

**Lemma.** The optimal control law for the LQG problem is  $\mathbf{u}_t^\star = -\mathbf{K}\hat{\mathbf{x}}_t = -\mathbf{R}^{-1}\mathbf{B}^*\bar{\mathbf{S}}\hat{\mathbf{x}}_t$ , where  $\hat{\mathbf{x}}_t$  is the unbiased minimum-variance estimate of  $\mathbf{x}_t$  given previous measurements and  $\bar{\mathbf{S}} \in \mathcal{P}^+$  solves the Riccati equation

$$\mathbf{S}\mathbf{A} + \mathbf{A}^*\mathbf{S} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{S} + \mathbf{Q} = \mathbf{0}.$$

The minimum cost  $J^\star$  achieved by the optimal control is:

$$\begin{aligned} J^\star &= \text{Tr}(\bar{\mathbf{S}}\bar{\Sigma}\mathbf{C}^*\mathbf{V}^{-1}\mathbf{C}\bar{\Sigma} + \bar{\Sigma}\mathbf{Q}) \\ &= \text{Tr}(\bar{\Sigma}\bar{\mathbf{S}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\bar{\mathbf{S}} + \bar{\Sigma}\mathbf{W}), \end{aligned}$$

where  $\bar{\Sigma} \in \mathcal{P}^+$  is the solution of the Riccati equation

$$\mathbf{A}\bar{\Sigma} + \bar{\Sigma}\mathbf{A}^* - \bar{\Sigma}\mathbf{C}^*\mathbf{V}^{-1}\mathbf{C}\bar{\Sigma} + \mathbf{W} = \mathbf{0}.$$