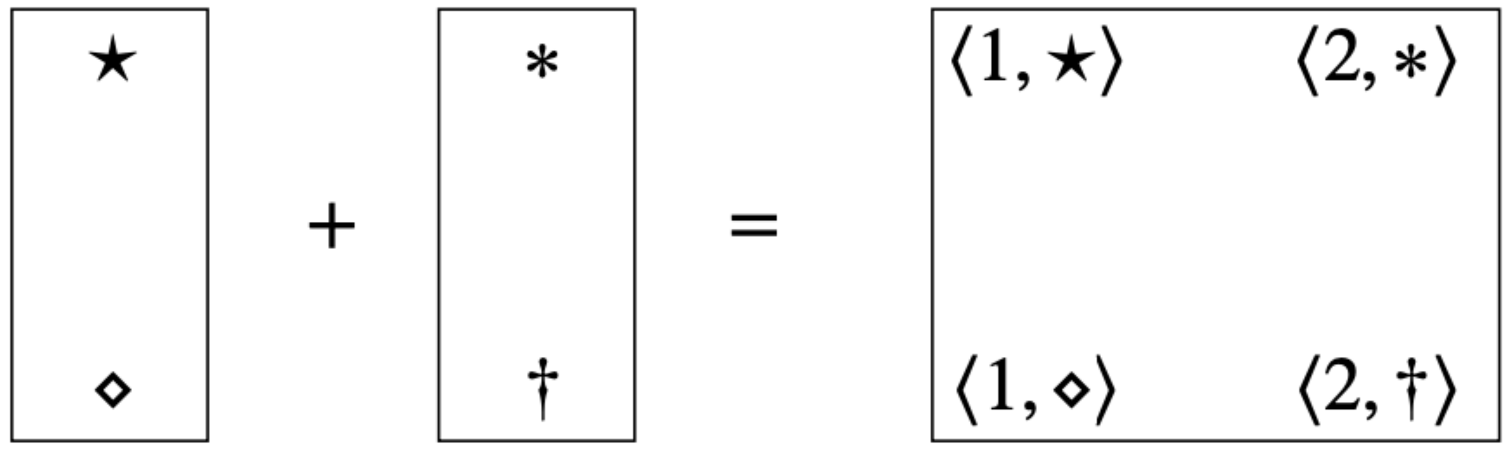
$$\frac{\langle a_1, b_1 \rangle \leq_{A \times B} \langle a_2, b_2 \rangle}{(a_1 \leq_A a_2) \wedge (b_1 \leq_B b_2)}$$



Disjoint union of posets

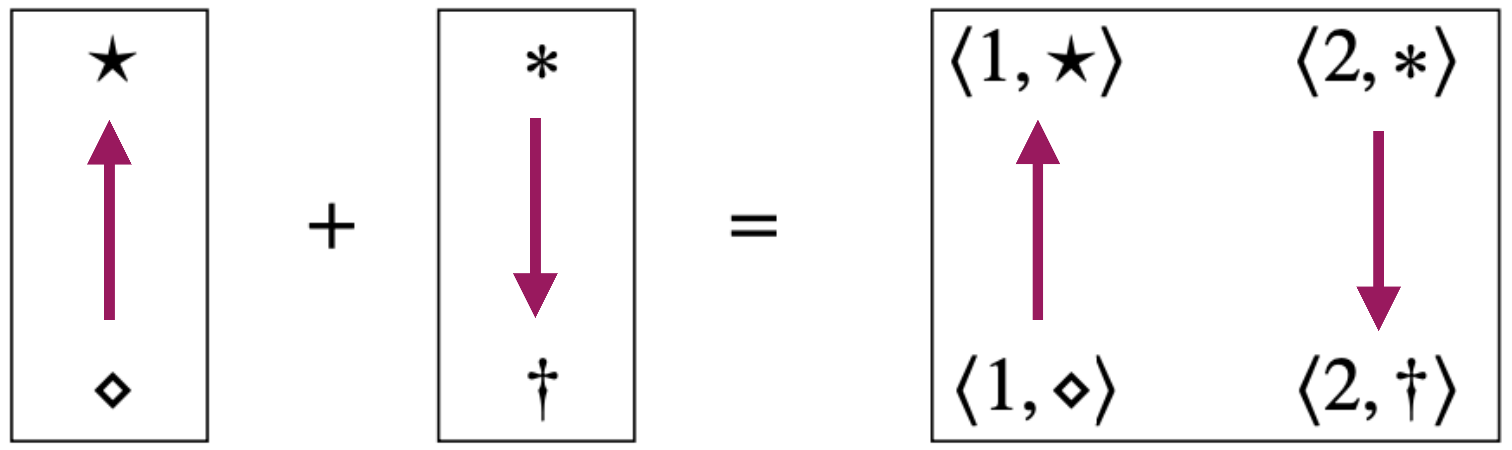
- Given two sets A and B, we have defined the disjoint union $A + B$.

