

Definition (Disjoint union category)

Given two categories \mathbf{C} and \mathbf{D} , their *disjoint union* $\mathbf{C} + \mathbf{D}$ is the category specified as follows:

1. *Objects*: Objects are elements of $\mathbf{Ob}_{\mathbf{C}} + \mathbf{Ob}_{\mathbf{D}}$; that is, objects are tuples of the form $\langle X, i \rangle$, with $i = 1$ or $i = 2$, depending on whether $X \in \mathbf{Ob}_{\mathbf{C}}$ or $X \in \mathbf{Ob}_{\mathbf{D}}$.
2. *Morphisms*: Given objects $\langle X, i \rangle, \langle Y, j \rangle \in \mathbf{Ob}_{\mathbf{C} + \mathbf{D}}$,

$$\mathbf{Hom}_{\mathbf{C} + \mathbf{D}}(\langle X, i \rangle; \langle Y, j \rangle) := \begin{cases} \mathbf{Hom}_{\mathbf{C}}(X; Y) & \text{if } i = j = 1, \\ \mathbf{Hom}_{\mathbf{D}}(X; Y) & \text{if } i = j = 2, \\ \emptyset & \text{else.} \end{cases}$$

3. *Identity morphisms*: Given $f : X \rightarrow Y \in \mathbf{Hom}_{\mathbf{C} + \mathbf{D}}(\langle X, i \rangle; \langle Y, i \rangle)$, one has

$$\mathbf{Id}_{\mathbf{C} + \mathbf{D}} := \begin{cases} \mathbf{Id}_{\mathbf{C}} & \text{if } i = 1, \\ \mathbf{Id}_{\mathbf{D}} & \text{if } i = 2. \end{cases}$$

4. *Composition*: Given $f : X \rightarrow Y \in \mathbf{Hom}_{\mathbf{C} + \mathbf{D}}(\langle X, i \rangle; \langle Y, i \rangle)$ and $g : Y \rightarrow Z \in \mathbf{Hom}_{\mathbf{C} + \mathbf{D}}(\langle Y, j \rangle; \langle Z, j \rangle)$, one has

$$f \circ_{\mathbf{C} + \mathbf{D}} g := \begin{cases} f \circ_{\mathbf{C}} g & \text{if } i = j = 1, \\ f \circ_{\mathbf{D}} g & \text{if } i = j = 2, \\ \text{does not exist} & \text{else.} \end{cases}$$