**Definition** (Operad)

An *operad* is defined by:

<u>Constituents</u>

1. *Objects:* A collection $\mathrm{Ob}_{\mathcal{O}}$;

2. *Morphisms:* Let $n \in \mathbb{N}$. For each finite string $[X_1, \ldots, X_n]$ of objects and each object $Y$, one specifies a set $\mathrm{Hom}_{\mathcal{O}}([X_1, \ldots, X_n]; Y)$, elements of which are morphisms $[X_1, \ldots, X_n] \to Y$;

3. *Identity morphisms:* For each object $X$, a morphism $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{O}}([X_1, \ldots, X_n]; Y)$;

4. *Composition operations:*

$$\mathrm{Hom}_{\mathcal{O}}\left([X_1^1, \ldots, X_{n_1}^1]; Y_1\right) \times \ldots \times \mathrm{Hom}_{\mathcal{O}}\left([X_1^m, \ldots, X_{n_m}^m]; Y_m\right) \times \mathrm{Hom}_{\mathcal{O}}([Y_1, \ldots, Y_m]; Z) \to \mathrm{Hom}_{\mathcal{O}}\left([X_1^1, \ldots, X_{n_m}^m]; Z\right)$$

$$\langle f_1, \ldots, f_m, g \rangle \mapsto [f_1, \ldots, f_m] \mathbin{\fatsemi} g.$$

<u>Conditions</u>

1. *Associativity:*

$$[[f_1^1, \ldots, f_{n_1}^1] \mathbin{\fatsemi} g_1, [f_1^2, \ldots, f_{n_2}^2] \mathbin{\fatsemi} g_2, \ldots, [f_1^m, \ldots, f_{n_m}^m] \mathbin{\fatsemi} g_m] \mathbin{\fatsemi} h = [f_1^1, \ldots, f_{n_m}^m] \mathbin{\fatsemi} ([g_1, \ldots, g_m] \mathbin{\fatsemi} h).$$

2. *Unitality:*

$$[\mathrm{Id}_{X_1}, \ldots, \mathrm{Id}_{X_n}] \mathbin{\fatsemi} f = f = f \mathbin{\fatsemi} \mathrm{Id}_Y, \quad \forall f : [X_1, \ldots, X_n] \to Y.$$