$$A + B = \{\langle 1, a\rangle \mid a \in A\} \cup \{\langle 2, b\rangle \mid b \in B\}$$

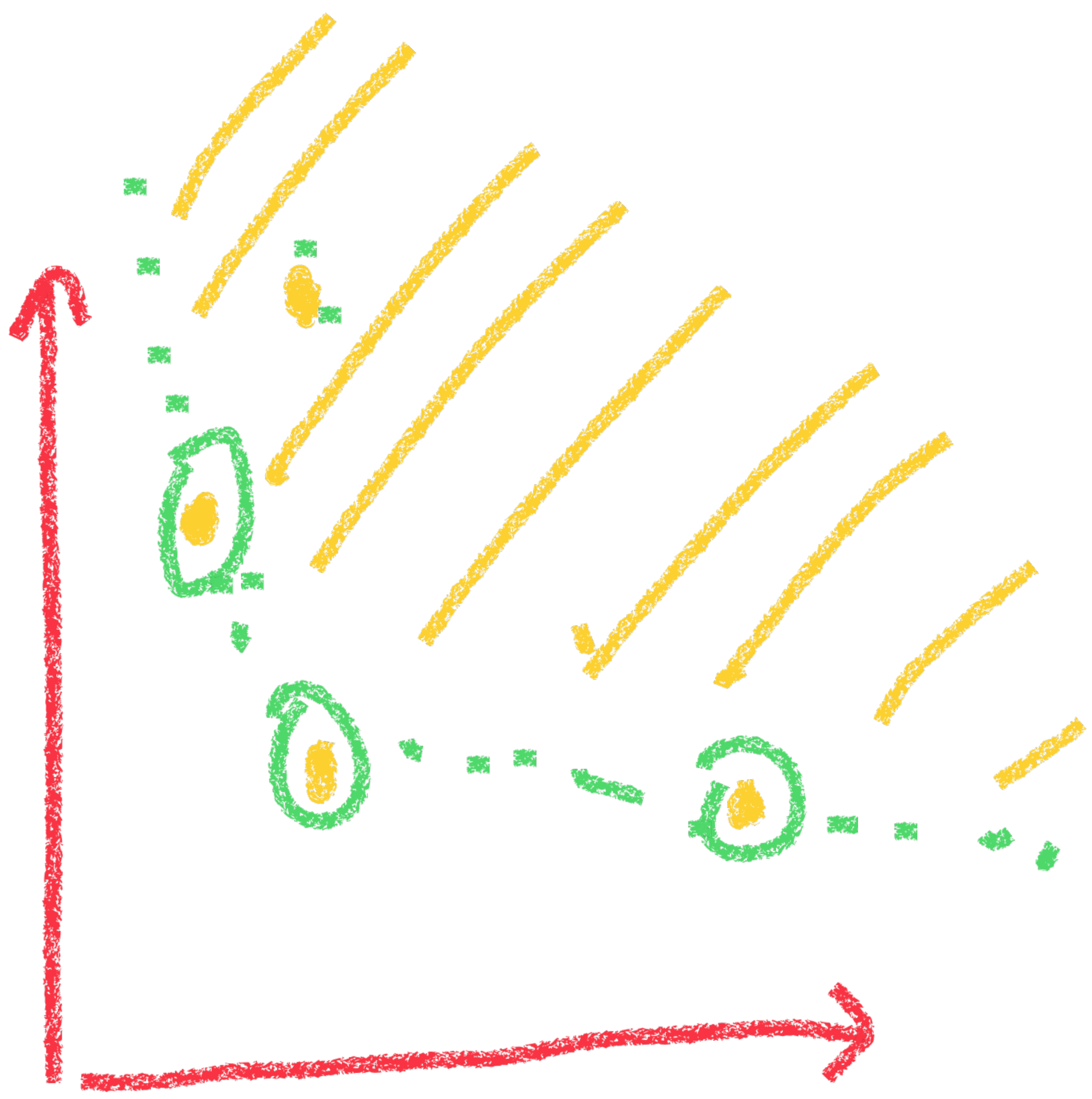


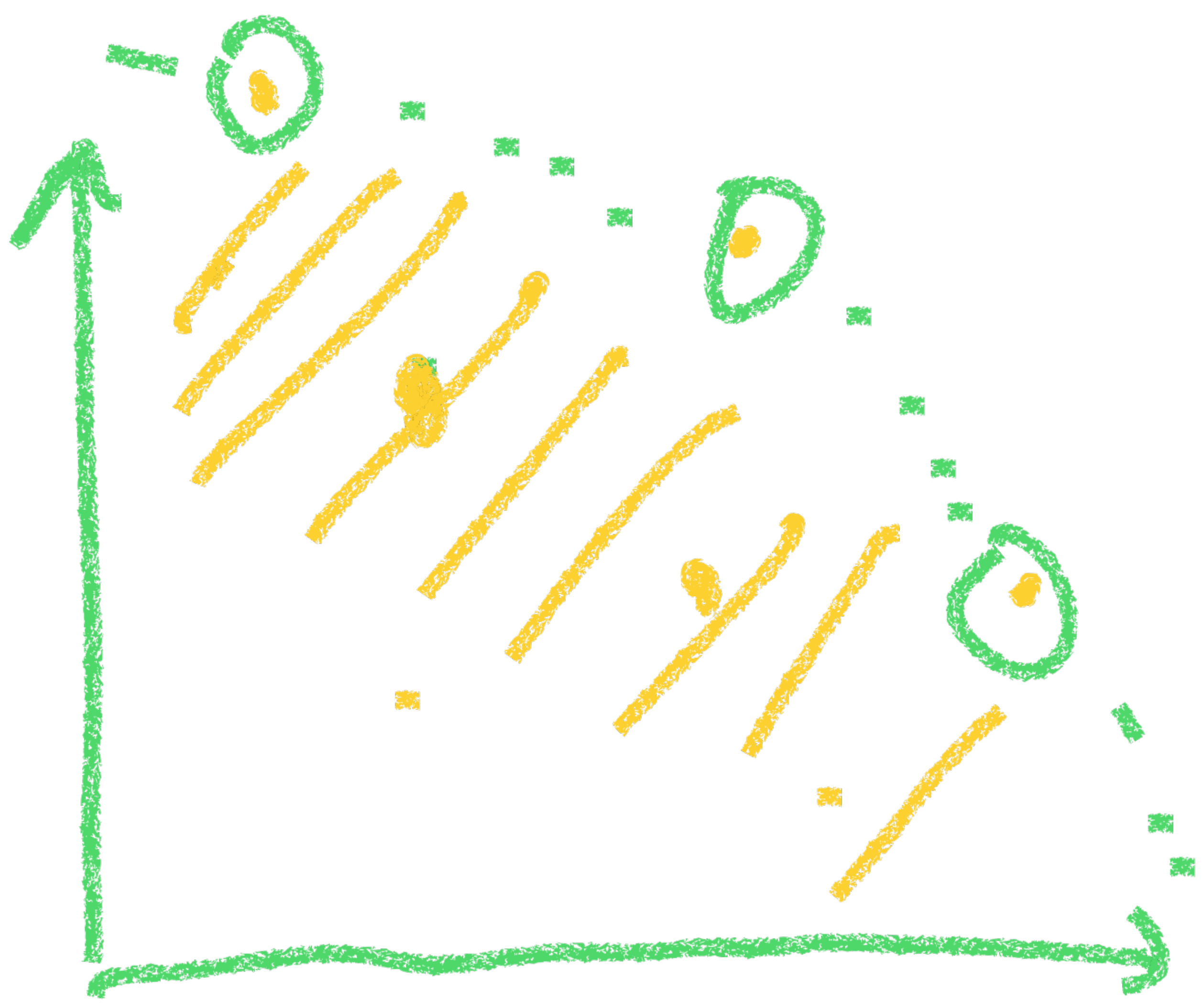
- If A and B are posets, we can give $A + B$ the structure of a poset.

$$\begin{aligned}
\leq_{A+B} &\colon (A + B) \times (A + B) \to \mathrm{Bool} \\
&\langle 1, a_1\rangle, \langle 1, a_2\rangle \mapsto (a_1 \leq_A a_2) \\
&\langle 2, b\rangle, \langle 1, a\rangle \mapsto \bot \\
&\langle 1, a\rangle, \langle 2, b\rangle \mapsto \bot \\
&\langle 2, b_1\rangle, \langle 2, b_2\rangle \mapsto (b_1 \leq_B b_2)
\end{aligned}$$



Questions from you

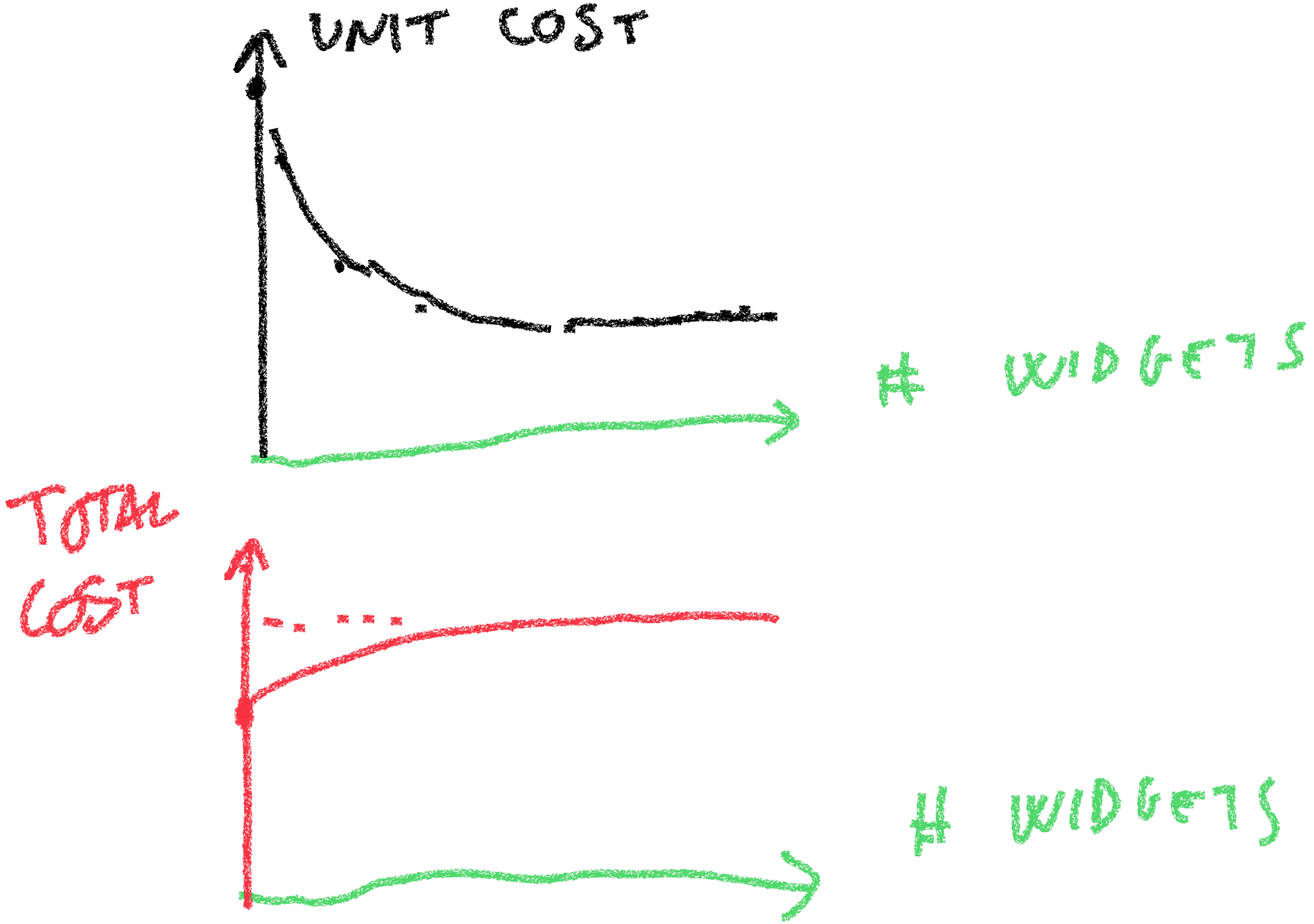




1. READ-ONLY + (COMMENTS)
   BOOK
   ON OVERLEAF

2. WIKI (THANKS TO DAVID T.)

3. BRENDAN FONG


UNIT COST
WIDGETS
TOTAL COST
WIDGETS